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Abstract

The problem of controlling a time-invariant system with parameter uncertainty is considered with incomplete state feedback. The controller is designed by minimaximizing (i) a quadratic performance criterion, (ii) a sensitivity (or loss) criterion, involving the state of the system, the control and the uncertainty vector. The resulting optimal controller is linear and optimal feedback gain matrix must satisfy a set of nonlinear algebraic equations. Some algorithms for algebraic minimax problems are presented.

I. INTRODUCTION

The design of a controller for a linear dynamic multi input-multi output system having parameter uncertainty is considered in this paper using a minimax technique. The problem is posed with the constraints that the controller be linear and require only the available outputs of the system. Control of this class of system with no parameter uncertainty has been treated by Levine and Athans.[1]. Minimax controller design for this class of problems using complete state feedback has been suggested by many authors.[2-6]

In this paper, the problem is treated initially by minimizing with respect to a feedback gain matrix and maximizing with respect to uncertainty, a quadratic performance index involving the system's state, the control and an uncertainty vector. The resulting controller is specified by the gain matrix which in turn must satisfy a set of algebraic nonlinear equations. This design procedure often leads to a pessimistic result either because the uncertainty does not act so perversely as assumed or because the control often makes an effort to reduce the cost where it is high even with perfect knowledge of parameters. To meet this objection, other criterion and in particular, a minimax sensitivity criterion[7] are also examined. The optimal feedback gain matrix for the regret criterion is shown to satisfy a set of nonlinear equations similar to those obtained for the standard criterion. Some recursive algorithms to solve these nonlinear equations and their convergence are discussed.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider an n^{th} order linear system with state vector $\underline{x}(t) \epsilon R_m$ and output vector $y(t) \epsilon R_q$ defined by

$$\dot{\underline{x}} = A_0\underline{x} + B_0\underline{u} + (A-A_0)\underline{x} + (B-B_0)\underline{u} \tag{1}$$

$$\underline{y} = C\underline{x} \tag{2}$$

with a controller

$$u = -Fy = -FC\underline{x} \tag{3}$$

where A_0, B_0 are nominal matrices. Using (3), (1) can be represented as

$$\dot{\underline{x}} = (A_0 - B_0FC)\underline{x} + [(A-A_0)-(B-B_0)FC]\underline{x} = (A_0-B_0FC)\underline{x} + (W-W_0)\underline{x}$$

$$= (A_0-B_0FC)\underline{x} + D\underline{\xi} \tag{4}$$

where $\underline{\xi}$ represents the effect of uncertainty.

Since the uncertainty is assumed to be limited, $\underline{\xi}$ will likewise be constrained. In order to place any restriction on the form of $(W-W_0)$, let $W-W_0 = DGC_1$. D, C_1 are fixed and G contains variable terms. An example is,

$$B=B_0,\ A-A_0= \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ (\alpha_0-\hat{\alpha}_0) & \cdots & (\alpha_{n-1}-\hat{\alpha}_{n-1}) & \end{bmatrix}$$

$$C_1=I,\ G = [\hat{\alpha}_0-\alpha_0,\ \hat{\alpha}_1-\alpha_1\ \ldots\ \hat{\alpha}_{n-1}-\alpha_{n-1}],\ D = [0,0,\ldots 1]^T$$

Thus the uncertainty vector is specified as

$$\underline{\xi} = GC_1\underline{x} \tag{5}$$

where G is the gain matrix associated with the uncertainty vector and C_1 has rank n or less. Both C, C_1 are assumed to have maximum rank, i.e. rank equal to number of rows.

Substitution of (5) in (4) gives

$$\dot{\underline{x}} = (A_0-B_0FC + DGC_1)\underline{x}. \tag{6}$$

In order to achieve a design through optimization, the feedback matrices F and G will initially be chosen to minimize and maximize, respectively, the performance criterion

$$J(F,G)= \left[\frac{1}{2}\int_0^\infty [\underline{x}^TQ\underline{x} + \underline{u}^TR\underline{u} - \underline{\xi}^TL\underline{\xi}]\ dt\right]_{\underline{u}=FC\underline{x},\ \underline{\xi}=GC\underline{x}} = J_0(\underline{u},\underline{\xi})\Big|_{\underline{u}=FC\underline{x},\ \underline{\xi}=GC_1\underline{x}}$$

$$= \frac{1}{2}\int_0^\infty [\underline{x}^T[Q + C^TF^TRFC - C_1^TG^TLGC_1]\ \underline{x}\ dt; \tag{7}$$

i.e.

(i) Find F and G such that

$$\min_F \max_G J(F,G) = \max_G \min_F J(F,G) \tag{8a}$$

We shall also consider the following minimax procedures to obtain optimal gain matrix F.

(ii) $\min_F J_0(u,\xi^*)\Big|_{\underline{u}=FC\underline{x}}$

where $\underline{\xi}^*$ is obtained from $\max_{\underline{\xi}^*} [\min_{\underline{u}^*} J_0(\underline{u}^*,0) - \frac{1}{2}\int_0^\infty \underline{\xi}^T L \underline{\xi}\, dt]$ (8b)

$\underline{u}^*$ minimizes J_0 assuming G=0.

(iii) $\min_F \max_G [J(F,G) - J_1^*(G)]$ (8c)

where $J_1^*(G) = \min_{\underline{u}} [J_0(\underline{u},\underline{\xi})]_{\underline{\xi}=GC_1\underline{x}}$

(iv) $\min_F \max_G [J(F,G) - J_2^*(G)]$ (8d)

where $J_2^*(G) = \min_F [J(F,G)]$ given G.

Criterion (ii) is less pessimistic in the sense that $\underline{\xi}$ is given the first play and in making its play assumes that $\underline{u}(\underline{x})$ is obtained by an optimal full state design (with $\underline{\xi}=0$) for the nominal plant. Matrix F is then chosen to minimize the criterion based on the announced strategy of $\underline{\xi}$.

In criterion (iii), (iv) the best control with perfect parameter information, i.e. $\underline{\xi}$ known, is obtained with full state feedback and output feedback respectively. Matrices F,G then minimize and maximize respectively the difference between the actual cost and cost with perfect parameter information.

