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FINAL REPORT ON A STUDY OF LOW-REYNOLDS-NUMBER NOZZLE FLOWS, INCLUDING RADIAL PRESSURE GRADIENTS

By: William J. Rae

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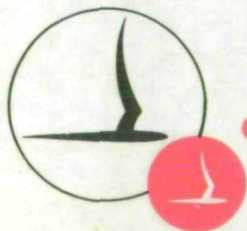
National Aeronautical and Space Administration
Langley Research Center
Hampton, Virginia 23365

FINAL REPORT

Contract No. NAS 1-10806

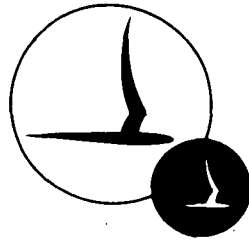
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Prepared by: William J. Rae
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ABSTRACT

An analysis is presented of the laminar, axisymmetric flow in a nozzle, including both axial and radial variations of the pressure. The system of equations derived is believed to contain all of the terms necessary for describing the flow through a relatively sharp throat (i. e. , one for which the longitudinal radius of curvature of the throat is comparable to, or less than, the transverse radius).

A finite-difference approximation of these equations is described, together with a computer program for finding numerical solutions. An instability has been found in the starting solution; a series of attempts to eliminate this instability is described.

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I. INTRODUCTION

Many of the auxiliary propulsion devices used for satellite attitude control and station-keeping require low thrust, typically on the order of millipounds or less. The nozzles used in these devices consequently are small in scale and use low reservoir pressures. Both of these factors tend to make the molecular mean free path significant compared to the nozzle dimensions, and thus the effect of viscosity is felt all through the flow. When this is the case, the conventional inviscid design relations no longer apply.

In an earlier contract (NASw-1668), a theoretical study was made, which produced a computer program capable of calculating the entire flow field, given only the nozzle geometry and certain properties of the propellant.^(1, 2) Calculations made during that theoretical study suggested that there might be a range of Reynolds numbers and nozzle expansion angles for which no supersonic flow would be possible. Instead, the effect of viscous shear was predicted to be strong enough to suppress supersonic flow.

Experiments carried out on a follow-on phase of the original contract^(3, 4) showed that this remarkable phenomenon does in fact occur, and that it is found in precisely the Reynolds number range where it had been predicted. This fundamental discovery provided the explanation for a variety of phenomena which had previously been considered anomalous in the testing of microthrust rockets.

In the experimental phase, the flow field was surveyed by an electron-beam fluorescence probe. This instrument made it possible to measure the distributions of temperature and density in the flow. Hence the complete pressure distribution could be inferred, using the equation of state. Examination of the measured pressures made it possible to evaluate the accuracy of one of the key assumptions of the analysis, namely the neglect of radial pressure gradients. (In the theoretical model, the pressure had been taken to depend only on axial position in the nozzle,

i. e. , to be constant over the cross-section at that position.) The results showed that a radial pressure variation on the order of 15 to 20 percent could occur, at stations some 10 to 20 throat radii downstream of the throat. It seemed probable that these gradients were caused, in large part, by the very sharp throat used (longitudinal radius of curvature equal to 0.5 of the transverse radius). In an effort to evaluate this effect further, the Langley Research Center awarded the present contract, whose main objective was to modify the previous computer program, so as to accommodate radial pressure gradients.

In Section II below, a general discussion is presented of the orders of magnitude of the new terms, which are required to account for the radial pressure gradient. The final equations chosen are given, along with certain conservation laws that must be enforced, and the initial and boundary conditions required.

The numerical methods which were applied to this set of equations are described in Section III. The finite-difference forms used are presented, and the sequence of steps used in the related computer program is discussed.

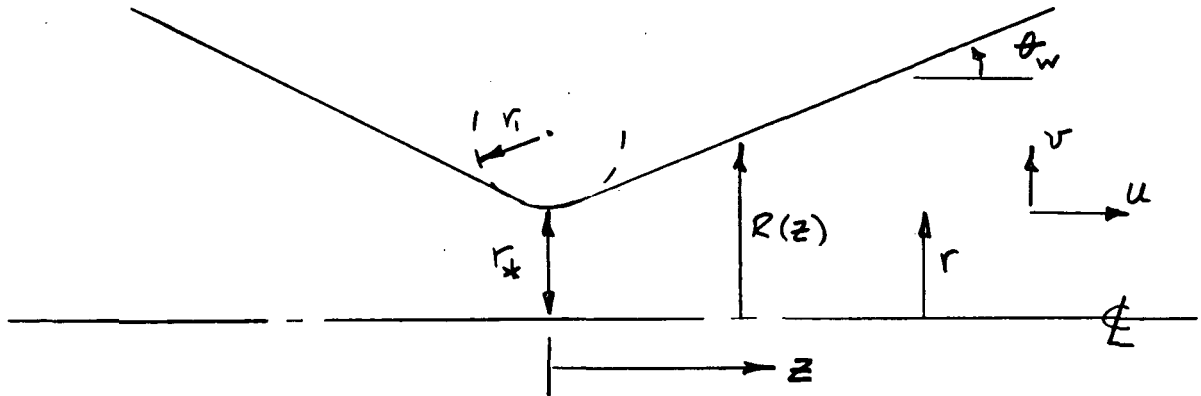
The computer program has been applied to the calculation of the flow for one particular case. These trial calculations revealed the existence of an instability in the numerical method, for which no satisfactory method of suppression was found. Consequently, the program in its present form cannot be used for direct calculations of nozzle flow fields. Despite this fact, it was nevertheless felt advisable to present in this report the final status of the program, the steps which led up to it, and a brief description of the efforts that were made to overcome the instability.

II. ANALYSIS

This section contains the derivation of the set of equations that were used to describe the flow field.

A. General Equations

In order to treat both two-dimensional and axisymmetric nozzle flows, the following coordinate system is used:



The coordinate r is the cylindrical radius in the axisymmetric case, or the transverse coordinate in the two-dimensional case. For the low Reynolds numbers considered here, the Navier-Stokes equations can be used. In their full generality, they are:

Continuity:

$$\frac{\partial}{\partial z} (\rho u) + \frac{1}{r^E} \frac{\partial}{\partial r} (\rho v r^E) = 0 \quad (1)$$

Axial Momentum:

$$\rho \left(u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} \right) = - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left[\mu \left(2 \frac{\partial u}{\partial z} - \frac{2}{3} \left\{ \frac{1}{r^e} \frac{\partial}{\partial r} (r^e v) + \frac{\partial u}{\partial z} \right\} \right) \right] + \frac{1}{r^e} \frac{\partial}{\partial r} \left[\mu r^e \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right) \right] \quad (2)$$

Radial Momentum:

$$\rho \left(u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} \right) = - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r} \left[\mu \left(2 \frac{\partial u}{\partial r} - \frac{2}{3} \left\{ \frac{1}{r^\epsilon} \frac{\partial}{\partial r} (r^\epsilon v) + \frac{\partial u}{\partial z} \right\} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right) \right] + 2\epsilon \frac{\mu}{r} \left[\frac{\partial v}{\partial r} - \frac{v}{r} \right] \quad (3)$$

Energy:

$$\rho \left(u \frac{\partial h}{\partial z} + v \frac{\partial h}{\partial r} \right) - u \frac{\partial p}{\partial z} - v \frac{\partial p}{\partial r} = \frac{1}{r^\epsilon} \frac{\partial}{\partial r} \left(r^\epsilon k \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \mu \left[2 \left\{ \left(\frac{\partial v}{\partial r} \right)^2 + \epsilon \left(\frac{v}{r} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right\} + \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right)^2 \right] - \frac{2}{3} \mu \left[\frac{\partial v}{\partial r} + \frac{\epsilon v}{r} + \frac{\partial u}{\partial z} \right]^2 \quad (4)$$

Here p denotes the pressure, ρ the density, μ the viscosity, h the static enthalpy, T the temperature, and k the thermal conductivity. The symbol H_0 , used below, denotes the total enthalpy in the reservoir. The parameter ϵ is one for axisymmetric flow, and zero for two-dimensional flow. (The numbers appearing above the various terms are for identification purposes in the analysis below.)

B. Present Approximation

The main area of interest in the present study is concerned with low Reynolds numbers and sharp throats. The Reynolds numbers, however, are not so low as to suppress expansion to supersonic conditions. Thus, judging from the results of the previous work^(1,2), the Reynolds number $B = \rho_0 \sqrt{2H_0} r^* / \mu_0$ must be on the order of 1000 or greater. In the vicinity of a sharp throat, the flow undergoes a very rapid acceleration, and this tends to thin the boundary layer, even at these low Reynolds numbers. For purposes of the present study, the boundary-layer thickness near the

throat can be assumed to be on the order of 5 to 10 percent of the transverse nozzle radius r_* .

The magnitudes of the axial and radial pressure gradients must be considered next. It can be shown (by using one-dimensional channel-flow theory, and expanding around the throat conditions) that the axial pressure gradient at the throat is inversely proportional to the geometric mean of the longitudinal and transverse radii:

$$\left(\frac{\partial p}{\partial z} \right)_* = O \left(\frac{p_0}{\sqrt{r_l r_*}} \right)$$

The radial pressure gradients, on the other hand, are governed by the centrifugal acceleration at the throat:

$$\left(\frac{\partial p}{\partial r} \right)_* = O \left(\frac{\rho_* u_*^2}{r_l} \right)$$

Here u_* denotes the centerline velocity at the throat. Note that the ratio of radial to axial pressure gradients varies as the square root of the radius ratio:

$$\left(\frac{\partial p / \partial r}{\partial p / \partial z} \right)_* = O \left(\sqrt{r_* / r_l} \right) \quad (5)$$

This is consistent with the experiments of Cuffel et al ⁽⁵⁾, who found these two pressure gradients to be about equal for $r_l / r_* = 0.625$.

The axial and radial velocity components have the orders:

$$u = O(u_*) \quad , \quad v = O(\theta_w u_*)$$

The length scale for significant axial gradients is on the order of the transverse radius, while radial gradients occur usually across the boundary-layer

thickness δ . Thus

$$\frac{\partial}{\partial z} = O\left(\frac{1}{r_*}\right), \quad \frac{\partial}{\partial r} = O\left(\frac{1}{\delta}\right)$$

If these order-of-magnitude estimates are now applied to the momentum and energy equations (all terms of the continuity equation can be retained without adding to the complexity of the problem), the results can be written in the following form, where the terms that have been dropped are identified by the numbers appearing in Eqs. 1-4 : (Here $B_* \equiv \rho_* u_* r_* / \mu_*$), and where the relative orders of magnitude of the various terms are shown.

Continuity:

$$\underbrace{\frac{1}{r_*} \frac{\partial}{\partial r} (\rho v r^*)}_1 + \underbrace{\frac{\partial}{\partial z} (\rho u)}_{\delta / r_* \theta_w} = 0 \quad (6)$$

Axial Momentum:

$$\underbrace{\rho u \frac{\partial u}{\partial z}}_1 + \underbrace{\rho v \frac{\partial u}{\partial r}}_{\theta_w r_* / \delta} = - \underbrace{\frac{\partial p}{\partial z}}_1 + \underbrace{\frac{1}{r_*} \frac{\partial}{\partial r} \left[\mu r^* \frac{\partial u}{\partial r} \right]}_{\frac{1}{B_*} \left(r_* / \delta \right)^2} \quad (7)$$

$$+ \underbrace{\textcircled{5} + \textcircled{6}}_{\frac{1}{B_*} \frac{\theta_w r_*}{\delta}} + \underbrace{\textcircled{4}}_{\frac{1}{B_*}}$$

Radial Momentum:

$$\begin{aligned}
 \underbrace{\rho u \frac{\partial v}{\partial z}}_{\theta_w} + \underbrace{\rho v \frac{\partial v}{\partial r}}_{r_* \theta_w^2 / \delta} &= \underbrace{-\frac{\partial p}{\partial r}}_1 + \frac{\partial}{\partial r} \left[\underbrace{\mu \left(2 \frac{\partial v}{\partial r} - \frac{2}{3} \left\{ \frac{1}{r \epsilon} \frac{\partial}{\partial r} (r^\epsilon v) + \frac{\partial u}{\partial z} \right\} \right)}_{\frac{\theta_w}{B_*} \left(\frac{r_*}{\delta} \right)^2} \right] + \underbrace{\frac{\partial u}{\partial z}}_{\frac{1}{B_*} \frac{r_*}{\delta}} \\
 &+ \underbrace{\frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial r} \right)}_{\frac{1}{B_*} \frac{r_*}{\delta}} + \underbrace{2 \epsilon \frac{\mu}{r} \left[\frac{\partial v}{\partial r} - \frac{v}{r} \right]}_{\frac{\theta_w}{B_*} \left(\frac{r_*}{\delta} \right)^2} + \underbrace{\textcircled{7}}_{\frac{\theta_w}{B_*}} \quad (8)
 \end{aligned}$$

Energy:

$$\begin{aligned}
 \underbrace{\rho u \frac{\partial h}{\partial z}}_{h_* / u_*^2} + \underbrace{\rho v \frac{\partial h}{\partial r}}_{\frac{\theta_w h_* r_*}{u_*^2 \delta}} - \underbrace{u \frac{\partial p}{\partial z}}_1 - \underbrace{v \frac{\partial p}{\partial r}}_{\theta_w} &= \underbrace{\frac{1}{r \epsilon} \frac{\partial}{\partial r} \left(r^\epsilon k \frac{\partial T}{\partial r} \right)}_{\frac{1}{B_*} \frac{h_*}{u_*^2} \left(\frac{r_*}{\delta} \right)^2} + \underbrace{\textcircled{6}}_{\frac{1}{B_*} \frac{h_*}{u_*^2}} \\
 + \underbrace{\mu \left[2 \left\{ \left(\frac{\partial v}{\partial r} \right)^2 + \epsilon \left(\frac{v}{r} \right)^2 \right\} - \frac{2}{3} \left\{ \frac{\partial v}{\partial r} + \epsilon \frac{v}{r} \right\}^2 + \left(\frac{\partial u}{\partial r} \right)^2 \right]}_{\frac{\theta_w^2}{B_*} \left(\frac{r_*}{\delta} \right)^2} &+ \underbrace{\frac{1}{B_*} \left(\frac{r_*}{\delta} \right)^2} \\
 + \underbrace{\textcircled{9} + \textcircled{14}}_{1/B_*} + \underbrace{\textcircled{10}}_{\theta_w^2 / B_*} + \underbrace{\textcircled{12} \cdot \textcircled{14}}_{\frac{\theta_w}{B_*} \cdot \frac{r_*}{\delta}} \quad (9)
 \end{aligned}$$

The terms that have been retained here are the same as those retained by Cheng et al ⁽⁶⁾ in their study of low Reynolds number external flows at high Mach number. In that work, as well as the present one, the terms which are kept are not all required to be uniformly of the same order. Some of the terms retained are expected to be numerically negligible in certain regions of the flow.

Throughout this development, only δ/r_* and $1/B_*$ have been assumed small. The quantity θ_w need not be small, although it will be so in many applications. Provided B_* is large enough, the limit $\theta_w \rightarrow 0$ simply reinforces the ordering used above. However, as shown in our previous work ⁽¹⁻⁴⁾, if B_* is not large enough, the limit $\theta_w \rightarrow 0$ will suppress supersonic flow, and the equations presented above may not be valid.

Finally, it should be noted that this ordering was specifically chosen for the throat region. At larger area ratios, the estimate used for $\partial p / \partial r$ will no longer apply, and this term will likely not contribute, numerically, in the proportion suggested above. The equations presented above include all the terms required in the region of large radial pressure gradient and reduce to those used previously ⁽¹⁻⁴⁾ in regions where this effect is unimportant. Thus, using the same terminology as Cheng et al ⁽⁶⁾, these can be called the "composite" equations for low Reynolds number nozzle flow.

C. Dimensionless Forms

The next step in preparing these equations for numerical evaluation is to choose convenient reference quantities for making the variables dimensionless. It is also convenient to make the radial scale uniform by defining a new independent radial variable:

$$x = z/r_*, \quad \sigma = r/r_*, \quad \eta = \frac{r}{R(z)} = \frac{\sigma}{\sigma_w(x)}, \quad D = \frac{A}{\rho_0} \quad (10)$$

$$U = \frac{u}{\sqrt{2H_0}}, \quad V = \frac{v}{\sqrt{2H_0}}, \quad P = p/p_0, \quad \Theta = h/H_0$$

$$W = V - \eta \sigma_w' U, \quad B = \rho_0 \sqrt{2H_0} r_*/\mu_0$$

The composite equations of motion take the form:

Continuity:

$$\frac{\partial}{\partial \eta} (D \eta^\epsilon W) + \sigma_w \frac{\partial}{\partial x} (D \eta^\epsilon U) + (1+\epsilon) \sigma_w' D \eta^\epsilon U = 0 \quad (11)$$

Axial Momentum:

$$D \left(U \frac{\partial U}{\partial x} + \frac{W}{\sigma_w} \frac{\partial U}{\partial \eta} \right) = - \frac{p_0}{2\rho_0 H_0} \left\{ \frac{\partial P}{\partial x} - \eta \frac{\sigma_w'}{\sigma_w} \frac{\partial P}{\partial \eta} \right\} \\ + \frac{1}{B \eta^\epsilon \sigma_w^2} \frac{\partial}{\partial \eta} \left\{ \eta^\epsilon \Theta^\omega \frac{\partial U}{\partial \eta} \right\} \quad (12)$$

Radial Momentum:

$$D \left(U \frac{\partial V}{\partial x} + \frac{W}{\sigma_w} \frac{\partial V}{\partial \eta} \right) = - \frac{p_0}{2\rho_0 H_0 \sigma_w} \frac{\partial P}{\partial \eta} \quad (13)$$

$$+ \frac{1}{B} \left\{ \frac{1}{\sigma_w} \frac{\partial}{\partial \eta} \left[\Theta^\omega \left(\frac{2}{\sigma_w} \frac{\partial V}{\partial \eta} - \frac{2}{3} \left\{ \frac{1}{\eta^\epsilon \sigma_w} \frac{\partial}{\partial \eta} (\eta^\epsilon V) + \frac{\partial U}{\partial x} - \eta \frac{\sigma_w'}{\sigma_w} \frac{\partial U}{\partial \eta} \right\} \right) \right] \right. \\ \left. + \frac{\partial}{\partial x} \left(\frac{\Theta^\omega}{\sigma_w} \frac{\partial U}{\partial \eta} \right) - \frac{\eta \sigma_w'}{\sigma_w^2} \frac{\partial}{\partial \eta} \left(\Theta^\omega \frac{\partial U}{\partial \eta} \right) + \frac{2\epsilon \Theta^\omega}{\eta \sigma_w^2} \left(\frac{\partial V}{\partial \eta} - \frac{V}{\eta} \right) \right\}$$

Energy:

$$\begin{aligned}
 D \left(u \frac{\partial \Theta}{\partial x} + \frac{w}{\sigma_w} \frac{\partial \Theta}{\partial n} \right) - \frac{1}{\rho_0 H_0} \left\{ u \frac{\partial P}{\partial x} + \frac{w}{\sigma_w} \frac{\partial P}{\partial n} \right\} = \\
 = \frac{1}{B \eta^\epsilon \sigma_w^2} \frac{\partial}{\partial \eta} \left(\frac{\eta^\epsilon \Theta^w}{R} \frac{\partial \Theta}{\partial \eta} \right) \\
 + \frac{2 \Theta^w}{B \sigma_w^2} \left\{ 2 \left[\left(\frac{\partial v}{\partial \eta} \right)^2 + \epsilon \left(\frac{v}{\eta} \right)^2 \right] - \frac{2}{3} \left[\frac{\partial v}{\partial \eta} + \epsilon \frac{v}{\eta} \right]^2 + \left(\frac{\partial v}{\partial \eta} \right)^2 \right\}
 \end{aligned} \tag{14}$$

The viscosity coefficient μ has been taken to vary as the w power of the static enthalpy, and also the Prandtl number R has been introduced:

$$\frac{\mu}{\mu_0} = \left(\frac{h}{H_0} \right)^w, \quad R = \frac{\mu c_p}{k} \tag{15}$$

D. Conservation Relations

Some useful relations can be derived by integrating the equations of motion across the channel, to yield expressions for the axial rate of change of the total mass, momentum, and energy flux.

