NASA TECHNICAL MEMORANDUM

NASA TM X-2628

NASA TM X-2628

PARTICLE TRACKABILITY CONSIDERATIONS FOR LASER DOPPLER VELOCIMETRY

by Allen M. Karchmer

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1972

1. Report No. NASA TM X-2628	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle PARTICLE TRACKABILITY CONSIDERATIONS FOR LASER DOPPLER VELOCIMETRY		5. Report Date September 1972
		6. Performing Organization Code
7. Author(s) Allen M. Karchmer		8. Performing Organization Report No. E-6992
		10. Work Unit No. 132-80
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

The technique for calculating single-particle trackability in a fluid flow in terms of rms velocities is discussed. The general equation for the motion of a single particle as a response to a fluid-flow driving force is presented, and a criterion for greatly simplifying the governing equation is developed. It is shown that in the case of large particle- to fluid-density ratio and low-frequency fluid motion compared to the natural time constant of the particle response, the governing equation can be reduced to a simple balance of inertia force and linear viscous drag (i.e., Stokes drag).

17. Key Words (Suggested by Author(s)) Laser-Doppler Turbulence Particle tracking		18. Distribution Statement Unclassified - unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 22	22. Price* $3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

PARTICLE TRACKABILITY CONSIDERATIONS FOR LASER DOPPLER VELOCIMETRY

by Allen M. Karchmer

Lewis Research Center

SUMMARY

The technique for calculating single-particle trackability in a fluid flow in terms of rms velocities is discussed. The general equation for the motion of a single particle as a response to a fluid-flow driving force is presented, and a criterion for greatly simplifying the governing equation is developed. It is shown that in the case of large particle-to fluid-density ratio and low-frequency fluid motion compared to the natural time constant of the particle response, the governing equation can be reduced to a simple balance of inertia force and linear viscous drag (i.e., Stokes drag).

INTRODUCTION

The advent of the laser Doppler velocimeter (LDV) has presented to the experimenter interested in making velocity measurements in flowing fluids an opportunity to make such measurements without the limitations and disadvantages of previous instruments. One of the more significant advantages of the LDV is its relative insensitivity to temperature and density, thus making the separation of compressibility effects from velocity signal unnecessary. There are a number of other advantages, but since there is an abundance of literature on this subject (ref. 1, e.g.), they are not listed in this report.

The principle behind velocity measurements in a fluid from Doppler-shifted laser light is well established and straightforward. Light from a laser beam is directed into a moving fluid. A portion of the beam is scattered from particles moving with the fluid. This scattered light is slightly shifted in frequency by the Doppler effect. The magnitude of the frequency shift is then measured and can be directly related to the velocities of the scattering centers. The scattering centers (either artifically introduced or naturally present) must satisfy two basic criteria: they must be efficient scatterers of the laser radiation and they must be accurately following, or tracking, the fluid flow. Several authors have shown that the scattering criteria require the LDV to be collecting radiation scattered in the Mie regime (ref. 1, e.g.). Basically, this amounts to having the diameter of the scattering center (i.e., the seeding particle) at least as large as the wavelength of the incident light λ. For an argon laser, λ is 0.488 micrometer (4880 Å); therefore, the particle diameters must be approximately 1/2 micrometer or larger.

This requirement puts an approximate lower limit on the size of the seeding particles used. So, except for particles whose density is the same as that of the fluid, there will always be some minimum tracking error. This error must be estimated for the particular flow of interest to obtain an overall estimate of the accuracy of any LDV measurements. This report presents techniques to enable individuals involved in laser-Doppler velocity measurements to estimate the errors involved in their measurements because of finite particle response.

SINGLE-PARTICLE RESPONSE TO FLUID FLOW

Particle Motion as Response to Driving Force

If the particle velocity is considered the output, or response, to an input driving force of a turbulent fluid velocity, the transfer function is defined through the relation

$$\tilde{v}_p(j\omega) = G(j\omega)\tilde{v}_f(j\omega) \tag{1}$$

where $\tilde{v}_p(j\omega)$ is the Fourier transform of the particle and $\tilde{v}_f(j\omega)$ is the Fourier transform of the fluid velocity. All symbols are defined in the appendix. The energy spectrum of the particle velocity is $|\tilde{v}_p(j\omega)|^2$ and is obtained from equation (1) as

$$|\tilde{v}_p(j\omega)|^2 = |G(j\omega)|^2 |\tilde{v}_f(j\omega)|^2 \tag{2}$$

In the time domain, the autocorrelation of the particle velocity is

$$R_p(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v_p(t)v_p(t + \tau)dt$$

With zero time delay

$$R_p(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v_p^2(t)dt$$

which is just the definition of the mean square. So,

$$R_p(0) = \langle v_p^2 \rangle \tag{3}$$

But, by the Wiener-Khintchine theorem, the autocorrelation is related to the energy spectrum by the inverse Fourier transform

$$R_p(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{v}_p(j\omega)|^2 e^{j\omega\tau} \, d\omega$$

So,

$$R_p(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v_p(j\omega)|^2 \, d\omega$$

So, from equation (3),

$$\langle v_p^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{v}_p(j\omega)|^2 \, d\omega \tag{4}$$

Similarly, the mean square of the fluid velocity is

$$\langle v_f^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{v}_f(j\omega)|^2 \, d\omega \tag{5}$$

Substituting equation (2) into (4):

$$\langle v_p^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 |\tilde{v}_f(j\omega)|^2 \, d\omega$$

