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Shear-Free, Twisting Einstein-Maxwell
Metrics in the Newman-Penrose Formalism

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ABSTRACT

The problem of finding algebraically special solutions to the vacuum Einstein-Maxwell equations is investigated using the spin coefficient formalism of Newman and Penrose. The general case in which the degenerate null vectors are not hypersurface orthogonal is reduced to a problem of solving five coupled differential equations that are no longer dependent on the affine parameter along the degenerate null directions.

It is shown that the most general regular, shear-free, non-radiating solution to these equations is the Kerr-Newman metric.

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1. Introduction

The spin coefficient formalism of Newman and Penrose [1] (hereafter referred to as (NP)) and its application by Newman and Unti [2] have proven to be extremely valuable in a new approach to the subject of equations of motion in asymptotically flat spaces [3-5]. It has also recently been shown that asymptotically flat spaces admit congruences of null geodesics which are asymptotically shear-free, but twisting [6].

For these (among other more general) reasons, it is felt that a presentation of the spin coefficient formulation of the shear-free, twisting solutions to the vacuum Einstein-Maxwell equations might be particularly appropriate at this time. The results should be useful in obtaining and studying equations of motion for charged spinning particles and may also prove to be helpful in resolving the still open problem [3,5] of finding a unique center of mass coordinate system.

The class of algebraically special solutions to the empty space Einstein equations admitting shear-free and diverging, but non-twisting geodesic rays are the well known Robinson-Trautman metrics [7]. Both these metrics [8] and their Einstein-Maxwell counterparts [9] have already been presented in the (NP) formalism.

The general class of degenerate solutions to the vacuum Einstein equations (admitting diverging and twisting, shear-free null geodesics), first outlined by Kerr [10], has been studied using the (NP) approach by Talbot [11]. These solutions along with the corresponding Einstein-

Maxwell ones have also been investigated by Debney, Kerr and Schild [12] and several explicit solutions have been given [13-15].

In this paper we use the (NP) spin coefficient formalism to present the solutions to the vacuum Einstein-Maxwell equations which admit congruences of shear-free, twisting null geodesics. The results show that the entire class of solutions can be expressed solely in terms of five functions (and their derivatives) that are independent of the affine parameter along the geodesics. These functions satisfy five coupled differential equations, the solution of which would then completely determine the metric.

In Section 2 we formulate the problem in the (NP) formalism. This is followed in Section 3 by further simplifications made possible by the use of coordinate-tetrad freedom. Section 4 contains a complete summary of all of the results and in Section 5 we show that the most general regular, shear-free, non-radiating solution to the remaining equations is the Kerr-Newman metric [16].

It is assumed that the reader is familiar with the operator edth ($\eth$) and the concept of spin s spherical harmonics [17,18], both of which will be used in this work. We will use the notation that $\eth$ applies to an arbitrary two-surface with the metric (in conformally flat form)

$$ds^2 = \frac{1}{P^2}\left[(dx^2)^2 + (dx^3)^2\right] = \frac{d\xi d\bar{\xi}}{P^2}, \quad \xi = (x^2 - i x^3),$$

and that $\eth_0$ applies to the unit sphere.

2. The Spin Coefficient Formulation of the Problem

Following Newman and Penrose [1] a null tetrad $Z_{m\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ [†] is introduced in a four dimensional Riemannian manifold with signature (+,-,-,-). The tetrad is composed of two real null vectors l_μ and n_μ and two complex null vectors m_μ and $\bar{m}_\mu$ satisfying the pseudo-orthogonality conditions

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \tag{2.1}$$

all other scalar products vanishing. Equation (2.1) implies the completeness relation

$$g^{\mu\nu} = Z_m^{\ \mu} Z_n^{\ \nu} \eta^{mn} = 2\left[l^{(\mu} n^{\nu)} - m^{(\mu} \bar{m}^{\nu)}\right], \tag{2.2}$$

where η^{mn} is the null Minkowski metric

$$\eta^{mn} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \eta_{mn} \tag{2.3}$$

used to raise and lower tetrad indices.

[†] Tetrad indices, ranging over the values 1,2,3,4, will be denoted by lower case Latin letters beginning with m. Tensor indices, ranging over 0,1,2,3 will be denoted by Greek letters.

From the tetrad we can define the Ricci rotation coefficients

$$\gamma^{mnp} = Z^{m}{}_{\mu;\nu} Z^{n\mu} Z^{p\nu} \tag{2.4}$$

and the spin coefficients,

$$\begin{aligned}
\kappa &= \gamma_{131} = l_{\mu;\nu} m^{\mu} l^{\nu}, & \nu &= -\gamma_{242} = -n_{\mu;\nu} \bar{m}^{\mu} n^{\nu}, \\
\rho &= \gamma_{134} = l_{\mu;\nu} m^{\mu} \bar{m}^{\nu}, & \mu &= -\gamma_{243} = -n_{\mu;\nu} \bar{m}^{\mu} m^{\nu}, \\
\sigma &= \gamma_{133} = l_{\mu;\nu} m^{\mu} m^{\nu}, & \lambda &= -\gamma_{244} = -n_{\mu;\nu} \bar{m}^{\mu} \bar{m}^{\nu}, \\
\tau &= \gamma_{132} = l_{\mu;\nu} m^{\mu} n^{\nu}, & \pi &= -\gamma_{241} = -n_{\mu;\nu} \bar{m}^{\mu} l^{\nu},
\end{aligned}$$
$$\begin{aligned}
\alpha &= \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}(l_{\mu;\nu} n^{\mu} \bar{m}^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} \bar{m}^{\nu}), \\
\beta &= \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}(l_{\mu;\nu} n^{\mu} m^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} m^{\nu}), \\
\gamma &= \tfrac{1}{2}(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}(l_{\mu;\nu} n^{\mu} n^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} n^{\nu}), \\
\epsilon &= \tfrac{1}{2}(\gamma_{121} - \gamma_{341}) = \tfrac{1}{2}(l_{\mu;\nu} n^{\mu} l^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} l^{\nu}).
\end{aligned} \tag{2.5}$$

Tetrad components of a tensor are defined by

$$A_{mn\ldots} = A^{\mu\nu\ldots} Z_{m\mu} Z_{n\nu} \ldots \tag{2.6}$$

and application to the Weyl tensor yields

$$
\begin{aligned}
\Psi_0 &= -C_{\mu\nu\rho\sigma}\, l^\mu m^\nu l^\rho m^\sigma, \\
\Psi_1 &= -C_{\mu\nu\rho\sigma}\, l^\mu n^\nu l^\rho m^\sigma, \\
\Psi_2 &= -C_{\mu\nu\rho\sigma}\, \bar{m}^\mu n^\nu l^\rho m^\sigma, \\
\Psi_3 &= -C_{\mu\nu\rho\sigma}\, \bar{m}^\mu n^\nu l^\rho n^\sigma, \\
\Psi_4 &= -C_{\mu\nu\rho\sigma}\, \bar{m}^\mu n^\nu \bar{m}^\rho n^\sigma.
\end{aligned}
\tag{2.7}
$$