Continuity:

Integration of the continuity equation leads to

$$\int_0^R \rho u r^\epsilon dr = \begin{cases} \dot{m}/2\pi, & \epsilon = 1 \\ \dot{m}_{2D}/2, & \epsilon = 0 \end{cases} \tag{16}$$

where $\dot{m}$ is the total mass flow (lbm/sec, for example), and $\dot{m}_{2D}$ is the total mass flow per unit width of the two-dimensional channel.

Momentum:

In order to express this relation in full generality, it is useful to start from the complete Navier-Stokes equation ⁽⁷⁾

$$\rho \frac{Du}{Dt} = \frac{1}{r^e} \left\{ \frac{\partial}{\partial r} (r^e \tau_{rz}) + \frac{\partial}{\partial z} (r^e \tau_{zz}) \right\} \quad (17)$$

where

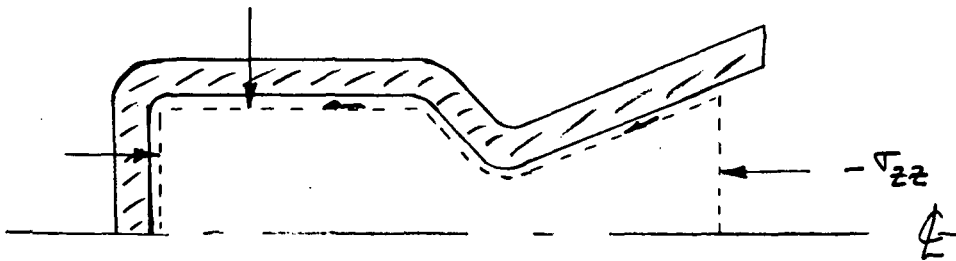
$$\tau_{rz} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right)$$

$$\tau_{zz} = -p + \tau_{zz} = -p + 2\mu \frac{\partial u}{\partial z} - \frac{2}{3}\mu \left\{ \frac{1}{r^e} \frac{\partial}{\partial r} (r^e v) + \frac{\partial u}{\partial z} \right\} \quad (18)$$

Integration of Eq. (17) gives, after a little algebra

$$\frac{\partial}{\partial z} \left\{ \int_0^R \rho u^2 r^e dr - \int_0^R r^e \tau_{zz} dr \right\} = R^e \tau_{rz} \Big|_{r=R} - R^e R' \tau_{zz} \Big|_{r=R} \quad (19)$$

The quantity on the left is the axial rate of change of the thrust F , while the two terms on the right are the shear and normal-stress influences on this rate of change. To see this, consider the momentum theorem applied to the fluid enclosed within the dotted control surface



The momentum theorem says that $F_z = \phi_z$, where F_z is the sum of the z -components of all the forces exerted on the fluid inside the control surface, and ϕ_z is the net flux of the z -component of momentum out of the control surface. But $F_z = F_w - N$, where F_w is the sum of

the z - component of all the forces exerted by the walls on the fluid, and N is the integrated normal stress exerted on the fluid over the exit-plane face of the control surface. The quantity F_w , in turn, is equal and opposite to the thrust F , i.e., to the sum of the z - component of all the forces exerted by the fluid on the walls. Thus, as in Eq. 19

$$|F| = \phi_z + N = 2\pi^\epsilon \left\{ \int_0^R \rho u^2 r^\epsilon dr - \int_0^R \tau_{zz} r^\epsilon dr \right\} \quad (20)$$

(In the two-dimensional case, F is the thrust per unit width of the nozzle.) In the present approximation, these become

$$\frac{1}{2\pi^\epsilon} \frac{dF}{dz} = R^\epsilon \left(\mu \frac{\partial u}{\partial r} \right)_{r=R} + p_w R^\epsilon R' \quad (21)$$

where

$$F = 2\pi^\epsilon \left\{ \int_0^R (p + \rho u^2) r^\epsilon dr \right\} \quad (22)$$

If p is independent of r , these reduce to the previous result (1, 2).

Energy:

In this case also, it is useful to start from the full Navier-Stokes equation

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} = \frac{1}{r^\epsilon} \frac{\partial}{\partial r} \left(r^\epsilon k \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \phi \quad (23)$$

where the dissipation function ϕ is given for axisymmetric flow by (see Ref. 7, Eqs. 2, 40, 3.34, 3.37):

$$\phi = \mu \left\{ \frac{4}{3} (e_{rr}^2 + e_{\theta\theta}^2 + e_{zz}^2 - e_{rr}e_{\theta\theta} - e_{rr}e_{zz} - e_{\theta\theta}e_{zz}) + e_{rz}^2 \right\}$$

where

$$e_{rr} = \frac{\partial v}{\partial r}, \quad e_{\theta\theta} = \frac{v}{r}, \quad e_{zz} = \frac{\partial u}{\partial z}, \quad e_{rz} = \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \quad (24)$$

To derive an expression for the variation of the total enthalpy, Eq. 23 is multiplied by r and integrated from 0 to R ; next, Eq. 17 is

multiplied by ru and integrated from 0 to R ; and finally, the radial momentum equation

$$\rho \frac{Dv}{Dt} = \frac{1}{r} \frac{\partial}{\partial r} (\tau_{rr}) + \frac{\partial}{\partial z} (\tau_{rz}) - \frac{\tau_{\theta\theta}}{r} \quad (25)$$

where

$$\tau_{rr} = -p + \tau_{rr} = -p + \mu \left\{ \frac{4}{3} \frac{\partial v}{\partial r} - \frac{2}{3} \frac{v}{r} - \frac{2}{3} \frac{\partial u}{\partial z} \right\} \quad (26)$$

$$\tau_{\theta\theta} = -p + \mu \left\{ -\frac{2}{3} \frac{\partial v}{\partial r} + \frac{4}{3} \frac{v}{r} - \frac{2}{3} \frac{\partial u}{\partial z} \right\}$$

is multiplied by rv , and integrated from 0 to R . These results are then combined, and after some algebra, they lead to the result

$$\begin{aligned} \frac{\partial}{\partial z} \int_0^R \rho u H r^e dr &= R^e \left[k \frac{\partial T}{\partial r} + u \tau_{rz} + v \tau_{rr} \right]_{r=R} \\ &+ \int_0^R r^e \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) dr + \int_0^R r^e \frac{\partial}{\partial z} (u \tau_{zz} + v \tau_{rz}) dr \end{aligned} \quad (27)$$

where

$$H = h + \frac{1}{2} (u^2 + v^2) \quad (28)$$

The first three terms on the right represent the loss of total enthalpy by heat transfer and sliding friction at the wall, while the two integral terms give the net loss (or gain), across an element of length dz and radius R , due to axial heat conduction and to the work done by the normal components of the shear stress tensor on the faces of the element. This equation is a different expression of the result given in Ref. 7, Eq. 3.21. In the present approximation, the integral terms are neglected. Thus the adiabatic-wall boundary condition is found by setting the first term on the right of Eq. 27 equal to zero.

In the dimensionless variables introduced above, these equations become:

Mass:

$$\sigma_w^{1+\epsilon} \int_0^1 D u \eta^\epsilon d\eta = A \equiv \begin{cases} \dot{m} / 2\pi \rho_0 r_*^2 \sqrt{2H_0} & , \epsilon=1 \\ \dot{m}_{2D} / 2\rho_0 r_* \sqrt{2H_0} & , \epsilon=0 \end{cases} \quad (29)$$

Momentum:

$$\frac{d\bar{F}}{dx} = P_w \frac{d}{dx} \sigma_w^2 + \frac{4\rho_0 H_0}{3p_0} [\Theta(1)]^\omega \left(\frac{\partial u}{\partial \eta} \right)_{\eta=1} \quad (30)$$

where

$$\bar{F} = \frac{F}{p_0 \pi^\epsilon r_*^{1+\epsilon}} = \sigma_w^{1+\epsilon} \left\{ 2 \int_0^1 P \eta^\epsilon d\eta + \frac{4\rho_0 H_0}{p_0} \int_0^1 D u^2 \eta^\epsilon d\eta \right\} \quad (31)$$

Energy:

$$\begin{aligned} \frac{d}{dx} \left\{ \sigma_w^{1+\epsilon} \int_0^1 D u \frac{H}{H_0} \eta^\epsilon d\eta \right\} &= \frac{\sigma_w^{\epsilon-1}}{B} \left[\Theta^\omega \left\{ \frac{1}{R} \frac{\partial \Theta}{\partial \eta} + 2u \left(r_w \frac{\partial V}{\partial x} - \sigma_w' \frac{\partial V}{\partial \eta} + \frac{\partial u}{\partial \eta} \right) \right. \right. \\ &\quad \left. \left. + 2V \left(\frac{4}{3} \frac{\partial V}{\partial \eta} - \frac{2\epsilon}{3} \frac{V}{\eta} - \frac{2}{3} \sigma_w \frac{\partial u}{\partial x} + \frac{2}{3} \sigma_w' \frac{\partial u}{\partial \eta} \right) \right\} \right]_{\eta=1} \end{aligned} \quad (32)$$

The relation connecting rates of change of pressure to area change, heat conduction, and shear along a streamtube plays a key role in any study of nozzle flow. In our previous work^(1,2), the form of this relation peculiar to the slender-channel equations was derived, and was used to guide passage of the solution through a saddle-point singularity that arose in that analysis.

In the present study, the streamtube relation appropriate to the full Navier-Stokes equations has been derived, and then specialized for the particular approximation used here. The details of the derivation are presented in Appendix A. The result is:

$$\begin{aligned}
 (1-M^2) \left\{ \cos \theta \frac{\partial p}{\partial z} + \sin \theta \frac{\partial p}{\partial r} \right\} = & \gamma p M^2 \left\{ \cos^2 \theta \left[-\mu \sin \theta \frac{\partial}{\partial z} \left(\frac{v}{u} \right) + \cos \theta \frac{\partial}{\partial r} \left(\frac{v}{u} \right) \right] \right. \\
 & \left. + \frac{\epsilon \cos \theta}{r} \cdot \frac{v}{u} \right\} \\
 & + \cos \theta \left\{ \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r\epsilon} \frac{\partial}{\partial r} (r\epsilon \tau_{rz}) \right\} \\
 & + \sin \theta \left\{ \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{2\mu}{r} \epsilon \left[\frac{\partial v}{\partial r} - \frac{v}{r} \right] \right\} \\
 & - \frac{g}{h} \left\{ \frac{1}{r\epsilon} \frac{\partial}{\partial r} (r\epsilon k \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + \phi \right\} \quad (33)
 \end{aligned}$$

where

$$M^2 = \frac{u^2 + v^2}{a^2}, \quad a^2 = \frac{\gamma p}{\rho} = (\gamma - 1) h, \quad g = (u^2 + v^2)^{1/2} \quad (34)$$

and where θ is the local flow inclination angle

$$\theta = \tan^{-1} \frac{v}{u} \quad (35)$$

The quantity in brackets on the left-hand side of Eq. 33, is the directional derivative of the pressure in the streamline direction. The first term on the right is the effect of area change, and contains the usual factor of $M^2/(1-M^2)$. The remaining terms on the right contain the effects of shear, heat conduction, and heat generation by the frictional conversion of directed to thermal kinetic energy.

In the dimensionless coordinate system, and with the present approximation to the viscous terms, this relation becomes

$$\begin{aligned}
& (1-M^2) \left\{ \cos \theta \frac{\partial P}{\partial x} + \frac{\sin \theta - \eta \tau_w' \cos \theta}{\tau_w} \frac{\partial P}{\partial \eta} \right\} = \\
& = \gamma P M^2 \left\{ \cos^2 \theta \left[-\sin \theta \frac{\partial}{\partial x} \left(\frac{V}{U} \right) + \frac{\eta \tau_w' \sin \theta + \cos \theta}{\tau_w} \frac{\partial}{\partial \eta} \left(\frac{V}{U} \right) \right] + \frac{\epsilon \cos \theta}{\eta \tau_w} \frac{V}{U} \right\} \\
& + \frac{2\gamma \cos \theta}{(\gamma-1) B \tau_w^2} \frac{1}{\eta^\epsilon} \frac{\partial}{\partial \eta} \left(\Theta^\omega \eta^\epsilon \frac{\partial U}{\partial \eta} \right) \\
& + \frac{2\gamma \sin \theta}{(\gamma-1) B \tau_w^2} \left\{ \frac{\partial}{\partial \eta} \left[\Theta^\omega \left(2 \frac{\partial V}{\partial \eta} - \frac{2}{3} \left\{ \frac{1}{\eta^\epsilon} \frac{\partial}{\partial \eta} (\eta^\epsilon V) + \tau_w \frac{\partial U}{\partial x} - \eta \tau_w' \frac{\partial U}{\partial \eta} \right\} \right) \right] \right. \\
& \quad \left. + \tau_w \left(\tau_w \frac{\partial}{\partial x} - \eta \tau_w' \frac{\partial}{\partial \eta} \right) \left(\frac{\Theta^\omega}{\tau_w} \frac{\partial U}{\partial \eta} \right) + \frac{2\epsilon \Theta^\omega}{\eta} \left[\frac{\partial V}{\partial \eta} - \frac{V}{\eta} \right] \right\} \\
& - \frac{2\gamma}{(\gamma-1) B \tau_w^2} \cdot \frac{Q}{\Theta} \left\{ \frac{1}{\eta^\epsilon} \frac{\partial}{\partial \eta} \left(\frac{\eta^\epsilon \Theta^\omega}{R} \frac{\partial \Theta}{\partial \eta} \right) + \Theta^\omega \left[4 \left(\frac{\partial V}{\partial \eta} \right)^2 + 4\epsilon \frac{V^2}{\eta^2} \right. \right. \\
& \quad \left. \left. - \frac{4}{3} \left(\frac{\partial V}{\partial \eta} + \frac{\epsilon V}{\eta} \right)^2 + 2 \left(\frac{\partial U}{\partial \eta} \right)^2 \right] \right\} \tag{36}
\end{aligned}$$

Under some circumstances, the terms involving $\frac{\partial}{\partial x} (V/U)$ can be difficult to evaluate numerically. When this is the case, it is advantageous to employ a modified version of the streamtube equation, in which the troublesome terms are expressed in terms of other quantities which are easy to evaluate. The key expression that is used is the equation for the rate of change of flow inclination angle along a streamline. Details of the

derivation of this equation and its use in the streamtube equation are given in Appendix B. The net result is the equation

$$\begin{aligned}
 (1-M^2) \left\{ \cos \theta \frac{\partial p}{\partial z} + \sin \theta \frac{\partial p}{\partial r} \right\} &= \gamma P M^2 \cos \theta \left\{ \frac{\partial}{\partial r} \left(\frac{v}{u} \right) + \frac{\epsilon}{r} \frac{v}{u} \right\} \\
 &+ \frac{1}{\cos \theta} \left\{ \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \right\} \\
 &- \frac{\sin^2 \theta}{\cos \theta} \frac{\partial p}{\partial z} + \sin \theta \frac{\partial p}{\partial r} \\
 &- \frac{q}{h} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (rk \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + \phi \right\}
 \end{aligned} \tag{37}$$

In the dimensionless variables, this becomes

$$\begin{aligned}
 (1-M^2) \left\{ \cos \theta \frac{\partial P}{\partial x} + \frac{\sin \theta - \eta \tau_w' \cos \theta}{\tau_w} \frac{\partial P}{\partial \eta} \right\} &= \\
 &= \frac{\gamma P M^2 \cos \theta}{\tau_w} \left\{ \frac{\partial}{\partial \eta} \left(\frac{V}{u} \right) + \frac{\epsilon}{\eta} \frac{V}{u} \right\} \\
 &- \frac{\sin^2 \theta}{\cos \theta} \left\{ \frac{\partial P}{\partial x} - \frac{\eta \tau_w'}{\tau_w} \frac{\partial P}{\partial \eta} \right\} + \frac{\sin \theta}{\tau_w} \frac{\partial P}{\partial \eta}
 \end{aligned}$$

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$$\begin{aligned}
& + \frac{2\gamma \Theta^w}{(\gamma-1) 8\pi w^2} \left\{ \left\langle \frac{w}{\Theta} \frac{\partial \Theta}{\partial \eta} \frac{\partial u}{\partial \eta} + \frac{\epsilon}{\eta} \frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right\rangle / \cos \theta \right. \\
& \quad - \frac{Q}{\Theta} \left\langle \frac{1}{R} \left[\frac{w}{\Theta} \left(\frac{\partial \Theta}{\partial \eta} \right)^2 + \frac{\epsilon}{\eta} \frac{\partial \Theta}{\partial \eta} + \frac{\partial^2 \Theta}{\partial \eta^2} \right] \right. \\
& \quad \left. \left. + 2 \left(\frac{\partial u}{\partial \eta} \right)^2 + \frac{\theta}{3} \left(\frac{\partial v}{\partial \eta} \right)^2 - \frac{\theta}{3} \frac{\epsilon v}{\eta} \frac{\partial v}{\partial \eta} + \frac{\theta}{3} \left(\frac{\epsilon v}{\eta} \right)^2 \right\rangle \right\}
\end{aligned} \tag{38}$$

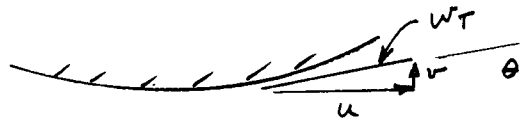
This equation is referred to in the discussion below as the modified stream-tube equation.