Hence,

$$\frac{\langle v_p^2 \rangle}{\langle v_f^2 \rangle} = \frac{\int_{-\infty}^{\infty} |G(j\omega)|^2 |\tilde{v}_f(j\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |\tilde{v}_f(j\omega)|^2 \, d\omega}$$

or

$$\frac{v_{p,rms}}{v_{f,rms}} = \sqrt{\frac{\langle v_p^2 \rangle}{\langle v_f^2 \rangle}} = \sqrt{\frac{\int_{-\infty}^{\infty} |G(j\omega)|^2 |\tilde{v}_f(j\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |\tilde{v}_f(j\omega)|^2 \, d\omega}} \tag{6}$$

Equation (6) is a weighting of the transfer function to the spectral distribution of energy in the flow; that is, the numerator of equation (6) is a measure of the particle's ability to track the flow (through the transfer function) with consideration given to the actual distribution of frequencies present in the flow (through the energy spectrum of the flow). For particle trackability through a frequency bandwidth ω_u,

$$\left. \frac{v_{p,rms}}{v_{f,rms}} \right|_{\omega_u} = \sqrt{\frac{\int_{0}^{\omega_u} |G(j\omega)|^2 |\tilde{v}_f(j\omega)|^2 \, d\omega}{\int_{0}^{\omega_u} |v_f(j\omega)|^2 \, d\omega}} \tag{7}$$

The energy spectrum of the main flow can be either measured directly or calculated from a measured autocorrelation by way of the Fourier transform

$$|\tilde{v}_f(j\omega)|^2 = \int_{-\infty}^{\infty} R_f(\tau) e^{-j\omega\tau} \, d\tau$$

Hence, to obtain the relative trackability of the particle, it is required to find the correct transfer function $G(j\omega)$. To determine the transfer function, the governing equation for the particle in the flow must be solved in the frequency domain.

Particle Motion in General Fluid Flow

The general equation for the motion of a small spherical particle in a fluid, derived by Basset, Oseen, and Boussinesq (ref. 2), is

$$\frac{\pi d^3}{6}\rho_p\dot{v}_p = \frac{\pi d^3}{6}\rho_f\dot{v}_f + \frac{1}{2}\frac{\pi d^3}{6}\rho_f(\dot{v}_f - \dot{v}_p) + \frac{3}{2}d^2\sqrt{\pi\rho_f\mu_f}\int_{t_o}^{t}\frac{\frac{d}{dt'}(v_f - v_p)dt'}{\sqrt{t - t'}} + D + P_g \tag{8}$$

where

$\frac{\pi d^3}{6}\rho_p$	mass of spherical particle of diameter d with density ρ_p
$\frac{\pi d^3}{6}\rho_f$	mass of equivalent spherical fluid particle with density ρ_f
μ_f	dynamic viscosity of fluid
D	viscous drag force
P_g	external potential force (the weight due to gravity minus the upward buoyant force)

The four basic assumptions in equation (8) are

(1) The particle is spherical.

(2) The particle is small when compared to the smallest wavelength of fluid motion.

(3) The flow field is not perturbed by the presence of the solid particle.

(4) Throughout the motion of the solid particle, the same fluid element remains in its neighborhood; that is, the pathlines of the solid particle and the fluid coincide.

For a 1-micrometer particle stationary in an airstream at 1 atmosphere (14.7 psia) and 27° C (80° F) with a mean velocity of 30 meters per second (98.4 ft/sec), the maximum particle Reynolds number is

$$Re = \frac{v_f d}{\nu} = \frac{(30\ \mathrm{m/sec})(1\times10^{-6}\ \mathrm{m})}{1.57\times10^{-5}\ \mathrm{m^2/sec}}$$

$$= 1.91$$

Strictly speaking, the Stokes flow regime, where drag force is proportional to velocity rather than to velocity squared, is valid only to a Reynolds number of about 1. However, even at a Reynolds number of 5, the deviation in drag coefficient from Stokes law is not great; and it can be assumed that the drag on the particle is proportional to the relative velocity between the particle and the fluid. The drag force, then, is given by Stokes law as

$$D = 3\pi\mu d(v_f - v_p)$$

The last term on the right side of equation (8) is any external potential force. In most applications this will simply be the weight due to gravity minus the upward buoyant force. In this case, the motion of the particle in steady state due to this force will be constant and equal to its terminal settling velocity in the given fluid. Because of the linearity of equation (8), this vertical component of the particle motion will not be affected by the general fluid flow field, and the total motion of the particle will be a linear superposition of its vertical settling velocity and its general motion induced by the flow field. The last term on the right side of equation (8), therefore, can be dropped without any loss of generality. The resulting equation is

$$\frac{\pi d^3}{6} \rho_p \dot{v}_p = 3\pi\mu_f d(v_f - v_p) + \frac{\pi d^3}{6} \rho_f \dot{v}_f + \frac{1}{2} \frac{\pi d^3}{6} \rho_f(\dot{v}_f - \dot{v}_p)$$

$$+ \frac{3}{2} d^2 \sqrt{\pi \rho_f \mu_f} \int_{t_o}^{t} \frac{\frac{d}{dt'}(v_f - v_p)dt'}{\sqrt{t - t'}} \qquad (9)$$

The meaning of each term in equation (9) is as follows: The single term on the left side is the mass of the particle times its acceleration; that is, it is the force required to accelerate the particle.

