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THEORETICAL MODELS AND NUMERICAL STUDIES OF WAVES IN A THREE-FLUID MEDIUM

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16. ABSTRACT

Theoretical models of wave propagation in a three-fluid plasma are developed and numerical solutions to certain aspects of the dispersion and propagation characteristics are obtained. Use of the PL/1-FORMAC Methematical Interpretator in the IBM-360-40 computer is discussed.

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PREFACE

Our knowledge of the ionosphere and its interactions with the neutral atmosphere has increased rapidly in the past decade due to the availability of a greater wealth of better observational data combined with the capability of advanced computer systems to handle complicated numerical problems.

Complex theoretical models and their numerical solutions on large computer systems are a requisite to the further advancement of our knowledge. This document contains a collection of papers describing theoretical modelling activities and numerical solutions obtained using the models by personnel of Mississippi State University. It is hoped that the document will provide readers a concept of the complexity of modelling a three-fluid plasma.

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CHAPTER I

NUMERICAL SOLUTION FOR PROPAGATION OF COUPLED LONGITUDINAL AND TRANSVERSE WAVES NORMAL TO THE APPLIED MAGNETIC FIELD IN A THREE-FLUID MEDIUM

by

Ronald W. McClendon and David L. Murphree

NOTE: Figures, references, and equations begin a new sequence in each Chapter. Also, the Appendices are lettered consecutively by Chapter, and each Chapter includes its own List of Symbols.

LIST OF SYMBOLS

e	Magnitude of Electric Charge
c	Velocity of Light in a Vacuum
γ	Specific Heat Ratio
D/Dt	Hydrodynamic Derivative
ν_{ab}	Effective Collision Frequency of Type a with Type b Particles
ν_a	$\nu_{ab} + \nu_{ac}$, Total Collision Frequency
$\overline{E}$	Electric Field Strength
$\overline{H}$	Magnetic Field Strength
$\overline{V}_{e,i,n}$	Fluid Velocity of Electrons, Ions, or Neutral Particles
$N_{e,i,n}$	Number Density of Electrons, Ions, or Neutral Particles
$\rho_{e,i,n}$	Mass Density of Electrons, Ions, or Neutral Particles
$P_{e,i,n}$	Partial Pressure of Electrons, Ions, or Neutral Particles
$\omega_{e,i}$	Electron or Ion Plasma Frequency
ω	Applied Frequency of the Wave
ω_c^e	Electron Cyclotron Frequency
ω_c^i	Ion Cyclotron Frequency
ω_{01}	$[\omega_e^2 + (\omega_c^e/2)^2 + \omega_c^e\omega_c^i]^{1/2} - 1/2\omega_c^e$
ω_{02}	$\omega_{01} + \omega_c^e$
ω_{03}	$\omega_i(\omega_c^e\omega_c^i)^{1/2}/(\omega_i^2 + \omega_c^e\omega_c^i)^{1/2}$
ω_0	$[\omega_e^2 + (\omega_c^e)^2]^{1/2}$
$\omega_{T,L}$	Cyclotron Frequency of the Electrons, Associated with Either the Transverse or Longitudinal Components of $\overline{H}_o$
k	Complex Wave Number
k_R	Real Part of the Wave Number
k_I	Imaginary Part of the Wave Number
n	kc/ω, Index of Refraction

m	m_i/m_e, Mass Ratio of Ions to Electrons
$U_{e,i,n}$	Acoustic Velocity of Electron, Ion, or Neutral Particle Species
U_P	$[\gamma(P_e + P_i)/(\rho_e + \rho_i)]^{1/2}$, Acoustic Velocity of the Electron-Ion Gas Mixture
V_a	$H^o/[4\pi(\rho_e + \rho_i + \rho_n)]^{1/2}$, Alfvén Velocity
V_a'	$H^o/[4\pi(\rho_e + \rho_i)]^{1/2}$, Alfvén Velocity in a Mixture of Just Charged Particle Fluids
i	$\sqrt{-1}$

Introduction

A three-fluid theory, using Maxwell's equations together with a set of coupled hydrodynamic equations for an interacting mixture of electrons, ions, and neutral molecules, has been employed by Tanenbaum and Mintzer[1] to examine small-amplitude oscillations in an infinite, homogeneous, partly ionized gas with a uniform external magnetic field. Plots of phase velocity versus frequency were obtained for the case of negligible collisional damping for wave propagation along and normal to the applied magnetic field. A set of approximate solutions to the dispersion relation was employed to yield the phase velocities for various frequency bands.

An investigation by Dahl and Murphree[2] yielded a solution to the complete three-fluid dispersion relation governing the propagation of longitudinal waves parallel to the applied magnetic field. Some differences were noticed between their phase velocity plot and the approximate solution given by Tanenbaum and Mintzer. In making a comparison, Dahl and Murphree were able to substantiate their results by requiring continuity in both phase velocity and e-folding distance curves with frequency change. Tanenbaum and Mintzer were limited in that their solution was just for the phase velocity and it was valid only in various frequency bands. Connecting the curves between the frequency bands was a possible source of error.

This paper will present a numerical solution to the complete three-fluid dispersion relation governing wave propagation normal to the magnetic field. Solutions have been determined for the complex wave numbers for a typical ionospheric condition without making any approximations to the dispersion relation. Obtaining the solution without making any approximations was made possible only by the use of a computer method to carry out the extremely large number of algebraic manipulations involved. These operations, accomplished by the PL/I-FORMAC interpreter in an IBM 360-40, could not have been performed manually. Plots of the phase velocity and damping characteristics of the resulting wave modes are presented for the frequency range $10^{-5} < \omega < 10^{9}$.

The phase velocity plot is compared to the approximate results of Tanenbaum and Mintzer. Although Tanenbaum and Mintzer had obtained phase velocity results for various frequency bands, a thorough investigation of the damping characteristics of the wave modes had not previously been performed. The purpose of the work described in this paper was to solve for the damping characteristics of the wave modes in addition to obtaining a complete numerical solution to the phase velocity over the entire frequency range. The resulting method allows for the complete solution of any wave propagation problem in a three-fluid medium.

Theory

A. General Dispersion Relation

The derivation of the general dispersion relation which governs wave propagation of small perturbations in a partially ionized gas with an applied magnetic field present will be outlined. This derivation is presented in Reference 1.

The following model is employed:

(1) The degree of ionization is fixed

(2) Each gas obeys the perfect gas law

(3) Damping caused by the frictional forces of each gas allows for the conservation of total momentum of the system

(4) No heat flow exists within the gases.

The set of equations given below describes the three-fluid mixture.

(1) Maxwell's equations:

$$\nabla \times \overline{E} = -\frac{1}{c}\frac{\partial \overline{H}}{\partial t}$$

$$\nabla \times \overline{H} = \frac{4\pi e}{c}(N_i\overline{V}_i - N_e\overline{V}_e) + \frac{1}{c}\frac{\partial \overline{E}}{\partial t}$$

(2) The continuity equation for each gas:

$$\frac{D}{Dt}\rho_{e,i,n} = -\rho_{e,i,n}\nabla \cdot \overline{V}_{e,i,n}$$

(3) The momentum equation for each gas:

$$\frac{D}{Dt}(\overline{V}_e) = -\frac{e}{m_e}\left(\overline{E} + \frac{\overline{V}_e \times \overline{H}}{c}\right) - \frac{\nabla P_e}{\rho_e} - \nu_{ei}(\overline{V}_e - \overline{V}_i) - \nu_{en}(\overline{V}_e - \overline{V}_n)$$

$$\frac{D}{Dt}(\overline{V}_i) = \frac{e}{m_i}\left(\overline{E} + \frac{\overline{V}_i \times \overline{H}}{c}\right) - \frac{\nabla P_i}{\rho_i} - \nu_{ie}(\overline{V}_i - \overline{V}_e) - \nu_{in}(\overline{V}_i - \overline{V}_n)$$

$$\frac{D}{Dt}(\overline{V}_n) = -\frac{\nabla P_n}{\rho_n} - \nu_{ne}(\overline{V}_n - \overline{V}_e) - \nu_{ni}(\overline{V}_n - \overline{V}_i)$$

(4) The adiabatic condition for each gas:

$$P_{e,i,n}N^{-\gamma}_{e,i,n} = \text{constant}$$

To facilitate the solution a 3-D cartesian coordinate system is chosen with x in the direction of wave propagation and z perpendicular to the plane formed by x and the applied magnetic field. The applied magnetic field vector can therefore be written $\overline{H}_o = (H_{ox}, H_{oy}, 0)$. The above plasma equations may be linearized by perturbing the quantities $\overline{H}$, $\overline{V}_{e,i,n}$, $\overline{E}$, $N_{e,i,n}$ and $P_{e,i,n}$ with small periodic oscillations of frequency ω. For example,

$$N_e = N_o + n_e e^{i(kx - \omega t)}$$

where N_o is the undisturbed electron density, n_e the amplitude of the perturbation, and N_e the resulting electron number density at any position x for any time t. The perturbation is considered to propagate only in the x-direction, i.e., one-dimensional propagation. The perturbed quantities are then substituted into the plasma equations yielding twenty-one equations and twenty-one unknowns. These equations can then be manipulated by substitution to yield a vector equation for the three components of the electron velocity.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} V_{ex} \\ V_{ey} \\ V_{ez} \end{bmatrix} = 0 \tag{1}$$

where

$$A_{11} = (C_1/m) - (C_3C_5/C_1) + (\omega^2\omega^2{}_T/mC_2)$$

$$A_{12} = A_{21} = -\ \omega^2\omega_T\omega_L/mC_2$$

$$A_{13} = i\omega\omega_T[(C_3/C_1) - (C_6/mC_2)]$$

$$A_{22} = (C_2/m) - (C_4C_6/C_2) + \omega^2\omega^2{}_L/mC_2$$

$$A_{23} = -A_{32} = -\ (i\omega\omega_L/C_2)[C_4 - (C_6/m)]$$

$$A_{31} = i\omega\omega_T[(C_5/mC_1) - (C_4/C_2)]$$

$$A_{33} = A_{22} + (\omega^2\omega^2{}_T/mC_1)$$

and

$$C_1 = \omega_e^2 - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_2)$$

$$C_2 = \omega_e^2(1 - n^2)^{-1} - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_1)$$

$$C_3 = \omega^2 - \omega_i^2 - k^2U_i^2 + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_2)$$

$$C_4 = \omega^2 - \omega_i^2(1 - n^2)^{-1} + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_1)$$

$$C_5 = \omega^2 - \omega_e^2 - k^2U_e^2 + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_2)$$

$$C_6 = \omega^2 - \omega_e^2(1 - n^2)^{-1} + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_1)$$

where

$$\gamma_1 = \omega^2 + i\omega\nu_n \qquad \omega_{T,L} = eH^o_{T,L}/m_e C$$

$$\gamma_2 = \gamma_1 - k^2U_n^2 \qquad U_{e,i,n} = (\gamma P_{e,i,n}/\rho_{e,i,n})^{\frac{1}{2}}$$

$$\omega_{e,i} = (4\pi e^2 N_o/m_{e,i})^{\frac{1}{2}} \qquad n = kc/\omega$$

B. Dispersion Relation for a Wave Propagating Normal to the Magnetic Field

Examining waves propagating normal to the magnetic field, the longitudinal component of $\overline{H}_o$ will be zero. It follows that the longitudinal component of the electron cyclotron frequency, ω_L, will also be zero. Applying this condition to Equation (1) we obtain

$$A_{11}V_{ex} + A_{13}V_{ez} = 0,$$

$$A_{22}V_{ey} = 0,$$

$$A_{31}V_{ex} + A_{33}V_{ez} = 0. \qquad (2)$$

From the second of these equations we see that a transverse wave can propagate provided that A_{22} as previously defined is zero. This solution is not examined in this analysis.

The first and third equation of Equations (2) form a set of linear homogeneous equations. The determinant of the matrix of coefficients must vanish in order that a nontrivial solution can exist. The dispersion relation for the case of a wave propagating normal to a magnetic field therefore takes the form

$$A_{11}A_{33} - A_{13}A_{31} = 0 \quad . \qquad (3)$$

Examining Equation (3) it is obvious that this dispersion relation is an extremely complicated equation in expanded form. The number of terms in this equation would make its use infeasible when working by hand.

This problem is considered later in the Discussion.

A computer method for carrying out the necessary algebra in order to obtain the dispersion relation $k(\omega)$ was then required. The PL/I-FORMAC Symbôlic Mathematics Interpreter[3] has the capability of symbolic manipulation of mathematical expressions. The expressions can contain variables, user-defined functions, constants up to 2295 digits, and symbolic constants, as well as functions such as SIN, COS, EXP, etc. A factored algebraic equation can be expanded and the coefficients of the various powers of any of the variables can be collected. This was especially useful in our solution for the coefficients of the dispersion relation.

Having this facility available, the terms C_ℓ, $\ell = 1,2,. . .,6$ were substituted into the appropriate A_{ij} terms of Equation (3). This procedure yielded a twelfth degree equation in k, the wave number.

$$A_1k^{12} + A_2k^{10} + A_3k^8 + A_4k^6 + A_5k^4 + A_6k^2 + A_7 = 0 \tag{4}$$

C. Newton-Raphson Iterative Solution

The dispersion relation could also be written in the form

$$A_1x^6 + A_2x^5 + A_3x^4 + A_4x^3 + A_5x^2 + A_6x + A_7 = 0 \tag{5}$$

where $x = k^2$ and the coefficients are complex numbers. A Newton-Raphson iterative procedure as shown below was chosen to solve for the roots of this equation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

or

$$x_{n+1} = x_n - \frac{A_1x_n^6 + A_2x_n^5 + A_3x_n^4 + A_4x_n^3 + A_5x_n^2 + A_6x_n + A_7}{6A_1x_n^5 + 5A_2x_n^4 + 4A_3x_n^3 + 3A_4x_n^2 + 2A_5x_n + A_6}$$

An initial estimate was made and the iteration was performed until a root of the sixth degree equation was found. This root was then factored out of the equation by complex synthetic division. For example let $x = \alpha$ be a root found from the previous equation. Now by synthetic division

$$\begin{array}{ccccccc|c} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & \alpha \\ & \alpha A_1 & \alpha B_2 & \alpha B_3 & \alpha B_4 & \alpha B_5 & \alpha B_6 & \\ \hline A_1 & B_2 & B_3 & B_4 & B_5 & B_6 & 0 & \end{array}$$

where $B_2 = A_2 + \alpha A_1$, etc. then,

$$(x - \alpha)(A_1x^5 + B_2x^4 + B_3x^3 + B_4x^2 + B_5x + B_6) = 0$$

Again the Newton-Raphson iterative procedure can be used to solve for a root of the resulting fifth degree equation. Let this root be $x = \beta$, therefore we have

$$(x - \alpha)(x - \beta)(A_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5) = 0$$

By continuing this approach, the six roots to this sixth degree polynomial can be found. It is important to choose realistic values for the initial estimates in the iterative solution. If the estimate is too far removed from the actual root, the iterative procedure will not be successful.

Discussion

The computer solution for the wave number k, which describes the characteristics of the wave, can be divided into two processes:

(1) substitution and expansion of the general dispersion relation for the specific case of a wave propagating normal to a magnetic field

(2) solution of the dispersion relation for the complex wave number.

The first process was performed largely by using the PL/I-FORMAC Symbolic Mathematics Interpreter. The main feature of the FORMAC interpreter as applied to our problem was its ability to carry out algebraic manipulations. The process of expanding the dispersion relation given in Equation (3) would be essentially impossible to accomplish by hand without assigning numerical values to the variables. This method would not be acceptable because a new dispersion relation would have to be derived manually for each change in the conditions and applied frequency.

A FORMAC program was written for the IBM 360 model 40 at Mississippi State University to carry out this operation. A simplified flow chart of this program is given in Appendix A. The input to this program consists of the applied frequency, magnetic field strength, and collision frequencies, plasma frequencies, acoustic velocities, number densities, and masses of the three species for the desired atmospheric condition.

Due to the size of some of the quantities exceeding the limitations of the computer, it was necessary to change the units of length and time to avoid an underflow or overflow condition. The length and time dimension was also included in the input, and the velocity and frequency terms were converted accordingly.

The dispersion relation given below was obtained from Equation (3) by substituting the expressions for A_{ij}.

$$C_1^2C_2^2 + 2\omega^2\omega_T^2C_1C_2 - mC_1^2C_4C_6 - mC_2^2C_3C_5 + m^2C_3C_4C_5C_6$$
$$- m^2\omega^2\omega_T^2C_3C_4 + \omega^4\omega_T^4 - \omega^2\omega_T^2C_5C_6 = 0 \tag{6}$$

If the expressions for the C's as previously defined were substituted into this equation by hand without assigning numerical values to the variables, the dispersion relation would be composed of several thousand terms. This process was initially attempted, but the complexity involved in expanding this equation by hand ruled this method impracticable.

Furthermore, after expansion the terms would have to be collected to form the real and imaginary components of the coefficients of the various powers of k. Equation (6) is the form of the dispersion relation before any substitutions or manipulations are performed in the FORMAC computer program. Since the available storage was not sufficient to hold the entire dispersion relation in expanded form, values for the applied frequency, cyclotron frequency, collision frequencies, plasma frequencies, and acoustic velocities were substituted. The dispersion relation was then expanded, evaluated, and the coefficients of the twelfth degree polynomial were collected and punched out on cards. This approach gives a dispersion relation for each of the chosen points on the frequency range.

To check the validity of the algebraic manipulations as performed by FORMAC, a test case was formulated. The expression $(A + iB)^n$ was expanded for each of the values $n = 1, 2, 3, \ldots, 10$. For example,

$$(A + iB)^3 = A^3 - 3B^2A + i3BA^2 - iB^3 \quad .$$

These ten expansions were then performed manually and the results of the FORMAC program were shown to be correct. Our use of FORMAC was similar to this example since complex expressions were raised to powers and multiplied by other complex expressions.

It had been hoped that for our problem, the coefficients of the wave number in the dispersion relation could remain in algebraic form. Storage problems forced the assignment of numerical values to the plasma parameters, consequently the coefficients of the various powers of k in the final dispersion relation were numerical constants for the given conditions. Algebraic manipulations were still required since the wave number k was included in the expressions that had to be multiplied and raised to powers. To demonstrate this need, the test case can be used again by holding A as a variable and assigning B a value of two. This corresponds to holding k as a variable and assigning numerical values to the other parameters in the problem considered in this paper. The resulting expression would be

$$A^3 + A^2(6i) - A(12) - 8i \quad .$$

The real and imaginary coefficients of the powers of A could then be collected. This is the process in extremely simplified form of the work performed by FORMAC to obtain the coefficients of the final dispersion relation. The analytical FORMAC compiler facilitated the analysis even when numerical values were substituted for the parameters because the requirement of manually rearranging the complex quantities in powers of k was eliminated.

The problem was now reduced to the solution of a sixth degree polynomial in k^2 with complex coefficients. The difficulty lay in the various sizes of the coefficients. A range of 10^{20} in the sizes of coefficients was not uncommon.

A digital computer program was written for the UNIVAC 1106 at Mississippi State University to solve for the roots of a polynomial equation with complex coefficients. The simplified flow chart for this program is given in Appendix C. The input to this program consists of the applied frequency, the length and time dimension, and the coefficients of the polynomial. Since we were working with a polynomial equal to zero (Eq. 5) it is permissible to multiply all the coefficients by some constant.

Due to the size of the coefficients it was necessary to do this to avoid an overflow or underflow condition. The quantity by which the coefficients were multiplied was also included in the input to the polynomial roots program. The roots of the polynomial equation are found by a Newton-Raphson iteration coupled with a synthetic division operation. The six roots to this sixth degree polynomial are equal to k^2. The square root of a complex number must now be taken to yield the wave number, k.

The square root operation was performed by the subroutine XPOCPX using the equation

$$(a + ib)^{1/n} = (\sqrt{a^2 + b^2})^{1/n} \{\cos(\frac{\phi + 2\pi K}{n}) + i\sin(\frac{\phi + 2\pi K}{n})\}$$

where

$K = 1, 2, \ldots, n - 1$, and ϕ is the argument of the complex number.

Taking the square root of the six complex numbers, we obtain twelve solutions for the wave number. Restricting our analysis to waves propagating in the positive x direction, we will have six wave numbers.

$$k_j = k_{R_j} + ik_{I_j} \qquad j = 1, 2, \ldots, 6$$

The other six will be same waves propagating in the negative x direction. Previously, the length and time units were changed from meters and seconds respectively to avoid an underflow or overflow condition. The wave number's dimension is 1/length so it must be converted back to 1/meters. The phase velocity for each wave is ω/k where ω is the applied frequency. The damping characteristics of the wave is contained in the imaginary part of the wave number, k_I.

Results

The solution was obtained using field parameters of a typical ionospheric condition at an altitude of 320 kilometers at 45° North latitude and 90° West longitude.

$\nu_{ni} = 1.1202 \times 10^{-4}$ coll/sec	$\omega_e = 2.8806 \times 10^7$ rad/sec
$\nu_{in} = 2.2541 \times 10^{-1}$ coll/sec	$\omega_i = 1.5731 \times 10^5$ rad/sec
$\nu_{en} = 1.3072 \times 10^1$ coll/sec	$U_{f,i,n} = 8.5097 \times 10^2$ m/sec
$\nu_{ei} = 5.7883 \times 10^1$ coll/sec	$U_e = 2.8158 \times 10^5$ m/sec

Perturbation frequencies ranging from 10^{-5} to 10^9 radians/second were examined. As mentioned previously, there are six values of k which represent waves propagating in the positive x direction, i.e. k_R positive where $k = k_R + ik_I$. Not all of the six mathematically possible solutions would necessarily represent physically possible wave modes. A value of k with a negative imaginary component would allow for an exponentially increasing amplitude of the perturbation with increasing distance from the source. A phase velocity plot of the approximated solutions of Tanenbaum and Mintzer (Fig. 4 of Reference 1) is presented as Fig. 1. This plot was obtained by Tanenbaum and Mintzer from a set of approximate solutions to the complete dispersion relation, each approximate solution valid in a given frequency range. From that analysis it appeared that there should be four physically possible solutions.

At relatively high frequencies there were, in fact, four physically possible solutions and two that were not physically possible. At low frequencies, however, there was a repeated root to the dispersion relation. This repeated root corresponded to a physically possible solution, leaving only one solution that was not physically possible.

The phase velocities and corresponding e-folding distances of the wave modes which exhibit decreasing amplitude with increasing distance from the source are plotted in Figs. 2 and 3 respectively. The waves which exhibit increasing amplitude with increasing distance from the source are plotted similarly in Figs. 4 and 5. In these four figures, both the horizontal and vertical axes are plotted on logarithmic scales.

Some difficulty was encountered in constructing smooth curves for these solutions due to the fact that we were limited in the number of points. As described earlier, the complexity of the problem prevented us from obtaining a single dispersion relation valid for all frequencies. Since the applied frequency was included in the input along with the atomospheric parameters, the FORMAC program solved for a dispersion relation good only for that particular frequency.

Phase Velocities

As seen in Figs. 2 and 3 at low frequencies, $\omega \approx \nu_{ni}$, only three of the four possible wave solutions have relatively large e-folding distances. The fourth wave mode has an e-folding distance less than 10^{-2} meters; therefore, at low frequencies it does not propagate. Mode 3, one of the three wave modes which is not damped out, has an extremely small phase velocity. Consequently, it appears that at low frequencies only two of the four possible wave solutions will propagate. Mode 1 has a phase velocity of U_n, the acoustic velocity in the neutral gas, and mode 2 has a lower phase velocity. In the numerical calculations, U_n and the acoustic velocity in the ion gas, U_i, were taken to be the same.

For $\nu_{ni} < \omega < \nu_{in}$, wave modes 2 and 3 increase in phase velocity with increasing frequency. Wave mode 1 which propagates at U_n for $\omega \approx \nu_{ni}$ continues to propagate at this constant value for higher applied frequencies. In this range the fourth wave mode continues to propagate at a constant phase velocity less than U_n.

As ω increases in range $\nu_{in} < \omega < \omega_e$, wave mode 2 increases to V_a', the Alfvén velocity in a medium composed of a mixture of just the charged particle fluids, and then decreases to U_n as ω approaches ω_i. The phase velocity of wave mode 3 increases to a value well above the speed of light, c, as the applied frequency passes ω_i. As ω nears ω_e the phase velocity of this mode decreases abruptly and approaches a phase velocity of U_e. In the range $\nu_{in} < \omega < \omega_e$ the e-folding distance for this solution is small and this wave mode does not propagate. In this frequency range wave mode 1 continues to propagate at a phase velocity of U_n as the frequency increases. The fourth wave mode starts to increase in phase velocity in this frequency range and has a phase velocity near c as ω passes ω_i. The phase velocity of this mode increases past c in the range $\omega_i < \omega < \omega_e$ and then starts to decrease and approaches c as ω approaches ω_e. However, as shown in Fig. 3, for the frequency range $\nu_{ni} < \omega < \omega_e$, this fourth wave mode has an e-folding distance less than 10^{-2} meters; therefore, for this mode there is no propagation of the disturbance.

For $\omega > \omega_e$, mode 4 has a phase velocity of the speed of light, mode 3 propagates at the acoustic velocity in the electron gas U_e, and wave modes 1 and 2 have a phase velocity U_n. However, wave mode 1 has an e-folding distance less than 10^{-2} meters, and therefore does not propagate.

These results will now be compared with the approximate solutions of Tanenbaum and Mintzer given in Fig. 4 of Reference 1 and presented as Fig. 1 of this paper. The points ω_{01}, ω_{02}, and ω_{03} on the frequency axis in the results by Tanenbaum and Mintzer were added to Fig. 2 to aid in this discussion. The wave mode, which increases to a phase velocity of U_n, i.e. mode 1, and continues to propagate at that velocity for increasing frequency, compares quite well with a solution found by Tanenbaum and Mintzer.

Wave mode 2, in which the phase velocity increases to V'_a then decreases, has the same shape as a solution of Tanenbaum and Mintzer for $\omega > \nu_{in}$, since U_n and U_i are assumed to be equal. In the range $\nu_{in} < \omega < \omega_i$, the approximate solution shows the phase velocity first increasing to V'_a and then at ω_{03} decreasing to U_p before finally decreasing to U_i at ω_i. As shown in Fig. 2, the distance between ω_{03} and ω_i for our calculations is quite small as compared to the distance between ν_{in} and ω_{03}, also the value of U_p is near U_n. Even with the very small region involved, mode 2 does appear to decrease to U_p at ω_{03} and then approach U_n at ω_i. Since we have taken U_i and U_n to be equal, our result for $\omega < \omega_i$ agrees well with the approximate solutions of Tanenbaum and Mintzer.

In the region $\omega < \nu_{in}$ the approximate solution has a wave mode with a phase velocity of V_a, the Alfvén velocity, before it starts to increase to V'_a at ν_{in}. Our solution does not agree with this result in this region. The phase velocity for our corresponding wave mode, i.e. mode 2 as shown in Fig. 2 is much lower than the Alfvén velocity.

Tanenbaum and Mintzer show two other approximate solutions in the frequency range near ω_e. One solution decreases at ω_{01}, to a phase velocity of c and at ω_o, it begins to decrease again to a value of U_e. The other curve decreases at ω_{02} to c and the phase velocity remains constant for higher frequencies.

From Fig. 2 we see that for our case ω_e and ω_o are for practical purposes the same point and ω_{01} and ω_{02} are extremely close. Considering the very narrow frequency range involved, Tanenbaum's result compares quite well with modes 3 and 4 of Fig. 2. At ω_{01} both mode 3 and 4 have phase velocities well above c. At ω_{01} mode 3 first decreases abruptly and then has a slight tendency to level as the phase velocity passes c. As the frequency increases the phase velocity decreases again before reaching a constant value of U_e. The phase velocity of mode 4 begins its decrease slightly after mode 3 and it approaches a constant value of c for higher frequencies.

e-folding Distances

The e-folding distance is defined as the distance from the source of the perturbation at which the amplitude of the wave is damped to 1/e of its initial amplitude, where e is the exponential factor. The solution for the e-folding distance to describe the damping characteristics of each wave mode was not obtained by Tanenbaum and Mintzer. Each of the e-folding solutions corresponds to one of the phase velocity solutions, since both quantities are taken from the imaginary and real components, respectively, of the same wave number, k. Note that corresponding solutions are symbolized in the same manner on each graph. Fig. 3 presents the plot of e-folding distance versus applied frequency.

The e-folding distance, corresponding to wave mode 1 in which the wave propagates at U_n, did not form a continuous curve. The points indicate a curve in the general shape as shown, but a smooth curve could not be drawn. It appears that for $\omega < \nu_{in}$ mode 1 is essentially undamped. The e-folding distance of this mode decreases with increasing frequency for $\omega > \nu_{in}$.

Wave mode 2, in which the phase velocity increases to V'_a and then decreases to U_n, has an e-folding distance that at low frequencies first decreases with increasing frequency. At a point between ν_{ni} and ν_{in} the e-folding distance starts to increase and reaches a maximum as the phase velocity approaches V'_a. At ω_{03} when the phase velocity starts to decrease the points on the e-folding distance plot become scattered. No attempt was made to draw a curve through these points.

Wave mode 4 has an e-folding distance less than 10^{-2} meters for the range $\omega < \omega_e$. At ω_e the e-folding distance increases abruptly indicating that for this mode a disturbance does not propagate except for frequencies above ω_e.

The remaining wave mode, mode 3, has a high e-folding distance for frequencies below ν_{ni}, but the phase velocity in this range is very low. The e-folding distance decreases and the phase velocity increases as the frequency is increased. At ω_{03} the e-folding distance increases to 10 meters and remains at this value until the frequency nears ω_e. At this point the e-folding distance increases to a significant value and the wave mode is essentially undamped for frequencies $\omega > \omega_e$. Both of these latter wave modes, mode 3 and mode 4, have frequency ranges in which the phase velocity increases above the speed of light. However, in these ranges the corresponding e-folding distances are small and the two waves are damped out.

Exponentially Growing Wave Modes

Figs. 4 and 5 contain the phase velocity and e-folding distance plots of the wave modes which exhibit increasing amplitude with increasing distance from the source. At low frequencies there is only one mode with a negative e-folding distance. The phase velocity for this mode increases with increasing frequency from a value below U_n at the lower end of the frequency spectrum until it approaches the speed of light. The phase velocity then remains constant at the speed of light for increasing frequency. The e-folding distance for this mode starts decreasing in magnitude from a large negative value at low frequencies. Near ν_{in} the e-folding distance begins to increase in magnitude and continues to increase with increasing frequency. Near an applied frequency of 10 radians/second another mode appears with a negative e-folding distance. This wave mode has a constant phase velocity of U_n for increasing frequency. The e-folding distance for this wave mode when plotted did not form a smooth curve and was not included in Fig. 5.

Conclusion

Complete solutions were obtained for the dispersion relation, based on the three-fluid plasma model, which governs the propagation of small perturbations normal to the applied magnetic field. The waves investigated were coupled longitudinal and transverse waves. A pure transverse wave can propagate normal to an applied magnetic field, but this case was not considered in this analysis. The solution for the complete dispersion relation governing coupled longitudinal and transverse wave propagation with no approximations made is advantageous because it gives the complete description of the wave propagation across the entire frequency spectrum considered. The resulting wave numbers consist of both real and imaginary parts which describe both the phase velocity and damping characteristics of each solution.

Obtaining the solution without making any approximations was made possible by use of a computer method to carry out the extremely large number of algebraic manipulations involved. These operations, accomplished by the PL/I-FORMAC interpreter in an IBM 360-40, could not have been performed manually.

The approximated phase velocity predictions made by Tanenbaum and Mintzer agree in most of the frequency ranges with our solution considering the limited number of points on the frequency range. Although Tanenbaum and Mintzer obtained approximated phase velocity results, they did not solve for the e-folding distances of the wave modes. Besides obtaining a complete numerical solution to the phase velocity over the entire frequency range, this paper presented an analysis of the damping characteristics in the form of the e-folding distance. Atmospheric conditions were chosen in this analysis but now that the method has been shown to be valid, any three-fluid problem of coupled longitudinal and transverse wave propagation normal to the applied magnetic field could be solved by using this approach.

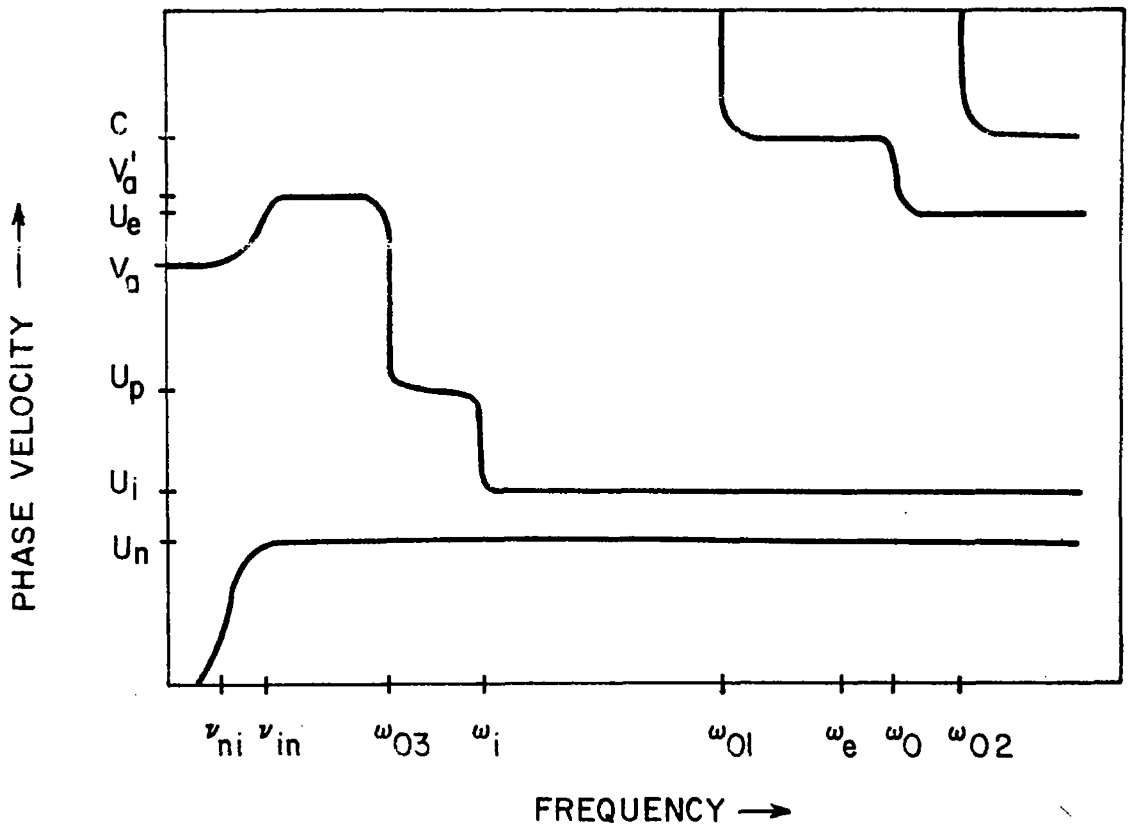


FIGURE I. APPROXIMATE SOLUTIONS FOR PHASE VELOCITY

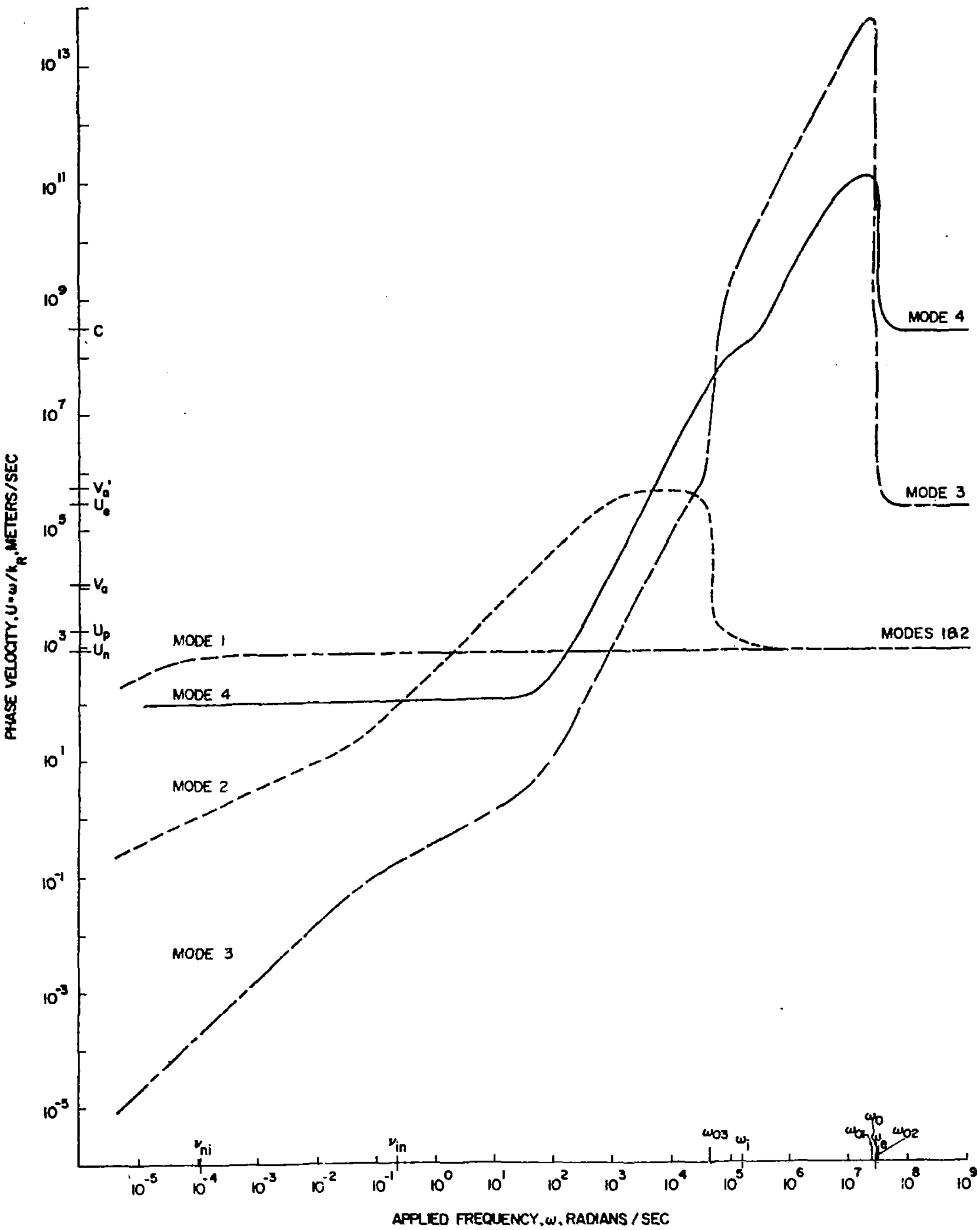


FIGURE 2. PHASE VELOCITY VERSUS FREQUENCY

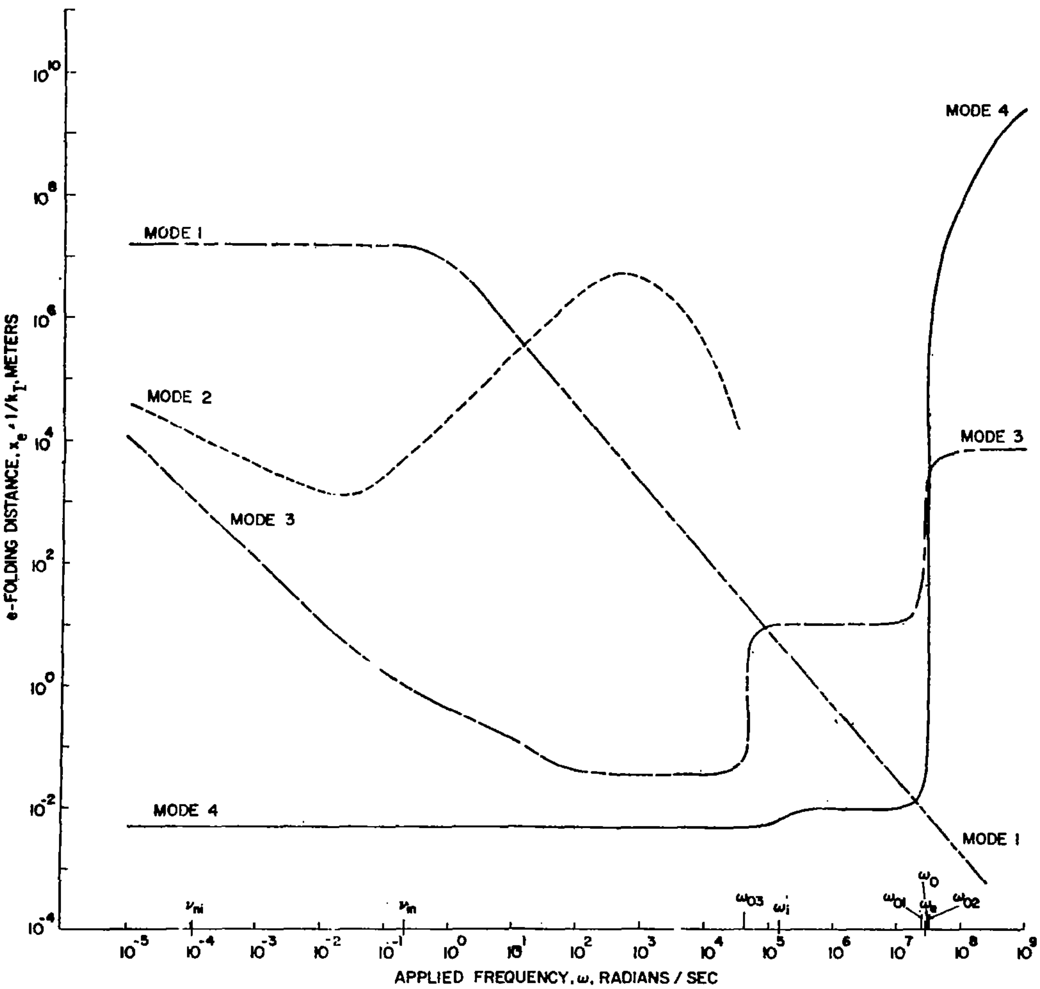


FIGURE 3. e-FOLDING DISTANCE VERSUS FREQUENCY

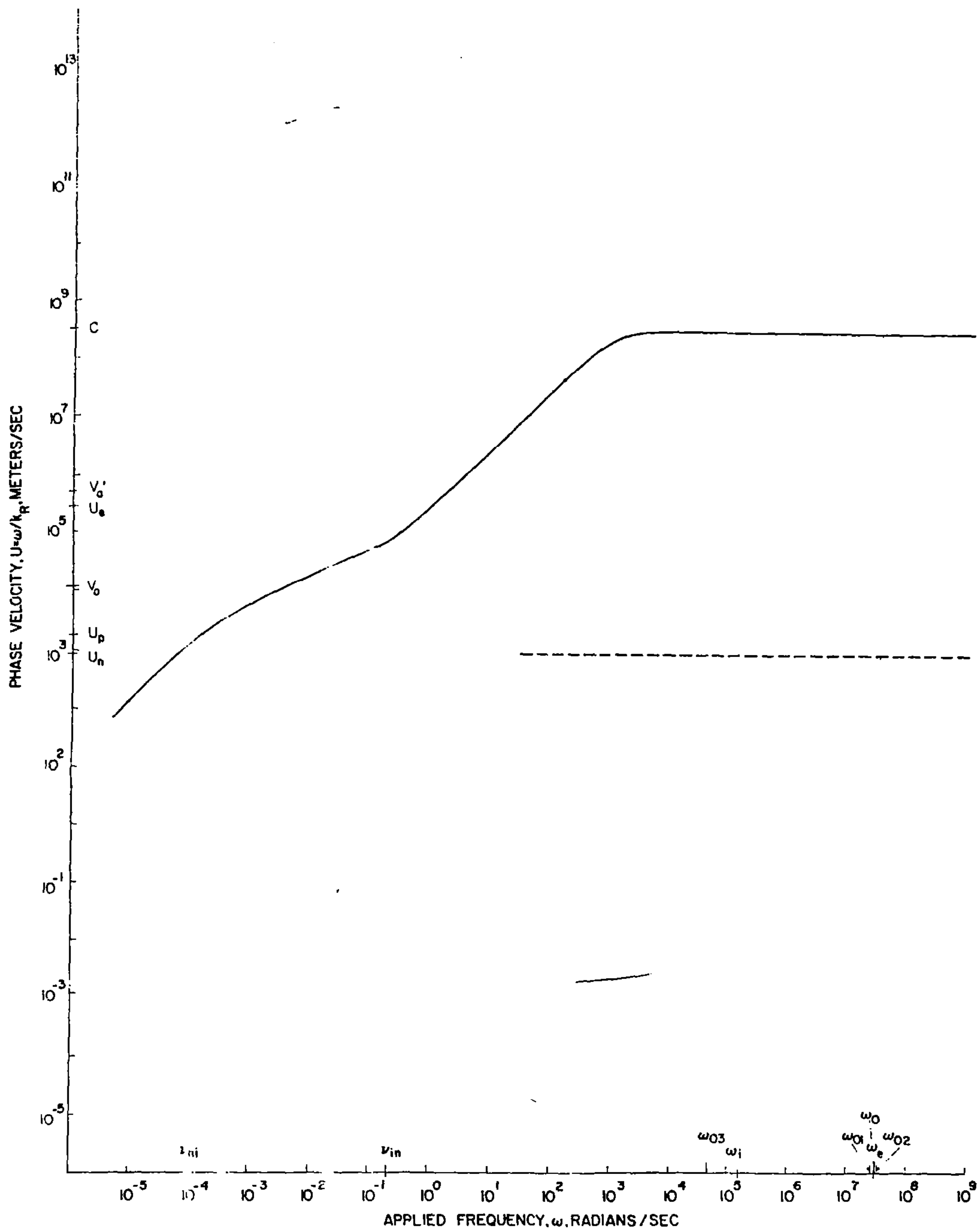


FIGURE 4. PHASE VELOCITY VERSUS FREQUENCY

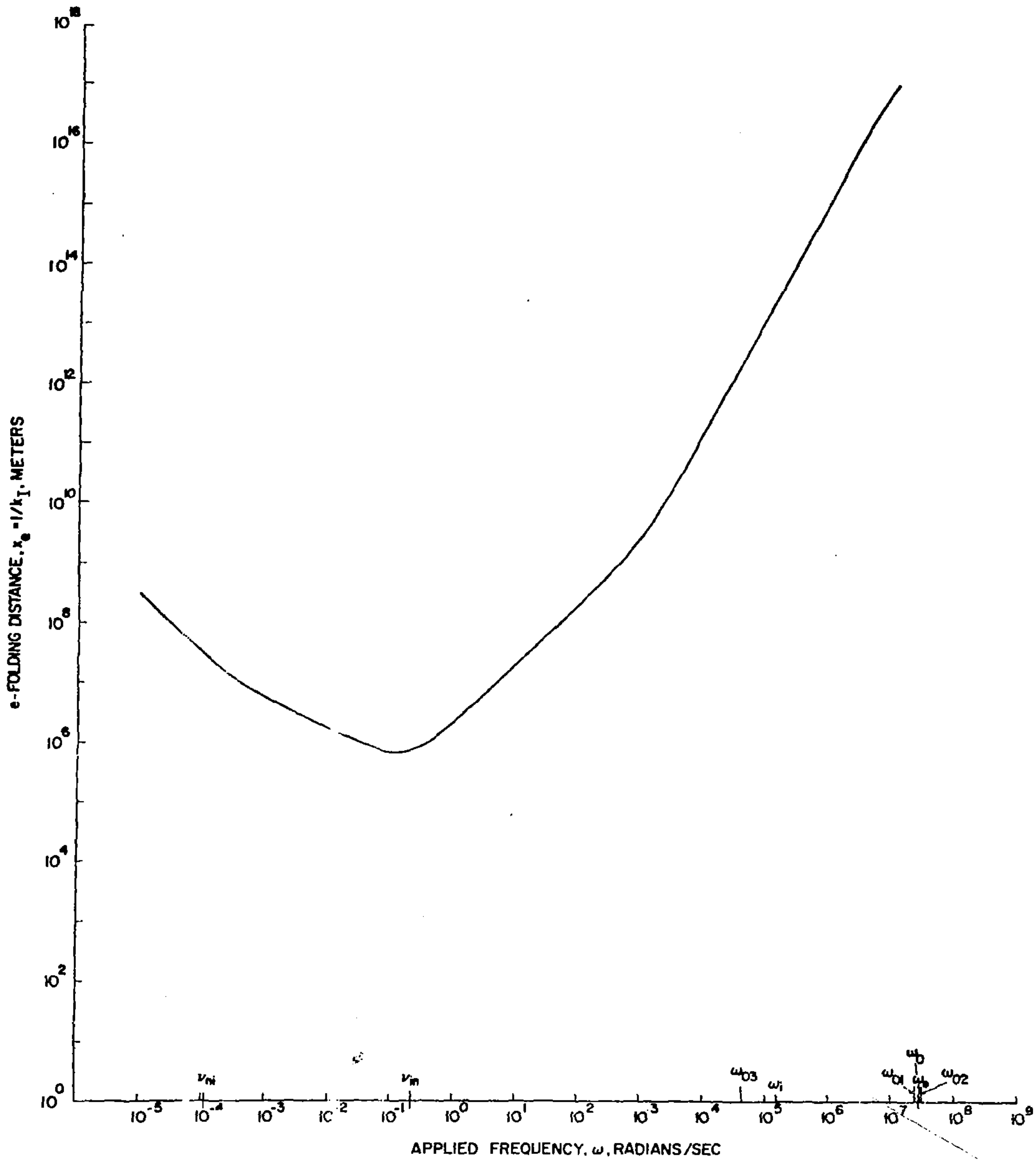
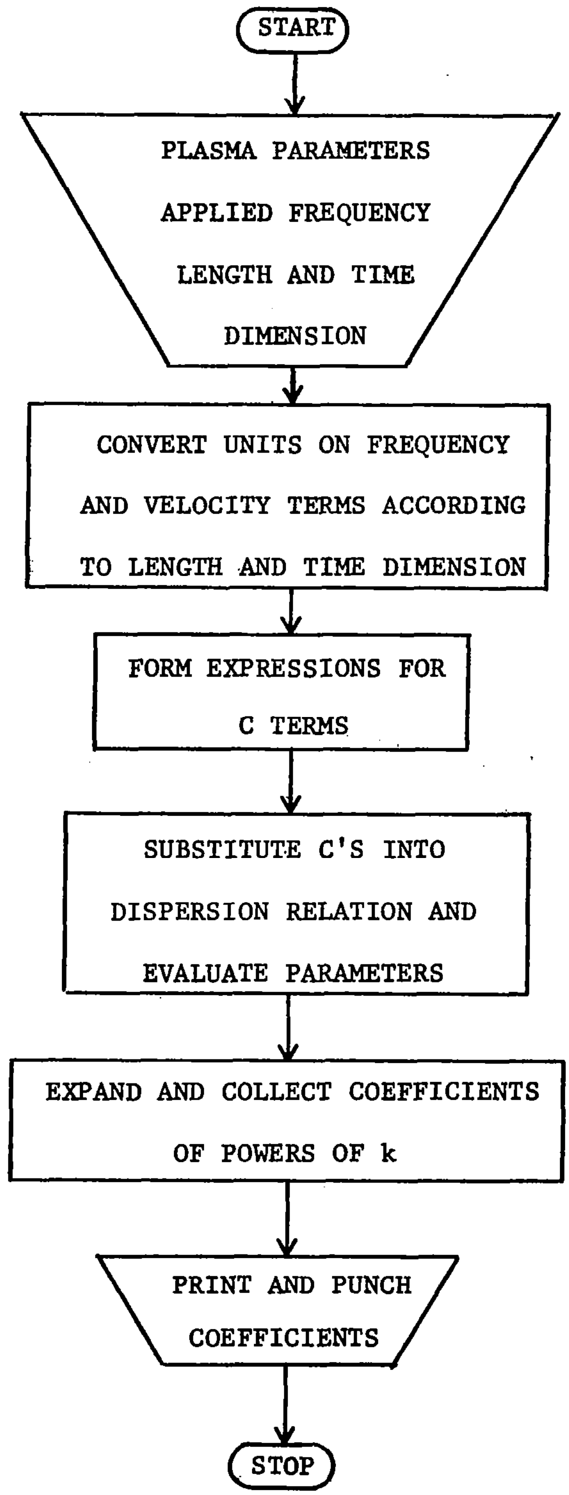


FIGURE 5. e-FOLDING DISTANCE VERSUS FREQUENCY

APPENDIX A

SIMPLIFIED FLOW CHART OF FORMAC COMPUTER PROGRAM


START
PLASMA PARAMETERS
APPLIED FREQUENCY
LENGTH AND TIME
DIMENSION
CONVERT UNITS ON FREQUENCY
AND VELOCITY TERMS ACCORDING
TO LENGTH AND TIME DIMENSION
FORM EXPRESSIONS FOR
C TERMS
SUBSTITUTE C'S INTO
DISPERSION RELATION AND
EVALUATE PARAMETERS
EXPAND AND COLLECT COEFFICIENTS
OF POWERS OF k
PRINT AND PUNCH
COEFFICIENTS
STOP