Similarly, for the Maxwell tensor we have

$$
\begin{aligned}
\phi_0 &= F_{\mu\nu}\, l^\mu m^\nu, \\
\phi_1 &= \tfrac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu), \\
\phi_2 &= F_{\mu\nu}\, \bar{m}^\mu n^\nu.
\end{aligned}
\tag{2.8}
$$

It is always possible to introduce a null tetrad and associated null coordinate system $x^0 = u$, $x^1 = r$, x^2, x^3, in such a way that l^μ is tangent to a congruence of null geodesics with r a standard affine parameter along each of the geodesics labeled by the remaining x^a.† Making this choice we find that

$$l^\mu = \frac{\partial x^\mu}{\partial r}, \tag{2.9}$$

$$l_{\mu;\nu} l^\nu = 0, \tag{2.10}$$

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† Lower case Latin letters from the beginning of the alphabet will range over the values 0,2,3.and capital Latin letters over 2,3.

and the tetrad has the form

$$
\begin{aligned}
l^{\mu} &= \delta_1^{\mu}, \\
n^{\mu} &= U\delta_1^{\mu} + X^a \delta_a^{\mu}, \\
m^{\mu} &= \omega\delta_1^{\mu} + \xi^a \delta_a^{\mu},
\end{aligned}
\tag{2.11}
$$

where U, X^a, ω and ξ^a are arbitrary functions of the coordinates.

With the help of the completeness relation (2.2) we can write

$$
\begin{aligned}
l_{\mu;\nu} = {} & (\gamma+\bar{\gamma}) l_\mu l_\nu + (\epsilon+\bar{\epsilon}) l_\mu n_\nu - (\alpha+\bar{\beta}) l_\mu m_\nu - (\bar{\alpha}+\beta) l_\mu \bar{m}_\nu \\
& - \bar{\tau} m_\mu l_\nu - \bar{\kappa} m_\mu n_\nu + \bar{\sigma} m_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu \\
& - \tau \bar{m}_\mu l_\nu - \kappa \bar{m}_\mu n_\nu + \sigma \bar{m}_\mu \bar{m}_\nu + \rho \bar{m}_\mu m_\nu
\end{aligned}
\tag{2.12}
$$

from which we see that (2.10) can be expressed in terms of spin coefficients as $\epsilon+\bar{\epsilon} = \kappa = 0$ and that the optical scalars which characterize the geometrical properties of the congruence are related to ρ and σ in the following way.

$$
\begin{aligned}
&\text{divergence:} && l^{\mu}{}_{;\mu} = -(\rho+\bar{\rho}), \\
&\text{curl (or twist):} && i\left[2 l_{[\mu;\nu]} l^{\mu;\nu}\right]^{\frac{1}{2}} = \rho-\bar{\rho}, \\
&\text{shear:} && \tfrac{1}{2}\left[2 l_{(\mu;\nu)} l^{\mu;\nu} - (l^{\mu}{}_{;\mu})^2\right]^{\frac{1}{2}} = \sigma\bar{\sigma}.
\end{aligned}
\tag{2.13}
$$

We now make our only assumption, namely that the congruence of null geodesics with tangent vector l^{μ} is shear-free, i.e.,

$$
\sigma = 0. \tag{2.14}
$$

By a corollary of the Goldberg-Sachs theorem [19] the solution will thus be algebraically special and characterized by the fact that l^μ will be tangent to a degenerate principal null direction of the Weyl tensor coincident with one of the principal null directions of the Maxwell tensor, or equivalently $\Psi_0 = \Psi_1 = \phi_0 = 0.$

Finally, we choose m^μ and n^μ to be parallely propagated along l^μ, i.e.,

$$m_{\mu;\nu} l^\nu = \kappa l_\mu - \pi n_\mu - (\epsilon - \bar{\epsilon}) \bar{m}_\mu = 0,$$

$$n_{\mu;\nu} l^\nu = (\epsilon + \bar{\epsilon}) l_\mu - \bar{\pi} m_\mu - \pi \bar{m}_\mu = 0,$$

so that $\epsilon = \pi = 0.$

The form of the tetrad and all of the above conditions will be preserved under the following freedom still remaining in the choice of the tetrad; the spatial rotation

$$l^{\mu *} = l^\mu, \quad n^{\mu *} = n^\mu, \quad m^{\mu *} = e^{iC^{0\dagger}} m^\mu, \tag{2.15}$$

depending on the real parameter $C^0 = C^0(x^a)$ and the null rotation

$$\begin{aligned} l^{\mu *} &= l^\mu, \quad m^{\mu *} = m^\mu + B^0 l^\mu, \\ n^{\mu *} &= n^\mu + \bar{B}^0 m^\mu + B^0 \bar{m}^\mu + B^0 \bar{B}^0 l^\mu, \end{aligned} \tag{2.16}$$

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†The superscript 0 indicates independence of r.

depending on the complex parameter $B^0 = B^0(x^a)$. The coordinate transformations

$$r' = r + R^0(x^a)\ , \qquad x^{a'} = x^a\ , \tag{2.17}$$

and
$$r' = r\ , \qquad x^{a'} = x^{a'}(x^a)\ , \tag{2.18}$$

are also still available along with the combined coordinate-tetrad transformation

$$\begin{gathered} l^{\mu *} = A^{0\,-1} l^{\mu}\ , \ n^{\mu *} = A^0 n^{\mu}\ , \ m^{\mu *} = m^{\mu} \\ r' = A^0 r\ , \ x^{a'} = x^a\ , \quad A^0 = A^0(x^a)\ . \end{gathered} \tag{2.19}$$

The (NP) formulation of the vacuum Einstein-Maxwell equations consists of four sets of first order differential equations for the four sets of variables; the spin coefficients (2.5), the Weyl tensor components (2.7), the Maxwell tensor components (2.8) and the tetrad components (or metric variables)(2.11).