III. MINIMAX PERFORMANCE CONTROL

(i) Direct Conflict of Interest

Let the Saddle point be defined by the following inequality

$$J(F^*,G) \le J(F^*,G^*) \le J(F,G^*). \tag{9}$$

It is clear from (6) and (8) that J is determined by the initial state $\underline{x}(t_0)$ as well as matrices F and G. That is,

$$J = J(F,G,\underline{x}(t_0)). \tag{10}$$

In order to make the optimum F and G independent of $\underline{x}(t_0)$, $\underline{x}(t_0)$ can be treated as a random vector in which case J may be replaced by

$$\hat{J}(F,G) = E[J(F,G,x(t_0)]. \tag{11}$$

E $(\cdot)$ denotes expectation with respect to $\underline{x}(t_0)$. The necessary condition that F and G should minimize and maximize $\hat{J}(F,G)$, respectively, requires [8]

$$\frac{\partial \hat{J}}{\partial F} = \frac{\partial}{\partial F} E[J(F,G,\underline{x}(t_0))] = E[\frac{\partial J}{\partial F}(F,G,\underline{x}(t_0))]=0 \tag{12a}$$

$$\frac{\partial \hat{J}}{\partial G} = \frac{\partial}{\partial G} E[J(F,G,\underline{x}(t_0))] = E[\frac{\partial J}{\partial G}(F,G,\underline{x}(t_0))]=0 \tag{12b}$$

The interchange of order of expectation (i.e. integration) and differentiation is critical here and is valid under rather general conditions.[9]

The partial derivatives of (12) will be evaluated by the application of the following well-known Lemma:

<u>Lemma 1</u>

If

$$J=J(\underline{x}(t_0)) = W(\underline{x}(t_0)) + \int_{t_0}^{t_f} L(\underline{x},t) \; dt$$

where

$\dot{\underline{x}} = f(\underline{x},t)$ and $W(\underline{x}(t_0))$ is the penalty on the initial states $\underline{x}(t_0)$

then

$$\frac{\partial J}{\partial x_i(t_0)} = \lambda_i(t_0) + \frac{\partial W}{\partial x_i}(\underline{x}(t_0))$$

where

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -\frac{\partial}{\partial \underline{x}} \; [L + \underline{\lambda}^T \underline{f}], \; \underline{\lambda}(t_f) = 0$$

This Lemma follows from the variational calculus where the first variation of J with respect to $\underline{x}(t_0)$ is $[\underline{\lambda}(t_0) + \partial W/\partial \underline{x}]^T \delta \underline{x}(t_0)$. See for example pp. 48-49 of [10].

In order to apply the lemma, the elements of F and G are treated as additional "states" which satisfy

$$\dot{F} = 0, \; \dot{G} = 0. \tag{13}$$

Vector multiplier $\underline{\lambda}_x$ will be used for the regular state constraint (6) and matrix multipliers $\Lambda_F(t)$ and $\Lambda_G(t)$ will be used for matrices F and G respectively. It is to be noted that the Hamiltonian H will be independent of $\Lambda_F(t)$ and $\Lambda_G(t)$ due to (13). Thus the Hamiltonian H for (6), (7) is

$$\begin{aligned} H &= \underline{\lambda}_x^T[(A_0-B_0FC_0 + DGC_1)\underline{x}] + \frac{1}{2}\underline{x}^T [Q + C^TF^TRFC - C_1^TG^TLGC_1]\underline{x} \\ &= \mathrm{Tr}[(A_0-B_0FC_0 + DGC_1)\underline{x}\,\underline{\lambda}_x^T + \frac{1}{2}(Q + C^TF^TRFC - C_1^TG^TLGC_1)\underline{xx}^T]. \end{aligned} \tag{14}$$

Tr denotes the trace and

$$\dot{\underline{\lambda}}_x = -\frac{\partial H}{\partial \underline{x}} = -(A_0-B_0FC + DGC_1)^T\underline{\lambda}_x - (Q + C^TF^TRFC-C_1^TG^TLGC)\underline{x}, \; \underline{\lambda}_x(t_f)=0; \tag{15}$$

$$\dot{\Lambda}_F(t) = -\frac{\partial H}{\partial F} = -RFC\underline{xx}^TC^T + B_0^T\underline{\lambda_x x}^TC^T, \quad \Lambda_F(t_f) = 0; \tag{16}$$

$$\dot{\Lambda}_G(t) = -\frac{\partial H}{\partial G} = -LGC_1\underline{xx}^TC_1^T - D^T\underline{\lambda_x x}^TC_1^T, \qquad \Lambda_G(t_f) = 0. \tag{17}$$

According to the lemma, the necessary condition (12), and integrated forms of (16) and (17), we obtain

$$0 = E[\frac{\partial J}{\partial F}] = E[\Lambda_F(t_0)] = E\int_{t_0}^{t_f} [RFC\underline{xx}^TC^T - B_0^T\underline{\lambda_x x}^TC^T]\, dt \tag{18}$$

$$0 = E[\frac{\partial J}{\partial G}] = E[\Lambda_G(t_0)] = -E\int_{t_0}^{t_f} [LGC_1\underline{xx}^TC_1^T - D^T\underline{\lambda_x x}^TC_1^T]\, dt \tag{19}$$

Thus if R and L are constants, (18) and (19) yield

$$F = R^{-1} \int_{t_0}^{t_f} B_0^T\, E[\underline{\lambda}_x \underline{x}^T] C^T\, dt \quad \left[\int_{t_0}^{t_f} C\, E[\underline{x}\underline{x}^T] C^T\, dt \right]^{-1} \tag{20}$$

$$G = L^{-1} \int_{t_0}^{t_f} D^T\, E[\underline{\lambda}_x x^T]\; C_1^T\, dt \quad \left[\int_{t_0}^{t_f} C_1\, E[\underline{x}\underline{x}^T] C_1^T\, dt \right]^{-1} \tag{21}$$

(20) and (21) can now be simplified. If $\underline{\lambda}_x = K(t)\underline{x}$ is assumed, then (6) and (15) give

$$-\dot{K} = A_*^T K + K A_* + Q + C^T F^T R F C - C_1^T G^T L G C_1, \qquad K(t_f) = 0 \tag{22}$$

$$\text{or } K(t) = \int_t^{t_f} \phi_*^T(\tau,t)\; [Q + C^T F^T R F C - C_1^T G^T L G C_1]\; \phi_*(\tau,t)\; d\tau \tag{23}$$

where ϕ_* is the transition matrix corresponding to $A_* \triangleq (A_0 - B_0 F C + D G C_1)$. Limiting attention to the time invariant case (Q, A_0, B_0, C, D, constant) with $t_f = \infty$, $t_0 = 0$, equation (20)-(23) yield

$$F = R^{-1} B_0^T K M C^T [C M C^T]^{-1} \tag{24}$$

$$G = L^{-1} D^T K M C_1^T\; [C_1 M C_1^T]^{-1} \tag{25}$$

where

$$K = \int_t^{\infty} e^{A_*^T(\tau - t)} (Q + C^T F^T R F C - C_1^T G^T L G C_1)\; e^{A_*(\tau - t)}\; d\tau$$