APPENDIX B

LISTING OF FORMAC COMPUTER PROGRAM

```
€XQT
MAP 0017-01/13-14:36
START=010463, PROG SIZE(I/D)=3919/2170
  READ:  GET DATA;
   /*  ROUTINE TO CHANGE TO APPROPRIATE UNITS OF TIME AND LENGTH.  */
    IF FACTIME=0 THEN GO TO SAME;
    IF FACTIME=1 THEN GO TO DESI;
    IF FACTIME=2 THEN GO TO CENTI;
    IF FACTIME=3 THEN GO TO MILLI;
    IF FACTIME=4 THEN GO TO MINUS4;
    IF FACTIME=5 THEN GO TO MINUS5;
    IF FACTIME=6 THEN GO TO MICRO;
    IF FACTIME=7 THEN GO TO MINUS7;
    IF FACTIME=8 THEN GO TO MINUS8;
  DESI: REW=REW*1.0E-1;  WE=WE*1.0E-1;  WI=WI*1.0E-1;  VEI=VEI*1.0E-1;
        VEN=VEN*1.0E-1;  VIN=VIN*1.0E-1;  VNI=VNI*1.0E-1;  UE=UE*1.0E-1;
        UI=UI*1.0E-1;  UN=UN*1.0E-1;  CL=CL*1.0E-1;
       GO TO SAME;
  CENTI:REW=REW*1.0E-2;  WE=WE*1.0E-2;  WI=WI*1.0E-2;  VEI=VEI*1.0E-2;
        VEN=VEN*1.0E-2;  VIN=VIN*1.0E-2;  VNI=VNI*1.0E-2;  UE=UE*1.0E-2;
        UI=UI*1.0E-2;  UN=UN*1.0E-2;  CL=CL*1.0E-2;
       GO TO SAME;
  MILLI: REW=REW*1.0E-3;  WE=WE*1.0E-3;  WI=WI*1.0E-3;  VEI=VEI*1.0E-3;
        VEN=VEN*1.0E-3;  VIN=VIN*1.0E-3;  VNI=VNI*1.0E-3;  UE=UE*1.0E-3;
        UI=UI*1.0E-3;  UN=UN*1.0E-3;  CL=CL*1.0E-3;
       GO TO SAME;
  MINUS4: REW=REW*1.0E-4;  WE=WE*1.0E-4;  WI=WI*1.0E-4;  VEI=VEI*1.0E-4;
        VEN=VEN*1.0E-4;  VIN=VIN*1.0E-4;  VNI=VNI*1.0E-4;  UE=UE*1.0E-4;
        UI=UI*1.0E-4;  UN=UN*1.0E-4;  CL=CL*1.0E-4;
       GO TO SAME;
  MINUS5: REW=REW*1.0E-5;  WE=WE*1.0E-5;  WI=WI*1.0E-5;  VEI=VEI*1.0E-5;
        VEN=VEN*1.0E-5;  VIN=VIN*1.0E-5;  VNI=VNI*1.0E-5;  UE=UE*1.0E-5;
        UI=UI*1.0E-5;  UN=UN*1.0E-5;  CL=CL*1.0E-5;
       GO TO SAME;
  MICRO: REW=REW*1.0E-6;  WE=WE*1.0E-6;  WI=WI*1.0E-6;  VEI=VEI*1.0E-6;
        VEN=VEN*1.0E-6;  VIN=VIN*1.0E-6;  VNI=VNI*1.0E-6;  UE=UE*1.0E-6;
        UI=UI*1.0E-6;  UN=UN*1.0E-6;  CL=CL*1.0E-6;
     GO TO SAME;
  MINUS7: REW=REW*1.0E-7;  WE=WE*1.0E-7;  WI=WI*1.0E-7;  VEI=VEI*1.0E-7;
    VEN=VEN*1.0E-7;  VIN=VIN*1.0E-7;  VNI=VNI*1.0E-7;  UE=UE*1.0E-7;
    UI=UI*1.0E-7;  UN=UN*1.0E-7;  CL=CL*1.0E-7;
      GO TO SAME;
  MINUS8: REW=REW*1.0E-8;  WE=WE*1.0E-8;  WI=WI*1.0E-8;  VEI=VEI*1.0E-8;
    VEN=VEN*1.0E-8;  VIN=VIN*1.0E-8;  VNI=VNI*1.0E-8;  UE=UE*1.0E-8;
    UI=UI*1.0E-8;  UN=UN*1.0E-8;  CL=CL*1.0E-8;
      GO TO SAME;
  SAME:    IF FACLEN=0  THEN GO TO OKAY;
           IF FACLEN=1  THEN GO TO M10;
           IF FACLEN=2  THEN GO TO M100;
           IF FACLEN=3  THEN GO TO KM;
           IF FACLEN=4  THEN GO TO M10TO4;
           IF FACLEN=5  THEN GO TO M10TO5;
           IF FACLEN=6  THEN GO TO MEGA;
           IF FACLEN=-1  THEN GO TO DECIM ;
           IF FACLEN=-2  THEN GO TO CENTIM ;
           IF FACLEN=-3  THEN GO TO MILLIM;
           IF FACLEN=-4 THEN GO TO NEG4;
   NEG4:   UE=UE*1.0E+4 ; UI=UI*1.0E+4 ; UN=UN*1.0E+4 ; CL=CL*1.0E+4 ;
```

```
    GO TO OKAY ;
  MILLIM: UE=UE*1.0E+3 ; UI=UI*1.0E+3 ; UN=UN*1.0E+3 ; CL=CL*1.0E+3 ;
    GO TO OKAY ;
  CENTIM: UE=UE*1.0E+2 ; UI=UI*1.0E+2 ; UN=UN*1.0E+2 ; CL=CL*1.0E+2 ;
    GO TO OKAY ;
  DECIM:  UE= UE*1.0E+1 ; UI = UI*1.0E+1; UN=UN*1.0E+1 ; CL=CL*1.0E+1 ;
   GO TO OKAY  ;
 M10:      UE=UE*1.0E-1; UI=UI*1.0E-1; UN=UN*1.0E-1; CL=CL*1.0E-1;
     GO TO OKAY;
 M100:     UE=UE*1.0E-2; UI=UI*1.0E-2; UN=UN*1.0E-2; CL=CL*1.0E-2;
     GO TO OKAY;
 KM:       UE=UE*1.0E-3; UI=UI*1.0E-3; UN=UN*1.0E-3; CL=CL*1.0E-3;
     GO TO OKAY;
 M10TO4:   UE=UE*1.0E-4; UI=UI*1.0E-4; UN=UN*1.0E-4; CL=CL*1.0E-4;
     GO TO OKAY;
 M10TO5:   UE=UE*1.0E-5; UI=UI*1.0E-5; UN=UN*1.0E-5; CL=CL*1.0E-5;
     GO TO OKAY;
 MEGA:     UE=UE*1.0E-6; UI=UI*1.0E-6; UN=UN*1.0E-6; CL=CL*1.0E-6;
 OKAY:   /*  CHANGING PL/1 VARIABLES TO FORMAC VARIABLES       */
     LET( WE='WE';WI='WI';UN='UN';UI='UI';UE='UE'; REW = 'REW' ;
          VEN='VEN';VEI='VEI';VIN='VIN';VNI='VNI';
          ME='ME';MI='MI';H='H';CE='CE';CL='CL';
          NE='NE';NN='NN';LIMIT='LIMIT');
   /*  BASIC EQUATIONS NEEDED TO RUN PROGRAM EFFICIENTLY   */
     LET( NI = NE;    M=MI/ME;
          VIE =(VEI*NE)/(M*NI);     VNE =(VEN*NE)/(M*NN);
          VE =VEI+VEN;   VI=VIE+VIN;   VN=VNE+VNI;
          UESQ=UE**2;   UISQ=UI**2;   UNSQ=UN**2;
          WESQ=WE**2;    WISQ=WI**2;    CLSQ=CL**2;
          MSQ=M**2;    MCL=M**3;    WTSQ=WT**2;
          WTQU=WTSQ**2;  WLSQ=WL**2;   WLQU=WLSQ**2;
          WSQ=W**2;    WCL=W*WSQ;    WQUAD=WSQ**2;
          W2WT2= WSQ*WTSQ;
          W2WL2 = WSQ * WLSQ ;
          W4WT4 = W2WT2**2;
          W4WL4 = W2WL2**2;
          W4WTL2 = WQUAD*WTSQ*WLSQ) ;
   VIE = ARITH(VIE);
   VNE = ARITH(VNE);
   NI = ARITH(NI);
  /*  PRINT OUT BASIC FIELD PARAMETERS AND INPUT DATA   */
  PUT LIST('COMPLETE THREE-FLUID THEORY DISPERSION EQUATION')PAGE;
  PUT EDIT('MAGNETIC FIELD STRENGTH = ',H,' WEBERS PER SQ. METER')
       (SKIP(6),A,E(12,5),A);
  PUT EDIT('PLASMA FREQUENCY OF ELECTRONS = ',WE) (SKIP(1),A,E(12,5));
  PUT EDIT('PLASMA FREQUENCY OF IONS       = ',WI) (SKIP(1),A,E(12,5));
  PUT EDIT('COLLISION FREQUENCY OF ELECTRONS WITH IONS      = ',VEI)
        (SKIP(1),A,E(12,5));
PUT EDIT('COLLISION FREQUENCY OF ELECTRONS WITH NEUTRALS = ',VEN)
        (SKIP(1),A,E(12,5));
PUT EDIT('COLLISION FREQUENCY OF IONS WITH ELECTRONS      = ',VIE)
        (SKIP(1),A,E(12,5));
  PUT EDIT('COLLISION FREQUENCY OF IONS WITH NEUTRALS       = ',VIN)
        (SKIP(1),A,E(12,5));
PUT EDIT('COLLISION FREQUENCY OF NEUTRALS WITH ELECTRONS = ',VNE)
        (SKIP(1),A,E(12,5));
PUT EDIT('COLLISION FREQUENCY OF NEUTRALS WITH IONS       = ',VNI)
```

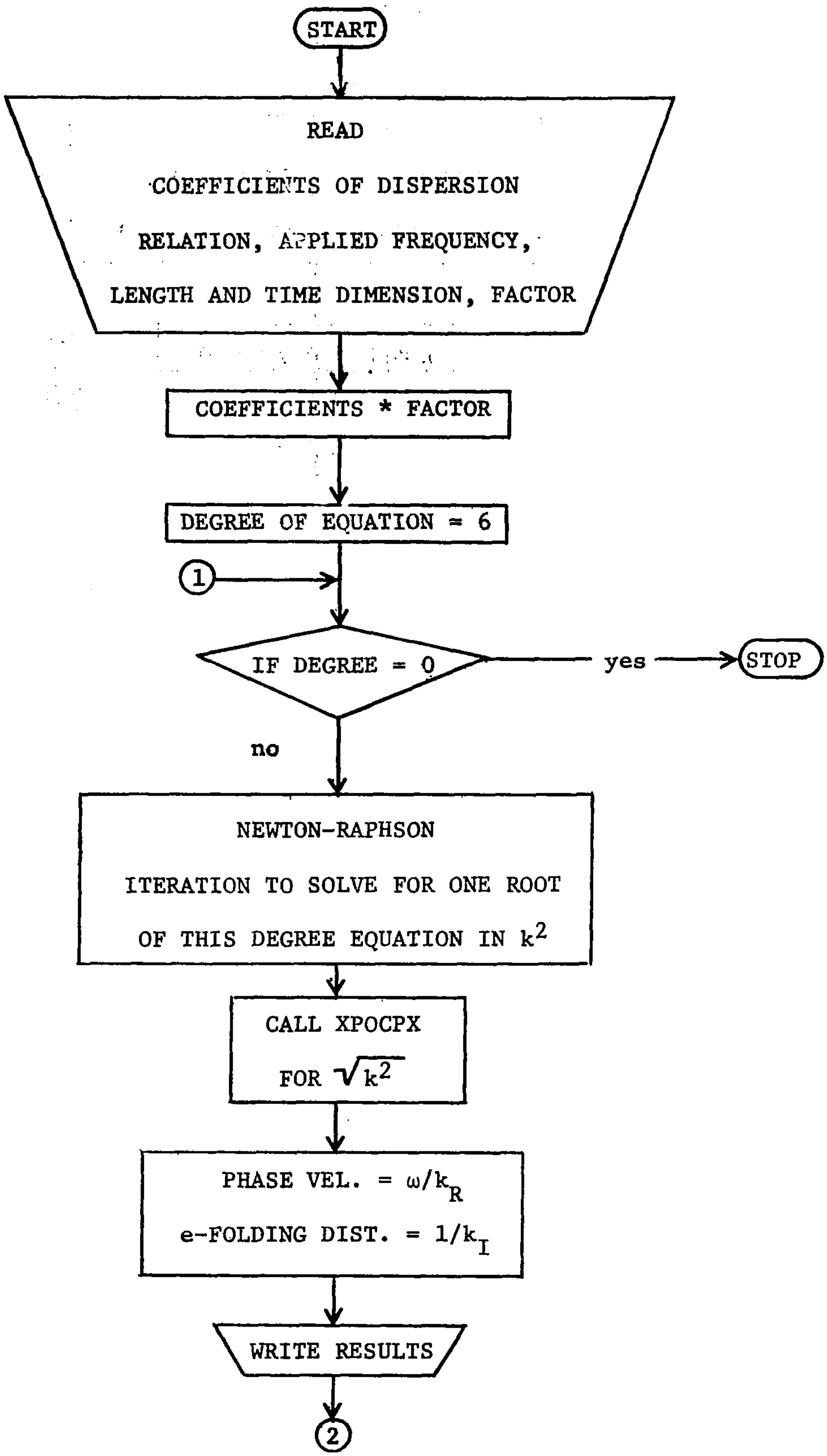
```
        (SKIP(1),A,E(12,5));
PUT EDIT('ELECTRON SOUND VELOCITY = ',UE,' METERS/SECOND')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('ION SOUND VELOCITY      = ',UI,' METERS/SECOND')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('NEUTRAL SOUND VELOCITY  = ',UN,' METERS/SECOND')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('ELECTRON NUMBER DENSITY =',NE,'NUMBER PER CUBIC METER')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('ION NUMBER DENSITY      =',NI,'NUMBER PER CUBIC METER')
        (SKIP(1),A,E(12,5),A);
PU.  . T('NEUTRAL NUMBER DENSITY  =',NN,'NUMBER PER CUBIC METER')
        (SKIP(1),A,E(12,5),A);
     /*  RELATIONS TO SHORTEN RUN TIME  */
     LET( SUB1 = WISQ*VN;   SUB2 = WESQ*VN;
          SUB3 = VE + VN;   SUB4 = VI + VN;
          SUB5 = (VE*VN)-(VEN*VNE) ;
          SUB6 = (VI*VN)-(VIN*VNI) ;
          SUB7 = (VEI*VN)+(VEN*VNI) ;
          SUB8 = WCU*CLSQ ;     SUB9 = WQUAD*CLSQ );
 /*  OBTAIN EQUATIONS FOR JUST THE NUMERATORS OF THE C'S
     SINCE THE DENOMINATOR WILL BE MULTIPLIED OUT IN THE
     FINAL DISPERSION RELATION.  THE FORM OF THE EQUATION
     IS AS FOLLOWS:
           C(I) = A(I)*K**4 + B(I)*K**2 + D(I)        */

     LET( A(1) = 0.0 ;
          A(2) = 0.0;
          A(3) = UISQ*UNSQ;
          A(4) = 0.0;
          A(5) = UESQ*UNSQ;
          A(6) = 0.0;
          B(1) = -(WESQ*UNSQ) + (#I*W*VEI*UNSQ)  ;
          B(2) = -(WSQ*CLSQ*SUB7) + (#I*WCU*CLSQ*VEI) ;
         B(3)=(WISQ*UNSQ)-WSQ*(UNSQ+UISQ)-(#I*W)*(VN*UISQ+VI*UNSQ);
          B(4) = (WSQ*CLSQ*SUB6)-(SUB9)-(#I*SUB8*SUB4);
         B(5)=(WESQ*UNSQ)-WSQ*(UNSQ+UESQ)-(#I*W)*(VN*UESQ+VE*UNSQ);
          B(6) = (WSQ*CLSQ*SUB5)-(SUB9)-(#I*SUB8*SUB3);
          D(1) = WSQ*(WESQ+SUB7)+ #I*(W*SUB2-WCU*VEI) ;
          D(2) = WSQ* D(1)   ;
         D(3)=WQUAD-WSQ*(WISQ+SUB6)+ #I*(WCU*SUB4-W*WISQ*VN);
          D(4) = WSQ * D(3)   ;
         D(5)=WQUAD-WSQ*(WESQ+SUB5)+ #I*(WCU*SUB3-W*WESQ*VN);
          D(6) = WSQ * D(5)  );
     /*   DENOMINATORS FOR C'S    */
     LET( CDEM(1) = (-UNSQ*K**2) + (WSQ+#I*W*VN) ;
         CDEM(2)=(-WSQ*CLSQ-#I*W*CLSQ*VN)*K**2 + (WQUAD+#I*WCU*VN);
          CDEM(3) = CDEM(1)   ;
          CDEM(4) = CDEM(2)   ;
          CDEM(5) = CDEM(1)   ;
          CDEM(6) = CDEM(2)    );
     /*  EQUATIONS FOR C'S WITH JUST THE NUMERATOR   */
EQS:   DO I=1 TO 6 BY 1 ;  LET(I='I');
     LET( C(I) = A(I)*K**4 + B(I)*K**2 + D(I)   );
 ATOMIZE( A(I); B(I);  D(I)   );
     END EQS;
   /*  PREPARING TO OBTAIN EACH TERM OF DISPERSION RELATION
```

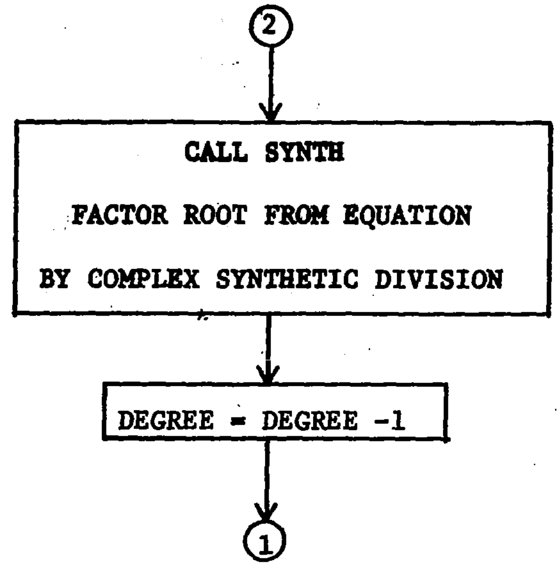
```
        AS FUNCTION OF W,WT,& WL.  COLLECTING LIKE QUANTITIES
        IN EACH TERM TO MAKE PROGRAM MORE EFFICIENT.    */
     LET( C1SQ = C(1)**2;    C2SQ = C(2)**2;
          C4SQ = C(4)**2;    C6SQ = C(6)**2;
          C1C2 = C(1)*C(2);   C3C4 = C(3)*C(4);
          C3C5 = C(3)*C(5);   C3C6 = C(3)*C(6);
          C4C5 = C(4)*C(5);   C4C6 = C(4)*C(6);
          C4C6SQ = C4C6**2;     C5C6 = C(5)*C(6);
          C2QU = C2SQ**2;    C1DC2D = CDEM(1)*CDEM(2);
          C2DSQ = CDEM(2)**2;    C1DC2DCU = C1DC2D*C2DSQ ;
          C1C2DSQ = C1DC2D**2;    C2DQUAD = C2DSQ**2   );
     /* INDIVIDUAL TERMS OF DISPERSION RELATION   */
    OPTSET(NOEXPND);
  LET(TERM(1) = C1SQ*C2SQ ;
      TERM(2) = C1DC2D*2.0*WSQ*WTSQ*C1C2   ;
      TERM(3) = -M*C1SQ*C4C6    ;
      TERM(4) = -M*C2SQ*C3C5   ;
      TERM(5) = MSQ*C3C4*C5C6 ;
      TERM(6) = -C1DC2D*MSQ*WSQ*WTSQ*C3C4     ;
      TERM(7) = C1C2DSQ*WQUAD*WTQU   ;
      TERM(8) = -C1DC2D*WSQ*WTSQ*C5C6      );
 /* ATOMIZING VARIABLES NO LONGER NEEDED.   */
ATOMIZE(C1SQ;C2SQ;C4SQ;C6SQ;C1C2;C3C4;C3C5;C3C6;C4C5;C4C6;
        C4C6SQ;C5C6;C2QU;C1DC2D;C2DSQ;C1DC2DCU;C1C2DSQ;
         C2DQUAD);
    OPTSET(EXPND);
     REWT = (CE*H)/(ME);
  /* THE FOLLOWING CHANGES REWT TO CORRECT UNITS OF TIME */
  IF FACTIME=0 THEN REWT=REWT;
  ELSE IF FACTIME=1 THEN REWT=REWT*1.0E-1   ;
  ELSE IF FACTIME=2 THEN REWT=REWT*1.0E-2   ;
  ELSE IF FACTIME=3 THEN REWT=REWT*1.0E-3   ;
  ELSE IF FACTIME=4 THEN REWT=REWT*1.0E-4   ;
  ELSE IF FACTIME=5 THEN REWT=REWT*1.0E-5   ;
  ELSE IF FACTIME=6 THEN REWT=REWT*1.0E-6   ;
  ELSE IF FACTIME=7 THEN REWT=REWT*1.0E-7   ;
  ELSE IF FACTIME=8 THEN REWT=REWT*1.0E-8   ;
      LET(REWT='REWT');
    OPTSET(NOEXPND);
  TERMS:   DO I=1 TO 8 BY 1 ; LET(I='I');
     LET(FTERM(I) = REPLACE(TERM(I),WT,REWT,W,REW));
      END TERMS ;
      LET(DISPER=0.0);
    OPTSET(EXPND);
         DO I=1 TO 8 BY 1 ;  LET(I='I');
    LET(DISPER=DISPER+FTERM(I));
    END;
  LET( FACTIME = 'FACTIME' ; FACLEN = 'FACLEN' ) ;
 /* NOW GET COEFFICIENTS OF K IN FORM TO BE PUNCHED OUT */
     LET( Z = HIGHPOW(DISPER,K) ;
          X = LOWPOW(DISPER,K)  );
KCOEF:  DO I = 2 TO 12 BY 2;  LET(I='I');
     LET( COEFK(I) = COEFF(DISPER,K**I) ;
          COEFKI(I) = COEFF(COEFK(I),#I) ;
          COEFKR(I) = COEFK(I) - #I*COEFKI(I)  );
          END KCOEF;
 /*  PUT IN CONSTANT TERM OF DISPERSION RELATION AS COEFK(0)  */
```

```
     LET( L = 0.0 ;
          COEFK(0) = REPLACE(DISPER,K,L) ;
          COEFKI(0) = COEFF(COEFK(0),#I)  ;
          COEFKR(0) = COEFK(0) - #I*COEFKI(0)  );
 /* PUNCH OUT ALL REAL AND IMAGINARY COEFFFICIENTS OF EACH
    POWER OF K, STARTING WITH HIGHEST POWER.  */
XPUNCH:   DO I= 12 BY -2 TO 0;   LET(I='I');
    LET( COER=COEFKR(I);
         COEI= COEFKI(I)    );
      PLCOER= ARITH(COER);
      PLCOEI= ARITH(COEI);
 PUT FILE(SYSPNCH)EDIT(PLCOER) (SKIP(1),E(13,5));
 PUT FILE(SYSPNCH)EDIT(PLCOEI) (SKIP(1),E(13,5));
  END XPUNCH;
   /*  DISPERSION RELATION FOR THE GIVEN VALUES OF
     INPUT VARIABLES,WT,AND W   */
    REW=10.0*REW  ;
    LET(REW='REW');
    IF REW<= LIMIT THEN GO TO TERMS;
    GO TO READ;
    QUIT:  PUT LIST('  SEE OTHER SIDE');
       END DISREL;
```

APPENDIX C

SIMPLIFIED FLOW CHART OF POLYNOMIAL ROOTS COMPUTER PROGRAM


START
READ
COEFFICIENTS OF DISPERSION
RELATION, APPLIED FREQUENCY,
LENGTH AND TIME DIMENSION, FACTOR
COEFFICIENTS * FACTOR
DEGREE OF EQUATION = 6
1
IF DEGREE = 0
yes
STOP
no
NEWTON-RAPHSON
ITERATION TO SOLVE FOR ONE ROOT
OF THIS DEGREE EQUATION IN k^2
CALL XPOCPX
FOR $\sqrt{k^2}$
PHASE VEL. = ω/k_R
e-FOLDING DIST. = $1/k_I$
WRITE RESULTS
2


2
CALL SYNTH
FACTOR ROOT FROM EQUATION
BY COMPLEX SYNTHETIC DIVISION
DEGREE = DEGREE -1
1

APPENDIX D

LISTING OF POLYNOMIAL ROOTS COMPUTER PROGRAM

```
£XQT
MAP 0017-01/13-11:20
START=010463, PROG SIZE(I/D)=3919/2170
      DIMENSION A1(20)
      DOUBLE PRECISION A(20),FRE,FIM,FPRE,FPIM,XRE,XIM,X2RE,X2IM,X3RE,
     1 X3IM,X4RE,X4IM,X5RE,X5IM,X6RE,X6IM,BRE,BIM,NUMRE,NUMIM,DEM,W,
     2 X8RE,X8IM,X7RE,X7IM,ERRORE,ERROIM,X(20),TEMP,FRT(35),FIT(35),
     3 FPRT(35),FPIT(35),X8TR(10),X8TI(10),X7TR(10),X7TI(10),X6TR(10),
     4 KR(5),KRE(5),KI(5),KIM(5),PHSVE(5),EFOLD(5),FACT
 4000 READ(5,106) W,LEN,TYME,FAC
      IF(LEN.EQ.99) GO TO 300
      READ(5,201) FACT
  201 FORMAT(D8.1)
      DO 420 I=1,14
      READ(5,129) A(I)
  129 FORMAT(D16.8)
  420 CONTINUE
      WRITE(6,105) W,FAC,LEN,TYME
      DO 20 I=1,7
      WRITE(6,113) A(2*I-1),A(2*I)
  113 FORMAT(2E13.5)
   20 CONTINUE
      DO 197 I=1,14
      A(I)=FACT*A(I)
  197 CONTINUE
      WRITE(6,166) FACT
  166 FORMAT('0',' ADJUSTED COEFFICIENTS      FACTOR=',D8.1)
      DO 67 I=1,7
      WRITE(6,167) A(2*I-1),A(2*I)
  167 FORMAT(2E13.5)
   67 CONTINUE
      K=6
    2 READ(5,101) BRE,BIM
      IF(BRE.EQ.99999.D0.AND.BIM.EQ.99999.D0) GO TO 300
      WRITE(6,10 ) BRE,BIM
      N = 1
      IF(K.EQ.8) GO TO 950
      IF(K.EQ.7) GO TO 951
      IF(K.EQ.6) GO TO 952
      IF(K.EQ.5) GO TO 953
      IF(K.EQ.4) GO TO 954
      IF(K.EQ.3) GO TO 955
      IF(K.EQ.2) GO TO 956
  950 CALL EIGHTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  951 CALL SEVEN(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  952 CALL SIXTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  953 CALL FIFTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  954 CALL FOURTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  955 CALL THIRD(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  956 CALL SECOND(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
  960 FRE= 0.0
      FIM= 0.0
```

```
      DO 15 I=1,M
      FRE = FRE+FRT(I)
      FIM = FIM+FIT(I)
 15   CONTINUE
      FPRE= 0.0
      FPIM= 0.0
      DO 16 I=1,L
      FPRE = FPRE+FPRT(I)
      FPIM = FPIM+FPIT(I)
 16   CONTINUE
      NUMRE = (FRE*FPRE) + (FIM*FPIM)
      NUMIM = (FIM*FPRE) - (FRE*FPIM)
      IF(NUMRE.EQ.0.D0.AND.NUMIM.EQ.0.D0) GO TO 4
      IF(FPRE.EQ.0.D0.OR.FPIM.EQ.0.D0) DEM= (FPRE**2)+(FPIM**2)
      IF(FPRE.EQ.0.D0.OR.FPIM.EQ.0.D0) GO TO 6
      DEM = (FPRE**2)*(1.D0+(FPIM/FPRE)**2)
  6   IF(FPRE.EQ.0.D0.AND.FPIM.EQ.0.D0) WRITE(6,102)
      IF(FPRE.EQ.0.D0.AND.FPIM.EQ.0.D0) GO TO 2
      FRE = NUMRE/DEM
      FIM = NUMIM/DEM
      BRE = BRE-FRE
      BIM = BIM-FIM
      N = N+1
      IF(N.GT.300) GO TO 4
      IF(K.EQ.8) GO TO 950
      IF(K.EQ.7) GO TO 951
      IF(K.EQ.6) GO TO 952
      IF(K.EQ.5) GO TO 953
      IF(K.EQ.4) GO TO 954
      IF(K.EQ.3) GO TO 955
      IF(K.EQ.2) GO TO 956
  4   ERRORE = 0.0
      ERROIM = 0.0
      DO 17 I=1,M
      ERRORE = ERRORE+FRT(I)
      ERROIM = ERROIM+FIT(I)
 17   CONTINUE
280   WRITE(6,103) BRE,BIM,ERRORE,ERROIM,N
      CALL XPOCFX(1,2,BRE,BIM,KR,KI)
      DO 90 I=1,2
      KRE(I) = KR(I)*(10.**(-LEN))/FAC
      KIM(I) = KI(I)*(10.**(-LEN))/FAC
      PHSVE(I) = W/KRE(I)
      EFOLD(I) = 1.0/KIM(I)
      WRITE(6,110) KR(I),KI(I)
      WRITE(6,112) KRE(I),KIM(I),PHSVE(I),EFOLD(I)
 90   CONTINUE
      K = K-1
      IF(K.EQ.1) GO TO 957
      IF(K.EQ.0) GO TO 4000
      CALL SYNDV(A,BRE,BIM,K)
      KJ=K+2
      DO 7000 I=1,KJ
      WRITE(6,115) A(2*I-1),A(2*I)
7000  CONTINUE
      GO TO 2
957   A(1) = A(1)
```

```
      A(2) = A(2)
      A(3) = A(3) + (BRE*A(1)-BIM*A(2))
      A(4) = A(4) + (BIM*A(1)+BRE*A(2))
      BRE =-(A(1)*A(3)+A(2)*A(4))/(A(1)**2+A(2)**2)
      BIM = (A(2)*A(3)-A(1)*A(4))/(A(1)**2+A(2)**2)
      ERRCRE=0.0
      ERRCIM=0.0
      N=1
      GO TO 280
  100 FORMAT(E13.6)
  101 FORMAT(2D15.7)
 102  FORMAT(' ***** DERIVATIVE OF F(X)=0 ***** ')
  103 FORMAT('0','   ROOT = ',2D14.6,'    ERROR = ',2D14.6,
     3 '     NUMBER OF ITERATIONS PERFORMED = ',I3)
  104 FORMAT('-','   INITIAL ESTIMATE = ',2D14.6)
  105 FORMAT('1','ROOTS OF THE DISPERSION RELATION BY A NEWTON-RAPHSON
     1 ITERATION TECHNIQUE WITH SYNTH DIVISION'/'0','APPLIED FREQUENCY
     C=',D12.6,
     2 '   LENGTH DIMENSION =',F3.1,'*10**',I2,'  METERS',
     1  '   TIME DIMENSION = 10**-',I2,'  SECONDS'/,
     1   '0',' DISPERSION RELATION COEFFICIENTS ')
  106 FORMAT(D15.7,2I2,F3.1)
  110 FORMAT('0','   K= ' , 2D14.6)
  112 FORMAT('0','  WAVE NUMBER = ',2D14.6,' 1/METERS '/'0',
     1   'PHASE VELOCITY = ',D14.6,'           E FOLDING DISTANCE =  ',
     1   D14.6)
  115 FORMAT(2E13.6)
  300 STOP
      END

      SUBROUTINE ATWLTF(N,X)
C   USED TO ARRANGE TERMS OF EQUATION IN ASCENDING ORDER.
      DOUBLE PRECISION X(50),TEMP
      DO 20 I=1,N
      IP1 = I+1
      DO 20 J=IP1,N
      IF(DABS(X(I)).LE.DABS(X(J))) GO TO 20
      TEMP = X(I)
      X(I) = X(J)
      X(J) = TEMP
   20 CONTINUE
      RETURN
      END

      SUBROUTINE XROCPX(M,N,A,B,XR,XM)
C     THIS SUBROUTINE CALCULATES THE M/NTH ROOTS OF A COMPLEX
C     NUMBER OF THE FORM 'C = A + I*B'.
      DOUBLE PRECISION A, B, BAR, BETA, COEF, K, PI, RM, RN,
     1  XR(25), ALFA, T, XM(25)
      DOUBLE PRECISION AA,BB
      AA=DABS(A)
      BB=DABS(B)
      RM = M
      RN = N
      PI = 3.14159265358979324D0
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) BAR=DSQRT(A**2+B**2)
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) GO TO 101
```

```
      BAR=DABS(A)*DSQRT(1.D0+(B/A)**2)
      IF(B.LE.1.D-32) GO TO 101
      IF(DABS(DLOG10(AA)-DLOG10(BB)).LE.4.D0)BAR=DSQRT(A**2+
     1  B**2)
  101 COEF = BAR**(RM/RN)
      T = DATAN2(B,A)
      IF (B .LT. 0.0D0) T = 2.0D0*PI - DABS(T)
      K = 0.0D0
      DO 100 I=1,N
      BETA = T + K*2.0D0*PI
      ALFA=BETA*RM/RN
      XR(I) = COEF*DCOS(ALFA)
      XM(I) = COEF*DSIN(ALFA)
  100 K = K + 1.0D0
      RETURN
      END

      SUBROUTINE SYNDV(A,BRE,BIM,K)
      DOUBLE PRECISION A(20),BRE,BIM
      J = 2*(K+1)+2
      DO 100 I=3,J,2
      A(1) = A(1)
      A(2) = A(2)
      A(I) = A(I) + (BRE*A(I-2)-BIM*A(I-1))
      A(I+1) = A(I+1) + (BIM*A(I-2)+BRE*A(I-1))
 100  CONTINUE
      RETURN
      END

      SUBROUTINE SECOND(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),FPIT(20)
      FRT(1) = A(1)*(BRE**2)-A(1)*(BIM**2)
      FRT(2) = -A(2)*2.0*BRE*BIM
      FRT(3) = A(3)*BRE
      FRT(4) = -A(4)*BIM
      FRT(5) = A(5)
      FIT(1) = A(1)*2.0*BRE*BIM
      FIT(2) = A(2)*(BRE**2)-A(2)*(BIM**2)
      FIT(3) = A(3)*BIM
      FIT(4) = A(4)*BRE
      FIT(5) = A(6)
      FPRT(1) = 2.0*A(1)*BRE
      FPRT(2) = -2.0*A(2)*BIM
      FPRT(3) = A(3)
      FPIT(1) = 2.0*A(1)*BIM
      FPIT(2) = 2.0*A(2)*BRE
      FPIT(3) = A(4)
      CALL ATWLTF(5,FRT)
      CALL ATWLTF(5,FIT)
      M = 5
      L = 3
      RETURN
      END

      SUBROUTINE THIRD(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),FPIT(20)
      FRT(1) = A(1)*(BRE**3)-A(1)*(BIM**2)*BRE
```

```
FRT(2) = -A(2)*3.0*BIM*(BRE**2) + A(2)*(BIM**3)
FRT(3) = A(3)*(BRE**2) - A(3)*(BIM**2)
FRT(4) = -A(4)*2.0*BRE*BIM
FRT(5) = A(5)*BRE
FRT(6) = -A(6)*BIM
FRT(7) = A(7)
FIT(1) = A(1)*3.0*BIM*(BRE**2) - A(1)*(BIM**3)
FIT(2) = A(2)*(BRE**3) - A(2)*3.0*BRE*(BIM**2)
FIT(3) = A(3)*2.0*BRE*BIM
FIT(4) = A(4)*(BRE**2) - A(4)*(BIM**2)
FIT(5) = A(5)*BIM
FIT(6) = A(6)*BRE
FIT(7) = A(8)
FPRT(1) = 3.0*(A(1)*(BRE**2)-A(1)*(BIM**2))
FPRT(2) = -6.0*A(2)*BRE*BIM
FPRT(3) = 2.0*A(3)*BRE
FPRT(4) = -2.0*A(4)*BIM
FPRT(5) = A(5)
FPIT(1) = 6.0*A(1)*BRE*BIM
FPIT(2) = 3.0*(A(2)*(BRE**2)-A(2)*(BIM**2))
FPIT(3) = 2.0*A(3)*BIM
FPIT(4) = 2.0*A(4)*BRE
FPIT(5) = A(6)
CALL ATWLTF(7,FRT)
CALL ATWLTF(7,FIT)
CALL ATWLTF(5,FPRT)
CALL ATWLTF(5,FPIT)
M = 7
L = 5
RETURN
END

SUBROUTINE FOURTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),FPIT(20)
FRT(1) = A(1)*(BRE**4)
FRT(2) = A(1)*(BIM**4)
FRT(3) = -6.0*A(1)*(BRE**2)*(BIM**2)
FRT(4) = A(2)*4.0*BRE*(BIM**3)
FRT(5) = -4.0*A(2)*BIM*(BRE**3)
FRT(6) = A(3)*(BRE**3)
FRT(7) = -3.0*A(3)*BRE*(BIM**2)
FRT(8) = A(4)*(BIM**3)
FRT(9) = -3.0*A(4)*BIM*(BRE**2)
FRT(10) = A(5)*(BRE**2) - A(5)*(BIM**2)
FRT(11) = -2.0*A(6)*BRE*BIM
FRT(12) = A(7)*BRE
FRT(13) = -A(8)*BIM
FRT(14) = A(9)
FIT(1) = 4.0*A(1)*BIM*(BRE**3)
FIT(2) = -4.0*A(1)*BRE*(BIM**3)
FIT(3) = A(2)*(BRE**4)
FIT(4) = A(2)*(BIM**4)
FIT(5) = -6.0*A(2)*(BRE**2)*(BIM**2)
FIT(6) = 3.0*A(3)*BIM*(BRE**2)
FIT(7) = -A(3)*(BIM**3)
FIT(8) = A(4)*(BRE**3)
FIT(9) = -3.0*A(4)*BRE*(BIM**2)
```

```
FIT(10) = 2.0*A(5)*BRE*BIM
FIT(11) = A(6)*(BRE**2) - A(6)*(BIM**2)
FIT(12) = A(7)*BIM
FIT(13) = A(8)*BRE
FIT(14) = A(10)
FPRT(1) = 4.0*A(1)*(BRE**3)
FPRT(2) = -12.0*A(1)*BRE*(BIM**2)
FPRT(3) = 4.0*A(2)*(BIM**3)
FPRT(4) = -12.0*A(2)*BIM*(BRE**2)
FPRT(5) = 3.0*A(3)*(BRE**2)
FPRT(6) = -3.0*A(3)*(BIM**2)
FPRT(7) = -6.0*A(4)*BRE*BIM
FPRT(8) = 2.0*A(5)*BRE
FPRT(9) = -2.0*A(6)*BIM
FPRT(10) = A(7)
FPIT(1) = -4.0*A(1)*(BIM**3)
FPIT(2) = 12.0*A(1)*BIM*(BRE**2)
FPIT(3) = 4.0*A(2)*(BRE**3)
FPIT(4) = -12.0*A(2)*BRE*(BIM**2)
FPIT(5) = 6.0*A(3)*BRE*BIM
FPIT(7) = -3.0*A(4)*(BIM**2)
FPIT(8) = 2.0*A(5)*BIM
FPIT(9) = 2.0*A(6)*BRE
FPIT(10) = A(8)
CALL ATWLTF(14,FRT)
CALL ATWLTF(14,FIT)
CALL ATWLTF(10,FPRT)
CALL ATWLTF(10,FPIT)
M = 14
L = 10
RETURN
END

SUBROUTINE FIFTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),FPIT(20)
FRT(1) = A(1)*(BRE**5)
FRT(2) = A(1)*5.0*BRE*(BIM**4)
FRT(3) = -10.0*A(1)*(BRE**3)*(BIM**2)
FRT(4) = -A(2)*(BIM**5)
FRT(5) = -A(2)*5.0*BIM*(BRE**4)
FRT(6) = 10.0*A(2)*(BRE**3)*(BIM**2)
FRT(7) = A(3)*(BRE**4)
FRT(8) = A(3)*(BIM**4)
FRT(9) = -6.0*A(3)*(BRE**2)*(BIM**2)
FRT(10) = 4.0*A(4)*BRE*(BIM**3)
FRT(11) = -4.0*A(4)*BIM*(BRE**3)
FRT(12) = A(5)*(BRE**3)
FRT(13) = -3.0*A(5)*BRE*(BIM**2)
FRT(14) = A(6)*(BIM**3)
FRT(15) = -3.0*A(6)*BIM*(BRE**2)
FRT(16) = A(7)*(BRE**2) - A(7)*(BIM**2)
FRT(17) = -2.0*A(8)*BRE*BIM
FRT(18) = A(9)*BRE
FRT(19) = -A(10)*BIM
FRT(20) = A(11)
FIT(1) = A(1)*(BIM**5)
FIT(2) = A(1)*5.0*BIM*(BRE**4)
```

```
FIT(3) = -10.0*A(1)*(BRE**2)*(BIM**3)
FIT(4) = A(2)*(BRE**5)
FIT(5) = 5.0*A(2)*BRE*(BIM**4)
FIT(6) = -10.0*A(2)*(BRE**3)*(BIM**2)
FIT(7) = 4.0*A(3)*BIM*(BRE**3)
FIT(8) = -4.0*A(3)*BRE*(BIM**3)
FIT(9) = A(4)*(BRE**4)
FIT(10) = A(4)*(BIM**4)
FIT(11) = -A(4)*(BRE**2)*(BIM**2)
FIT(12) = -A(5)*(BIM**3)
FIT(13) = 3.0*A(5)*BIM*(BRE**2)
FIT(14) = A(6)*(BRE**3)
FIT(15) = -3.0*A(6)*BRE*(BIM**2)
FIT(16) = 2.0*A(7)*BRE*BIM
FIT(17) = A(8)*((BRE**2)-(BIM**2))
FIT(18) = A(9)*BIM
FIT(19) = A(10)*BRE
FIT(20) = A(12)
FPRT(1) = 5.0*A(1)*(BRE**4)
FPRT(2) = 5.0*A(1)*(BIM**4)
FPRT(3) = -30.0*A(1)*(BRE**2)*(BIM**2)
FPRT(4) = 20.0*A(2)*BRE*(BIM**3)
FPRT(5) = -20.0*A(2)*BIM*(BRE**3)
FPRT(6) = 4.0*A(3)*(BRE**3)
FPRT(7) = -12.0*A(3)*BRE*(BIM**2)
FPRT(8) = 4.0*A(4)*(BIM**3)
FPRT(9) = -12.0*A(4)*BIM*(BRE**2)
FPRT(10) = 3.0*A(5)*(BRE**2)
FPRT(11) = -3.0*A(5)*(BIM**2)
FPRT(12) = -6.0*A(6)*BRE*BIM
FPRT(13) = 2.0*A(7)*BRE
FPRT(14) = -2.0*A(8)*BIM
FPRT(15) = A(9)
FPIT(1) = 20.0*A(1)*BIM*(BRE**3)
FPIT(2) = -20.0*A(1)*BRE*(BIM**3)
FPIT(3) = 5.0*A(2)*(BRE**4)
FPIT(4) = 5.0*A(2)*(BIM**4)
FPIT(5) = -30.0*A(2)*(BRE**2)*(BIM**2)
FPIT(6) = -4.0*A(3)*(BIM**3)
FPIT(7) = 12.0*A(3)*BIM*(BRE**2)
FPIT(8) = 4.0*A(4)*(BRE**3)
FPIT(9) = -12.0*A(4)*BRE*(BIM**2)
FPIT(10) = 6.0*A(5)*BRE*BIM
FPIT(11) = 3.0*A(6)*(BRE**2)
FPIT(12) = -3.0*A(6)*(BIM**2)
FPIT(13) = 2.0*A(7)*BIM
FPIT(14) = 2.0*A(8)*BRE
FPIT(15) = A(10)
CALL ATWLTF(20,FRT)
CALL ATWLTF(20,FIT)
CALL ATWLTF(15,FPRT)
CALL ATWLTF(15,FPIT)
M = 20
L = 15
RETURN
END
```

```
SUBROUTINE SIXTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
DOUBLE PRECISION BRE,BIM,A(20),FRT(30),FIT(30),FPRT(30),FPIT(30)
FRT(1) = A(1)*(BRE**6)
FRT(2) = -A(1)*(BIM**6)
FRT(3) = 15.0*A(1)*(BRE**2)*(BIM**4)
FRT(4) = -15.0*A(1)*(BRE**4)*(BIM**2)
FRT(5) = -6.0*A(2)*BRE*(BIM**5)
FRT(6) = -6.0*A(2)*BIM*(BRE**5)
FRT(7) = 20.0*A(2)*(BRE**3)*(BIM**3)
FRT(8) = A(3)*(BRE**5)
FRT(9) = 5.0*A(3)*BRE*(BIM**4)
FRT(10) = -10.0*A(3)*(BRE**3)*(BIM**2)
FRT(11) = -A(4)*(BIM**5)
FRT(12) = -5.0*A(4)*BIM*(BRE**4)
FRT(13) = 10.0*A(4)*(BRE**2)*(BIM**3)
FRT(14) = A(5)*(BRE**4)
FRT(15) = A(5)*(BIM**4)
FRT(16) = -6.0*A(5)*(BRE**2)*(BIM**2)
FRT(17) = 4.0*A(6)*BRE*(BIM**3)
FRT(18) = -4.0*A(6)*BIM*(BRE**3)
FRT(19) = A(7)*(BRE**3)
FRT(20) = -3.0*A(7)*BRE*(BIM**2)
FRT(21) = A(8)*(BIM**3)
FRT(22) = -3.0*A(8)*BIM*(BRE**2)
FRT(23) = A(9)*((BRE**2)-(BIM**2))-2.0*A(10)*BRE*BIM
FRT(24) = (A(11)*BRE) - (A(12)*BIM)
FRT(25) = A(13)
FIT(1) = 6.0*A(1)*BRE*(BIM**5)
FIT(2) = 6.0*A(1)*BIM*(BRE**5)
FIT(3) = -20.0*A(1)*(BRE**3)*(BIM**3)
FIT(4) = A(2)*(BRE**6)
FIT(5) = -A(2)*(BIM**6)
FIT(6) = 15.0*A(2)*(BRE**2)*(BIM**4)
FIT(7) = -15.0*A(2)*(BRE**4)*(BIM**2)
FIT(8) = A(3)*(BIM**5)
FIT(9) = 5.0*A(3)*BIM*(BRE**4)
FIT(10) = -10.0*A(3)*(BRE**2)*(BIM**3)
FIT(11) = A(4)*(BRE**5)
FIT(12) = 5.0*A(4)*BRE*(BIM**4)
FIT(13) = -10.0*A(4)*(BRE**3)*(BIM**2)
FIT(14) = 4.0*A(5)*BIM*(BRE**3)
FIT(15) = -4.0*A(5)*BRE*(BIM**3)
FIT(16) = A(6)*(BRE**4)
FIT(17) = A(6)*(BIM**4)
FIT(18) = -6.0*A(6)*(BRE**2)*(BIM**2)
FIT(19) = -A(7)*(BIM**3)
FIT(20) = 3.0*A(7)*BIM*(BRE**2)
FIT(21) = A(8)*(BRE**3)
FIT(22) = -3.0*A(8)*BRE*(BIM**2)
FIT(23) = (2.0*A(9)*BRE*BIM) + A(10)*((BRE**2)-(BIM**2))
FIT(24) = (A(11)*BIM) + (A(12)*BRE)
FIT(25) = A(14)
FPRT(1) = 6.0*A(1)*(BRE**5)
FPRT(2) = 30.0*A(1)*BRE*(BIM**4)
FPRT(3) = -60.0*A(1)*(BRE**3)*(BIM**2)
FPRT(4) = -6.0*A(2)*(BIM**5)
FPRT(5) = -30.0*A(2)*BIM*(BRE**4)
```

```
FPRT(6) = 60.0*A(2)*(BRE**2)*(BIM**3)
FPRT(7) = 5.0*A(3)*(BRE**4)
FPRT(8) = 5.0*A(3)*(BIM**4)
FPRT(9) = -30.0*A(3)*(BRE**2)*(BIM**2)
FPRT(10) = 20.0*A(4)*BRE*(BIM**3)
FPRT(11) = -20.0*A(4)*BIM*(BRE**3)
FPRT(12) = 4.0*A(5)*(BRE**3)
FPRT(13) = -12.0*A(5)*BRE*(BIM**2)
FPRT(14) = 4.0*A(6)*(BIM**3)
FPRT(15) = -12.0*A(6)*BIM*(BRE**2)
FPRT(16) = 3.0*A(7)*(BRE**2)
FPRT(17) = -3.0*A(7)*(BIM**2)
FPRT(18) = -6.0*A(8)*BRE*BIM
FPRT(19) = 2.0*(A(9)*BRE-A(10)*BIM)
FPRT(20) = A(11)
FPIT(1) = 6.0*A(1)*(BIM**5)
FPIT(2) = 30.0*A(1)*BIM*(BRE**4)
FPIT(3) = -60.0*A(1)*(BRE**2)*(BIM**3)
FPIT(4) = 6.0*A(2)*(BRE**5)
FPIT(5) = 30.0*A(2)*BRE*(BIM**4)
FPIT(6) = -60.0*A(2)*(BRE**3)*(BIM**2)
FPIT(7) = 20.0*A(3)*BIM*(BRE**3)
FPIT(8) = -20.0*A(3)*BRE*(BIM**3)
FPIT(9) = 5.0*A(4)*(BRE**4)
FPIT(10) = 5.0*A(4)*(BIM**4)
FPIT(11) = -30.0*A(4)*(BRE**2)*(BIM**2)
FPIT(12) = -4.0*A(5)*(BIM**3)
FPIT(13) = 12.0*A(5)*BIM*(BRE**2)
FPIT(14) = 4.0*A(6)*(BRE**3)
FPIT(15) = -12.0*A(6)*BRE*(BIM**2)
FPIT(16) = 6.0*A(7)*BRE*BIM
FPIT(17) = 3.0*A(8)*((BRE**2)-(BIM**2))
FPIT(18) = 2.0*A(9)*BIM
FPIT(19) = 2.0*A(10)*BR
FPIT(20) = A(12)
CALL ATWLTF(25,FRT)
CALL ATWLTF(25,FIT)
CALL ATWLTF(20,FPRT)
CALL ATWLTF(20,FPIT)
N = 25
L = 20
RETURN
END

SUBROUTINE SEVEN(BRE,BIM,A,FRT,FIT,FPRT,FPIT,N,L)
DOUBLE PRECISION BRE,BIM,A(20),FRT(35),FIT(35),FPRT(30),FPIT(30)
FRT(1) = A(1)*(BRE**7)
FRT(2) = -7.0*A(1)*BRE*(BIM**6)
FRT(3) = -21.0*A(1)*(BRE**5)*(BIM**2)
FRT(4) = 35.0*A(1)*(BRE**3)*(BIM**4)
FRT(5) = A(2)*(BIM**7)
FRT(6) = -7.0*A(2)*BIM*(BRE**6)
FRT(7) = -21.0*A(2)*(BRE**2)*(BIM**5)
FRT(8) = 35.0*A(2)*(BRE**4)*(BIM**3)
FRT(9) = A(3)*(BRE**6)
FRT(10) = -A(3)*(BIM**6)
FRT(11) = 15.0*A(3)*(BRE**2)*(BIM**4)
```

```
FRT(12) = -15.0*A(3)*(BRE**4)*(BIM**2)
FRT(13) = -6.0*A(4)*BRE*(BIM**5)
FRT(14) =  20.0*A(4)*(BRE**3)*(BIM**3)
FRT(15) = -6.0*A(4)*BIM*(BRE**5)
FRT(16) = A(5)*(BRE**5)
FRT(17) = 5.0*A(5)*BRE*(BIM**4)
FRT(18) = -10.0*A(5)*(BRE**3)*(BIM**2)
FRT(19) = -A(6)*(BIM**5)
FRT(20) = -5.0*A(6)*BIM*(BRE**4)
FRT(21) = 10.0*A(6)*(BRE**2)*(BIM**3)
FRT(22) = A(7)*(BRE**4)
FRT(23) = A(7)*(BIM**4)
FRT(24) = -6.0*A(7)*(BRE**2)*(BIM**2)
FRT(25) = 4.0*A(8)*BRE*(BIM**3)
FRT(26) = -4.0*A(8)*BIM*(BRE**3)
FRT(27) = A(9)*(BRE**3)
FRT(28) = -3.0*A(9)*BRE*(BIM**2)
FRT(29) = A(10)*(BIM**3)
FRT(30) = -3.0*A(10)*BIM*(BRE**2)
FRT(31) = A(11)*((BRE**2)-(BIM**2))
FRT(32) = -2.0*A(12)*BRE*BIM
FRT(33) = (A(13)*BRE) - (A(14)*BIM)
FRT(34) = A(15)
FIT(1) = -A(1)*(BIM**7)
FIT(2) = 7.0*A(1)*BIM*(BRE**6)
FIT(3) = -35.0*A(1)*(BRE**4)*(BIM**3)
FIT(4) = 21.0*A(1)*(BRE**2)*(BIM**5)
FIT(5) = A(2)*(BRE**7)
FIT(6) = -21.0*A(2)*(BRE**5)*(BIM**2)
FIT(7) = 35.0*A(2)*(BRE**3)*(BIM**4)
FIT(8) = -7.0*A(2)*BRE*(BIM**6)
FIT(9) = 6.0*A(3)*BRE*(BIM**5)
FIT(10) = -20.0*A(3)*(BRE**3)*(BIM**3)
FIT(11) = 6.0*A(3)*BIM*(BRE**5)
FIT(12) = A(4)*(BRE**6)
FIT(13) = -A(4)*(BIM**6)
FIT(14) = -15.0*A(4)*(BRE**4)*(BIM**2)
FIT(15) = 15.0*A(4)*(BRE**2)*(BIM**4)
FIT(16) = A(5)*(BIM**5)
FIT(17) = 5.0*A(5)*BIM*(BRE**4)
FIT(18) = -10.0*A(5)*(BRE**2)*(BIM**3)
FIT(19) = A(6)*(BRE**5)
FIT(20) = 5.0*A(6)*BRE*(BIM**4)
FIT(21) = -10.0*A(6)*(BRE**3)*(BIM**2)
FIT(22) = -4.0*A(7)*BRE*(BIM**3)
FIT(23) = 4.0*A(7)*BIM*(BRE**3)
FIT(24) = A(8)*(BRE**4)
FIT(25) = A(8)*(BIM**4)
FIT(26) = -6.0*A(8)*(BRE**2)*(BIM**2)
FIT(27) = -A(9)*(BIM**3)
FIT(28) = 3.0*A(9)*BIM*(BRE**2)
FIT(29) = A(10)*(BRE**3)
FIT(30) = -3.0*A(10)*BRE*(BIM**2)
FIT(31) = 2.0*A(11)*BRE*BIM
FIT(32) = A(12)*((BRE**2)-(BIM**2))
FIT(33) = (A(13)*BIM) + (A(14)*BRE)
FIT(34) = A(16)
```

```
FPRT(1) = 7.0*A(1)*(BRE**6)
FPRT(2) = -7.0*A(1)*(BIM**6)
FPRT(3) = -105.0*A(1)*(BRE**4)*(BIM**2)
FPRT(4) = 105.0*A(1)*(BRE**2)*(BIM**4)
FPRT(5) = -42.0*A(2)*BIM*(BRE**5)
FPRT(6) = 140.0*A(2)*(BRE**3)*(BIM**3)
FPRT(7) = -42.0*A(2)*BRE*(BIM**5)
FPRT(8) = 6.0*A(3)*(BRE**5)
FPRT(9) = -60.0*A(3)*(BRE**3)*(BIM**2)
FPRT(10) = 30.0*A(3)*BRE*(BIM**4)
FPRT(11) = -6.0*A(4)*(BIM**5)
FPRT(12) = -30.0*A(4)*BIM*(BRE**4)
FPRT(13) = 60.0*A(4)*(BRE**2)*(BIM**3)
FPRT(14) = 5.0*A(5)*(BRE**4)
FPRT(15) = 5.0*A(5)*(BIM**4)
FPRT(16) = -30.0*A(5)*(BRE**2)*(BIM**2)
FPRT(17) = -20.0*A(6)*BIM*(BRE**3)
FPRT(18) = 20.0*A(6)*BRE*(BIM**3)
FPRT(19) = 4.0*A(7)*(BRE**3)
FPRT(20) = -12.0*A(7)*BRE*(BIM**2)
FPRT(21) = 4.0*A(8)*(BIM**3)
FPRT(22) = -12.0*A(8)*BIM*(BRE**2)
FPRT(23) = 3.0*A(9)*((BRE**2)-(BIM**2))
FPRT(24) = -6.0*A(10)*BRE*BIM
FPRT(25) = 2.0*((A(11)*BRE)-(A(12)*BIM))
FPRT(26) = A 13)
FPIT(1) = 42.0*A(1)*BRE*(BIM**5)
FPIT(2) = -140.0*A(1)*(BRE**3)*(BIM**3)
FPIT(3) = 42.0*A(1)*BIM*(BRE**5)
FPIT(4) = 7.0*A(2)*(BRE**6)
FPIT(5) = -7 0*A(2)*(BIM**6)
FPIT(6) = -105.0*A(2)*(BRE**4)*(BIM**2)
FPIT(7) = 105.0*A(2)*(BRE**2)*(BIM**4)
FPIT(8) = 6.0*A(3)*(BIM**5)
FPIT(9) = 30.0*A(3)*BIM*(BRE**4)
FPIT(10) = -60.0*A(3)*(BRE**2)*(BIM**3)
FPIT(11) = 6.0*A(4)*(BRE**5)
FPIT(12) = -60.0*A(4)*(BRE**3)*(BIM**2)
FPIT(13) = 30.0*A(4)*BRE*(BIM**4)
FPIT(14) = 20.0*A(5)*BIM*(BRE**3)
FPIT(15) = -20.0*A(5)*BRE*(BIM**3)
FPIT(16) = 5.0*A(6)*(BRE**4)
FPIT(17) = 5.0*A(6)*(BIM**4)
FPIT(18) = -30.0*A(6)*(BRE**2)*(BIM**2)
FPIT(19) = -4.0*A(7)*(BIM**3)
FPIT(20) = 12.0*A(7)*BIM*(BRE**2)
FPIT(21) = 4.0*A(8)*(BRE**3)
FPIT(22) = -12.0*A(8)*BRE*(BIM**2)
FPIT(23) = 6.0*A(9)*BRE*BIM
FPIT(24) = 3.0*A(10)*((BRE**2)-(BIM**2))
FPIT(25) = (A(11)*BIM -(A(12)*BRE)
FPIT(26) = A(14)
CALL ATWLTF(34,FRT)
CALL ATWLTF(34,FIT)
CALL ATWLTF(26,FPRT)
CALL ATWLTF(26,FPIT)
M = 34
```