The first term on the right side of equation (9) is the viscous drag on the particle, assuming Stokes flow. The second term is the force on the particle due to the pressure gradient in the fluid. This pressure gradient is caused by the acceleration of the fluid. And the resulting force is, in a true sense, a buoyant force in a direction opposing acceleration; that is, in so far as the particle is concerned, the effect of an accelerating fluid field is the same as a gravity field. The third term is the force required to accelerate the apparent mass of the particle relative to the fluid. The last term, derived by Basset, accounts for the deviation of the flow from steady state.

Rewriting equation (9) gives

$$\dot{v}_p + \alpha\beta v_p = \alpha\beta v_f + \beta\dot{v}_f + \beta\left(\frac{3\alpha}{\pi}\right)^{1/2}\int_{t_o}^{t}\frac{\frac{d}{dt'}(v_f - v_p)dt'}{\sqrt{t - t'}} \tag{10}$$

where

$$\alpha = \frac{12\nu}{D^2} \qquad \left(\nu = \frac{\mu_f}{\rho_f}\right)$$

and

$$\beta = \frac{3\rho_f}{2\rho_p + \rho_f} = \frac{3}{2\left(\frac{\rho_p}{\rho_f}\right) + 1}$$

Simplification of Flow Equation

Equation (9) suggests that when $\rho_p/\rho_f \rightarrow \infty$ the second, third, and fourth terms of the right side of equation (9) become negligibly small. The equivalent limit in equation (10) is $\beta \rightarrow 0$ with $\alpha\beta$ simultaneously remaining finite. This would leave only the first term on the right side of equation (10); that is, the only driving force would be a linear drag force. It is worth investigating the specific characteristics of the flow that would be required for this assumption to be valid. To do this, equation (10) must be nondimensionalized. As a reference velocity, let

$$v_{ref} = \overline{\omega}d$$

where $\overline{\omega}$ is some frequency associated with the main flow. As a reference time, let

$$t_{ref} = (\overline{\omega})^{-1}$$

The nondimensional velocities and time then become

$$V_p = \frac{v_p}{v_{ref}} = \frac{v_p}{\overline{\omega}d}$$

and

$$V_f = \frac{v_f}{v_{ref}} = \frac{v_f}{\overline{\omega}d}$$

So,

$$\begin{aligned} v_p &= \overline{\omega}dV_p \\ v_f &= \overline{\omega}dV_f \end{aligned} \tag{11}$$

Also,

$$\theta = \frac{t}{t_{ref}} = t\overline{\omega}$$

Therefore,

$$t = \frac{\theta}{\overline{\omega}} \tag{12}$$

Substituting equations (11) and (12) into equation (10) gives

$$\overline{\omega}^2 d \frac{dV_p}{d\theta} + \alpha\beta\overline{\omega}dV_p = \alpha\beta\overline{\omega}dV_f + \beta\overline{\omega}^2 d \frac{dV_f}{d\theta} + \beta\left(\frac{3\alpha}{\pi}\right)^{1/2}\overline{\omega}^{3/2} d \int_{\theta_o}^{\theta} \frac{\frac{d}{d\theta'}(V_f - V_p)}{\sqrt{\theta - \theta'}}$$

Rearranging gives

$$\frac{dV_p}{d\theta} + \frac{\beta}{\left(\frac{\overline{\omega}}{\alpha}\right)} V_p = \frac{\beta}{\left(\frac{\overline{\omega}}{\alpha}\right)} V_f + \beta \frac{dV_f}{d\theta} + \frac{\beta}{\left(\frac{\overline{\omega}}{\alpha}\right)^{1/2}} \left(\frac{3}{\pi}\right)^{1/2} \int_{\theta_o}^{\theta} \frac{\frac{d}{d\theta'}(V_f - V_p)d\theta'}{\sqrt{\theta - \theta'}} \tag{13}$$

It is now clear when the last two terms on the right side of equation (10) can be neglected and only the linear drag force retained. In the limit as $\beta \rightarrow 0$ (i.e., $\rho_p/\rho_f \rightarrow \infty$) and at the same time the dimensionless frequency $(\overline{\omega}/\alpha)$ becomes very small, the second and third terms on the right side of equation (13) become negligible in comparison with the remaining terms. More specifically, if $(\overline{\omega}/\alpha\beta) = \mathcal{O}(1)$ or less as $\beta \rightarrow 0$, the predominant driving force is the linear drag force. As the flow departs from this condition, the next to become signficant is the Basset force, and finally the drag force proportional to acceleration. In other words, for low-frequency flow with a large particle-to fluid-density ratio, only the linear viscous drag force need be considered.

Returning to equation (10) with only the viscous drag force retained,

$$\dot{v}_p + \alpha\beta v_p = \alpha\beta v_f \tag{14}$$

Taking the Fourier transform of equation (14) yields

$$j\omega\tilde{v}_p(j\omega) + \alpha\beta\tilde{v}_p(j\omega) = \alpha\beta\tilde{v}_f(j\omega)$$

So,

$$\tilde{v}_p(j\omega) = \frac{\alpha\beta\tilde{v}_f(j\omega)}{\alpha\beta + j\omega}$$

The complex transfer function is

$$G_1(j\omega) = \frac{\tilde{v}_p(j\omega)}{\tilde{v}_f(j\omega)} = \frac{\alpha\beta}{\alpha\beta + j\omega} = \frac{(\alpha\beta)^2 - \alpha\beta j\omega}{(\alpha\beta)^2 + \omega^2}$$