E. Boundary Conditions

Along the nozzle walls, the velocity component tangent to the wall is taken to be determined by the slip boundary condition

$$w_T = \frac{2 - \alpha_u}{\alpha_u} \frac{\mu}{p} \sqrt{\frac{\pi}{2} \frac{RT}{M}} \frac{\partial w_T}{\partial \eta} \tag{39}$$

where α_u denotes the velocity accommodation coefficient, μ , p , and T the viscosity, pressure, and temperature, R the universal gas constant, M the molecular weight, η the coordinate normal to the surface, and w_T the tangential velocity component:



At the wall, $\eta = u \tan \theta$, so w_T becomes

$$w_T = u / \cos \theta$$

The derivative $\partial w_T / \partial \eta$, expressed in terms of u , v , z , and r , becomes

$$\frac{\partial w_T}{\partial \eta} = -\cos^2 \theta \frac{\partial u}{\partial r} - \sin \theta \cos \theta \left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z} \right) + \sin^2 \theta \frac{\partial v}{\partial z}$$

If these equations are now written in terms of the dimensionless variables, the result is

$$\frac{u}{\cos \theta_w} = -\frac{2-\alpha_u}{\alpha_u} \sqrt{\frac{\pi Y}{Y-1}} \frac{\cos \theta_w \Theta_w^{\omega+1/2}}{B P \tau_w} \left\{ \frac{\partial u}{\partial \eta} + \tau_w' \frac{\partial v}{\partial \eta} - \tau_w \sin \theta_w \cos \theta_w \frac{\partial u}{\partial x} - \tau_w \sin^2 \theta_w \frac{\partial v}{\partial x} \right\} \quad (40)$$

This can be compared with Eq. (3-21) of the final report on the previous program (Ref. 1), where the factor of $\cos \theta_w$ on the left-hand side was taken equal to unity.

If the wall temperature is prescribed, the thermal boundary condition becomes:

$$\begin{aligned} T - T_w &= \frac{2-\alpha_T}{\alpha_T} \sqrt{\frac{\pi}{2} \frac{RT}{m}} \frac{2\gamma}{P_r(\gamma+1)} \frac{\mu}{p} \frac{\partial T}{\partial \eta} \\ &= \frac{2-\alpha_T}{\alpha_T} \sqrt{\frac{\pi}{2} \frac{RT}{m}} \frac{2\gamma}{P_r(\gamma+1)} \frac{\mu}{p} \left\{ -\cos \theta_w \frac{\partial T}{\partial r} + \sin \theta_w \frac{\partial T}{\partial z} \right\} \end{aligned}$$

where α_T denotes the thermal accommodation coefficient. This can be written, in terms of the dimensionless variables, as:

$$\Theta_{\eta \rightarrow 1} - \Theta_w(x) = \frac{2-\alpha_T}{\alpha_T} \sqrt{\frac{\pi \gamma}{\gamma-1}} \frac{2\gamma}{P_r(\gamma+1)} \frac{\Theta_w^{\omega+1/2}}{B P \tau_w} \left\{ \frac{-1}{\cos \theta_w} \frac{\partial \Theta}{\partial \eta} + \tau_w \sin \theta_w \frac{\partial \Theta}{\partial x} \right\} \quad (41)$$

For the case of an adiabatic wall, the thermal boundary condition is that which produces no change in the axial flux of total enthalpy. This condition is found from Eq. 27, using the present approximations for the shear components:

$$k \frac{\partial T}{\partial r} + u \mu \left(\frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right) + v \mu \left(\frac{4}{3} \frac{\partial v}{\partial r} - \frac{2\epsilon}{3} \frac{v}{r} - \frac{2}{3} \frac{\partial u}{\partial z} \right) = 0 \quad (42)$$

In terms of the dimensionless variables, and using the relation $V_w = \sigma_w' u_w$, this becomes:

$$\begin{aligned} \frac{1}{R} \frac{\partial \Theta}{\partial \eta} + u \frac{\partial u}{\partial \eta} \left(2 + \frac{4}{3} \sigma_w'^2 \right) - \frac{4}{3} \sigma_w \sigma_w' u \frac{\partial u}{\partial x} \\ + \frac{2}{3} \sigma_w' u \frac{\partial V}{\partial \eta} + 2 \sigma_w u \frac{\partial V}{\partial x} - \frac{4\epsilon}{3} V^2 = 0 \end{aligned} \quad (43)$$

F. Initial Conditions

In order to start the calculation, profiles of velocity and temperature in the reservoir must be specified. Several versions have been used in the present study. One version is based on the solution for slow viscous flow in a converging cone, studied by Ackerberg⁽⁸⁾, Oka⁽⁹⁾, and others. Their solution, which includes radial pressure gradient, has the following form, far upstream of the apex of the cone:

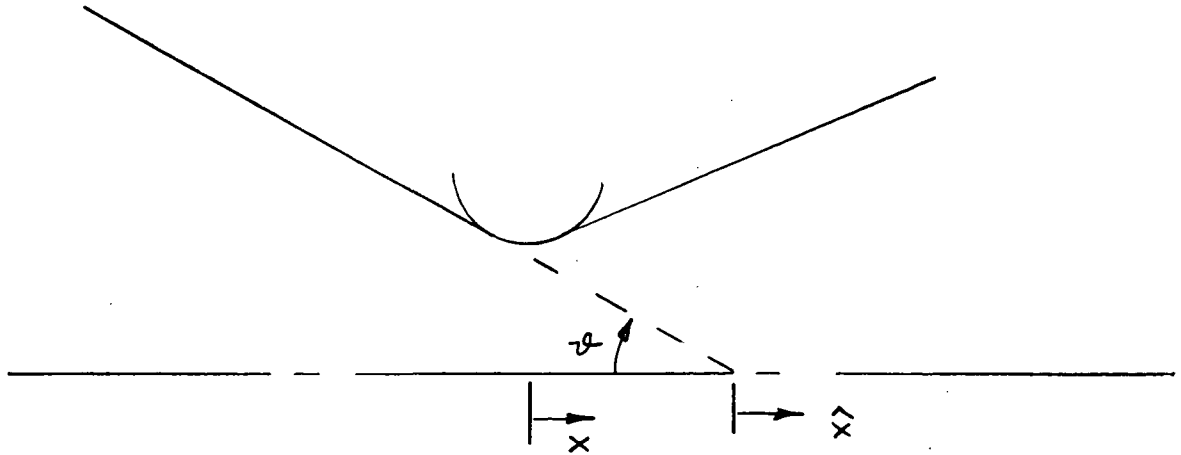
$$u = \frac{3A \cos \vartheta}{2(\tau^2 + \hat{x}^2)} \cdot \frac{(\cos 2\vartheta - \cos 2\vartheta_w)}{(1 - \cos \vartheta_w)^2 (1 + 2 \cos \vartheta_w)}$$

$$\Theta = 1, \quad V = \sigma_w' u, \quad \cos \vartheta = -\hat{x} / \sqrt{\tau^2 + \hat{x}^2}$$

(44)

$$P = 1 - \frac{A}{B} \cdot \frac{4\gamma}{\gamma-1} \cdot \frac{1}{(\tau^2 + \hat{x}^2)^{3/2}} \cdot \frac{3 \cos^2 \vartheta - 1}{(1 - \cos \vartheta_w)^2 (1 + 2 \cos \vartheta_w)}$$

where $\sigma = \eta \sigma_w$, and $\hat{x}$ denotes distance measured from the apex of the cone:



A second set of initial conditions was used, in some of the calculations for the axisymmetric case. This set was found by substituting in Eqs. 11-14 the expansions:

$$U(x, \eta) = \sum_{N=2} \frac{U_N(\eta)}{\sigma_w^N} \quad W(x, \eta) = \sum_{N=3} \frac{W_N(\eta)}{\sigma_w^N} \quad (45)$$

$$P(x, \eta) = 1 + \sum_{N=3} \frac{\pi_N(\eta)}{\sigma_w^N} \quad \Theta(x, \eta) = 1 + \sum_{N=4} \frac{T_N(\eta)}{\sigma_w^N}$$

The leading terms in this solution (see Appendix C for details) are

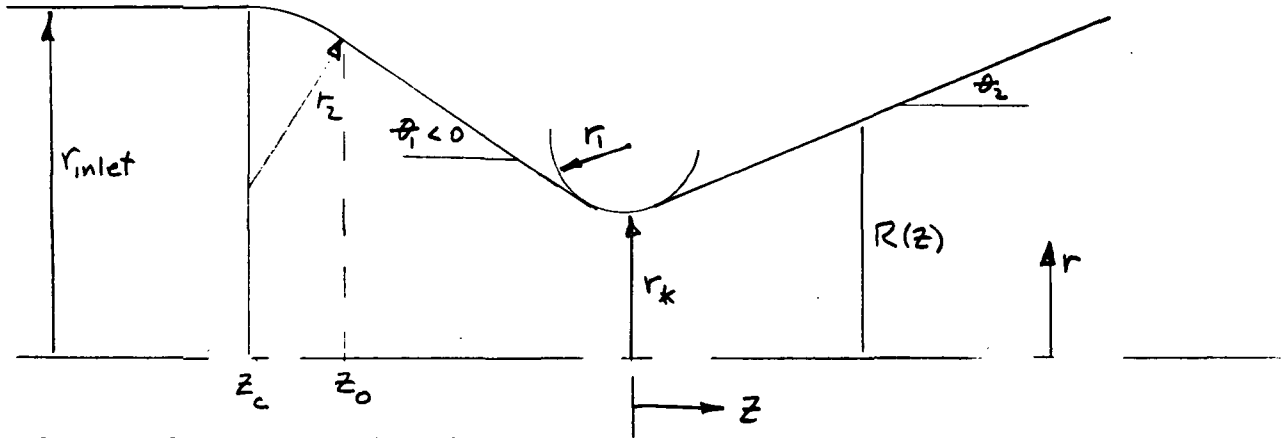
$$U_2(\eta) = U_2(0) \frac{\cos^3 \eta - \cos^3 \theta_w}{1 - \cos^3 \theta_w}$$

$$\pi_3(\eta) = \frac{2\sigma_w' U_2(0)}{(\gamma-1)B} \left\{ \frac{2\cos^3 \theta_w}{1 - \cos^3 \theta_w} + 2 \frac{U_2(\eta)}{U_2(0)} - \frac{3\eta^2 \sigma_w'^2}{(1 + \eta^2 \sigma_w'^2)^{3/2} (1 - \cos^3 \theta_w)} \right\}$$

$$U_2(0) = \frac{2A \tan^2 \theta_w (1 - \cos^3 \theta_w)}{2 - 3 \cos \theta_w + \cos^3 \theta_w} \quad (46)$$

G. Nozzle Geometry

The nozzle geometry used was the same as that of the previous contract, with the exception that a cylindrical inlet section was added:



The coordinates are given by

$$x \leq \frac{z_c}{r_*} : \quad \sigma_w = \frac{r_{inlet}}{r_*}, \quad \theta_w = 0$$

$$\frac{z_c}{r_*} \leq x \leq x_0 : \quad \sigma_w = \frac{r_{inlet}}{r_*} - \frac{r_2}{r_*} (1 - \cos \theta_w)$$

$$\theta_w = -\sin^{-1} \left(\frac{x - z_c/r_*}{r_2/r_*} \right)$$

$$\frac{z_c}{r_*} = x_0 + \frac{r_2}{r_*} \sin \theta_1$$

$$x_0 \leq x \leq R_1 \sin \theta_1 :$$

$$\sigma_w = x \tan \theta_1 + 1 + R_1 (1 - \cos \theta_1 - \sin \theta_1 \tan \theta_1)$$

$$\sigma_w' = \tan \theta_1$$

$$R_1 \sin \theta_1 \leq x \leq R_1 \sin \theta_2 :$$

$$\sigma_w = 1 + R_1 - \sqrt{R_1^2 - x^2}$$

$$\sigma'_w = x / \sqrt{R_1^2 - x^2}$$

$$x \geq R_1 \sin \theta_2 :$$

$$\sigma_w = x \tan \theta_2 + 1 + R_1 (1 - \cos \theta_2 - \sin \theta_2 \tan \theta_2)$$

$$\sigma'_w = \tan \theta_2 \quad (47)$$

where

$$R_1 \equiv r_1 / r_*$$

III. NUMERICAL EVALUATIONS

There are many approaches that can be used for solving the equations of motion derived in Section I. These include analytical approximations, integral methods, and finite-difference calculations. The last of these approaches was used in the present work, since the previous study ⁽¹⁾ had shown that a relatively simple and efficient numerical method was available. The subsections that follow contain the finite-difference approximation used, some details of the method of solution, and the results of our attempts to make calculations with the computer program.

A. Finite-Difference Equations

If the equation of state, $P = D\Theta$, is used to eliminate the density in favor of the pressure and temperature, then the solution is expressed in terms of the four dependent variables P , $\mathcal{U}$, Θ , and V . For this there are four equations: the streamtube relation, the energy and axial momentum equations, and the radial momentum equation. As before, a rectangular grid is introduced, with index K counting axial distance, and index L the radial distance:

$$X = X_0 + K \Delta X, \quad \eta = (L-1) \Delta \eta$$

The interval $\Delta \eta$ has been taken as 0.01, so that L ranges from 1 to 101.

The finite-difference representations of the differential equations were written using the Crank-Nicholson implicit scheme (see Ref. 10). Because the differential equations are nonlinear, there are many options that can be followed. The choices made have generally followed those which were found to give satisfactory results in the previous program.

Streamtube Relation:

$$(1 - \bar{M}^2) \left\{ \cos \bar{\Theta} \frac{P_{K+1}^L - P_K^L}{\Delta X} + \frac{\sin \bar{\Theta} - \eta \bar{\Gamma}_w'}{\bar{\Gamma}_w} \cdot \frac{P_{K+1}^{L+1} - P_{K+1}^{L-1} + P_K^{L+1} - P_K^{L-1}}{4 \Delta \eta} \right\} = R_K^L \quad (48)$$

where the notation F_K^L means $F(K\Delta x, L\Delta n)$; where $M = \frac{q}{a} = \frac{Q}{Q}$

$$Q = a/\sqrt{2H_0} = \sqrt{\frac{\gamma-1}{2}} \Theta$$

and where

$$\begin{aligned} R_K^L = & \frac{\gamma \bar{P} \bar{M}^2 \cos \bar{\Theta}}{\bar{\tau}_w} \left\{ \frac{\partial}{\partial n} \left(\frac{\bar{V}}{\bar{u}} \right) + \frac{\epsilon}{\eta} \left(\frac{\bar{V}}{\bar{u}} \right) \right\} \\ & - \frac{\sin^2 \bar{\Theta}}{\cos \bar{\Theta}} \left\{ \left(\frac{\partial \bar{P}}{\partial x} \right) - \frac{\eta \bar{\tau}_w'}{\bar{\tau}_w} \left(\frac{\partial \bar{P}}{\partial n} \right) \right\} + \frac{\sin \bar{\Theta}}{\bar{\tau}_w} \left(\frac{\partial \bar{P}}{\partial n} \right) \\ & + \frac{2\gamma \bar{\Theta}^\omega}{(\gamma-1) B \bar{\tau}_w^2} \left\{ \left\langle \frac{\omega}{\bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial n} \right) \left(\frac{\partial \bar{u}}{\partial n} \right) + \frac{\epsilon}{\eta} \left(\frac{\partial \bar{u}}{\partial n} \right) + \left(\frac{\partial^2 \bar{u}}{\partial n^2} \right) \right\rangle / \cos \bar{\Theta} \right. \\ & \left. - \frac{\bar{Q}}{\bar{\Theta}} \left\langle \frac{1}{\bar{u}} \left[\frac{\omega}{\bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial n} \right)^2 + \frac{\epsilon}{\eta} \left(\frac{\partial \bar{\Theta}}{\partial n} \right) + \left(\frac{\partial^2 \bar{\Theta}}{\partial n^2} \right) \right] \right. \right. \\ & \left. \left. + 2 \left(\frac{\partial \bar{u}}{\partial n} \right)^2 + \frac{\theta}{3} \left(\frac{\partial \bar{V}}{\partial n} \right)^2 - \frac{\theta}{3} \frac{\epsilon \bar{V}}{\eta} \left(\frac{\partial \bar{V}}{\partial n} \right) + \frac{\theta}{3} \left(\frac{\epsilon \bar{V}}{\eta} \right)^2 \right\rangle \right\} \end{aligned} \quad (49)$$

The overbar denotes quantities evaluated halfway across the step, i.e.:

$$\bar{P} = \frac{1}{2} (P_K^L + P_{K+1}^L), \quad \bar{\tau}_w = \tau_w \Big|_{(K+\frac{1}{2})\Delta x}, \quad \bar{\tau}_w' = \tau_w' \Big|_{(K+\frac{1}{2})\Delta x}$$

when terms in R_K^L contain quantities at $K+1$ that are being solved for (e.g., P_{K+1}^L in $\bar{P}$), they are found either by extrapolation, or are taken from a previous iteration.