$$= \int_0^{\infty} e^{A_*^T \sigma}\; (Q + C^T F^T R F C - C_1^T G^T L G C_1)\; e^{A_* \sigma}\; d\sigma\,, \tag{26a}$$

or

$$K(A_0 - B_0 F C + D G C_1) + (A_0 - B_0 F C + D G C_1)^T K + Q + C^T F^T R F C - C_1^T G^T L G C_1 = 0; \tag{26b}$$

and

$$M \triangleq \int_0^\infty E[\underline{x}\underline{x}^T]\, dt = \int_0^\infty e^{A_* t}\, E[\underline{x}(t_0)\underline{x}^T(t_0)]\, e^{A_*^T t}\, dt, \tag{27a}$$

or

$$(A_0 - B_0FC + DGC_1)\, M + M(A_0 - B_0FC + DGC_1)^T + E[\underline{x}(t_0)\, \underline{x}^T(t_0)] = 0 \tag{27b}$$

$[CMC^T]^{-1}$ and $[C_1MC_1^T]^{-1}$ exist because C, C_1 have maximum rank and M is positive definite.

If $E[\underline{x}(t_0)] \triangleq \underline{x}_0$, $E[(\underline{x}(t_0)-\underline{x}_0)\ (\underline{x}(t_0)-\underline{x}_0)^T] \triangleq X_0$ (28)

Then

$$E[\underline{x}(t_0)\underline{x}^T(t_0)] = X_0 + \underline{x}_0\underline{x}_0^T \tag{29}$$

is positive-definite for $X_0 \neq 0$. Thus M is a positive definite solution of (27b) if $X_0 \neq 0$, M is positive semi-definite if $X_0 = 0$.

The optimal cost can be seen to satisfy

$$\hat{J} = \frac{1}{2} E\, [\, \underline{x}^T(t_0)\ K\underline{x}(t_0)] = \frac{1}{2} \mathrm{Tr}\ [KE(\underline{x}(t_0)\underline{x}^T(t_0))] = \frac{1}{2} \mathrm{Tr}\ [K(X_0 + \underline{x}_0\underline{x}_0^T)]$$
$$= \frac{1}{2} \mathrm{Tr}\ [K] \text{ when } E\,[\underline{x}(t_0)\underline{x}^T(t_0)] = I\,.$$

Remark 1

It can be easily seen that $\min_F \max_G \hat{J}(F,G) = \max_G \min_F \hat{J}(F,G)$

IV. COMPUTATION OF F* AND G*

The feedback gain matrices F and G are specified by (24) and (25), where K&M are given by (26b) and (27b) respectively. These equations must be solved numerically and the following algorithm similar to [11] can be conveniently used for this purpose.

F_{n+1}, G_{n+1} and M_{n+1} are computed using:

$$F_{n+1} = R^{-1}B_0^T K_{n+1} M_{n+1} C^T (CM_{n+1}C^T)^{-1} \tag{30}$$

$$G_{n+1} = L^{-1}D^T K_{n+1} M_{n+1} C_1^T (C_1 M_{n+1} C_1^T)^{-1} \tag{31}$$

$$(A_0 - B_0 F_{n+1} C + DG_{n+1} C_1) M_{n+1} + M_{n+1} (A_0 - B_0 F_{n+1} C + DG_{n+1} C_1)^T + I = 0 \tag{32}$$

where K_{n+1} is given by the following equation:

$$(A_0 - B_0 F_n C + DG_n C_1)^T K_{n+1} + K_{n+1} (A_0 - B_0 F_n C + DG_n C_1) + Q + C^T F_n^T R F_n C$$

$$- C_1^T G_n^T L G_n C_1 = 0. \tag{33}$$

The iteration starts with an initial guess of F_0 and G_0 such that $(A_0 - B_0 F_0 C + DG_0 C_1)$ is stable and also $(Q + C^T F_0^T R F_0 C - C_1^T G_0^T L G_0 C)$ is positive definite. Thus K_1 is the positive definite solution of (33). With this value of K_1, (30)-(32) can be solved simultaneously to get F_1, G_1, M_1 which, in turn, give new estimate, K_2 and the iteration proceeds.

Lemma 2

If $(B_0 R^{-1} B_0^T - DL^{-1} D^T) \geq 0$ and $C = C_1$, the above algorithm will converge in the sense that $Tr[K_n - K_{n+1}] \geq 0$ for all n.

Proof:

The proof closely follows [11].

M_n can be expressed as

$$M_n = \int_0^\infty \phi_{*n} \phi_{*n}^T \, dt \triangleq \psi_n \psi_n^T \tag{34}$$

If $C = C_1$, then

$$(B_0 F_n - DG_n) C = (B_0 R^{-1} B_0^T - DL^{-1} D^T) K_n M_n C^T (CM_n C^T)^{-1} C \tag{35}$$

and

$$C^T(F_n^T R F_n - G_n^T L G_n)C = C^T(CM_nC^T)^{-1} CM_nK_n(B_0R^{-1}B_0^T - DL^{-1}D^T)\, K_nM_n$$

$$C^T(CM_nC^T)^{-1}C \tag{36}$$

Substitution of (35) and (36) into (32) and (33) yield equations identical to those of [11] for which $Tr[K_n - K_{n+1}] \geq 0$ is proven except that $B_0R^{-1}B_0^T - DL^{-1}D^T$ replaces $B_0R^{-1}B_0^T$. Thus Lemma 2 holds. Proof of convergence under less restrictive assumptions is under study.

V. MINIMAX PERFORMANCE CONTROL

(ii) Indirect Conflict of Interest

In the previous formulation, the feedback matrix F has been chosen in a most favourable way after the uncertainty vector was allowed to take its 'worst'value. This will lead to a very conservative design approach. On the other hand, it may be assumed that nature is not perverse enough to alter its strategy with that of the control. Under this situation of indirect conflict of interest, the previous formulation may be modified as follows.

The game is, as usual, defined by

$$\dot{\underline{x}} = A_0\underline{x} + B_0\underline{u} + D\underline{\xi}. \tag{37}$$

To start with, let us assume $\underline{\xi} \equiv 0$. The optimal control $\underline{u}_0^*$ is obtained by minimizing

$$J = \frac{1}{2}\int_0^\infty (\underline{x}^T Q\underline{x} + \underline{u}^T R\underline{u})\, dt \tag{38}$$

subject to (37).