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L = 26
RETURN
END
```

References

1. Tanenbaum, B. Samuel and Mintzer, David, "Wave Propagation in a Partly Ionized Gas," The Physics of Fluids, Vol. 5, Number 10, October 1962.

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3. Xenakis, John, "PL/I-FORMAC Symbolic Mathematics Interpreter," IBM Corporation, Program Information Department, August 1969.

4. Froberg, Carl-Erik, Introduction to Numerical Analysis, London, Addison-Wesley Publishing Company, Inc., 1965.

CHAPTER II

NUMERICAL SOLUTION FOR PROPAGATION OF LONGITUDINAL WAVES ALONG THE APPLIED MAGNETIC FIELD IN A THREE-FLUID PARTIALLY IONIZED GAS

by

Thomas L. Dahl and David L. Murphree

NOTE: Figures, references and equations begin a new sequence in each Chapter. Also, the Appendices are lettered consecutively by Chapter, and each Chapter includes its own list of symbols.

LIST OF SYMBOLS

e	Magnitude of Electric Charge
c	Velocity of Light in a Vacuum
γ	Specific Heat Ratio
D/Dt	Hydrodynamic Derivative
ν_{ab}	Effective Collision Frequency of Type a with Type b Particles
ν_a	$\nu_{ab}+\nu_{ac}$, Total Collision Frequency
$\bar{E}$	Electric Field Strength
$\bar{H}$	Magnetic Field Strength
$\bar{V}_{e,i,n}$	Fluid Velocity of Electrons, Ions, or Neutral Particles
$N_{e,i,n}$	Number Density of Electrons, Ions, or Neutral Particles
$\rho_{e,i,n}$	Mass Density of Electrons, Ions, or Neutral Particles
$P_{e,i,n}$	Partial Pressure of Electron, Ion, or Neutral Particles
$\omega_{e,i}$	Electron or Ion Plasma Frequency
ω	Applied Frequency of the Wave
$\omega_{T,L}$	Cyclotron Frequency of the Electrons, Associated with Either the Transverse or Longitudinal Components of $\bar{H}_o$
k	Complex Wave Number
k_R	Real Part of the Wave Number
k_I	Imaginary Part of the Wave Number
n	kc/ω, Index of Refraction
m	m_i/m_e, Mass Ratio of Ions to Electrons
$U_{e,i,n}$	Acoustic Velocity of Electron, Ion, or Neutral Particle Species
U_f	$[\gamma(P_i+P_e+P_n)/(\rho_i+\rho_e+\rho_n)]^{1/2}$, Acoustic Velocity of the Entire Gas
U_p	$[\gamma(P_e+P_i)/(\rho_e+\rho_i)]^{1/2}$, Acoustic Velocity of the Electron-Ion Gas Mixture
i	$\sqrt{-1}$

I. INTRODUCTION

A three-fluid theory, using Maxwell's equations together with a set of coupled hydrodynamic equations for an interacting mixture of electrons, ions, and neutral molecules, has been employed by Tanenbaum and Mintzer[1] to examine small-amplitude oscillations in an infinite, homogenous, partly ionized gas with a uniform external magnetic field. Plots of phase velocity versus frequency were obtained for the case of negligible collisional damping for wave propagation along and normal to the applied magnetic field. A set of approximate solutions to the dispersion relation was employed to yield the phase velocities for various frequency bands. Employing the same physical model, Tanenbaum and Meskan[2] later presented the complete dispersion equation with no approximations for propagation of longitudinal waves along the magnetic field.

This paper will present a numerical solution to the complete dispersion relation derived by Tanenbaum and Meskan[2] governing the propagation of longitudinal waves along the magnetic field. Solutions have been determined for the complex wave number for a typical ionospheric condition. Plots of the phase velocity and damping characteristics of the three resulting wave modes are presented for the frequency range $10^{-5} < \omega < 10^{8}$ radians/second. Since the governing equations are linear, the longitudinal perturbation is described by superimposing the three wave solutions determined from the solution of the dispersion relation.

II. THEORY

A. Dispersion Relation

The derivation of the dispersion relation which governs wave propagation of small longitudinal perturbations along a magnetic field in a partially ionized gas composed of interacting electrons, ions, and neutral particles will be outlined. This derivation is presented in References 1 and 2.

Assume that:

(1) The degree of ionization is fixed

(2) Each gas obeys the perfect gas law

(3) Damping caused by the frictional forces of each gas allows for the conservation of total momentum of the system

(4) No heat flow exists within the gases

Therefore, the following set of equations describe the three-fluid mixture.

(1) Maxwell's equations:

$$\nabla x \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t}$$

$$\nabla x \bar{H} = \frac{4\pi e}{c}(N_i\bar{V}_i - N_e\bar{V}_e) + \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$$

(2) The continuity equation for each gas:

$$\frac{D}{Dt}\rho_{e,i,n} = -\rho_{e,i,n}\,\nabla \cdot \bar{V}_{e,i,n}$$

(3) The momentum equation for each gas:

$$\frac{D}{Dt}(\bar{V}_e) = -\frac{e}{m_e}\left(\bar{E} + \frac{\bar{V}_e \times \bar{H}}{c}\right) - \frac{\nabla P_e}{\rho_e} - \nu_{ei}(\bar{V}_e - \bar{V}_i) - \nu_{en}(\bar{V}_e - \bar{V}_n)$$

$$\frac{D}{Dt}(\bar{V}_i) = \frac{e}{m_i}\left(\bar{E} + \frac{\bar{V}_i \times \bar{H}}{c}\right) - \frac{\nabla P_i}{\rho_i} - \nu_{ie}(\bar{V}_i - \bar{V}_e) - \nu_{in}(\bar{V}_i - \bar{V}_n)$$

$$\frac{D}{Dt}(\bar{V}_n) = -\frac{\nabla P_n}{\rho_n} - \nu_{ne}(\bar{V}_n - \bar{V}_e) - \nu_{ni}(\bar{V}_n - \bar{V}_i)$$

(4) The adiabatic condition for each gas:

$$P_{e,i,n}\,N^{-\gamma}_{e,i,n} = \text{constant}$$

The coordinate system is aligned such that the applied magnetic field vector $\bar{H}_o = (H_{ox}, H_{oy}, 0)$. The above plasma equations may be linearized by perturbing the quantities $\bar{H}$, $\bar{V}_{e,i,n}$, $\bar{E}$, $N_{e,i,n}$ and $P_{e,i,n}$ with small, periodic oscillations of frequency ω. For example,

$$N_e = N_o + n_e e^{i(kx-\omega t)}$$

where N_o is the undisturbed electron density, n_e the amplitude of the perturbation, and N_e the resulting electron number density at any position x at any time t. The perturbation is considered to propagate only in the x-direction, i.e., one-dimensional propagation. The wave number, $k(\omega)$, is one of the allowed set of wave numbers whose value, as a function of frequency, we wish to determine. The resulting twenty-one equations (with twenty-one unknowns) may be manipulated by substitution until all the variables, except $\bar{V}_e$, are eliminated. The resulting expressions for $\bar{V}_e$ may be written most conveniently as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} V_{ex} \\ V_{ey} \\ V_{ez} \end{bmatrix} = 0 \tag{1}$$

where

$$A_{11} = (C_1/m) - (C_3C_5/C_1) + (\omega^2\omega^2{}_T/mC_2)$$

$$A_{12} = A_{21} = -\omega^2\omega_T\omega_L/mC_2$$

$$A_{13} = i\omega\omega_T[(C_3/C_1) - (C_6/mC_2)]$$

$$A_{22} = (C_2/m) - (C_4C_6/C_2) + \omega^2\omega^2{}_L/mC_2$$

$$A_{23} = -A_{32} = -(i\omega\omega_L/C_2)[C_4-(C_6/m)]$$

$$A_{31} = i\omega\omega_T[(C_5/mC_1) - (C_4/C_2)]$$

$$A_{33} = A_{22} + (\omega^2\omega^2{}_T/mC_1)$$

and

$$C_1 = \omega^2{}_e - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_2)$$

$$C_2 = \omega^2{}_e(1-n^2)^{-1} - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_1)$$

$$C_3 = \omega^2-\omega^2{}_i-k^2U^2{}_i + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_2)$$

$$C_4 = \omega^2-\omega^2{}_i(1-n^2)^{-1} + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_1)$$

$$C_5' = \omega^2-\omega^2{}_e-k^2U^2{}_e + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_2)$$

$$C_6 = \omega^2-\omega^2{}_e(1-n^2)^{-1} + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_1)$$

where

$$\gamma_1 = \omega^2 + i\omega\nu_n \qquad \omega_{T,L} = eH^o{}_{T,L}/m_eC$$

$$\gamma_2 = \gamma_1 - k^2U^2{}_n \qquad U_{e,i,n} = (\gamma P_{e,i,n}/\rho_{e,i,n})^{\frac{1}{2}}$$

$$\omega_{e,i} = (4\pi e^2N_o/m_{e,i})^{\frac{1}{2}} \qquad n = kc/\omega$$

Therefore, the dispersion relation which governs the perturbation in the electron velocity may be determined by expanding the matrix expression, Equation (1).

Restricting attention to propagation of longitudinal waves along the magnetic field, i.e., $\omega_T = 0$, we obtain

$$A_{11}V_{ex} = 0 \quad .$$

For V_{ex} to exist

$$A_{11} = 0 \quad ,$$

which when written completely yields the dispersion relation governing the propagation of longitudinal waves along the applied magnetic field

$$k^6 + Ak^4 + Bk^2 + C = 0 \quad , \tag{2}$$

where

$$A = -\left[\frac{\omega^2-\omega^2{}_e}{U^2{}_e} + \frac{\omega^2-\omega^2{}_i}{U^2{}_i} + \frac{\omega^2}{U^2{}_n}\right] -i\omega\left[\frac{\nu_{ei}+\nu_{en}}{U^2{}_e} + \frac{\nu_{ie}+\nu_{in}}{U^2{}_i} + \frac{\nu_{ni}+\nu_{ne}}{U^2{}_n}\right]$$

$$B = \frac{1}{U^2{}_eU^2{}_n}\left[\omega^4-\omega^2\omega^2{}_e-\omega^2(\nu_{ei}\nu_{ni}+\nu_{ei}\nu_{ne}+\nu_{en}\nu_{ni})\right]$$
$$+\frac{1}{U^2{}_iU^2{}_n}\left[\omega^4-\omega^2\omega^2{}_i-\omega^2(\nu_{ie}\nu_{ni}+\nu_{ne}\nu_{in}+\nu_{ne}\nu_{ie})\right]$$
$$+\frac{1}{U^2{}_iU^2{}_e}\left[\omega^4-\omega^2(\omega^2{}_e+\omega^2{}_i)-\omega^2(\nu_{in}\nu_{en}+\nu_{ie}\nu_{en}+\nu_{in}\nu_{ei})\right]$$
$$+\frac{i}{U^2{}_eU^2{}_n}\left[\omega^3(\nu_{ei}+\nu_{en}+\nu_{ni}+\nu_{ne})-\omega\omega^2{}_e(\nu_{ni}+\nu_{ne})\right]$$
$$+\frac{i}{U^2{}_iU^2{}_n}\left[\omega^3(\nu_{in}+\nu_{ie}+\nu_{ni}+\nu_{ne})-\omega\omega^2{}_e(\nu_{ni}+\nu_{ne})\frac{\rho_e}{\rho_i}\right]$$

$$+ \frac{i}{U^2{}_iU^2{}_e} \left[\omega^3(\nu_{ei}+\nu_{en}+\nu_{in}+\nu_{ie}) - \omega\omega^2{}_e(\nu_{ni}+\nu_{ne}) \frac{\rho_n}{\rho_i} \right]$$

$$C = -\omega^6 + \omega^4(\omega^2{}_i + \omega^2{}_e) + \omega^4[(\nu_{ei}\nu_{ni} + \nu_{ei}\nu_{ne} + \nu_{en}\nu_{ni})$$
$$+ (\nu_{ie}\nu_{ne} + \nu_{ie}\nu_{ni} + \nu_{in}\nu_{ne}) + (\nu_{ei}\nu_{in} + \nu_{en}\nu_{in} + \nu_{en}\nu_{ie})]$$
$$+ i \frac{[-\omega^5(\nu_i+\nu_e+\nu_n) + \omega^3\omega^2{}_e\nu_n \ (1 + \rho_e/\rho_i + \rho_n/\rho_i)]}{U^2{}_iU^2{}_eU^2{}_n}$$

Equation (2) is the relationship between the complex wave number, k, and the wave frequency, ω, of the propagating wave. Solution of this dispersion relationship will determine the value of the longitudinal perturbations at any position x and any time t. Note that the dispersion relation is actually a cubic equation in k^2, and will be solved algebraically using Cardan's solution for cubics.[3] This set of answers will then be exacted using the Newton-Raphson[4] method of iteration. This dispersion relation does not hold for high frequencies, i.e., $\omega \gg \omega_e$. In this case, some of the waves described by the dispersion relation have wave lengths comparable to or less than the mean free paths of the heavy particles and the Debye length of the electrons.

B. Cardan's Solution of Cubics[3]

We seek solution to the dispersion relation, Equation (2), of the form

$$x^3 + Ax^2 + Bx + C = 0$$

where the coefficients are complex numbers previously defined and $x = k^2$. To remove the quadratic term, define the transforming equation

$$x = y - A/3 \tag{3}$$

and obtain the reduced cubic equation,

$$y^3 + py + q = 0$$

where

$$p = (B - A^2/3)$$
$$q = (2A^3 - 9AB + 27C)/27$$

In order to solve the reduced cubic, transform again by letting

$$y = z - \frac{p}{3z} \tag{4}$$

and obtain

$$z^6 + qz^3 - p^3/27 = 0 \quad ,$$

which is a quadratic in z^3 with the roots

$$z^3 = [-q \pm (q^2 + 4p^3/27)^{\frac{1}{2}}]/2 \tag{5}$$

By choosing either the plus or the minus sign, equation (5) may be solved for three values of z. Then by combining (3) and (4), obtain

$$x = z - p/3z - A/3 \quad ,$$

into which the three values of z may be substituted in order to yield the three roots of the cubic. The Cardan's method is strictly algebraic and presents the advantage of an ordered extraction of the roots. However, the computer operations produced enough error that the Newton-Raphson method is required to exact the solution.

C. Newton-Raphson Method[4]

Using each of the roots found by Cardan's method as an approximation, substitute into the iterative process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ,$$

or

$$x_{n+1} = x_n - \frac{x^3{}_n + Ax^2{}_n + Bx + C}{3x^2{}_n + 2Ax_n + B} \quad .$$

Care must be taken to insure that the approximations do not fall in regions that iterate to misleading results, such as two equal roots or the interchange in order of extraction.

III. DISCUSSION

The objective is to find the complex wave number, k, which is a function of the applied frequency, ω. That is, solve

$$x^3 + Ax^2 + Bx + C = 0$$

where $x = k^2$ and the complex coefficients are functions of wave frequency, collision frequencies, plasma frequencies, and acoustic velocities of all species. Digital computer programs were written for the IBM 360 Model 40 at Mississippi State University and the Univac 1108 at The NASA-Slidell Computer Facility at Slidell, Louisiana. The simplified flow chart in Appendix A may be helpful in the discussion of the program procedure which follows.

After the plasma properties have been determined, and the coefficients of the dispersion relation calculated, Cardan's method of solution of cubic equations is employed. Note that after obtaining the reduced cubic

$$(z^3)^2 + qz^3 - p^3/27 = 0 \qquad ,$$

the Newton-Raphson iteration is performed to insure the proper choice of z^3. The three resulting values of z are substituted into the following relation to yield a set of roots to the cubic dispersion relation.

$$x_i = z_i - p/3z_i - A/3, \qquad i = 1,3$$

Since the value of x_i is usually small, the terms on the right of the above equation are of such magnitudes as to cancel each other, instead of summing to the proper x. Consequently, considerable accuracy is lost during the complex subtraction and division in the above expression. Therefore, these roots are treated only as approximations, and again the Newton-Raphson iteration is performed on the cubic

$$f(x) = x^3 + Ax^2 + Bx + C \qquad .$$

The resulting roots are checked first by comparing the value of the cubic to zero and second by comparing the sum of the roots to -A, i.e., $x_1 + x_2 + x_3 = -A$. Finally, the six wave numbers, k, are found by taking the square roots of x_i, $i = 1,3$.

The values of $k_j = k_{R_j} + k_{I_j}$, $j = 1,6$ and the corresponding ω, complétely describe the nature of the wave propagation at any position x and any time t, i.e.

$$Y = Y_o + ye^{i(kx-\omega t)}$$
$$= Y_o + ye^{-k_I x}e^{i(k_R x-\omega t)}$$

where Y is some field quantity, Y_o the undisturbed value, and y the amplitude of the perturbation at the source. The phase velocity is defined $U_j = \omega/k_{R_j}$, and the wave damping is determined by k_I. Only the three positive values of k, which represent waves propagating in the positive x direction will be discussed, since the three negative values represent the same wave form in the negative x direction.

Because of the wide separation in the magnitude of the real and imaginary components of the coefficients and resulting Cardan terms, the complex operations were reduced so that all calculations take place in real arithmetic. This allows the greatest use of the double precision features of the Fortran language. Special care must be taken in all operations to obtain maximum accuracy. Whenever possible, expressions were written in their lowest power form where addition and multiplication are the dominant operations. For instance, $\sqrt{A^2 - B^2}$ was written as $\sqrt{(A+B)(A-B)}$. Special subprograms were written to perform the complex operations of division, multiplication, squaring, cubing, square and cube root extraction, and Newton-Raphson iteration.

Only the subroutines performing square and cube root extraction converted the complex numbers into polar form. These operations were performed in the XPOCPX subroutine by the expression

$$(a+ib)^{1/n} = \left(\sqrt{a^2+b^2}\right)^{1/n} \left\{\cos\left(\frac{\phi+2\pi K}{n}\right) + i \sin\left(\frac{\phi+2\pi K}{n}\right)\right\}$$

where K = 1, ··, n-1 and ϕ is the argument of the complex number. All other functions merely manipulated on the coefficients of the operating complex numbers.

Consequently, the product and the square of complex numbers, i.e. the CPROD and CDSQ subroutines, were taken simply and accurately since

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$$

A similar approach was taken to the division operation, the CDDIV subroutine, since

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{ac+bd+i(bc-ad)}{c^2+d^2} \quad ,$$

and also the cubing operation, CDCB subroutine, because

$$\begin{aligned}(a+ib)^3 &= (a^2-b^2+2abi)(a+ib)\\ &= a(a+\sqrt{3}\ b)(a-\sqrt{3}\ b) + ib(\sqrt{3}\,a+b)(\sqrt{3}\,a-b) \quad .\end{aligned}$$

By exercising the caution discussed above and utilizing the iterations described previously, roots of the highest possible accuracy are obtained from the dispersion relation.

IV. RESULTS

The program was tested for a plasma with the following parameters, which were calculated as a typical ionospheric condition at an altitude of 320 kilometers at 45° North latitude and 90° West longitude.

$\nu_{ni} = 1.1202 \times 10^{-4}$ coll./sec.	$\omega_e = 2.8806 \times 10^{7}$ rad./sec.
$\nu_{in} = 2.2541 \times 10^{-1}$ coll./sec.	$\omega_i = 1.5731 \times 10^{5}$ rad./sec.
$\nu_{en} = 1.3072 \times 10^{1}$ coll./sec.	$U_{f,i,n} = 8.5097 \times 10^{2}$ m./sec.
$\nu_{ei} = 5.7883 \times 10^{1}$ coll./sec.	$U_e = 2.8158 \times 10^{5}$ m./sec.

A complete solution to the dispersion relation was determined for applied frequencies ranging from 10^{-5} to 10^{8} radians/second. The mathematically possible solutions for the phase velocities and corresponding e-folding distances are plotted in Figures 1 and 2, respectively. In these two figures, both the horizontal and vertical axes are plotted on logarithmic scales. The propagating and damping characteristics of each solution for $\omega = 1$ radian/second are plotted in Figure 3.

Phase Velocities

As seen in Figure 1, the phase velocity predictions agree in many respects with the approximated solutions of Tanenbaum and Mintzer, which are presented in Figure 1 of Reference 1.

1. At low frequencies, $\omega \simeq \nu_{ni} < \nu_{in}$, only two of the possible three wave solutions propagate through the plasma, one at the acoustic velocity of the entire fluid, $U_f = (\gamma P/\rho)^{\frac{1}{2}}$; and the other at a much lower velocity. In the numerical calculations, U_f, U_i, and U_n were taken as the same. The third wave mode has a phase velocity greater than the speed of light for $\omega < \omega_e$. However, as shown in Figure 2, the e-folding distance for the third solution is less than 10^{-2} meters for this frequency range; therefore, there is no propagation for this mode of the disturbance.
2. When $\omega \simeq \nu_{ni} < \nu_{in}$, one of the propagating wave modes increases exponentially in phase velocity with increasing frequency until $\omega \simeq \nu_{in}$. The other propagating mode propagates at the constant phase velocity U_f.
3. When $\nu_{in} < \omega < \omega_i$, the wave mode which was propagating with exponentially increasing phase velocity with increasing frequency now propagates at the acoustic velocity of the electron-ion gas mixture
$$U_p = \left[\frac{\gamma(P_e + P_i)}{\rho_e + \rho_i}\right]^{\frac{1}{2}}$$
4. When $\omega_i < \omega < \omega_e$, the two existing wave modes propagate at the ion and neutral acoustic velocity, U_i and U_n.
5. When $\omega > \omega_e$, the third wave mode now has a finite e-folding distance and propagates at the electron acoustic velocity.

The phase velocity plot presented as Figure 1 of Reference 1 was obtained from a set of approximate solutions to the complete dispersion relation, each approximate solution valid in a given frequency range. That analysis indicated that the wave mode which propagates at the phase velocity U_p in the range $\nu_{in} < \omega < \omega_i$ was the same mode which propagates at U_f for low frequencies, $\omega < \nu_{in}$. The present analysis which considers the complete dispersion relation for longitudinal oscillations along the magnetic field shows that this is not the case. The wave solution with an exponentially increasing phase velocity with increasing frequency for $\omega < \nu_{in}$ is the wave mode which propagates at a phase velocity U_p for $\nu_{in} < \omega < \omega_i$.

Both the real and the imaginary components of the wave number, k_R and k_I, are interdependent during all mathematical operations performed to obtain the solutions of the complete dispersion relation for continuously increasing frequency. Consequently, both components of the wave number are calculated simultaneously at each frequency during a continuous transfer across the frequency spectrum considered. Therefore, information on obtaining the continuous curve describing the phase velocity behavior of a given wave mode with increasing frequency is contained in requiring continuity of both k_R and k_I with frequency change. The qualitative plots presented in Figure 1 of Reference 1 could not reveal the correct smooth transition of the phase velocity of a given wave mode with frequency since the phase velocity was calculated in several discrete frequency ranges and the k_I solutions were not considered.

E-Folding Distances

The e-folding distance is defined as the distance from the source of the perturbation at which the amplitude of the wave is damped to 1/e of its initial amplitude, where e is the exponential factor. Each of the e-folding solutions corresponds to one of the phase velocity solutions, since both quantities are taken from the imaginary and real components, respectively, of the same wave number, k. Note that corresponding solutions are symbolized in the same manner on each graph. Some interesting facts are observed in Figure 2, the plot of e-folding distance versus wave frequency.

1. When $\omega < \nu_{in}$, the wave mode which propagates at a phase velocity of U_f is essentially undamped. The e-folding distance for this wave mode decreases exponentially to a constant value when $\omega \simeq \nu_{in}$. This solution shows very little damping throughout the frequency spectrum considered.
2. When $\omega < \nu_{in}$, the wave mode which propagates with an exponentially increasing phase velocity has a decreasing e-folding distance with increasing frequency until $\omega \simeq \nu_{in}$. The e-folding distance for this wave mode then remains constant with increasing frequency until $\omega \simeq \omega_i$. The phase velocity of this wave mode is U_p for $\nu_{in} < \omega < \omega_i$. At $\omega \simeq \omega_i$, the e-folding distance for this wave mode decreases to a new constant value. The phase velocity of this wave also decreases to a new constant value at $\omega \simeq \omega_i$.
3. For $\omega < \omega_e$, the third wave solution shows almost instantaneous damping near the source of the disturbance, i.e. an e-folding distance of less than 10^{-2} meters. The corresponding phase velocity for this wave mode was greater than the speed of light. A slight increase in the e-folding distance for this third mode of wave propagation occurs at $\omega \simeq \omega_i$. At $\omega \simeq \omega_e$, the e-folding distance for this wave mode increases almost immediately to the e-folding value of the second solution.

Propagation

Figure 3 shows the propagation described completely by the three complex wave numbers at $\omega = 1$ radian/second. The influence of the different k_R and k_I on the wave length and attenuation of the possible wave solutions can be seen. The wave configurations in Figure 3 occurred at the time when the wave with the slowest phase velocity had traversed a distance twice the maximum wave length given by the three solutions. The actual perturbation of the electron velocity along the magnetic field would propagate as the superposition of the three solutions since the governing equations were linearized by small perturbation theory. For illustrative purposes, the initial perturbation was taken as V_{ex_o} for all solutions. The superposition is shown as a solid line, while the other solutions are symbolized as before. The solution which is damped almost immediately at the source of the disturbance is shown as a point on the vertical axis at $x = 0$.

V. CONCLUSION

Complete solutions were obtained for the dispersion relation based on the three-fluid plasma model which governs the propagation of small longitudinal perturbations along the magnetic field. The complete solution for the dispersion relation is advantageous because it gives the complete description of the wave propagation. The resulting wave numbers consist of both real and imaginary parts which describe both the velocity and damping characteristics of each solution. Since the governing equations are linearized, the complete perturbation can be described by superimposing the wave solutions for the three-fluid plasma model.

An iteration procedure was necessary for accurate results because the magnitudes of some important terms in Cardan's method are such that the terms cannot be subtracted or added effectively by the computer. Numerically, the roots of the cubic dispersion relation are very accurate except at the high frequencies above ω_e. Here, although the roots can still be trusted numerically, their magnitudes are not as precise as the roots of lower frequencies.

The complex phase velocity solutions closely agree with the approximated predictions made by Tanenbaum and Mintzer in the various frequency ranges. However, the order of extraction of the solutions over the considered frequency range does not agree. The approximate solution could not reveal the correct order since it did not continuously transfer across the entire frequency spectrum nor was the information contained in requiring continuity in the imaginary part of the wave number utilized.

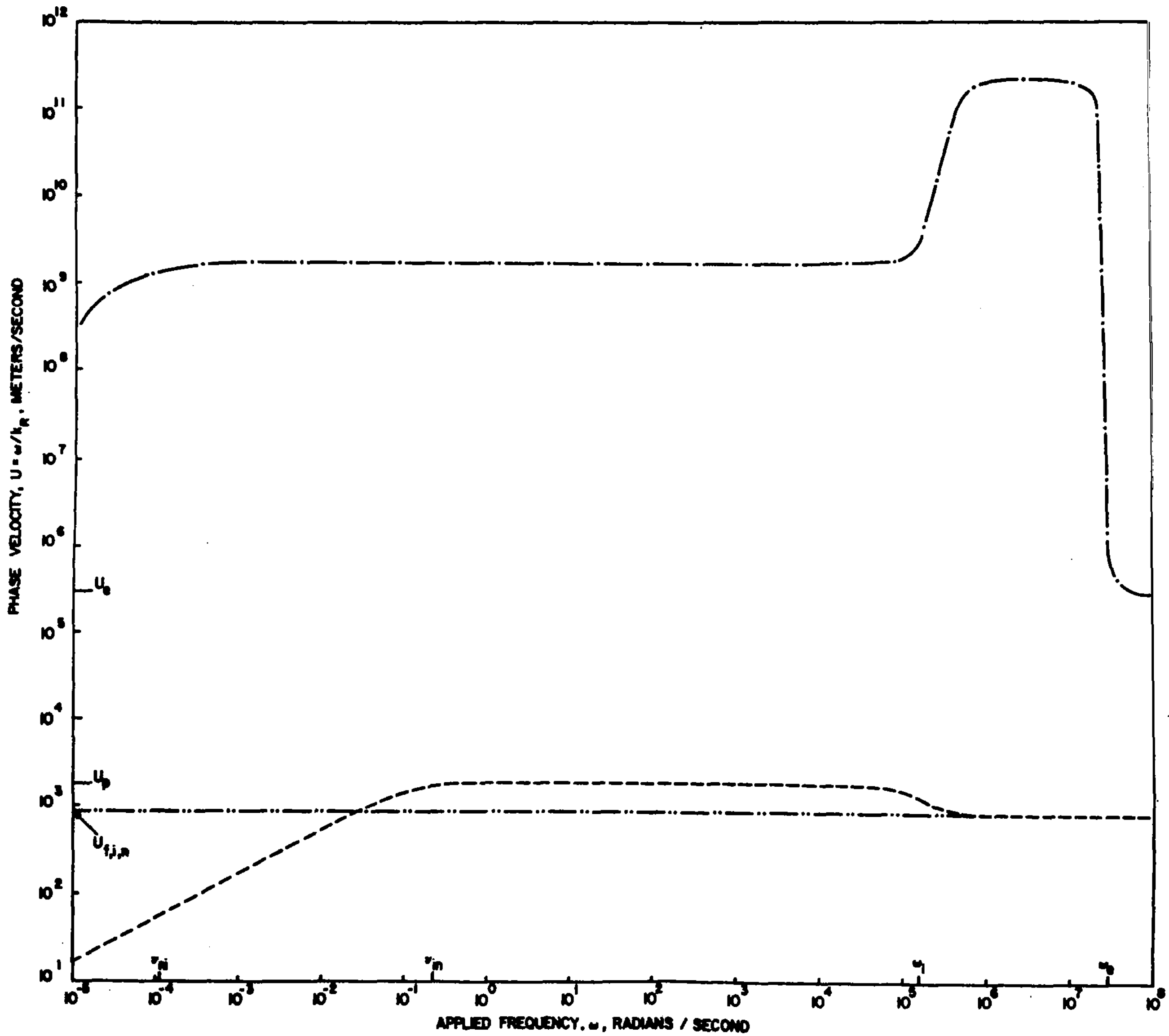


Figure 1. Plot of Phase Velocities versus Frequency.

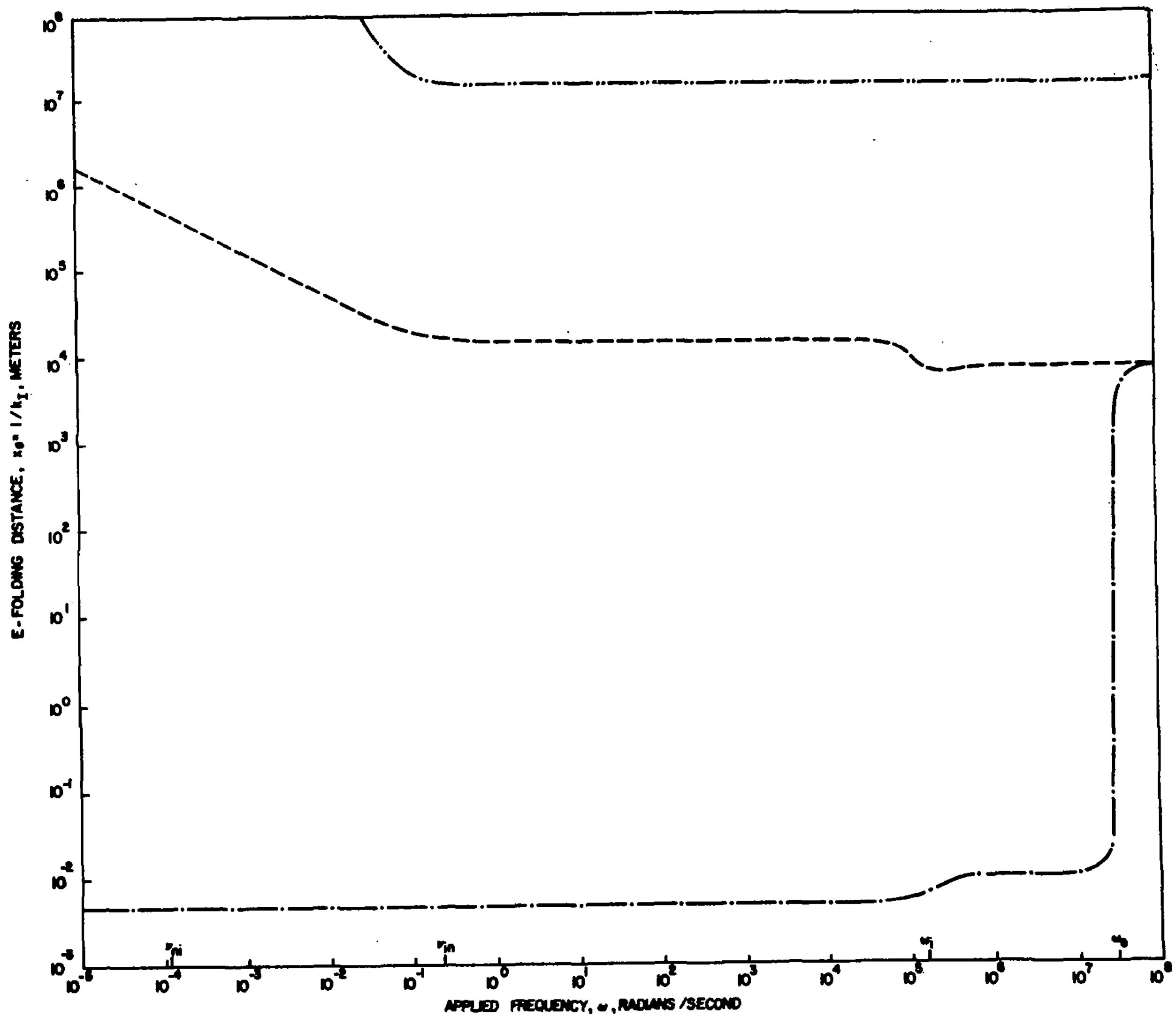


Figure 2. Plot of E-Folding Distances versus Frequency.

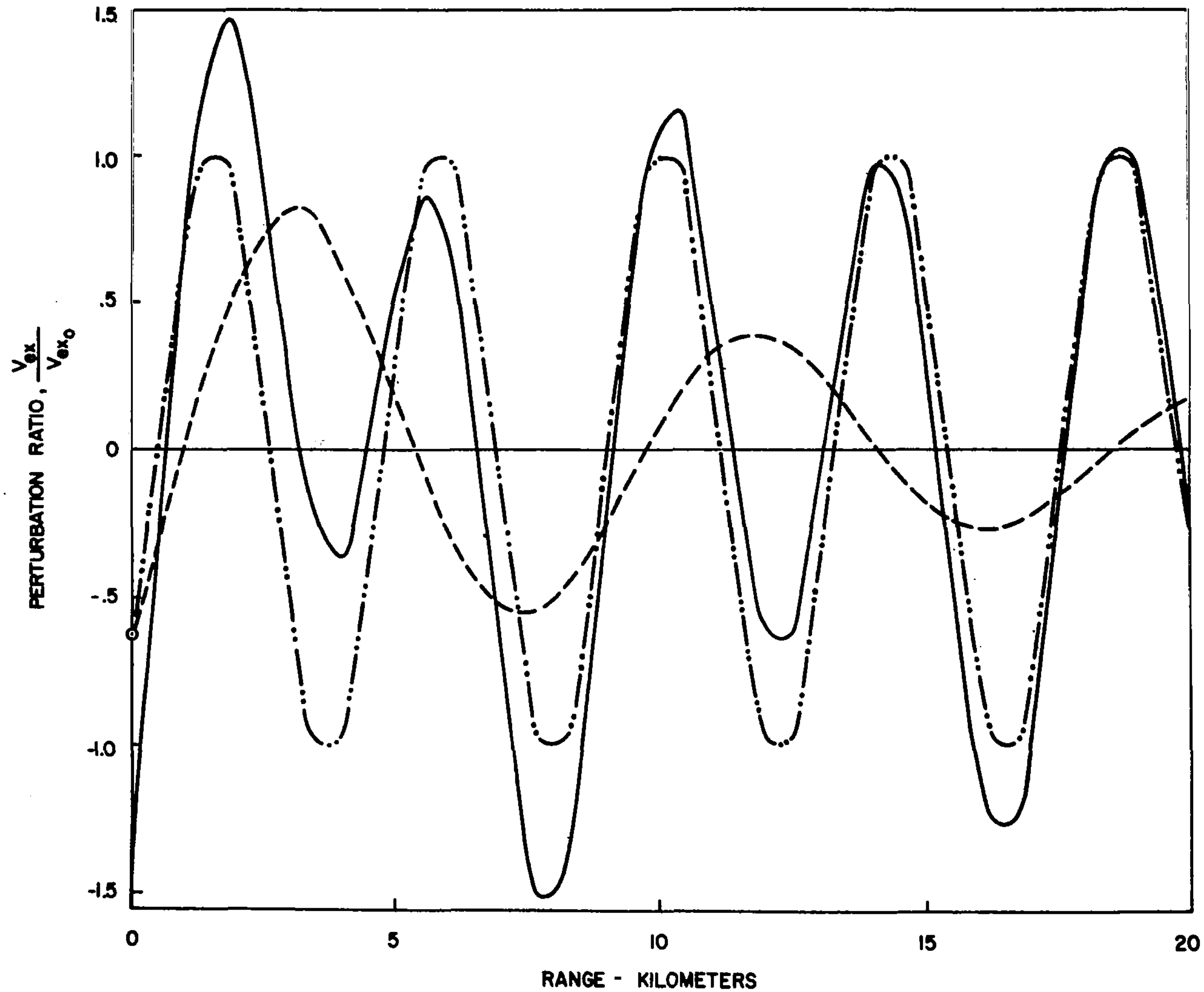
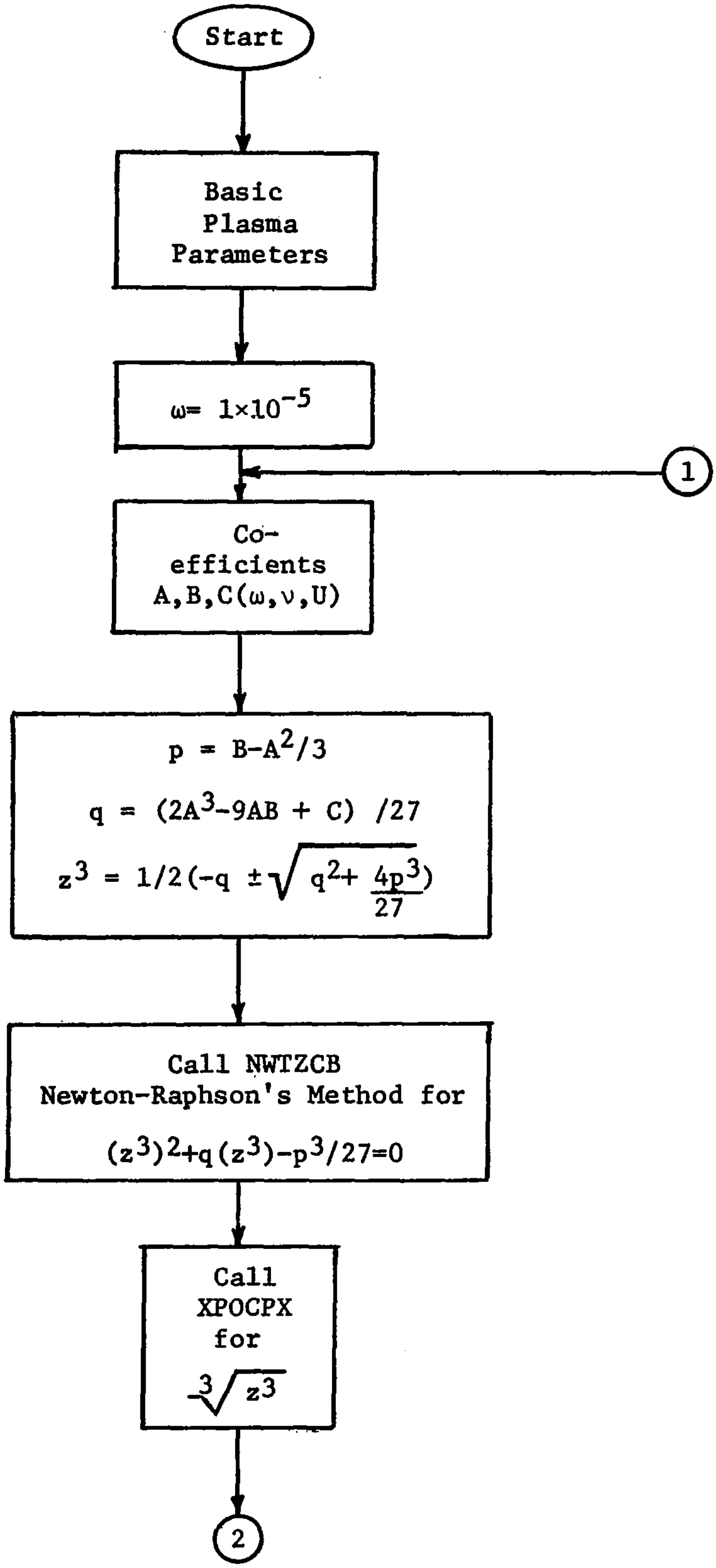
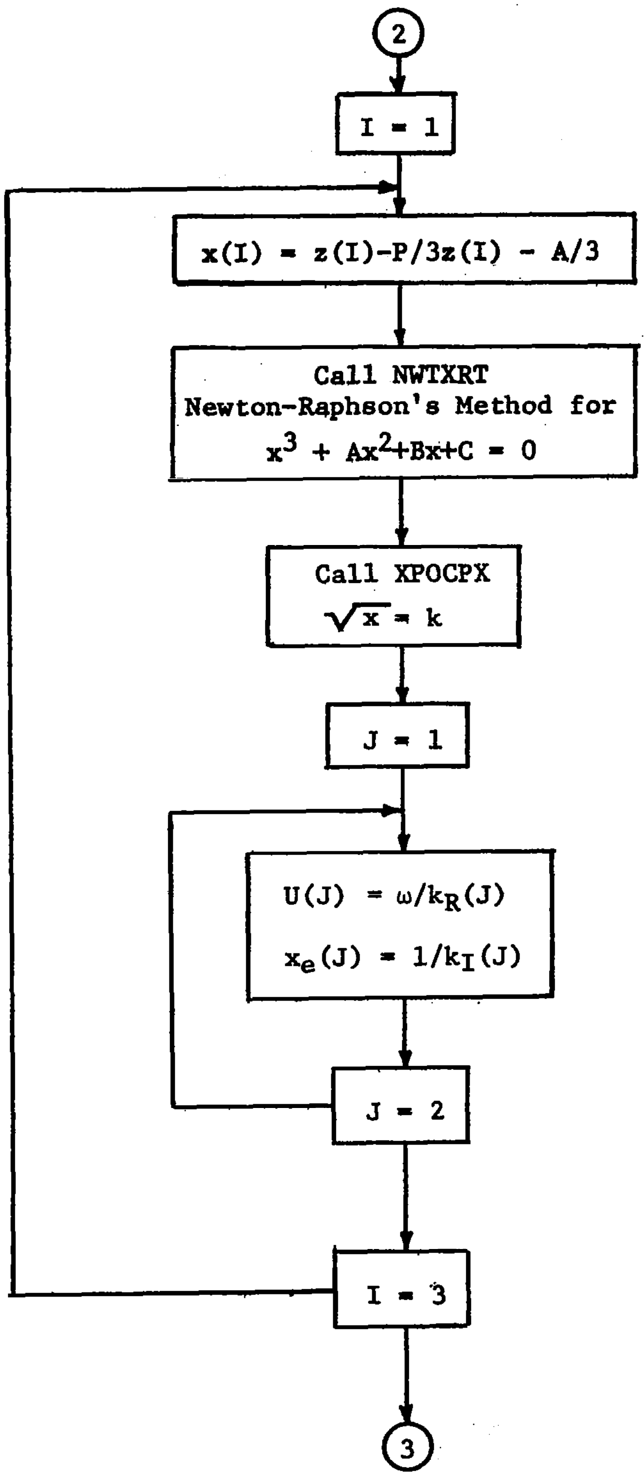


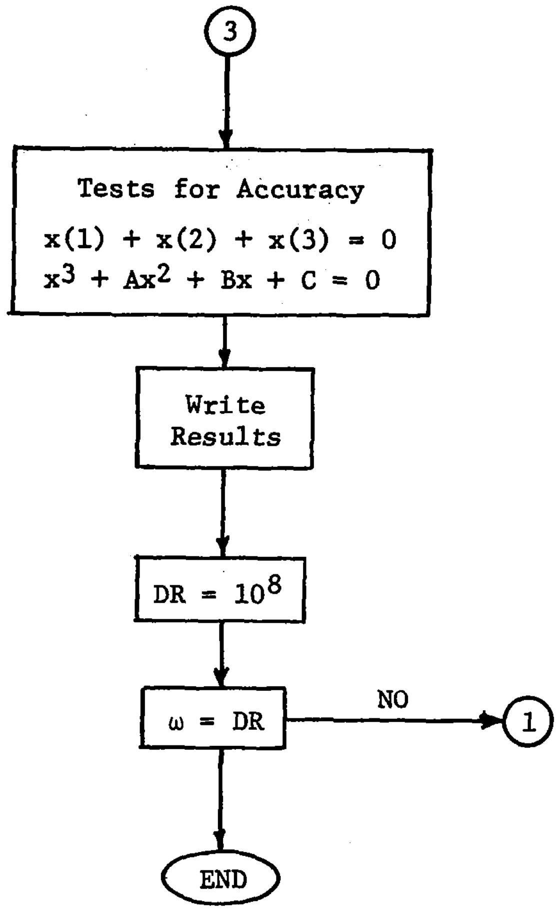
Figure 3. Plot of Wave Propagation at ω = 1 radian/second.

Appendix A. Simplified Flow Chart of Computer Program.





2
I = 1
x(I) = z(I)-P/3z(I) - A/3
Call NWTXRT
Newton-Raphson's Method for
$x^3 + Ax^2+Bx+C = 0$
Call XPOCPX
$\sqrt{x} = k$
J = 1
$U(J) = \omega/k_R(J)$
$x_e(J) = 1/k_I(J)$
J = 2
I = 3
3


3
Tests for Accuracy
x(1) + x(2) + x(3) = 0
x3 + Ax2 + Bx + C = 0
Write
Results
DR = 108
ω = DR
NO
1
END

Appendix B. Computer Program.

```
    **************COMPUTATION OF IONOSPHERIC PARAMETERS*************
    ME = ELECTRON MASS
    MI = ION NASS
    XMW = MOLECULAR WEIGHT
    r = ELECTRON CHARGE
    K = BOLTZMAN  CONSTANT
    DN = DIAMETER OF NEUTRAL ATOM
    DIAMETER OF NEUTRAL PARTICLE AND ION IS 3.0E-10 METERS
    GAMMA = SPECIFIC HEAT RATIO FOR COMPOSITE GAS
    AL = ALTITUDE IN KILOMETERS
    B = MAGNETIC FIELD STRENGTH IN WEBERS PER SQUARE METER
    ALL NO. DENSITIES ARE IN NO. PER CUBIC METER EXCEPT FOR NEC
    NO = NO. DENSITY FOR ATOMIC OXYGEN
    N E== NO. DENSITY FOR HELIUM
    NO2 = NO. DENSITY FOR MOLECULAR OXYGEN
    NN2 = NO. DENSITY FOR MOLECULAR NITROGEN
    NN = NEUTRAL NUMBER DENSITY IN 1/METERS CUBED
    NI = ION NUMBER DENSITY IN 1/METERS CUBED
    NE = ELECTRON NUMBER DENSITY IN 1/METERS CUBED
    NEC = ELECTRON NUMBER DENSITY IN 1/CENTIMETERS CUBED
    TI = ION TEMPERATURE IN DEGREES KELVIN
    TE = ELECTRON TEMPERATURE IN DEGREES KELVIN
    F = DEGREE OF NON-IONIZATION
    WI = CYCLOTRON FREQUENCY OF IONS
    WE = CYCLOTRON FREQUENCY OF ELECTRONS
    PFI = PLASMA FREQUENCY OF IONS
    PFE = PLASMA FREQUENCY OF ELECTRONS
    UI,UN = ION AND NEUTRAL SOUND VELOCITY, RESPECTIVELY
    UE = ELECTRON SOUND VELOCITY
    UF = ACOUSTIC VELOCITY FOR THE ENTIRE FLUID
    QIN = COLLISION CROSS SECTION OF IONS WITH NEUTRALS
    QEN = COLLISION CROSS SECTION OF ELECTRONS WITH NEUTRALS
    VEI = COLLISION FREQUENCY OF ELECTRONS WITH IONS
    VEN = COLLISION FREQUENCY OF ELECTRONS WITH NEUTRALS
    VIN=COLLISION FREQUENCY FOR IONS PASSING THRU NEUTRAL GAS
    VNI=COLLISION FREQUENCY FOR NEUTRALS PASSING THRU ION GAS
    CI = MEAN THERMAL SPEED OF IONS IN METERS PER SECOND
    CE = MEAN THERMAL SPEED OF ELECTRONS IN METERS PER SECOND
    TAUE = TIME BETWEEN COLLISIONS OF ELECTRONS WITH HEAVY PARTICLES
    TAUIN = TIME BETWEEN COLLISIONS OF IONS WITH NEUTRALS
    REAL NHE,LAM1,LAM2,K1,NEC,KR1,KR2,KI1,KI2
    DOUBLE PRECISION UE,UI,UN,VEI,VEN,VIN,NE,NI,NN,ME,MI,MN,RHOE,RHOI,
   1  RHON,VNI,PFE,PFI,NO,NO2,NN2
    AL=320.0
    C1=1.6E-19
    DN=3.E-10
    K1=1.38E-23
    B1=5.1E-5
    IF(AL-300.) 1,2,2
  1 NE  =    (EXP(0.0069*(AL-300.)))*3.E11
```

```
      GO TO 3
    2 NE = EXP(0.0069*(300.-AL))*3.E11
    3 NI = NE
      NEC = NE*1.E-6
      NN = 10.**(18.0-((AL-120.)/18.6)**0.5)
      F = 1.0-NE/NN
      TI = 1000.-650.*EXP(0.035*(120.-AL))
      TE = TI+200.+2200.*SIN(0.00523*(AL-100.))
      IF(AL-275.) 4,5,5
    4 XMW = 27.7-0.0457*(AL-100.)
      GO TO 6
    5 XMW = 19.4-0.0222*(AL-275.)
    6 MN = XMW*1.66E-27
      MI = MN
      ME = 9.11E-31
      A1= 9.42 + 1.5*ALOG(TE) - 0.5*ALOG(NEC)
      VEI = NEC*A1/(0.38*(TE**1.5))
      QEN = 3.14*(DN**2)/4.0
      CE = (8.*K1*TE/(3.14*ME))**0.5
      VEN = CE*QEN*NN
      VE=VEI+VEN
      TAUE = 1./(VEN+VEI)
      QIN = 3.14*DN*DN
      CJ = (8.*K1*TI/(3.14*MI))**0.5
      VIN = (2.**0.5)*CJ*QIN*NN
      VNI = (2.**0.5)*CJ*QIN*NI
      TAUIN = 1./VIN
      WI =C1*B1/MI
      WEC=C1*B1/ME
      SIGMA = NE*C1*C1/(ME*(VEN+VEI))
      B20 = WEC*TAUE
      B10 = F*WEC*TAUE*WI*TAUIN
      RO=MN*NN+MI*NI
      DELTA = (12.5E-7)*SIGMA
      AX =B1/(RO*12.56E-7)**0.5
      IF (AL-165.) 20,21,21
   20 ALX=AL-120.
      NO = 10.**(16.8-0.0222*ALX)
      NHE = 10.**(13.5-0.0089*ALX)
      NO2 = 10.**(16.8-0.0356*ALX)
      NN2 = 10.**(17.5-0.0334*ALX)
      GO TO 22
   21 ALX = AL - 165.
      NO = 10.**(15.8-0.00827*ALX)
      NHE = 10.**(13.1-0.00184*ALX)
      NO2 = 10.**(15.1-0.0145*ALX)
      NN2 = 10.**(16.0-0.0129*ALX)
   22 XM1 = (16.*NO+4.*NHE)/(NO+NHE)
      XM2 = (28.*NN2 + 32.*NO2)/(NN2 +NO2)
      XP = (NO2+NN2)/(NO2+NO+NN2+NHE)
```

```
      GAM1 = 1.67
      IF (TI-550.)23,24,24
   23 GAM2 = 1.40
      GO TO 25
   24 GAM2 = 1.39
   25 GAMMA = (GAM1*XM2/XM1 + XP*GAM2*(GAM1-1.)/(1.-XP)/(GAM2-1.))/
     1  (XM2/XM1 + XP/(1.-XP)*(GAM1-1.)/(GAM2-1.))
      RI = 8.31E3/XMW
      RE = 1.48E7
      UI = (GAMMA*RI*TI)**0.5
      UE = (GAM1*RE*TE)**0.5
      UF=UI
      EPS=8.85E-12
      PFI=(NI*C1*C1/MI/EPS)**0.5
      PFE=(NE*C1*C1/ME/EPS)**0.5
      THIS STACK OF CARDS CONVERTS ALL QUANTITIES TO GAUSSIAN UNITS
      C1=C1*3.E9
      B1=B1*1.E4
      NE=NE*1.D-6
      NI=NI*1.D-6
      NN=NN*1.D-6
      MI=MI*1.D3
      MN=MN*1.D3
      ME=ME*1.D3
      QEN=QEN*1.E4
      CE=CE*1.E2
      QIN=QIN*1.E4
      CJ=CJ*1.E2
      SIGMA=SIGMA*9.E9
      RO=RO*1.E-3
      DELTA=DELTA*9.E16
      AX=AX*1.E2
      NO=NO*1.E-6
      NHE=NHE*1.E-6
      NO2=NO2*1.E-6
      RI=RI*1.E-3
      NN2=NN2*1.E-6
      RE=RE*1.E-3
      UI=UI*1.D2
      UN=UI
      UE=UE*1.D2
      UF=UF*1.D2
   12 CONTINUE
      RHOE=NE*ME
      RHOI=NI*MI
      RHON=NN*MN
      CALL TRWVLG(PFE,PFI,RHOE,RHOI,RHON,VEI,VEN,VIN,VNI,UI,UE,UN)
      STOP
      END
```

```
      SUBROUTINE TRWVLG(WE,WI,RHOE,RHOI,RHON,VEI,VEN,VIN,VNI,UI,UE,UN)
      DOUBLE PRECISION KRA(3),KIA(3),BDKR(3),BDXR(3),BDXI(3),BX2R(3),
     1 BX2I(3),BX3R(3),BX3I(3),BCX2R(3),BCX2I(3),BCNR(3),BCNI(3),
     2 BCXR(3),BCXI(3)
      DOUBLE PRECISION UE,UI,UN,VEI,VEN,VIN,RHOE,RHOI,
     1RHON,VIE,VNI,VNE,WI,WE,WI2,VE,VI,VN,UPR2,
     2  WE2, W2, W3, W4, W5, W6
      DOUBLE PRECISION A3, A2, ASQR, ASQI, ACBR, ACBI, B3, B2, ATMSBR,
     1  ATMSBI, P1R, P1I, PR, PIM, QR, QI, QSQR, QSQI, PSQR, PSQI, PCBR,
     2  PCBI, AMADR, AMADI, RADR(2), RADI(2), ZCBR, ZCBI,
     3 ZR(3),ZI(3),BETR(3),BETI(3),XZR(3),XZI(3)
      DOUBLE PRECISION XSQR(3),XSQI(3),XCBR(3),XCBI(3),AXSQR(3),AXSQI(3)
     1,BXR(3),BXI(3),CRDNR(3),CRDNI(3),XTSKR(3),XTSKI(3),XRRTO(3),
     3XIRTO(3)
      DOUBLE PRECISION NUMR,NUMI,ATSTR,ATSTI,RTAR,RTAI,A(10),B(10),
     1W,UE2,UI2,UN2,DR,PRD3,PID3,KR(2),KI(2),U(2),XE(2),C3,C2,XR(3)
      DIMENSION PHVY(4,250)

      PROGRAM USES CARDAN'S METHOD TO SOLVE CUBIC EQUATIONS
      OF THE FORM 'X**3 + AX**2 + BX + C = 0'.