By defining the intrinsic (or directional) derivatives acting on a scalar φ by

$$\begin{aligned} D\varphi &= \varphi_{;\mu} l^{\mu} = \frac{\partial \varphi}{\partial r}\ , \\ \Delta\varphi &= \varphi_{;\mu} n^{\mu} = U\frac{\partial \varphi}{\partial r} + X^a \frac{\partial \varphi}{\partial x^a}\ , \\ \delta\varphi &= \varphi_{;\mu} m^{\mu} = \omega \frac{\partial \varphi}{\partial r} + \xi^a \frac{\partial \varphi}{\partial x^a}\ , \\ \bar{\delta}\varphi &= \varphi_{;\mu} \bar{m}^{\mu} = \bar{\omega} \frac{\partial \varphi}{\partial r} + \bar{\xi}^a \frac{\partial \varphi}{\partial x^a}\ , \end{aligned} \tag{2.20}$$

we could now write down the (NP) equations. In order to save space, however, we simply point out that the appropriate equations can be found in (NP)[1][†] and are considerably simplified by the assumption (2.14) that the space admits shear-free null geodesics, characterized with our choice of tetrad by

$$\kappa = \epsilon = \sigma = \pi = 0 = \phi_0 = \Psi_0 = \Psi_1 . \tag{2.21}$$

In the next section we write down only those equations necessary to show how the remaining coordinate-tetrad freedom can be used to make even further simplifications.

[†]In (NP) the spin coefficient equations are given by (4.2a)-(4.2r) with $\phi_{mn} = k\,\phi_m \phi_n$ (k is twice the Newtonian gravitational constant G), the equations for the components of the Weyl tensor by (A3) and for the Maxwell tensor by (A1). The metric equations are obtained by applying the commutators (4.4) in (NP) to each of the coordinates, u, r, x^A, respectively.

3. Coordinate-tetrad Freedom

All of the (NP) equations can be divided into two groups, the radial equations (in which the operator D appears explicitly) and the non-radial equations (in which D does not appear). The radial equations can be integrated directly with respect to r, thereby introducing, in each case, functions of integration depending on the remaining variables x^a. Substitution of the results of these integrations into the non-radial equations then yields relations among the functions of integration and the five differential equations mentioned in the introduction.

Because this procedure, although lengthy and quite tedious, is entirely straight forward, the complete details (which may, in any case, be found elsewhere [20]) will not be given here. In fact, the solution of the following few equations alone (in addition to demonstrating the use of coordinate-tetrad freedom) should provide a sufficiently clear understanding of what is involved in the process. A complete summary of all of the results will be given in the next section.

The equations whose explicit solutions we need to know now in order to make further simplifications are the following:

$$D\rho = \rho^2 , \tag{3.1a}$$

$$D\tau = \tau\rho , \tag{3.1b}$$

$$D\xi^a = \xi^a\bar{\rho} , \tag{3.1c}$$

$$DX^a = \tau\bar{\xi}^a + \bar{\tau}\xi^a , \tag{3.1d}$$

$$D\alpha = \alpha\rho\,, \tag{3.1e}$$

$$D\beta = \beta\bar{\rho}\,, \tag{3.1f}$$

$$D\omega = \bar{\rho}\omega - (\bar{\alpha}+\beta)\,, \tag{3.1g}$$

$$D\phi_1 = 2\phi_1\rho\,, \tag{3.1h}$$

$$D\Psi_2 = 3\Psi_2\rho + 2k\phi_1\bar{\phi}_1\rho\,, \tag{3.1i}$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 + k\phi_1\bar{\phi}_1\,, \tag{3.1j}$$

$$D\mu = \mu\bar{\rho} + \Psi_2\,, \tag{3.1k}$$

$$DU = \tau\bar{\omega} + \bar{\tau}\omega - (\gamma+\bar{\gamma})\,, \tag{3.1ℓ}$$

$$\delta\tau = \bar{\lambda}\rho + \tau(\tau+\beta-\bar{\alpha})\,, \tag{3.2a}$$

$$\delta X^a - \Delta\xi^a = (\tau-\bar{\alpha}-\beta)X^a + \bar{\lambda}\bar{\xi}^a + (\mu-\gamma+\bar{\gamma})\xi^a. \tag{3.2b}$$

Equations (3.1a) and (3.1b) have the solutions $\rho = -(r+R^0)^{-1}$ and $\tau = \tau^0\rho$. The coordinate transformation (2.17) can be used to make the real part of R^0 vanish so that we may write

$$\rho = -(r+i\Sigma)^{-1}\,, \tag{3.3}$$

with Σ a real function independent of r. Since $\tau^{0*} = \tau^0 + B^0$ under (2.16) this transformation can be used to put $\tau^0 = 0$. Hence,

$$\tau = 0\,, \tag{3.4}$$

and examination of the non-radial equation (3.2a) immediately tells us that

$$\lambda = 0 , \qquad \text{also.} \tag{3.5}$$

Incorporating these simplifications into the remaining radial equations (3.1) yields the following results:

$$(3.1c) \rightarrow \quad \xi^a = \xi^{a0}\bar{\rho} , \tag{3.6a}$$

$$(3.1d) \rightarrow \quad X^a = X^{a0} , \tag{3.6b}$$

$$(3.1e) \rightarrow \quad \alpha = \alpha^0 \rho , \tag{3.6c}$$

$$(3.1f) \rightarrow \quad \beta = \beta^0 \bar{\rho} , \tag{3.6d}$$

$$(3.1g) \rightarrow \quad \omega = \omega^0 \bar{\rho} + (\bar{\alpha}^0 + \beta^0), \tag{3.6e}$$

$$(3.1h) \rightarrow \quad \Phi_1 = \Phi_1^0 \rho^2 , \tag{3.6f}$$

$$(3.1i) \rightarrow \quad \Psi_2 = \Psi_2^0 \rho^3 + 2k\, \Phi_1^0 \bar{\Phi}_1^0 \bar{\rho}\, \rho^3 , \tag{3.6g}$$

$$(3.1j) \rightarrow \quad \gamma = \gamma^0 + \tfrac{1}{2} \Psi_2^0 \rho^2 + k\, \Phi_1^0 \bar{\Phi}_1^0 \bar{\rho}\, \rho^2 , \tag{3.6h}$$

$$(3.1k) \rightarrow \mu = \mu^0 \rho + \tfrac{1}{2} \Psi_2^0 (\rho^2 + \rho\bar{\rho}) + k\, \Phi_1^0 \bar{\Phi}_1^0 \bar{\rho}\, \rho^2 , \tag{3.6i}$$

$$(3.1l) \rightarrow \quad U = U^0 - (\gamma + \bar{\gamma}^0) r - \tfrac{1}{2}(\Psi_2^0 \rho + \bar{\Psi}_2^0 \bar{\rho}) - k\, \Phi_1^0 \bar{\Phi}_1^0 \rho\bar{\rho} . \tag{3.6j}$$

The results given by (3.6) may now be substituted into the non-radial equation (3.2b). Equating the coefficient of the r^{-1} term equal to zero yields

$$(\bar{\alpha}^0 + \beta^0) X^{a0} + \xi^{b0} X^{a0}{}_{,b} - X^{b0} \xi^{a0}{}_{,b} - 2\bar{\gamma}^0 \xi^{a0} = 0 , \tag{3.7}$$

which is the only new information contained in (3.2b) not contained in any other non-radial equation.