Thus the resulting control is given by

$$\underline{u}_0^* = -\, R^{-1}B_0^T P_0\underline{x} \tag{39}$$

where P_0 is given by

$$A_0^T P_0 + P_0 A_0 + Q - P_0 B_0 R^{-1} B_0^T P_0 = 0 \tag{40}$$

Substitution of (39) in (37) yields

$$\dot{\underline{x}} = (A_0 - B_0 R^{-1} B_0^T P_0)\underline{x} + D\underline{\xi}. \tag{41}$$

To include the effect of uncertainty at this stage, the performance criterion (38) is modified as

$$J = \frac{1}{2}\int_0^\infty [\underline{x}^T(Q + P_0 B_0 R^{-1} B_0^T P_0)\underline{x} - \underline{\xi}^T L \underline{\xi}] \; dt \tag{42}$$

The 'worst' value of $\underline{\xi}$ is obtained by maximizing (42) with respect to $\underline{\xi}$, subject to (41), and is given by

$$\underline{\xi}_0^* = L^{-1} D^T P_1 \underline{x} \tag{43}$$

where P_1 is the solution of

$$(A_0 - B_0 R^{-1} B_0^T P_0)^T P_1 + P_1(A_0 - B_0 R^{-1} B_0^T P_0) + Q + P_0 B_0 R^{-1} B_0^T P_0 - P_1 D L^{-1} D^T P_1 = 0 \tag{44}$$

Using the estimate of $\underline{\xi}$ as in (43), the original system is reduced to

$$\dot{\underline{x}} = (A_0 + DL^{-1}D^T P_1)\underline{x} + B_0\underline{u} \tag{45}$$

with the controller

$$\underline{u} = -Fy = -FC\underline{x} \tag{46}$$

Now F can be chosen to minimize $E[J] = \frac{1}{2} E \int_0^\infty \underline{x}^T[Q + C^T F^T RFC]\underline{x} \; dt$ subject to (45). The optimal F is given by

$$F = -R^{-1} B_0^T PMC^T (CMC^T)^{-1} \tag{47}$$

where P and M are given by

$$(A_0 + DL^{-1}D^T P_1 - B_0 FC)^T P + P(A_0 + DL^{-1}D^T P_1 - B_0 FC) + Q + C^T F^T RFC = 0 \tag{48}$$

$$(A_0 + DL^{-1}D^T P_1 - B_0 FC)M + M(A_0 + DL^{-1}D^T P_1 - B_0 FC)^T + I = 0. \tag{49}$$

Remark 2

(a) To be more general, u* in (39) and $\underline{\xi}_0^*$ in (43) may be constrained to the form

$$\underline{u}^*_0 = - F^*_0 C\underline{x}, \ \underline{\xi}^*_0 = G^*_0 C_1 \underline{x}$$

(b) It should be noted that this formulation assumes that the existence of matrices P_0, P_1 and F that stabilize $(A_0 + DL^{-1}D^TP_1 - B_0FC)$ and $(A_0 - B_0R^{-1}B_0^TP_0 - DL^{-1}D^TP_1)$. Under this condition, (47)-(49) can be solved using basically the same algorithm as described in Section IV.

VI. MINIMAX SENSITIVITY (OR LOSS) CONTROL

If G as defined in (5) were known, the ideal optimal control would be obtained by minimizing

$$J = \frac{1}{2} \int_0^\infty \{\underline{x}^T(Q - C_1^TG^TLGC_1)\underline{x} + \underline{u}^TR\underline{u}\} \ dt \tag{50}$$

with respect to $\underline{u}$ subject to

$$\dot{\underline{x}} = (A_0 + DGC_1)\underline{x} + B\underline{u}. \tag{51}$$

The resulting 'ideal' optimal control u* is given by

$$\underline{u}^* = -R^{-1}B_0^TP\underline{x}^* \tag{52}$$

where P and $\underline{x}^*$ satisfy respectively

$$P(A_0 + DGC_1) + (A_0 + DGC_1)^TP + Q - PB_0R^{-1}B_0^TP - C_1^TG^TLGC_1 \triangleq N(P,G) = 0 \tag{53}$$

and

$$\dot{\underline{x}}^* = (A_0 + DGC_1 - B_0R^{-1}B_0^TP)\underline{x}^*. \tag{54}$$

The resulting cost,

$$J_1^*(G) = \frac{1}{2} \int_0^\infty \underline{x}^{*T} \ [Q + PB_0R^{-1}B_0^TP - C_1^TG^TLGC_1]\underline{x}^* \ dt \tag{55}$$

is the best that can be achieved with complete state feedback and perfect parameter information (G). Now we consider a performance sensitivity or "regret loss" criterion.

$$S(F,G) = f[J(F,G), \ J_1^* (G)] \tag{56}$$

is a performance sensitivity function if [7,4, 2]

1) $f(\cdot)$ is continuous jointly in its two agruments

2) $f > 0 \rightarrow \quad J(F,G) > J_1^*(G)$

3) $f = 0 \rightarrow \quad J(F,G) = J_1^*(G)$

In this paper, attention will be confined to the following sensitivity function

$$\hat{S}(F,G) = J(F,G) - J_1^*(G) \tag{57}$$

The immediate problem is now to minimize and maximize $\hat{S}$ with respect to F and G respectively, subject to (6), (53) and (54).

(57) modified to include the equality constraint (53) is

$$S = Tr[N(P,G)P_1] + \frac{1}{2} E \int_0^\infty \underline{x}^T[Q + C^T F^T RFC - C_1^T G^T LGC_1]\underline{x} \; dt$$

$$- \frac{1}{2} E \int_0^\infty \underline{x}^{*T} [Q + PBR^{-1}B^T P - C_1^T G^T LGC_1]\underline{x}^* \; dt \tag{58}$$

where P_1 is a matrix Lagrange multiplier.