      W    :APPLIED FREQUENCY
      WE   :PLASMA FREQUENCY OF ELECTRONS
      WI   :PLASMA FREQUENCY OF IONS
      RHOI:MASS DENSITY OF IONS
      RHOE:MASS DENSITY OF ELECTRONS
      RHON:MASS DENSITY OF NEUTRALS
      VAB :EFFECTIVE COLLISION FREQUENCY
           OF 'A' WITH  'B'
      UI   :ACOUSTIC VELOCITY OF IONS
      UE   :ACOUSTIC VELOCITY OF ELECTRONS
      UN   :ACOUSTIC VELOCITY OF NEUTRALS
      VIE= RHOE*VEI/RHOI
      VNE=RHOE*VEN/RHON
      WI2=WI**2
      WE2=WE**2
      UI2=UI**2
      UE2=UE**2
      UN2=UN**2
      VE=VEI+VEN
      VI=VIE+VIN
      VN=VNE+VNI
      UPR2 = (UI*UN*UE)**(-2 )
      UTOT=UI*UN*UE
 **********************
      W=1.D8
      JJ=0
1000  WRITE (3,104) W
104   FORMAT (//'   W=',1PD11.4)
      JJ=JJ+1
```

```
 L=JJ
 W2 = W**2
 W3 = W**3
 W4 = W**4
 W5 = W**5
 W6 = W**6
 A3=(WE/UE)**2+(WI/UI )**2-((W/UE)**2+(W/UI)**2+( W/UN)**2)

 A2= -W*(VE*(UI*UN)**2 +VI*(UE*UN)**2 +VN*(UE*UI)**2)*UPR2

 B3 =  (W4*(UN2+UI2+UE2)- W2*(UE2*(VNI*VIE +VNE*VIN +VNE*VIE) +
1UI2*(VEI*VNI+VEI*VNE +VEN*VNI) +UN2*(VIN*VEN+ VIE*VEN + VIN*VEI)+
2WE2*UN2 +WI2*UN2 +WE2*UI2 +WI2*UE2))*UPR2

 B2 = (W3*( UN2*(VEI+VEN+VIN+VIE)+UI2*(VEI+VEN+VNI+VNE)+UE2*(VIN+
1 VIE+VNI+VNE)) -W*WE2*( (VNI+VNE)*(UN2*RHON+UE2*RHOE)/RHOI +
1 (VNI+VNE)*UI2))*UPR2

 C3=(W4*(VEI*VNI +VEI*VNE +VEN*VNE +VNI*VIE +VNE*VIN +VNE*VIE
1+VIN*VEN +VIE*VEN +VIN*VEI +WI2 +WE2)-W6 )*UPR2

 C2 =(-W5*(VIN+VIE+VEI+VEN+VNI+VNE)+W3*WE2*(VN+((RHOE+RHON)*VN/
1  RHOI)))*UPR2

 CALL CDSQ(A3,A2,ASQR,ASQI)
 CALL CDCB(A3,A2,ACBR,ACBI)
 CALL CPROD(A3,A2,B3,B2,ATMSBR,ATMSBI)

 DEFINE P'S AND Q'S.

 P1R = ASQR/3.0D0
 P1I = ASQI/3.0D0
 PR = B3 - P1R
 PIM = B2 - P1I
 PRD3=PR/3.D0
 PID3=PIM/3.D0
 QR= 2.D0*ACBR/27.D0 - ATMSBR/3.D0 +C3
 QI= 2.D0*ACBI/27.D0 - ATMSBI/3.D0 +C2

 DEFINE Q**2, P**3 AND AMAD.

 CALL CDSQ(QR,QI,QSQR,QSQI)
 CALL CDSQ(PR,PIM,PSQR,PSQI)
 CALL CDCB(PR,PIM,PCBR,PCBI)
 AMADR = QSQR + 4.0D0*PCBR/27.0D0
 AMADI = QSQI + 4.0D0*PCBI/27.0D0
 CALL XPOCPX(1,2,AMADR,AMADI,RADR,RADI)
 ZCBR = 0.5D0*(-QR + RADR(1))
 ZCBI = 0.5D0*(-QI + RADI(1))
```

```
      CALL NWTZCB(ZCBR,ZCBI,QR,QI,(PCBR/27.D0),(PCBI/27.D0),ZCBR,
     1ZCBI,NUMR,NUMI)
      CALL XPOCPX(1,3,ZCBR,ZCBI,ZR,ZI)
      DO 9 I=1,3
      XZR(I) = 3.0D0*ZR(I)
      XZI(I) = 3.0D0*ZI(I)
      CALL CDDIV (PR,PIM,XZR(I),XZI(I),BETR(I),BETI(I))
      XR(I)=ZR(I)-BETR(I)-A3/3.D0
      XI(I)=ZI(I)-BETI(I)-A2/3.D0
    9 CONTINUE
      ATSTR=XR(1)+XR(2)+XR(3)
      ATSTI=XI(1)+XI(2)+XI(3)
      RTAR=-ATSTR/A3
      RTAI=-ATSTI/A2
      IF(RTAR.EQ.1.D0.AND.RTAI.EQ.1.D0) GO TO 301
      DO 145 I=1,3
      IF(DABS(XR(I)/ZR(I)).LE.1.D-13) XR(I)=0.D0
      IF(DABS(XI(I)/ZI(I)).LE.1.D-13) XI(I)=0.D0
      WRITE(3,222) XR(I),XI(I)
  222 FORMAT('    XR=',D23.16,' XI=',D23.16)
      CALL NWTXRT(XR(I),XI(I),A3,A2,B3,B2,C3,C2,XR(I),XI(I),NUMR,NUMI)
      WRITE(3,223) XR(I),XI(I),NUMR,NUMI
  223 FORMAT('    XR=',D23.16,' XI=',D23.16,' ZERO=',2D14.6)
  145 CONTINUE
      ATSTR=XR(1)+XR(2)+XR(3)
      ATSTI=XI(1)+XI(2)+XI(3)
      RTAR=-ATSTR/A3
      RTAI=-ATSTI/A2
  301 CONTINUE
      DO 302 I=1,3
      CALL CDSQ(XR(I),XI(I),XSQR(I),XSQI(I))
      CALL CDCB(XR(I),XI(I),XCBR(I),XCBI(I))
      CALL CPROD(A3,A2,XSQR(I),XSQI(I),AXSQR(I),AXSQI(I))
      CALL CPROD(B3,B2,XR(I),XI(I),BXR(I),BXI(I))
      CRDNR(I)=XCBR(I)+AXSQR(I)+BXR(I)+C3
      CRDNI(I)=XCBI(I)+AXSQI(I)+BXI(I)+C2
  302 CONTINUE

      NOTE:  K = DSQRT(XR), UU = W/K AND XE = 1/KI.
      NOTE:  EACH 'CALL' GIVES TWO K'S.

      PHVY(4,JJ)=W
      DO 10 I=1,3
      CALL XPOCPX(1,2,XR(I),XI(I),KR,KI)
      PKR(I)=KR(1)
      PKI(I)=KI(1)
      DO 15 J=1,2
      U(J) = W/KR(J)
   15 XE(J) = 1.0D0/KI(J)
      PHVY(I,JJ)=DLOG10(DABS(U(1)))*10.
```

```
      WRITE (3,155) XR(I),XI(I),KR(1),KI(1),U(1),XE(1),KR(2),KI(2),U(2),
     1  XE(2)
155   FORMAT(' XR=',D23.16,' XI=',D23.16,' KR =',D13.6,'   KI =',
     1  D13.6,2X,'U =',D13.6,2X,'XE =',D13.6/55X,'KR =',D13.6, '   KI =',
     2  D13.6,2X,'U =',D13.6,2X,'XE =',D13.6)
   10 CONTINUE
      WRITE(3,158) ATSTR,ATSTI,RTAR,RTAI
  158 FORMAT(' AR=',D23.16,' AI=',D23.16,' TEST=',2D23.16)
      DO 33 I=1,3
      WRITE(3,156)XR(I),XI(I),XCBR(I),XCBI(I),XSQR(I),XSQI(I),
     1CRDNR(I),CRDNI(I)
  156 FORMAT(' ',' X=',2D14.6,' X3=',2D14.6,' X2=',2D14.6,' CRDN=',2D14.6
     1 )
   33 CONTINUE
      DR=1.D12
      IF(W.GE.DR) GO TO 9000
      IF (W .GE. 1.0D 9 .AND. W .LT. 1.0D23) W = W * 1.0D 1
      IF (W .GE. 1.0D 8 .AND. W .LT. 1.0D 9) W = W + 1.0D 8
      IF (W .GE. 1.0D 7 .AND. W .LT. 1.0D 8) W = W + 1.0D 7
      IF (W .GE. 1.0D 6 .AND. W .LT. 1.0D 7) W = W + 1.0D 6
      IF (W .GE. 1.0D 5 .AND. W .LT. 1.0D 6) W = W + 1.0D 5
      IF (W .GE. 1.0D 4 .AND. W .LT. 1.0D 5) W = W + 1.0D 4
      IF (W .GE. 1.0D 3 .AND. W .LT. 1.0D 4) W = W + 1.0D 3
      IF (W .GE. 1.0D 2 .AND. W .LT. 1.0D 3) W = W + 1.0D 2
      IF (W .GE. 1.0D 1 .AND. W .LT. 1.0D 2) W = W + 1.0D 1
      IF (W .GE. 1.0D 0 .AND. W .LT. 1.0D 1) W = W +1.0D 0
      IF (W.GE. 1.0D-1 .AND. W .LT. 1.0D 0) W = W + 1.0D-1
      IF (W .GE. 1.0D-2 .AND. W .LT. 1.0D-I) W = W+ 1.0D-2
      IF (W .GE. 1.0D-3 .AND. W .LT. 1.0D-2) W = W + 1.0D-3
      IF (W .GE. 1.0D-4 .AND. W .LT. 1.0D-3) W = W + 1.0D-4
      IF (W .GE. 1.0D-5 .AND. W .LT. 1.0D-4) W = W + 1.0D-5
      IF (W .GE. 1.0D-6 .AND. W .LT. 1.0D-5) W = W + 1.0D-6
      IF (W .GE. 1.0D-7 .AND. W .LT. 1.0D-6) W = W + 1.0D-7
      GO TO 1000
9000  RETURN
      DEBUG UNIT(3),SUBCHK
      END


      SUBROUTINE CDSQ(A,B,SQ1,SQ2)
      DOUBLE PRECISION A, B, SQ1, SQ2
      SQ1=(A-B)*(A+B)
      SQ2 = 2.0D0*A*B
      RETURN
      DEBUG UNIT(3),SUBCHK
      END
```

```
      SUBROUTINE XPOCPX(M,N,A,B,XR,XM)
      THIS SUBROUTINE CALCULATES THE M/NTH ROOTS OF A COMPLEX NUMBER
      OF THE FORM 'C = A + I*B'.
      DOUBLE PRECISION A, B, BAR, BETA, COEF, K, PI, RM, RN, T, XM(25),
     1  XR(25),ALFA
      DOUBLE PRECISION AA,BB
      AA=DABS(A)
      BB=DABS(B)
      RM = M
      RN = N
      PI = 3.14159265358979324D0
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) BAR=DSQRT(A**2+B**2)
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) GO TO 1010
      BAR=DABS(A)*DSQRT(1.D0+(B/A)**2)
      IF(B.LE.1.D-32) GO TO 1010
      IF(DABS(DLOG10(AA)-DLOG10(BB)).LE.4.D0)BAR=DSQRT(A**2+B**2)
 1010 COEF = BAR**(RM/RN)
      T = DATAN2(B,A)
      IF (B .LT. 0.0D0) T = 2.0D0*PI - DABS(T)
      K = 0.0D0
      DO 100 I=1,N
      BETA = T + K*2.0D0*PI
      ALFA=BETA*RM/RN
      XR(I) = COEF*DCOS(ALFA)
      XM(I) = COEF*DSIN(ALFA)
100   K = K + 1.0D0
      RETURN
      DEBUG UNIT(3),SUBCHK
      END
```

```
      SUBROUTINE  CDDIV(A,B,C,D,E,F)
      DOUBLE PRECISION A,B,C,D,E,F,DEM
      DOUBLE PRECISION CC,DD
      CC=DABS(C)
      DD=DABS(D)
      IF(CC.EQ.0.D0.OR.DD.EQ.0.D0) DEM=C**2+D**2
      IF(CC.EQ.0.D0.OR.DD.EQ.0.D0) GO TO 1010
      DEM=(C**2)*(1.D0+(D/C)**2)
      IF(D.LE.1.D-32) GO TO 1010
      IF(DABS(DLOG10(CC)-DLOG10(DD)).LE.4.D0)DEM=(C**2+D**2)
 1010 E=(A*C+B*D)/DEM
      F=(B*C-D*A)/DEM
      RETURN
      DEBUG UNIT(3),SUBCHK
      END
```

```
      SUBROUTINE NWTXRT(XR,XI,AR,AI,BR,BI,CR,CI,XNEWR,XNEWI,NUMR,NUMI)
      DOUBLE PRECISION XR,XI,AR,AI,BR,BI,CR,CI,XNEWR,XNEWI,
     1 XCBR,XCBI,XSQR,XSQI,AXSQR,AXSQI,AXR,AXI,BXR,BXI,
     2 NUMR,NUMI,DEMR,DEMI,DLTR,DLTI
      DOUBLE PRECISION TESTR,TESTI
      N=0
    5 CONTINUE
      N=N+1
      IF(N.GT.200) GO TO 15
      CALL CDCB(XR,XI,XCBR,XCBI)
      CALL CDSQ(XR,XI,XSQR,XSQI)
      CALL CPROD(AR,AI,XSQR,XSQI,AXSQR,AXSQI)
      CALL CPROD(AR,AI,XR,XI,AXR,AXI)
      CALL CPROD(BR,BI,XR,XI,BXR,BXI)
      NUMR=XCBR+AXSQR+BXR+CR
      NUMI=XCBI+AXSQI+BXI+CI
      DEMR=3.D0*XSQR+2.D0*AXR+BR
      DEMI=3.D0*XSQI+2.D0*AXI+BI
      IF(DEMR.EQ.0.D0.AND .DEMI.EQ.0.D0) WRITE(3,101)
      IF(DEMR.EQ.0.D0.AND .DEMI.EQ.0.D0) GO TO 15
      IF(NUMR.EQ.0.D0.AND.NUMI.EQ.0.D0) GO TO 15
      CALL CDDIV(NUMR,NUMI,DEMR,DEMI,DLTR,DLTI)
      XNEWR=XR-DLTR
      XNEWI=XI-DLTI
      TESTR=DABS(XR/DLTR)
      TESTI=DABS(XI/DLTI)
      IF(TESTR.LT.1.D-15.AND.TESTI.LT.1.D-15) GO TO 15
      XR=XNEWR
      XI=XNEWI
      GO TO 5
   15 RETURN
  101 FORMAT(' **** THE DERIVATIVE OF  F(X**3)=0   ****')
      DEBUG UNIT(3),SUBCHK
      END
```

```
      SUBROUTINE NWTZCB(XR,XI,QR,QI,SR,SI,XNEWR,XNEWI,NUMR,NUMI)
      DOUBLE PRECISION  XR,XI,QR,QI,SR,SI,XNEWR,XNEWI,
     1 XSQR,XSQI,QTMSXR,QTMSXI,NUMR,NUMI,DEMR,
     2 DEMI,DLTR,DLTI
      DOUBLE PRECISION TESTR,TESTI
      N=0
    5 CONTINUE
      N=N+1
      IF(N.GT.100) GO TO 15
      CALL CDSQ(XR,XI,XSQR,XSQI)
      CALL CPROD(XR,XI,QR,QI,QTMSXR,QTMSXI)
      NUMR=XSQR + QTMSXR - SR
      NUMI=XSQI + QTMSXI - SI
      DEMR=2.D0*XR + QR
      DEMI=2.D0*XI + QI
      IF(DEMR.EQ.0.D0.AND .DEMI.EQ.0.D0) WRITE(3,101)
      IF(DEMR.EQ.0.D0.AND .DEMI.EQ.0.D0) GO TO 15
      IF(NUMR.EQ.0.D0.AND.NUMI.EQ.0.D0) GO TO 15
      CALL CDDIV(NUMR,NUMI,DEMR,DEMI,DLTR,DLTI)
      XNEWR=XR-DLTR
      XNEWI=XI-DLTI
      TESTR=DABS(XR/DLTR)
      TESTI=DABS(XI/DLTI)
      IF(TESTR.LT.1.D-15.AND.TESTI.LT.1.D-15) GO TO 15
      XR=XNEWR
      XI=XNEWI
      GO TO 5
   15 RETURN
  101 FORMAT(' **** THE DERIVATIVE OF  F(Z**3)=0  ****')
      END

      SUBROUTINE CPROD(A,B,C,D,PRODR,PRODI)
      DOUBLE PRECISION A, B, C, D, PRODR, PRODI
      PRODR = A*C - B*D
      PRODI = B*C + A*D
      RETURN
      END