Under the transformation (2.18) X^{a0} and ξ^{a0} transform as

$$X^{a0'} = \frac{\partial x^{a'}}{\partial x^{a}} X^{a0},$$

$$\xi^{a0'} = \frac{\partial x^{a'}}{\partial x^{a}} \xi^{a0},$$

so that the $u' = u'(x^a)$ transformation can be used to put X^{a0} into the form

$$X^{a0} = (1, X^{20}, X^{30})$$

and the $x^{A'} = x^{A'}(x^a)$ transformation to put ξ^{a0} into the form

$$\xi^{a0} = (L, P, iP),$$

where X^{A0}, L and P are arbitrary functions of the x^a.

After introducing the complex variable ζ defined by

$$\zeta = x^2 - i x^3 \tag{3.8}$$

we see that the form of ξ^{a0} will still be preserved under further transformations of the type

$$\zeta' = \zeta'(u, \zeta, \bar{\zeta}) \tag{3.9}$$

provided that $\bar{\zeta}'$ satisfies

$$L \dot{\bar{\zeta}}' + 2P \frac{\partial \bar{\zeta}'}{\partial \zeta} \tag{3.10}$$

where a dot above a quantity denotes $\frac{\partial}{\partial u}$.

The complex quantity $X \equiv X^{20} - i\, X^{30}$ will transform under (3.9) as

$$X' = \dot{\xi}' + X \frac{\partial \bar{\xi}'}{\partial \xi} + \bar{X} \frac{\partial \bar{\xi}'}{\partial \bar{\xi}} \tag{3.11}$$

and we will now show that (3.9) can be used to put X' = 0 while still maintaining the form of ξ^{a0}. It is clear from (3.10) and (3.11) that this is equivalent to requiring the system of linear equations

$$\begin{aligned} A_1(\bar{\xi}') &= 0 \;, \\ A_2(\bar{\xi}') &= 0 \;, \end{aligned} \tag{3.12}$$

to have a non-trivial solution, where

$$A_1 \equiv \frac{\partial}{\partial u} + X \frac{\partial}{\partial \xi} + \bar{X} \frac{\partial}{\partial \bar{\xi}} \;,$$

$$A_2 \equiv L \frac{\partial}{\partial u} + 2P \frac{\partial}{\partial \xi} \;.$$

Since the possession of a common integral by the equations (3.12) causes them to vanish identically, the linear equation formed by

$$A_3(\bar{\xi}') \equiv A_1(A_2(\bar{\xi}')) - A_2(A_1(\bar{\xi}')) = 0 \tag{3.13}$$

must also be satisfied by this integral, where

$$\begin{aligned} A_3 = \; & \left(\dot{L} + X \frac{\partial L}{\partial \xi} + \bar{X} \frac{\partial L}{\partial \bar{\xi}} \right) \frac{\partial}{\partial u} \\ & + \left[2\left(\dot{P} + X \frac{\partial P}{\partial \xi} + \bar{X} \frac{\partial P}{\partial \bar{\xi}} \right) - L \dot{X} - 2P \frac{\partial X}{\partial \xi} \right] \frac{\partial}{\partial \xi} \\ & - \left[L \dot{\bar{X}} + 2P \frac{\partial \bar{X}}{\partial \xi} \right] \frac{\partial}{\partial \bar{\xi}} \;. \end{aligned}$$

By including (3.13) with the equations (3.12) we have constructed a complete system of three homogeneous linear partial differential equations for one function ($\bar{\xi}'$) of three independent variables $(u, \xi, \bar{\xi})$. It is obvious that if all three equations are linearly independent the only possible solution is the trivial one, $\bar{\xi}'$ = constant. On the other hand, if the three equations are linearly dependent the original system (3.12) is then guaranteed one non-trivial solution. (For complete details on this, see, for example, Forsyth [21].)

The linear dependence of the equations is easily demonstrated by virtue of equation (3.7), whose three components may now be written as

$$\bar{\alpha}^0 + \beta^0 - 2\bar{\gamma}^0 L = \dot{L} + X\frac{\partial L}{\partial \xi} + \bar{X}\frac{\partial L}{\partial \bar{\xi}} , \tag{3.14a}$$

$$(\bar{\alpha}^0 + \beta^0)X - 4\bar{\gamma}^0 P = 2\left(\dot{P} + X\frac{\partial P}{\partial \xi} + \bar{X}\frac{\partial P}{\partial \bar{\xi}}\right) - L\dot{X} - 2P\frac{\partial X}{\partial \xi} , \tag{3.14b}$$

$$(\bar{\alpha}^0 + \beta^0)\bar{X} = -L\dot{\bar{X}} - 2P\frac{\partial \bar{X}}{\partial \xi} . \tag{3.14c}$$

Multiplying (3.14a), (3.14b) and (3.14c) by $\dot{\bar{\xi}}'$, $\frac{\partial \bar{\xi}'}{\partial \xi}$ and $\frac{\partial \bar{\xi}'}{\partial \bar{\xi}}$, respectively, and adding yields

$$A_3(\bar{\xi}') = (\bar{\alpha}^0 + \beta^0)A_1(\bar{\xi}') - 2\bar{\gamma}^0 A_2(\bar{\xi}') \tag{3.15}$$

so that the original system (3.12) does indeed have a non-trivial solution. Thus (3.9) can be used to put X' = 0 without disturbing the form of ξ^{a0} and additional transformations of this type must now be restricted to

$$\xi' = \xi'(\xi) \tag{3.16}$$

where ξ' is an analytic function of ξ .