Thus the problem reduces to minimizing and maximizing (58) w.r.t. F and G, subject to (6) and (54) and

$$\dot{F} = 0, \; \dot{G} = 0, \; \dot{P} = 0 \tag{59}$$

Thus the Hamiltonian H for this case is given by

$$H = \frac{1}{2} Tr[(Q + C^T F^T RFC - C_1^T G^T LGC_1)\underline{xx}^T] - \frac{1}{2} Tr[(Q + PBR^{-1}B_0^T P - C_1^T G^T LGC_1)\underline{x}^* x^{*T}]$$

$$+ Tr[(A - BFC + DGC_1)\underline{x}\underline{\lambda}_x^T] + Tr[(A - BR^{-1}B_0^T P + DGC_1)\underline{x}^*\underline{\lambda}_{x^*}^T] \tag{60}$$

with

$$\dot{\underline{\lambda}}_x = -\frac{\partial H}{\partial \underline{x}} = -(Q + C^TF^TRFC - C_1^TG^TLGC_1)\underline{x} - (A-BFC + DGC_1)^T\underline{\lambda}_x, \quad \underline{\lambda}_x(\infty)=0; \tag{61}$$

$$\dot{\underline{\lambda}}_{x*} = -\frac{\partial H}{\partial \underline{x}^*} = (Q + PBR^{-1}B_0^TP - C_1^TG^TLGC_1)\underline{x}^* - (A-BR^{-1}B_0^TP + DGC_1)^T\underline{\lambda}_{x*} \quad \underline{\lambda}_{x*}(\infty)=0; \tag{62}$$

$$\dot{\Lambda}_F = -\frac{\partial H}{\partial F} = -RFC\underline{x}\underline{x}^TC^T + B_0^T\underline{\lambda}_x\underline{x}^TC^T, \qquad \Lambda_F(\infty)=0; \tag{63}$$

$$\dot{\Lambda}_G = -\frac{\partial H}{\partial G} = LGC_1\underline{x}\underline{x}^TC_1^T - LGC_1\underline{x}^*\underline{x}^{*T}C_1^T - D^T\underline{\lambda}_x x^TC_1 - D^T\underline{\lambda}_{x*}\underline{x}^{*T}G^T, \quad \Lambda_G(\infty)=0; \tag{64}$$

$$\dot{\Lambda}_P = -\frac{\partial H}{\partial P} = BR^{-1}B_0^T(P\underline{x}^*x^{*T} + \underline{\lambda}_{x*}x^{*T}), \qquad \Lambda_P(\infty)=0, \tag{65}$$

Now according to Lemma 1 and the necessary conditions (12), we obtain after integrating (63) — (65)

$$0=E[\Lambda_F(0)] = E\int_0^\infty [RFCxx^TC^T - B_0^T\underline{\lambda}_x x^TC^T]\, dt \tag{66}$$

$$0=E[\Lambda_G(0) + \frac{\partial}{\partial G}Tr(N(P,G)P_1] = D^TP\ (P_1 + P_1^T)\ C_1^T$$

$$- LGE\int_0^\infty C_1xx^TC_1^T\, dt + LGE\int_0^\infty C_1\underline{x}^*\underline{x}^{*T}C_1^T\, dt + D^T\int_0^\infty E(\underline{\lambda}_x\underline{x}^TC_1^T)\, dt$$

$$+ D^T\int_0^\infty E[\underline{\lambda}_{x*}x^{*T}C_1^T]\, dt \tag{67}$$

$$0=E[\Lambda_P(0) + \frac{\partial}{\partial P}Tr(N(P,G)P_1)] = (A_0 + DGC_1 - B_0R^{-1}B_0^TP)(P_1 + P_1^T) -$$

$$B_0R^{-1}B_0^TP\int_0^\infty E(\underline{x}^*\underline{x}^{*T})\, dt - B_0R^{-1}B_0^T\int_0^\infty(\underline{\lambda}_{x*}\underline{x}^{*T})\, dt \tag{68}$$

We claim as usual that

$$\underline{\lambda}_x = \tilde{K}\underline{x},\ \underline{\lambda}_{x*} = \tilde{K}^*\underline{x}^* \tag{69}$$

as may be verified by (61) and (62) provided $\tilde{K}$, $\tilde{K}^*$ satisfy

$$(A_0-B_0FC + DGC_1)^T\tilde{K} + \tilde{K}(A_0-B_0FC + DGC_1) + Q + C^TF^TRFC-C_1^TG^TLGC_1=0 \quad (70)$$

$$(A_0-B_0R^{-1}B_0^TP + DGC_1)^T\tilde{K}^* + \tilde{K}^*(A_0-B_0R^{-1}B_0^TP + DGC_1) - (Q + PBR^{-1}B_0^TP-C_1^TG^TLGC_1)=0 \quad (71)$$

Furthermore

$M \triangleq \int_0^\infty E[\underline{x}\underline{x}^T]\, dt$, $M^* \triangleq \int_0^\infty E[\underline{x}^*\underline{x}^{*T}]\, dt$ are given by

$$(A_0-B_0FC + DGC_1) M + M(A_0-B_0FC + DGC_1)^T + E[\underline{X}(t_0)\underline{x}^T(t_0)]=0 \quad (72)$$

$$(A_0-B_0R^{-1}B_0^TP + DGC_1) M^* + M^*(A_0-B_0R^{-1}B_0^TP + DGC_1)^T + E[\underline{x}^*(t_0)\underline{x}^{*T}(t_0)] = 0 \quad (73)$$

It can be easily seen from (53) and (71) that $\tilde{K}^* = -P$ (74)

Using (69) and (74), (68) reduces to $(P_1 + P_1^T) = 0$ for

$(A_0 + DGC_1 - B_0R^{-1}B_0^TP)$ is stable (75)

Thus (67) and (66) give

$$G = L^{-1}D^T(\tilde{K}M + \tilde{K}^*M^*)\, C_1^T[C_1(M-M^*)C_1^T]^{-1} \quad (76)$$

$$F = R^{-1}B_0^T\tilde{K}MC^T(CMC^T)^{-1} \quad (77)$$

Solution for F requires simultaneous solutions of (70)-(73) with (76)-(77).

Remark 3

It can be readily verified that (a) the optimal cost $S = \frac{1}{2}\, Tr[\tilde{K}+\tilde{K}^*]$, when

$E[\underline{x}(t_0)\underline{x}^T(t_0)] = E[\underline{x}^*(t_0)\underline{x}^{*T}(t_0)]=I$ (78)

(b) $\min_F \max_G \hat{S}(F,G) = \max_G \min_F \hat{S}(F,G)$ (79)

It should be note from (52) that u* was allowed to be linear function of all state variables. A more general and perhaps more realistic formulation would be to constrain u* to the form

$$\underline{u}^* = -F_1^* C \underline{x}^* \quad (80)$$

Now if G were known F_1 would be chosen to minimize

$$J = \frac{1}{2}\int_0^\infty \underline{x}^T [Q - C_1^T G L G C_1 + C^T F_1^T R F_1 C]\underline{x}\, dt \quad (81)$$

$$\text{subject to } \dot{\underline{x}} = (A - B_0 F_1 C + DGC_1)\underline{x} \quad (82)$$