Axial Momentum:

$$\begin{aligned}
 & \frac{\eta \bar{P}}{\bar{\Theta}} \left[\bar{u} \frac{u_{k+1}^L - u_k^L}{\Delta x} + \frac{\bar{w}}{\bar{v}_w} \frac{u_k^{L+1} - u_k^{L-1} + u_{k+1}^{L+1} - u_{k+1}^{L-1}}{4\Delta\eta} \right] = \\
 & = \frac{-\bar{p}_0}{2\rho_0 H_0} \eta \left[\frac{p_{k+1}^L - p_k^L}{\Delta x} - \eta \frac{\bar{v}_w'}{\bar{v}_w} \frac{p_k^{L+1} - p_k^{L-1} + p_{k+1}^{L+1} - p_{k+1}^{L-1}}{4\Delta\eta} \right] \\
 & + \frac{\bar{\Theta}^\omega}{B\bar{v}_w^2} \left[\frac{u_k^{L+1} - u_k^{L-1} + u_{k+1}^{L+1} - u_{k+1}^{L-1}}{4\Delta\eta} \right. \\
 & + \frac{\omega\eta}{\bar{\Theta}} \frac{(u_k^{L+1} - u_k^{L-1})(\Theta_{k+1}^{L+1} - \Theta_{k+1}^{L-1}) + (u_{k+1}^{L+1} - u_{k+1}^{L-1})(\Theta_k^{L+1} - \Theta_k^{L-1})}{8(\Delta\eta)^2} \\
 & \left. + \eta \frac{u_k^{L+1} - 2u_k^L + u_k^{L-1} + u_{k+1}^{L+1} - 2u_{k+1}^L + u_{k+1}^{L-1}}{2(\Delta\eta)^2} \right] \quad (50)
 \end{aligned}$$

Energy:

$$\begin{aligned}
 & \frac{\eta \bar{P}}{\bar{\Theta}} \left[\bar{u} \frac{\Theta_{k+1}^L - \Theta_k^L}{\Delta x} + \frac{\bar{W}}{\bar{\sigma}_w} \frac{\Theta_k^{L+1} - \Theta_k^{L-1} + \Theta_{k+1}^{L+1} - \Theta_{k+1}^{L-1}}{4 \Delta \eta} \right] = \\
 & = \frac{\rho_0 \eta}{\rho_0 H_0} \left[\bar{u} \frac{P_{k+1}^L - P_k^L}{\Delta x} + \frac{\bar{W}}{\bar{\sigma}_w} \frac{P_k^{L+1} - P_k^{L-1} + P_{k+1}^{L+1} - P_{k+1}^{L-1}}{4 \Delta \eta} \right] \\
 & + \frac{\bar{\Theta}^2}{8 \bar{R} \bar{\sigma}_w^2} \left\{ \frac{\Theta_k^{L+1} - \Theta_k^{L-1} + \Theta_{k+1}^{L+1} - \Theta_{k+1}^{L-1}}{4 \Delta \eta} + \frac{\omega \eta}{\bar{\Theta}} \frac{(\Theta_k^{L+1} - \Theta_k^{L-1})(\Theta_{k+1}^{L+1} - \Theta_{k+1}^{L-1})}{4 (\Delta \eta)^2} \right. \\
 & \quad \left. + \eta \frac{\Theta_k^{L+1} - 2\Theta_k^L + \Theta_k^{L-1} + \Theta_{k+1}^{L+1} - 2\Theta_{k+1}^L + \Theta_{k+1}^{L-1}}{2 (\Delta \eta)^2} \right. \\
 & \quad \left. + 2 \bar{P}_n \eta \left[\frac{(\mathcal{U}_k^{L+1} - \mathcal{U}_k^{L-1})(\mathcal{U}_{k+1}^{L+1} - \mathcal{U}_{k+1}^{L-1})}{4 (\Delta \eta)^2} + \frac{4}{3} \left(\frac{V_k^{L+1} - V_k^{L-1}}{2 \Delta \eta} \right)^2 \right. \right. \\
 & \quad \left. \left. - \frac{4}{3} \frac{V_k^L}{\eta} \frac{V_k^{L+1} - V_k^{L-1}}{2 \Delta \eta} + \frac{4}{3} \left(\frac{V_k^L}{\eta} \right)^2 \right] \right\} \quad (51)
 \end{aligned}$$

Radial Momentum:

$$\begin{aligned}
 & \eta \frac{\bar{P}}{\bar{\Theta}} \left[\bar{u} \frac{V_{k+1}^L - V_k^L}{\Delta x} + \frac{\bar{W}}{\bar{\sigma}_w} \frac{V_k^{L+1} - V_k^{L-1} + V_{k+1}^{L+1} - V_{k+1}^{L-1}}{4 \Delta \eta} \right] = \\
 & = - \frac{\gamma-1}{2\gamma} \frac{\eta}{\bar{\sigma}_w} \frac{P_k^{L+1} - P_k^{L-1} + P_{k+1}^{L+1} - P_{k+1}^{L-1}}{4 \Delta \eta} \\
 & + \frac{\bar{\Theta}^2}{8 \bar{R} \bar{\sigma}_w^2} \left\{ \frac{V_k^{L+1} - V_k^{L-1} + V_{k+1}^{L+1} - V_{k+1}^{L-1}}{4 \Delta \eta} \left[\frac{4}{3} \eta \frac{\omega}{\bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial \eta} \right) + \frac{4\epsilon}{3} \right] + \frac{2\epsilon \bar{V} \omega}{3 \bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial \eta} \right) \right. \\
 & \quad \left. - \frac{4\epsilon \bar{V}}{3 \eta} + \frac{4}{3} \eta \frac{V_k^{L+1} - 2V_k^L + V_k^{L-1} + V_{k+1}^{L+1} - 2V_{k+1}^L + V_{k+1}^{L-1}}{2 (\Delta \eta)^2} \right. \\
 & \quad \left. - \frac{\omega}{\bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial \eta} \right) \left(\frac{2}{3} \eta \bar{\sigma}_w \left(\frac{\partial \bar{u}}{\partial x} \right) + \frac{\eta^2 \bar{\sigma}_w'}{3} \left(\frac{\partial \bar{u}}{\partial \eta} \right) \right) + \frac{\eta \bar{\sigma}_w \omega}{3} \left(\frac{\mathcal{U}_{k+1}^{L+1} - \mathcal{U}_{k+1}^{L-1} - \mathcal{U}_k^{L+1} + \mathcal{U}_k^{L-1}}{2 \Delta \eta \Delta x} \right) \right. \\
 & \quad \left. - \frac{\eta \bar{\sigma}_w'}{3} \left(\frac{\partial \bar{u}}{\partial \eta} \right) - \frac{\eta^2 \bar{\sigma}_w'}{3} \left(\frac{\partial^2 \bar{u}}{\partial \eta^2} \right) + \frac{\eta \bar{\sigma}_w \omega}{\bar{\Theta}} \left(\frac{\partial \bar{\Theta}}{\partial x} \right) \left(\frac{\partial \bar{u}}{\partial \eta} \right) \right\} \quad (52)
 \end{aligned}$$

The boundary conditions are written as:

Velocity:

$$\begin{aligned}
 u_{k+1}^{101} = & \frac{-\frac{2-\alpha_u}{\alpha_u} \sqrt{\frac{\pi \gamma}{\gamma-1}} \cos \theta_w (\Theta_k^{101})^{w+1/2}}{B P_{k+1}^{101} \sigma_{w,k+1}} \left\{ \frac{u_k^{101} - u_k^{100} + u_{k+1}^{101} - u_{k+1}^{100}}{2 \Delta \eta} \right. \\
 & + \sigma_w' \frac{V_k^{101} - V_k^{100}}{\Delta \eta} - \sigma_w \sin \theta_w \cos \theta_w \frac{u_{k+1}^{101} - u_k^{101}}{\Delta x} \\
 & \left. - \sigma_w \sin^2 \theta_w \frac{V_k^{101} - V_{k-1}^{101}}{\Delta x} \right\} \quad (53)
 \end{aligned}$$

Here P , σ_w , and θ_w are all evaluated at the end of the step. The thermal boundary condition, for a prescribed wall temperature, is

$$\begin{aligned}
 \Theta_{k+1}^{101} - \Theta_w(x_{k+1}) = & \frac{-\frac{2-\alpha_T}{\alpha_T} \sqrt{\frac{\pi \gamma}{\gamma-1}} \frac{2 \gamma}{B (\gamma+1)} (\Theta_k^{101})^{w+1/2}}{B P_{k+1}^{101} \sigma_{w,k+1} \cos \theta_w} \times \\
 & \times \left\{ \frac{\Theta_{k+1}^{101} - \Theta_{k+1}^{100} + \Theta_k^{101} - \Theta_k^{100}}{2 \Delta \eta} - \sigma_w \cos \theta_w \sin \theta_w \frac{\Theta_{k+1}^{101} - \Theta_k^{101}}{\Delta x} \right\} \quad (54)
 \end{aligned}$$

while for an adiabatic flow:

$$\begin{aligned}
 & \frac{\Theta_k^{101} - \Theta_k^{100} + \Theta_{k+1}^{101} - \Theta_{k+1}^{100}}{B \cdot 2 \Delta \eta} + \left(2 + \frac{4}{3} \sigma_w'^2\right) u_k^{101} \frac{u_k^{101} - u_k^{100} + u_{k+1}^{101} - u_{k+1}^{100}}{2 \Delta \eta} \\
 & - \frac{4}{3} \sigma_w \sigma_w' u_k^{101} \frac{u_{k+1}^{101} - u_k^{101}}{\Delta x} + \frac{2}{3} \sigma_w' u_k^{101} \frac{V_k^{101} - V_k^{100}}{\Delta \eta} \\
 & + 2 \sigma_w u_k^{101} \frac{V_k^{101} - V_{k-1}^{101}}{\Delta x} - \frac{4}{3} (V_k^{101})^2 = 0 \quad (55)
 \end{aligned}$$

B. Method of Solution

In the present work, there is a set of four equations to be solved, which are very similar to those treated in the previous work ⁽¹¹⁾. In that work, only two equations (the energy and axial momentum equations) had to be solved by a finite-difference procedure. Following the solution of that pair, the continuity equation was used to determine the radial velocity, and an integrated form of either the streamtube equation or the continuity equation was then used to find the pressure.

In the present work, the energy and axial momentum equations are changed only slightly. However, the radial momentum equation must be solved numerically for the radial velocity, and then the streamtube equation must be solved numerically for the pressure distribution across the channel.

Ideally, one could solve all four of these equations simultaneously. However, a great savings in algebraic complexity can be had by solving the equations one at a time, using, for the values of the other three dependent variables, either the values most recently computed, or an extrapolation from the results at the previous station. After all four equations have been solved in this way, the whole process is repeated, and the iterations continue until some preassigned measure of convergence is satisfied. This process has been used in many comparable numerical studies of viscous flows - see, for example, the thesis of Novack ⁽¹¹⁾.

A slight variation of this process was used in the present work: the energy and axial momentum equations were first solved as a pair, and then the radial momentum and streamtube equations were solved, one at a time. This was done so that the basic programming of the previous work ⁽¹¹⁾, in which the first two equations were solved simultaneously, could be used.

The energy and axial momentum equations are written as:

$$\begin{aligned}
& - \begin{pmatrix} a_{11}^L & a_{12}^L \\ a_{21}^L & a_{22}^L \end{pmatrix} \begin{pmatrix} U_{k+1}^{L+1} \\ \Theta_{k+1}^{L+1} \end{pmatrix} + \begin{pmatrix} b_{11}^L & b_{12}^L \\ b_{21}^L & b_{22}^L \end{pmatrix} \begin{pmatrix} U_{k+1}^L \\ \Theta_{k+1}^L \end{pmatrix} - \begin{pmatrix} c_{11}^L & c_{12}^L \\ c_{21}^L & c_{22}^L \end{pmatrix} \begin{pmatrix} U_{k+1}^{L-1} \\ \Theta_{k+1}^{L-1} \end{pmatrix} = \\
& = \begin{pmatrix} d_1^L \\ d_2^L \end{pmatrix}, \quad L = 2, 3, \dots, 100
\end{aligned} \tag{56}$$

The matrix coefficients are given in Appendix D. The solution then proceeds exactly as on pages 22-25 of Ref. 1, with the exception that the boundary conditions are slightly different. In particular:

$$M_{11} = 1 + \frac{2}{K_1} - 2\sigma_w \sin \theta_w \cos \theta_w \frac{\Delta \eta}{\Delta x}; \quad M_{12} = 0$$

$$\begin{aligned}
M_1 = U_K^{101} \left\{ 1 + 2\sigma_w \sin \theta_w \cos \theta_w \frac{\Delta \eta}{\Delta x} \right\} - U_K^{101} + 2V_K^{101} \left(\sigma_w' - \sigma_w \sin^2 \theta_w \frac{\Delta \eta}{\Delta x} \right) \\
- 2\sigma_w' V_K^{100} + 2\sigma_w \sin^2 \theta_w \frac{\Delta \eta}{\Delta x} V_{K-1}^{101}
\end{aligned} \tag{57}$$

$$K_1 = \frac{\frac{2-\alpha_u}{\alpha_u} \sqrt{\frac{\pi \gamma}{\gamma-1}} \cos \theta_w (\Theta_K^{101})^{\omega+1/2}}{B P_{K+1}^{101} \sigma_w)_{K+1} \Delta \eta}$$

For an adiabatic wall,

$$M_{21} = U_{k+1}^{101} (1 - M_{11}) U_K^{101} \cdot R \cdot \left\{ \left(2 + \frac{4}{3} \sigma_w' \right) - \frac{8}{3} \sigma_w \sigma_w' \frac{\Delta \eta}{\Delta x} \right\}$$

$$M_{21} = 1$$

$$\begin{aligned} n_2 = & \Theta_K^{101} - \Theta_K^{100} + R \left(2 + \frac{4}{3} \sigma_w'^2 \right) \mathcal{U}_K^{101} (\mathcal{U}_K^{101} - \mathcal{U}_K^{100}) \\ & + \frac{8}{3} R \sigma_w \sigma_w' (\mathcal{U}_K^{101})^2 \frac{\Delta \eta}{\Delta x} + \frac{4}{3} R \sigma_w' \mathcal{U}_K^{101} (V_K^{101} - V_K^{100}) \end{aligned} \quad (58)$$

$$+ 4 R \sigma_w \mathcal{U}_K^{101} \frac{\Delta \eta}{\Delta x} (V_K^{101} - V_{K-1}^{101}) - \frac{8}{3} R \Delta \eta (V_K^{101})^2$$

When heat transfer is allowed, these are replaced by:

$$M_{21} = 0 \quad M_{22} = 1 + \frac{2}{K_2} - 2 \sigma_w \cos \theta_w \sin \theta_w \frac{\Delta \eta}{\Delta x}$$

$$n_2 = \Theta_K^{101} - \Theta_K^{100} - \frac{2}{K_2} \Theta_{w,K+1} + 2 \sigma_w \cos \theta_w \sin \theta_w \frac{\Delta \eta}{\Delta x} \cdot \Theta_K^{101}$$

$$K_2 = \frac{\frac{2 - \alpha_T}{\alpha_T} \sqrt{\frac{\pi \gamma}{\gamma - 1}} \frac{2 \gamma}{R (\gamma + 1)} (\Theta_K^{101})^{\omega + 1/2}}{B P_{K+1}^{101} \sigma_{w,K+1} \cos \theta_w \Delta \eta} \quad (59)$$

Next, the radial momentum equation is solved. It is first written in the form

$$-A_K^L V_{K+1}^{L+1} + B_K^L V_{K+1}^L - C_{K+1}^{L-1} = D_K^L, \quad L = 2, 3, \dots, 100 \quad (60)$$

with boundary conditions $V_{K+1}^1 = 0$, $V_{K+1}^{101} = \tau_w' u_{K+1}^{101}$. The coefficients A_K^L , B_K^L , C_K^L and D_K^L , which are given in Appendix D, all involve known quantities, some of which (such as $\bar{p}$, for instance) may be at station $K+1$. These average quantities are always evaluated using the data from the most recently completed phase.

The solution of this equation is found in the usual way (Ref. 10), by applying the recursion formula

$$V_{K+1}^L = E^L V_{K+1}^{L+1} + F^L, \quad L = 1, 2, \dots, 100 \quad (61)$$

Substitution of this, with L replaced by $L-1$, into Eq. 60, followed by comparison with Eq. 61 itself, gives the recursion relations:

$$E^L = \frac{A_K^L}{B_K^L - C_K^L E^{L-1}}, \quad F^L = \frac{D_K^L + C_K^L F^{L-1}}{B_K^L - C_K^L E^{L-1}} \quad (62)$$

The boundary condition on the axis is enforced by taking $E^1 = F^1 = 0$, and these equations are then used recursively to find E^L and F^L for L ranging from 2 to 100. The wall value V_{K+1}^{101} is then assigned from the wall boundary condition, and Eq. 61 is then used to calculate the values of V for $L = 100$ down to $L = 2$.

The solution of the streamtube equation can be accomplished explicitly, by recognizing that the left-hand side of Eq. 48 is just the directed derivative along a streamline - i.e., it can be written as

$$\frac{dP}{dS} = \frac{RHS}{(1-M^2) \cos \theta} \quad (63)$$

where S denotes distance along a streamline, and RHS stands for the right-hand side of Eq. 48.

A computer program for performing the numerical steps described above has been written, and is listed in Appendix E. In its present form, it will only do a calculation for a specific mass flow. If it had been able to run stably, it would have been expanded to incorporate the necessary logic for iterating on the mass flow, and for crossing the saddle-point where the local value of $1 - M^2$ goes through zero. However, as pointed out in the section below, this program contains a very serious instability, which it has not been possible to overcome.