      SUBROUTINE CDCB(X,Y,X3,Y3)
      DOUBLE PRECISION X,Y,X3,Y3,C
      DOUBLE PRECISION CY,CX
      C=DSQRT(3.D0)
      CY=C*Y
      CX=C*X
      X3=X*(X+CY)*(X-CY)
      Y3=Y*(CX+Y)*(CX-Y)
      RETURN
      END
```

REFERENCES

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CHAPTER III

EXPANSION AND NUMERICAL SOLUTION OF THE GENERAL DISPERSION RELATION FOR SMALL AMPLITUDE PERTURBATIONS IN A THREE-FLUID PLASMA

by

Raymond L. Brown and David L. Murphree

NOTE: Figures, references and equations begin a new sequence in each Chapter. Also, the Appendices are lettered consecutively by Chapter, and each Chapter includes its own List of Symbols.

LIST OF SYMBOLS

C	Velocity of Light in a Vacuum
C_p	Specific Heat at Constant Pressure
C_v	Specific Heat at Constant Volume
e	Charge of Electron
$\vec{E}$	Electrical Field Strength
$\vec{H}$	Magnetic Field Strength
$\vec{H}^o$	Steady Applied Magnetic Field Strength
h	Perturbation Magnetic Field Strength
i	$\sqrt{-1}$
K	Wave Number (K_r = real part; K_i = imaginary part)
$m_{e,i,n}$	Mass of Electron, Ion, and Neutral Particle
m	Mass Ratio of Ions to Electrons
n	Index of Refraction
$N_{e,i,n}$	Number Density of Electrons, Ions, and Neutral Particles
$P_{e,i,n}$	Partial Pressures of the Electron, Ion, or Neutral Particle Gas
$\vec{U}_{e,i,n}$	Acoustic Velocity of Electron, Ion, or Neutral Particle Species
$\vec{U}_a$	Acoustic Velocity of the Entire Gas
$\vec{U}_p$	Acoustic Velocity of the Electron-Ion Gas Mixture
V_a	Alfven Velocity of Entire Gas
V'_a	Alfven Velocity for Charged Particle Fluids
Θ	Angle Between Direction of Wave Propagation and Applied Magnetic Field
$\rho_{e,i,n}$	Mass Density of Electrons, Ions, or Neutral Gas
ω	Applied Frequency of Wave
$\omega_{e,i}$	Electron or Ion Plasma Frequency

ω_c	Cyclotron Frequency of Electron
$\omega_{T,L}$	Cyclotron Frequency of Electrons Associated with Transverse or Longitudinal Component of H
ν_{ei}	Effective Collision Frequency of Electrons with Ions
ν_{en}	Effective Collision Frequency of Electrons with Neutrals
ν_e	Total Collision Frequency of Electrons $(\nu_{ei} + \nu_{en})$
ν_{ie}	Effective Collision Frequency of Ions with Electrons
ν_{in}	Effective Collision Frequency of Ions with Neutrals
ν_i	Total Collision Frequency of Ions $(\nu_{ie} + \nu_{in})$
ν_{ne}	Effective Collision Frequency of Neutrals with Electrons
ν_{ni}	Effective Collision Frequency of Neutrals with Ions
ν_n	Total Collision Frequency of Neutrals $(\nu_{ne} + \nu_{ni})$
γ	Specific Heat Ratio
ω_0	$[\omega_e^2 + (\omega_c^e)^2]^{\frac{1}{2}}$
ω_{01}	$[\omega_e^2 + (\omega_c^e/2)^2 + \omega_c^e\omega_c^i]^{\frac{1}{2}} - \frac{1}{2}\omega_c^e$
ω_{02}	$[\omega_{01} + \omega_c^e]$
ω_{03}	$\omega_i(\omega_c^e\omega_c^i)^{\frac{1}{2}}/(\omega_i^2 + \omega_c^e\omega_c^i)^{\frac{1}{2}}$

Introduction

Tanenbaum and Mintzer,[1] and Tanenbaum and Meskan[2] have conducted an intensive study of wave propagation modes employing the three-fluid theory for a partly ionized gas. Their study of small amplitude oscillations in an infinite, homogeneous, partly ionized gas with a uniform external magnetic field employed Maxwell's equations together with a set of coupled hydrodynamic equations for an interacting mixture of electrons, ions, and neutral molecules to obtain the dispersion relations for wave propagation perpendicular and parallel to the magnetic field. All the work done by Tanenbaum, Mintzer, and Meskan used approximate equations to obtain the possible wave modes for propagation of longitudinal waves parallel to the field and for propagation of coupled longitudinal and transverse waves perpendicular to the field. No attempt, not even approximate, was made to obtain the general dispersion relation for propagation at any angle relative to the magnetic field.

Dahl and Murphree[3] considered the case of longitudinal waves propagating parallel to the magnetic field, but this study also used some approximations.

McClendon and Murphree[4] conducted a study considering the propagation of coupled longitudinal and transverse waves with a transverse magnetic field. This study involved no approximations, and the wave modes obtained were the exact solutions to the dispersion relation for the case of coupled longitudinal and transverse waves with an applied transverse magnetic field.

This paper will present the complete solution of the general dispersion relation using numerical techniques. The solutions of the general dispersion relation, which are the complex wave numbers, are plotted in terms of phase velocity and e-folding distance versus disturbance frequency for the frequency range, $10^{-5} \leq \omega \leq 10^{+9}$ rad/sec. Comparisons with the previous works are included, and also, all discrepancies are noted.

Theory

The main objective of this wave propagation study is the determination of the phase velocities and e-folding distances for all possible wave modes versus the disturbance frequency.

The approach used in this study consisted of the small perturbation theory applied to a three-fluid partly ionized gas with the three fluids being electron, ion, and neutral gases.

The assumptions were made as follows:

1. Fixed degree of ionization
2. Adiabatic
3. All gases obey Ideal Gas Law
4. The frictional forces between the gases, which cause damping effects, allow for the conservation of momentum of the total system.

These assumptions are reasonable for any plasma which is near equilibrium and not too dense.[1]

The plasma can now be described completely by using Maxwell's equations, the conservation of mass, momentum, and energy equations for each species of gas (electron, ion, and neutral), and the equations of state for each gas.

(1) Maxwell's Equations;

$$\nabla X \vec{E} = - \frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

$$\nabla X \vec{H} = \frac{4\pi e}{c} (N_i \vec{V}_i - N_e \vec{V}_e) + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

(2) Continuity Equation for each gas,

$$\frac{D\rho_{e,i,n}}{Dt} = -\rho_{e,i,n}\, \nabla \cdot \vec{V}_{e,i,n}$$

(3) Momentum Equation for each gas,

$$\frac{D\vec{V}_e}{Dt} = -\frac{e}{m_e}\left(\vec{E} + \frac{\vec{V}_e X \vec{H}}{c}\right) - \frac{\nabla P_e}{\rho_e} - \nu_{ei}(\vec{V}_e - \vec{V}_i) - \nu_{en}(\vec{V}_e - \vec{V}_n)$$

$$\frac{D\vec{V}_i}{Dt} = \frac{e}{m_e}\left(\vec{E} + \frac{\vec{V}_i X \vec{H}}{c}\right) - \frac{\nabla P_i}{\rho_i} - \nu_{ie}(\vec{V}_i - \vec{V}_e) - \nu_{in}(\vec{V}_i - \vec{V}_n)$$

$$\frac{D\vec{V}_n}{Dt} = -\frac{\nabla P_n}{\rho_n} - \nu_{ne}(\vec{V}_n - \vec{V}_e) - \nu_{ni}(\vec{V}_n - \vec{V}_i)$$

(4) Adiabatic Condition (Ideal Gas),

$$P_{e,i,n}\, N_{e,i,n}^{-\gamma} = \text{CONSTANT}$$

where, $\gamma = C_p/C_v$

To obtain the dispersion relation, a small periodic oscillation of frequency ω is applied to the plasma, and the co-ordinate system is aligned such that propagation is in the X-direction and the applied magnetic field is given by $\vec{H}^o = (H_L^o, H_T^o, 0)$. To include the periodic oscillations applied to the plasma, the variables are put in the form,

$$\vec{E}(\vec{r},t) = \vec{E}\, e^{i(kx - \omega t)}$$

$$\vec{H}(\vec{r},t) = \vec{H}^o + h\, e^{i(kx - \omega t)}$$

$$\vec{V}_{e,i,n}(\vec{r},t) = \vec{V}_{e,i,n}\, e^{i(kx - \omega t)}$$

$$N_{e,i}(\vec{r},t) = N_o + n_{e,i}\, e^{i(kx - \omega t)}$$

$$N_n(\vec{r},t) = N_1 + n_n\, e^{i(kx - \omega t)}$$

$$P_{e,i}(\vec{r},t) = P_o + p_{e,i}\, e^{i(kx - \omega t)}$$

$$P_n(\vec{r},t) = P_1 + p_n\, e^{i(kx - \omega t)}$$

Now, substituting successively each equation into the other, the twenty-one equations with twenty-one unknowns can be reduced to three equations with three unknowns. For the present study, the variables retained are V_{ex}, V_{ey}, and V_{ez}. The resulting equations are given below, and Appendix A contains the substitution procedure followed in obtaining these final three equations which were derived previously by Tanenbaum and Mintzer.[1]

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} V_{ex} \\ V_{ey} \\ V_{ez} \end{pmatrix} = 0$$

where,

$$A_{11} = (c_1/m) - (c_3\ c_5/c_1) + (\omega^2\omega_T^2/mc_2)$$

$$A_{12} = A_{21} = -\ \omega^2\omega_T\omega_L/mc_2$$

$$A_{13} = i\omega\omega_T[(c_3/c_1) - (c_6/mc_2)]$$

$$A_{22} = (c_2/m) - (c_4\ c_6/c_2) + (\omega^2\omega_L^2/mc_2)$$

$$A_{23} = -A_{32} = -\ (i\omega\omega_L/c_2)\ [c_4 - (c_6/m)]$$

$$A_{31} = i\omega\omega_T[(c_5/mc_1) - (c_4/c_2)]$$

$$A_{33} = A_{22} + (\omega^2\omega_T^2/mc_1)$$

and the c's are given by,

$$c_1 = \omega_e^2 - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_2)$$

$$c_2 = \omega_e^2(1 - n^2)^{-1} - i\omega\nu_{ei} + \omega^2(\nu_{en}\nu_{ni}/\gamma_1)$$

$$c_3 = \omega^2 - \omega_i^2 - K^2U_i^2 + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_2)$$

$$c_4 = \omega^2 - \omega_i^2(1 - n^2)^{-1} + i\omega\nu_i + \omega^2(\nu_{in}\nu_{ni}/\gamma_1)$$

$$c_5 = \omega^2 - \omega_e^2 - K^2U_e^2 + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_2)$$

$$c_6 = \omega^2 - \omega_e^2(1 - n^2)^{-1} + i\omega\nu_e + \omega^2(\nu_{en}\nu_{ne}/\gamma_1)$$

where,

$$\gamma_1 = \omega^2 + i\omega\nu_n \qquad \omega_{T,L} = e\, H^o_{t,L}/m_e c$$

$$\gamma_2 = \gamma_1 - K^2U_n^2 \qquad U_{e,i,n} = (\gamma P_{e,i,n}/\rho_{e,i,n})^{\frac{1}{2}}$$

$$\omega_{e,i} = \left(\frac{4\pi e^2 N_o}{m_{e,i}}\right)^{\frac{1}{2}} \qquad n = Kc/\omega$$

For a non-trivial solution to exist for the matrix equation,

$$\underset{\sim}{A}\,\underset{\sim}{V_e} = 0 \tag{1}$$

the determinant of the coefficients must be zero.

$$|A| = 0 \tag{2}$$

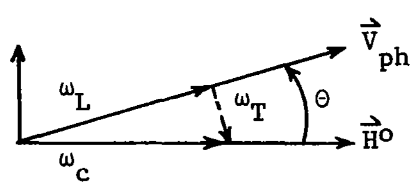
Expansion of the determinant $|A|$ yields the following dispersion relation in terms of the c's (See Appendix B for expansion procedure of $|A|$.),

$$\begin{aligned}
&2m^2c_2^2c_3c_4c_5c_6 - m^3c_3c_4^2c_5c_6^2 - mc_2^4c_3c_5 - 2mc_1^2c_2^2c_4c_6 + m^2c_1^2c_4^2c_6^2 \\
&+ c_1^2c_2^4 + 2\omega^2\omega_T^2c_1c_2^3 - \omega^2\omega_T^2c_2^2c_5c_6 + m\omega^2\omega_T^2c_4c_5c_6^2 - m^2\omega^2\omega_T^2c_2^2c_3c_4 \\
&- 2m\omega^2\omega_T^2c_1c_2c_4c_6 + m^3\omega^2\omega_T^2c_3c_4^2c_6 + 2\omega^2\omega_L^2c_1^2c_2^2 - \omega^2\omega_L^2c_1^2c_6^2 \\
&- m^2\omega^2\omega_L^2c_1^2c_4^2 - 2m\omega^2\omega_L^2c_2^2c_3c_5 + m\omega^2\omega_L^2c_3c_5c_6^2 + m^3\omega^2\omega_L^2c_4^2c_5c_3 \\
&+ 2\omega^4\omega_T^2\omega_L^2c_1c_2 - m\omega^4\omega_T^2\omega_L^2c_4c_5 - m\omega^4\omega_T^2\omega_L^2c_3c_6 + \omega^4\omega_T^4c_2^2 - m\omega^4\omega_T^4c_4c_6 \\
&+ \omega^4\omega_L^4c_1^2 - m\omega^4\omega_L^4c_3c_5 = 0
\end{aligned} \tag{3}$$

When fully expanded, by putting in the equations for the c's, the full dispersion relation would contain thousands of terms, and therefore, it is impractical, and probably impossible, to write out in full. But examination of the equations for the c's and the dispersion relation shows that it is an eight-ordered equation in K-square, and therefore, theoretically eight wave solutions are possible.

The general dispersion relation which has been obtained is a relation for the wave number (K) in terms of the disturbance frequency (ω) for a given set of field conditions and for propagation at some given angle relative to the magnetic field direction. By solving for K as a function of ω, the phase velocity (ω/K_r) and e-folding distance ($1/K_i$) can be obtained for all possible modes of propagation at any angle relative to the magnetic field.

The direction of propagation can be controlled by the longitudinal and transverse components of the cyclotron frequency. This is accomplished by varying the angle Θ between the direction of propagation and the direction of the applied magnetic field.



$$\omega_c = \frac{e\,\vec{H}^o}{c\,m_e} \qquad \omega_T = \omega_c \mathrm{SIN}(\Theta) \qquad \omega_L = \omega_c \mathrm{COS}(\Theta)$$

Once the dispersion relation has been solved, the fluctuations in $\vec{V}_e$ are known for given time and position by using the following relations,

$$V_{ex} = V_{ex})_o \; e^{-iK_i x} \; \mathrm{COS}(K_r x - \omega t)$$

$$V_{ey} = V_{ey})_o \; e^{-iK_i x} \; \mathrm{COS}(K_r x - \omega t)$$

$$V_{ez} = V_{ez})_o \; e^{-iK_i x} \; \mathrm{COS}(K_r x - \omega t)$$

Where, $V_{ex})_o$, $V_{ey})_o$, and $V_{ez})_o$ are initial values.

Appendix C contains the equations relating the other parameters ($\vec{H}$, $\vec{E}$, $N_{e,i,n}$, V_{ix}, V_{iy}, V_{iz}, V_{nx}, V_{ny}, V_{nz}) to the known solutions of V_{ex}, V_{ey}, and V_{ez}.

Discussion

A. Problems Involved

Referring to the dispersion relation (3), it can be seen that two major problems will be encountered when an attempt is made to obtain the wave number (K) in terms of the disturbance frequency (ω) for given field conditions. The first obstacle is the expansion of the equation, which up until the present has been considered almost impossible, and the second obstacle is the solution of the eighth-ordered equation once the expansion has been accomplished.

From a cursory observation of the equation this would not seem to be as complex as might have been indicated above, but a closer observance shows that it is a totally impossible problem to approach by hand or with conventional computer techniques.

To explain why the afore mentioned approach cannot be used, consider the first term of the general dispersion relation,

$$2m^2 c_2^2 c_3 c_4 c_5 c_6 \quad .$$

Since c_2 contains eight separate terms containing several different variables, then c_2^2 will contain at least thirty-six separate terms. Also, c_3 and c_5 contain thirteen terms a piece, and c_4 and c_6 each contain twelve terms. Allowing for the fact that a few of the terms in the different c's might be the same, a conservative estimate of the number of terms that would be present if the above expression was completely expanded is one hundred thousand (100,000) terms. Consider also that there are sixteen different variables present, and any one term of the 100,000 terms could contain one or more of these variables. This should be satisfactory in explaining why the general dispersion relation was never obtained, or for that matter, could ever be obtained by a person writing it out by hand. While the immensity of the equation is still being considered, it can be seen by careful observation that even computers with the most advance high speed printers would take at least ten hours and fifteen thousand sheets of paper to print out the equation in its entirety. An equation of this magnitude could only serve to obscure any information which might be beneficial, and therefore, the total dispersion relation will never be printed out in the expanded form.

Since the total dispersion relation containing the sixteen different variables cannot be written out completely as at first considered, the next approach to be considered was to put in all values for the field parameters except ω, ω_T, and ω_L. This allows for the formation of a dispersion relation for a given set of field conditions, while retaining the ability to vary the disturbance frequency and direction of propagation relative to the magnetic field direction.

Still the final dispersion relation would contain too many terms to consider expansion by hand or by use of conventional computer programming languages which require that all variables be assigned a numerical value. The problem of how to expand the dispersion while retaining some of the variables was solved when the PL/1 FORMAC SYMBOLIC MATH FORMULA-MANIPULATION INTERPRETER was obtained from IBM Corporation. The capabilities of this system can best be explained by use of a direct quote from the FORMAC language manual[5]:

> The PL/1-FORMAC interpreter is an extension of the OS/360 PL/1 (F) Compiler. It consists of two modules of assembled routines, each module having about 70K bytes, which are added to a Systems Subroutine Library. FORMAC provides for the symbolic manipulation of mathematical expressions; e.g., the expression SIN(X), can be differentiated resulting in the expression COS(X). Expressions can contain variables, user-defined functions, constants to 2295 digits, and symbolic constants representing pi, e, and i (the square root of -1), as well as functions such as SIN, COS, EXP, etc. Expressions can be differentiated,

evaluated, replaced, compared, and parsed. Since FORMAC is a superset of PL/1, the facilities of PL/1 are available for program structure, loop control, I/O, etc. FORMAC can be used to obtain symbolic solutions in problem areas which heretofore could only be approached numerically.

B. FORMAC Program for Expanding Dispersion Relation

As explained in the quote from the FORMAC manual, the use of FORMAC is confined to the OS/360 IBM Computer, and PL/1 facilities are used for input-output and all loop control and program structure.

The IBM OS/360-40 at Mississippi State University was used for all FORMAC runs. Due to the limited storage space at this facility the program was not the most efficient program, as far as time was concerned.

The complete print-out of DISREL, the dispersion relation expanding program, is given in Appendix D, and a brief outline of the program follows.

DISREL (FORMAC EXPANSION PROGRAM)

(1) Read in all field parameters except cyclotron frequency and disturbance frequency as PL/1 variables.

(2) Multiply by length and time factors to obtain minimum range for coefficients K^n's in the dispersion relation.

(3) Transfer all PL/1 variables to FORMAC variables.

(4) Define as variables all repeated multiplications, etc., to make program more efficient.

(5) Obtain equations for numerators of the c's.

(6) Obtain equations for denominators of the c's.

(7) Define additional repeated terms to make program more efficient.

(8) Obtain each of the twenty-five terms of the dispersion

relation. ω, ω_T, and ω_L are still undefined numerically.

(9) Atomize all variables which are not needed for further computation to release the storage space which they occupy in the computer.

(10) Obtain cyclotron frequency and multiply by time factor.

(11) Obtain ω_T and ω_L for given angle of propagation with respect to magnetic field direction.

(12) Obtain each of the twenty-five terms of the dispersion relation with all variables now having numerical values and sum to give total dispersion relation.

(13) Arrange coefficients of dispersion relation in descending order of powers of K.

(14) Punch out the coefficients to be used in program ROOTS.

The units used for expressing field parameters were MKS, but leaving the variables in terms of meters and seconds usually gave a very large range for the coefficients of the K^n's. For the frequencies 10^{-5} to 10^{-2} and 10^6 to 10^9, the range from the smallest to the largest coefficient was of the order 10^{60}, and of course, this would present quite a problem when trying to solve for the roots of the equation. With careful selection of time and length multiplication factors, the range of the coefficients was decreased down to the order of 10^{14} to 10^{28}, depending on disturbance frequency.

The equations for the numerator and denominator of each of the c's was obtained separately since the final dispersion relation was to be multiplied through by a common denominator to eliminate all denominator terms.

Each of the twenty-five terms of the general dispersion relation was then obtained in expanded form, but ω, ω_T, and ω_L were not defined numerically. Then for each given ω over the frequency range 10^{-5} to 10^{9}, the angle of propagation relative to the magnetic field, Θ, was varied from 0^{o} to 90^{o}. ω_T and ω_L were then determined since they are dependent on the angle of propagation. With the defining of ω, ω_T, and ω_L over the ranges given above, each of the twenty-five terms of the general dispersion relation was obtained with all numerical values except K, the wave number for whose value the dispersion relation is to be solved. An example term is given below in both forms to show the importance of the FORMAC capabilities.

$$
\begin{aligned}
\text{TERM(13)} = {} & -.115\text{E-}09\ \#\text{I}\ \text{WL}^2\ \text{K}^2\ -.171\text{E-}04\ \text{WL}^2\ \text{K}^2\ +5.17\ \#\text{I}\ \text{WL}^2\ \text{K}^4 \\
& +767985.\ \text{WL}^2\ \text{K}^4\ +964289.\ \#\text{I}\ \text{WL}^2\ \text{K}^6\ -1112279.\ \text{WL}^2\ \text{K}^6 \\
& -1396589.\ \#\text{I}\ \text{WL}^2\ \text{K}^8\ +100037.\ \text{WL}^2\ \text{K}^8\ +505672.\ \#\text{I}\ \text{WL}^2\ \text{K}^{10} \\
& +438393.\ \text{WL}^2\ \text{K}^{10}\ -.792\ \#\text{I}\ \text{WL}^2\ \text{K}^{12}\ -158732.\ \text{WL}^2\ \text{K}^{12} \\
& +.638\text{E-}21\ \#\text{I}\ \text{WL}^2\ +.948\text{E-}16\ \text{WL}^2
\end{aligned}
$$

$$
\begin{aligned}
\text{FTERM(13)} = {} & -.924\text{E-}04\ \#\text{I}\ \text{K}^2\ +4158831.\ \#\text{I}\ \text{K}^4\ +7.75\text{E+}11\ \#\text{I}\ \text{K}^6\ -1.12\text{E+}12 \\
& \#\text{I}\ \text{K}^8\ +4.06\text{E+}11\ \#\text{I}\ \text{K}^{10}\ -637647.\ \#\text{I}\ \text{K}^{12}\ -13.7\ \text{K}^2\ +6.2\text{E+}11 \\
& \text{K}^4\ -8.94\text{E+}11\ \text{K}^6\ +8.04\text{E+}10\ \text{K}^8\ +3.52\text{E+}11\ \text{K}^{10}\ -1.27\text{E+}11 \\
& \text{K}^{12}\ +.513\text{E-}15\ \#\text{I}\ +.762\text{E-}10
\end{aligned}
$$

With the summation of the terms, the dispersion relation for a given set of field conditions at a given angle to the magnetic field is obtained for some given disturbance frequency. For example, consider the dispersion relation for propagation parallel to the magnetic field for the case $\omega = 10^2$,

$$\begin{aligned}\text{DISPER} = {} & 22161.\ \#I\ K^2\ +4.26E{+}08\ \#I\ K^4\ +1.04E{+}14\ \#I\ K^6\ +3.8E{+}18\ \#I \\ & K^8\ -5.42E{+}18\ \#I\ K^{10}\ +1.96E{+}18\ \#I\ K^{12}\ +1.22E{+}12\ \#I\ K^{14} \\ & +274263.\ \#I\ K^{16}\ +9496042.\ K^2\ +2.55E{+}12\ K^4\ +4.49E{+}16\ K^6 \\ & +6.66E{+}21\ K^8\ -9.46E{+}21\ K^{10}\ +3.49E{+}21\ K^{12}\ +2.17E{+}15\ K^{14} \\ & +4.88E{+}08\ K^{16}\ +.0017\ \#I\ +261.112\end{aligned}$$

This equation can be treated as an eighth-ordered polynomial in terms of K-square when extracting the roots, instead of a sixteenth-ordered equation in K.

C. ROOTS, Program Used to Solve Dispersion Relation

Having expanded the dispersion relation and obtained an eighth-ordered equation in terms of K-square, the only remaining step is to solve for the roots of the equation.

In selecting a method for solution of the dispersion relation and in using computer defined functions, the large range of the coefficients becomes the dominating feature. Taking into account this large and variable range, and also considering that the dispersion relation has both complex coefficients and roots, the method of solution chosen was the Newton-Raphson Iteration Technique.[6] Since both overflow and underflow occurred after only a few iterations, no computer defined functions were used, and all work was done in the double precision mode.

The entire program with all the subroutines is given in Appendix E, and the main program ROOTS can be explained briefly as follows:

ROOTS (Root Extracting Program)

(1) Read in all coefficients of dispersion relation.

(2) Read in time and length factors to be used to change units back to meters and seconds once the root has been found.

(3) Read in first root estimate.

(4) Call subroutine for $f(x)$ and $f'(x)$, depending on order of equation. $(x = K^2)$

(5) Using the $f(x)$ and $f'(x)$, a new estimate is obtained using the Newton-Raphson formula,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

(6) The above iteration is continued until the relative error is less than or equal to 10^{-20}.

(7) Call square-root program to obtain solution to K^2.

(8) Multiply by appropriate length factor so that K_r and K_i will have meters as unit of length.

(9) Obtain phase velocity (ω/K_r) and e-folding distance $(1/K_i)$.

(10) Call synthetic division subroutine and divide out root.

(11) Repeat above procedure starting with step (3) until all eight roots have been obtained.

There are ten subroutines used in association with the main program and the function of each is as follows:

1. ATWLTF - Arranges terms in ascending order. This allows for the retaining of as many significant digits as possible and is therefore very beneficial due to the large and varied ranges of values obtained from the dispersion relation.

2. XPOCPX - Calculates the square-root of a complex number. It is used to obtain K_1 and K_2, once K-square has been obtained by Newton-Raphson method.

3. SYNDV - Synthetic division subroutine used to divide out each root as it is obtained. This reduces the order of the equation and assures that the Newton-Raphson iteration will not continue to iterate back to the same root.

4. SECOND, THIRD, FOURTH, FIFTH, SIXTH, SEVENTH, EIGHTH - Subroutines used to obtain f(x) and f'(x) depending on the order of the equation being solved.

Results

The expansion and solution of the general dispersion relation was accomplished by employing the IBM 360-40 Computer for the FORMAC program (DISREL) and the UNIVAC 1106 Computer for the program ROOTS.

The ionospheric field conditions considered for this study were,

$\nu_{ni} = 1.1202 \times 10^{-4}$ coll/sec	$\omega_e = 2.8806 \times 10^{7}$ rad/sec
$\nu_{ne} = 1.9375 \times 10^{-7}$ coll/sec	$\omega_i = 1.5731 \times 10^{5}$ rad/sec
$\nu_{in} = 2.2541 \times 10^{-1}$ coll/sec	$\omega_c^e = 8.968 \times 10^{6}$ rad/sec
$\nu_{ie} = 1.7263 \times 10^{-3}$ coll/sec	$\vec{H} = 5.1 \times 10^{-5}$ webers/m^2
$\nu_{ei} = 5.7783 \times 10^{1}$ coll/sec	$U_{i,n} = 8.5097 \times 10^{2}$ m/sec
$\nu_{en} = 1.3072 \times 10^{1}$ coll/sec	$U_e = 2.8158 \times 10^{5}$ m/sec

These are the same conditions employed by Dahl and Murphree[3] and McClendon and Murphree[4], but they do not coincide with the field conditions used in Tanenbaum and Mintzer's[1] and Tanenbaum and Meskan's[2] qualitative and approximate analysis.

Before a strict analysis is made of the results of this study, a brief explanation is needed in regards to the plots shown in Figures 1 through 14. The curves in these figures are shown to be relatively smooth, but at some angles, especially at $\Theta = 0^{o}$, there were several e-folding distance points which were considerably different, and the phase velocity curves had one or two points that varied from the curves shown for $\Theta = 30^{o}$ and 45^{o}. The wave solutions whose phase velocities and e-folding distances varied from the curves shown were usually the sixth, seventh, or eighth root extracted from the dispersion relation, but in some cases the roots which were extracted third or fourth gave an erroneous e-folding distance. Since all numbers involved are quite large and varied, and since only five digits were retained from the FORMAC expansion program, it is possible that the error build-up in the program ROOTS is too extensive. The possibility that the points thought to be erroneous could be correct also exists since most occurred near significant points, such as collision frequencies, cyclotron frequencies, and electron or ion plasma frequency. So to make a statement concerning these points which do not coincide with the curves shown would require that additional dispersion relations be obtained for both the ionospheric field conditions used in this study and for other different ionospheric field conditions.

Another point to be made before discussing the results is that all roots which yielded a negative K_i and a positive K_r were omitted. The reason being that this represents a wave with increasing amplitude, and since our system assumed constant energy, this result is not physically possible. Only the physically possible wave modes are shown, and in all cases, the three or four roots which resulted in physically impossible wave modes were complex conjugate of the acceptable modes of propagation shown in Figures 1 through 14.

A. <u>Propagation Parallel to Magnetic Field Direction ($\Theta = 0^{o}$)</u>

The wave solutions presented consist of both the pure longitudinal and the pure transverse wave modes with an applied longitudinal magnetic field. The term dominant wave mode refers to the modes of propagation whose e-folding distances are much greater than the e-folding distances of the other wave modes. Figures 1 and 2 present the solutions.

$\underline{\omega \leq \nu_{ie}}$: There are five possible wave solutions in this range, and Mode I can be neglected since it has an extremely small e-folding distance and phase velocity. The other four solutions are really two double roots, with Mode II increasing up to the phase velocity of $U_{i,n}$. Mode III, the other double root, increases up to the phase velocity equal to the acoustic velocity of the electron-ion gas mixture (U_p).

$\underline{\nu_{ie} < \omega < \nu_{ei}}$: Five solutions exist, and Mode I, the single root solution, can still be neglected due to small e-folding distance. The double root solution, Mode II, with phase velocity equal to $U_{i,n}$ is now the dominant wave mode since the e-folding distance for Mode III has decreased significantly. Mode III is a double root solution and the phase velocity has increased from U_p up to V_a'.

$\underline{\nu_{ei} < \omega < \omega_i}$: Only four possible wave modes exist within this range. Mode I, the solution neglected in the previous ranges, can still be neglected due to small e-folding distance. Mode II, the double root solution with phase velocity $U_{i,n}$, is just a single root in this range, and it is the dominant wave mode. Mode III, the other double root solution, has now split into two separate solutions, but neither wave solution is significant due to small e-folding distances. These are Mode III and Mode IV.

$\underline{\omega_i < \omega < \omega_e}$: Again five wave solutions exist, and although the phase velocity has increased tremendously for Mode I, it can still be neglected since the e-folding distance is small. Mode II, the double root solution with phase velocity $U_{i,n}$, exists, but one wave solution has a very small e-folding distance and can be neglected. The other two roots which exist consist of Mode III, whose phase velocity levels off at the speed of light, and Mode IV, whose phase velocity approaches the speed of light. Mode IV can be neglected due to small e-folding distance.

$\underline{\omega > \omega_e}$: Mode I, the dominant wave mode for this range, is propagating at the acoustic electron velocity (U_e). Mode III and Mode IV have phase velocities equal to the speed of light. Mode II, a double root solution with phase velocity $U_{i,n}$, also exists, and one wave can be neglected due to small e-folding distance.

B. <u>Propagation at Acute Angle to Magnetic Field Direction ($0^\circ < \Theta < 90^\circ$)</u>

All of the modes of propagation for the angles shown in Figures 3 through 12 are quite similar, and therefore, only one case need be discussed. The case to be considered is $\Theta = 45^\circ$.

$\underline{\omega < \nu_{ie}}$: Five wave solutions exist. Mode I, a single root solution, can be neglected due to extremely small phase velocity and e-folding distance. Mode II, one double root solution, levels off at phase velocity $U_{i,n}$, while Mode III, the other double root solution, increases in phase velocity to U_p.

$\underline{\nu_{ie} < \omega < \nu_{en}}$: Same set of solutions exist as in previous range. Mode I, the single root solution, can still be neglected. Mode II, the double root solution with phase velocity $U_{i,n}$, is the dominant mode of propagation. Mode III, the other double root solution, has an increase in phase velocity from U_p to V'_a and a decrease in e-folding distance.

$\underline{\nu_{en} < \omega < \omega_i}$: Four wave solutions exist, and Mode I, the single root solution, can still be neglected due to small e-folding distance. Mode II, the double root solution with phase velocity $U_{i,n}$, has become a single root solution and has a large decrease in e-folding distance. Mode III, the double root solution starting with phase velocity V'_a, has split into two separate solutions which have small e-folding distances. These are Mode III and IV.

$\underline{\omega_i < \omega < \omega_e}$: Four wave solutions exist, and Mode I, the single root solution which has been neglected in all previous ranges, is still negligible due to small e-folding distance. Mode II, the single root solution with phase velocity $U_{i,n}$, is also negligible. Mode IV, the solution with phase velocity greater than the speed of light, is also damped out. Mode III, the only solution that is not damped out, levels off at phase velocity equal to the speed of light.

$\underline{\omega > \omega_e}$: Four solutions exist, and Mode I, the single root solution neglected for $\omega < \omega_e$, has leveled off at a phase velocity of U_e. The Mode II wave with phase velocity of $U_{i,n}$ is damped out. Mode III and Mode IV exist with phase velocity equal to the speed of light.

C. Propagation Perpendicular to the Magnetic Field Direction ($\Theta = 90^{\circ}$)

The wave mode solutions for the case of an applied transverse magnetic field consist of both the pure transverse wave and the coupled longitudinal and transverse wave. Figures 13 and 14 present the solutions.

$\underline{\omega \leq \nu_{ie}}$: Seven possible wave solutions exist, and six of these are given by three sets of double roots. Mode I, the single root solution, can be neglected since both the phase velocity and e-folding distance are small. Mode II, the double root solution which levels off at phase velocity $U_{i,n}$, is the most dominant wave mode. The other two sets of double root solutions, Mode III and Mode IV, have increasing phase velocity and decreasing e-folding distance as ω increases.

$\underline{\nu_{ie} < \omega < \nu_{in}}$: Seven wave solutions exist, and Mode I, the single root solution, can be neglected in this range. Mode II, the double root solution with phase velocity $U_{i,n}$, is still the dominant wave mode. Mode III, the double root solution which increases in phase velocity to U_p, is also a prominent mode of propagation in this range. Mode IV, the third double root solution, can be neglected since the e-folding distance has become quite small.

$\underline{\nu_{in} < \omega < \nu_{ei}}$: Only six solutions exist, and as in the previous ranges, Mode I is negligible. Mode II, the wave solution with phase velocity $U_{i,n}$, is a single root solution in this range. Mode III, a double root solution, has an increase in phase velocity from U_p to greater than V_a'. Mode IV, a double root solution, is still negligible.

$\underline{\nu_{ei} < \omega < \omega_i}$: Five possible wave mode solutions exist, and Mode I, the single root solution, is still negligible. Mode IV is the only double root solution which still exists, but it is negligible. Mode II, the wave solution with phase velocity $U_{i,n}$, and Mode III, the wave solution with phase velocity greater than the speed of light, will both be damped out. But they are more dominant than the other wave modes.

$\underline{\omega_i < \omega < \omega_e}$: Six wave solutions are possible. The Mode II wave propagating at velocity $U_{i,n}$ becomes a double root solution, but one wave can be neglected due to small e-folding distance. Mode I and Mode IV, the single and double root solutions neglected previously, can still be neglected due to small e-folding distances. Mode III is dominant.

$\underline{\omega > \omega_e}$: Five solutions exist. Mode II is the only double root solution, and one wave solution can be neglected since it is only a standing wave. Mode II has phase velocity of $U_{i,n}$. Mode III, the solution with phase velocity equal to the speed of light, still exists.

Mode IV, the double root solution neglected previously, is now a single root solution with phase velocity equal to the speed of light. Mode I, the solution which has been neglected for all $\omega < \omega_e$, exists and has phase velocity of U_e.

D. Comparison of Results Obtained by Others and the Present Study

Figures 15 through 20 show the phase velocities of the wave solutions which exist. All other modes of propagation have been neglected due to extreme damping or some other condition, such as increasing wave amplitude in constant energy system.

Referring to Figures 15, 16, and 17, a comparison of the results for the case of longitudinal waves propagating parallel to the magnetic field can be made. Figure 15 shows the physically possible wave modes which are not damped out as found by Tanenbaum and Mintzer,[1] and Tanenbaum and Meskan.[2] Figure 16 shows the results obtained by Dahl and Murphree,[3] and Figure 17 shows the results of this study.

$\underline{\omega \le \nu_{in}}$: Mode I has phase velocity increasing up to $U_{i,n}$ for all three studies. Mode II has phase velocity increasing up to U_p for Tanenbaum's, et al.,[1,2] study and Dahl's, et al.,[3] study, but increases to a velocity slightly greater than U_p for the present study.

$\underline{\nu_{in} < \omega < \omega_i}$: Mode I for all three studies has phase velocity remaining at $U_{i,n}$. Mode II phase velocity remains at U_p for Tanenbaum's, et al.,[1,2] study and Dahl's, et al.,[3] study, but increases up to $U > U_e$ for the present study.

$\underline{\omega \approx \omega_i}$: Mode I exists in all three studies and the phase velocity is equal to $U_{i,n}$. Mode II exists only in the solution by Dahl, et al.,[3] and the phase velocity is still equal to U_p.

$\underline{\omega_i < \omega < \omega_e}$: Mode II phase velocity decreases down to $U_{i,n}$, which is the same phase velocity as Mode I which still exists. This is true for all three studies.

$\underline{\omega > \omega_e}$: Mode I and Mode II exist for all three studies and have phase velocity equal to $U_{i,n}$. Also, Mode III exists for all three studies and has phase velocity decreasing down to and leveling off at U_e, the acoustic velocity of the electron gas.

Referring to Figures 18, 19, and 20, a comparison can be made of the results obtained by Tanenbaum and Mintzer,[1] McClendon and Murphree,[4] and the present study for the case of coupled longitudinal and transverse waves propagating perpendicular to the magnetic field.

$\underline{\omega \leq \nu_{in}}$: Mode I exists for all three studies and has phase velocity which increases up to and levels off at $U_{i,n}$. Mode II also exists in all three studies and for Tanenbaum's, et al.,[1] study the phase velocity increases from V_a up to V_a'. For McClendon's, et al.,[4] study and the present study the phase velocity increases up to a value less than $U_{i,n}$.

$\underline{\nu_{in} < \omega < \omega_i}$: Mode I has phase velocity of $U_{i,n}$ for all three studies. Mode II for Tanenbaum's, et al.,[1] study has phase velocity of V_a'. For McClendon's, et al.,[4] study the phase velocity also increases up to and levels off at V_a'. For the present study the phase velocity increases up to V_a', but for $\omega_i > \omega > 10$, this mode of propagation is damped out.

$\underline{\omega \approx \omega_i}$: Mode I still has phase velocity of $U_{i,n}$ for all three studies. Mode II phase velocity decreases down to $U_{i,n}$ for all three studies.

$\underline{\omega_i < \omega < \omega_e}$: Mode I and Mode II exist for all three studies and have phase velocity of $U_{i,n}$. Mode III exists only for the present study and has phase velocity which decreases down to and levels off at the speed of light.

$\underline{\omega > \omega_e}$: Modes I and II for all three studies still have phase velocity of $U_{i,n}$. Mode III has phase velocity equal to the speed of light for the present study. For Tanenbaum's, et al.,[1] study and McClendon's, et al.,[4] study, Mode III has phase velocity decreasing down to and leveling off at the speed of light. Mode IV for all three studies has phase velocity decreasing down to and leveling off at U_e.

Conclusion

The significance of this study is that the general dispersion relation considering small amplitude oscillations in a three-fluid medium has been expanded and solved without any approximations being made. Prior to this study, no solution, approximate or otherwise, had been obtained for the general dispersion relation.

With this capability of being able to solve equations of extreme algebraic complexity, the only obstacle which remains in the way of adding additional conditions to the original continuity, momentum, and energy equations is the tedious substitution procedure needed to eliminate all variables until only three equations with three unknowns remain. Any of the variables could be retained, and in this study, the three variables were the components of the electron fluid velocity.

With the solution of the general dispersion relation for the complex wave number, the modes of wave propagation are known for any condition desired. The real and imaginary parts of the wave number provide a complete description of the wave propagation by giving the velocity and damping characteristics of each wave solution.

Round-off error, due to the sources mentioned previously, exists in the results of this study, but most of these inaccuracies can be eliminated by use of more efficient and accurate computer techniques.

Figures 15 through 20 show there is very good agreement between the results of this study and the work done previously by Tanenbaum, et al.,[1,2] Dahl, et al.,[3] and McClendon, et al.,[4] for the solutions which exist and have large e-folding distances. Exact agreement was not expected since there was error build-up in the method employed in this study and in McClendon's, et al.,[4] work. Although no error build-up should occur for the methods employed by Tanenbaum, et al.,[1,2] and by Dahl, et al.,[3] the dispersion relation which they solved was an approximation. The equation used by Dahl, et al.,[3] and by Tanenbaum, et al.,[1,2] is a third ordered equation in K-square. The exact dispersion relation is a fourth ordered equation in K-square and has quite a number of terms more than those shown in the dispersion relation used by both Tanenbaum, et al.,[1,2] and Dahl, et al.[3]

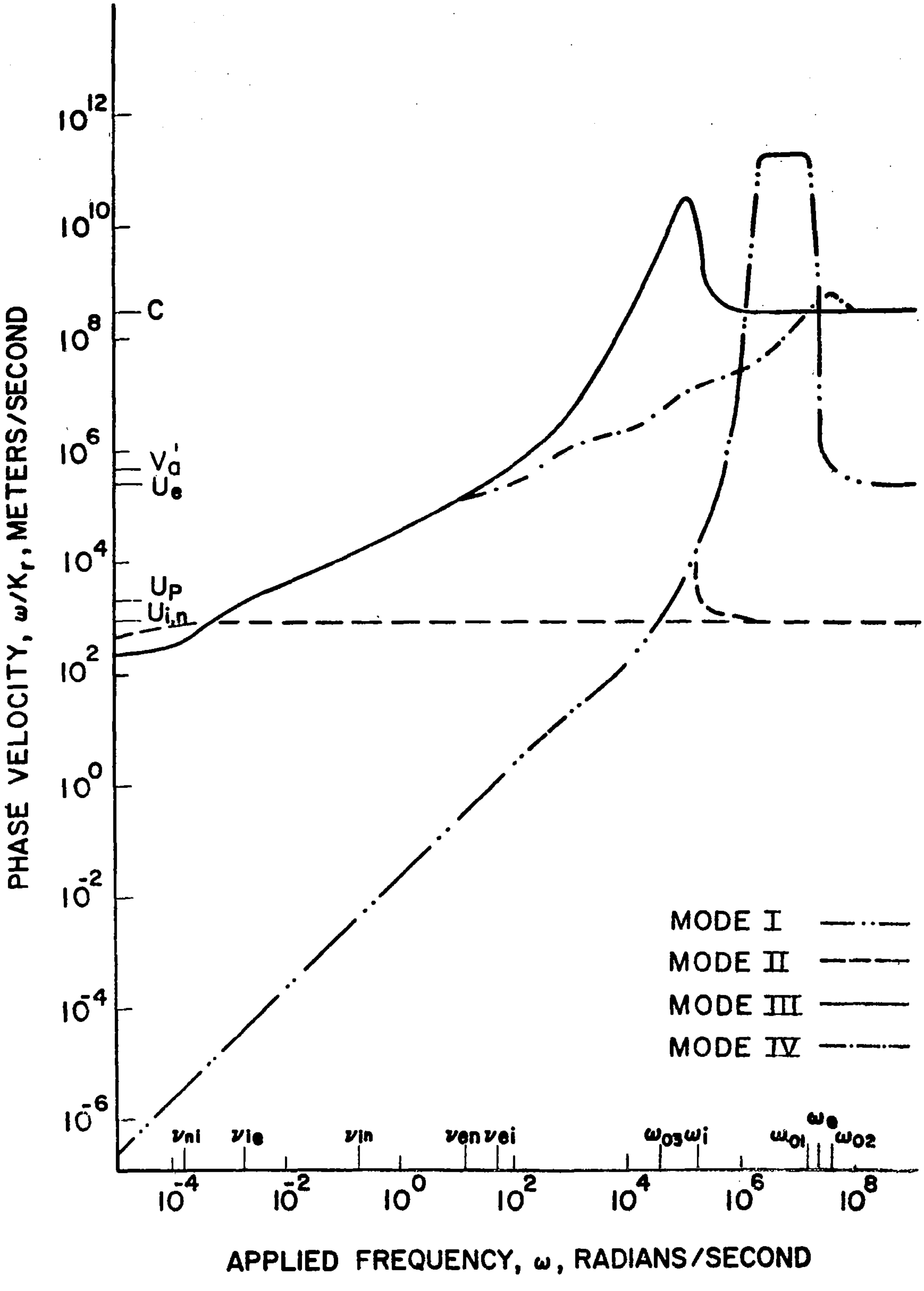


FIGURE 1. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 0°$

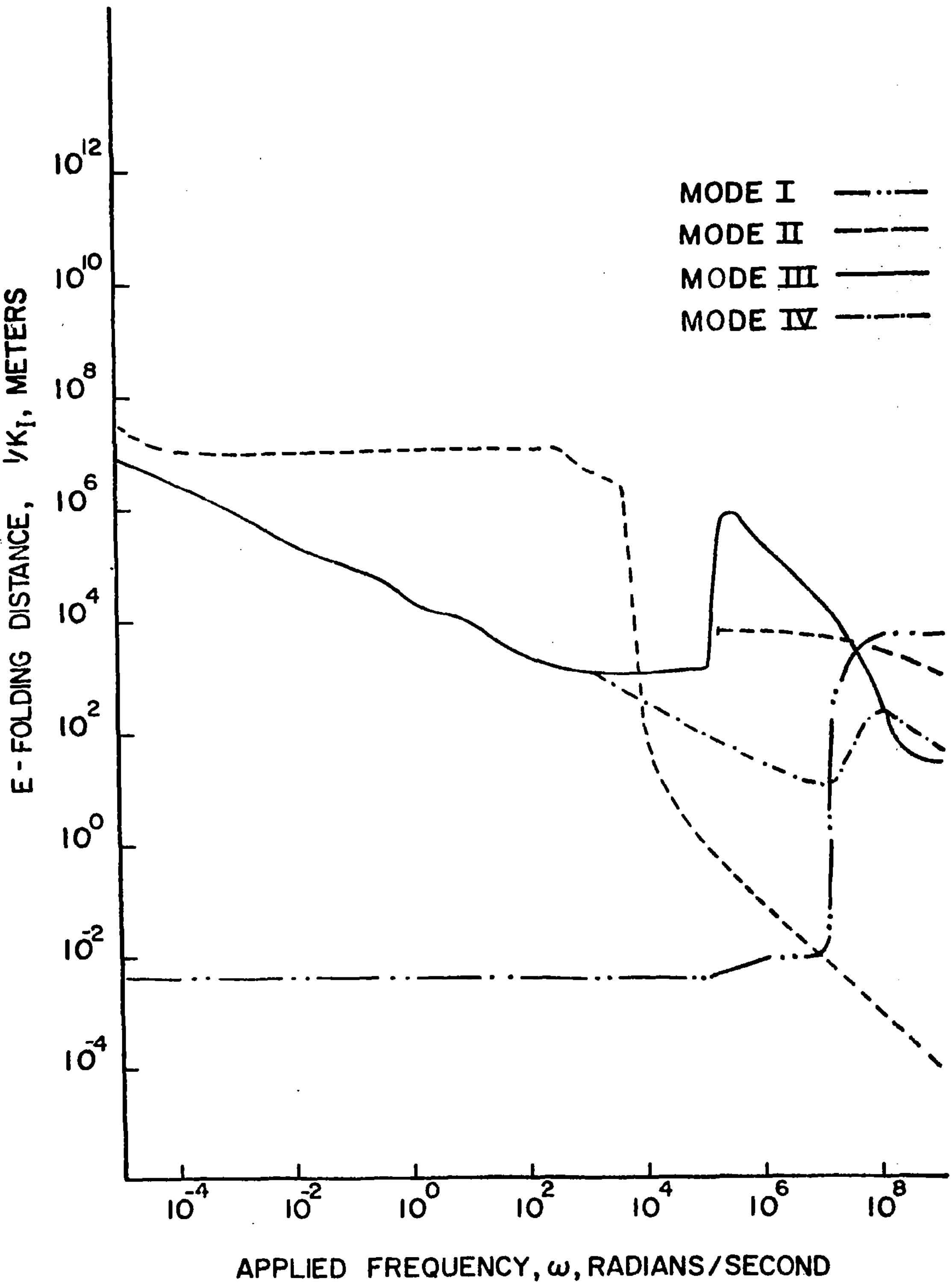


FIGURE 2. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 0°$

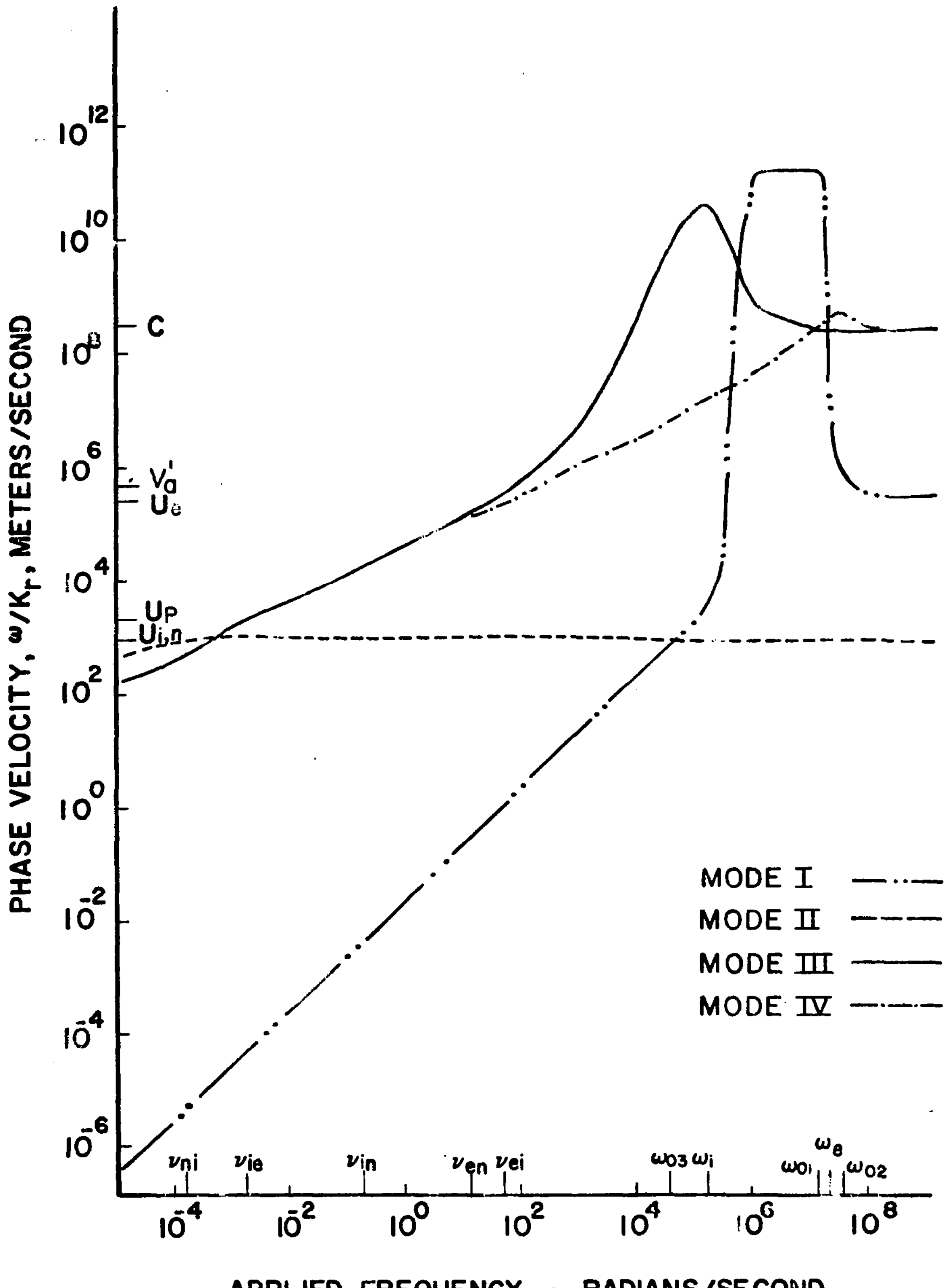


FIGURE 3 PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 15°$

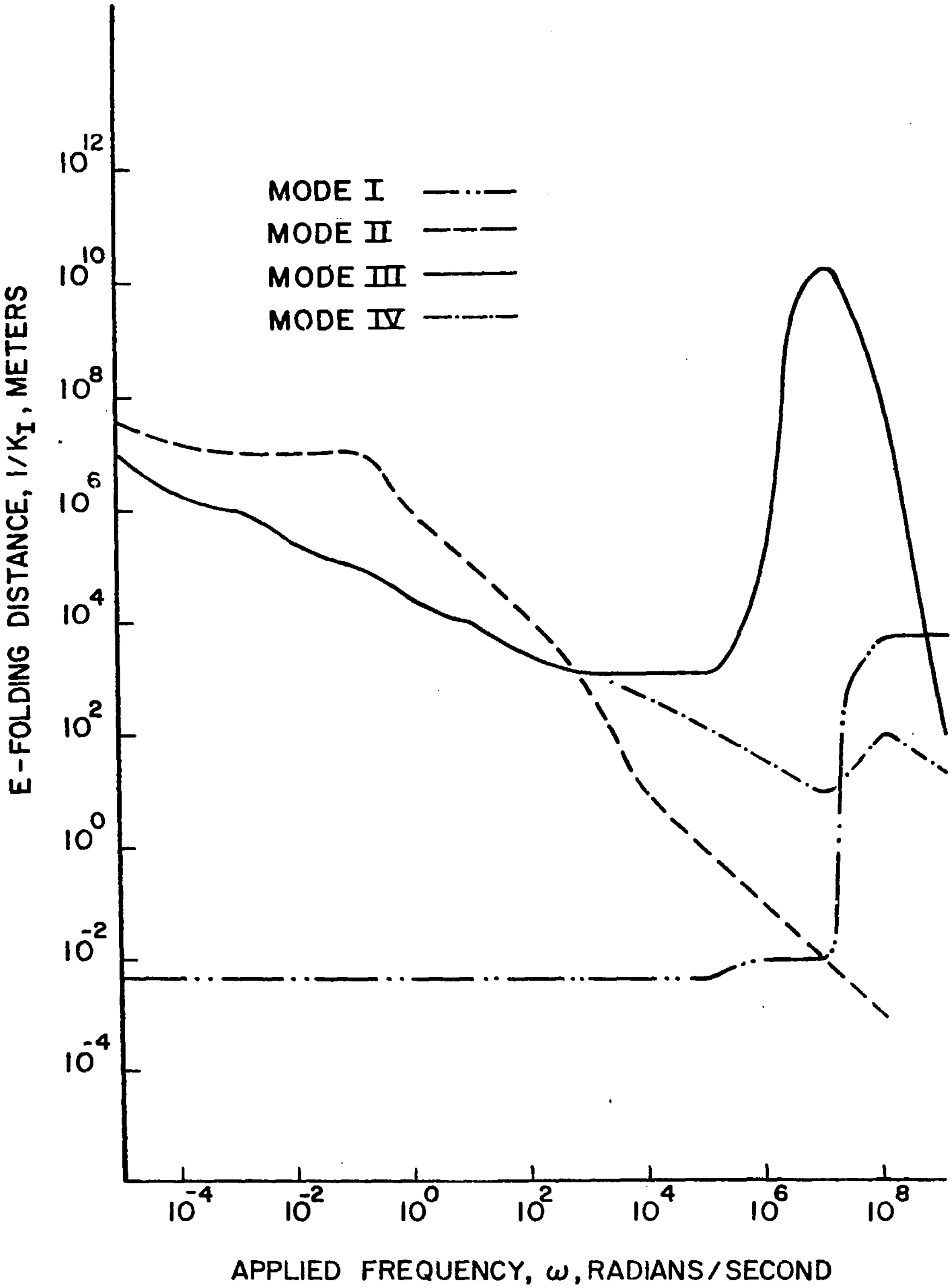


FIGURE 4. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 15°$

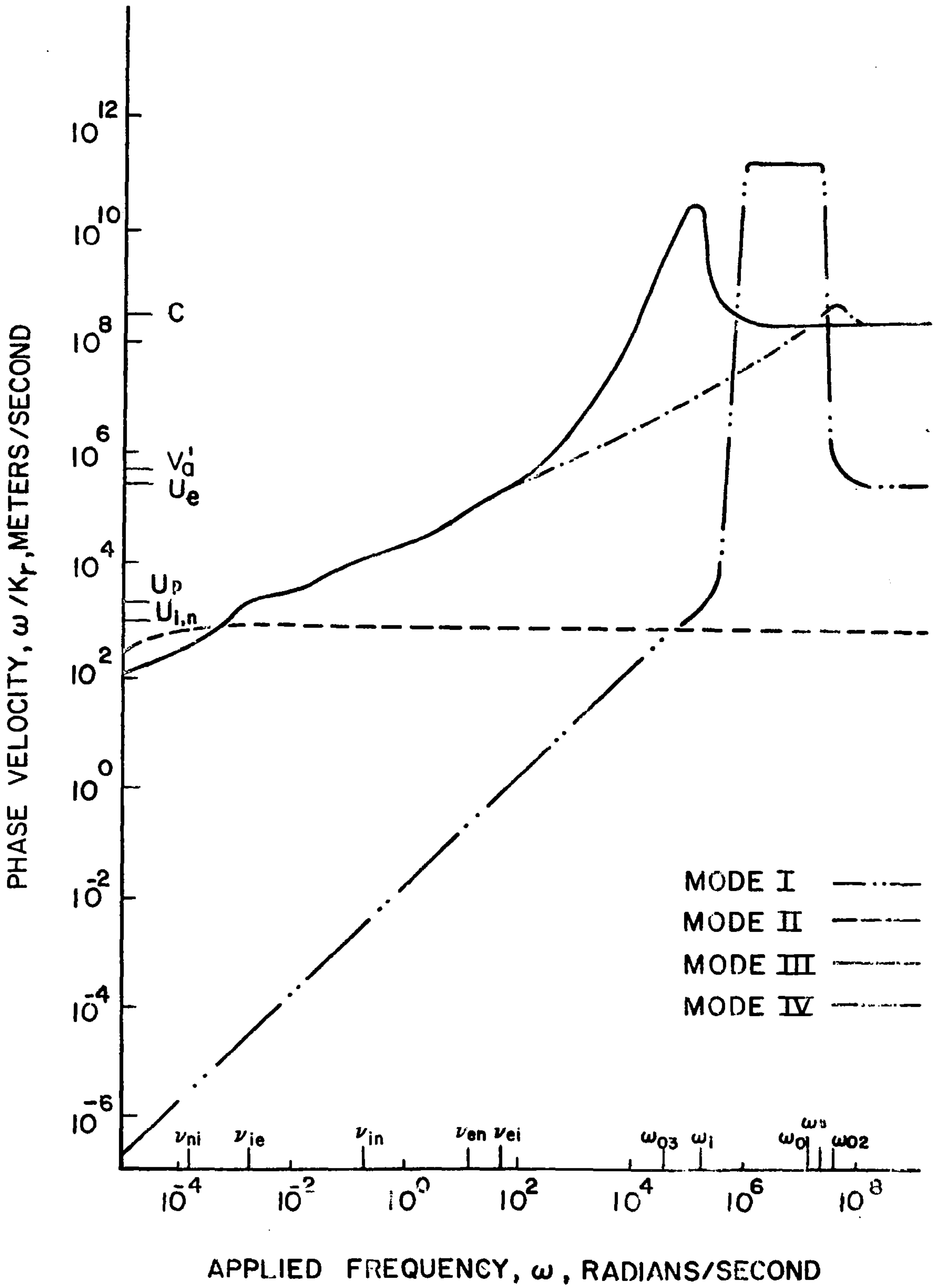


FIGURE 5. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 30°$

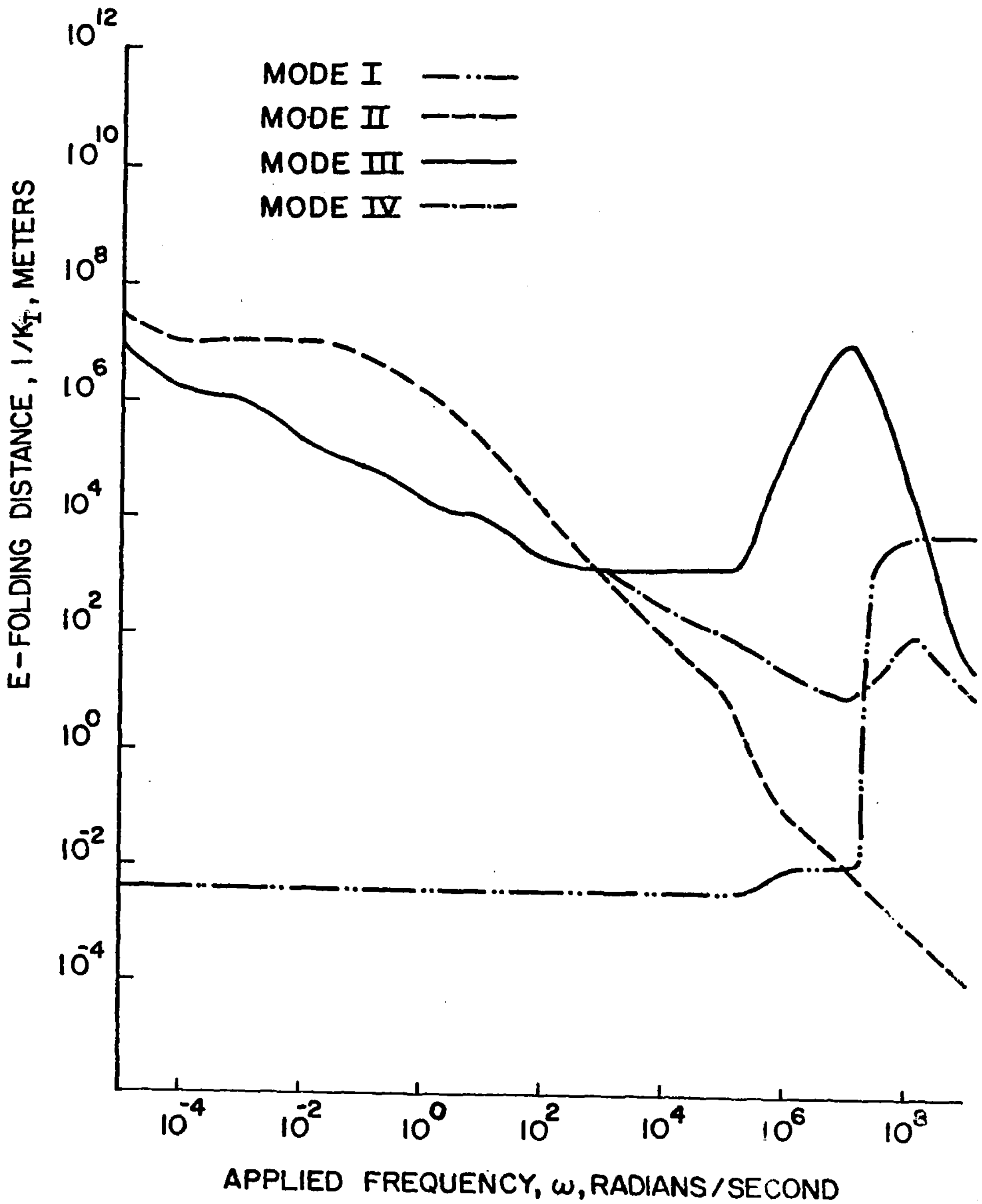


FIGURE 6. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 30°$

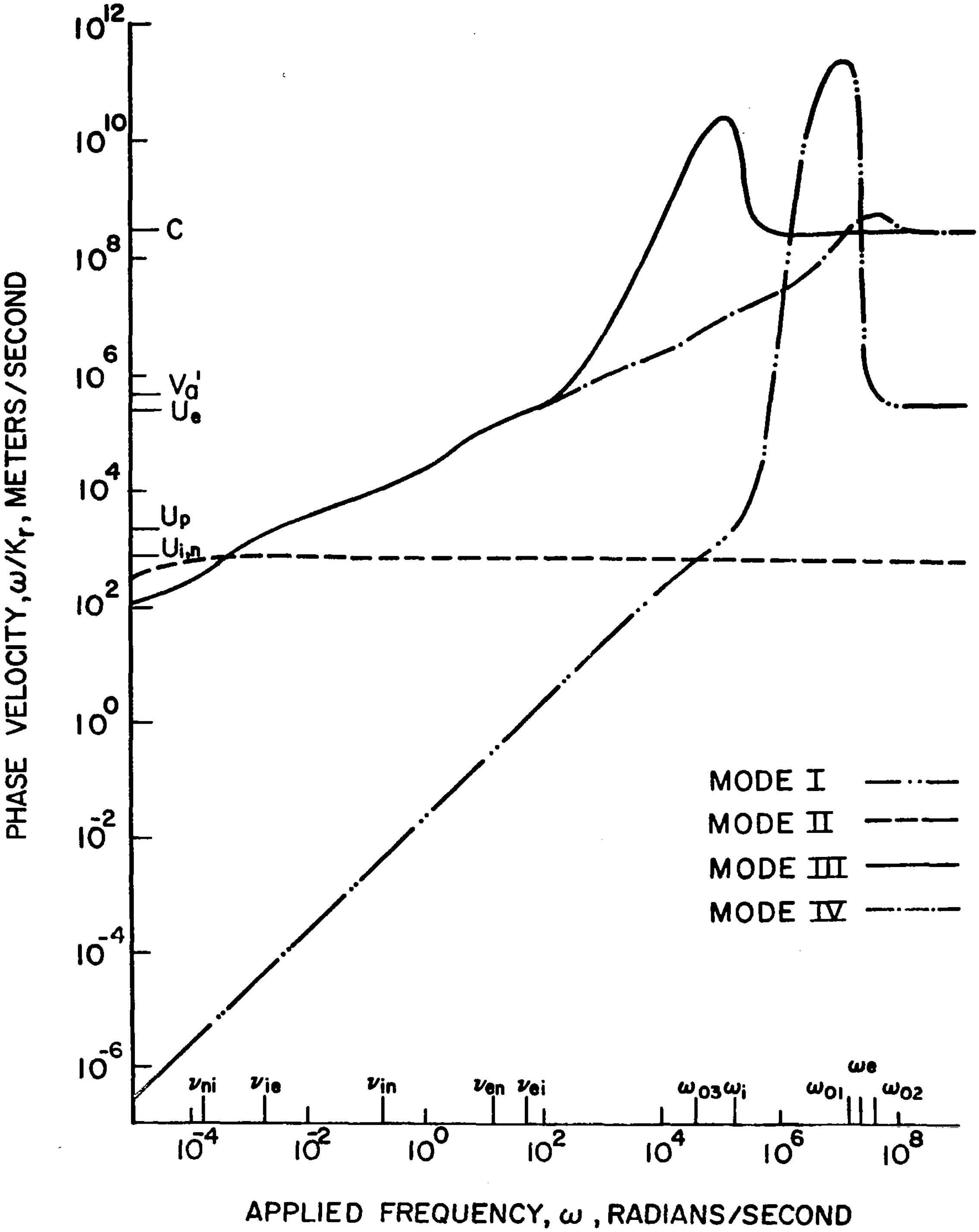


FIGURE 7. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 45°$

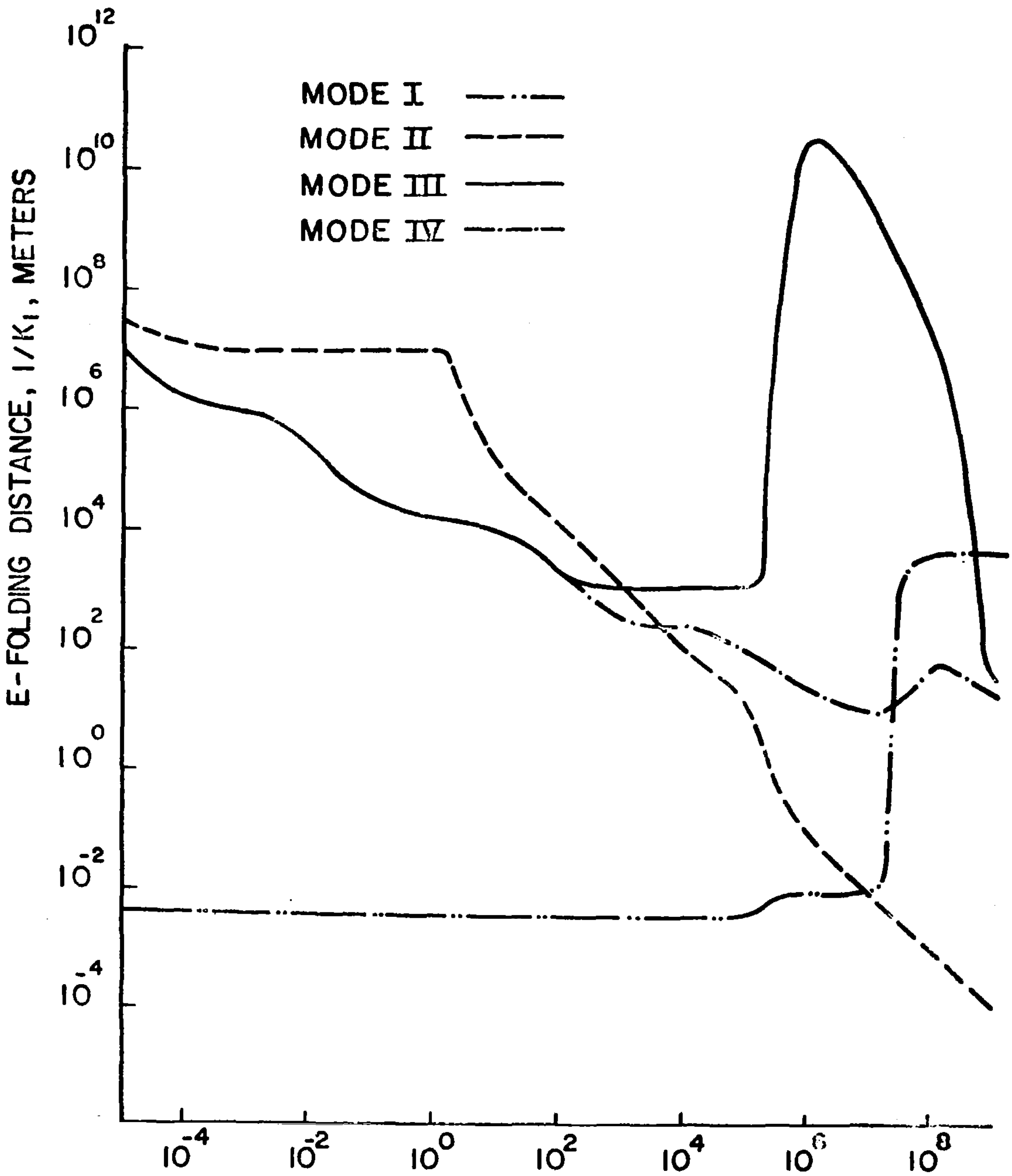


FIGURE 8. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 45°$

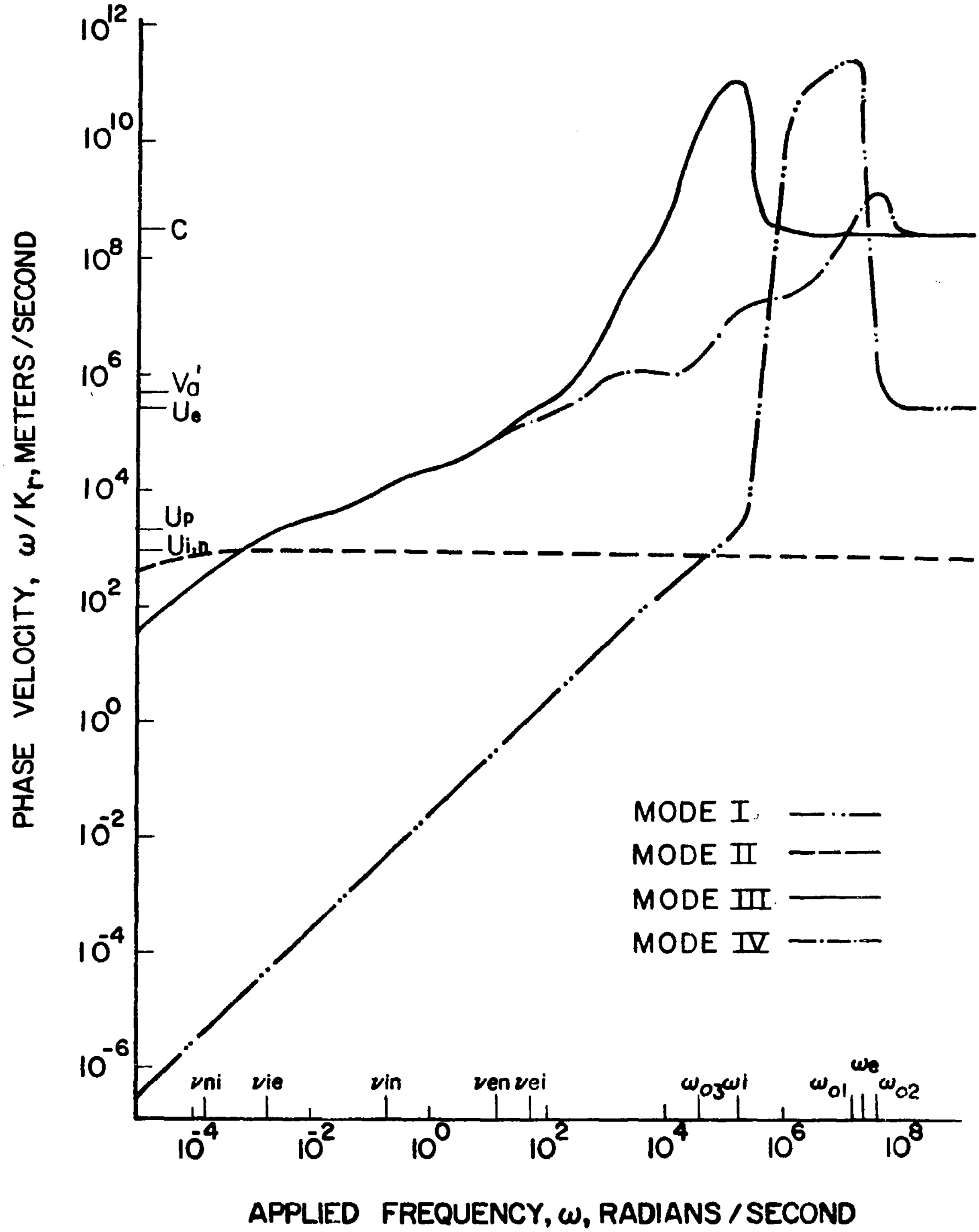


FIGURE 9. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 60°$

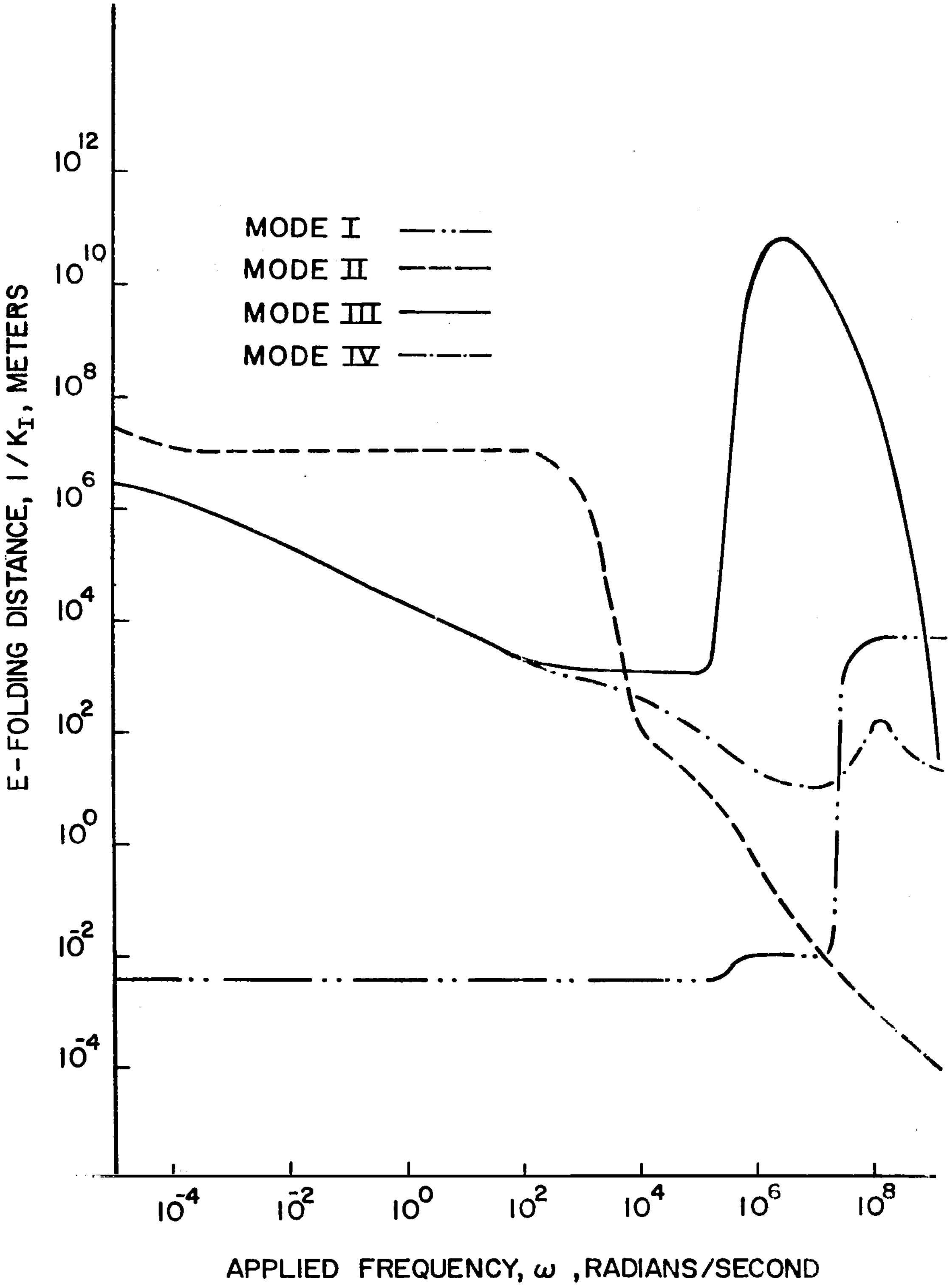


FIGURE 10. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 60°$

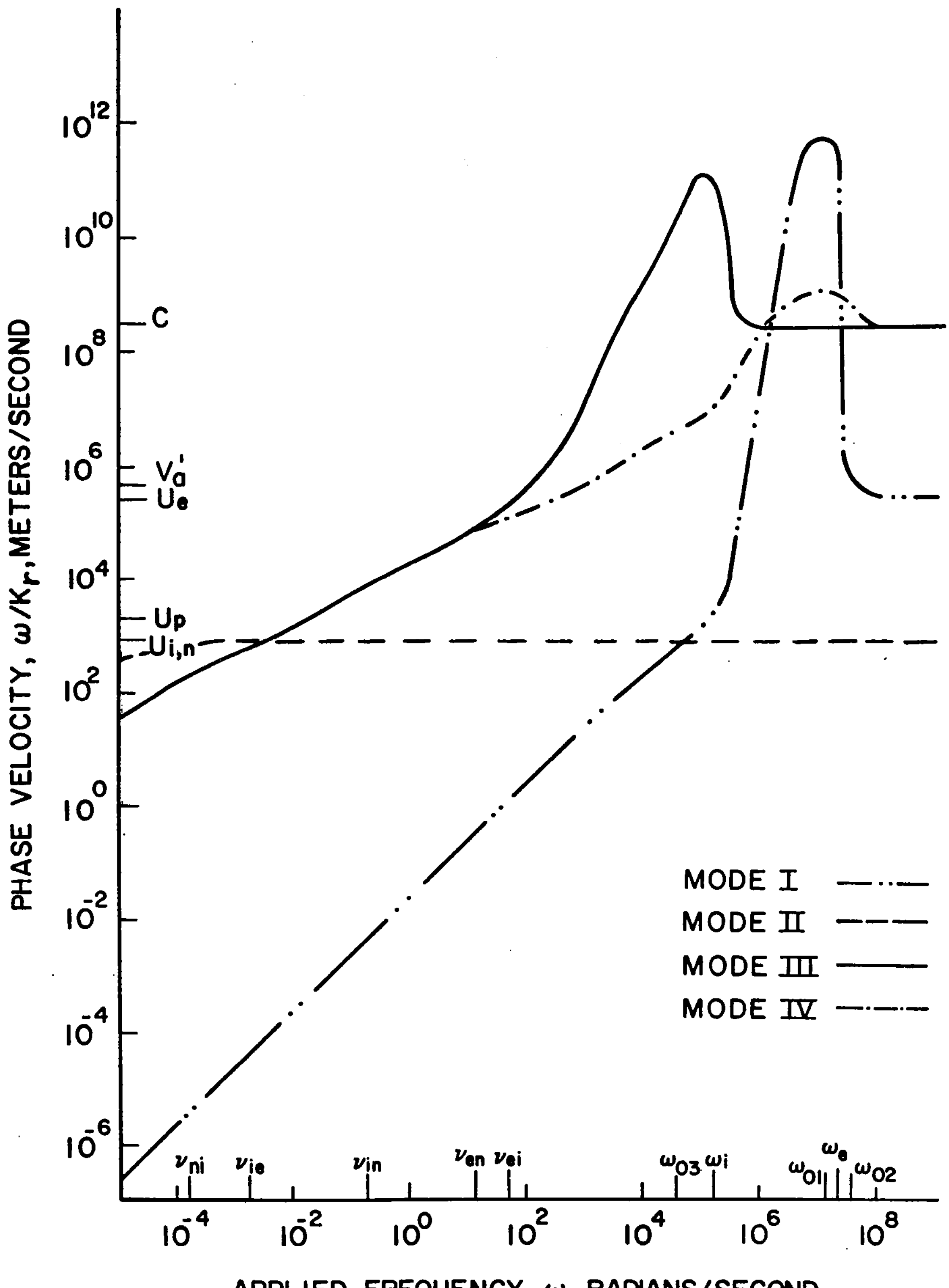


FIGURE 11. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 75°$

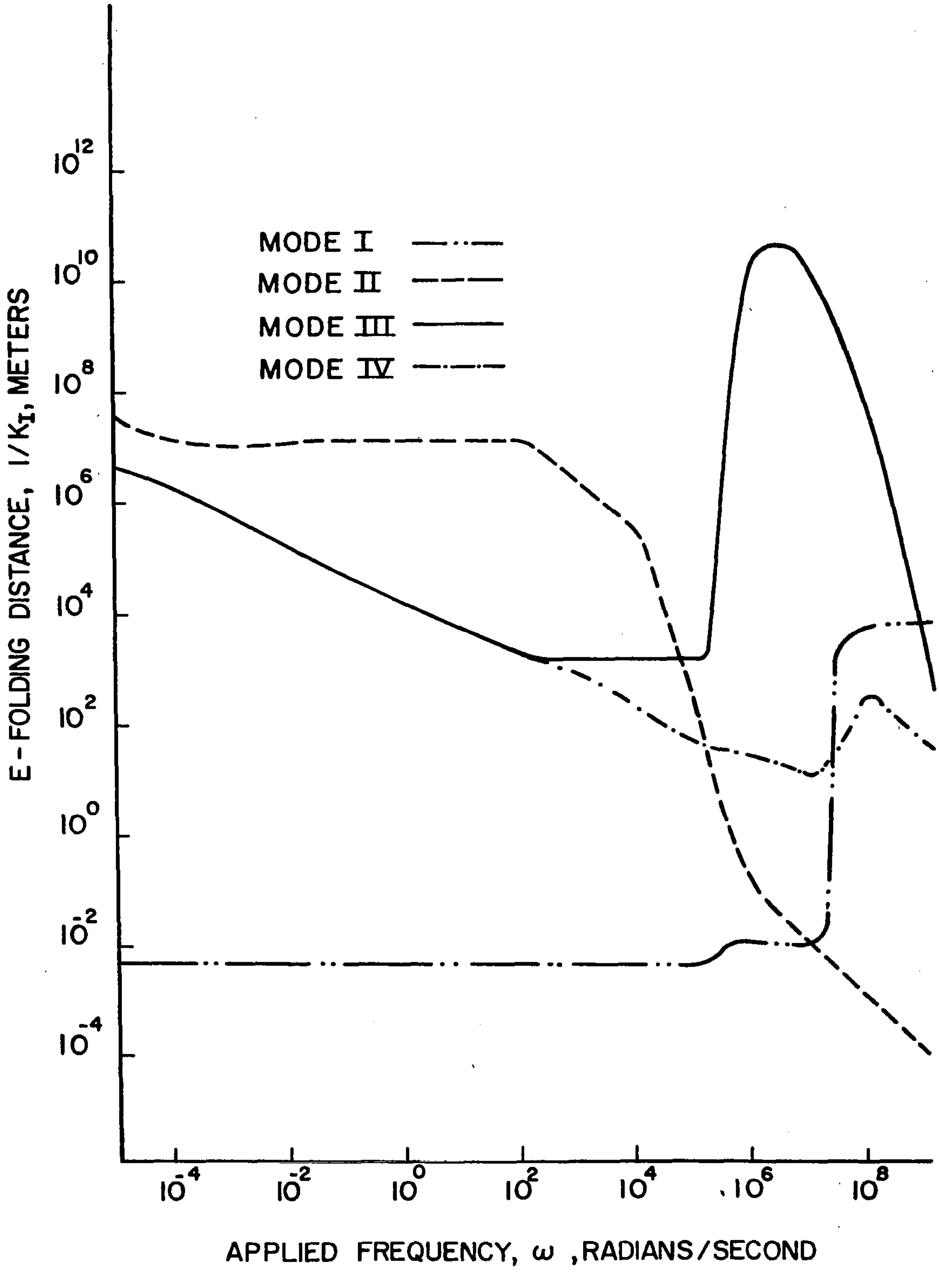


FIGURE 12. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 75°$

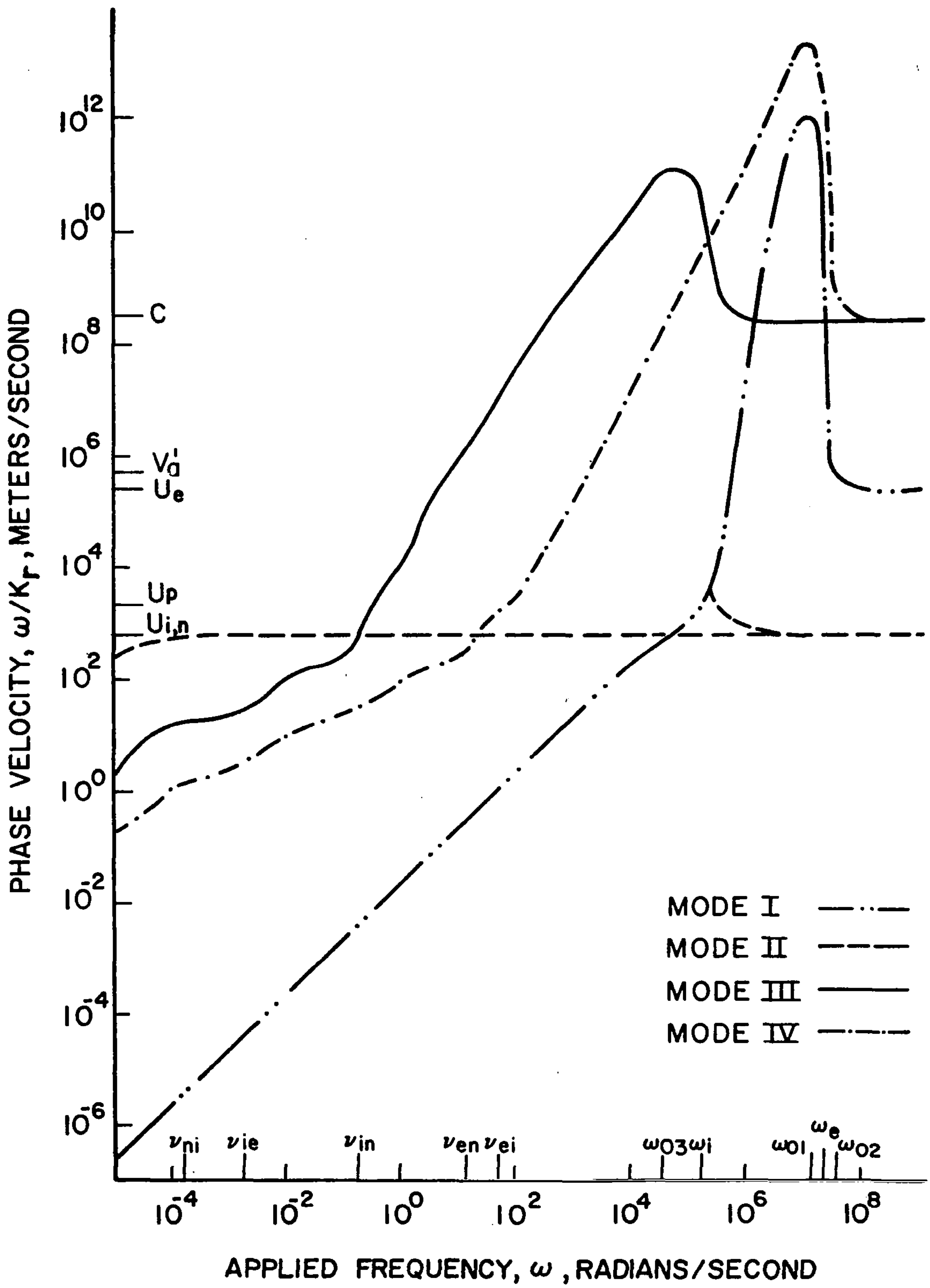


FIGURE 13. PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 90°$

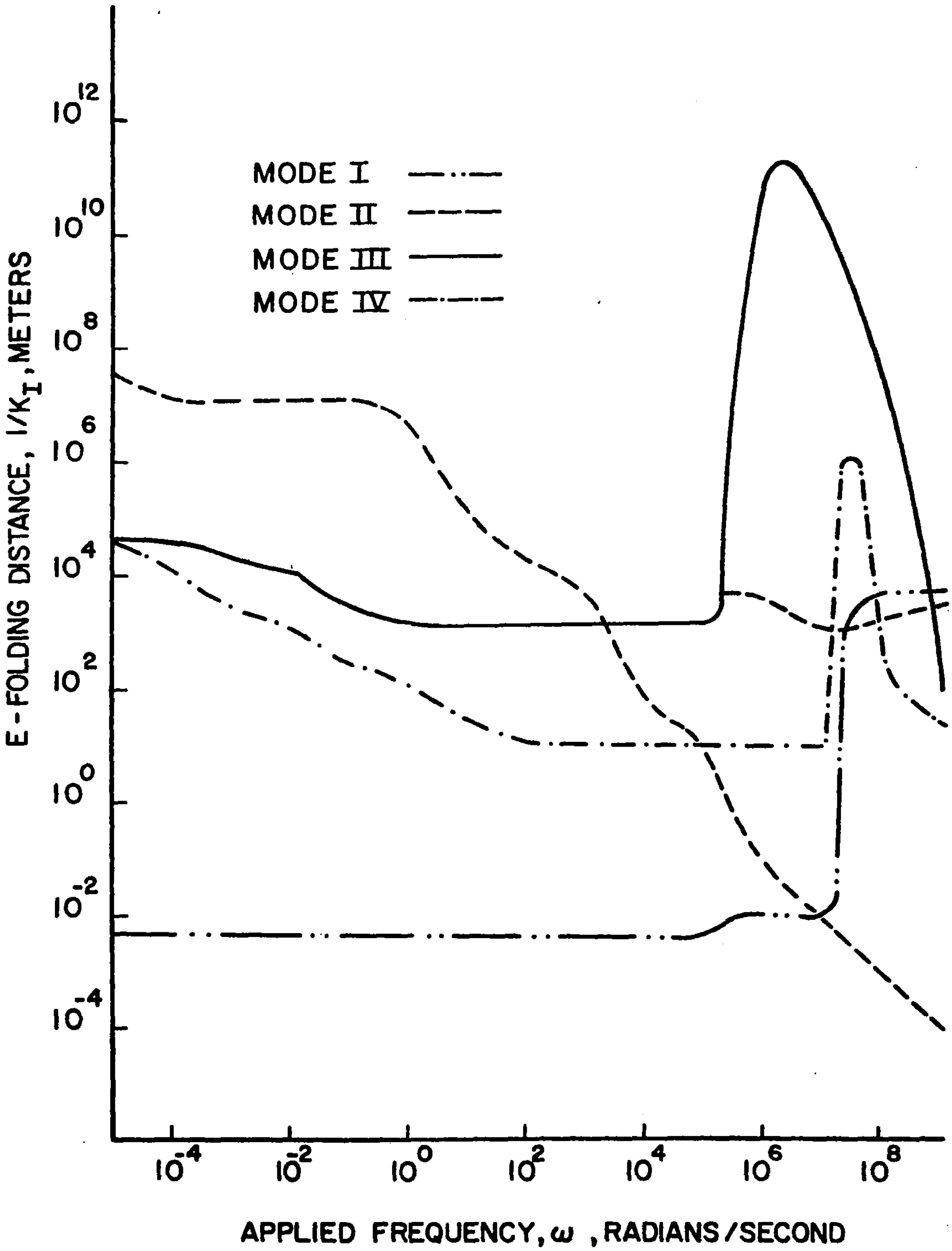


FIGURE 14. E-FOLDING DISTANCE VERSUS FREQUENCY FOR $\theta = 90°$

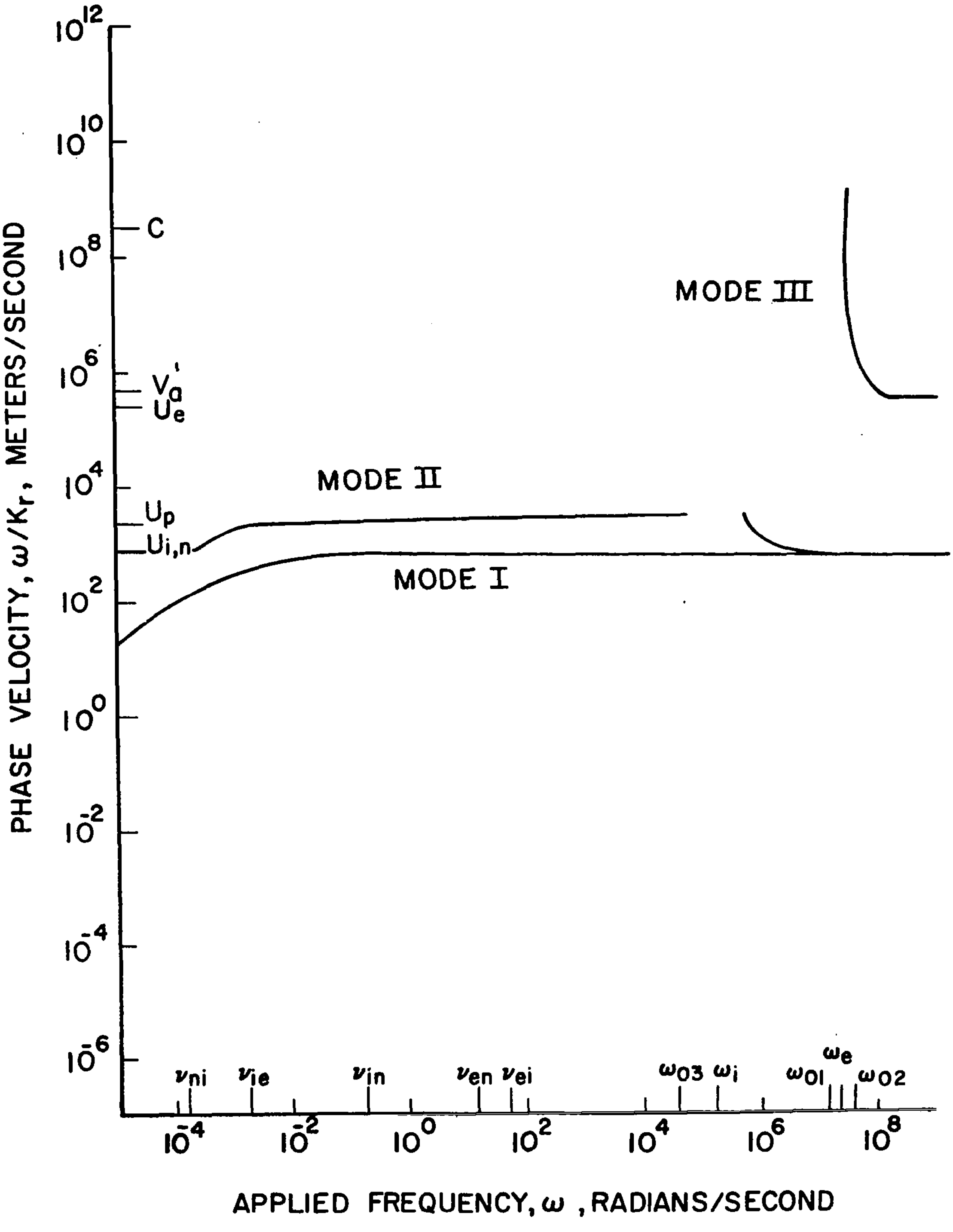


FIGURE 15. TANENBAUM'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 0°$

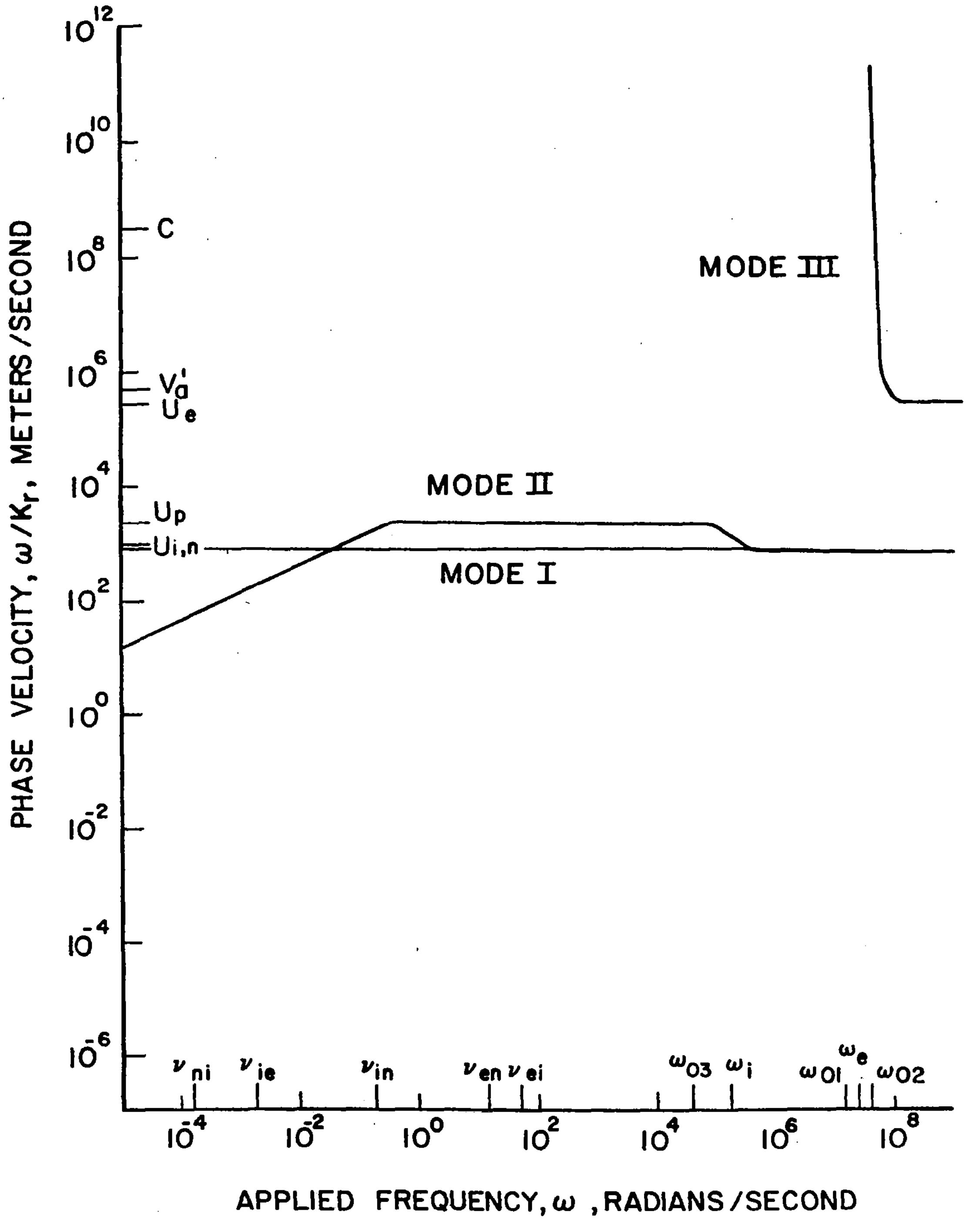


FIGURE 16. DAHL'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 0^\circ$

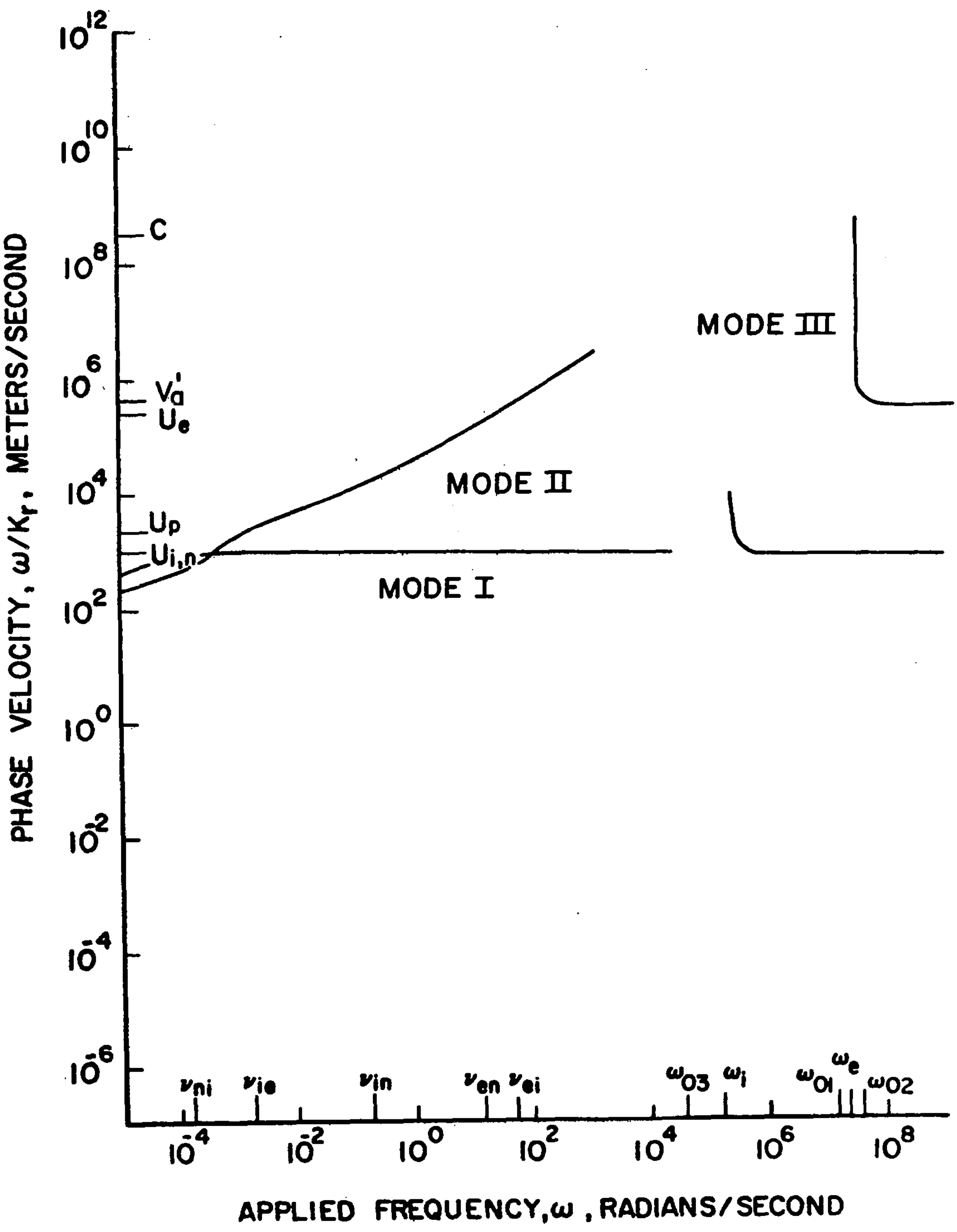


FIGURE 17. BROWN'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 0°$

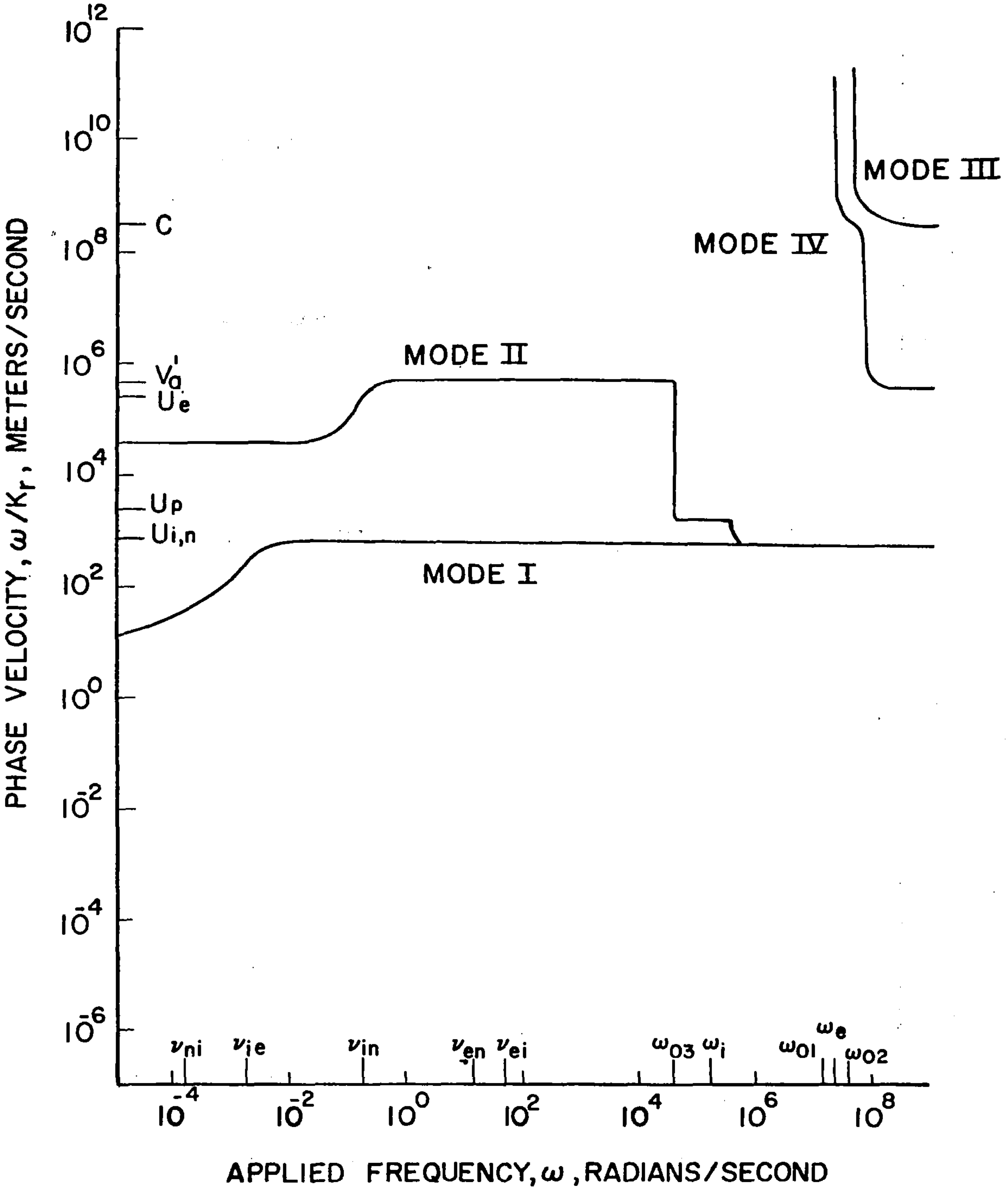


FIGURE 18. TANENBAUM'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 90^{\circ}$

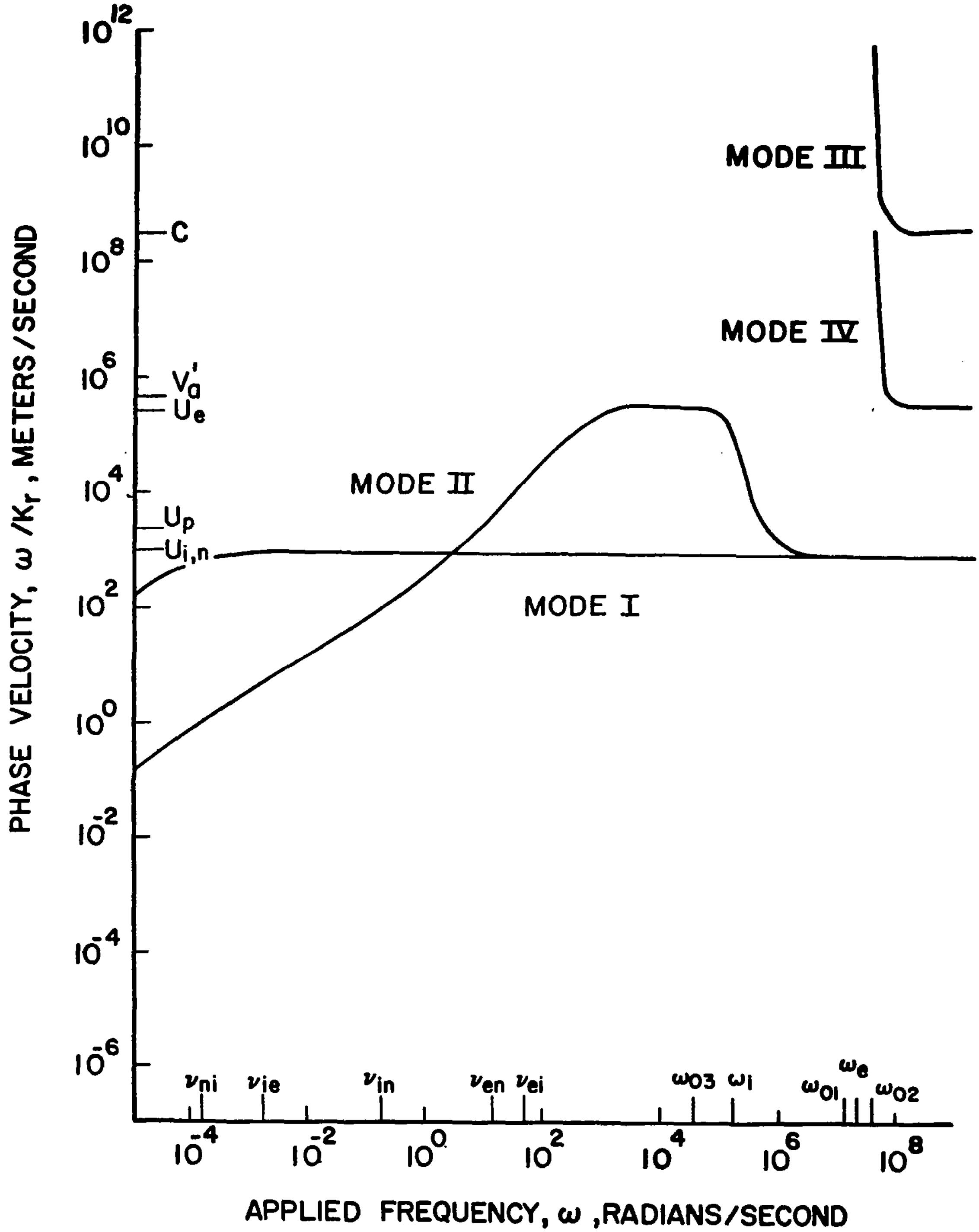


FIGURE 19. McCLENDON'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 90°$

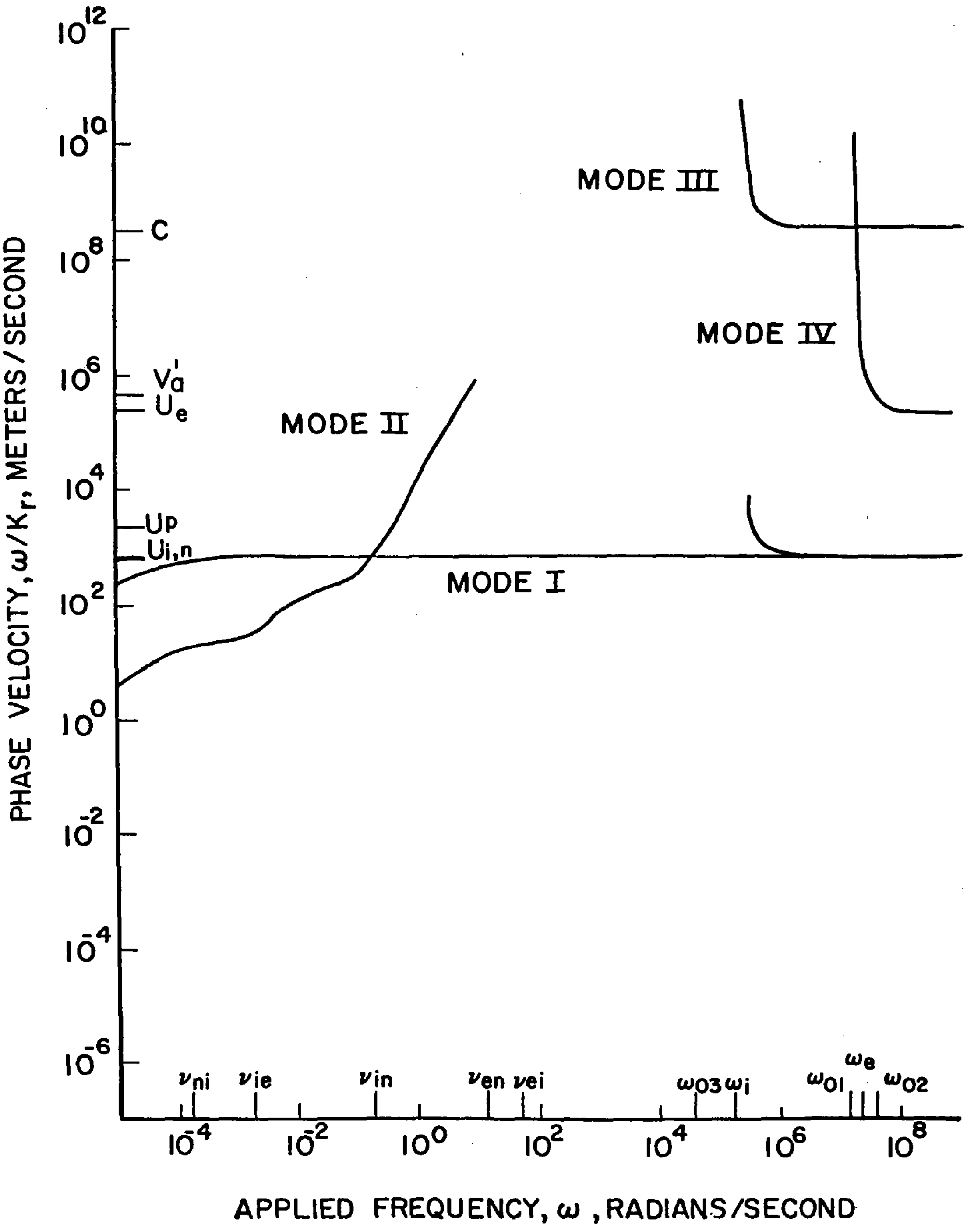


FIGURE 20. BROWN'S PHASE VELOCITY VERSUS FREQUENCY FOR $\theta = 90^\circ$

APPENDIX A

The substitution procedure employed in the derivation of the determinant of the coefficients of V_{ex}, V_{ey}, and V_{ez},

Maxwell's Equations,

$$\nabla X \vec{E} = \frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$\nabla X \vec{H} = \frac{4\pi e}{c}(N_i \vec{V}_i - N_e \vec{V}_e) + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Continuity Equations,

$$\frac{D\rho_{e,i,n}}{Dt} = -\rho_{e,i,n}\, \nabla \cdot \vec{V}_{e,i,n}$$

Momentum Equations,

$$\frac{D\vec{V}_e}{Dt} = \frac{-e}{m_e}(\vec{E} + \frac{\vec{V}_e X \vec{H}}{c}) - \frac{\nabla P_e}{\rho_e} - \nu_{ei}(\vec{V}_e - \vec{V}_i)$$
$$- \nu_{en}(\vec{V}_e - \vec{V}_n)$$

$$\frac{D\vec{V}_i}{Dt} = \frac{e}{m_i}(\bar{E} + \frac{\vec{V}_i X \vec{H}}{c}) - \frac{\nabla P_i}{\rho_i} - \nu_{ie}(\vec{V}_i - \vec{V}_e)$$
$$- \nu_{in}(\vec{V}_i - \vec{V}_n)$$

$$\frac{D\vec{V}_n}{Dt} = -\frac{\nabla P_n}{\rho_n} - \nu_{ne}(\vec{V}_n - \vec{V}_e) - \nu_{ni}(\vec{V}_n - \vec{V}_i)$$

Ideal Gas Law,

$$P_{e,i,n}\, N_{e,i,n}^{-\gamma} = \text{CONSTANT}$$

Now consider small perturbation,

$$\vec{E} = \vec{E}' \qquad \vec{H} = \vec{H}_o + \vec{h}$$
$$\vec{V}_{e,i,n} = \vec{V}'_{e,i,n} \qquad P_{e,i} = P_o + p_{e,i}$$
$$N_n = N_1 + n_n \qquad P_n = P_1 + p_n$$
$$N_{e,i} = N_o + n_{e,i}$$

also, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

Now, Maxwell's Equations,

$$\nabla X\vec{E} = \nabla X\vec{E}' = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{H}_o + \vec{h})$$

$$\nabla X\vec{E}' = -\frac{1}{c}\frac{\partial \vec{h}}{\partial t}$$

$$(-\frac{\partial E_z}{\partial x}\hat{j} + \frac{\partial E_y}{\partial x}\hat{k}) = -\frac{1}{c}\frac{\partial}{\partial t}(h_x\hat{i} + h_y\hat{j} + h_z\hat{k})$$

$$(1) \qquad \frac{\partial h_x}{\partial t} = 0$$

$$(2) \qquad \frac{1}{c}\frac{\partial h_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$(3) \qquad -\frac{1}{c}\frac{\partial h_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

$$\nabla X(\vec{H}_o + \vec{h}) = \frac{4\pi e}{c}[(N_o + n_i)\vec{V}_i' - (N_o + n_e)\vec{V}_e'] + \frac{1}{c}\frac{\partial \vec{E}'}{\partial t}$$

$$(\nabla X\vec{h}) = \frac{4\pi e N_o}{c}(\vec{V}_i' - \vec{V}_e') + \frac{1}{c}\frac{\partial \vec{E}'}{\partial t}$$

$$(-\frac{\partial h_z}{\partial x}\hat{j} + \frac{\partial h_y}{\partial x}\hat{k}) = \frac{4\pi e N_o}{c}[(V_{ix}' - V_{ex}')\hat{i} + (V_{iy}' - V_{ey}')$$

$$\hat{j} + (V_{iz}' - V_{ez}')\hat{k}] + \frac{1}{c}\frac{\partial}{\partial t}[E_x'\hat{i} + E_y'\hat{j} + E_z'\hat{k}]$$

$$(4) \qquad 4\pi e N_o(V_{ix}' - V_{ex}') + \frac{\partial E_x'}{\partial t} = 0$$

$$(5) \qquad 4\pi e N_o(V_{iy}' - V_{ey}') + \frac{\partial E_y'}{\partial t} + c\frac{\partial h_z}{\partial x} = 0$$

$$(6) \qquad 4\pi e N_o(V_{iz}' - V_{ez}') + \frac{\partial E_z'}{\partial t} - c\frac{\partial h_y}{\partial x} = 0$$

Continuity Equations,

$$\frac{\partial}{\partial t}[m_e(N_o + n_e)] + \vec{V}_e' \cdot \nabla[m_e(N_o + n_e)] = - m_e(N_o + n_e)\nabla \cdot \vec{V}_e'$$

$$m_e\frac{\partial n_e}{\partial t} = - m_e N_o(\nabla \cdot \vec{V}_e')$$

$$\frac{\partial n_e}{\partial t} = -N_o(\nabla \cdot \vec{V}'_e)$$

$$\frac{\partial n_i}{\partial t} = -N_o(\nabla \cdot \vec{V}'_i)$$

$$\frac{\partial n_n}{\partial t} = -N_1(\nabla \cdot \vec{V}'_n)$$

$$(7) \qquad \frac{\partial n_e}{\partial t} = -N_o \frac{\partial V'_{ex}}{\partial X}$$

$$(8) \qquad \frac{\partial n_i}{\partial t} = -N_o \frac{\partial V'_{ix}}{\partial X}$$

$$(9) \qquad \frac{\partial n_n}{\partial t} = -N_1 \frac{\partial V'_{nx}}{\partial X}$$

Momentum Equations,

$$\frac{\partial \vec{V}'_e}{\partial t} + \vec{V}'_e \cdot \nabla \vec{V}'_e = \frac{-e}{m_e}\left[\vec{E}' + \frac{\vec{V}'_e \, X(\vec{H}_o + \vec{h})}{c}\right]$$

$$- \frac{\nabla(P_o + p_e)}{m_e(N_o + n_e)} - \nu_{ei}(\vec{V}'_e - \vec{V}'_i) - \nu_{en}(\vec{V}'_e - \vec{V}'_n)$$

$$\frac{\partial \vec{V}'_e}{\partial t} = \frac{-e}{m_e}\left[\vec{E}' + \frac{\vec{V}'_e \text{ X } \vec{H}_o}{c}\right] - \frac{\nabla p_e}{m_e N_o} - \nu_{ei}(\vec{V}'_e - \vec{V}'_i) - \nu_{en}(\vec{V}'_e - \vec{V}'_n)$$

$$\frac{\partial}{\partial t}[V'_{ex}\hat{i} + V'_{ey}\hat{j} + V'_{ez}\hat{k}] = -\frac{e}{m_e}\ [(E'_x\hat{i} + E'_y\hat{j} + E'_z\hat{k})$$

$$+ \frac{(V'_{ey}H_{oz} - V'_{ez}H_{oy})}{c}\hat{i} + \frac{(V'_{ez}H_{ox} - V'_{ex}H_{oz})}{c}\hat{j}$$

$$+ \frac{(V'_{ex}H_{oy} - V'_{ey}H_{ox})}{c}\hat{k}] - \frac{1}{\rho_e}\frac{\partial p_e}{\partial X}\hat{i}$$

$$- \nu_{ei}\ [(V'_{ex} - V'_{ix})\hat{i} + (V'_{ey} - V'_{iy})\hat{j} + (V'_{ez} - V'_{iz})\hat{k}]$$

$$- \nu_{en}\ [(V'_{ex} - V'_{nx})\hat{i} + (V'_{ey} - V'_{nx})\hat{j} + (V'_{ez} - V'_{nz})\hat{k}]$$

$$(10) \qquad \frac{\partial V'_{ex}}{\partial t} + \frac{e}{m_e}\ [E'_x + \frac{1}{c}(V'_{ey}H_{oz} - V'_{ez}H_{oy})] + \frac{1}{m_e N_o}\frac{\partial p_e}{\partial X}$$

$$+ \nu_{ei}(V'_{ex} - V'_{ix}) + \nu_{en}(V'_{ex} - V'_{nx}) = 0$$

$$(11) \quad \frac{\partial V'_{ey}}{\partial t} + \frac{e}{m_e} [E'_y + \frac{1}{c}(V'_{ez}H_{ox} - V'_{ex}H_{oz})]$$

$$+ \nu_{ei}(V'_{ey} - V'_{iy}) + \nu_{en}(V'_{ey} - V'_{ny}) = 0$$

$$(12) \quad \frac{\partial V'_{ez}}{\partial t} + \frac{e}{m_e} [E'_z + \frac{1}{c}(V'_{ex}H_{oy} - V'_{ey}H_{oz})]$$

$$+ \nu_{ei}(V'_{ez} - V'_{iz}) + \nu_{en}(V'_{ez} - V'_{nz}) = 0$$

Likewise,

$$(13) \quad \frac{\partial V'_{ix}}{\partial t} - \frac{e}{m_i} [E'_x + \frac{1}{c}(V'_{iy}H_{oz} - V'_{iz}H_{oy})] + \frac{1}{m_i N_o} \frac{\partial p_i}{\partial X}$$

$$+ \nu_{ie}(V'_{ix} - V'_{ex}) + \nu_{in}(V'_{ix} - V'_{nx}) = 0$$

$$(14) \quad \frac{\partial V'_{iy}}{\partial t} - \frac{e}{m_i} [E'_y + \frac{1}{c}(V'_{iz}H_{ox} - V'_{ix}H_{oz})] + \nu_{ie}(V'_{iy} - V'_{ey})$$

$$+ \nu_{in}(V'_{iy} - V'_{ny}) = 0$$

$$(15) \quad \frac{\partial V'_{iz}}{\partial t} - \frac{e}{m_i} [E'_z + \frac{1}{c}(V'_{ix}H_{oy} - V'_{iy}H_{ox})] + \nu_{ie}(V'_{iz} - V'_{ez})$$

$$+ \nu_{in}(V'_{iz} - V'_{nz}) = 0$$

$$(16) \quad \frac{\partial V'_{nx}}{\partial t} + \frac{1}{m_n N_1} \frac{\partial p_n}{\partial X} + \nu_{ne}(V'_{nx} - V'_{ex}) + \nu_{ni}(V'_{nx} - V'_{ix}) = 0$$

$$(17) \quad \frac{\partial V'_{ny}}{\partial t} + \nu_{ne}(V'_{ny} - V'_{ey}) + \nu_{ni}(V'_{ny} - V'_{iy}) = 0$$

$$(18) \quad \frac{\partial V'_{nz}}{\partial t} + \nu_{ne}(V'_{nz} - V'_{ez}) + \nu_{ni}(V'_{nz} - V'_{iz}) = 0$$

Using the relationship,

$$\frac{1}{mN} \frac{\partial P}{\partial X} = \frac{U^2}{N} \frac{\partial n}{\partial X}$$

And considering the form of the solution to be,

$$e^{i(KX - \omega t)}$$

The eighteen equations now have the form,

(1) $h_x = 0$ (2) $h_y = -nE_z$

(3) $h_z = nE_y$ (4) $E_x = \dfrac{4\pi eN_o i}{\omega}(V_{ex} - V_{ix})$

(5) $E_y = \dfrac{4\pi eN_o i}{\omega(1-n^2)}(V_{ey} - V_{iy})$

(6) $E_z = \dfrac{4\pi eN_o i}{\omega(1-n^2)}(V_{ez} - V_{iz})$

(7) $n_e = \dfrac{kN_o}{\omega} V_{ex}$ (8) $n_i = \dfrac{kN_o}{\omega} V_{ix}$ (9) $n_n = \dfrac{kN_1}{\omega} V_{nx}$

(10) $(\nu_e - i\omega)V_{ex} - \nu_{ei}V_{ix} - \nu_{en}V_{nx} - \omega_T V_{ez} + \dfrac{e}{m_e} E_x + \dfrac{iU_e^2 n_e k}{N_o} = 0$

(11) $(\nu_e - i\omega)V_{ey} - \nu_{ei}V_{iy} - \nu_{en}V_{ny} + \omega_L V_{ez} + \dfrac{eE_y}{m_e} = 0$

(12) $(\nu_e - i\omega)V_{ez} - \nu_{ei}V_{iz} - \nu_{en}V_{nz} + \omega_T V_{ex} - \omega_L V_{ey} + \dfrac{eE_z}{m_e} = 0$

(13) $(\nu_i - i\omega)V_{ix} - \nu_{ie}V_{ex} - \nu_{in}V_{nx} + \dfrac{\omega_T}{m} V_{iz} - \dfrac{eE_x}{m_i} + \dfrac{iU_i^2 n_i k}{N_o} = 0$

(14) $(\nu_i - i\omega)V_{iy} - \nu_{ie}V_{ey} - \nu_{in}V_{ny} - \dfrac{\omega_L}{m} V_{iz} - \dfrac{eE_y}{m_i} = 0$

(15) $(\nu_i - i\omega)V_{iz} - \nu_{ie}V_{ez} - \nu_{in}V_{nz} - \dfrac{\omega_T V_{ix}}{m} + \dfrac{\omega_L V_{iy}}{m} - \dfrac{eE_z}{m_i} = 0$

(16) $(\nu_n - i\omega)V_{nx} - \nu_{ne}V_{ex} - \nu_{ni}V_{ix} + \dfrac{iU_n^2 n_n k}{N_1} = 0$

(17) $(\nu_n - i\omega)V_{ny} - \nu_{ne}V_{ey} - \nu_{ni}V_{iy} = 0$

(18) $(\nu_n - i\omega)V_{nz} - \nu_{ne}V_{ez} - \nu_{ni}V_{iz} = 0$

Now, put in the values of E (Eqs. 4,5,&6) and $n_{e,i,n}$ (Eqs. 7,8,&9) and obtain,

$$(10)\qquad \left(\frac{iU_e^2k^2}{\omega} + \frac{i\omega_e^2}{\omega} + \nu_e - i\omega\right)V_{ex} - \nu_{en}V_{nx} - \left(\nu_{ei} + \frac{i\omega_e^2}{\omega}\right)V_{ix} - \omega_T V_{ez} = 0$$

$$(11)\qquad \left(\nu_e - i\omega + \frac{i\omega_e^2}{\omega(1-n^2)}\right)V_{ey} - \left(\nu_{ei} + \frac{i\omega_e^2}{\omega(1-n^2)}\right)V_{iy} - \nu_{en}V_{ny} + \omega_L V_{ez} = 0$$

$$(12)\qquad \left(\nu_e - i\omega + \frac{i\omega_e^2}{\omega(1-n^2)}\right)V_{ez} - \left(\nu_{ei} + \frac{i\omega_e^2}{\omega(1-n^2)}\right)V_{iz} - \nu_{en}V_{nz} + \omega_T V_{ex} - \omega_L V_{ey} = 0$$

$$(13)\qquad \left(\frac{iU_i^2k^2}{\omega} + \frac{i\omega_i^2}{\omega} + \nu_i - i\omega\right)V_{ix} - \left(\nu_{ie} + \frac{i\omega_i^2}{\omega}\right)V_{ex} - \nu_{in}V_{nx} + \frac{\omega_T}{m}V_{iz} = 0$$

$$(14)\qquad \left(\frac{i\omega_i^2}{\omega(1-n^2)} + \nu_i - i\omega\right)V_{iy} - \left(\nu_{ie} + \frac{i\omega_i^2}{\omega(1-n^2)}\right)V_{ey} - \nu_{in}V_{ny} - \frac{\omega_L}{m}V_{iz} = 0$$

$$(15)\qquad \left(\frac{i\omega_i^2}{\omega(1-n^2)} + \nu_i - i\omega\right)V_{iz} - \left(\nu_{ie} + \frac{i\omega_i^2}{\omega(1-n^2)}\right)V_{ez} - \nu_{in}V_{nz} - \frac{\omega_T}{m}V_{ix} + \frac{\omega_L}{m}V_{iy} = 0$$

$$(16)\qquad \left(\frac{iU_n^2k^2}{\omega} + \nu_n - i\omega\right)V_{nx} - \nu_{ne}V_{ex} - \nu_{ni}V_{ix} = 0$$

$$(17)\qquad (\nu_n - i\omega)V_{ny} - \nu_{ne}V_{ey} - \nu_{ni}V_{iy} = 0$$

$$(18)\qquad (\nu_n - i\omega)V_{nz} - \nu_{ne}V_{ez} - \nu_{ni}V_{iz} = 0$$

Now solve equation (16) for V_{nx}, equation (17) for V_{ny}, and equation (18) for V_{nz},

$$(16a)\qquad V_{nx} = \frac{i\omega}{\gamma_2}(\nu_{ne}V_{ex} + \nu_{ni}V_{ix})$$

(17a) $$V_{ny} = \frac{i\omega}{\gamma_1} (\nu_{ne} V_{ey} + \nu_{ni} V_{iy})$$

(18a) $$V_{nz} = \frac{i\omega}{\gamma_1} (\nu_{ne} V_{ez} + \nu_{ni} V_{iz})$$

Put equation (16a) into equation (10) and solve for V_{ix},

$$V_{ix} = \left[\frac{-i\omega}{\omega_e^2 - i\omega\nu_{ei} + \frac{\omega^2 \nu_{en}\nu_{ni}}{\gamma_2}} \right] \left[\left(\frac{U_e^2 k^2 - \omega^2 + \omega_e^2 - \frac{\omega^2 \nu_{en}\nu_{ne}}{\gamma_2}}{\omega} i \right. \right.$$

$$\left. + \nu_e \right) V_{ex} - \omega_T V_{ez} \Big]$$

(10a) $$V_{ix} = - \frac{c_5}{c_1} V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez}$$

Put equation (17a) into equation (11) and solve for V_{iy},

$$V_{iy} = \left[\frac{-i\omega}{\frac{\omega_e^2}{(1-n^2)} + \frac{\omega^2 \nu_{en}\nu_{ni}}{\gamma_1} - i\omega\nu_{ei}} \right] \left[\left(\nu_e - i\omega \right. \right.$$

$$\left. + \frac{i\omega_e^2}{\omega(1-n\ )} - \frac{i\omega\nu_{en}\nu_{ne}}{\gamma_1} \right) V_{ey} + \omega_L V_{ez} \Big]$$

(11a) $$V_{iy} = - \frac{c_6}{c_2} V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez}$$

Put equation (18a) into equation (12) and solve for V_{iz},

$$V_{iz} = \left[\frac{-i\omega}{\frac{\omega_e^2}{(1-n^2)} + \frac{\omega^2 \nu_{en}\nu_{ni}}{\gamma_1} - i\omega\nu_{ei}} \right] \left[\omega_T V_{ex} - \omega_L V_{ey} \right.$$

$$\left. + \left(\nu_e + i \frac{\frac{\omega_e^2}{(1-n^2)} - \frac{\omega^2 \nu_{en}\nu_{ne}}{\gamma_1} - \omega^2}{\omega} \right) V_{ez} \right]$$

(12a) $$V_{iz} = - \frac{c_6}{c_2} V_{ez} - \frac{i\omega\omega_T}{c_2} V_{ex} + \frac{i\omega\omega_L}{c_2} V_{ey}$$

Put equations (10a), (16a), and (12a) into equation (13),

$$(\nu_i - i\omega + \frac{i\omega_i^2}{\omega} + \frac{iU_i^2 k^2}{\omega}) \left(- \frac{c_5}{c_1} V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez}\right)$$

$$- (\nu_{ie} + \frac{i\omega_i^2}{\omega})V_{ex} + \frac{\omega_T}{m} (- \frac{c_6}{c_2} V_{ez} - \frac{i\omega\omega_T}{c_2} V_{ex} + \frac{i\omega\omega_L}{c_2} V_{ey})$$

$$- \nu_{in} \left\{ \frac{i\omega}{\gamma_2} \ [\nu_{ne} V_{ex} + \nu_{ni} \ (- \frac{c_5}{c_1} V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez})] \right\} = 0$$

$$\text{(13a)} \quad \left[- \frac{c_5 c_3}{c_1} + \frac{c_1}{m} + \frac{\omega^2\omega_T^2}{mc_2} \right] V_{ex} - \frac{\omega^2\omega_L\omega_T}{mc_2} V_{ey}$$

$$+ \left[\frac{i\omega\omega_T c_3}{c_1} - \frac{i\omega\omega_T c_6}{mc_2} \right] V_{ez} = 0$$

Put equations (11a), (12a), and (17a) into equation (14),

$$(\nu_i - i\omega + \frac{i\omega_i^2}{\omega(1-n^2)}) \ (- \frac{c_6}{c_2} V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez})$$

$$- (\nu_{ie} + \frac{i\omega_i^2}{\omega(1-n^2)}) \ V_{ey} - \frac{\omega_L}{m} \ (- \frac{c_6}{c_2} V_{ez} - \frac{i\omega\omega_T}{c_2} V_{ex} + \frac{i\omega\omega_L}{c_2} V_{ey})$$

$$- \nu_{in} \left[\frac{i\omega\nu_{ne}}{\gamma_1} V_{ey} + \frac{i\omega\nu_{ni}}{\gamma_1} \ (- \frac{c_6}{c_2} V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez}) \right] = 0$$

$$\text{(14a)} \quad (- \frac{\omega^2\omega_L\omega_T}{mc_2})V_{ex} + (\frac{c_2}{m} - \frac{c_6 c_4}{c_z} + \frac{\omega^2\omega_L^2}{mc_2}) \ V_{ey}$$

$$+ \frac{i\omega\omega_L}{c_2} \ (\frac{c_6}{m} - c_4) \ V_{ez} = 0$$

Put equations (10a), (11a), (12a), and (18a) into equation (15),

$$(\nu_i - i\omega + \frac{i\omega_i^2}{\omega(1-n^2)}) \ (- \frac{c_6}{c_2} V_{ez} - \frac{i\omega\omega_T}{c_2} V_{ex} + \frac{i\omega\omega_L}{c_2} V_{ey})$$

$$- (\nu_{ie} + \frac{i\omega_i^2}{\omega(1-n^2)}) \ V_{ez} - \frac{\omega_T}{m} \ (- \frac{c_5}{c_1} V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez})$$

$$+ \frac{\omega_L}{m} \ (- \frac{c_6}{c_2} V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez}) - (\frac{i\omega\nu_{ne}\nu_{in}}{\gamma_1}) \ V_{ez}$$

$$- \frac{i\omega\nu_{in}\nu_{ni}}{\gamma_1} \ (- \frac{c_6}{c_2} V_{ez} - \frac{i\omega\omega_T}{c_2} V_{ex} + \frac{i\omega\omega_L}{c_2} V_{ey}) = 0$$

$$\text{(15a)} \quad (i\omega\omega_T) \ (\frac{c_5}{mc_1} - \frac{c_4}{c_2}) \ V_{ex} + (\frac{i\omega\omega_L}{c_2}) \ (c_4 - \frac{c_6}{m}) \ V_{ey}$$

$$+ \ (\frac{c_2}{m} - \frac{c_6 c_4}{c_2} + \frac{\omega^2\omega^2{}_L}{mc_2} + \frac{\omega^2\omega^2{}_T}{mc_1}) \ V_{ez} = 0$$

The original set of eighteen equations containing eighteen unknowns has now been combined into three equations with three unknowns. Equations (13a), (14a), and (15a) are three coupled equations for V_{ex}, V_{ey}, and V_{ez}. In matrix form these equations are given by,

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} V_{ex} \\ V_{ey} \\ V_{ez} \end{pmatrix} = 0$$

Where,

$$A_{11} = \frac{c_1}{m} - \frac{c_3 c_5}{c_1} + \frac{\omega^2 \omega_T^2}{mc_2}$$

$$A_{12} = - \frac{\omega^2 \omega_L \omega_T}{mc_2}$$

$$A_{13} = i\omega\omega_T \left(\frac{c_3}{c_1} - \frac{c_6}{mc_2}\right)$$

$$A_{21} = - \frac{\omega^2 \omega_L \omega_T}{mc_2}$$

$$A_{22} = \frac{c_2}{m} - \frac{c_4 c_6}{c_2} + \frac{\omega^2 \omega_L^2}{mc_2}$$

$$A_{23} = - \frac{i\omega\omega_L}{c_2} \left(c_4 - \frac{c_6}{m}\right)$$

$$A_{31} = i\omega\omega_T \left(\frac{c_5}{mc_1} - \frac{c_4}{c_2}\right)$$

$$A_{32} = \frac{i\omega\omega_L}{c_2} (c_4 - \frac{c_6}{m}$$

$$A_{33} = \left(\frac{c_2}{m} - \frac{c_4 c_6}{c_2} + \frac{\omega^2 \omega_L^2}{mc_2} + \frac{\omega^2 \omega_T^2}{mc_1}\right)$$

APPENDIX B

The dispersion relation in terms of the C's is obtained by expanding the determinant derived in Appendix A. This is the determinant of the coefficients of V_{ex}, V_{ey}, and V_{ez}, and to have a nontrivial solution the value of the determinant must be zero.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = 0$$

$$A_{12}A_{23}(A_{31}-A_{13}) + A_{22}(A_{11}A_{33}-A_{13}A_{31}) + A_{11}A_{23}^2 - A_{33}A_{12}^2 = 0$$

1st TERM,

$$(A_{31}-A_{13}) = i\omega\omega_T \left[\frac{c_5}{mc_1} - \frac{c_4}{c_2}\right] - i\omega\omega_T \left[\frac{c_3}{c_1} - \frac{c_6}{mc_2}\right]$$

$$= i\omega\omega_T \left[\frac{c_5}{mc_1} + \frac{c_6}{mc_2} - \frac{c_4}{c_2} - \frac{c_3}{c_1}\right]$$

$$A_{12}A_{23} = \left(\frac{-\omega^2\omega_T\omega_L}{mc_2}\right)\left(\frac{-i\omega\omega_L}{c_2}\right)\left(c_4 - \frac{c_6}{m}\right)$$

$$= (i\omega^3\omega^2{}_L\omega_T)\left[\frac{c_4}{mc_2^2} - \frac{c_6}{mc_2^2}\right]$$

$$A_{12}A_{23}(A_{31}-A_{13}) = (i\omega\omega_T)(i\omega^3\omega^2{}_L\omega_T)\left[\frac{c_4}{mc_2^2} - \frac{c_6}{m^2c_2^2}\right]$$

$$\left[\frac{c_5}{mc_1} + \frac{c_6}{mc_2} - \frac{c_4}{c_2} - \frac{c_3}{c_1}\right]$$

$$= -\omega^4\omega^2{}_T\omega_L^2\left[\frac{c_4c_5}{m^2c_1c_2^2} + \frac{c_4c_6}{m^2c_2^3} - \frac{c_4^2}{mc_2^3} - \frac{c_3c_4}{mc_1c_2^2}\right.$$

$$\left. - \frac{c_5c_6}{m^3c_1c_2^2} - \frac{c_6^2}{m^3c_2^3} + \frac{c_4c_6}{m^2c_2^3} + \frac{c_3c_6}{m^2c_1c_2^2}\right]$$

$$= \omega^4\omega_T^2\omega_L^2 \left[\frac{c_6^2}{m^3c_2^3} + \frac{c_5c_6}{m^3c_1c_2^2} - \frac{2c_4c_6}{m^2c_2^3} + \frac{-c_4c_5-c_3c_6}{m^2c_1c_2^2} + \frac{c_4^2}{mc_2^3} + \frac{c_3c_4}{mc_1c_2^2}\right]$$

2nd TERM,

$$A_{11}A_{33} = \left[\frac{c_1}{m} - \frac{c_3c_5}{c_1} + \frac{\omega^2\omega_T^2}{mc_2}\right]\left[\frac{c_2}{m} - \frac{c_4c_6}{c_2} + \frac{\omega^2\omega_L^2}{mc_2} + \frac{\omega^2\omega_T^2}{mc_1}\right]$$

$$= \left[\frac{c_1c_2}{m^2} - \frac{c_1c_4c_6}{mc_2} + \frac{c_1\omega^2\omega^2{}_L}{m^2c_2} + \frac{\omega^2\omega^2{}_T}{m^2} - \frac{c_2c_3c_5}{mc_1} + \frac{\omega^2\omega_T^2}{m^2} + \frac{c_3c_4c_5c_6}{c_1c_2} - \frac{c_3c_5\omega^2\omega_L^2}{mc_1c_2} - \frac{c_3c_5\omega^2\omega_T^2}{mc_1^2} - \frac{c_4c_6\omega^2\omega_T^2}{mc_2^2} + \frac{\omega^4\omega_T^2\omega_L^2}{m^2c_2^2} + \frac{\omega^4\omega_T^4}{m^2c_1c_2}\right]$$

$$-A_{13}A_{31} = -(i\omega\omega_T)\left[\frac{c_3}{c_1} - \frac{c_6}{mc_2}\right](i\omega\omega_T)\left[\frac{c_5}{mc_1} - \frac{c_4}{c_2}\right]$$

$$= \omega^2\omega_T^2\left[\frac{c_3c_5}{mc_1^2} - \frac{c_3c_4}{c_1c_2} - \frac{c_5c_6}{m^2c_1c_2} + \frac{c_4c_6}{mc_2^2}\right]$$

$$(A_{11}A_{33}-A_{13}A_{31}) = \left[\frac{c_3c_4c_5c_6-\omega^2\omega_T^2c_3c_4}{c_1c_2} + \frac{c_1c_2+2\omega^2\omega_T^2}{m^2} - \frac{c_2c_3c_5}{mc_1} - \frac{c_1c_4c_6}{mc_2} + \frac{\omega^2\omega_L^2c_1}{m^2c_2} + \frac{\omega^4\omega_T^2\omega_L^2}{m^2c_2^2} - \frac{\omega^2\omega_L^2c_3c_5}{mc_1c_2} + \frac{\omega^4\omega_T^4-\omega^2\omega_T^2c_5c_6}{m^2c_1c_2}\right]$$

$$A_{22}(A_{11}A_{33}-A_{13}A_{31}) = \left[\frac{c_1c_2^2+2\omega^2\omega_T^2c_2+2\omega^2\omega_L^2c_1}{m^3} + \frac{3\omega^4\omega_T^2\omega_L^2}{m^3c_2} - \frac{\omega^2\omega_T^2c_3c_4}{mc_1} + \frac{\omega^4\omega_T^4-\omega^2\omega_T^2c_5c_6}{m^3c_1} + \frac{\omega^4\omega_L^4c_1}{m^3c_2^3} + \frac{\omega^6\omega_L^4\omega_T^2}{m^3c_2^3} - \frac{2c_1c_4c_6}{m^2} + \frac{-2\omega^2\omega_L^2c_3c_5-c_2^2c_3c_5}{m^2c_1} - \frac{2\omega^2\omega_T^2c_4c_6}{m^2c_2} - \frac{2\omega^2\omega_L^2c_1c_4c_6}{m^2c_2^2} + \frac{\omega^6\omega_T^4\omega_L^2-\omega^4\omega_T^2\omega_L^2c_5c_6}{m^3c_1c_2^2} - \frac{\omega^4\omega_T^2\omega_L^2c_4c_6}{m^2c_2^3} + \frac{c_1c_4^2c_6^2}{mc_2^2}\right.$$

$$+ \frac{\omega^2\omega_T^2 c_4 c_5 c_6^2 - \omega^4\omega_T^4 c_4 c_6 - \omega^4\omega_L^4 c_3 c_5}{m^2 c_1 c_2^2} + \frac{2c_3c_4c_5c_6}{mc_1}$$

$$+ \frac{2\omega^2\omega_L^2 c_3c_4c_5c_6 - \omega^4\omega_T^2\omega_L^2 c_3 c_4}{mc_1c_2^2} + \frac{\omega^2\omega_T^2 c_3 c_4^2 c_6 - c_3 c_4^2 c_5 c_6^2}{c_1 c_2^2} \Bigg]$$

3rd TERM,

$$A_{11}A_{23}^2 = \left[\frac{c_1}{m} - \frac{c_3c_5}{c_1} + \frac{\omega^2\omega_T^2}{mc_2}\right] (i\omega\omega_L)^2 \left[\frac{c_4}{c_2} - \frac{c_6}{mc_2}\right]^2$$

$$= (-\omega^2\omega_L^2) \left[\frac{c_1}{m} - \frac{c_3c_5}{c_1} + \frac{\omega^2\omega_T^2}{mc_2}\right]\left[\frac{c_4^2}{c_2^2} - \frac{2c_4c_6}{mc_2^2} + \frac{c_6^2}{m^2c_2^2}\right]$$

$$= \Bigg[-\frac{\omega^2\omega_L^2c_1c_6^2}{m^3c_2^2} - \frac{\omega^4\omega_T^2\omega_L^2c_6^2}{m^3c_2^3} + \frac{2\omega^2\omega_L^2c_1c_4c_6}{m^2c_2^2}$$

$$+ \frac{2\omega^4\omega_L^2\omega_T^2c_4c_6}{m^2c_2^3} + \frac{\omega^2\omega_L^2c_3c_5c_6^2}{m^2c_1c_2^2} - \frac{\omega^2\omega_L^2c_1c_4^2}{mc_2^2}$$

$$- \frac{\omega^4\omega_T^2\omega_L^2c_4^2}{mc_2^3} - \frac{2\omega^2\omega_L^2c_3c_4c_5c_6}{mc_1c_2^2} + \frac{\omega^2\omega_L^2c_3c_4^2c_5}{c_1c_2^2}\Bigg]$$

4th TERM,

$$-A_{33}A_{12}^2 = -\left[\frac{c_2}{m} - \frac{c_4c_6}{c_2} + \frac{\omega^2\omega_L^2}{mc_2} + \frac{\omega^2\omega_T^2}{mc_1}\right] \left(\frac{-\omega^2\omega_T\omega_L}{mc_2}\right)^2$$

$$= \left[-\frac{\omega^4\omega_T^2\omega_L^2}{m^3c_2} - \frac{\omega^6\omega_L^4\omega_T^2}{m^3c_2^3} - \frac{\omega^6\omega_T^4\omega_L^2}{m^3c_1c_2^2} + \frac{\omega^4\omega_T^2\omega_L^2c_4c_6}{m^2c_2^3}\right]$$

Combine all terms and multiply by $(m^3\ c_1\ c_2^2)$,

DISPERSION RELATION

$$2m^2c_2^2c_3c_4c_5c_6 - m^3c_3c_4^2c_5c_6 - mc_2^4c_3c_5 - 2mc_1^2c_2^2c_4c_6 + m^2c_1^2c_4^2c_6^2$$

$$+ c_1^2c_2^4 + 2\omega^2\omega_T^2c_1c_2^3 - \omega^2\omega_T^2c_2^2c_5c_6 + m\omega^2\omega_T^2c_4c_5c_6^2 - m^2\omega^2\omega_T^2c_2^2c_3c_4$$

$$- 2m\omega^2\omega_T^2c_1c_2c_4c_6 + m^3\omega^2\omega_T^2c_3c_4^2c_6 + 2\omega^2\omega_L^2c_1^2c_2^2 - \omega^2\omega_L^2c_1^2c_6^2 - m^2\omega^2\omega_L^2c_1^2c_4^2$$

$$- 2m\omega^2\omega_L^2c_2^2c_3c_5 + m\omega^2\omega_L^2c_3c_5c_6^2 + m^3\omega^2\omega_L^2c_3c_4^2c_5 + 2\omega^4\omega_T^2\omega_L^2c_1c_2$$

$$- m\omega^4\omega_T^2\omega_L^2 c_4 c_5 - m\omega^4\omega_T^2\omega_L^2 c_3 c_6 + \omega^4\omega_T^4 c_2^2 - m\omega^4\omega_T^4 c_4 c_6 + \omega^4\omega_L^4 c_1^2$$

$$- m\omega^4\omega_L^4 c_3 c_5 = 0$$

APPENDIX C

Once the wave number k has been obtained for given ω, then the values for V_{ex}, V_{ey}, and V_{ez} are known for any given time and position. To determine the values for the other parameters (h_x, h_y, h_z, E_x, E_y, E_z, $N_{e,i,n}$, V_{ix}, V_{iy}, V_{iz}, V_{nx}, V_{ny}, V_{nz}), the following relations are used,

$$V_{ex} = V_{ex\,o}\, e^{-k_i x} \cos(k_r x - \omega t)$$

$$V_{ey} = V_{ey\,o}\, e^{-k_i x} \cos(k_r x - \omega t)$$

$$V_{ez} = V_{ez\,o}\, e^{-k_i x} \cos(k_r x - \omega t)$$

$$V_{ix} = -\frac{c_5}{c_1} V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez}$$

$$V_{iy} = -\frac{c_6}{c_2} V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez}$$

$$V_{iz} = -\frac{c_6}{c_2} V_{ez} + \frac{i\omega\omega_L}{c_2} V_{ey} - \frac{i\omega\omega_T}{c_2} V_{ex}$$

$$V_{nx} = \frac{i\omega}{\gamma_2} \left[(\gamma_{ne} - \frac{c_5}{c_1}) V_{ex} + \frac{i\omega\omega_T}{c_1} V_{ez} \right]$$

$$V_{ny} = \frac{i\omega}{\gamma_1} \left[(\gamma_{ne} - \frac{c_6}{c_2}) V_{ey} - \frac{i\omega\omega_L}{c_2} V_{ez} \right]$$

$$V_{nz} = \frac{i\omega}{\gamma_1} \left[(\gamma_{ne} - \frac{c_6}{c_2}) V_{ez} + \frac{i\omega\omega_L}{c_2} V_{ey} - \frac{i\omega\omega_T}{c_2} V_{ex} \right]$$

$$E_x = \frac{i4\pi e N_o}{\omega(1-n^2)} (V_{ex} - V_{ix})$$

$$E_y = \frac{i4\pi e N_o}{\omega(1-n^2)} (V_{ey} - V_{iy})$$

$$E_z = \frac{i4\pi eN_o}{\omega(1-n^2)} (V_{ez} - V_{iz})$$

$$N_e = \frac{kN_o}{\omega} V_{ex}$$

$$N_i = \frac{kN_o}{\omega} V_{ix}$$

$$N_n = \frac{kN_o}{\omega} V_{nx}$$

$$h_x = 0$$

$$h_y = -\frac{kc}{\omega} E_z$$

$$h_z = \frac{kc}{\omega} E_y$$

APPENDIX D

LISTING OF FORMAC EXPANSION PROGRAM