In this case, F_1^* is given by

$$F_1^* = R^{-1} B_0^T K^* M^* C^T (CM^* C^T)^{-1} \quad (83)$$

where K* and M* satisfy

$$N_1(F_1^*, G, K^*) \triangleq (A_0 + DGC_1 - B_0 F_1^* C)^T K^* + K^*(A_0 + DGC_1 - B_0 F_1^* C) + Q$$
$$+ C^T F_1^{*T} R F_1^* C - C_1^T G^T L G C_1 = 0 \quad (84)$$

$$N_2(F_1^*, G, M^*) \triangleq (A_0 + DGC_1 - B_0 F_1^* C) M^* + M^*(A_0 + DGC_1 - B_0 F_1^* C)^T + I = 0 \quad (85)$$

$\underline{x}^*$ satisfies

$$\dot{\underline{x}}^* = (A + DGC_1 - BF_1^* C)\underline{x}^* \quad (86)$$

and

$$J_2^*(G) = \min_{F_1} J \text{ is given by}$$

$$J_2^*(G) = \frac{1}{2}\int_0^\infty \underline{x}^{*T}[Q + C^T F_1^T R F_1 C - C_1^T G^T L G C_1]\underline{x}^* dt. \quad (87)$$

We may define the criterion analogous to (58) as

$$S = J(F,G) - J_2^*(G) = \mathrm{Tr}[N_1(F_1^*, G, K^*)P_1 + N_2(F_1^*, G, M^*)P_2] \quad (88)$$
$$+ \frac{1}{2}E\int_0^\infty \underline{x}^T[Q + C^T F^T R F C - C_1^T G^T L G C_1]\underline{x}dt - \frac{1}{2}E\int_0^\infty \underline{x}^{*T}[Q + C^T F_1^{*T} R F_1^* C - C_1^T G^T L G C_1]\underline{x}^* dt$$

where P_1 and P_2 are matrix Lagrange multipliers.

Now minimizing and maximizing (88) w.r.t. F and G respectively subject to (6) (86), it can be shown in a similar way that F_1^*, K*, M* satisfy (83), (84), (85) respectively and F, G, K, M satisfy

$$F = R^{-1}B_0^T KMC^T(CMC^T)^{-1} \tag{89}$$

$$G = L^{-1}D^T(KM + K^*M^*)\ C_1^T\ [C_1(M-M^*)\ C_1^T]^{-1} \tag{90}$$

$$(A-BFC + DGC_1)^T K + K\ (A-BFC + DGC_1) + Q + C^TF^TRFC - C_1^TG^TLGC_1 = 0 \tag{91}$$

$$(A-BFC + DGC_1)\ M + M\ (A-BFC + DGC_1)^T + I=0 \tag{92}$$

VII. COMPUTATION OF F, F_1^*, G

An algorithm similar to that mentioned earlier can be used to solve the feedback matrices. As before, at iteration n, positive definite matrix K_n and negative definite matrix K_n^* are obtained from:

$$(A-BF_nC + DG_nC)^TK_{n+1} + K_{n+1}(A-BF_nC + DG_nC) + Q + C^TF_n^TRF_nC - C_1^TG_n^TLG_nC_1 = 0 \tag{93}$$

$$(A-BF_{1n}^*C + DG_nC_1)^TK_{n+1}^* + K^*_{n+1}\ (A-BF^*_{1n}C + DG_nC_1)$$

$$-(Q + C^TF_{1n}^{*T}\ RF^*_{1n}C - C_1^TG_n^TLG_nC_1)=0 \tag{94}$$

F_n, F^*_{1n}, G_n, M_n,M^*_n are then obtained by simultaneous solution of (83)-(85) and (89)-(92), with K,K* replaced by K_n^*, K_n^*. The algorithm starts with initial guesses F_0,F_{10}^*, G_0 such that $(A_0-B_0F_0C + DG_0C)$ and $(A_0-B_0F_{10}^*C + DG_0C)$ are stable and also $(Q + C^TF_0^TRF_0C - C_1^TG_0^TLG_0C)$ and $(Q + C^TF_{10}^{*T}RF_{10}^*C-C^TG_0^TLG_0C)$ are positive definite.

VIII. SOME STABILITY BOUNDS IN TERMS OF PARAMETER VARIATION

The perturbed system (1) can be represented as

$$\underline{\dot{x}} = A_0\underline{x} + B_0\underline{u} + (A-A_0)\underline{x} + (B-B_0)\underline{u} \tag{95}$$

$$= [(A_0-B_0FC) + \Delta A + \Delta BFC]\underline{x} \tag{96}$$

where F is given by (24). The following analysis is also true for F, given by (47), (77) and (83).

Define the Liapunov function V(x) as

$$V(\underline{x}) = \frac{1}{2}\underline{x}^T K\underline{x} \tag{97}$$

where K, a positive definite matrix, satisfies (26). The time derivative $\dot{V}(\underline{x})$ of $V(\underline{x})$, evaluated along the trajectory (96), is given by

$$\dot{V}(\underline{x}) = -\frac{1}{2}\underline{x}^T[-(A_0-B_0FC)^T K-K(A_0-B_0FC)-2K\Delta A-2K\Delta BFC]\underline{x} \tag{98}$$

Using (26), (98) reduces to

$$\dot{V}(\underline{x}) = -\frac{1}{2}\underline{x}^T[(Q+C^TF^TRFC-C_1^TG^TLGC_1)-2K\Delta A + 2K(DGC_1K^{-1} + \Delta BFCK^{-1})K]\underline{x} \tag{99}$$

Let the norms of $\underline{x}$ amd matrix A are defined as follows

$$||\underline{x}|| \triangleq (\underline{x}^T\underline{x})^{1/2}$$

$$||A|| \triangleq \sup_{||\underline{x}||=1} ||A\underline{x}|| \text{ so that } ||A|| = \lambda_{max}^{1/2}[A^TA]$$

where $\lambda_{max}(\cdot)$ is the maximum eigenvalue of a symmetric positive definite matrix $(\cdot)$. Restricting terms in the brackett in (99) to be at least p.s.d. to guarantee stability of perturbed system (96), the bounds on ΔA and ΔB can be found as

$$||\Delta A|| \le \frac{\lambda_{min}(Q + C^TF^TRFC-C_1^TG^TLGC_1)}{2||K||} \tag{100}$$

$$||\Delta B|| \le \frac{||DGC_1K^{-1}||}{||FCK^{-1}||} \tag{101}$$

It should be noted that $(Q + C^T F^T RFC - C_1^T G^T LGC_1)$ is at least positive semidefinite under the condition mentioned in Lemma 2.