Under the tetrad transformation (2.15), $P^* = Pe^{iC^0}$,
so that a suitable choice of C^0 can be used to put $P = \overline{P}$. Under (3.16), however $P' = P\dfrac{\partial \xi'}{\partial \xi}$ so that whenever (3.16) is used it must always be combined with another spatial rotation (2.15) in order to keep P real. The complete transformation is

$$\begin{aligned} &u' = u\ ,\ r' = r\ , \qquad \xi' = \xi'(\xi)\ , \\ &l^{\mu *} = l^{\mu}\quad ,\quad n^{\mu *} = n^{\mu}, \\ &m^{\mu *} = e^{i\lambda} m^{\mu} \equiv \left[\frac{\partial \overline{\xi}'}{\partial \overline{\xi}} \Big/ \frac{\partial \xi'}{\partial \xi}\right]^{\frac{1}{2}} m^{\mu}. \end{aligned} \tag{3.17}$$

Because it will prove to be useful later on we introduce a new function V at this time defined in terms of P by

$$P = P_0 V\ , \quad P_0 = \tfrac{1}{2}(1 + \xi\overline{\xi})\ . \tag{3.18}$$

V and some other important variables transform under (3.17) as

$$\begin{aligned} V' = KV\ ,\ \Sigma' = \Sigma\ ,\ L' = e^{i\lambda} L\ , \\ \phi_1^{0\prime} = \phi_1^0\ ,\ \Psi_2^{0\prime} = \Psi_2^0\ , \end{aligned} \tag{3.19}$$

where K = $\dfrac{P_0}{P_0'}\left[\dfrac{\partial \xi'}{\partial \xi} \cdot \dfrac{\partial \overline{\xi}'}{\partial \overline{\xi}}\right]^{\frac{1}{2}}$.

The only other freedom left is the combined coordinate-tetrad transformation (2.19), under which X^{oo} transforms as $X^{oo'} = A^{o}X^{oo}$. Therefore, in order to keep $X^{oo} = 1$ it is necessary to combine (2.19) with an appropriate coordinate transformation in u . The complete transformation that accomplishes this is given by

$$u' = G(u, \zeta, \bar{\zeta}) \quad , r' = r\dot{G}^{-1} \quad , \quad \zeta' = \zeta ,$$
$$l^{\mu *} = \dot{G} l^{\mu} \quad , m^{\mu *} = m^{\mu} \quad , n^{\mu *} = \dot{G}^{-1} n^{\mu} , \tag{3.20}$$

under which

$$V' = V\dot{G}^{-1} \quad , \quad \Sigma' = \Sigma \dot{G}^{-1} ,$$
$$L' = L + \dot{G}^{-1} \eth G , \tag{3.21}$$
$$\phi_1^{0\prime} = \phi_1^0 \dot{G}^{-2} , \quad \psi_2^{0\prime} = \psi_2^0 \dot{G}^{-3} .$$

4. Summary of the Final Results

In this section we summarize the shear-free, twisting solutions to the vacuum Einstein-Maxwell equations.

A. Tetrad Components of the Weyl Tensor:

$$\Psi_0 = \Psi_1 = 0 \,, \tag{4.1a}$$

$$\Psi_2 = \Psi_2^0 \rho^3 + 2k\,\Phi_1^0 \bar{\Phi}_1^0 \bar{\rho} \rho^3 \,, \tag{4.1b}$$

$$\Psi_3 = \Psi_3^0 \rho^2 + \Psi_3^1 \rho^3 + \Psi_3^2 \rho^4 + k\,\bar{\Phi}_1^0 \bar{\rho}\,(D\Phi_2) \,, \tag{4.1c}$$

$$\begin{aligned}\Psi_4 = {} & \Psi_4^0 \rho + \Psi_4^1 \rho^2 + \tfrac{1}{2}\Psi_4^2 \rho^3 + \tfrac{1}{3}\Psi_4^3 \rho^4 + \tfrac{1}{4}\Psi_4^4 \rho^5 \\ & + k\,\bar{\Phi}_1^0 \bar{\rho}\left[\Psi_4^5 \rho^2 + \Psi_4^6 \rho^3 + \Psi_4^7 \rho^4 + \Psi_4^8 \rho^5\right] ,\end{aligned} \tag{4.1d}$$

with

$$\Psi_3^1 = \bar{\eth}\Psi_2^0 + \bar{L}\dot{\Psi}_2^0 + 3\dot{\bar{L}}\Psi_2^0 - 3\bar{L}\frac{\dot{V}}{V}\Psi_2^0 \,, \tag{4.2a}$$

$$\Psi_3^2 = 3iW\Psi_2^0 \,, \tag{4.2b}$$

$$\Psi_4^1 = \bar{\eth}\Psi_3^0 + \bar{L}\dot{\Psi}_3^0 + 4\dot{\bar{L}}\Psi_3^0 - 3\bar{L}\frac{\dot{V}}{V}\Psi_3^0 \,, \tag{4.2c}$$

$$\Psi_4^2 = \bar{\eth}\Psi_3^1 + \bar{L}\dot{\Psi}_3^1 + 5\dot{\bar{L}}\Psi_3^1 - 4\bar{L}\frac{\dot{V}}{V}\Psi_3^1 + 4iW\Psi_3^0 \,, \tag{4.2d}$$

$$\Psi_4^3 = \bar{\eth}\Psi_3^2 + \bar{L}\dot{\Psi}_3^2 + 6\dot{\bar{L}}\Psi_3^2 - 5\bar{L}\frac{\dot{V}}{V}\Psi_3^2 + 6iW\Psi_3^1 \,, \tag{4.2e}$$

$$\Psi_4^4 = 8iW\Psi_3^2 \,, \tag{4.2f}$$

$$\psi_4^5 = \bar{\eth}\,\phi_2^0 + \bar{L}\,\dot{\phi}_2^0 + 3\dot{\bar{L}}\,\phi_2^0 - 2\bar{L}\,\frac{\dot{V}}{V}\,\phi_2^0\,, \tag{4.2g}$$

$$\psi_4^6 = \bar{\eth}\,\phi_2^1 + \bar{L}\,\dot{\phi}_2^1 + 4\dot{\bar{L}}\,\phi_2^1 - 3\bar{L}\,\frac{\dot{V}}{V}\,\phi_2^1 + 2iW\phi_2^0\,, \tag{4.2h}$$

$$\psi_4^7 = \bar{\eth}\,\phi_2^2 + \bar{L}\,\dot{\phi}_2^2 + 5\dot{\bar{L}}\,\phi_2^2 - 4\bar{L}\,\frac{\dot{V}}{V}\,\phi_2^2 + 4iW\phi_2^1\,, \tag{4.2i}$$

$$\psi_4^8 = 6iW\phi_2^2\,, \tag{4.2j}$$

$$W \equiv \bar{\eth}\Sigma + \bar{L}\dot{\Sigma} + \dot{\bar{L}}\,\Sigma - \bar{L}\,\frac{\dot{V}}{V}\,\Sigma\,. \tag{4.2k}$$