Thus,

$$|G_1(j\omega)| = \sqrt{G_1(j\omega)G_1^*(j\omega)} = \sqrt{\left[\frac{(\alpha\beta)^2 - \alpha\beta j\omega}{(\alpha\beta)^2 + \omega^2}\right]\left[\frac{(\alpha\beta)^2 + \alpha\beta j\omega}{(\alpha\beta)^2 + \omega^2}\right]} = \frac{\alpha\beta}{\sqrt{(\alpha\beta)^2 + \omega^2}} \tag{15}$$

and from equation (6),

$$\frac{v_{p,rms}}{v_{f,rms}} = \sqrt{\frac{\int_{-\infty}^{\infty}\left[\frac{(\alpha\beta)^2}{(\alpha\beta)^2 + \omega^2}\right]|\tilde{v}_f(j\omega)|^2\,d\omega}{\int_{-\infty}^{\infty}|v_f(j\omega)|^2\,d\omega}} \tag{16}$$

For the assumed model, as given by equation (14), equation (16) is valid regardless of the nature of the fluid flow field: turbulent or laminar, time varying or constant.

The simplest case is that of a sinusoidally varying flow field

$$v_f(t) = v_f \cos \omega_o t \tag{17}$$

The energy spectrum of the flow field is given by the Fourier transform of equation (17)

$$\mathscr{F}\left[v_f(t)\right] = \mathscr{F}\left[v_f \cos \omega_o t\right] = \pi\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right] = \tilde{v}_f(j\omega)$$

where δ is the Dirac-delta function. So, from equation (16),

$$\frac{v_{p,rms}}{v_{f,rms}} = \sqrt{\frac{\pi^2 \int_{-\infty}^{\infty} \left[\frac{(\alpha\beta)^2}{(\alpha\beta)^2 + \omega^2}\right]\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]^2 d\omega}{\pi^2 \int_{-\infty}^{\infty} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]^2 d\omega}}$$

$$= \sqrt{\frac{\frac{(\alpha\beta)^2}{(\alpha\beta)^2 + \omega_0^2} \int_{-\infty}^{\infty} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]^2 d\omega}{\int_{-\infty}^{\infty} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]^2 d\omega}} = \frac{\alpha\beta}{\sqrt{(\alpha\beta)^2 + \omega_0^2}} = |G_1(j\omega)| \tag{18}$$

Therefore, for a simple sinusoidal velocity field, the particle trackability, or the true frequency response, is the magnitude of the transfer function. Although no real flow will be described by an equation like (17), the magnitude of the transfer function as a function of frequency does serve somewhat as a measure of relative particle trackability under different flow conditions, in the absence of any spectral knowledge of the fluid flow. The quantity $1/(\alpha\beta)$ can be considered as a crude time constant: the smaller $1/(\alpha\beta)$, the better the particles will track for given fluid flow conditions.

A plot of equation (18) as a function of $(\omega_0/\alpha\beta)$ is shown in figure 1. To more graphically illustrate the effect on trackability of particle size and frequency, equation (18) is plotted in figure 2 for the specific case of aluminum oxide (Al_2O_3) particles ($\rho_p = 3.5\ g/cm^3$) in an airstream ($\rho_f = 1.2\ kg/m^3$; $\mu_f = 1.85\times10^{-5}$ N-sec/m^2) with sinusoidally varying velocity, for various particle diameters. From the figure, it can be seen that there is a severe drop in particle trackability with frequency. For example, the velocity of a 0.5-micrometer particle will track within 1.5 percent of the true fluctuating velocity of the fluid at 10 kilohertz, with correspondingly greater errors for larger particles and higher frequencies.

More realistically, the fluid flow is not oscillating at a single frequency but presents a broad spectrum of frequencies for the particle to follow. It is not important that the particle can track the fluctuating velocity of the fluid within 20 percent at 50 kilohertz if, for example, most of the turbulent energy occurs at frequencies below 10 kilohertz.

The effect of this weighting of the transfer function with the energy spectrum of the flow is best shown by example. Laurence (ref. 3) has shown that the energy spectrum