```
DISREL:   PROC OPTIONS(MAIN);
        FORMAC≠OPTIONS;
  OPTSET(PRINT);
  OPTSET(EXPND;LINELENGTH=120);
      DCL SYSPNCH OUTPUT STREAM FILE;
      DCL(L,ME,MI,NE,NN,NI,M,MSQ,MCU)  FLOAT DECIMAL;
      ON ENDFILE(SYSIN) GO TO QUIT;
 READ:  GET DATA;
/*ROUTINE TO CHANGE TO APPROPRIATE UNITS OF TIME AND LENGTH*/
   IF FACTIME=0 THEN GO TO SAME;
   IF FACTIME=1 THEN GO TO DESI;
   IF FACTIME=2 THEN GO TO CENTI;
   IF FACTIME=3 THEN GO TO MILLI;
   IF FACTIME=4 THEN GO TO MINUS4;
   IF FACTIME=5 THEN GO TO MINUS5;
   IF FACTIME=6 THEN GO TO MICRO;
   IF FACTIME=7 THEN GO TO MINUS7;
   IF FACTIME=8 THEN GO TO MINUS8;
   IF FACTIME=9 THEN GO TO MINUS9;
   IF FACTIME=10 THEN GO TO MINUS10;
DESI:  W=W*1.0E-1;  WE=WE*1.0E-1;  WI=WI*1.0E-1;
    VEI=VEI*1.0E-1; VEN=VEN*1.0E-1; VIN=VIN*1.0E-1;
    VNI=VNI*1.0E-1; UE=UE*1.0E-1; UI=UI*1.0E-1; UN=UN*1.0E-1;
    CL=CL*1.0E-1;
     GO TO SAME;
CENTI: W=W*1.0E-2;  WE=WE*1.0E-2;  WI=WI*1.0E-2;
    VEI=VEI*1.0E-2; VEN=VEN*1.0E-2; VIN=VIN*1.0E-2 ;
    VNI=VNI*1.0E-2; UE=UE*1.0E-2; UI=UI*1.0E-2; UN=UN*1.0E-2;
    CL=CL*1.0E-2;
     GO TO SAME;
MILLI: W=W*1.0E-3;  WE=WE*1.0E-3;  WI=WI*1.0E-3;
    VEI=VEI*1.0E-3; VEN=VEN*1.0E-3; VIN=VIN*1.0E-3;
    VNI=VNI*1.0E-3; UE=UE*1.0E-3; UI=UI*1.0E-3; UN=UN*1.0E-3;
    CL=CL*1.0E-3;
     GO TO SAME;
MINUS4:W=W*1.0E-4;  WE=WE*1.0E-4;  WI=WI*1.0E-4;
    VEI=VEI*1.0E-4; VEN=VEN*1.0E-4; VIN=VIN*1.0E-4;
    VNI=VNI*1.0E-4; UE=UE*1.0E-4; UI=UI*1.0E-4; UN=UN*1.0E-4;
    CL=CL*1.0E-4;
     GO TO SAME;
MINUS5:W=W*1.0E-5;  WE=WE*1.0E-5;  WI=WI*1.0E-5;
    VEI=VEI*1.0E-5; VEN=VEN*1.0E-5; VIN=VIN*1.0E-5;
    VNI=VNI*1.0E-5; UE=UE*1.0E-5; UI=UI*1.0E-5; UN=UN*1.0E-5;
    CL=CL*1.0E-5;
     GO TO SAME;
MICRO: W=W*1.0E-6;  WE=WE*1.0E-6;  WI=WI*1.0E-6;
    VEI=VEI*1.0E-6; VEN=VEN*1.0E-6; VIN=VIN*1.0E-6;
    VNI=VNI*1.0E-6; UE=UE*1.0E-6; UI=UI*1.0E-6; UN=UN*1.0E-6;
    CL=CL*1.0E-6;
     GO TO SAME;
```