B. Tetrad Components of the Maxwell Tensor:

$$\phi_0 = 0\,, \tag{4.3a}$$

$$\phi_1 = \phi_1^0\rho^2\,, \tag{4.3b}$$

$$\phi_2 = \phi_2^0\rho + \phi_2^1\rho^2 + \phi_2^2\rho^3\,, \tag{4.3c}$$

with

$$\phi_2^1 = \bar{\eth}\,\phi_1^0 + \bar{L}\,\dot{\phi}_1^0 + 2\dot{\bar{L}}\,\phi_1^0 - 2\bar{L}\,\frac{\dot{V}}{V}\,\phi_1^0\,, \tag{4.4a}$$

$$\phi_2^2 = 2iW\phi_1^0\,. \tag{4.4b}$$

C. Spin Coefficients:

$$\kappa = \epsilon = \pi = \sigma = \tau = \lambda = 0\,, \tag{4.5a}$$

$$\rho = -(r + i\,\Sigma)^{-1}\,, \tag{4.5b}$$

$$\alpha = \alpha^0\rho\,, \tag{4.5c}$$

$$\beta = \beta^0 \bar\rho \ , \tag{4.5d}$$

$$\gamma = \gamma^0 + \tfrac{1}{2}\psi_2^0 \rho^2 + k\phi_1^0 \bar\phi_1^0 \bar\rho\rho^2 , \tag{4.5e}$$

$$\mu = \mu^0 \rho + \tfrac{1}{2}\psi_2^0 (\rho^2 + \rho\bar\rho) + k\phi_1^0 \bar\phi_1^0 \bar\rho\rho^2 , \tag{4.5f}$$

$$\nu = \nu^0 + \psi_3^0 \rho + \tfrac{1}{2}\psi_3^1 \rho^2 + \tfrac{1}{3}\psi_3^2 \rho^3 + k\bar\phi_1^0 \bar\rho(\phi_2) . \tag{4.5g}$$

D. Metric Variables:

$$\omega = \omega^0 \bar\rho + \dot{L} - L\frac{\dot V}{V} \ , \tag{4.6a}$$

$$U = U^0 + \frac{\dot V}{V} r - Re(\psi_2^0 \rho) - k\phi_1^0 \bar\phi_1^0 \rho\bar\rho \ , \tag{4.6b}$$

$$X^a = (1, 0, 0) , \tag{4.6c}$$

$$\xi^a = \bar\rho\, (L, P, iP) . \tag{4.6d}$$

E. Components of the Metric Tensor:

$$g^{00} = -2L\bar L \rho\bar\rho \ , \tag{4.7a}$$

$$g^{01} = 1 - 2Re(\bar L \omega\rho) , \tag{4.7b}$$

$$g^{0A} = 2P\rho\bar\rho(-Re\, L, Im\, L) , \tag{4.7c}$$

$$g^{11} = 2(U - \omega\bar\omega) , \tag{4.7d}$$

$$g^{1A} = 2P[-Re(\omega\rho), Im(\omega\rho)] , \tag{4.7e}$$

$$g^{AB} = -2P^2 \rho\bar\rho\, \delta^{AB} . \tag{4.7f}$$

F. The Line Element:

$$ds^2 = 2(l_\mu dx^\mu)\{dr + Re[\omega(r - i\Sigma)\frac{d\zeta}{P}] - U(l_\nu dx^\nu)\} - \frac{(r^2+\Sigma^2)}{2}\frac{d\zeta d\bar{\zeta}}{P^2}, \tag{4.8a}$$

$$(l_\mu dx^\mu) = du + Re\left(\frac{L d\zeta}{P}\right). \tag{4.8b}$$

G. Relations Among the Functions of Integration:

$$\alpha^0 = \tfrac{1}{2}\left(\bar{\eth} \ln P - \bar{L}\frac{\dot{V}}{V} + 2\dot{\bar{L}}\right), \tag{4.9a}$$

$$\beta^0 = -\tfrac{1}{2}\left(\bar{\eth} \ln P + L\frac{\dot{V}}{V}\right), \tag{4.9b}$$

$$\gamma^0 = \bar{\gamma}^0 = -\tfrac{1}{2}\frac{\dot{V}}{V}, \tag{4.9c}$$

$$\omega^0 = -i\left(\eth\Sigma + L\dot{\Sigma} + 2\dot{L}\Sigma - 2L\frac{\dot{V}}{V}\Sigma\right), \tag{4.9d}$$

$$2i\Sigma = \eth\bar{L} + L\dot{\bar{L}} - \bar{\eth}L - \bar{L}\dot{L}, \tag{4.9e}$$

$$\mu^0 = V^2\left[\eth_0 N + \frac{L}{V}\dot{N}\right], \tag{4.9f}$$

$$U^0 = -\tfrac{1}{2}(\mu^0 + \bar{\mu}^0), \tag{4.9g}$$

$$\nu^0 = V\dot{N}, \tag{4.9h}$$

$$\psi_3^0 = V^3\left[\eth_0 R + \frac{L}{V}\dot{R}\right], \tag{4.9i}$$

$$\psi_4^0 = V^2\dot{R}, \tag{4.9j}$$

with

$$N \equiv \bar{\eth}_0 \ln P_0 V + \frac{\dot{\bar{L}}}{V}, \tag{4.10a}$$

$$R \equiv \bar{\eth}_0 N + \frac{\bar{L}}{V}\dot{N} + N^2 - 2N\bar{\eth}_0 \ln P_0, \tag{4.10b}$$

$$V \equiv \frac{P}{P_0} \equiv \frac{P}{\tfrac{1}{2}(1+\zeta\bar{\zeta})}. \tag{4.10c}$$

H. Differential Equations:

$$\eth \phi_1^0 + L \dot{\phi}_1^0 + 2 \dot{L} \phi_1^0 - 2 L \frac{\dot{V}}{V} \phi_1^0 = 0, \tag{4.11a}$$

$$\eth \phi_2^0 + L \dot{\phi}_2^0 + \dot{L} \phi_2^0 - 2 L \frac{\dot{V}}{V} \phi_2^0 = \dot{\phi}_1^0 - 2 \frac{\dot{V}}{V} \phi_1^0, \tag{4.11b}$$

$$\eth \Psi_2^0 + L \dot{\Psi}_2^0 + 3 \dot{L} \Psi_2^0 - 3 L \frac{\dot{V}}{V} \Psi_2^0 = 2k \phi_1^0 \bar{\phi}_2^0, \tag{4.11c}$$

$$\eth \Psi_3^0 + L \dot{\Psi}_3^0 + 2 \dot{L} \Psi_3^0 - 3 L \frac{\dot{V}}{V} \Psi_3^0 = \dot{\Psi}_2^0 - 3 \frac{\dot{V}}{V} \Psi_2^0 + k \phi_2^0 \bar{\phi}_2^0, \tag{4.11d}$$

$$\Psi_2^0 - \bar{\Psi}_2^0 = 2i \left[\mathrm{Re} \left(\eth W + L \dot{W} + \dot{L} W - 2 L \frac{\dot{V}}{V} W \right) - 2 \Sigma U^0 \right]. \tag{4.11e}$$

5. The Kerr-Newman Metric

In this section we use the results of this paper to show that the most general regular, non-radiating, shear-free solution to the vacuum Einstein-Maxwell equations is the Kerr-Newman metric [16].