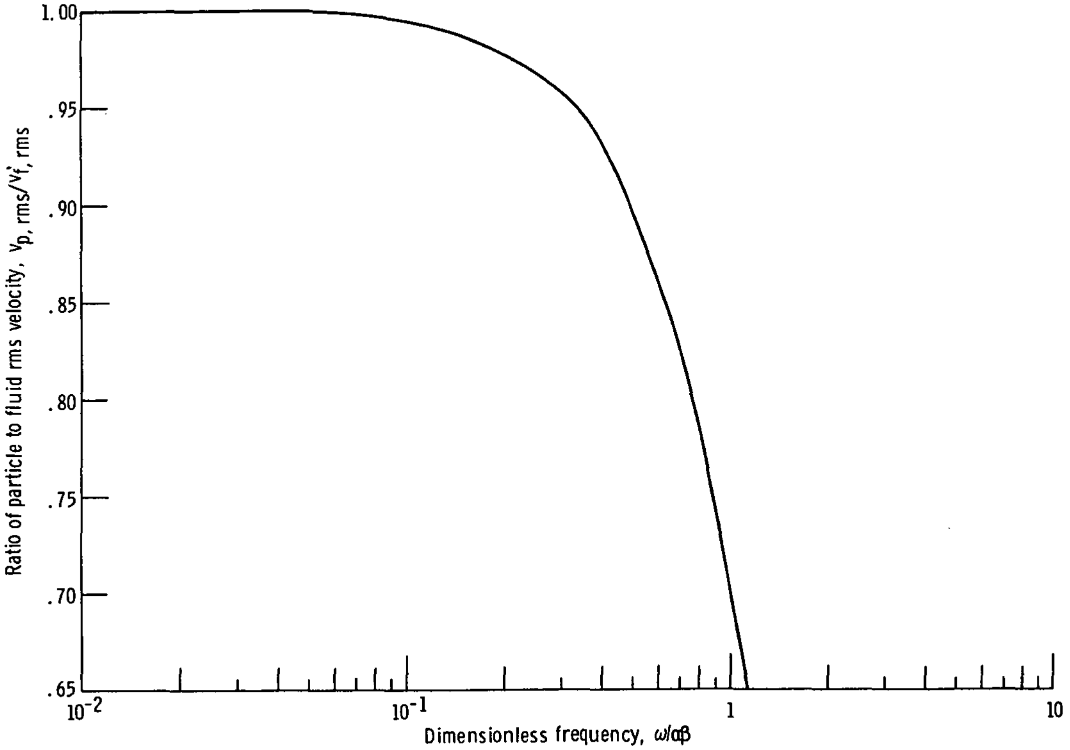

Figure 1. - Single-particle response to sinusoidal velocity.

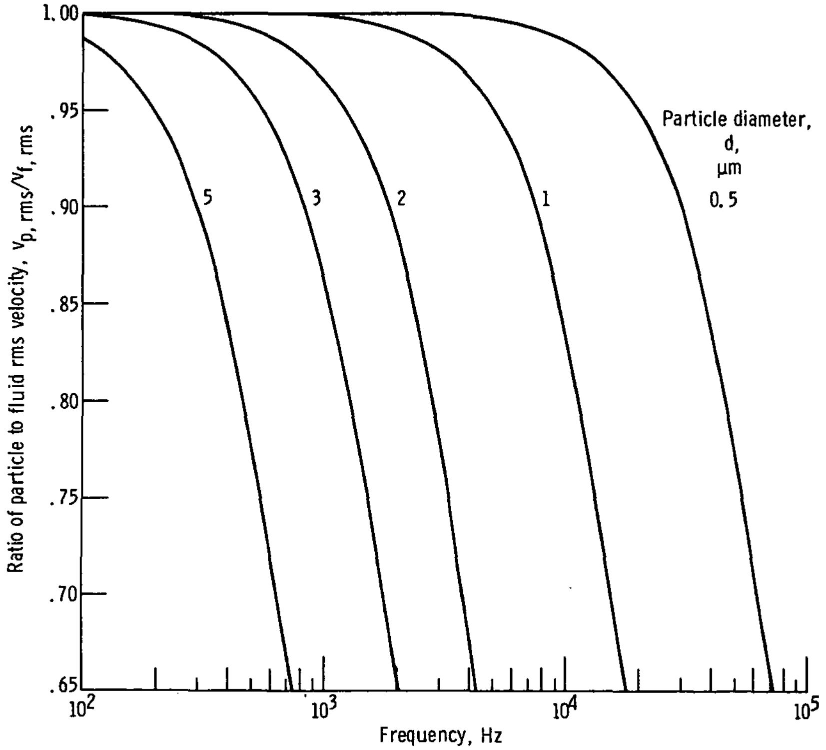

Figure 2. - Single-particle response to sinusoidal velocity for Al_2O_3 particles of various diameters.

of free jet turbulence can be given by an expression of the form

$$|\tilde{v}_f(j\omega)|^2 = \frac{(1 + B)\pi}{4\hat{\omega}} \frac{\left(\frac{\omega}{\hat{\omega}}\right)^2}{\left[1 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]\left[B^2 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]} \tag{19}$$

where $\hat{\omega}$ is some characteristic frequency and B is a constant. Both $\hat{\omega}$ and B can be obtained from the autocorrelation of the flow at the point of interest. Specifically, $\hat{\omega}$ and B are obtained from the equations

$$R_f(\tau)_{min} = -B^{(1+B)/(1-B)} \tag{20a}$$

and

$$\hat{\omega} = \frac{\ln B}{\tau_0(B - 1)} \tag{20b}$$

where $R_f(\tau)_{min}$ is the minimum value of the autocorrelation and τ_0 is the value of the time delay at the first zero crossing of the autocorrelation.

Putting equation (19) into (7) and using equation (15) yield

$$\left.\frac{v_{p,rms}}{v_{f,rms}}\right|_{\omega_u} = \sqrt{\frac{\displaystyle\int_0^{\omega_u} \left[\frac{(\alpha\beta)^2}{(\alpha\beta)^2 + \omega^2}\right]\left\{\frac{\left(\frac{\omega}{\hat{\omega}}\right)^2}{\left[1 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]\left[B^2 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]}\right\} d\omega}{\displaystyle\int_0^{\omega_u} \left\{\frac{\left(\frac{\omega}{\hat{\omega}}\right)^2}{\left[1 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]\left[B^2 + \left(\frac{\omega}{\hat{\omega}}\right)^2\right]}\right\} d\omega}} \tag{21}$$

Equation (21) was evaluated numerically for the specific case of Al_2O_3 particles ($\rho_p = 3.5\ g/cm^3$) in an airstream ($\rho_f = 1.2\ kg/m^3$, $\mu_f = 1.85 \times 10^{-5}\ N\text{-}sec/m^2$) using the Laurence data for a point 1.14 diameters downstream of the nozzle exit plane and 1/2 nozzle diameter from the jet axis.