```
MINUS7:  W=W*1.0E-7;  WE=WE*1.0E-7;  WI=WI*1.0E-7;
    VEI=VEI*1.0E-7; VEN=VEN*1.0E-7; VIN=VIN*1.0E-7;
    VNI=VNI*1.0E-7; UE=UE*1.0E-7; UI=UI*1.0E-7; UN=UN*1.0E-7;
    CL=CL*1.0E-7;
     GO TO SAME;
MINUS8:  W=W*1.0E-8;  WE=WE*1.0E-8;  WI=WI*1.0E-8;
    VEI=VEI*1.0E-8; VEN=VEN*1.0E-8; VIN=VIN*1.0E-8;
    VNI=VNI*1.0E-8; UE=UE*1.0E-8; UI=UI*1.0E-8; UN=UN*1.0E-8;
    CL=CL*1.0E-8;
     GO TO SAME;
MINUS9:  W=W*1.0E-9;  WE=WE*1.0E-9;  WI=WI*1.0E-9;
    VEI=VEI*1.0E-9; VEN=VEN*1.0E-9; VIN=VIN*1.0E-9;
    VNI=VNI*1.0E-9; UE=UE*1.0E-9; UI=UI*1.0E-9; UN=UN*1.0E-9;
    CL=CL*1.0E-9;
     GO TO SAME;
MINUS10: W=W*1.0E-10;  WE=WE*1.0E-10;  WI=WI*1.0E-10;
    VEI=VEI*1.E-10; VEN=VEN*1.0E-10; VIN=VIN*1.0E-10;
    VNI=VNI*1.0E10; UE=UE*1.0E-10; UI=UI*1.0E-10;
    UN=UN*1.0E-10; CL=CL*1.0E-10;
 SAME:   IF FACLEN=0  THEN GO TO OKAY;
         IF FACLEN=-1 THEN GO TO P10;
         IF FACLEN=-2 THEN GO TO P100;
        IF FACLEN=-3 THEN GO TO P1000;
         IF FACLEN=1  THEN GO TO M10;
         IF FACLEN=2  THEN GO TO M100;
         IF FACLEN=3  THEN GO TO KM;
         IF FACLEN=4  THEN GO TO M10TO4;
         IF FACLEN=5  THEN GO TO M10TO5;
         IF FACLEN=6  THEN GO TO MEGA;
P10:    UE=UE*1.0E01; UI=UI*1.0E01;  UN=UN*1.0E01;
    CL=CL*1.0E01;
    GO TO OKAY;
P100:   UE=UE*1.0E02;  UI=UI*1.0E02;  UN=UN*1.0E02;
    CL=CL*1.0E02;
    GO TO OKAY;
P1000:   UE=UE*1.0E03;  UI=UI*1.0E03;  UN=UN*1.0E03;
    CL=CL*1.0E03;
    GO TO OKAY;
M10:     UE=UE*1.0E-1;  UI=UI*1.0E-1;  UN=UN*1.0E-1;
    CL=CL*1.0E-1;
     GO TO OKAY;
M100:    UE=UE*1.0E-2;  UI=UI*1.0E-2;  UN=UN*1.0E-2;
    CL=CL*1.0E-2;
     GO TO OKAY;
KM:      UE=UE*1.0E-3;  UI=UI*1.0E-3;  UN=UN*1.0E-3;
    CL=CL*1.0E-3;
     GO TO OKAY;
M10TO4:  UE=UE*1.0E-4;  UI=UI*1.0E-4;  UN=UN*1.0E-4;
    CL=CL*1.0E-4;
     GO TO OKAY;
```

```
M10T05:  UE=UE*1.0E-5;  UI=UI*1.0E-5;  UN=UN*1.0E-5;
     CL=CL*1.0E-5;
      GO TO OKAY;
MEGA:     UE=UE*1.0E-6;  UI=UI*1.0E-6;  UN=UN*1.0E-6;
     CL=CL*1.0E-6;
 OKAY:   /*  CHANGING PL/1 VARIABLES TO FORMAC VARIABLES  */
       LET( WE=¤WE¤;WI=¤WI¤;UN=¤UN¤;UI=¤UI¤;UE=¤UE¤;W=¤W¤;
            VEN=¤VEN¤;VEI=¤VEI¤;VIN=¤VIN¤;VNI=¤VNI¤;
            ME=¤ME¤;MI=¤MI¤;H=¤H¤;CE=¤CE¤;CL=¤CL¤;
            NE=¤NE¤;NN=¤NN¤);
   /*  BASIC EQUATIONS NEEDED TO RUN PROGRAM EFFICIENTLY  */
       LET( NI = NE;   M=MI/ME;
            VIE =(VEI*NE)/(M*NI);   VNE =(VEN*NE)/(M*NN);
            VE =VEI+VEN;  VI=VIE+VIN;  VN=VNE+VNI;
            UESQ=UE**2;  UISQ=UI**2;  UNSQ=UN**2;
            WESQ=WE**2;   WISQ=WI**2;   CLSQ=CL**2;
            MSQ=M**2;   MCU=M**3;   WTSQ=WT**2;
            WTQU=WTSQ**2;  WLSQ=WL**2;  WLQU=WLSQ**2;
            WSQ=W**2;   WCU=W*WSQ;   WQUAD=WSQ**2;
            W2WT2= WSQ*WTSQ;
            W2WL2 = WSQ * WLSQ ;
            W4WT4 = W2WT2**2;
            W4WL4 = W2WL2**2;
            W4WTL2 = WQUAD*WTSQ*WLSQ) ;
    VIE = ARITH(VIE);
    VNE = ARITH(VNE);
    NI = ARITH(NI);
   /*  PRINT OUT BASIC FIELD PARAMETERS AND INPUT DATA  */
PUT LIST('COMPLETE THREE-FLUID THEORY DISPERSION EQUATION')
    PAGE;
PUT EDIT('MAGNETIC FIELD STRENGTH=',H,'WEBERS PER SQ. METER')
       (SKIP(6),A,E(12,5),A);
 PUT EDIT('PLASMA FREQUENCY OF ELECTRONS = ',WE) (SKIP(1),A,
  E(12,5));
 PUT EDIT('PLASMA FREQUENCY OF IONS       = ',WI) (SKIP(1),A,
  E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF ELECTRONS WITH IONS=',VEI)
          (SKIP(1),A,E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF ELECTRONS WITH NEUTRALS =',
   VEN)   (SKIP(1),A,E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF IONS WITH ELECTRONS=',VIE)
          (SKIP(1),A,E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF IONS WITH NEUTRALS =',VIN)
          (SKIP(1),A,E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF NEUTRALS WITH ELECTRONS =',
     VNE) (SKIP(1),A,E(12,5));
 PUT EDIT('COLLISION FREQUENCY OF NEUTRALS WITH IONS =',VNI)
          (SKIP(1),A,E(12,5));
 PUT EDIT('ELECTRON SOUND VELOCITY = ',UE,' METERS/SECOND')
          (SKIP(1),A,E(12,5),A);
```

```
PUT EDIT('ION SOUND VELOCITY        = ',UI,' METERS/SECOND')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('NEUTRAL SOUND VELOCITY    = ',UN,' METERS/SECOND')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('ELECTRON NUMBER DENSITY =',NE,'NUMBER PER CUBIC
 METER') (SKIP(1),A,E(12,5),A);
PUT EDIT('ION NUMBER DENSITY=',NI,'NUMBER PER CUBIC METER')
        (SKIP(1),A,E(12,5),A);
PUT EDIT('NEUTRAL NUMBER DENSITY  =',NN,'NUMBER PER CUBIC
 METER') (SKIP(1),A,E(12,5),A);
     /*  RELATIONS TO SHORTEN RUN TIME   */
     LET( SUB1 = WISQ*VN;    SUB2 = WESQ*VN;
          SUB3 = VE + VN;    SUB4 = VI + VN;
          SUB5 = (VE*VN)-(VEN*VNE) ;
          SUB6 = (VI*VN)-(VIN*VNI) ;
          SUB7 = (VEI*VN)+(VEN*VNI) ;
          SUB8 = WCU*CLSQ ;     SUB9 = WQUAD*CLSQ );
 /*  OBTAIN EQUATIONS FOR JUST THE NUMERATORS OF THE C'S
     SINCE THE DENOMINATOR WILL BE MULTIPLIED OUT IN THE
     FINAL DISPERSION RELATION.  THE FORM OF THE EQUATION
     IS AS FOLLOWS:
           C(I) = A(I)*K**4 + B(I)*K**2 + D(I)          */

     LET( A(1) = 0.0 ;
          A(2) = 0.0;
          A(3) = UISQ*UNSQ;
          A(4) = 0.0;
          A(5) = UESQ*UNSQ;
          A(6) = 0.0;
          B(1) = -(WESQ*UNSQ) + (#I*W*VEI*UNSQ)  ;
          B(2) = -(WSQ*CLSQ*SUB7) + (#I*WCU*CLSQ*VEI) ;
        B(3)=(WISQ*UNSQ)-WSQ*(UNSQ+UISQ)-(#I*W)*(VN*UISQ+
             VI*UNSQ);
          B(4) = (WSQ*CLSQ*SUB6)-(SUB9)-(#I*SUB8*SUB4);
        B(5)=(WESQ*UNSQ)-WSQ*(UNSQ+UESQ)-(#I*W)*(VN*UESQ+
             VE*UNSQ);
          B(6) = (WSQ*CLSQ*SUB5)-(SUB9)-(#I*SUB8*SUB3);
          D(1) = WSQ*(WESQ+SUB7)+ #I*(W*SUB2-WCU*VEI) ;
          D(2) = WSQ* D(1)   ;
        D(3)=WQUAD-WSQ*(WISQ+SUB6)+ #I*
            (WCU*SUB4-W*WISQ*VN);
          D(4) = WSQ * D(3)   ;
        D(5)=WQUAD-WSQ*(WESQ+SUB5)+ #I*
            (WCU*SUB3-W*WESQ*VN);
          D(6) = WSQ * D(5)   );
     /*   DENOMINATORS FOR C'S    */
     LET( CDEM(1) = (-UNSQ*K**2) + (WSQ+#I*W*VN) ;
        CDEM(2)=(-WSQ*CLSQ-#I*W*CLSQ*VN)*K**2 +
            (WQUAD+#I*WCU*VN);
          CDEM(3) = CDEM(1)   ;
```

```
            CDEM(4) = CDEM(2)  ;
            CDEM(5) = CDEM(1)  ;
            CDEM(6) = CDEM(2)  );
       /*  EQUATIONS FOR C'S WITH JUST THE NUMERATOR  */
 EQS:   DO I=1 TO 6 BY 1 ;  LET(I=¤I¤);
       LET( C(I) = A(I)*K**4 + B(I)*K**2 + D(I)  );
  ATOMIZE( A(I); B(I);  D(I)  );
       END EQS;
     /*  PREPARING TO OBTAIN EACH TERM OF DISPERSION RELATION
         AS FUNCTION OF W,WT,& WL.  COLLECTING LIKE QUANTITIES
         IN EACH TERM TO MAKE PROGRAM MORE EFFICIENT.   */
       LET( C1SQ = C(1)**2;   C2SQ = C(2)**2;
            C4SQ = C(4)**2;   C6SQ = C(6)**2;
            C1C2 = C(1)*C(2);  C3C4 = C(3)*C(4);
            C3C5 = C(3)*C(5);  C3C6 = C(3)*C(6);
            C4C5 = C(4)*C(5);  C4C6 = C(4)*C(6);
            C4C6SQ = C4C6**2;    C5C6 = C(5)*C(6);
            C2QU = C2SQ**2;   C1DC2D = CDEM(1)*CDEM(2);
            C2DSQ = CDEM(2)**2;   C1DC2DCU = C1DC2D*C2DSQ ;
            C1C2DSQ = C1DC2D**2;   C2DQUAD = C2DSQ**2  );
       /* INDIVIDUAL TERMS OF DISPERSION RELATION  */
       LET( TERM(1) = 2.0*MSQ*C2SQ*C3C5*C4C6  ;
            TERM(2) = MCU*C3C5*C4C6SQ  ;
            TERM(3) = M*C2QU*C3C5  ;
            TERM(4) = 2.0*M*C1SQ*C2SQ*C4C6  ;
            TERM(5) = MSQ*C1SQ*C4C6SQ ;
            TERM(6) = C1SQ*C2QU  ;
            TERM(7) = 2.0*W2WT2*C1C2*C2SQ*C1DC2D  ;
            TERM(8) = W2WT2*C2SQ*C5C6*C1DC2D  ;
            TERM(9) = M*W2WT2*C4C6*C5C6*C1DC2D  ;
            TERM(10) = MSQ*W2WT2*C2SQ*C3C4*C1DC2D  ;
            TERM(11) = 2.0*M*W2WT2*C1C2*C4C6*C1DC2D  ;
            TERM(12) = MCU*W2WT2*C3C4*C4C6*C1DC2D  ;
            TERM(13) = 2.0*W2WL2*C1SQ*C2SQ*C2DSQ  ;
            TERM(14) = W2WL2*C1SQ*C6SQ*C2DSQ  ;
            TERM(15) = MSQ*W2WL2*C1SQ*C4SQ*C2DSQ) ;
       LET( TERM(16) = 2.0*M*W2WL2*C2SQ*C3C5*C2DSQ  ;
            TERM(17) = M*W2WL2*C3C5*C6SQ*C2DSQ  ;
            TERM(18) = MCU*W2WL2*C3C4*C4C5*C2DSQ  ;
            TERM(19) = 2.0*W4WTL2*C1C2*C1DC2DCU  ;
            TERM(20) = M*W4WTL2*C4C5*C1DC2DCU  ;
            TERM(21) = M*W4WTL2*C3C6*C1DC2DCU  ;
            TERM(22) = W4WT4*C2SQ*C1C2DSQ  ;
            TERM(23) = M*W4WT4*C4C6*C1C2DSQ  ;
            TERM(24) = W4WL4*C1SQ*C2DQUAD  ;
            TERM(25) = M*W4WL4*C3C5*C2DQUAD  );
  /* ATOMIZING VARIABLES NO LONGER NEEDED.  */
 ATOMIZE(C1SQ;C2SQ;C4SQ;C6SQ;C1C2;C3C4;C3C5;C3C6;C4C5;C4C6;
         C4C6SQ;C5C6;C2QU;C1DC2D;C2DSQ;C1DC2DCU;C1C2DSQ;
          C2DQUAD);
```

```
 /*  DETERMINING CYCLOTRON FREQUENCY OF ELECTRON FOR USE IN
     VARYING PROPAGATION DIRECTION RELATIVE TO MAGNETIC
     FIELD.      */
     WC = (CE*H)/ME   ;   /* H IN WEBERS PER SQ. METER  */
 /* THE FOLLOWING CHANGES WC TO CORRECT UNITS OF TIME   */
  IF FACTIME=0 THEN WC=WC;
  ELSE IF FACTIME=1 THEN WC=WC*1.0E-1;
  ELSE IF FACTIME=2 THEN WC=WC*1.0E-2;
  ELSE IF FACTIME=3 THEN WC=WC*1.0E-3;
  ELSE IF FACTIME=4 THEN WC=WC*1.0E-4;
  ELSE IF FACTIME=5 THEN WC=WC*1.0E-5;
  ELSE IF FACTIME=6 THEN WC=WC*1.0E-6;
  ELSE IF FACTIME=7 THEN WC=WC*1.0E-7;
  ELSE IF FACTIME=8 THEN WC=WC*1.0E-8;
  ELSE IF FACTIME=9 THEN WC=WC*1.0E-9;
  ELSE IF FACTIME=10 THEN WC=WC*1.0E-10;
     LET( WC=¤WC¤);
 /*  PUTTING IN VALUES FOR WT AND WL AND OBTAINING
     DISPERSION RELATION.  ALSO, OBTAIN COEFFICIENTS OF
     POWERS OF K TO BE PUNCHED OUT FOR USE IN ROOTS
     PROGRAM.      */
CYCLOTRON:  DO J= 0 TO 90 BY 15 ; LET(J=¤J¤);
   PUT SKIP(6)  ;
PUT LIST('ANGLE BETWEEN DIRECTION OF PROPAGATION AND
   MAGNETIC FIELD');
PUT EDIT('IS EQUAL TO ',J,' DEGREES') (A,F(5,2),A);
     IF J= 0!J= 90 THEN GO TO NAT;
     IF J=15 THEN GO TO FIFTEEN;
     IF J= 30 THEN GO TO THIRTY;
     IF J=45 THEN GO TO ANGLE45;
     IF J= 60 THEN GO TO SIXTY;
     IF J=75 THEN GO TO ANGLE75;
FIFTEEN:   LET( REWT = WC*0.25882   ;
                REWL = WC*0.96593   );
     GO TO DIS;
THIRTY:  LET( REWT = WC*SIND(J)   ;
              REWL = WC*0.86603   );
         GO TO DIS;
ANGLE45:  LET( REWT = WC*0.70711   ;
               REWL = WC*0.70711   );
     GO TO DIS;
SIXTY:  LET( REWT = WC*0.86603   ;
             REWL = WC*COSD(J)   );
        GO TO DIS;
ANGLE75:  LET( REWT = WC*0.96593   ;
               REWL = WC*0.25882   );
     GO TO DIS;
NAT:    LET( REWT = WC*SIND(J)   ;
             REWL = WC*COSD(J)   );
DIS:    LET( DISPER = 0.0   );
```

```
TERMS:  DO I= 1 TO 25 BY 1 ; LET(I=¤I¤);
     LET(FTERM(I) = REPLACE(TERM(I),WT,REWT,WL,REWL));
     LET( DISPER = DISPER +FTERM(I)    );
        ATOMIZE(FTERM(I));
          END TERMS;        /*  NOW HAVE DISPERSION RELATION
                             FOR GIVEN VALUE OF WT, WL, & W. */
 /* NOW GET COEFFICIENTS OF K IN FORM TO BE PUNCHED OUT */
     LET( Z = HIGHPOW(DISPER,K) ;
          X = LOWPOW(DISPER,K)   );
KCOEF:  DO I = 2 TO 16 BY 2;  LET(I=¤I¤);
     LET( COEFK(I) = COEFF(DISPER,K**I) ;
          COEFKI(I) = COEFF(COEFK(I),#I) ;
          COEFKR(I) = COEFK(I) - #I*COEFKI(I)   );
          END KCOEF;
/*PUT IN CONSTANT TERM OF DISPERSION RELATION AS COEFK(0)*/
     LET( L = 0.0 ;
          COEFK(0) = REPLACE(DISPER,K,L) ;
          COEFKI(0) = COEFF(COEFK(0),#I)  ;
          COEFKR(0) = COEFK(0) - #I*COEFKI(0)  );
 /* PUNCH OUT ALL REAL AND IMAGINARY COEFFFICIENTS OF EACH
    POWER OF K, STARTING WITH HIGHEST POWER.  */
XPUNCH:   DO I= 16 BY -2 TO 0;   LET(I=¤I¤);
    LET( COER=COEFKR(I);
         COEI= COEFKI(I)     );
      PLCOER= ARITH(COER);
      PLCOEI= ARITH(COEI);
 PUT FILE(SYSPNCH)EDIT(PLCOER) (SKIP(1),E(13,5));
 PUT FILE(SYSPNCH)EDIT(PLCOEI) (SKIP(1),E(13,5));
  END XPUNCH;
     END CYCLOTRON;
     GO TO READ;
 PUT SKIP(1);
   QUIT: PUT LIST('THAT'S ALL FOLKS');
  END DISREL;
```

APPENDIX E

LISTING OF PROGRAM ROOTS

```
      DIMENSION A1(20)
      DOUBLE PRECISION A(20),FRE,FIM,FPRE,FPIM,XRE,XIM,X2RE,
     1X2IM,X3RE,X3IM,X4RE,X4IM,X5RE,X5IM,X6RE,X6IM,BRE,BIM,
     2NUMRE,NUMIM,DEM,W,X8RE,X8IM,X7RE,X7IM,ERRORE,ERROIM,
     3X(20),TEMP,FRT(35),FIT(35),FPRT(35),FPIT(35),X8TR(10),
     4X8TI(10),X7TR(10),X7TI(10),X6TR(10),KR(5),KRE(5),KI(5),
     5KIM(5),PHSVE(5),EFOLD(5)
 4000 READ(5,106) W,LEN,TYME,FAC
      DO 1000 I=1,18
      READ(5,100) A1(I)
      A(I) = A1(I)
 1000 CONTINUE
      WRITE(6,105) W,FAC,LEN,TYME
      DO 20 I=1,9
      WRITE(6,115) A(2*I-1),A(2*I)
   20 CONTINUE
      K = 8
    2 READ(5,101) BRE,BIM
      IF(BRE.EQ.99999.D0.AND.BIM.EQ.99999.D0) GO TO 300
      WRITE(6,104) BRE,BIM
      N = 1
      IF(K.EQ.8) GO TO 950
      IF(K.EQ.7) GO TO 951
      IF(K.EQ.6) GO TO 952
      IF(K.EQ.5) GO TO 953
      IF(K.EQ.4) GO TO 954
      IF(K.EQ.3) GO TO 955
      IF(K.EQ.2) GO TO 956
  950 CALL EIGHTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  951 CALL SEVEN(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  952 CALL SIXTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  953 CALL FIFTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  954 CALL FOURTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  955 CALL THIRD(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      GO TO 960
  956 CALL SECOND(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
  960 FRE= 0.0
      FIM= 0.0
      DO 15 I=1,M
      FRE = FRE+FRT(I)
      FIM = FIM+FIT(I)
   15 CONTINUE
      FPRE= 0.0
      FPIM= 0.0
      DO 16 I=1,L
```

```
      FPRE = FPRE+FPRT(I)
      FPIM = FPIM+FPIT(I)
   16 CONTINUE
      NUMRE = (FRE*FPRE) + (FIM*FPIM)
      NUMIM = (FIM*FPRE) - (FRE*FPIM)
      IF(NUMRE.EQ.0.D0.AND.NUMIN.EQ.0.D0) GO TO 4
      IF(FPRE.EQ.0.D0.OR.FPIM.EQ.0.D0) DEM= (FPRE**2)+
     1(FPIM**2)
      IF(FPRE.EQ.0.D0.OR.FPIM.EQ.0.D0) GO TO 6
      DEM = (FPRE**2)*(1.D0+(FPIM/FPRE)**2)
    6 IF(FPRE.EQ.0.D0.AND.FPIM.EQ.0.D0) WRITE(6,102)
      IF(FPRE.EQ.0.D0.AND.FPIM.EQ.0.D0) GO TO 2
      FRE = NUMRE/DEM
      FIM = NUMIM/DEM
      BRE = BRE-FRE
      BIM = BIM-FIM
      N = N+1
      IF(N.GT.300) GO TO 4
      IF(K.EQ.8) GO TO 950
      IF(K.EQ.7) GO TO 951
      IF(K.EQ.6) GO TO 952
      IF(K.EQ.5) GO TO 953
      IF(K.EQ.4) GO TO 954
      IF(K.EQ.3) GO TO 955
      IF(K.EQ.2) GO TO 956
    4 ERRORE = 0.0
      ERROIM = 0.0
      DO 17 I=1,M
      ERRORE = ERRORE+FRT(I)
      ERROIM = ERROIM+FIT(I)
   17 CONTINUE
  280 WRITE(6,103) BRE,BIM,ERRORE,ERROIM,N
      CALL XPOCPX(1,2,BRE,BIM,KR,KI)
      DO 90 I=1,2
      KRE(I) = KR(I)*(10.**(-LEN))/FAC
      KIM(I) = KI(I)*(10.**(-LEN))/FAC
      PHSVE(I) = W/KRE(I)
      EFOLD(I) = 1.0/KIM(I)
      WRITE(6,110) KR(I),KI(I)
      WRITE(6,112) KRE(I),KIM(I),PHSVE(I),EFOLD(I)
   90 CONTINUE
      K = K-1
      IF(K.EQ.1) GO TO 957
      IF(K.EQ.0) GO TO 4000
      CALL SYNDV(A,BRE,BIM,K)
      KJ=K+2
      DO 7000 I=1,KJ
      WRITE(6,115) A(2*I-1),A(2*I)
 7000 CONTINUE
      GO TO 2
```

```
957   A(1) = A(1)
      A(2) = A(2)
      A(3) = A(3) + (BRE*A(1)-BIM*A(2))
      A(4) = A(4) + (BIM*A(1)+BRE*A(2))
      BRE =-(A(1)*A(3)+A(2)*A(4))/(A(1)**2+A(2)**2)
      BIM = (A(2)*A(3)-A(1)*A(4))/(A(1)**2+A(2)**2)
      ERRORE=0.0
      ERROIM = 0.0
      N = 1
      GO TO 280
  100 FORMAT(E13.6)
  101 FORMAT(2D15.7)
  102 FORMAT(' ***** DERIVATIVE OF F(X)=0 ***** ')
  103 FORMAT('0',' ROOT = ',2D14.6,'   ERROR = ',2D14.6,
     3 '     NUMBER OF ITERATIONS PERFORMED = ',I3)
  104 FORMAT('-','   INITIAL ESTIMATE = ',2D14.6)
 105  FORMAT('1','ROOTS OF THE DISPERSION RELATION BY A
     1NEWTON-RAPHSON ITERATION TECHNIQUE '/'0',' APPLIED
     2 FREQUENCY = ',D12.6,'   LENGTH DIMENSION =',F3.1.'*
     310**',I2,'  METERS',
     1 '   TIME DIMENSION = 10**-',I2,'  SECONDS'/,
     1   '0',' DISPERSION RELATION COEFFICIENTS ')
  106 FORMAT(D15.7,2I2,F3.1)
  110 FORMAT('0','   K= ' , 2D14.6)
  112 FORMAT('0','  WAVE NUMBER = ',2D14.6,' 1/METERS '/'0',
     1   'PHASE VELOCITY = ',D14.6,'          E FOLDING
     2 DISTANCE = ',
     1   D14.6)
  115 FORMAT(2E13.6)
  300 STOP
      END



      SUBROUTINE ATWLTF(N,X)
C   USED TO ARRANGE TERMS OF EQUATION IN ASCENDING ORDER.
      DOUBLE PRECISION X(50),TEMP
      DO 20 I=1,N
      IP1 = I+1
      DO 20 J=IP1,N
      IF(DABS(X(I)).LE.DABS(X(J))) GO TO 20
      TEMP = X(I)
      X(I) = X(J)
      X(J) = TEMP
   20 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE XPOCPX(M,N,A,B,XR,XM)
C     THIS SUBROUTINE CALCULATES THE M/NTH ROOTS OF A
C     COMPLEX NUMBER OF THE FORM 'C = A + I*B'.
      DOUBLE PRECISION A, B, BAR, BETA, COEF, K, PI, RM, RN,
     1  XR(25), ALFA, T, XM(25)
      DOUBLE PRECISION AA,BB
      AA=DABS(A)
      BB=DABS(B)
      RM = M
      RN = N
      PI = 3.14159265358979324D0
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) BAR=DSQRT(A**2+B**2)
      IF(AA.EQ.0.D0.OR.BB.EQ.0.D0) GO TO 101
      BAR=DABS(A)*DSQRT(1.D0+(B/A)**2)
      IF(B.LE.1.D-32) GO TO 101
      IF(DABS(DLOG10(AA)-DLOG10(BB)).LE.4.D0)BAR=DSQRT(A**2+
     1  B**2)
  101 COEF = BAR**(RM/RN)
      T = DATAN2(B,A)
      IF (B .LT. 0.0D0) T = 2.0D0*PI - DABS(T)
      K = 0.0D0
      DO 100 I=1,N
      BETA = T + K*2.0D0*PI
      ALFA=BETA*RM/RN
      XR(I) = COEF*DCOS(ALFA)
      XM(I) = COEF*DSIN(ALFA)
  100 K = K + 1.0D0
      RETURN
      END
```

```
      SUBROUTINE SYNDV(A,BRE,BIM,K)
      DOUBLE PRECISION A(20),BRE,BIM
      J = 2*(K+1)+2
      DO 100 I=3,J,2
      A(1) = A(1)
      A(2) = A(2)
      A(I) = A(I) + (BRE*A(I-2)-BIM*A(I-1))
      A(I+1) = A(I+1) + (BIM*A(I-2)+BRE*A(I-1))
100   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SECOND(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
```

```
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),
     1FPIT(20)
      FRT(1) = A(1)*((BRE+BIM)*(BRE-BIM))
      FRT(2) = -A(2)*2.0*BRE*BIM
      FRT(3) = A(3)*BRE
      FRT(4) = -A(4)*BIM
      FRT(5) = A(5)
      FIT(1) = A(1)*2.0*BRE*BIM
      FIT(2) = A(2)*((BRE+BIM)*(BRE-BIM))
      FIT(3) = A(3)*BIM
      FIT(4) = A(4)*BRE
      FIT(5) = A(6)
      FPRT(1) = 2.0*A(1)*BRE
      FPRT(2) = -2.0*A(2)*BIM
      FPRT(3) = A(3)
      FPIT(1) = 2.0*A(1)*BIM
      FPIT(2) = 2.0*A(2)*BRE
      FPIT(3) = A(4)
      CALL ATWLTF(5,FRT)
      CALL ATWLTF(5,FIT)
      M = 5
      L = 3
      RETURN
      END
```

```
      SUBROUTINE THIRD(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),
     *FPRT(20),FPIT(20)
      FRT(1)=(A(1)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT(3.0)
     1*BIM))
      FRT(2)=(A(2)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-DSQRT(3.0)
     1*BRE))
      FRT(3) = A(3)*((BRE+BIM)*(BRE-BIM))
      FRT(4) = -A(4)*2.0*BRE*BIM
      FRT(5) = A(5)*BRE
      FRT(6) = -A(6)*BIM
      FRT(7) = A(7)
      FIT(1)=(A(1)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT(3.0)*
     1BRE-BIM))
      FIT(2)=(A(2)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT(3.0)
     1*BIM))
      FIT(3) = A(3)*2.0*BRE*BIM
      FIT(4) = A(4)*((BRE+BIM)*(BRE-BIM))
      FIT(5) = A(5)*BIM
      FIT(6) = A(6)*BRE
      FIT(7) = A(8)
      FPRT(1)= (3.0*A(1))*((BRE+BIM)*(BRE-BIM))
```

```
      FPRT(2) = -6.0*A(2)*BRE*BIM
      FPRT(3) = 2.0*A(3)*BRE
      FPRT(4) = -2.0*A(4)*BIM
      FPRT(5) = A(5)
      FPIT(1) = 6.0*A(1)*BRE*BIM
      FPIT(2) = (3.0*A(2))*((BRE+BIM)*(BRE-BIM))
      FPIT(3) = 2.0*A(3)*BIM
      FPIT(4) = 2.0*A(4)*BRE
      FPIT(5) = A(6)
      CALL ATWLTF(7,FRT)
      CALL ATWLTF(7,FIT)
      CALL ATWLTF(5,FPRT)
      CALL ATWLTF(5,FPIT)
      M = 7
      L = 5
      RETURN
      END



      SUBROUTINE FOURTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FPRT(20),FIT(20),
     1FPIT(20)
      FRT(1)=(A(1)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-DSQRT
     1(6.0)*BIM))
      FRT(2) = A(1)*(BIM**4)
      FRT(3)= (4.0*A(2)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
      FRT(4)=(A(3)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT(3.0)
     1*BIM))
      FRT(5)=(A(4)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-DSQRT(3.0)
     1*BRE))
      FRT(6) = A(5)*((BRE+BIM)*(BRE-BIM))
      FRT(7)  = -2.0*A(6)*BRE*BIM
      FRT(8)  = A(7)*BRE
      FRT(9)  = -A(8)*BIM
      FRT(10) = A(9)
      FIT(1)=(4.0*A(1)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
      FIT(2)=(A(2)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-DSQRT
     1(6.0)*BIM))
      FIT(3)=(A(3)*BIM)*((DSQRT(3.0)*BRE+BIM)*DSQRT(3.0)*
     1BRE-BIM))
      FIT(4) = A(2)*(BIM**4)
      FIT(5)=(A(4)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT(3.0)
     1*BIM))
      FIT(6)= A(6)*((BRE+BIM)*(BRE-BIM))
      FIT(7)  = A(7)*BIM
      FIT(8)  = A(8)*BRE
      FIT(9)  = A(10)
      FIT(10) = 2.0*A(5)*BRE*BIM
```

```
 FPRT(1)=(4.0*A(1)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
1DSQRT(3.0)*BIM))
 FPRT(2)=(4.0*A(2)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-
1DSQRT(3.0)*BRE))
 FPRT(3) = (3.0*A(3))*((BRE+BIM)*(BRE-BIM))
 FPRT(4) = 2.0*A(5)*BRE
 FPRT(5) = -2.0*A(6)*BIM
 FPRT(6)  = A(7)
 FPRT(7) = -6.0*A(4)*BRE*BIM
 FPIT(1)=(4.0*A(1)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT
1(3.0)*BRE-BIM))
 FPIT(2)=(4.0*A(2)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
1DSQRT(3.0)*BIM))
 FPIT(3) = (3.0*A(4))*((BRE+BIM)*(BRE-BIM))
 FPIT(4) = 2.0*A(5)*BIM
 FPIT(5) = 6.0*A(3)*BRE*BIM
 FPIT(6)  = A(8)
 FPIT(7) = 2.0*A(6)*BRE
 CALL ATWLTF(10,FRT)
 CALL ATWLTF(10,FIT)
 CALL ATWLTF(7,FPRT)
 CALL ATWLTF(7,FPIT)
 M = 10
 L = 7
 RETURN
 END



 SUBROUTINE FIFTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
 DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),
 1FPIT(20)
 FRT(1)=(A(1)*BRE**3)*((BRE+DSQRT(10.0)*BIM)*(BRE-
1DSQRT(10.0)*BIM))
 FRT(2)= (BIM**4)*(5.0*A(1)*BRE-A(2)*BIM)
 FRT(3)=(5.0*A(2)*BIM*BRE**2)*((DSQRT(2.0)*BIM+BRE)*
1(DSQRT(2.0)*BIM-BRE))
 FRT(4)=(A(6)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-DSQRT
1(3.0)*BRE))
 FRT(5)=(A(3)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-DSQRT
1(6.0)*BIM))
 FRT(6)= (4.0*A(4)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
 FRT(7)=(A(5)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT
1(3.0)*BIM))
 FRT(8) = A(3)*(BIM**4)
 FRT(9)= A(7)*((BRE+BIM)*(BRE-BIM))
 FRT(10) = -2.0*A(8)*BRE*BIM
 FRT(11) = A(9)*BRE
 FRT(12) = -A(10)*BIM
```

```
 FRT(13) = A(11)
 FIT(1)=(A(2)*BRE**3)*((BRE+DSQRT(10.0)*BIM)*(BRE-
1DSQRT(10.0)*BIM))
 FIT(2)= (BIM**4)*(A(1)*BIM+5.0*A(2)*BRE)
 FIT(3)=(5.0*A(1)*BIM*BRE**2)*((BRE+DSQRT(2.0)*BIM)*
*(BRE-DSQRT(2.0)*BIM))
 FIT(4)= (4.0*A(3)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
 FIT(5)= (A(4)*BRE**2)*((BRE+BIM)*(BRE-BIM))
 FIT(6)  = 2.0*A(7)*BRE*BIM
 FIT(7)=(A(5)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT(3.0)*
1BRE-BIM))
 FIT(8)=(A(6)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT
1(3.0)*BIM))
 FIT(9)= A(8)*((BRE+BIM)*(BRE-BIM))
 FIT(10) = A(4)*(BIM**4)
 FIT(11) = A(9)*BIM
 FIT(12) = A(10)*BRE
 FIT(13) = A(12)
 FPRT(1)=(5.0*A(1)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*
1(BRE-DSQRT(6.0)*BIM))
 FPRT(2) = 5.0*A(1)*(BIM**4)
 FPRT(3)=(20.0*A(2)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
 FPRT(4)=(4.0*A(3)*BRE)*((BRE+DSQRT(3.0)*BIM)*
1(BRE-DSQRT(3.0)*BIM))
 FPRT(5)=(4.0*A(4)*BIM)*((BIM+DSQRT(3.0)*BRE)*
1(BIM-DSQRT(3.0)*BRE))
 FPRT(6)= (3.0*A(5))*((BRE+BIM)*(BRE-BIM))
 FPRT(7)  = -6.0*A(6)*BRE*BIM
 FPRT(8)  = 2.0*A(7)*BRE
 FPRT(9)  = -2.0*A(8)*BIM
 FPRT(10) = A(9)
 FPIT(1)=(20.0*A(1)*BIM*BRE)*((BRE+BIM)*(BRE-BIM))
 FPIT(2)=(5.0*A(2)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*
1(BRE-DSQRT(6.0)*BIM))
 FPIT(3)=(4.0*A(3)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT
1(3.0)*BRE-BIM))
 FPIT(4) = 5.0*A(2)*(BIM**4)
 FPIT(5)=(4.0*A(4)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
1DSQRT(3.0)*BIM))
 FPIT(6)= (3.0*A(6))*((BRE+BIM)*(BRE-BIM))
 FPIT(7)  = 2.0*A(7)*BIM
 FPIT(8)  = 2.0*A(8)*BRE
 FPIT(9)  = A(10)
 FPIT(10) = 6.0*A(5)*BRE*BIM
 CALL ATWLTF(13,FRT)
 CALL ATWLTF(13,FIT)
 CALL ATWLTF(10,FPRT)
 CALL ATWLTF(10,FPIT)
 M = 13
 L = 10
```

```
      RETURN
      END

      SUBROUTINE SIXTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(30),FIT(30),FPRT(30),
     1FPIT(30)
      FRT(1)=(A(1)*BIM**4)*((DSQRT(15.0)*BRE+BIM)*(DSQRT
     1(15.0)*BRE-BIM))
      FRT(2)=(A(1)*BRE**4)*((BRE+DSQRT(15.0)*BIM)*(BRE-
     1DSQRT(15.0)*BIM))
      FRT(3)=(A(3)*BRE**3)*((BRE+DSQRT(10.0)*BIM)*(BRE-
     1DSQRT(10.0)*BIM))
      FRT(4)=(A(4)*BIM**3)*((DSQRT(10.0)*BRE+BIM)*(DSQRT
     1(10.0)*BRE-BIM))
      FRT(5) = -6.0*A(2)*BRE*(BIM**5)
      FRT(6) = -6.0*A(2)*BIM*(BRE**5)
      FRT(7) = 20.0*A(2)*(BRE**3)*(BIM**3)
      FRT(8)=(A(5)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
     1DSQRT(6.0)*BIM))
      FRT(9) = 5.0*A(3)*BRE*(BIM**4)
      FRT(10) = (4.0*A(6)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
      FRT(11)=(A(7)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
     1DSQRT(3.0)*BIM))
      FRT(12) = -5.0*A(4)*BIM*(BRE**4)
      FRT(13)=(A(8)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-DSQRT
     1(3.0)*BRE))
      FRT(14) = A(9)*((BRE+BIM)*(BRE-BIM))
      FRT(15) = A(5)*(BIM**4)
      FRT(16) = -2.0*A(10)*BRE*BIM
      FRT(17) = A(11)*BRE
      FRT(18) = -A(12)*BIM
      FRT(19) = A(13)
      FIT(1) = 6.0*A(1)*BRE*(BIM**5)
      FIT(2) = 6.0*A(1)*BIM*(BRE**5)
      FIT(3) = -20.0*A(1)*(BRE**3)*(BIM**3)
      FIT(4)=(A(2)*BRE**4)*((BRE+DSQRT(15.0)*BIM)*(BRE-
     1DSQRT(15.0)*BIM))
      FIT(5)=(A(2)*BIM**4)*((DSQRT(15.0)*BRE+BIM)*(DSQRT
     1(15.0)*BRE-BIM))
      FIT(6)=(A(3)*BIM**3)*((BIM+DSQRT(10.0)*BRE)*(BIM-
     1DSQRT(10.0)*BRE))
      FIT(7)=(A(4)*BRE**3)*((BRE+DSQRT(10.0)*BIM)*(BRE-
     1DSQRT(10.0)*BIM))
      FIT(8)=(4.0*A(5)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
      FIT(9) = 5.0*A(3)*BIM*(BRE**4)
      FIT(10)=(A(6)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
     1DSQRT(6.0)*BIM))
```