IX. EXAMPLE

Following example will be considered to illustrate various theoretical formulations discussed earlier.

Let the system be described by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = A\underline{x} + \underline{b}_o u \tag{102}$$

$$y = [0\ 1]\ \underline{x} = c\underline{x} \tag{103}$$

with controller

$$u = -fy = -f\ [0\ 1]\ \underline{x} \tag{104}$$

'a' in (102) is the uncertain parameter. Let the nominal system correspond to the one with a = 0. Thus (102) can be written as

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi$$

$$= A_0\underline{x} + \underline{b}_0 u + \underline{d}\ \xi \tag{105}$$

with ξ constrained to be

$$\xi = gx_2 = gy = gc\underline{x} \tag{106}$$

where g is the gain (i.e. an estimate of the uncertainty) to be determined.

Consider the following performance criterion

$$\hat{J} = \min_f \max_g\ E\ 1/2 \int_0^\infty [\underline{x}^T Q\underline{x} + Ru^2 - L\xi^2]dt \tag{107}$$

with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $R = 1$, $E[\underline{x}(0)\underline{x}^T(0)] = I$

f optimal for the nominal system (i.e. with no parameter uncertainty) is determined to be o.816. f and g for different values of L are obtained through minimax procedures (i), (ii) and (iii) and using algorithms of sections IV and VII. Simultaneous nonlinear algebraic equations, e.g., equations (30) - (32) of minimax procedure (i) and (ii) have been solved at each iteration using a conjugate gradient technique. The computed values of f for different values of L are tabulated for various minimax procedures.

TABLE I

L^{-1} \ f	Minimax Performance Control		Minimax Sensitivity Control
	Criterion (i)	Criterion (ii)	Criterion (iii)
0.1	.878	.914	.824
0.2	.947	.96	.844
0.3	1.03	1.1	1.08
0.5	1.265	1.277	1.354
0.7	1.69	1.71	1.815

To study the effect of uncertainty, J is computed for different values of 'a' using f as tabulated above and

$$J = \frac{1}{2}\ \mathrm{Tr}\ K = \frac{1}{2} E \int_0^{\infty} [\underline{x}^T Q \underline{x} + R f^2 x_2^2] dt \tag{108}$$

where K is the solution of

$$(A-b_0\ fc)^T K+K(A-b_0\ fc)+Q+c^T f^2 c = 0 \tag{109}$$

and are plotted as shown in Figures (1)-(2). In Figure 1, cost J is plotted as a function of the uncertain parameter 'a', using the feedback gain as determined in minimax performance sensitivity criterion (iii), for different values of L. For comparison, we have also plotted the 'optimal' cost as a function of parameter 'a' if it were known. In Figure 2, different design criterion are compared as "a" varies from nominal. It can be seen that the minimax procedure effects the design of f in such a way that the system will operate acceptably over a wider range of parameters than a purely nominal design.

For any particular parameter set, however, the nominal design may be superior. It is also evident from Figure (1)-(2) that the penalty on the uncertainty should be relaxed to accommodate larger parameter variation. For limited parameter variation, different design approaches give nearly identical performance whereas the minimax performance sensitivity control offers better design when the parameter variation is large.

X. CONCLUSION

The problem of controlling a system with parameter uncertainty is treated using only available output feedback. Since the controller is designed with incomplete state feedback, the uncertainty is likewise constrained. To achieve a design via optimization, a quadratic cost function involving the system state, the control and the uncertainty vector, is defined and the optimal feedback matrices relating the control and the uncertainty are chosen to minimize and maximize, respectively, the performance criterion. The resulting controller is linear, the optimal feedback matrix being specified by a set of simultaneous nonlinear equations. The above procedure usually leads to a conservative design. To meet this objection, a sensitivity or loss criterion is defined. Minimaximization of the sensitivity function with respect to feedback matrices yields a linear controller. The optimal feedback matrices must satisfy a set of nonlinear simultaneous algebraic equations. Some algorithms to solve these algebraic minimax problems and their convergence properties are discussed. An example is treated to illustrate the various formulations presented in this paper. It should be noted that the restriction imposed on the control and the uncertainty can be relaxed by generating the required optimal control as initial condition response of a linear dynamical system with suitable order. In particular, it is assumed throughout the paper that the nominal system is stabilizable with output feedback. In order to relax the restriction on the uncertainty vector, a more general dynamical controller as reported in [13] may be examined in this manner. Detailed results on this will be reported in a future paper.