The result is plotted in figure 3. Some caution should be exercised in the interpretation of figure 3. The figure does not indicate, for example, that a 1-micrometer particle will faithfully reproduce the rms velocity of this flow within $2\frac{1}{2}$ percent at 1 megahertz. It does indicate, however, that for this particular flow, if information is acquired through the entire spectrum to 1 megahertz, the measured rms velocity of the particle will be within $2\frac{1}{2}$ percent of the true rms of the fluid in that bandwidth. The reason for this is simply that the energy spectrum for this point, as given by equation (19), contains little significant energy beyond the breakpoint of the corresponding transfer function given by equation (15); that is, the energy spectrum drops more rapidly with frequency than the transfer function.

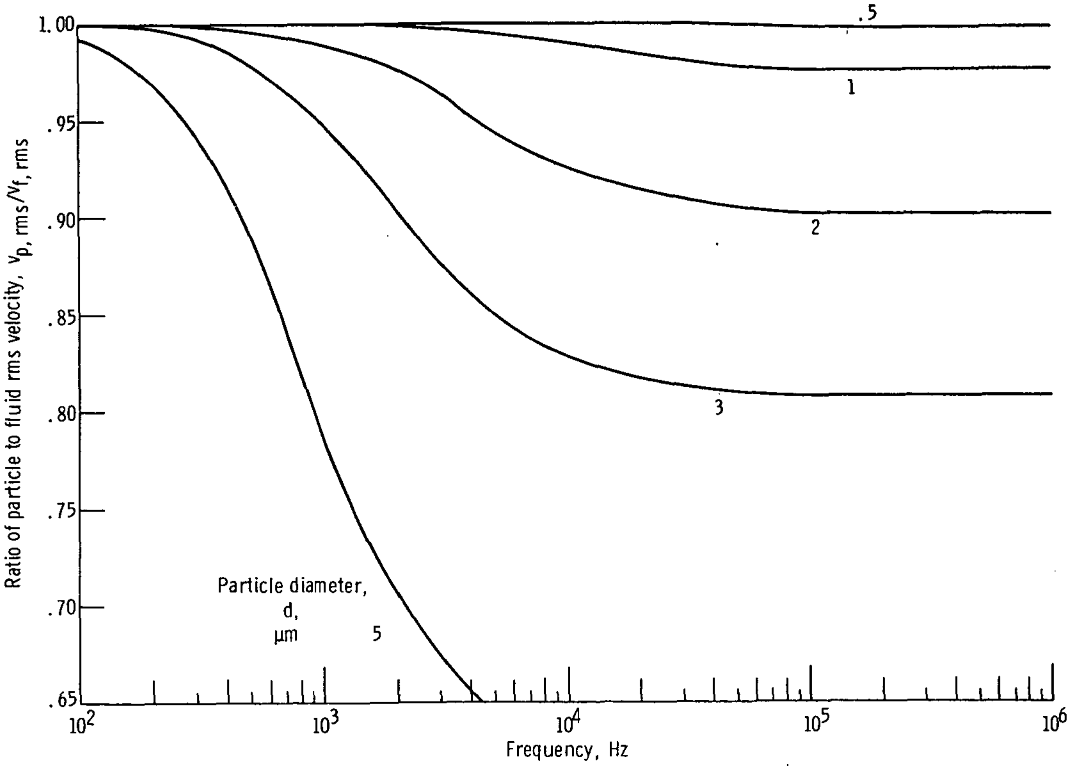

Figure 3. - Accuracy of reproduction of rms velocity in a 8.9-centimeter (3.5-in.) diameter free jet for Al_2O_3 particles of various diameters. (Data from ref. 3.)

Comparison with Complete Solution

Within the limits of the assumptions, equation (21) is a fair representation of the trackability of a single particle in a turbulent flow. For best trackability, however, it is obvious that the particle density should be as close as possible to the fluid density. For example, the same Al_2O_3 particles in a water flow give a ρ_p/ρ_f of 3.5. Under such circumstances, the assumption of $\rho_p/\rho_f >> 1$ would no longer be valid, and the full equation (10) would have to be solved to determine trackability.

The solution to equation (10) for a stationary flow of infinite extent (i.e., no solid boundaries) in terms of the transfer function, as presented in reference 2, is

$$G_2(j\omega) = \frac{\left[\alpha + \left(\frac{3\alpha\omega}{2}\right)^{1/2}\right] + j\left[\omega + \left(\frac{3\alpha\omega}{2}\right)^{1/2}\right]}{\left[\alpha + \left(\frac{3\alpha\omega}{2}\right)^{1/2}\right] + j\left[\frac{\omega}{\beta} + \left(\frac{3\alpha\omega}{2}\right)^{1/2}\right]}$$

and

$$\left|G_2(j\omega)\right| = \sqrt{G_2(j\omega)G_2^*(j\omega)} = \sqrt{\frac{\Omega^{(1)}\left(\frac{\omega}{\alpha}\right)}{\Omega^{(2)}\left(\frac{\omega}{\alpha}, \beta\right)}} \qquad (22)$$

where

$$\Omega^{(1)}\left(\frac{\omega}{\alpha}\right) = \left(\frac{\omega}{\alpha}\right)^2 + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{3/2} + 3\left(\frac{\omega}{\alpha}\right) + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{1/2} + 1 \qquad (22a)$$

and

$$\Omega^{(2)}\left(\frac{\omega}{\alpha}, \beta\right) = \frac{1}{\beta^2}\left(\frac{\omega}{\alpha}\right)^2 + \frac{\sqrt{6}}{\beta}\left(\frac{\omega}{\alpha}\right)^{3/2} + 3\left(\frac{\omega}{\alpha}\right) + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{1/2} + 1 \qquad (22b)$$

As the flow becomes very slowly varying and approaches dc (i.e., $\omega/\alpha \rightarrow 0$), then

$|G_2(j\omega)| \rightarrow 1$. Further, as $\beta \rightarrow 1$ (i.e., $\rho_p/\rho_f \rightarrow 1$), then $|G_2(j\omega)| \rightarrow 1$ for all values of ω/α. In other words, for slowly varying flows or for neutral density particles, perfect trackability is approached. However, if $\rho_p/\rho_f < 1$, then $\beta > 1$ and $|G_2(j\omega)| > 1$. So, under these circumstances the particle motion overshoots the fluid motion. Hence, attempting LDV measurements with very light particles (relative to the fluid) can introduce errors just as large as with relatively heavy particles.