By regular solution we simply mean one for which the metric has no angular singularities, or equivalently, that the functions V and φ are both expandable in spherical harmonics such that $0 < V < \infty$ and $\dot{\varphi}$ has no zeros, where φ is a potential for L [6] defined by

$$L = -\frac{V\,\eth_0\varphi}{\dot{\varphi}}\ , \tag{5.1}$$

or,

$$\mathcal{D}_0\varphi = 0\,, \tag{5.2}$$

where

$$\mathcal{D}_0 \equiv \eth_0 + \left(\frac{L}{V}\right)\frac{\partial}{\partial u}\ . \tag{5.3}$$

Using the operator $\mathcal{D}_0$, the differential equations (4.11) can be rewritten as

$$\mathcal{D}_0\left[\left(\frac{\phi_1^0}{V^2}\right)\chi^2\right] = 0\,, \tag{5.4a}$$

$$\mathcal{D}_0\left[\left(\frac{\phi_2^0}{V^2}\right)\chi\right] = \frac{\chi}{V}\left(\frac{\phi_1^0}{V^2}\right)^{\cdot}\,, \tag{5.4b}$$

$$\mathcal{D}_0\left[\left(\frac{\Psi_2^0}{V^3}\right)\chi^3\right] = 2k\left(\frac{\phi_1^0}{V^2}\right)\left(\frac{\bar{\phi}_2^0}{V^2}\right)\chi^3\,, \tag{5.4c}$$

$$\mathcal{D}_0\left[R\chi^2\right] = \chi^2\left[\frac{1}{V}\left(\frac{\Psi_2^0}{V^3}\right)^{\cdot} + k\left(\frac{\phi_2^0}{V^2}\right)\left(\frac{\bar{\phi}_2^0}{V^2}\right)\right]\,, \tag{5.4d}$$

$$Im\left\{\left(\frac{\Psi_2^0}{V^3}\right) - \frac{1}{\chi}\mathcal{D}_0\left[\frac{\chi}{\bar{\chi}}\,\bar{\mathcal{D}}_0\left(i\frac{\Sigma}{V}\bar{\chi}\right)\right] - 2i\frac{\Sigma}{V}\mathcal{D}_0 N\right\} = 0\,, \tag{5.4e}$$

where

$$N \equiv \bar{\eth}_0 (\ln P_0 \bar{\chi}) \,, \tag{5.5a}$$

$$R \equiv \frac{\bar{\eth}_0^{\,2} \bar{\chi}}{\bar{\chi}} \,, \tag{5.5b}$$

$$\frac{\Sigma}{V} = V \, Im \left[\eth_0 \left(\frac{\bar{L}}{V} \right) \right] \,, \tag{5.5c}$$

and a new quantity

$$\chi \equiv \frac{V}{\dot{\varphi}} \tag{5.6}$$

has been introduced. The functions χ , $\left(\frac{\phi_1^\circ}{V^2}\right)$, $\left(\frac{\phi_2^\circ}{V^2}\right)$, $\left(\frac{\psi_2^\circ}{V^3}\right)$, R, N and $\frac{\Sigma}{V}$ all transform as scalars under (3.20) so that the equations (5.4) are all in a form that is now manifestly invariant under that transformation. Also, from (5.2), we see that we are free to replace φ at any time by

$$\tilde{\varphi} = F(\varphi) \,, \tag{5.7}$$

where F is an arbitrary regular function of φ .

We now assume that there is no outgoing electromagnetic radiation, i.e.,

$$\phi_2^0 = 0 \,, \tag{5.8}$$

and that there is no Bondi news (hence, no outgoing gravitational radiation) i.e.,

$$R = 0 \,. \tag{5.9}$$

It can be shown (not easily) that if the solution of this paper is re-expressed in a Bondi-type frame then (5.9) would read $\dot{\sigma}^{o}_{B} = 0$.†

Under the above assumptions we see immediately that (5.4c) has the general solution $\left(\frac{\Psi_2^o}{V^3}\right) = M(\varphi)\,\chi^{-3}$ in which (5.7) can be used to put M = 1, so that

$$\left(\frac{\Psi_2^o}{V^3}\right) = \chi^{-3}. \tag{5.10}$$

(5.4d) then reveals that

$$\dot{\chi} = 0 \tag{5.11}$$

and the only freedom remaining in the choice of φ is

$$\tilde{\varphi} = a\,\varphi, \tag{5.12}$$

where a is an arbitrary complex constant.

From (5.11) and the definition of R (5.5b), we see that (5.9) becomes

$$\eth_0^2\,\chi = 0\,, \tag{5.13}$$

which under the condition of regularity has the general solution

$$\chi = \chi_0 + \chi_1\,. \tag{5.14}$$

---

†For more details concerning the relationship between asymptotically shear-free, but twisting, congruences and twist-free, shearing ones in asymptotically flat spaces the reader is referred to [6].