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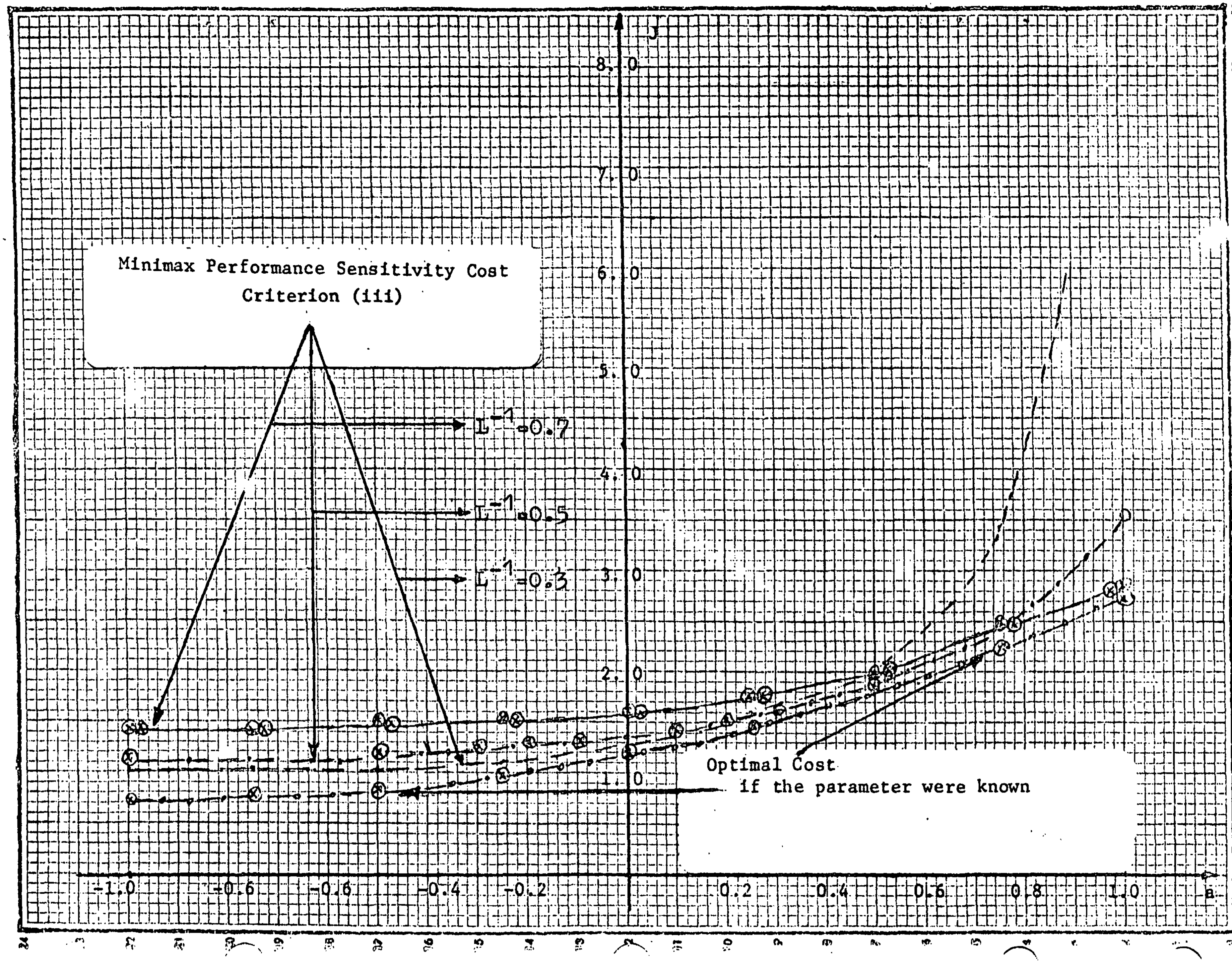
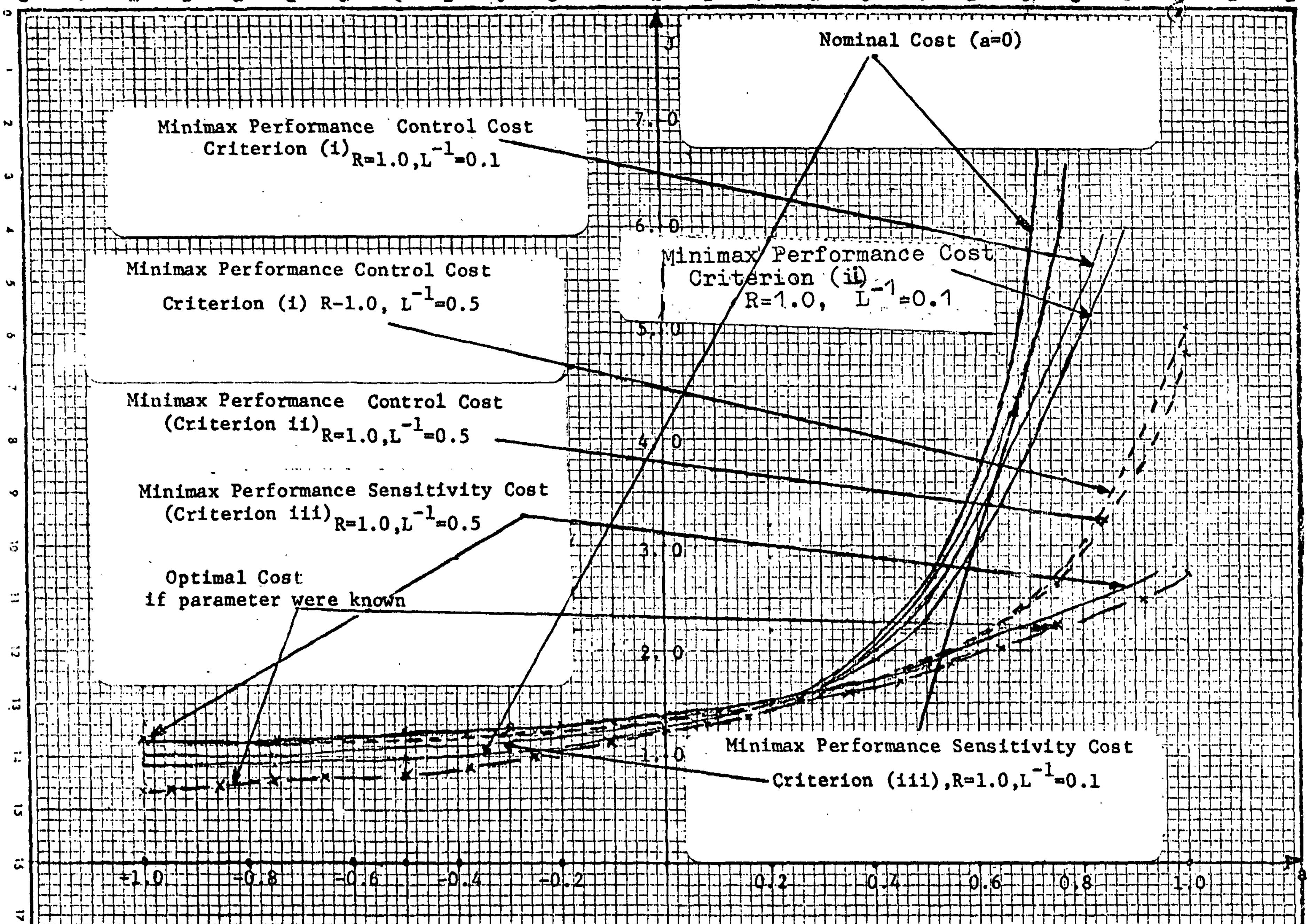

J
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
Minimax Performance Sensitivity Cost
Criterion (iii)
L^{-1}=0.7
L^{-1}=0.5
L^{-1}=0.3
Optimal Cost
if the parameter were known
-1.0
-0.6
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1.0
a

FIGURE-1



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Nominal Cost (a=0)
Minimax Performance Control Cost Criterion (i) R=1.0, L^-1=0.1
Minimax Performance Cost Criterion (ii) R=1.0, L^-1=0.1
Minimax Performance Control Cost Criterion (i) R-1.0, L^-1=0.5
Minimax Performance Control Cost (Criterion ii) R=1.0, L^-1=0.5
Minimax Performance Sensitivity Cost (Criterion iii) R=1.0, L^-1=0.5
Optimal Cost if parameter were known
Minimax Performance Sensitivity Cost Criterion (iii), R=1.0, L^-1=0.1
7.0
6.0
5.0
4.0
3.0
2.0
1.0
-1.0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1.0
a

FIGURE-2