That the solution to equation (10) approaches the solution to equation (14) when the ratio of particle density to fluid density gets very large can be seen by applying this limit to the respective transfer functions. First, rewrite equation (22) as

$$|G_2(j\omega)| = \sqrt{\frac{\Omega^{(1)}\left(\frac{\omega}{\alpha}\right)}{\Omega^{(2)}\left(\frac{\omega}{\alpha}\right)}} = \sqrt{\frac{\left(\frac{\omega}{\alpha}\right)^2 + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{3/2} + 3\left(\frac{\omega}{\alpha}\right) + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{1/2} + 1}{\frac{1}{\beta^2}\left(\frac{\omega}{\alpha}\right)^2 + \frac{\sqrt{6}}{\beta}\left(\frac{\omega}{\alpha}\right)^{3/2} + 3\left(\frac{\omega}{\alpha}\right) + \sqrt{6}\left(\frac{\omega}{\alpha}\right)^{1/2} + 1}}$$

or

$$|G_2(j\omega)| = \sqrt{\frac{\beta^2\left(\frac{\omega}{\alpha\beta}\right)^2 + \sqrt{6}\,\beta^{3/2}\left(\frac{\omega}{\alpha\beta}\right)^{3/2} + 3\beta\left(\frac{\omega}{\alpha\beta}\right) + \sqrt{6}\,\beta^{1/2}\left(\frac{\omega}{\alpha\beta}\right)^{1/2} + 1}{\left(\frac{\omega}{\alpha\beta}\right)^2 + \sqrt{6}\,\beta^{1/2}\left(\frac{\omega}{\alpha\beta}\right)^{3/2} + 3\beta\left(\frac{\omega}{\alpha\beta}\right) + \sqrt{6}\,\beta^{1/2}\left(\frac{\omega}{\alpha\beta}\right)^{1/2} + 1}}$$

As $\rho_p/\rho_f \rightarrow \infty$, then $\beta \rightarrow 0$ and $\omega/\alpha\beta$ is finite. So,

$$\lim_{\substack{\beta \rightarrow 0 \\ \frac{\omega}{\alpha\beta}\ \text{finite}}} |G_2(j\omega)| = \sqrt{\frac{1}{\left(\frac{\omega}{\alpha\beta}\right)^2 + 1}} = \frac{\alpha\beta}{\sqrt{(\alpha\beta)^2 + \omega^2}}$$

which is the same as equation (15). The solution, then, can be written as

$$\left.\frac{v_{p,rms}}{v_{f,rms}}\right|_{\omega_u} = \sqrt{\frac{\displaystyle\int_0^{\omega_u} \frac{\Omega^{(1)}(\omega/\alpha)}{\Omega^{(2)}(\omega/\alpha,\beta)} \left|\tilde{v}_f(j\omega)\right|^2 d\omega}{\displaystyle\int_0^{\omega_u} \left|\tilde{v}_f(j\omega)\right|^2 d\omega}} \tag{23}$$

Equation (23) is an exact solution to equation (10); and either equation (23) can be numerically evaluated if an expression for $\tilde{v}_f(j\omega)$ is available, or the transfer function $\sqrt{\Omega^{(1)}/\Omega^{(2)}}$ alone can be used as a rough measure of trackability. When $\tilde{v}_f(j\omega)$ is available, though, the numerical computation of equation (23) is somewhat lengthy even with a high-speed computer. For an accurate error analysis, equation (23) would have to be evaluated at each point of interest in the flow under consideration, and many such computations require a large amount of computing time. The same computation using the transfer function for the simplified analysis (i.e., eq. (16)) is considerably faster and it is therefore of some interest to determine when equation (16) can be used with accuracy and when equation (23) must be used.

The analysis of the nondimensional equation (13) indicated that for $\rho_p/\rho_f \rightarrow \infty$, the reduced equation (14) could be used if $\mathscr{O}(\omega/\alpha) = \mathscr{O}(\beta)$ (i.e., at low frequency). A plot