The subscripts refer to the ℓ values of the subscripted quantities so that in this case, for example,

$$\chi_0 = b^0 {}_0Y_{00}(\zeta,\bar{\zeta}), \tag{5.15a}$$

$$\chi_1 = \sum_{m=-1}^{1} b^m {}_0Y_{1m}(\zeta,\bar{\zeta}), \tag{5.15b}$$

where b^0 and b^m are complex constants satisfying

$$\chi^2 + \chi\,\eth_0\bar{\eth}_0\chi - \eth_0\chi\,\bar{\eth}_0\chi \equiv (b^0)^2 - \sum_{m=-1}^{1}(b^m)^2 \equiv c \neq 0,$$

so that (5.12) can be used to put c = 1, resulting in

$$\chi^2 + \chi\,\eth_0\bar{\eth}_0\chi - \eth_0\chi\,\bar{\eth}_0\chi = 1. \tag{5.16}$$

By choosing $G(u,\zeta,\bar{\zeta}) = Re\,\varphi$ the transformation (3.20) can be used to put

$$\varphi = u + i\beta(u,\zeta,\bar{\zeta}) \ , \ \beta = \bar{\beta}, \tag{5.17}$$

so that χ now has the form

$$\chi = \frac{V}{1+i\dot{\beta}} \ . \tag{5.18}$$

After defining the quantity

$$S \equiv \eth_0(LV) + L\dot{L} \tag{5.19}$$

and noting that

$$R = \frac{1}{V}\left[\left(\frac{\bar{S}}{V} \right)^{\cdot} + \bar{\gamma}_0^{\,2} V \right], \tag{5.20}$$

equation (5.4e) can be written in the much simpler form

$$Im\left[\chi^{-3} - \gamma_0^2 \left(\frac{\bar{S}}{V} \right) \right] = 0 . \tag{5.21}$$

In order to maintain regularity, transformation (3.17) must be restricted to the fractional linear transformation

$$\zeta' = \frac{a\zeta + b}{c\zeta + d} , \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, \tag{5.22}$$

under which

$$\left[\chi^{-3} - \gamma_0^2 \left(\frac{\bar{S}}{V} \right) \right]' = K^{-3} \left[\chi^{-3} - \gamma_0^2 \left(\frac{\bar{S}}{V} \right) \right] \tag{5.23}$$

This means that the operator $\Pi_{(0,1)}$ of reference [4] (which commutes with (5.22)) can be applied to (5.21) with the following result. (The proof of this may be found in the appendix.)

$$0 = \Pi_{(0,1)}\, Im\left[\chi^{-3} - \gamma_0^2 \left(\frac{\bar{S}}{V} \right) \right] = Im\, \chi . \tag{5.24}$$

Thus, χ is real, which means that

$$\dot{\beta} = 0 , \tag{5.25}$$

and (5.22) can be used to put

$$\chi = V = 1 . \tag{5.26}$$

Furthermore, $L = -i\,\eth_0 \beta$, $S = -i\,\eth_0^2 \beta$ and (5.21) becomes simply

$$\eth_0^2\, \bar{\eth}_0^2\, \beta = 0\,, \tag{5.27}$$

so that L and Σ are purely $\ell = 1$ quantities.

Finally (5.4a) and (5.4b) yield

$$\phi_1^0 = e \quad = \text{constant} \tag{5.28}$$

and the solution is indeed the Kerr-Newman metric.

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Appendix

Let $\tilde{\eta}(\zeta,\bar{\zeta})$ be a spin zero function on the sphere, expandable in spherical harmonics

$$\tilde{\eta} = \tilde{\eta}_0 + \tilde{\eta}_1 + O(\ell=2),^{\dagger} \tag{A1}$$

such that under (5.22)

$$\tilde{\eta}' = K^{-3}\,\tilde{\eta}. \tag{A2}$$

Then from [4] or [22] we find that

$$\Pi_{(0,1)}\tilde{\eta} \equiv {}_0Y_{00}(\zeta,\bar{\zeta})\int {}_0\overline{Y}_{00}(\zeta',\bar{\zeta}')\,\tilde{\eta}(\zeta',\bar{\zeta}')\,d\Omega' - \frac{1}{3}\sum_{m=-1}^{1} {}_0Y_{1m}(\zeta,\bar{\zeta})\int {}_0\overline{Y}_{1m}(\zeta',\bar{\zeta}')\,\tilde{\eta}(\zeta',\bar{\zeta}')\,d\Omega', \tag{A3}$$

where $d\Omega'$ is the area element of the unit sphere.

Substitution of (A1) into (A3) then yields

$$\Pi_{(0,1)}\tilde{\eta} = \tilde{\eta}_0 - \frac{1}{3}\tilde{\eta}_1 , \tag{A4}$$

where the orthogonality properties of the spherical harmonics have been used.

Now let

$$\tilde{\eta} = \chi^{-3} - \eth_0^2\left(\frac{\bar{S}}{V}\right). \tag{A5}$$

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† The subscripts again refer to the ℓ-values of the quantities and $O(\ell = 2)$ means that the expression is expandable in harmonics with $\ell \geq 2$.

Since

$$\left(\frac{\bar{S}}{V}\right)' = K^{-1} e^{-2i\lambda} \left(\frac{\bar{S}}{V}\right) \tag{A6}$$

under (5.22), $\left(\frac{\bar{S}}{V}\right)$ has spin weight minus two so that

$$\eth_0^2 \left(\frac{\bar{S}}{V}\right) = O(\ell = 2), \tag{A7}$$

and

$$\Pi_{(0,1)}\, \eth_0^2 \left(\frac{\bar{S}}{V}\right) = 0. \tag{A8}$$

By applying $\eth_0 \bar{\eth}_0$ to $\chi = \chi_0 + \chi_1$, we obtain

$$\eth_0 \bar{\eth}_0 \chi \equiv -2\chi + 2\chi_0 \tag{A9}$$

and this together with (5.16) yields

$$\eth_0 \chi\, \bar{\eth}_0 \chi = -\chi^2 + 2\chi_0 \chi - 1. \tag{A10}$$

Dividing (5.16) by χ^3 and using (A9) and (A10) then yields the identity

$$\chi^{-3} \equiv \chi_0 + \chi_0\, \eth_0 \bar{\eth}_0 \ln \chi + \tfrac{1}{2}\, \eth_0 [\chi^{-2} \bar{\eth}_0 \chi] \tag{A11}$$

$$= \chi_0 + O(\ell = 1)$$

from which we immediately conclude that

$$(\chi^{-3})_0 = \chi_0 . \tag{A12}$$

Similarly, by forming $\mathfrak{F}_0^2 \left[\chi^2 \bar{\mathfrak{F}}_0 (\chi^{-2} \bar{\mathfrak{F}}_0 \chi)\right]$

and again using (A9) and (A10) we obtain

$$\chi^{-3} \equiv 4\chi_0 - 3\chi - \frac{1}{4}\mathfrak{F}_0^2\left[\chi^2 \bar{\mathfrak{F}}_0 (\chi^{-2} \bar{\mathfrak{F}}_0 \chi)\right]$$

$$= \chi_0 - 3\chi_1 + O(\ell = 2), \tag{A13}$$

so that

$$(\chi^{-3})_1 = -3\chi_1 \, . \tag{A14}$$

Substitution of (A13) and (A14) into (A4) then yields

$$\Pi_{(0,1)} \left[\chi^{-3} - \mathfrak{F}_0^2 \left(\frac{\bar{S}}{V}\right)\right] = \chi \, . \tag{A15}$$