C. Results

The computer program described in the section above was run a number of times, for an axisymmetric, adiabatic-wall case, with the values:

$$\begin{aligned} \theta_1 &= -30^\circ, \quad R_1 = 0.5, \quad \theta_2 = +20^\circ \\ B &= 1230, \quad A = 0.12 \\ \gamma &= 1.4, \quad R_2 = 0.75, \quad \omega = 0.75, \quad \alpha_u = \alpha_r = 1.0 \end{aligned} \quad (64)$$

The calculations were started at $x_0 = -5.3$; a step size $\Delta x = 0.1$ was used. (The input values of x_1 and x_2 were ± 0.3 , so that a value of $\Delta x = 0.01$ would have been used in the interval $-0.3 \leq x \leq +0.3$, if the calculation had gone that far.) These values were chosen to match a calculation that had been done with the previous program, and had been compared with experiment by Rothe^(3,4).

From the outset, these runs displayed a characteristic instability. The radial pressure distribution, which is perfectly smooth at the initial station, would begin to show small departures from a monotonic behavior. These would be reflected in an even larger variation in $\partial P / \partial r$, which in turn would cause oscillations in V , through the radial momentum equation. These oscillations in V would then be fed into the streamtube equation, where they would cause even greater oscillations in P . The amplification factor in this sequence was always greater than one - no way was found to fully suppress this amplification, and the result was always that the calculation would be terminated, after a few steps, when one or more variables

exceeded the overflow capacity of the IBM 370/165 computer used for the calculations. A typical result is shown in Figure 1. The smooth profile used as a starting condition at $x = x_0 = -5.3$ has already developed a small oscillation at the end of the first step, and after three steps it has grown beyond any reasonable bound.

Ideally, this research program should have been interrupted as soon as the severity of the instability became clear, and a major re-direction of effort made, to thoroughly understand the problem. Unfortunately, the duration and funding level were insufficient to allow this approach. Instead, a series of ad hoc variations in the computer program was made, in an attempt to suppress the oscillation. Approximately 120 separate runs were made in this effort. Some of these were partially successful, but none of them was completely so. In view of this fact, there does not seem to be any merit in presenting detailed results for each attempt. Instead, a brief sketch of some of the major efforts is given in the following paragraphs, in the hope that they may provide some guidance for any future attempts to overcome the problem.

Algebraic Rearrangement - Certain forms of the streamtube equation contain terms which are inversely proportional to the square of the axial Mach number M^{-2} . Since this quantity is very small in the reservoir, it makes the associated terms quite large. Several rearrangements of the terms in this equation were used, in order to avoid taking small differences of large numbers.

Filtering - There are many techniques for numerically smoothing an oscillatory solution. To test the applicability of such a technique, several runs were made in which the pressure at any radial station was taken to be the average of the values just calculated at the stations above and below it.

Wall Pressure Boundary Condition - During some early runs, the wall pressure was found by extrapolation from the two points nearest the wall. To gain greater accuracy, a finite-difference approximation of the streamtube relation at the wall was written, and used as a boundary condition.

Variations in the Order of Calculation - In all runs, the values of U and Θ are calculated first. In several, the order of calculating P and V was reversed.

Revision of the ∇ -Equation - There is a group of terms in the ∇ -equation, involving velocity components and their derivatives, which is an expression for the divergence of the velocity field. This quantity is small in the nearly incompressible flow conditions upstream of the throat, and it was suspected that errors might be entering because of imperfect cancellation of the various velocity gradients. Thus a new form of the ∇ -equation was derived, differenced, programmed, and run, in which these terms were replaced by the equivalent gradients of the density field.

Use of the $\mathcal{W}$ -Equation - The quantity $\mathcal{W}$, defined by

$$\mathcal{W} = \nabla - \gamma \tau'_w \mathcal{U}$$

is very small. In the leading terms of the asymptotic solution far upstream, it is identically zero, and the calculations made during the previous program always showed it to be small. Furthermore, it responds to the axial pressure gradient as well as to the radial one. Thus it was thought that this equation would be less susceptible to the instability. This was indeed found to be true - the oscillation did build up more slowly, but it was still fatal.

Original Continuity Equation - The streamtube equation, which is derived from the continuity, energy, and axial momentum equations, is fairly complicated, although it has the virtue of showing clearly the contribution of various terms to the pressure gradient. It was hoped that a simpler equation might afford less opportunity for the instability to build up, and accordingly the streamtube equation was replaced by

$$\frac{\partial}{\partial x} (D \gamma \mathcal{W}) + \tau_w \frac{\partial}{\partial x} (D \gamma \mathcal{U}) + 2 \tau'_w D \gamma \mathcal{U} = 0$$

This equation was also programmed and run, but did not improve the situation.

Use of $\mathcal{W} \equiv 0$ - Several runs were made in which the radial momentum equation was bypassed altogether, and the radial velocity calculated from the approximation $\nabla = \gamma \tau'_w \mathcal{U}$. One of these runs went as far as $x = -2.5$ before instability. This shows that the instability involves a coupling between V and P .

New Streamtube Equation - By working in coordinates along, and normal to the streamlines, it is possible to derive the form of the streamtube equation given in Eq. 37. Its use has not eliminated the problem, but it has been retained in the form of the program presented here, since it yields results

which are somewhat easier to interpret physically.

Smoothing of $\sqrt{\nu}$ Near the Centerline - Much of the instability comes from calculating the streamline direction $\sqrt{\nu}/u$, where both ν and u are very small. It was found that instabilities near the centerline could be removed by fairing $\sqrt{\nu}$ in such a way that the quantity $\sqrt{\nu}/u$, for $0 \leq \eta \leq 0.05$ was set equal to its calculated value at $\eta = 0.05$. A wide variety of comparable fairing formulas was used to fair $\sqrt{\nu}$ near the wall, all without success.

Elimination of $\frac{\partial}{\partial x}(\sqrt{\nu}/u)$ - Small errors in ν and u (both of which are themselves quite small) produce large errors in the term $\frac{\partial}{\partial x}(\sqrt{\nu}/u)$. By suitable algebraic manipulation, this term can be replaced in favor of $\partial P/\partial x$ and $\partial P/\partial \eta$, which in turn can be approximated by their values at the previous iteration. This version of the streamtube equation is the one given in this report.

Fairing of the Pressure Profile - After the P -profile has been calculated, it can then be faired by a polynomial. Many such polynomials were used (up to a quartic) with various constraints on the fitting points. Some of these showed promise: in one calculation, fitting of a single cubic in η produced a stable result down to $X = -2.7$. However, it was found that the $\sqrt{\nu}$ -profile resulting from this fairing of the P -profile was physically unacceptable (too large negative, especially near the wall).

Iterations on Pressure Gradient - In the previous work (1, 2), where $\partial P/\partial \eta$ was zero, the axial pressure gradient was found by iterating on it until mass conservation was satisfied. An attempt was made to do the same type of iteration in this program, where 100 streamtubes must all be satisfied simultaneously. The runs made with this technique were not successful.

The situation can be summarized by stating that a relatively large number of fixes has been tried, some of which were partially successful. (In particular, the instability near the axis, which was present at the time the calculations for Fig. 1 were made, has been completely removed.) However, none of the approaches used was capable of removing the instability near the wall.

IV. CONCLUDING REMARKS

The study reported above was initiated with the aim of predicting the radial pressure variations that exist at low Reynolds numbers in nozzles with very sharp throats. A composite set of equations was chosen, which appear to contain all of the terms necessary for describing these effects. The particular computational efforts exerted to evaluate these equations encountered a severe instability, which it was not possible to overcome.

It is tempting to speculate on what may have caused the instability, and on what other approaches might have proven effective. However, the evidence gathered does not afford a safe basis for such speculation. It is clear that some form of unstable feedback between the P and V calculations is responsible for the problem, but it is not possible to say how this feedback is affected by such factors as the difference scheme used, the step size, the sequence of performing the iterations, and the choice of terms retained. The present studies have shown that further progress on the problem of radial pressure variations in low Reynolds number nozzle flows must await the development of a deeper understanding of the roles played by these factors.

V. REFERENCES

1. Rae, W.J., "Final Report on a Study of Low-Density Nozzle Flows with Application to Microthrust Rockets", CAL Tech. Report No. AI-2590-A-1, Dec. 1969, Cornell Aeronautical Laboratory, Inc., Buffalo, New York.
2. Rae, W.J., "Some Numerical Results on Viscous Low-Density Nozzle Flows in the Slender-Channel Approximation", AIAA Journal, 9, No. 5 (May 1971) 811-820.
3. Rothe, D.E., "Experimental Study of Viscous Low-Density Nozzle Flows", CAL Tech. Report No. AI-2590-A-2, June 1970, Cornell Aeronautical Laboratory, Inc., Buffalo, New York.
4. Rothe, D.E., "Electron-Beam Studies of Viscous Flow in Supersonic Nozzles", AIAA Journal, 9, No. 5 (May 1971) 804-811.
5. Cuffel, R.F., Back, L.H., and Massier, P.F., "Transonic Flow-field in a Supersonic Nozzle with Small Throat Radius of Curvature", AIAA Journal, 7, No. 7 (July 1969) 1364-1366.
6. Cheng, H.K., Chen, S.Y., Mobley, R., and Huber, C.R., "The Viscous Hypersonic Slender-Body Problem: A Numerical Approach Based on a System of Composite Equations", RAND Corp. Memorandum No. RM-6193-PR, May 1970.
7. Pai, S.I., Viscous Flow Theory, Vol. I, Laminar Flow, Van Nostrand Co., Princeton, N.J. (1956).
8. Ackerberg, R.C., "The Viscous Incompressible Flow Inside a Cone", J. Fluid Mech. 21, 47-81 (1965).
9. Oka, S., "The Steady Slow Motion of a Viscous Fluid Through a Tapered Tube", J. Phys. Soc. Japan 19 (1964) 1481-4.
10. Richtmeyer, R.D. and Morton, K.W., Difference Method for Initial-Value Problems. 2nd Ed. (Interscience Publishers, New York, 1967).
11. Novack, B.B., "Numerical Analysis of High Mach Number Slip Flow", PhD Thesis, Univ. of Southern California (1971).

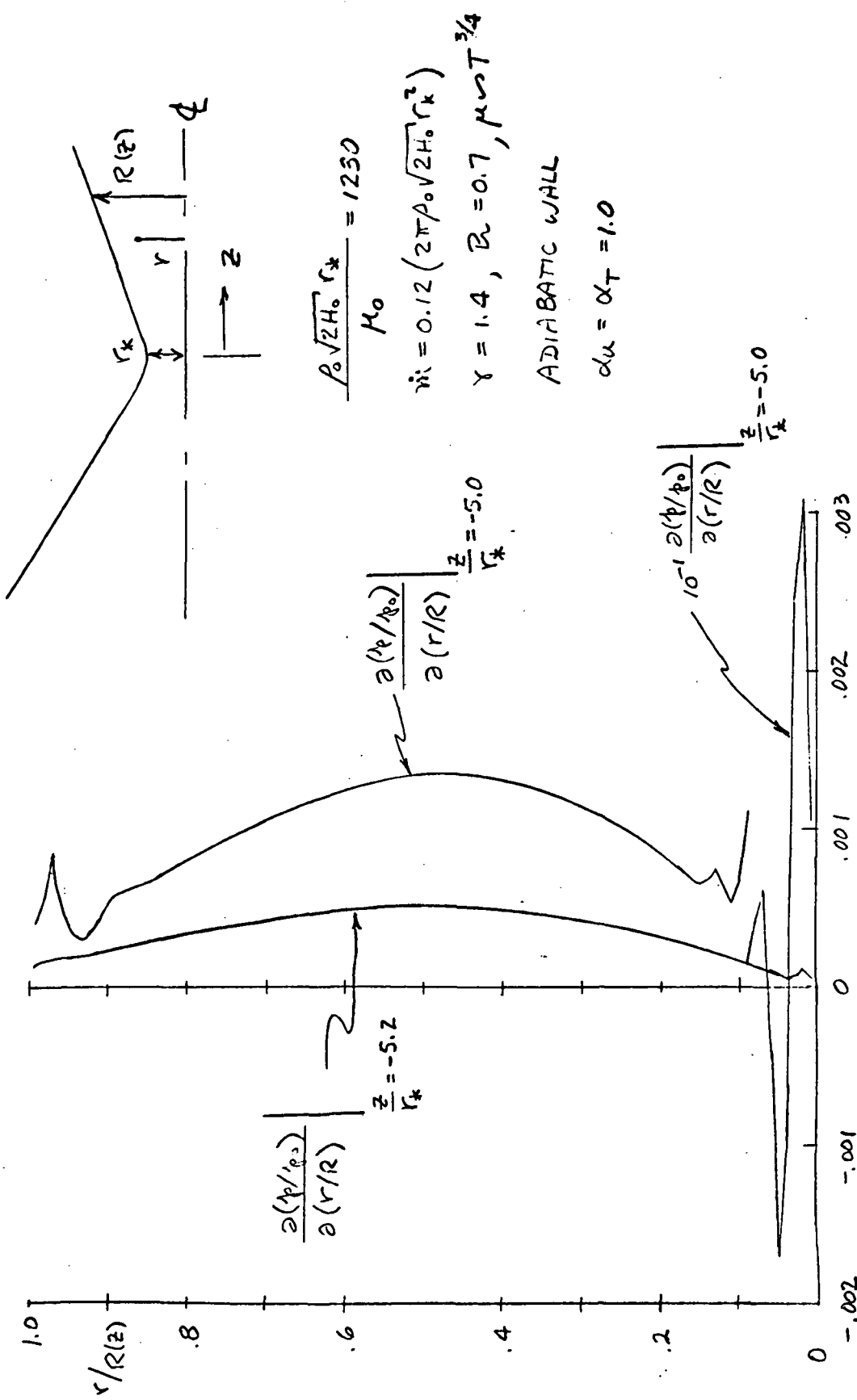


Figure 1. Radial Pressure Gradient Oscillation

APPENDIX A

DERIVATION OF THE STREAMTUBE EQUATION

To derive Eq. (33), derivatives of the density are first replaced in favor of derivatives of the pressure and static enthalpy

$$\frac{\partial \rho}{\partial z} = \frac{\gamma}{\gamma-1} \left\{ \frac{1}{h} \frac{\partial p}{\partial z} - \frac{p}{h^2} \frac{\partial h}{\partial z} \right\} \quad (\text{A-1})$$

This expression is substituted into the continuity equation. In the resulting expression, the energy and axial momentum equations (Eq. 23 and 17) are then used to replace $\partial u/\partial z$ and $\partial h/\partial z$ in favor of $\partial p/\partial z$. The result, after a little rearrangement, is

$$\begin{aligned} & \left(\frac{u^2}{a^2} - 1 \right) \frac{\partial p}{\partial z} + \frac{v^2}{a^2} \cdot \frac{u}{v} \frac{\partial p}{\partial r} + \frac{\rho u^2}{r} \frac{\partial}{\partial r} \left(r \frac{v}{u} \right) \\ & + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \\ & - \frac{u}{h} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (rk \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (k \frac{\partial T}{\partial z}) + \phi \right\} = 0 \end{aligned} \quad (\text{A-2})$$

If the radial pressure gradient is neglected, this reduces to Eq. (3-32) of the previous work⁽¹⁾.

In the present generalization, we would expect that the total Mach number, and not just its axial component, would appear as the factor multiplying the pressure gradient. To obtain this form, it is necessary to use the radial momentum equation. This is done by first noting the identity: (see next page)

$$\left(\frac{u^2}{a^2} - 1\right) \frac{\partial p}{\partial z} + \frac{v^2}{a^2} \frac{u}{v} \frac{\partial p}{\partial r} =$$

$$= \frac{\gamma p}{\cos \theta} \frac{u^2}{a^2} \left\{ \frac{g^2 a^2}{g^2} \cdot \frac{1}{\gamma p} \left[\frac{u}{g} \frac{\partial p}{\partial z} + \frac{v}{g} \frac{\partial p}{\partial r} \right] \right. \\ \left. + \frac{1}{g^2} \frac{v}{u} \left[-\frac{v}{g} \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{u}{g} \frac{1}{\rho} \frac{\partial p}{\partial r} \right] \right\} \quad (\text{A-3})$$

The pressure derivatives appearing in the last term of this expression are then replaced using the momentum equations (17 and 25). After considerable rearrangement, the form shown in Eq. (33) is obtained.

APPENDIX B

DERIVATIVE OF THE MODIFIED STREAMTUBE EQUATION

The derivation of the modified streamtube equation begins with a derivation of the expression for the rate of change of the flow inclination angle along a streamline. In an inviscid flow, this quantity responds only to the pressure gradient normal to a streamline:

$$-\frac{\partial p}{\partial n} = \sin \theta \frac{\partial p}{\partial z} - \cos \theta \frac{\partial p}{\partial r} \quad (\text{B-1})$$

The momentum equations are now used to replace the pressure gradients. The result is:

$$\sin \theta \frac{\partial p}{\partial z} - \cos \theta \frac{\partial p}{\partial r} = \sin \theta \left\{ \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \right\}$$

$$- \cos \theta \left\{ \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{2\mu}{r} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \right\}$$

$$+ \rho u^2 \left\{ \cos \theta \frac{\partial}{\partial z} \left(\frac{v}{u} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{v}{u} \right) \right\}$$

The last term on the right-hand side can be recognized as the rate of change of θ in the streamline direction.

If this equation is now used to eliminate $\frac{\partial}{\partial z} (v/u)$ from Eq. (33), the result is Eq. (37).