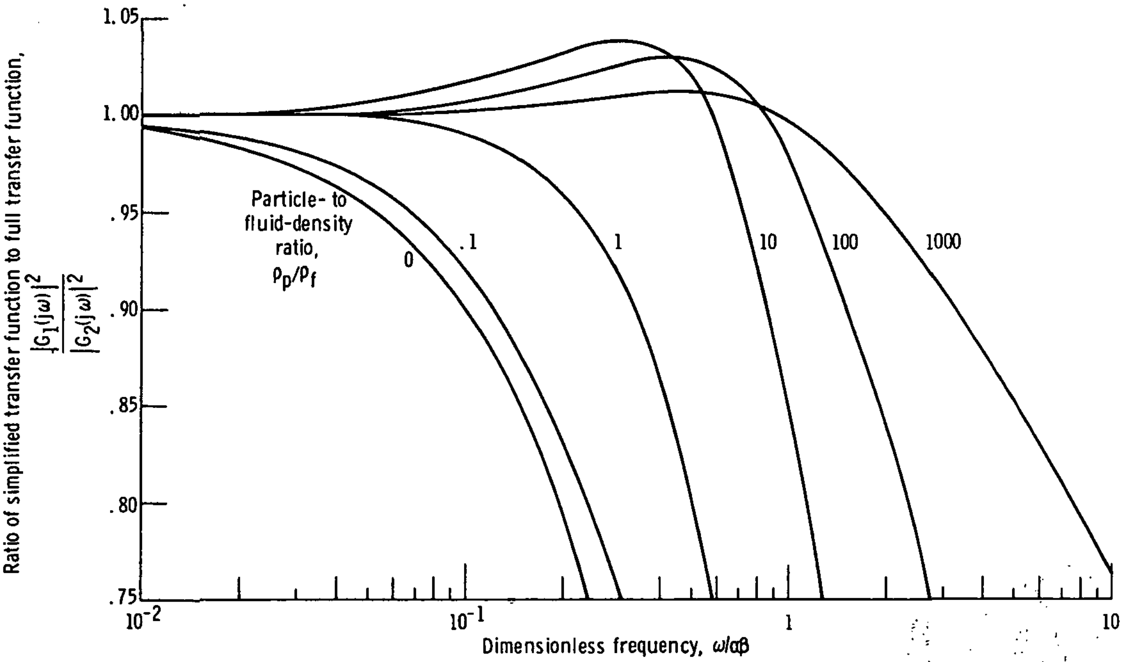

Figure 4. - Variation of transfer function ratio with dimensionless frequency for various particle- to fluid-density ratios.

of the ratio of the simplified transfer function (eq. (15)) to the full transfer function (eq. (22)) against $\omega/\alpha\beta$ is shown in figure 4 for various values of ρ_p/ρ_f. The value of $\omega/\alpha\beta$ at which a given curve deviates significantly from unity represents the frequency above which the full equation (23) must be used. These curves, however, must always be used in conjunction with some spectral knowledge of the fluid flow. For a given fluid and seeding material, a curve in figure 4 might indicate that the simplified transfer function can be used to 100 kilohertz. Nevertheless, if the general fluid flow contains significant energy above 100 kilohertz, equation (23) must be used.

CONCLUDING REMARKS

It was shown that for the limiting case of large particle- to fluid-density ratio and low dimensionless frequency, the reduced flow equation can be used to compute particle trackability. The question then arises as to why one would use particles whose density is large compared to the fluid it must track, since the larger the deviation from neutral density, the larger the tracking error. One answer is that there is not always complete freedom in the choice of seeding particles. For example, many (if not most) applications would require seeding materials to be nontoxic, noncorrosive, nonabrasive, etc. This would eliminate a large number of smokes and similar aerosols. Similarly, if the flow environment is severe, such as high temperature, then many of the synthetically produced materials, such as polystyrene particles are eliminated. Finally, when cost, availability, and ease of flow seeding are considered, the number of practical seeding materials is greatly reduced and, for a given application, only a relatively dense seeding material might be suitable.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, June 26, 1972,
132-80.

APPENDIX - SYMBOLS

B	constant defined by eq. (20a)
D	drag force
d	particle diameter
$\mathscr{F}$	Fourier transform
f	frequency, Hz
$G(j\omega)$	complex transfer function
$G_1(j\omega)$	complex transfer function for eq. (15)
$G_2(j\omega)$	complex transfer function for eq. (11)
j	$\sqrt{-1}$
$\mathcal{O}$	order of magnitude
P_g	force due to gravity
$R(\tau)$	autocorrelation function
Re	Reynolds number
T	limit of integration, sec
t	time, sec
t'	dummy variable of integration, eq. (8)
V	dimensionless velocity
v	velocity
$\dot{v}$	dv/dt
$\tilde{v}$	velocity in frequency domain
α	time constant, $12\nu/d^2$
β	density ratio, $3\rho_f \Big/ \left[2\rho_p + \rho_f\right]$
δ	Dirac-delta function
θ	dimensionless time
θ'	dummy variable of integration, eq. (13)
λ	wavelength
μ	dynamic viscosity
ν	kinematic viscosity

ρ	density
τ	delay time, sec
τ_0	delay time at zero crossing of autocorrelation function
$\Omega^{(1)},\Omega^{(2)}$	functions defined by eq. (22a) and (22b)
ω	circular frequency, rad/sec
$\hat{\omega}$	characteristic circular frequency defined by eq. (20b), rad/sec
$\bar{\omega}$	reference circular frequency, rad/sec

Subscripts:

f	fluid
p	particle
o	initial time
ref	reference
rms	root-mean-square
u	upper limit

Superscripts:

*	complex conjugate
$\langle\ \rangle$	time average

REFERENCES

1. Rolfe, E.; Silk, J. K.; Booth, S.; Meister, K.; and Young, R. M.: Laser Doppler Velocity Instrument. NASA CR-1199, 1968.

2. Soo, S. L.: Fluid Dynamics of Multiphase Systems. Blaisdell Publishing Co., 1967.

3. Laurence, James C.: Intensity, Scale, and Spectra of Turbulence in Mixing Region of Free Subsonic Jet. NACA TR 1292, 1956.

NASA-Langley, 1972 —— 12 E-6992

> *"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."*
>
> —NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546