APPENDIX C

DERIVATION OF THE INITIAL CONDITIONS

If the expansions shown in Eq. (45) are substituted into the full equations of motion, the first few of the set of ordinary differential equations to be solved are (for the axisymmetric case):

Continuity:

$$W_2 = 0$$

$$(\eta W_3)' - \eta U_3 \sigma_w' = 0 \quad (C-1)$$

Axial Momentum:

$$(\eta U_2)' + \frac{(\gamma-1) B \sigma_w'}{2\gamma} \left\{ 3\eta \pi_3 + \eta^2 \pi_3' \right\} = 0 \quad (C-2)$$

Radial Momentum:

$$-\frac{\gamma-1}{2\gamma} B \pi_3' + \frac{4}{3} (\eta V_2)' - \frac{4}{3} \frac{V_2}{\eta} - \eta \sigma_w' U_2' - \frac{\eta^2 \sigma_w'}{3} U_2'' = 0 \quad (C-3)$$

These can be looked upon as three equations for U_2 , V_2 , and π_3 . If the relation $V_2 = \eta \sigma_w' U_2$ is now used in Eq. (C-3), the result can be integrated once, to give

$$-2K = -\frac{\gamma-1}{2\gamma} B \pi_3 + \sigma_w' (\eta U_2' + 2U_2) \quad (C-4)$$

where K is a constant of integration. This result is now used to eliminate π_3 from Eq. (C-2), giving a result that can be integrated once as

$$\eta U_2' + 3\sigma_w' K \eta^2 + \sigma_w'^2 (\eta^3 U_2)' = \text{const} = 0 \quad (C-5)$$

where the constant has been set equal to zero, by the boundary condition on the axis. The constant K can be evaluated, by letting $\eta \rightarrow 0$ in (C-5), as:

$$K = -\tau_w' u_2(0) - \frac{u_2''(0)}{3\tau_w'} \quad (C-6)$$

next, Eq. (C-5) is integrated, using an integrating factor, and the resulting constant is evaluated by conditions on the axis. The result is the first of eqs. (46). The quantity π_3 is then found directly from (C-4). Finally, the value of $u_2(0)$ is found from the condition that the total mass flow be conserved. In particular, this requires that

$$\int_0^1 u_2(\eta) \cdot \eta d\eta = A \quad (C-7)$$

APPENDIX D
MATRIX-COEFFICIENT EXPRESSIONS

The matrix coefficients appearing in Eq. (56) are:

$$Q_{11}^L = -\frac{\eta \bar{P} \bar{W}}{\bar{T} \bar{T}_w \cdot 4\Delta\eta} + \frac{(\bar{T})^\omega}{B \bar{T}_w^2} \left\{ \frac{\epsilon}{4\Delta\eta} + \frac{\omega\eta}{\bar{T}} \frac{(T_K^{L+1} - T_K^{L-1})}{8(\Delta\eta)^2} + \frac{\eta}{2(\Delta\eta)^2} \right\}$$

$$a_{12}^L = \frac{(\bar{T})^\omega}{B \bar{T}_w^2} \frac{\omega\eta}{\bar{T}} \frac{(\mathcal{U}_K^{L+1} - \mathcal{U}_K^{L-1})}{8(\Delta\eta)^2}$$

$$a_{21}^L = \frac{(\bar{T})^\omega}{B \bar{T}_w^2} \frac{\eta}{2(\Delta\eta)^2} (\mathcal{U}_K^{L+1} - \mathcal{U}_K^{L-1})$$

$$a_{22}^L = -\frac{\eta \bar{P} \bar{W}}{\bar{T} \bar{T}_w \cdot 4\Delta\eta} + \frac{(\bar{T})^\omega}{B \bar{T}_w^2 R} \left\{ \frac{\epsilon}{4\Delta\eta} + \frac{\omega\eta}{\bar{T}} \frac{(T_K^{L+1} - T_K^{L-1})}{4(\Delta\eta)^2} + \frac{\eta}{2(\Delta\eta)^2} \right\}$$

$$b_{11}^L = \frac{\eta \bar{P}}{\bar{T}} \frac{\bar{\mathcal{U}}}{\Delta x} + \frac{(\bar{T})^\omega}{B \bar{T}_w^2} \cdot \frac{\eta}{(\Delta\eta)^2}$$

$$b_{12}^L = 0$$

$$b_{21}^L = 0$$

$$b_{22}^L = \frac{\eta \bar{P}}{\bar{T}} \frac{\bar{\mathcal{U}}}{\Delta x} + \frac{(\bar{T})^\omega}{B \bar{T}_w^2 R} \cdot \frac{\eta}{(\Delta\eta)^2}$$

$$C_{11}^L = \frac{\eta \bar{P} \bar{W}}{\bar{T} \bar{T}_W 4\Delta\eta} - \frac{(\bar{T})^\omega}{B \bar{T}_W^2} \left\{ \frac{\epsilon}{4\Delta\eta} + \frac{\omega\eta}{\bar{T}} \frac{(T_K^{L+1} - T_K^{L-1})}{8(\Delta\eta)^2} - \frac{\eta}{2(\Delta\eta)^2} \right\}$$

$$C_{12}^L = - \frac{(\bar{T})^\omega}{B \bar{T}_W^2} \left\{ \frac{\omega\eta}{\bar{T}} \frac{(\alpha_K^{L+1} - \alpha_K^{L-1})}{8(\Delta\eta)^2} \right\}$$

$$C_{21}^L = - \frac{(\bar{T})^\omega}{B \bar{T}_W^2} \frac{\eta}{2(\Delta\eta)^2} (\alpha_K^{L+1} - \alpha_K^{L-1})$$

$$C_{22}^L = \frac{\eta \bar{P} \bar{W}}{\bar{T} \bar{T}_W 4\Delta\eta} - \frac{(\bar{T})^\omega}{B \bar{T}_W^2 R} \left\{ \frac{\epsilon}{4\Delta\eta} + \frac{\omega\eta}{\bar{T}} \frac{(T_K^{L+1} - T_K^{L-1})}{4(\Delta\eta)^2} - \frac{\eta}{2(\Delta\eta)^2} \right\}$$

$$d_1 = \frac{\eta \bar{P}}{\bar{T}} \left\{ \frac{\bar{\alpha} \alpha_K^L}{\Delta x} - \frac{\bar{W} (\alpha_K^{L+1} - \alpha_K^{L-1})}{\bar{T}_W 4\Delta\eta} \right\} - \frac{\gamma-1}{2\gamma} \eta \cdot \text{GRADP} \\ + \frac{(\bar{T})^\omega}{B \bar{T}_W^2} \left\{ \frac{\epsilon (\alpha_K^{L+1} - \alpha_K^{L-1})}{4\Delta\eta} + \frac{\eta}{2(\Delta\eta)^2} (\alpha_K^{L+1} - 2\alpha_K^L + \alpha_K^{L-1}) \right\}$$

$$d_2 = \frac{\eta \bar{P}}{\bar{T}} \left\{ \frac{\bar{\alpha} T_K^L}{\Delta x} - \frac{\bar{W} (T_K^{L+1} - T_K^{L-1})}{\bar{T}_W 4\Delta\eta} \right\} + \frac{\gamma-1}{\gamma} \eta \cdot \text{QGRADP} \\ + \frac{(\bar{T})^\omega}{B \bar{T}_W^2 R} \left\{ \epsilon \frac{(T_K^{L+1} - T_K^{L-1})}{4\Delta\eta} + \frac{\eta}{2(\Delta\eta)^2} (T_K^{L+1} - 2T_K^L + T_K^{L-1}) \right. \\ \left. + \text{VDISS} \right\}$$

where

$$\text{GRADP} = \frac{P_{K+1}^L - P_K^L}{\Delta x} - \frac{\eta \bar{\tau}_w'}{\bar{\tau}_w} \frac{P_K^{L+1} - P_K^{L-1} + P_{K+1}^{L+1} - P_{K+1}^{L-1}}{4\Delta\eta}$$

$$\text{QGRADP} = \bar{u} \frac{P_{K+1}^L - P_K^L}{\Delta x} + \frac{\bar{w}}{\bar{\tau}_w} \frac{P_K^{L+1} - P_K^{L-1} + P_{K+1}^{L+1} - P_{K+1}^{L-1}}{4\Delta\eta}$$

$$\text{VDISS} = \frac{\theta}{3} \bar{u} \eta \left\{ \frac{(V_K^{L+1} - V_K^{L-1})^2}{4(\Delta\eta)^2} - \epsilon \frac{V_K^L}{\eta} \frac{V_K^{L+1} - V_K^{L-1}}{2\Delta\eta} + \epsilon \left(\frac{V_K^L}{\eta} \right)^2 \right\}$$

The coefficients appearing in the radial momentum equation are:

$$A_K^L = -\eta \frac{\bar{p}}{\bar{\tau}} \frac{\bar{w}}{4\bar{\tau}_w} + \frac{(\bar{\tau})^\omega}{B\bar{\tau}_w^2} \left\{ \frac{\eta}{3} \frac{\omega}{\bar{\tau}} \left(\overline{\frac{\partial \theta}{\partial \eta}} \right) + \frac{\epsilon}{3} + \frac{2}{3} \frac{\eta}{\Delta\eta} \right\}$$

$$B_K^L = \eta \frac{\bar{p}}{\bar{\tau}} \bar{u} \frac{\Delta\eta}{\Delta x} + \frac{(\bar{\tau})^\omega}{B\bar{\tau}_w^2} \left\{ \frac{\epsilon \omega \Delta\eta}{3 \bar{\tau}} \left(\overline{\frac{\partial \theta}{\partial \eta}} \right) + \frac{2\epsilon}{3} \frac{\Delta\eta}{\eta} + \frac{4}{3} \frac{\eta}{\Delta\eta} \right\}$$

$$C_K^L = \eta \frac{\bar{p}}{\bar{\tau}} \frac{\bar{w}}{4\bar{\tau}_w} - \frac{(\bar{\tau})^\omega}{B\bar{\tau}_w^2} \left\{ \frac{\eta}{3} \frac{\omega}{\bar{\tau}} \left(\overline{\frac{\partial \theta}{\partial \eta}} \right) + \frac{\epsilon}{3} - \frac{2}{3} \frac{\eta}{\Delta\eta} \right\}$$

$$\begin{aligned}
D_K^L = & \frac{\eta \bar{P}}{\bar{T}} \left\{ \frac{\bar{U} V_K^L \Delta \eta}{\Delta x} - \frac{\bar{W}}{4 \bar{T}_w} (V_K^{L+1} - V_K^{L-1}) \right\} - \frac{\gamma-1}{2\gamma} \frac{\eta}{\bar{T}_w} \left(\frac{\partial P}{\partial \eta} \right) \Delta \eta \\
& + \frac{(\bar{T})^4}{3B \bar{T}_w^2} \left\{ (V_K^{L+1} - V_K^{L-1}) \left(\frac{\eta \omega}{\bar{T}} \left(\frac{\partial \Theta}{\partial \eta} \right) + \epsilon \right) - \frac{\epsilon V_K^L \omega}{\bar{T}} \left(\frac{\partial \Theta}{\partial \eta} \right) \Delta \eta \right. \\
& \quad - \frac{2 \epsilon V_K^L \Delta \eta}{\eta} + \frac{2 \eta}{\Delta \eta} (V_K^{L+1} - 2 V_K^L + V_K^{L-1}) \\
& \quad - \frac{\omega \Delta \eta}{\bar{T}} \left(\frac{\partial \Theta}{\partial \eta} \right) \left[2 \eta \bar{T}_w \left(\frac{\partial \bar{U}}{\partial x} \right) + \eta^2 \bar{T}_w' \left(\frac{\partial \bar{U}}{\partial \eta} \right) \right] \\
& \quad + \frac{\eta \bar{T}_w}{2 \Delta x} (\mathcal{U}_{K+1}^{L+1} - \mathcal{U}_{K+1}^{L-1} - \mathcal{U}_K^{L+1} + \mathcal{U}_K^{L-1}) - \eta \bar{T}_w \Delta \eta \left(\frac{\partial \bar{U}}{\partial \eta} \right) \\
& \quad \left. - \eta \bar{T}_w' \left(\frac{\partial^2 \bar{U}}{\partial \eta^2} \right) \Delta \eta + \frac{3 \eta \bar{T}_w \omega}{\bar{T}} \left(\frac{\partial \Theta}{\partial x} \right) \left(\frac{\partial \bar{U}}{\partial \eta} \right) \Delta \eta \right\}
\end{aligned}$$

APPENDIX E

COMPUTER PROGRAM LISTING

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C
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FINITE-DIFFERENCE SOLUTION FOR LOW REYNOLDS NUMBER
NOZZLE FLOW, INCLUDING RADIAL PRESSURE GRADIENT
IMPLICIT REAL*8 (A-H,O-Z)
COMMON A,ALPHU,ALPHT,AC,A11,A12,A21,A22,AN1,AN2,ABDA1,ABDA2,AKL,
*   AINT,AINTM1,ANEW
COMMON B,BKL,B11,B21,B22,BOTTOM,B12
COMMON CPKU,CFBOT,CS2TW,C,CKL,C11,C12,C21,C22
COMMON DELX,DELETA,DELXS,DPDXW,DVU,DU,DT,DDU,DDT,DV,DKL,DENOM,
*   D1,D2,DA1,DA2,DTBR,DPBR,DDV,DUXBR,DUBR,DDUBR,DTXBR,DPLAST
COMMON EM11,EM12,EM21,EM22,ERR,EP
COMMON FN1,FN2
COMMON GAMMA,G4BGM1,GDX,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,
*   G13,G14,G15,G16,G17,G18,G19,G20,GRADP
COMMON HFLUX,HTR,HDELX
COMMON OMEGA,O3,O4,O5,OHMPH
COMMON PR,PI3,PGRT,PG1,PG2,PHW,PART1,PART2,PG
COMMON QGRADP,QQ11,QQ12,QQ21,QQ22,Q11,Q12,Q21,Q22
COMMON R1,RADDEG,RTPB2,R12,RT2GM1,R1ST1,R1ST2,RTZ8Z7,RM,RMT1,RMT2,
*   RMT
COMMON SA1,SC1,SW2,SW3,SW4,S,SG2,SIGMAW,SIGBAR,SCALE,SWLRT,SW2K
COMMON THETA1,THETA2,TW2,TCFTOP,THETAW,THETAP,TANTHP,TANBAR,TOP,
*   TCF,THW,TUHW,TLHW,TCL,T51,T99,TERR
COMMON UCFTOP,UCF,UHW,UUHW,ULHW,UCL,U51,U99,UERR
COMMON VDISS,VWKM1,VEE,VHW,VUHW,VLHW,V51,VERR
COMMON W
COMMON XPRINT,X0,X1,X2,XMAX,XTW1,XTW2,XPLAST,X,XSQ,XHW,XORG,XEF
COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y10
COMMON Z1,Z2,Z5,Z7,Z8,Z10,Z11,Z12,Z13,Z15,Z23,Z31,Z32,Z33,Z34,
*   Z35,Z42,Z43,Z41,ZR1,ZR2,ZR3,ZR4,ZR5,ZR6,ZR7,ZR8,ZR9,ZR10,
*   ZM,ZR11,ZR12,ZR13,ZR14
COMMON IBC,IETAPR,IX,IXTHW,ITER,I,ITERU
COMMON K,KX,KI
COMMON L,LC
COMMON NU,NUEPS
COMMON ETA(101),U(101,2),V(101,2),P(101,2),T(101,2),VV(101,2),
*   E11(101),E12(101),E22(101),E21(101),F1(101),F2(101)
CALL INPUT
CALL INIT
1 CALL START
5 VWKM1=V(101,1)
DO 4 L=1,101
E11(L)=2.000*U(L,2)-U(L,1)
E12(L)=2.000*V(L,2)-V(L,1)
E21(L)=2.000*P(L,2)-P(L,1)
E22(L)=2.000*T(L,2)-T(L,1)
U(L,1)=U(L,2)
T(L,1)=T(L,2)
V(L,1)=V(L,2)
P(L,1)=P(L,2)
U(L,2)=E11(L)
V(L,2)=E12(L)
P(L,2)=E21(L)
4 T(L,2)=E22(L)
ITER=1

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      GO TO (21,22,23,23), IX
21  IF(X.LT.X1) GO TO 23
      IX=2
      KI=KI*10
      DELX=DELX/10.000
      Z41=Z34*DELX
      GO TO 23
22  IF(MOD(KI,10).NE.0) GO TO 23
      IF (X.LT.X2) GO TO 23
      IX=3
      KI=KI/10
      DELX=DELXS
      Z41=Z34*DELX
23  KI=KI+1
      SW2K=SIGMAW**2
      X=X0+DFLOAT(KI)*DELX
      KX=KX+1
      HDELX=0.500*DELX
      XHW=X-HDELX
      GO TO (41,42,14), IXTHW
41  IF(X.LE.XTW1) GO TO 14
      IXTHW=2
42  IF(X.GT.XTW2) GO TO 43
      THETA=03+04*DCOS(05*(X-XTW1))
      GO TO 14
43  IXTHW=3
      THETA=TW2
14  CALL GEOM(XHW)
      SIGBAR=SIGMAW
      TANBAR=TANTHP
C    CALCULATION OF U AND T
415  E11(1)=1.000
      E12(1)=0
      E21(1)=0
      E22(1)=1.000
      F1(1)=0
      F2(1)=0
      UCL=U(1,2)
      U51=U(51,2)
      U99=U(99,2)
      TCL=T(1,2)
      T51=T(51,2)
      T99=T(99,2)
      V51=V(51,2)
      Y1=Z35*SIGBAR**2
      Y2=2.000*Y1
      Y3=4.000*Y1
      Y4=8.000*Y1
      Y5=Z34*SIGBAR
      Y6=Z41*SIGBAR
      Y7=Y1*PR
      Y8=4.000*Y7
      Y10=B*PR*SIGBAR**2
      DO 6 L=2,100
      PHW=0.500*(P(L,1)+P(L,2))
      UHW=0.500*(U(L,1)+U(L,2))
      VHW=0.500*(V(L,1)+V(L,2))
      THW=0.500*(T(L,1)+T(L,2))

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TUHW=0.5D0*(T(L+1,1)+T(L+1,2))
TLHW=0.5D0*(T(L-1,1)+T(L-1,2))
UUHW=0.5D0*(U(L+1,1)+U(L+1,2))
ULHW=0.5D0*(U(L-1,1)+U(L-1,2))
VUHW=0.5D0*(V(L+1,1)+V(L+1,2))
VLHW=0.5D0*(V(L-1,1)+V(L-1,2))
W=VHW-ETA(L)*UHW*TANBAR
G1=PHW*ETA(L)
G2=W*G1
G3=Y5*THW
G4=THW**Z32
G5=Z33*THW
G6=OMEGA*ETA(L)
G7=T(L+1,1)-T(L-1,1)
G8=G6*G7
G9=4.0D0*ETA(L)
G10=G9*THW
G11=U(L+1,1)-U(L-1,1)
G12=THW**OMEGA
G13=ETA(L)*G12
G14=DELETA*THW
G15=2.0D0*ETA(L)
G16=G15*THW
G17=DELX*THW
G18=G1*UHW
G19=DELX*W
G20=Y6*THW
GRADP=(P(L,2)-P(L,1))/DELX-ETA(L)*TANBAR*(P(L+1,1)
* -P(L-1,1)+P(L+1,2)-P(L-1,2))/Y5
QGRADP=UHW*(P(L,2)-P(L,1))/DELX+W*(P(L+1,1)-P(L-1,1)+P(L+1,2)
* -P(L-1,2))/Y5
VDISS=8.0D0*PR*ETA(L)*(((VUHW-VLHW)/Z33)**2
*+EP*( - VHW *( VUHW -VLHW )/(ETA(L)*Z33)
* +(VHW/ETA(L))**2))/3.0D0
A11=-G2/G3+G4*(G5*EP+G8+G10)/Y4
A12=G6*G4*G11/Y4
A21=G13*G11/Y2
A22=-G2/G3+G4*(G14*EP+G8+G16)/Y8
B11=G18/G17+G13/Y1
B12=0
B21=0
B22=G18/G17+G13/Y7
C11=G2/G3-G4*(G5*EP+G8-G10)/Y4
C12=-A12
C21=-A21
C22=G2/G3-G4*(G14*EP+G8-G16)/Y8
D1=G1*(Y5*U(L,1)*UHW-G19*G11)/G20-Z7*ETA(L)*GRADP/Z8
* +G12*(DELETA*G11*EP+G15*(U(L+1,1)-2.0D0*U(L,1)+U(L-1,1)))/Y3
D2=G1*(Y5*UHW*T(L,1)-G19*G7)/G20+Z7*ETA(L)*QGRADP/GAMMA
* +G12*(DELETA*G7*EP+G15*(T(L+1,1)-2.0D0*T(L,1)+T(L-1,1)))/Y8
* +G12*VDISS/Y10
QQ11= B22-C21*E12(L-1)-C22*E22(L-1)
QQ12=-B12+C11*E12(L-1)+C12*E22(L-1)
QQ21=-B21+C21*E11(L-1)+C22*E21(L-1)
QQ22= B11-C11*E11(L-1)-C12*E21(L-1)
DENOM=QQ11*QQ22-QQ21*QQ12
Q11=QQ11/DENOM
Q12=QQ12/DENOM

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Q21=QQ21/DENOM
Q22=QQ22/DENOM
PART1=C11*F1(L-1)+C12*F2(L-1)+D1
PART2=C21*F1(L-1)+C22*F2(L-1)+D2
E11(L)=Q11*A11+Q12*A21
E12(L)=Q11*A12+Q12*A22
E21(L)=Q21*A11+Q22*A21
E22(L)=Q21*A12+Q22*A22
F1(L) =Q11*PART1+Q12*PART2
6 F2(L) =Q21*PART1+Q22*PART2
CALL GEOM(X)
SW2=SIGMAW**2
TOP=DCOS(THETAP)*T(101,1)**OHMPH
BOTTOM=CFBOT*P(101,2)*SIGMAW
UCF=TOP*UCFTOP/BOTTOM
TCF=(TOP*TCFTOP)/(BOTTOM*(DCOS(THETAP))**2)
EM11=E11(100)-1.000-2.000/UCF+2.000*DCOS(THETAP)*DSIN(THETAP)
* *SIGMAW*DELETA/DELX
EM12=E12(100)
FN1=U(101,1)*(1.000+2.000*SIGMAW*DCOS(THETAP)*DSIN(THETAP)*DELETA
* /DELX)-U(100,1)+2.000*V(101,1)*(TANTHP-(DELETA*SIGMAW*(DSIN(
* THETAP))**2)/DELX)-2.000*TANTHP*V(100,1)+2.000*DELETA*VWKM1
* *SIGMAW*(DSIN(THETAP)**2)/DELX-F1(100)
GO TO (403,7),IBC
403 EM21=E21(100)
EM22=E22(100)-1.000-2.000/TCF+2.000*SIGMAW*DELETA*DCOS(THETAP)*
* DSIN(THETAP)/DELX
FN2=T(101,1)*(1.000+2.000*SIGMAW*DELETA*DSIN(THETAP)*DCOS(THETAP)
* /DELX)-T(100,1)-2.000*THETAW/TCF-F2(100)
GO TO 8
7 EM21=E21(100)-(2.000*SIGMAW*DELETA*DSIN(THETAP)*DCOS(THETAP)/DELX
* -2.000/UCF)*U(101,1)*PR*(2.000+4.000*(TANTHP**2)/3.000
* -8.000*SIGMAW*TANTHP*DELETA/(3.000*DELX))
EM22=E22(100)-1.000
FN2=T(101,1)-T(100,1)+PR*(2.000+4.000*(TANTHP**2)/3.000)*U(101,1)*
* (U(101,1)-U(100,1))+8.000*PR*SIGMAW*TANTHP*DELETA*
* (U(101,1)**2)/(3.000*DELX)+4.000*PR*TANTHP*U(101,1)*(V(101,1)
* -V(100,1))/3.000+4.000*PR*SIGMAW*U(101,1)*DELETA*(V(101,1)
* -VWKM1)/DELX-8.000*PR*DELETA*(V(101,1)**2)/3.000-F2(100)
8 DENOM=EM11*EM22-EM12*EM21
U(101,2)=(EM22*FN1-EM12*FN2)/DENOM
T(101,2)=(EM11*FN2-EM21*FN1)/DENOM
DO 9 I=1,100
U(101-I,2)=E11(101-I)*U(102-I,2)+E12(101-I)*T(102-I,2)+F1(101-I)
9 T(101-I,2)=E21(101-I)*U(102-I,2)+E22(101-I)*T(102-I,2)+F2(101-I)

C
AINT=4.000*P(2,2)*ETA(2)*U(2,2)/T(2,2)
DO 10 L=3,100
SCALE=4-2*MCD(L,2)
10 AINT=AIN+SCALE*P(L,2)*ETA(L)*U(L,2)/T(L,2)
AINT=(AINT+P(101,2)*U(101,2)/T(101,2))*Z13
ANEW=AIN*SW2
C
CALCULATION OF V
ZR11 = 3.000*B*SIGBAR**2
ZR12 = 4.000*SIGBAR
E11(1)=0
F1(1)=0
DO 404 L=2,100

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```

PHW=0.5D0*(P(L,1)+P(L,2))
UHW=0.5D0*(U(L,1)+U(L,2))
VHW=0.5D0*(V(L,1)+V(L,2))
THW=0.5D0*(T(L,1)+T(L,2))
W = VHW-ETA(L)*UHW*TANBAR
G12=THW**OMEGA
DTBR=(T(L+1,1)-T(L-1,1)+T(L+1,2)-T(L-1,2))/Z34
AKL=-ETA(L)*PHW*W/(THW*ZR12)+G12*(ETA(L)*OMEGA*DTBR/THW+EP
* +2.0D0*ETA(L)/1.0D-02)/ZR11
BKL=ETA(L)*PHW*UHW*DELETA/(THW*DELX)+G12*(OMEGA*DTBR*DELETA*EP/THW
*+2.0D0*EP*DELETA/ETA(L)+4.0D0*ETA(L)/1.0D-02)/ZR11
CKL=ETA(L)*PHW*W/(ZR12*THW)-G12*(ETA(L)*OMEGA*DTBR/THW+EP
* -2.0D0*ETA(L)/1.0D-02)/ZR11
DV=V(L+1,1)-V(L-1,1)
DPBR=(P(L+1,1)-P(L-1,1)+P(L+1,2)-P(L-1,2))/Z34
DDV=V(L+1,1)-2.0D0*V(L,1)+V(L-1,1)
DUXBR=(U(L,2)-U(L,1))/DELX
DUBR = (U(L+1,1)-U(L-1,1)+U(L+1,2)-U(L-1,2))/Z34
DDUBR=(U(L+1,1)-2.0D0*U(L,1)+U(L-1,1)+U(L+1,2)-2.0D0*U(L,2)
* +U(L-1,2))/Z42
DTXBR=(T(L,2)-T(L,1))/DELX
DKL=ETA(L)*PHW*(UHW*V(L,1)*DELETA/DELX-W*DV/ZR12)/THW-ZR13*ETA(L)
* *DPBR*DELETA/SIGBAR+G12*(DV*(ETA(L)*OMEGA*DTBR/THW+EP
* -EP*V(L,1)*OMEGA*DTBR*DELETA/THW-2.0D0*EP*V(L,1)*DELETA/ETA(L)
* +2.0D0*ETA(L)*DDV/DELETA-OMEGA*DTBR*DELETA*(2.0D0*ETA(L)*SIGBAR
* *DUXBR+TANBAR*DUBR*ETA(L)**2)/THW+SIGBAR*ETA(L)*(U(L+1,2)
* -U(L-1,2)-U(L+1,1)+U(L-1,1))/(2.0D0*DELX)-ETA(L)*TANBAR*DUBR
* *DELETA-TANBAR*DDUBR*DELETA*ETA(L)**2+3.0D0*ETA(L)*SIGBAR*OMEGA
* *DTXBR*DUBR*DELETA/THW)/ZR11
DENOM = BKL-CKL*E11(L-1)
E11(L)=AKL/DENOM
404 F1(L)=(DKL+CKL*F1(L-1))/DENOM
V(101,2)=TANTHP*U(101,2)
DO 405 I=1,100
405 V(101-I,2)=E11(101-I)*V(102-I,2)+F1(101-I)
DO 71 L=2,6
V(L,2) = (ETA(L)*U(L,2)*V(7,2))/(6.0D-02*U(7,2))
71 CONTINUE
C CALCULATION OF P
DO 63 L=1,101
63 F2(L)=P(L,2)
61 PRF = Z8/(Z7*8*SIGBAR**2)
L =1
UHW = (U(1,1)+U(1,2))/2.0D0
PHW = (P(1,1)+P(1,2))/2.0D0
THW=0.5D0*(T(1,1)+T(1,2))
AC=DSQRT(THW)/RT2GM1
ZM = UHW/AC
DVDN = (V(2,2)+V(2,1))/2.0D-02
D2UDN2 = (U(3,2)-2.0D0*U(2,2)+U(1,2)+U(3,1)-2.0D0*U(2,1)+U(1,1))
* /2.0D-04
D2TDN2 = (T(3,2)-2.0D0*T(2,2)+T(1,2)+T(3,1)-2.0D0*T(2,1)+T(1,1))
* /2.0D-04
RHS = (GAMMA*PHW*ZM**2*(1.0D0+EP)*DVDN/(SIGBAR*UHW)
* +PRF*(D2UDN2-(UHW/THW)*(D2TDN2/PR+8.0D0*DVDN**2/3.0D0)))/
* (1.0D0-ZM**2)
P(1,2) = P(1,1)+RHS*DELX
DO 501 L=2,100

```

```

THW = (T(L,1)+T(L,2))/2.000
PHW = (P(L,1)+P(L,2))/2.000
UHW = (U(L,1)+U(L,2))/2.000
VHW = (V(L,1)+V(L,2))/2.000
AC=DSQRT(THW)/RT2GM1
ZM = UHW/AC
RM = VHW/AC
TM = DSQRT(ZM**2+RM**2)
Q = TM*AC
CS = UHW/Q
SN = VHW/Q
DVUN = (V(L+1,2)/U(L+1,2)-V(L-1,2)/U(L-1,2)
*      +V(L+1,1)/U(L+1,1)-V(L-1,1)/U(L-1,1))/4.00-02
PA = GAMMA*PHW*TM**2*CS*(DVUN+EP*VHW/(UHW*ETA(L)))/SIGBAR
DTDN = (T(L+1,2)-T(L-1,2)+T(L+1,1)-T(L-1,1))/4.00-02
DUDN = (U(L+1,2)-U(L-1,2)+U(L+1,1)-U(L-1,1))/4.00-02
DVDN = (V(L+1,2)-V(L-1,2)+V(L+1,1)-V(L-1,1))/4.00-02
D2TDN2 = (T(L+1,2)-2.000*T(L,2)+T(L-1,2)
*      +T(L+1,1)-2.000*T(L,1)+T(L-1,1))/2.00-04
D2UDN2 = (U(L+1,2)-2.000*U(L,2)+U(L-1,2)
*      +U(L+1,1)-2.000*U(L,1)+U(L-1,1))/2.00-04
PB = (OMEGA*DTDN*DUDN/THW+D2UDN2+EP*DUDN/ETA(L))/CS
DPDX=(P(L,2)-P(L,1))/DELX
DPDN=(P(L+1,1)-P(L-1,1)+F2(L+1)-F2(L-1))/4.00-02
PC=-DPDX*SN**2/CS+DPDN*(SN+SN**2*ETA(L)*TANBAR/CS)/SIGBAR
PF = (OMEGA*DTDN**2/THW+EP*DTDN/ETA(L)+D2TDN2)/PR
PH = 2.000*DUDN**2+8.000*(DVDN**2-EP*VHW*DVDN/ETA(L)
*      +EP*(VHW/ETA(L))**2)/3.000
PI = Q*(PF + PH)/THW
PJ = PRF*THW**OMEGA*(PB -PI)
RHS = PA + PJ +PC
DNDX = (SN-ETA(L)*TANBAR*CS)/(SIGBAR*CS)
ETA1 = ETA(L)-DNDX*DELX
IET=ETA1/DELETA
LI = IET + 1
IF (LI.GT.100) LI=100
IF (LI.LT.1) LI=1
501 P(L,2) = (ETA(LI+1)-ETA1)*P(LI,1)/DELETA + (ETA1-ETA(LI))*P(LI+1
*      ,1)/DELETA + RHS*DELX*DSQRT(1.000+DNDX**2)/((1.000-TM**2)*CS)
L= 101
PHW = (P(L,1)+P(L,2))/2.000
THW = (T(L,1)+T(L,2))/2.000
UHW = (U(L,1)+U(L,2))/2.000
VHW = (V(L,1)+V(L,2))/2.000
AC=DSQRT(THW)/RT2GM1
ZM = UHW/AC
RM = VHW/AC
TM = DSQRT(ZM**2+RM**2)
Q = TM*AC
CS = UHW/Q
SN = VHW/Q
DVUN = (TANTHP-V(100,2)/U(100,2)+TANTHP-V(100,1)/U(100,1))/2.00-02
PA = GAMMA*PHW*TM**2*CS*(DVUN+EP*VHW/(UHW*ETA(L)))/SIGBAR
DTDN=(T(101,2)-T(100,2)+T(101,1)-T(100,1))/2.00-02
DUDN=(U(101,2)-U(100,2)+U(101,1)-U(100,1))/2.00-02
DVDN=(V(101,2)-V(100,2)+V(101,1)-V(100,1))/2.00-02
D2TDN2=(T(101,2)-2.000*T(100,2)+T(99,2)
*      +T(101,1)-2.000*T(100,1)+T(99,1))/2.00-04

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D2UDN2=(U(101,2)-2.000*U(100,2)+U(99,2)
*   +U(101,1)-2.000*U(100,1)+U(99,1))/2.00-04
PB= (OMEGA*DTDN*DUDN/THW+D2UDN2+EP*DUDN/ETA(L))/CS
DPDX=(P(L,2)-P(L,1))/DELX
UPDN=(P(101,1)-P(100,1)+F2(101)-F2(100))/2.00-02
PC=-DPDX*SN**2/CS+DPDN*(SN+SN**2*ETA(L)*TANBAR/CS)/SIGBAR
PF = (OMEGA*DTDN**2/THW+EP*DTDN/ETA(L)+D2UDN2)/PR
PH = 2.000*DUDN**2+8.000*(DVDN**2-EP*VHW*DVDN/ETA(L)
*   +EP*(VHW/ETA(L))**2)/3.000
PI = Q*(PF + PH)/THW
PJ = PRF*THW**OMEGA*(PB -PI)
RHS = PA + PJ +PC
P(101,2)=P(101,1)+RHS*DELX/((1.000-TM**2)*CS)

```

C

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503 UERR=DSQRT(((UCL-U(1,2))**2+(U51-U(51,2))**2+(U99-U(99,2))**2)
*   /3.000)
TERR=DSQRT(((TCL-T(1,2))**2+(T51-T(51,2))**2+(T99-T(99,2))**2)
*   /3.000)
VERR=DSQRT(((V51-V(51,2))**2)
ERR=DSQRT(((UERR/U(1,1))**2+(TERR/T(1,1))**2+(VERR/V(51,1))**2)
*   /3.000)
304 ITER=ITER+1
ITP=ITER-1
410 WRITE (6,200) ITP,X,SIGMAW,SIGBAR,TANTHP,TANBAR,DPDXW,ANEW,ITERU,
*   PERR,UERR,TERR,VERR,ERR
DO 407 L=1,101
E11(L)=V(L,2)/U(L,2)
407 WRITE (6,201)ETA(L),U(L,2),T(L,2),P(L,2),V(L,2),E11(L)
IF(ITER-3)14,14,416
416 IF(X.GE.XMAX) GO TO 13
GO TO 5
13 STOP
200 FORMAT(1H1,3X,7HITER = ,I2,5X,4HX = ,1PE13.6,5X,9HSIGMAW = ,
*   1PE13.6,5X,9HSIGBAR = ,1PE13.6,7/4X,9HTANTHP = ,1PE13.6,5X,
*   9HTANBAR = ,1PE13.6,5X,8HDPDXW = ,1PE13.6,5X, 7HANEW = ,
*   1PE13.6,3X,8HITERU = ,I2,7/3X,7HPERR = ,1PE13.6,5X,7HUERR = ,
*   1PE13.6,5X,7HTERR = ,1PE13.6,5X,7HVERR = ,1PE13.6,5X,
*   6HERR = ,1PE13.6,
*   7/3X,3HETA,9X,6HU(L,2),14X,6HT(L,2),14X,6HP(L,2),
*   14X,6HV(L,2),14X,3HV/U)
201 FORMAT (F7.2,5E20.8)
END

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SUBROUTINE GEOM (Y)
  IMPLICIT REAL*8 (A-H,O-Z)
  COMMON A,ALPHU,ALPHT,AC,A11,A12,A21,A22,AN1,AN2,ABDA1,ABDA2,AKL,
*    AINT,AINTM1,ANEW
  COMMON B,BKL,B11,B21,B22,BOTTOM,B12
  COMMON CPKU,CFBOT,CS2TW,C,CKL,C11,C12,C21,C22
  COMMON DELX,DELETA,DELXS,DPDXW,DVU,DU,DT,DDU,DDT,DV,DKL,DENOM,
*    D1,D2,DA1,DA2,DTBR,DPBR,DDV,DUXBR,DUBR,DDUBR,DTXBR,DPLAST
  COMMON EM11,EM12,EM21,EM22,ERR,EP
  COMMON FN1,FN2
  COMMON GAMMA,G4BGM1,GDX,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,
*    G13,G14,G15,G16,G17,G18,G19,G20,GRADP
  COMMON HFLUX,HTR,HDELX
  COMMON OMEGA,O3,O4,O5,OHMPH
  COMMON PR,PI3,PGRT,PG1,PG2,PHW,PART1,PART2,PG
  COMMON QGRADP,QQ11,QQ12,QQ21,QQ22,Q11,Q12,Q21,Q22
  COMMON R1,RADDEG,RTPB2,R12,RT2GM1,R1ST1,R1ST2,RTZ8Z7,RM,RMT1,RMT2,
*    RMT
  COMMON SA1,SC1,SW2,SW3,SW4,S,SG2,SIGMAW,SIGBAR,SCALE,SWLRT,SW2K
  COMMON THETA1,THETA2,TW2,TCFTOP,THETAW,THETAP,TANTHP,TANBAR,TOP,
*    TCF,THW,TUHW,TLHW,TCL,T51,T99,TERR
  COMMON UCFTOP,UCF,UHW,UUHW,ULHW,UCL,U51,U99,UERR
  COMMON VDISS,VWKM1,VEE,VHW,VUHW,VLHW,V51,VERR
  COMMON W
  COMMON XPRINT,X0,X1,X2,XMAX,XTW1,XTW2,XPLAST,X,XSQ,XHW,XORG,XEF
  COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y10
  COMMON Z1,Z2,Z5,Z7,Z8,Z10,Z11,Z12,Z13,Z15,Z23,Z31,Z32,Z33,Z34,
*    Z35,Z42,Z43,Z41,ZR1,ZR2,ZR3,ZR4,ZR5,ZR6,ZR7,ZR8,ZR9,ZR10,
*    ZM,ZR11,ZR12,ZR13,ZR14
  COMMON IBC,IETAPR,IX,IXTHW,ITER,I,ITERU
  COMMON K,KX,KI
  COMMON L,LC
  COMMON NU,NUEPS
  COMMON ETA(101),U(101,2),V(101,2),P(101,2),T(101,2),VV(101,2),
*    E11(101),E12(101),E22(101),E21(101),F1(101),F2(101)
  IF (Y.GT.R1ST1) GO TO 1
  SIGMAW=Y*Z1+Z10
  THETAP=THETA1
  GO TO 3
1 IF(Y.GE.R1ST2) GO TO 2
  C=DSQRT(R12-Y**2)
  SIGMAW=Z11-C
  THETAP=DATAN(Y/C)
  GO TO 3
2 SIGMAW=Y*Z2+Z12
  THETAP=THETA2
3 TANTHP=DTAN(THETAP)
  RETURN
  END

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SUBROUTINE INIT
  IMPLICIT REAL*8 (A-H,O-Z)
  COMMON A,ALPHU,ALPHT,AC,A11,A12,A21,A22,AN1,AN2,ABDA1,ABDA2,AKL,
*   AINT,AINTM1,ANEW
  COMMON B,BKL,B11,B21,B22,BOTTOM,B12
  COMMON CPKU,CFBOT,CS2TW,C,CKL,C11,C12,C21,C22
  COMMON DELX,DELETA,DELXS,DPDXW,DVU,DU,DT,DDU,DDT,DV,DKL,DENOM,
*   D1,D2,DA1,DA2,DTBR,DPBR,DDV,DUXBR,DUBR,DDUBR,DTXBR,DPLAST
  COMMON EM11,EM12,EM21,EM22,ERR,EP
  COMMON FN1,FN2
  COMMON GAMMA,G4BGM1,GDX,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,
*   G13,G14,G15,G16,G17,G18,G19,G20,GRADP
  COMMON HFLUX,HTR,HDELX
  COMMON OMEGA,O3,O4,O5,OHMPH
  COMMON PR,PI3,PGRT,PG1,PG2,PHW,PART1,PART2,PG
  COMMON QGRADP,QQ11,QQ12,QQ21,QQ22,Q11,Q12,Q21,Q22
  COMMON R1,RADDEG,RTPB2,R12,RT2GM1,R1ST1,R1ST2,RTZBZ7,RM,RMT1,RMT2,
*   RMT
  COMMON SA1,SC1,SW2,SW3,SW4,S,SG2,SIGMAW,SIGBAR,SCALE,SWLRT,SW2K
  COMMON THETA1,THETA2,TW2,TCFTOP,THETAW,THETAP,TANTHP,TANBAR,TOP,
*   TCF,THW,TUHW,TLHW,TCL,T51,T99,TERK
  COMMON UCFTOP,UCF,UHW,UUHW,ULHW,UCL,U51,U99,UERR
  COMMON VDISS,VWKM1,VEE,VHW,VUHW,VLHW,V51,VERR
  COMMON W
  COMMON XPRINT,X0,X1,X2,XMAX,XTW1,XTW2,XPLAST,X,XSQ,XHW,XORG,XFF
  COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y10
  COMMON Z1,Z2,Z5,Z7,Z8,Z10,Z11,Z12,Z13,Z15,Z23,Z31,Z32,Z33,Z34,
*   Z35,Z42,Z43,Z41,ZR1,ZR2,ZR3,ZR4,ZR5,ZR6,ZR7,ZR8,ZR9,ZR10,
*   ZM,ZR11,ZR12,ZR13,ZR14
  COMMON IBC,IETAPR,IX,IXTHW,ITER,I,ITERU
  COMMON K,KX,KI
  COMMON L,LC
  COMMON NU,NUEPS
  COMMON ETA(101),U(101,2),V(101,2),P(101,2),T(101,2),VV(101,2),
*   E11(101),E12(101),E22(101),E21(101),F1(101),F2(101)
  IF(XTW1.GE.XTW2) GO TO 1
  U3=(1.000+TW2)*0.500
  O4=(1.000-TW2)*0.500
  O5=3.1415926500/(XTW2-XTW1)
1 RADDEG=57.295779500
  RTPB2=1.25331413700
  X1=X1-1.00-05
  X2=X2-1.00-05
  XPLAST=X0
  R12=R1**2
  THETA1=THETA1/RADDEG
  THETA2=THETA2/RADDEG
  DELETA=1.00-02
  Z1=DTAN(THETA1)
  Z2=DTAN(THETA2)
  Z5=DCOS(THETA1)
  Z7=GAMMA-1.000
  Z8=2.000*GAMMA
  RT2GM1=DSQRT(2.000/Z7)
  G4BGM1=4.000*GAMMA/Z7

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R1ST1=R1*DSIN(THETA1)
R1ST2=R1*DSIN(THETA2)
Z10=1.000+R1*(1.000-Z5)-Z1*R1ST1
Z11=1.000+R1
Z12=1.000+R1*(1.000-DCOS(THETA2))-Z2*R1ST2
DELXS=DELX
Z13=DELETA/3.000
RTZ8Z7=DSQRT(Z8/Z7)
Z15=RTZ8Z7*Z5/B
Z23=DELETA/12.000
SA1=((2.000-ALPHU)*RTPB2)/ALPHU
SC1=((2.000-ALPHT)*RTPB2*Z8)/(ALPHT*PR*(GAMMA+1.000))
CPKU=SA1*Z15
UCFTOP=RTZ8Z7*SA1
CFBOT=B*DELETA
TCFTOP=RTZ8Z7*SC1
QHMPH=OMEGA+0.500
Z31=1.00-04
Z32=OMEGA-1.000
Z33=2.00-02
Z34=4.00-02
Z35=B*Z31
Z42=2.000*Z31
Z43=Z33/(3.000*Z7)
ZR5=2.000*PR*DELETA
ZR6=OMEGA/(PR*4.000*DELETA**2)
ZR7=PR*DELETA**2
ZR8=2.000/(3.000*DELETA**2)
ZR9=4.000/(3.000*DELETA)
ZR10=OMEGA/(Z33**2)
ZR13=Z7/Z8
ZR14=0.500*DELETA
GO TO (2,3),NUEPS
2 EP=0
GO TO 4
3 EP=1.000
4 CONTINUE
RETURN
END

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SUBROUTINE INPUT
IMPLICIT REAL*8 (A-H,O-Z)
COMMON A,ALPHU,ALPHT,AC,A11,A12,A21,A22,AN1,AN2,ABDA1,ABDA2,AKL,
*   AINT,AINTM1,ANEW
COMMON B,BKL,B11,B21,B22,BOTTOM,B12
COMMON CPKU,CFBOT,CS2TW,C,CKL,C11,C12,C21,C22
COMMON DELX,DELETA,DELXS,DPCXW,DVU,DU,DT,DDU,DDT,DV,DKL,DENOM,
*   D1,D2,DA1,DA2,DTBR,DPBR,DDV,DUXBR,DUBR,DDUBR,DTXBR,DPLAST
COMMON EM11,EM12,EM21,EM22,ERR,EP
COMMON FN1,FN2
COMMON GAMMA,G48GM1,GDX,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,
*   G13,G14,G15,G16,G17,G18,G19,G20,GRADP
COMMON HFLUX,HTR,HDELX
COMMON OMEGA,O3,O4,O5,OHMPH
COMMON PR,P13,PGRT,PG1,PG2,PHW,PART1,PART2,PG
COMMON QGRADP,QQ11,QQ12,QQ21,QQ22,Q11,Q12,Q21,Q22
COMMON R1,RADDEG,RTPB2,R12,RT2GM1,R1ST1,R1ST2,RTZ8Z7,RM,RMT1,RMT2,
*   RMT
COMMON SA1,SC1,SW2,SW3,SW4,S,SG2,SIGMAW,SIGBAR,SCALE,SWLRT,SW2K
COMMON THETA1,THETA2,TW2,TCFTOP,THETAW,THETAP,TANTHP,TANBAR,TOP,
*   TCF,THW,TUHW,TLHW,TCL,T51,T99,TERR
COMMON UCFTOP,UCF,UHW,UUHW,ULHW,UCL,U51,U99,UERR
COMMON VDISS,VWKM1,VEE,VHW,VUHW,VLHW,V51,VERR
COMMON W
COMMON XPRINT,X0,X1,X2,XMAX,XTW1,XTW2,XPLAST,X,XSQ,XHW,XORG,XEF
COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y10
COMMON Z1,Z2,Z5,Z7,Z8,Z10,Z11,Z12,Z13,Z15,Z23,Z31,Z32,Z33,Z34,
*   Z35,Z42,Z43,Z41,ZR1,ZR2,ZR3,ZR4,ZR5,ZR6,ZR7,ZR8,ZR9,ZR10,
*   ZM,ZR11,ZR12,ZR13,ZR14
COMMON IBC,IETAPR,IX,IXTHW,ITER,I,ITERU
COMMON K,KX,KI
COMMON L,LC
COMMON NU,NUEPS
COMMON ETA(101),U(101,2),V(101,2),P(101,2),T(101,2),VV(101,2),
*   E11(101),E12(101),E22(101),E21(101),F1(101),F2(101)
C READ(5,100) IBC,IETAPR,NU
  NU = 0,1 FOR TWO-DIMENSIONAL, AXISYMMETRIC FLOW, RESPECTIVELY
  READ(5,101) R1,THETA1,THETA2,TW2,GAMMA,OMEGA,PR,B,A,XPRINT,
*   DELX,X0,X1,X2,XMAX,ALPHU,ALPHT,XTW1,XTW2
  IF (THETA1.GT.0.0) THETA1=-THETA1
  NUEPS = NU + 1
  WRITE(6,200)
  WRITE(6,201)
  WRITE(6,202) R1,THETA1,THETA2,TW2,GAMMA,OMEGA,PR,B,A,XPRINT,
*   DELX,X0,X1,X2,XMAX,ALPHU,ALPHT,XTW1,XTW2
  GO TO (1,2),IBC
1 WRITE(6,203)
  GO TO 3
2 WRITE(6,205)
3 GO TO (4,5), NUEPS
4 WRITE (6,206)
  GO TO 6
5 WRITE (6,207)
6 CONTINUE
  RETURN

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100 FORMAT (19I4)
101 FURMAT (5E14.8)
200 FORMAT(1H1)
201 FORMAT(13X,2HR1,16X,6HTHETA1,14X,6HTHETA2,15X,3HTW2,16X,5HGAMMA,
*      /11X,5HOMEGA,17X,2HPR,18X,1HB,19X,1HA,17X,6HXPRINT/12X,
*      4HDELX,17X,2HXO,18X,2HX1,18X,2HX2,17X,4HXMAX/11X,5HALPHU,
*      15X,5HALPHT,16X,4HXTW1,16X,4HXTW2/)
202 FORMAT(1P5E20.6)
203 FORMAT(//1X,22H HEAT TRANSFER ALLOWED//)
205 FORMAT(//1X,20H ADIABATIC WALL CASE//)
206 FORMAT (//1X,20HTWO-DIMENSIONAL CASE//)
207 FORMAT (//1X,17HAXISYMMETRIC CASE//)
      END

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SUBROUTINE START
IMPLICIT REAL*8 (A-H,O-Z)
COMMON A,ALPHU,ALPHT,AC,A11,A12,A21,A22,AN1,AN2,ABDA1,ABDA2,AKL,
*   AINT,AINTM1,ANEW
COMMON B,BKL,B11,B21,B22,BOTTOM,B12
COMMON CPKU,CFBOT,CS2TW,C,CKL,C11,C12,C21,C22
COMMON DELX,DELETA,DELXS,DPDXW,DVU,DU,DT,DDU,DDT,DV,DKL,DENDM,
*   D1,D2,DA1,CA2,DTBR,DPBR,DDV,DUXBR,DUBR,DDUBR,DTXBR,DPLAST
COMMON EM11,EM12,EM21,EM22,FRR,EP
COMMON FN1,FN2
COMMON GAMMA,G4BGM1,GDX,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12,
*   G13,G14,G15,G16,G17,G18,G19,G20,GRADP
COMMON HFLUX,HTR,HDELX
COMMON OMEGA,O3,O4,O5,OHMPH
COMMON PR,PI3,PGRT,PG1,PG2,PHW,PART1,PART2,PG
COMMON QGRAOP,QQ11,QQ12,QQ21,QQ22,Q11,Q12,Q21,Q22
COMMON R1,RADDEG,RTPB2,R12,RT2GM1,R1ST1,R1ST2,RTZ8Z7,RM,RMT1,RMT2,
*   RMT
COMMON SA1,SC1,SW2,SW3,SW4,S,SG2,SIGMAW,SIGBAR,SCALE,SWLRT,SW2K
COMMON THETA1,THETA2,TW2,TCFTOP,THETAW,THETAP,TANTHP,TANBAR,TOP,
*   TCF,THW,TUHW,TLHW,TCL,T51,T99,TERR
COMMON UCFTOP,UCF,UHW,UUHW,ULHW,UCL,U51,U99,UERR
COMMON VDISS,VWKM1,VEE,VHW,VUHW,VLHW,V51,VERR
COMMON W
COMMON XPRINT,X0,X1,X2,XMAX,XTW1,XTW2,XPLAST,X,XSQ,XHW,XORG,XEF
COMMON Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y10
COMMON Z1,Z2,Z5,Z7,Z8,Z10,Z11,Z12,Z13,Z15,Z23,Z31,Z32,Z33,Z34,
*   Z35,Z42,Z43,Z41,ZR1,ZR2,ZR3,ZR4,ZR5,ZR6,ZR7,ZR8,ZR9,ZR10,
*   ZM,ZR11,ZR12,ZR13,ZR14
COMMON IBC,IETAPR,IX,IXTHW,ITER,I,ITERU
COMMON K,KX,KI
COMMON L,LC
COMMON NU,NUEPS
COMMON ETA(101),U(101,2),V(101,2),P(101,2),T(101,2),VV(101,2),
*   E11(101),E12(101),F22(101),E21(101),F1(101),F2(101)
DELX=DELXS
Z41=Z34*DELX
CALL GEOM(X0)
SW2=SIGMAW**2
SW3=SIGMAW*SW2
SW4=SIGMAW*SW3
X=X0
K=1
KX=1
KI=0
IX=1
IF(X1.GE.X2)IX=4
ZR1=(3.000*A)/(2.000*((1.000-Z5)**2)*(1.000+2.000*Z5))
ZR2=(4.000*GAMMA*A)/(B*Z7*((1.000-Z5)**2)*(1.000+2.000*Z5))
CS2TW=2.000*(Z5**2)-1.000
CS3TW = Z5**3
Z16=(2.000*A*Z1**2)/(2.000-3.000*Z5+CS3TW)
PI3=32.000*GAMMA*A/(3.000*Z7*B*Z1)
XORG=-Z10/Z1
U31=8.000*CPKU*A/SW3

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ZU=4.000*A/SW2
XEF=X-XORG
DO 2 L=1,101
ETA(L)=DFLOAT(L-1)/100.000
U(L,2)=ZU*(1.000-ETA(L)**2)+U31
V(L,2)=Z1*ETA(L)*U(L,2)
P(L,2)=1.000-PI3*(XEF/(SIGMAW*DSQRT(XEF**2+SW2*ETA(L)**2)))**3
2 T(L,2)=1.000
X=X0-DELX
CALL GEOM(X)
SW2=SIGMAW**2
SW3=SW2*SIGMAW
ZU=4.000*A/SW2
U31=8.000*CPKU*A/SW3
XEF=X-XORG
DO 4 L=1,101
U(L,1)=ZU*(1.000-ETA(L)**2)+U31
V(L,1)=Z1*ETA(L)*U(L,1)
P(L,1)=1.000-PI3*(XEF/(SIGMAW*DSQRT(XEF**2+SW2*ETA(L)**2)))**3
4 T(L,1)=1.000
X=X0
HFLUX=A
HTR=0
THETAW=1.000
IXTHW=1
IF(XTW1.GE.XTW2)IXTHW=3
LC=0
201 FORMAT(1X,F7.2,3F20.8)
RETURN
END

```