```
 FIT(11)=(A(7)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT
1(3.0)*BRE-BIM))
 FIT(12) = 5.0*A(4)*BRE*(BIM**4)
 FIT(13)=(A(8)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
1DSQRT(3.0)*BIM))
 FIT(14) = 2.0*A(9)*BRE*BIM
 FIT(15) = A(10)*((BRE+BIM)*(BRE-BIM))
 FIT(16) = A(11)*BIM
 FIT(17) = A(6)*(BIM**4)
 FIT(18) = A(12)*BRE
 FIT(19) = A(14)
 FPRT(1) = 6.0*A(1)*(BRE**5)
 FPRT(2) = 30.0*A(1)*BRE*(BIM**4)
 FPRT(3) = -60.0*A(1)*(BRE**3)*(BIM**2)
 FPRT(4) = -6.0*A(2)*(BIM**5)
 FPRT(5) = -30.0*A(2)*BIM*(BRE**4)
 FPRT(6) = 60.0*A(2)*(BRE**2)*(BIM**3)
 FPRT(7)=(5.0*A(3)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*
1 (BRE-DSQRT(6.0)*BIM))
 FPRT(8) = 5.0*A(3)*(BIM**4)
 FPRT(9)=(20.0*A(4)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
 FPRT(10)=(4.0*A(5)*BRE)*((BRE+DSQRT(3.0)*BIM)*
1(BRE-DSQRT(3.0)*BIM))
 FPRT(11)=(4.0*A(6)*BIM)*((BIM+DSQRT(3.0)*BRE)*
1(BIM-DSQRT(3.0)*BRE))
 FPRT(12)= (3.0*A(7))*((BRE+BIM)*(BRE-BIM))
 FPRT(13) = -6.0*A(8)*BRE*BIM
 FPRT(14) = 2.0*(A(9)*BRE-A(10)*BIM)
 FPRT(15) = A(11)
 FPIT(1) = 6.0*A(1)*(BIM**5)
 FPIT(2) = 30.0*A(1)*BIM*(BRE**4)
 FPIT(3) = -60.0*A(1)*(BRE**2)*(BIM**3)
 FPIT(4) = 6.0*A(2)*(BRE**5)
 FPIT(5) = 30.0*A(2)*BRE*(BIM**4)
 FPIT(6) = -60.0*A(2)*(BRE**3)*(BIM**2)
 FPIT(7)=(20.0*A(3)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
 FPIT(8)=(5.0*A(4)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*
1(BRE-DSQRT(6.0)*BIM))
 FPIT(9)=(4.0)*A(5)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT
1(3.0)*BRE-BIM))
 FPIT(10) = 5.0*A(4)*(BIM**4)
 FPIT(11)=(4.0*A(6)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE
1    -DSQRT(3.0)*BIM))
 FPIT(12)= (3.0*A(8))*((BRE+BIM)*(BRE-BIM))
 FPIT(13) = 6.0*A(7)*BRE*BIM
 FPIT(14) = 2.0*A(9)*BIM + 2.0*A(10)*BRE
 FPIT(15) = A(12)
 CALL ATWLTF(19,FRT)
 CALL ATWLTF(19,FIT)
 CALL ATWLTF(15,FPRT)
```

```
      CALL ATWLTF(15,FPIT)
      M = 19
      L = 15
      RETURN
      END


      SUBROUTINE SEVEN(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(35),FIT(35),FPRT(30),
     1FPIT(30)
      FRT(1) = A(1)*(BRE**7)
      FRT(2) = -7.0*A(1)*BRE*(BIM**6)
      FRT(3) = -21.0*A(1)*(BRE**5)*(BIM**2)
      FRT(4) = 35.0*A(1)*(BRE**3)*(BIM**4)
      FRT(5) = A(2)*(BIM**7)
      FRT(6) = -7.0*A(2)*BIM*(BRE**6)
      FRT(7) = -21.0*A(2)*(BRE**2)*(BIM**5)
      FRT(8) = 35.0*A(2)*(BRE**4)*(BIM**3)
      FRT(9) = A(3)*(BRE**6)
      FRT(10) = -A(3)*(BIM**6)
      FRT(11) = 15.0*A(3)*(BRE**2)*(BIM**4)
      FRT(12) = -15.0*A(3)*(BRE**4)*(BIM**2)
      FRT(13) = -6.0*A(4)*BRE*(BIM**5)
      FRT(14) =  20.0*A(4)*(BRE**3)*(BIM**3)
      FRT(15) = -6.0*A(4)*BIM*(BRE**5)
      FRT(16) = A(5)*(BRE**5)
      FRT(17) = 5.0*A(5)*BRE*(BIM**4)
      FRT(18) = -10.0*A(5)*(BRE**3)*(BIM**2)
      FRT(19) = -A(6)*(BIM**5)
      FRT(20) = -5.0*A(6)*BIM*(BRE**4)
      FRT(21) = 10.0*A(6)*(BRE**2)*(BIM**3)
      FRT(22)=(A(7)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
     1DSQRT(6.0)*BIM))
      FRT(23) = A(7)*(BIM**4)
      FRT(24)=(4.0*A(8)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
      FRT(25)=(A(9)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT
     1(3.0)*BIM))
      FRT(26)=(A(10)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-DSQRT
     1(3.0)*BRE))
      FRT(27) = A(11)*((BRE+BIM)*(BRE-BIM))
      FRT(28) = -2.0*A(12)*BRE*BIM
      FRT(29) = (A(13)*BRE) - (A(14)*BIM)
      FRT(30) = A(15)
      FIT(1) = -A(1)*(BIM**7)
      FIT(2) = 7.0*A(1)*BIM*(BRE**6)
      FIT(3) = -35.0*A(1)*(BRE**4)*(BIM**3)
      FIT(4) = 21.0*A(1)*(BRE**2)*(BIM**5)
      FIT(5) = A(2)*(BRE**7)
```

```
 FIT(6) = -21.0*A(2)*(BRE**5)*(BIM**2)
 FIT(7) = 35.0*A(2)*(BRE**3)*(BIM**4)
 FIT(8) = -7.0*A(2)*BRE*(BIM**6)
 FIT(9) = 6.0*A(3)*BRE*(BIM**5)
 FIT(10) = -20.0*A(3)*(BRE**3)*(BIM**3)
 FIT(11) = 6.0*A(3)*BIM*(BRE**5)
 FIT(12) = A(4)*(BRE**6)
 FIT(13) = -A(4)*(BIM**6)
 FIT(14) = -15.0*A(4)*(BRE**4)*(BIM**2)
 FIT(15) = 15.0*A(4)*(BRE**2)*(BIM**4)
 FIT(16) = A(5)*(BIM**5)
 FIT(17) = 5.0*A(5)*BIM*(BRE**4)
 FIT(18) = -10.0*A(5)*(BRE**2)*(BIM**3)
 FIT(19) = A(6)*(BRE**5)
 FIT(20) = 5.0*A(6)*BRE*(BIM**4)
 FIT(21) = -10.0*A(6)*(BRE**3)*(BIM**2)
 FIT(22)=(4.0*A(7)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
 FIT(23)=(A(8)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-DSQRT
1(6.0)*BIM))
 FIT(24)=(A(9)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT(3.0)*
1BRE-BIM))
 FIT(25) = A(8)*(BIM**4)
 FIT(26)=(A(10)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT
1(3.0)*BIM))
 FIT(27)= A(12) * ((BRE+BIM)*(BRE-BIM))
 FIT(28) = 2.0*A(11)*BRE*BIM
 FIT(29) = (A(13)*BIM) + (A(14)*BRE)
 FIT(30) = A(16)
 FPRT(1) = 7.0*A(1)*(BRE**6)
 FPRT(2) = -7.0*A(1)*(BIM**6)
 FPRT(3) = -105.0*A(1)*(BRE**4)*(BIM**2)
 FPRT(4) = 105.0*A(1)*(BRE**2)*(BIM**4)
 FPRT(5) = -42.0*A(2)*BIM*(BRE**5)
 FPRT(6) = 140.0*A(2)*(BRE**3)*(BIM**3)
 FPRT(7) = -42.0*A(2)*BRE*(BIM**5)
 FPRT(8) = 6.0*A(3)*(BRE**5)
 FPRT(9) = -60.0*A(3)*(BRE**3)*(BIM**2)
 FPRT(10) = 30.0*A(3)*BRE*(BIM**4)
 FPRT(11) = -6.0*A(4)*(BIM**5)
 FPRT(12) = -30.0*A(4)*BIM*(BRE**4)
 FPRT(13) = 60.0*A(4)*(BRE**2)*(BIM**3)
 FPRT(14)=(5.0*A(5)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
1DSQRT(6.0)*BIM))
 FPRT(15) = 5.0*A(5)*(BIM**4)
 FPRT(16)=(20.0*A(6)*BRE*BIM)*((BIM+BRE)*(BIM-BRE))
 FPRT(17)=(4.0*A(7)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
1DSQRT(3.0)*BIM))
 FPRT(18)=(4.0*A(8)*BIM)*((BIM+DSQRT(3.0)*BRE)*(BIM-
1DSQRT(3.0)*BRE))
 FPRT(19)=(3.0*A(9))*((BRE+BIM)*(BRE-BIM))
```

```
      FPRT(20) = -6.0*A(10)*BRE*BIM
      FPRT(21) = 2.0*((A(11)*BRE)-(A(12)*BIM))
      FPRT(22) = A(13)
      FPIT(1) = 42.0*A(1)*BRE*(BIM**5)
      FPIT(2) = -140.0*A(1)*(BRE**3)*(BIM**3)
      FPIT(3) = 42.0*A(1)*BIM*(BRE**5)
      FPIT(4) = 7.0*A(2)*(BRE**6)
      FPIT(5) = -7.0*A(2)*(BIM**6)
      FPIT(6) = -105.0*A(2)*(BRE**4)*(BIM**2)
      FPIT(7) = 105.0*A(2)*(BRE**2)*(BIM**4)
      FPIT(8) = 6.0*A(3)*(BIM**5)
      FPIT(9) = 30.0*A(3)*BIM*(BRE**4)
      FPIT(10) = -60.0*A(3)*(BRE**2)*(BIM**3)
      FPIT(11) = 6.0*A(4)*(BRE**5)
      FPIT(12) = -60.0*A(4)*(BRE**3)*(BIM**2)
      FPIT(13) = 30.0*A(4)*BRE*(BIM**4)
      FPIT(14)=(20.0*A(5)*BRE*BIM)*((BRE+BIM)*(BRE-BIM))
      FPIT(15)=(5.0*A(6)*BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
     1DSQRT(6.0)*BIM))
      FPIT(16)=(4.0*A(7)*BIM)*((DSQRT(3.0)*BRE+BIM)*(DSQRT
     1(3.0)*BRE-BIM))
      FPIT(17) = 5.0*A(6)*(BIM**4)
      FPIT(18)=(4.0*A(8)*BRE)*((BRE+DSQRT(3.0)*BIM)*(BRE-
     1DSQRT(3.0)*BIM))
      FPIT(19)=(3.0*A(10))*((BRE+BIM)*(BRE-BIM))
      FPIT(20) = 6.0*A(9)*BRE*BIM
      FPIT(21) = (A(11)*BIM)-(A(12)*BRE)
      FPIT(22) = A(14)
      CALL ATWLTF(30,FRT)
      CALL ATWLTF(30,FIT)
      CALL ATWLTF(22,FPRT)
      CALL ATWLTF(22,FPIT)
      M = 30
      L = 22
      RETURN
      END




      SUBROUTINE EIGHTH(BRE,BIM,A,FRT,FIT,FPRT,FPIT,M,L)
      DOUBLE PRECISION BRE,BIM,A(20),FRT(20),FIT(20),FPRT(20),
     1FPIT(20),X8TR(10),X8TI(10),X7TR(10),X7TI(10),X6TR(10),
     2X8RE,X8IM,X7RE,X7IM,X6RE,X6IM,X5RE,X5IM,X4RE,X4IM,X3RE,
     3X3IM,X2RE,X2IM,XRE,XIM
C     TERMS CONTAINING EACH POWER OF K
    3 X8TR(1) = -28.0*(BIM**6)*(BRE**2)
      X8TR(2) = 70.0*(BIM**4)*(BRE**4)
      X8TR(3) = -28.0*(BIM**2)*(BRE**6)
```

```
      X8TR(4) = BRE**8
      X8TR(5) = BIM**8
      X8TI(1) = -8.0*(BIM**7)*BRE
      X8TI(2) = 56.0*(BIM**5)*(BRE**3)
      X8TI(3) = -56.0*(BIM**3)*(BRE**5)
      X8TI(4) = 8.0*(BIM)*(BRE**7)
      X7TR(1) = -7.0*(BIM**6)*BRE
      X7TR(2) = 35.0*(BIM**4)*(BRE**3)
      X7TR(3) = -21.0*(BIM**2)*(BRE**5)
      X7TR(4) = BRE**7
      X7TI(1) = 21.0*(BIM**5)*(BRE**2)
      X7TI(2) = -35.0*(BIM**3)*(BRE**4)
      X7TI(3) = 7.0*BIM*(BRE**6)
      X7TI(4) = -(BIM**7)
      X6TR(1) = 15.0*(BIM**4)*(BRE**2)
      X6TR(2) = -15.0*(BIM**2)*(BRE**4)
      X6TR(3) = BRE**6
      X6TR(4) = -(BIM**6)
      CALL ATWLTF(5,X8TR)
      CALL ATWLTF(4,X8TI)
      CALL ATWLTF(4,X7TR)
      CALL ATWLTF(4,X7TI)
      CALL ATWLTF(4,X6TR)
      X8RE=0.0
      X8IM=0.0
      X7RE=0.0
      X7IM=0.0
      X6RE=0.0
      DO 35 I=1,5
      X8RE = X8RE + X8TR(I)
35    CONTINUE
      DO 36 I=1,4
      X8IM = X8IM + X8TI(I)
      X7IM = X7IM + X7TI(I)
      X6RE = X6RE + X6TR(I)
36    CONTINUE
      X6IM = 6.0D0*(BIM**5)*BRE - 20.0D0*(BIM**3)*(BRE**3)
     2      + 6.0D0* BIM*(BRE**5)
      X5RE=5.0*(BIM**4)*BRE+(BRE**3)*((BRE+DSQRT(10.0)*BIM)*
     1(BRE-DSQRT(10.0)*BIM))
      X5IM=(BIM**3)*((BIM+DSQRT(10.0)*BRE)*(BIM-DSQRT(10.0)*
     1BRE))+5.0*BIM*(BRE**4)
      X4RE=(BIM**4)+(BRE**2)*((BRE+DSQRT(6.0)*BIM)*(BRE-
     1DSQRT(6.0)*BIM))
      X4IM=(4.0*BIM*BRE)*((BRE+BIM)*(BRE-BIM))
      X3RE= BRE*((BRE+DSQRT(3.0)*BIM)*(BRE-DSQRT(3.0)*BIM))
      X3IM = BIM*((DSQRT(3.0)*BRE+BIM)*(DSQRT(3.0)*BRE-BIM))
      X2RE = (BRE+BIM)*(BRE-BIM)
      X2IM = 2.0D0*BIM*BRE
      XRE = BRE
```

```
      XIM= BIM
C    TERMS OF REAL PART OF F(X)
      FRT(1) = A(1)*X8RE
      FRT(2) = -A(2)*X8IM
      FRT(3) = A(3)*X7RE
      FRT(4) = -A(4)*X7IM
      FRT(5) = A(5)*X6RE
      FRT(6) = -A(6)*X6IM
      FRT(7) = A(7)*X5RE
      FRT(8) = -A(8)*X5IM
      FRT(9) = A(9)*X4RE
      FRT(10)= -A(10)*X4IM
      FRT(11)= A(11)*X3RE
      FRT(12)= -A(12)*X3IM
      FRT(13)= A(13)*X2RE
      FRT(14)= -A(14)*X2IM
      FRT(15)= A(15)*XRE
      FRT(16)= -A(16)*XIM
      FRT(17)= A(17)
C    TERMS OF IMAGINARY PART OF F(K)
      FIT(1) = A(1)*X8IM
      FIT(2) = A(2)*X8RE
      FIT(3) = A(3)*X7IM
      FIT(4) = A(4)*X7RE
      FIT(5) = A(5)*X6IM
      FIT(6) = A(6)*X6RE
      FIT(7) = A(7)*X5IM
      FIT(8) = A(8)*X5RE
      FIT(9) = A(9)*X4IM
      FIT(10)= A(10)*X4RE
      FIT(11)= A(11)*X3IM
      FIT(12)= A(12)*X3RE
      FIT(13)= A(13)*X2IM
      FIT(14)= A(14)*X2RE
      FIT(15)= A(15)*XIM
      FIT(16)= A(16)*XRE
      FIT(17) = A(18)
C    TERMS OF REAL PART OF F'(K)
      FPRT(1) = 8.0*A(1)*X7RE
      FPRT(2) = -8.0*A(2)*X7IM
      FPRT(3) = 7.0*A(3)*X6RE
      FPRT(4) = -7.0*A(4)*X6IM
      FPRT(5) = 6.0*A(5)*X5RE
      FPRT(6) = -6.0*A(6)*X5IM
      FPRT(7) = 5.0*A(7)*X4RE
      FPRT(8) = -5.0*A(8)*X4IM
      FPRT(9) = 4.0*A(9)*X3RE
      FPRT(10)= -4.0*A(10)*X3IM
      FPRT(11)= 3.0*A(11)*X2RE
      FPRT(12)= -3.0*A(12)*X2IM
```

```
      FPRT(13)= 2.0*A(13)*XRE
      FPRT(14)= -2.0*A(14)*XIM
      FPRT(15)= A(15)
C   TERMS OF IMAGINARY PART OF F'(K)
      FPIT(1) = 8.0*A(1)*X7IM
      FPIT(2) = 8.0*A(2)*X7RE
      FPIT(3) = 7.0*A(3)*X6IM
      FPIT(4) = 7.0*A(4)*X6RE
      FPIT(5) = 6.0*A(5)*X5IM
      FPIT(6) = 6.0*A(6)*X5RE
      FPIT(7) = 5.0*A(7)*X4IM
      FPIT(8) = 5.0*A(8)*X4RE
      FPIT(9) = 4.0*A(9)*X3IM
      FPIT(10)= 4.0*A(10)*X3RE
      FPIT(11)= 3.0*A(11)*X2IM
      FPIT(12)= 3.0*A(12)*X2RE
      FPIT(13)= 2.0*A(13)*XIM
      FPIT(14)= 2.0*A(14)*XRE
      FPIT(15)= A(16)
      CALL ATWLTF(17,FRT)
      CALL ATWLTF(17,FIT)
      CALL ATWLTF(15,FPRT)
      CALL ATWLTF(15,FPIT)
      M = 17
      L = 15
      RETURN
      END
```

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CHAPTER IV

ANALYSIS OF THE MIXED INITIAL-BOUNDARY VALUE PROBLEM FOR A THREE-FLUID PLASMA WITH A NUMERICAL EXAMPLE

by

Billy H. Johnson and David L. Murphree

NOTE: Figures, references and equations begin a new sequence in each Chapter. Also, the Appendices are lettered consecutively by Chapter, and each Chapter includes its own list of symbols.

LIST OF SYMBOLS

a_i	thermal speed of ions
a_e	thermal speed of electrons
a_n	thermal speed of neutrals
$\vec{B}$	magnetic field, weber/m^2, $B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$
c	speed of light, 3×10^8 m/sec
e	fundamental electronic charge, 1.6×10^{-19} coulomb
$\vec{E}$	electric field, V/m, $E_1\hat{i} + E_2\hat{j} + E_3\hat{k}$
$\vec{J}$	current density
K	Boltzman's constant, 1.38×10^{-23} joule/^{o}K
K_o	permittivity of vacuum, 8.85×10^{-12} coul^2 - sec^2/m^3 - Kg
m_i	mass of an argon ion, 6.73×10^{-26} Kg
m_e	mass of an electron, 9.1×10^{-31} Kg
m_n	mass of a neutral particle, 6.73×10^{-26} Kg
N_i	ion number density
N_e	electron number density
N_n	neutral number density
P_i	ion pressure
P_e	electron pressure
P_n	neutral species pressure
T_i	ion temperature
T_e	electron temperature
T_n	neutral species temperature
$\vec{V}_i$	velocity of ions, m/sec, $u_i\hat{i} + v_i\hat{j} + w_i\hat{k}$
$\vec{V}_e$	velocity of electrons, m/sec, $u_e\hat{i} + v_e\hat{j} + w_e\hat{k}$

$\vec{V}_n$	velocity of neutrals, m/sec, $u_n\hat{i} + v_n\hat{j} + w_n\hat{k}$
$W^{(i)}$	finite difference approximation of $V^{(i)}$, $i = 1,2,\cdots,17$
$\lambda^{(i)}$	eigenvalues of [A], $i = 1,2,\cdots,17$
$\Lambda^{(i)}$	finite difference approximation of $\lambda^{(i)}$
μ_o	permeability of vacuum, $4\pi \times 10^{-7}$ volt-sec/amp-m
ρ_i	ion mass density
ρ_e	electron mass density
ρ_n	neutral species mass density
ν_{ab}	effective collision frequency of species a with species b
ν'_{ab}	average collision frequency of species a with species b
$C \cap D$	intersection of sets C and D
$C \cup D$	union of sets C and D
$f \varepsilon \xi^2$	f plus its first and second derivatives are continuous

Subscripts

k,ℓ	x and t co-ordinates, $\phi_{k,\ell} = \phi(k\Delta x, \ell\Delta t)$

Notations

[ ]	matrix, either square or column
$\rightarrow$	3-D vector
$\overline{\phi}$	column matrix
$\overline{\phi\phi'}$	line segment from point ϕ to point ϕ'

INTRODUCTION

The physical problem of this study is to analyze how introducing a disturbance, which results in waves propagating, affects a one-dimensional three-fluid plasma flow. Previous studies of waves in plasmas have fully explored the use of the linearized governing equations in terms of finding various velocities as functions of the frequency. In addition, studies using a fully ionized two-fluid model have been performed using the nonlinear equations. However, in neither case have the governing equations been actually solved given an initial equilibrium state and some forced disturbance at a particular point as a function of time. The purpose of the present study is to solve the governing nonlinear equations of the three-fluid plasma given an initial equilibrium state and some forced disturbance at a particular point as a function of time. The equations are then linearized and solved using the same initial state and forced disturbance.

A partially ionized gas flowing in the x-direction which is composed of ions, electrons, and neutrals, all of which interact with each other through collisions, is considered. Thus a three-fluid mathematical model of the gas is chosen, where the three fluids are the ion, electron and neutral species. As previously stated, two separate models are considered. First, the nonlinear equations are used and then the linearized system of equations is considered. This linear system is obtained from the nonlinear equations. The numerical values of such quantities as collision frequencies, temperatures, and initial number densities, for the purpose of later solving a particular example, are considered to be those which might exist in an ionized gas at atmospheric pressure. Additional assumptions are that each fluid obeys the ideal gas law and viscosity effects are negligible. As will be seen later, each fluid is also assumed to be isothermal.

With this model of the plasma the dependent variables, which are all assumed to be functions of only the two independent variables x and t, become u_i, u_e, u_n, v_i, v_e, v_n, w_i, w_e, and w_n, which are the velocity components of the ions, electrons and neutrals in the x, y, and z directions; N_i, N_e and N_n, which are the number densities of the ions, electrons and neutrals; and E_1, E_2, E_3, B_2 and B_3, which are the components of the electric and magnetic fields.

Writing the continuity and momentum equations for the ions, electrons and neutrals; Maxwell's equations; and the equations of state for each fluid, all in scalar form, a system of 17 quasi-linear first order partial differential equations which can be written in matrix form as shown, is obtained.

$$[U_t] + [A][U_x] = [B]$$

where

$$[U_t] = [\frac{\partial U}{\partial t}] \text{ and } [U_x] = [\frac{\partial U}{\partial x}] \quad .$$

Similarly, a single matrix equation can be formed from the linearized system of equations. Since it is shown that the eigenvalues of [A] are all real and the corresponding eigenvectors are linearly independent, the system is classified as a hyperbolic one.

A numerical solution of first the nonlinear system of equations and then the linear system, using the same initial values for the 17 variables listed previously plus a forced disturbance at x = 0, is to be found. A comparison of these two solutions should then yield some insight into the limitations of a linearized approach. The disturbance imposed, which is the specification of boundary values for some of the variables, is not arbitrary if the existence of a unique solution is desired.

A finite difference technique is employed to find the numerical solutions of the two systems of equations for a particular example problem. The original system can be transformed in such a manner that the matrix [A] becomes a diagonal matrix by performing a transformation of the form

$$[U] = [T][V]$$

where [T] is an orthogonal matrix whose columns are composed of the unit orthogonal eigenvectors of [A]. It can be seen from this transformed system that the sum of the time and space derivative in each equation can be considered to be a time derivative along the characteristic associated with that equation. Two-point forward differences along the characteristic are then used to replace this time derivative. It is shown that this difference is the equivalent of the replacement of the original time derivative by forward differences and the spatial derivative by either backward or forward differences, depending upon whether the associated characteristic is positive or negative. The consistency, convergence and stability of the difference equations are then investigated.

LITERATURE REVIEW

As previously stated, many studies, especially with the linearized equations, have been done in the area of plasma waves. Perhaps Tanenbaum and Mintzer[1] have presented the best linear analysis by assuming a partially ionized three-fluid mathematical model of the plasma. Harmonic plane wave solutions were assumed and expressions relating wave number and frequency were then obtained. A rather comprehensive listing of other studies in this area is included in Ref. 1.

Other authors, among them Adlam and Allen[2], Baños and Vernon[3], Montgomery[4] and Saffman[5,6], have studied nonlinear waves in collision-free, fully ionized plasmas; i.e., the governing equations are not linearized. The major differences between these studies lie in the assumed direction of the applied magnetic field and in the neglect or retention of pressure terms in the momentum equations. In all the nonlinear studies encountered the physical situation was that of a disturbance traveling with a constant speed into an undisturbed plasma. Thus, the governing equations could be written in the reference frame of the wave and all time dependence was eliminated.

No work in the literature has been found where the multi-fluid equations, either linear or nonlinear, describing a plasma flow were completely solved subject to particular initial values and boundary conditions. Thus, quite naturally, no comparison of the linear and nonlinear solutions of such a problem has been possible.

Several authors have studied the mathematical aspects of the problem considered in the present study. Lax[7] has a very good discussion on the solution of hyperbolic systems of first order partial differential equations, as does Jeffrey and Tanuiti[8]. The latter, as well as Courant, Rees, and Isaacson[9], gives a discussion of the development of a numerical solution of the pure initial-value problem for quasi-linear systems. Thomée[10] employs a difference scheme to illustrate the existence and uniqueness of a solution of the mixed initial-boundary value problem for quasi-linear systems. The present study utilizes this particular scheme, with a discussion on how it is arrived at, to obtain a solution of an example problem. The convergence analysis given in Ref. 9 is applied to the particular diagonal system of equations with which the present study is concerned and the consistency and stability analysis is taken from Isaacson and Keller[11].

SECTION I

The problem of this study is to determine numerically how a disturbance affects a one-dimensional three-fluid plasma flow. The mathematical model is formulated using first the three-fluid nonlinear governing equations and then the corresponding linearized system of equations. A comparison between the two cases is made for a particular example to obtain some insight into the limitations of a linearized approach. In this section the basic governing equations are obtained, subject to the various assumptions imposed upon them. This scalar system of equations is then cast into a single matrix equation and classified as to its particular type. A discussion of the mathematical problem follows.

1. Basic Equations and Assumptions

In this study the model of the fluid used is commonly called a three-fluid model. In this model the plasma is assumed to be composed of ions, electrons, and neutral particles which interact with each other through collisions. The degree of ionization is assumed to be fixed so that the three-fluid mixture can be described completely by Maxwell's equations, the transport equations expressing conservation of mass and momentum for each of the ion, electron and neutral species, and the equations of state for the three gases. The following assumptions are made:

(1) Each gas obeys the ideal gas law.

(2) Collisional effects among the three interacting gases allow for conservation of the total momentum of the system.

(3) Each gas is inviscid.

(4) Each gas is isothermal.

The following set of equations[1] is therefore used to describe the plasma.

(1) The Maxwell equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_o \vec{J} + K_o \mu_o \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{1}{K_o} (N_i - N_e)$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

(2) The mass transport equations:

$$\frac{D\rho_e}{Dt} + \rho_e \nabla \cdot \vec{V}_e = 0$$

$$\frac{D\rho_i}{Dt} + \rho_i \nabla \cdot \vec{V}_i = 0$$

$$\frac{D\rho_n}{Dt} + \rho_n \nabla \cdot \vec{V}_n = 0$$

(3) The momentum transport equations:

$$\frac{D\vec{V}_e}{Dt} = - \frac{e}{m_e} (\vec{E} + \vec{V}_e \times \vec{B}) - \frac{\nabla P_e}{\rho_e} - \nu_{ei}(\vec{V}_e - \vec{V}_i) - \nu_{en}(\vec{V}_e - \vec{V}_n)$$

$$\frac{D\vec{V}_i}{Dt} = \frac{e}{m_i} (\vec{E} + \vec{V}_i \times \vec{B}) - \frac{\nabla P_i}{\rho_i} - \nu_{ie}(\vec{V}_i - \vec{V}_e) - \nu_{in}(\vec{V}_i - \vec{V}_n)$$

$$\frac{D\vec{V}_n}{Dt} = - \frac{\nabla P_n}{\rho_n} - \nu_{ne}(\vec{V}_n - \vec{V}_e) - \nu_{ni}(\vec{V}_n - \vec{V}_i)$$

(4) The equations of state:

$$P_e = N_e K T_e$$

$$P_i = N_i K T_i$$

$$P_n = N_n K T_n$$

where, as stated previously, T_e, T_i, and T_n are all equal to constants.

The collision frequencies which are used (ν_{ei} for electron-ion collisions, ν_{in} for ion-neutral collisions, etc.) are effective collision frequencies for momentum transfer between particles. If ν'_{ab} is the average number of collisions per second which each particle of type a has with particles of type b, then the effective collision frequency for momentum transfer can be expressed as[1]

$$\nu_{ab} = m_b \nu'_{ab} / (m_a + m_b)$$

As a consequence of this definition

$$\rho_a \nu_{ab} = \rho_b \nu_{ba} \quad ;$$

hence, there is no loss of total momentum in collisions among the particles in the fluid.

If the additional assumption is now made that all dependent variables are functions of only x and t, and if

$$\zeta_1 = \ln N_i$$

$$\zeta_2 = \ln N_e$$

$$\zeta_3 = \ln N_n \quad ,$$

then the vector equations can be expanded into the scalar system below.

$$\zeta_{1_t} + u_i \zeta_{1_x} + u_{i_x} = 0 \tag{1.1}$$

$$\zeta_{2_t} + u_e \zeta_{2_x} + u_{e_x} = 0 \tag{1.2}$$

$$\zeta_{3_t} + u_n \zeta_{3_x} + u_{n_x} = 0 \tag{1.3}$$

$$u_{e_t} + a_e^2 \zeta_{2_x} + u_e u_{e_x} = -\frac{e}{m_e}(E_1 + v_e B_3 - w_e B_2)$$

$$+ \nu_{ei}(u_i - u_e) + \nu_{en}(u_n - u_e) \qquad (1.5)$$

$$u_{n_t} + a_n^2 \zeta_{3_x} + u_n u_{n_x} = \nu_{ni}(u_i - u_n) + \nu_{ne}(u_e - u_n) \qquad (1.6)$$

$$v_{i_t} + u_i v_{i_x} = \frac{e}{m_i}(E_2 + w_i B_1 - u_i B_3) + \nu_{ie}(v_e - v_i) + \nu_{in}(v_n - v_i) \qquad (1.7)$$

$$v_{e_t} + u_e v_{e_x} = -\frac{e}{m_e}(E_2 + w_e B_1 - u_e B_3) + \nu_{ei}(v_i - v_e)$$

$$+ \nu_{en}(v_n - v_e) \qquad (1.8)$$

$$v_{n_t} + u_n v_{n_x} = \nu_{ni}(v_i - v_n) + \nu_{ne}(v_e - v_n) \qquad (1.9)$$

$$w_{i_t} + u_i w_{i_x} = \frac{e}{m_i}(E_3 + u_i B_2 - v_i B_1) + \nu_{ie}(w_e - w_i)$$

$$+ \nu_{in}(w_n - w_i) \qquad (1.10)$$

$$w_{e_t} + u_e w_{e_x} = -\frac{e}{m_e}(E_3 + u_e B_2 - v_e B_1) + \nu_{ei}(w_i - w_e)$$

$$+ \nu_{en}(w_n - w_e) \qquad (1.11)$$

$$w_{n_t} + u_n w_{n_x} = \nu_{ni}(w_i - w_n) + \nu_{ne}(w_e - w_n) \qquad (1.12)$$

$$E_{1_t} + (u_e + u_i)E_{1_x} = c^2 \mu_o e(u_e \exp\zeta_1 - u_i \exp\zeta_2) \qquad (1.13)$$

$$E_{2_t} + c^2 B_{3_x} = c^2 \mu_o e(v_e \exp\zeta_2 - v_i \exp\zeta_1) \qquad (1.14)$$

$$E_{3_t} - c^2 B_{2_x} = c^2 \mu_o e(w_e \exp\zeta_2 - w_i \exp\zeta_1) \qquad (1.15)$$

$$B_{2_t} - E_{3_x} = 0 \tag{1.16}$$

$$B_{3_t} + E_{2_x} = 0 \tag{1.17}$$

where $a_i^2 = \dfrac{KT_i}{m_i}$, $a_e^2 = \dfrac{KT_e}{m_e}$, $a_n^2 = \dfrac{KT_n}{m_n}$.

The reason for assuming that each gas is isothermal and then for writing the equations in terms of the natural log of the number densities will be discussed in a later section involving the transformation of the equations into a diagonal form.

The system of equations (1.1 - 1.17) is composed of the full non-linear governing equations. It is now desired to linearize the system by replacing each dependent variable by some constant part plus a fluctuating component; i.e.,

$$\phi(x,t) = \phi_o + \phi'(x,t)$$

where the fluctuating component ϕ' is assumed to be small enough that second order terms can be neglected. In order to have the same form as the non-linear system, let

$$\zeta_1' = N_i'/N_{i_o} \quad ,$$

$$\zeta_2' = N_e'/N_{e_o}$$

and

$$\zeta_3' = N_n'/N_{n_o} \quad .$$

Thus, the linear system becomes:

$$\zeta'_{1_t} + u_{i_o}\zeta'_{1_x} + u'_{i_x} = 0 \tag{1.18}$$

$$\zeta'_{2_t} + u_{e_o}\zeta'_{2_x} + u'_{e_x} = 0 \tag{1.19}$$

$$\zeta'_{3_t} + u_{n_o}\zeta'_{3_x} + u'_{n_x} = 0 \tag{1.20}$$

$$\begin{aligned} u'_{i_t} + a_i^2\zeta'_{1_x} + u_{i_o}u'_{i_x} = \frac{e}{m_i}(E_{1_o} &+ v_{i_o}B_{3_o} - w_{i_o}B_{2_o} + E'_1 \\ &+ v_{i_o}B'_3 + B_{3_o}v'_i - w_{i_o}B'_2 - B_{2_o}w'_i) \\ &+ \nu_{ie}(u_{e_o} - u_{i_o} + u'_e - u'_i) \\ &+ \nu_{in}(u_{n_o} - u_{i_o} + u'_n - u'_i) \end{aligned} \tag{1.21}$$

$$\begin{aligned} u'_{e_t} + a_e^2\zeta'_{2_x} + u_{e_o}u'_{e_x} = -\frac{e}{m_e}(E_{1_o} &+ v_{e_o}B_{3_o} - w_{e_o}B_{2_o} + E'_1 \\ &+ v_{e_o}B'_3 + B_{3_o}v'_e - w_{e_o}B'_2 - B_{2_o}w'_e) \\ &+ \nu_{ei}(u_{i_o} - u_{e_o} + u'_i - u'_e) \\ &+ \nu_{en}(u_{n_o} - u_{e_o} + u'_n - u'_e) \end{aligned} \tag{1.22}$$

$$\begin{aligned} u'_{n_t} + a_n^2\zeta'_{3_x} + u_{n_o}u'_{n_x} = \nu_{ni}(u_{i_o} &- u_{n_o} + u'_i - u'_n) \\ &+ \nu_{ne}(u_{e_o} - u_{n_o} + u'_e - u'_n) \end{aligned} \tag{1.23}$$

$$\begin{aligned} v'_{i_t} + u_{i_o}v'_{i_x} = \frac{e}{m_i}(E_{2_o} &+ w_{i_o}B_1 - u_{i_o}B_{3_o} + E'_2 + B_1w'_i - u_{i_o}B'_3 \\ &- B_{3_o}u'_i) + \nu_{ie}(v_{e_o} - v_{i_o} + v'_e - v'_i) \\ &+ \nu_{in}(v_{n_o} - v_{i_o} + v'_n - v'_i) \end{aligned} \tag{1.24}$$

$$v'_{e_t} + u_{e_o} v'_{e_x} = -\frac{e}{m_e}(E_{2_o} + w_{e_o} B_1 - u_{e_o} B_{3_o} + E'_2 + B_1 w'_e$$

$$- u_{e_o} B'_3 - B_{3_o} u'_e) + \nu_{ei}(v_{i_o} - v_{e_o} + v'_i - v'_e)$$

$$+ \nu_{en}(v_{n_o} - v_{e_o} + v'_n - v'_e) \tag{1.25}$$

$$v'_{n_t} + u_{n_o} v'_{n_x} = \nu_{ni}(v_{i_o} - v_{n_o} + v'_i - v'_n)$$

$$+ \nu_{ne}(v_{e_o} - v_{n_o} + v'_e - v'_n) \tag{1.26}$$

$$w'_{i_t} + u_{i_o} w'_{i_x} = \frac{e}{m_i}(E_{3_o} + u_{i_o} B_{2_o} - B_1 v_{i_o} + E'_3 + u_{i_o} B'_2 + B_{2_o} u'_i - B_1 v'_i)$$

$$+ \nu_{ie}(w_{e_o} - w_{i_o} + w'_e - w'_i) + \nu_{in}(w_{n_o} - w_{i_o}$$

$$+ w'_n - w'_i) \tag{1.27}$$

$$w'_{e_t} + u_{e_o} w'_{e_x} \quad \frac{e}{m_e}(E_{3_o} + u_{e_o} B_{2_o} - B_1 v_{e_o} + E'_3 + u_{e_o} B'_2 + B_{2_o} u'_e$$

$$- B_1 v'_e) + \nu_{ei}(w_{i_o} - w_{e_o} + w'_i - w'_e)$$

$$+ \nu_{en}(w_{n_o} - w_{e_o} + w'_n - w'_e) \tag{1.28}$$

$$w'_{n_t} + u_{n_o} w'_{n_x} = \nu_{ni}(w_{i_o} - w_{n_o} + w'_i - w'_n)$$

$$+ \nu_{ne}(w_{e_o} - w_{n_o} + w'_e - w'_n) \tag{1.29}$$

$$E'_{1_t} + (u_{i_o} + u_{e_o})E'_{1_x} = c^2 \mu_o e(u_{e_o} N_{i_o} + u_{e_o} N_{i_o} \zeta'_1 + N_{i_o} u'_e$$

$$- N_{e_o} u_{i_o} - u_{i_o} N_{e_o} \zeta'_2 - N_{e_o} u'_i) \tag{1.30}$$

$$E'_{2_t} + c^2 B'_{3_x} = c^2 \mu_o e(v_{e_o} N_{e_o} + v_{e_o} N_{e_o} \zeta'_2 + N_{e_o} v'_e$$

$$- N_{i_o} v_{i_o} - v_{i_o} N_{i_o} \zeta'_1 - N_{i_o} v'_i) \tag{1.31}$$

$$E'_{3_t} - c^2 B'_{2_x} = c^2 \mu_o e(w_{e_o} N_{e_o} + w_{e_o} N_{e_o} \zeta'_2 + N_{e_o} w'_e$$

$$- N_{i_o} w_{i_o} - w_{i_o} N_{i_o} \zeta'_1 - N_{i_o} w'_i) \tag{1.32}$$

$$B'_{2_t} - E'_{3_x} = 0 \tag{1.33}$$

$$B'_{3_t} + E'_{2_x} = 0 \tag{1.34}$$

2. Matrix Formulation and Classification of the Equations

Equations (1.1 - 1.17) comprise a non-linear scalar system of seventeen first-order partial differential equations involving the seventeen unknowns ζ_1, ζ_2, ζ_3, u_i, u_e, u_n, v_i, v_e, v_n, w_i, w_e, w_n, E_1, E_2, E_3, B_2, and B_3. B_1 is a constant and thus is known. The above equations are now cast into a single matrix equation so that they can be transformed into a diagonal system. It can easily be seen that these equations may be written as

$$[U_t] + [A][U_x] = [B] \tag{1.35}$$

with

$$[A] = [A]([U]) \text{ and } [B] = [B]([U])$$

where [U] and [B] are column matrices and [A] is a square 17 x 17 matrix, as shown. Here,

$$[U]_t = [U_t]$$

and

$$[U]_x = [U_x]$$

where the t and x of course refer to partial differentiation.

$$[U] = \begin{bmatrix} \zeta_1(x,t) \\ \zeta_2(x,t) \\ \zeta_3(x,t) \\ u_i(x,t) \\ u_e(x,t) \\ u_n(x,t) \\ v_i(x,t) \\ v_e(x,t) \\ v_n(x,t) \\ w_i(x,t) \\ w_e(x,t) \\ w_n(x,t) \\ E_1(x,t) \\ E_2(x,t) \\ E_3(x,t) \\ B_2(x,t) \\ B_3(x,t) \end{bmatrix}$$

$$[A] = \begin{bmatrix}
u_i & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_e & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_n & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_i^2 & 0 & 0 & u_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_e^2 & 0 & 0 & u_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_n^2 & 0 & 0 & u_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u_e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_i+u_e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e/m_i(E_1 + v_iB_3 - w_iB_2) + \nu_{ie}(u_e - u_i) + \nu_{in}(u_n - u_i) \\ -e/m_e(E_1 + v_eB_3 - w_eB_2) + \nu_{ei}(u_i - u_e) + \nu_{en}(u_n - u_e) \\ \nu_{ni}(u_i - u_n) + \nu_{ne}(u_e - u_n) \\ e/m_i(E_2 + B_1w_i - u_iB_3) + \nu_{ie}(v_e - v_i) + \nu_{in}(v_n - v_i) \\ -e/m_e(E_2 + B_1w_e - u_eB_3) + \nu_{ei}(v_i - v_e) + \nu_{en}(v_n - v_e) \\ \nu_{ni}(v_i - v_n) + \nu_{ne}(v_e - v_n) \\ e/m_i(E_3 + u_iB_2 - v_iB_1) + \nu_{ie}(w_e - w_i) + \nu_{in}(w_n - w_i) \\ -e/m_e(E_3 + u_eB_2 - v_eB_1) + \nu_{ei}(w_i - w_e) + \nu_{en}(w_n - w_e) \\ \nu_{ni}(w_i - w_n) + \nu_{ne}(w_e - w_n) \\ c^2\mu_o e(u_e e^{\zeta_1} - u_i e^{\zeta_2}) \\ c^2\mu_o e(v_e e^{\zeta_2} - v_i e^{\zeta_1}) \\ c^2\mu_o e(w_e e^{\zeta_2} - w_i e^{\zeta_1}) \\ 0 \\ 0 \end{bmatrix}$$

Similarly, the linearized system of equations (1.18 - 1.34) could also be written as a single matrix equation. A system of equations such as equation (1.35) is hyperbolic if the coefficient matrix [A] contains only real eigenvalues and is diagonalizable[7]; i.e., if one can find a matrix [T] such that

$$[T]^{-1}[A][T] = [D] \quad ,$$

where [D] is a real diagonal matrix. Such a [T] can be found if the eigenvectors of [A] are linearly independent. Setting $\det[A - \lambda I] = 0$, and solving for the roots of the resulting polynomial, the eigenvalues of [A] are found to be

$$\begin{aligned}
\lambda_1 &= -c \\
\lambda_2 &= -c \\
\lambda_3 &= u_e - a_e \\
\lambda_4 &= u_i - a_i \\
\lambda_5 &= u_n - a_n \\
\lambda_6 &= u_i + a_i \\
\lambda_7 &= u_n + a_n \\
\lambda_8 &= u_i \\
\lambda_9 &= u_i \\
\lambda_{10} &= u_e \\
\lambda_{11} &= u_e \\
\lambda_{12} &= u_n \\
\lambda_{13} &= u_n \\
\lambda_{14} &= u_i + u_e \\
\lambda_{15} &= u_e + a_e \\
\lambda_{16} &= c \\
\lambda_{17} &= c
\end{aligned}$$

and the corresponding eigenvectors, as obtained from

$$[A - \lambda_i I][X_i] = [0]$$

where $[X_i]$ is an eigenvector corresponding to the eigenvalue λ_i, are given on the following page.

$$\overline{X}_1 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -c, 0, 0, 1)$$

$$\overline{X}_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, c, 1, 0)$$

$$\overline{X}_3 = (0, -\frac{1}{a_e}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_4 = (-\frac{1}{a_i}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_5 = (0, 0, -\frac{1}{a_n}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_6 = (\frac{1}{a_i}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_7 = (0, 0, \frac{1}{a_n}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_8 = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_9 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_{10} = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_{11} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_{12} = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_{13} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$$

$$\overline{X}_{14} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)$$

$$\overline{X}_{15} = (0, \frac{1}{a_e}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$\overline{X}_{16} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, c, 0, 0, 1)$$

$$\overline{X}_{17} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -c, 1, 0)$$

Forming a matrix whose columns (or rows) are composed of these eigenvectors and then evaluating the determinant of that matrix, one finds its value is not zero; thus, the eigenvectors above are linearly independent. Therefore, since the eigenvalues of [A] are real and the corresponding eigenvectors are linearly independent, the matrix [A] can be diagonalized to yield a real diagonal matrix, and thus the system of equations, represented by equation (1.35), is hyperbolic.

3. The Mathematical Problem

It has been stated that the physical problem involved in this study is to impose some disturbance upon a plasma initially in equilibrium and then determine the effect at later times and spatial positions. Thus, a mixed initial-boundary value problem is being considered where values of variables at t = 0 are known over some range of x and values of some variables are known at x = 0 as a function of time. As discussed in Ref. 7, the number of dependent variables which may be assigned values on the boundary x = 0 is not arbitrary. This is discussed in more detail in the next chapter after the equations have been cast in a diagonal form. In summary, the mathematical problem is to find a solution of

$$[U_t] + [A][U_x] = [B]$$

given the initial values $[U(x,o)]$ and some of the boundary values $[U(o,t)]$.

The existence and uniqueness of a solution is discussed in a later section. In this discussion and in discussions about the solution technique the nonlinear system will be considered; however, the same discussion could be applied to the linearized equations.

SECTION II

The first part of this section is concerned with the transformation of the matrix [A] into diagonal form. Next the boundary conditions; i.e., the imposed disturbance, are discussed in general and then, along with the initial values, for a particular example model. In the concluding section the conditions which must be met in order for a unique solution to exist are considered.

1. Tranformation of the System into a Diagonal Form

It is advantageous to transform [A] into diagonal form for several reasons. First the existence theorem[10] (discussed in the last section of this chapter) applies to such a diagonal system. In addition, a diagonal form is needed in order to determine what variables, or relations such that these variables can be calculated, can be specified on the x = 0 boundary as a function of time.

Previously when the system was classified as being hyperbolic it was stated that one could find a matrix [T] such that

$$[T]^{-1}[A][T] = [D] \tag{2.1}$$

where [D] is a real diagonal matrix. The determination of the transformation matrix [T] proceeds as follows: In Section I the eigenvalues and corresponding eigenvectors of [A] were listed. The first step in the formation of [T] is to find the corresponding set of unit orthogonal eigenvectors; i.e., the set such that

$$\overline{X}_i^T \, \overline{X}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{2.2}$$

Since it has been shown that the eigenvectors of [A] are linearly independent, the Gram-Schmidt orthogonalization procedure[12] can be used to find a set of unit orthogonal eigenvectors. In Appendix A such a procedure is illustrated. Once these are found, [T], which is shown on the following page, with

$$D_1 = \frac{1}{\sqrt{1 + a_i^2}} \quad , \quad D_2 = \frac{1}{\sqrt{1 + a_e^2}}$$

$$D_3 = \frac{1}{\sqrt{1 + a_n^2}} \quad , \quad D_4 = \frac{1}{\sqrt{1 + c^2}} \quad ,$$

is formed by letting its columns be composed of these unit orthogonal eigenvectors. This method of forming [T] creates an orthogonal matrix; i.e.,

$$[T]^{-1} = [T]^T \tag{2.3}$$

which proves to be useful later. In addition, with [T] formed in the manner explained, [D] is a diagonal matrix containing the eigenvalues of [A] as its diagonal elements. A discussion of such a similarity transformation can be found in any book on matrices. With the [T] matrix having been formed, the discussion of the transformation of equation (1.35) into a diagonal form can be continued.

First make the transformation

$$[U] = [T][V] \tag{2.4}$$

in equation (1.35), which yields

$$[T]_t[V] + [T][V]_t + [A]([T]_x[V] + [T][V]_x) = [B]$$

or

$$[T][V]_t + [A][T][V]_x = [B] - [T]_t[V] - [A][T]_x[V] \quad .$$

$$
[T] = \begin{bmatrix}
0 & 0 & 0 & -a_iD_1 & 0 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_eD_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_2 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_nD_3 & 0 & D_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D_1 & 0 & a_iD_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_eD_2 & 0 & 0 \\
0 & 0 & 0 & 0 & D_3 & 0 & a_nD_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-D_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & cD_4 & 0 \\
0 & D_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -cD_4 \\
0 & cD_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_4 \\
cD_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_4 & 0
\end{bmatrix}
$$

Pre-multiplying the preceding equation by $[T]^{-1}$ one obtains

$$[V]_t + [T]^{-1}[A][T][V]_x = [T]^{-1}[B] - [T]^{-1}[T]_t[V] - [T]^{-1}[A][T]_x[V] \quad .$$

Then making use of equation (2.1) yields

$$[V]_t + [D][V]_x = [C] \tag{2.5}$$

where

$$[C] = [T]^{-1}[B] - [T]^{-1}[T]_t[V] - [T]^{-1}[A][T]_x[V] \quad .$$

Equation (2.5) is the desired diagonal form of equation (1.35), except that [C] contains derivatives. However, from an inspection of [T] it is seen that

$$[T]_x = [T]_t = [0] \quad . \tag{2.6}$$

Therefore, using this along with the fact that [T] is an orthogonal matrix yields

$$[C] = [T]^T[B] \quad . \tag{2.7}$$

Thus, instead of having to find $[T]^{-1}$ in order to obtain [C] one merely has to take the transpose of [T]. If the basic equations had not been written in terms of the natural log of the number densities, which could only be done by assuming that each species was isothermal, then equation (2.6) would not be true. Therefore, the expression for [C] would have contained derivatives, which would have resulted in a severe complication.

In conclusion, the original system of equations given by equation (1.35) has been changed into the diagonal system below

$$[V]_t + [D][V]_x = [C] \tag{2.8}$$

with, in general,

$$[D] = [D](x,t,[V]) \text{ and } [C] = [C](x,t,[V])$$

where

$$[V] = [T]^T[U]$$

and

$$[C] = [T]^T[B] \quad .$$

[V], [C], and [D] are shown in the following along with [U] in terms of [V].

$$[V] = \begin{bmatrix} V_1\ (x,t) \\ V_2\ (x,t) \\ V_3\ (x,t) \\ V_4\ (x,t) \\ V_5\ (x,t) \\ V_6\ (x,t) \\ V_7\ (x,t) \\ V_8\ (x,t) \\ V_9\ (x,t) \\ V_{10}(x,t) \\ V_{11}(x,t) \\ V_{12}(x,t) \\ V_{13}(x,t) \\ V_{14}(x,t) \\ V_{15}(x,t) \\ V_{16}(x,t) \\ V_{17}(x,t) \end{bmatrix} = \begin{bmatrix} D_4(cB_3 - E_2) \\ D_4(E_3 + cB_2) \\ D_2(u_e - a_e\zeta_2) \\ D_1(u_i - a_i\zeta_1) \\ D_3(u_n - a_n\zeta_3) \\ D_1(\zeta_1 + a_iu_i) \\ D_3(\zeta_3 + a_nu_n) \\ v_i \\ w_i \\ v_e \\ w_e \\ v_n \\ w_n \\ E_1 \\ D_2(\zeta_2 + a_eu_e) \\ D_4(cE_2 + B_3) \\ D_4(B_2 - cE_3) \end{bmatrix}$$

[C] is a column matrix with the following as its elements.

$$C_1 = - D_4c^2\mu_oe\{V_{10}\exp[D_2(V_{15} - a_eV_3)] - V_8\exp[D_1(V_6 - a_iV_4)]\}$$

$$C_2 = D_4c^2\mu_oe\{V_{11}\exp[D_2(V_{15} - a_eV_3)] - V_9\exp[D_1(V_6 - a_iV_4)]\}$$

$$C_3 = D_2\{-\frac{e}{m_e}[V_{14} + V_{10}D_4(cV_1 + V_{16}) - V_{11}D_4(cV_2 + V_{17})] + \nu_{ei}[D_1(V_4 + a_iV_6) - D_2(V_3 + a_eV_{15})] + \nu_{en}[D_3(V_5 + a_nV_7) - D_2(V_3 + a_eV_{15})]\}$$

$$C_4 = D_1\{\frac{e}{m_i}[V_{14} + V_8 D_4(cV_1 + V_{16}) - V_9 D_4(cV_2 + V_{17})] + \nu_{ie}[D_2 (V_3 + a_e V_{15}) - D_1(V_4 + a_i V_6)] + \nu_{in}[D_3(V_5 + a_n V_7) - D_1 (V_4 + a_i V_6)]\}$$

$$C_5 = D_3\{\nu_{ni}[D_1(V_4 + a_i V_6) - D_3(V_5 + a_n V_7)] + \nu_{ne}[D_2(V_3 + a_e V_{15}) - D_3(V_5 + a_n V_7)]\}$$

$$C_6 = a_i C_4$$

$$C_7 = a_n C_5$$

$$C_8 = \frac{e}{m_i}[D_4(cV_{16} - V_1) + B_1 V_9 - D_1 D_4(V_4 + a_i V_6)(cV_1 + V_{16})] + \nu_{ie}(V_{10} - V_8) + \nu_{in}(V_{12} - V_8)$$

$$C_9 = \frac{e}{m_i}[D_4(V_2 - cV_{17}) + D_1 D_4(V_4 + a_i V_6)(cV_2 + V_{17}) - B_1 V_8] + \nu_{ie}(V_{11} - V_9) + \nu_{in}(V_{13} - V_9)$$

$$C_{10} = -\frac{e}{m_e}[D_4(cV_{16} - V_1) + B_1 V_{11} - D_2 D_4(V_3 + a_e V_{15})(cV_1 + V_{16})] + \nu_{ei}(V_8 - V_{10}) + \nu_{en}(V_{12} - V_{10})$$

$$C_{11} = -\frac{e}{m_e}\ D_4(V_2 - cV_{17}) + D_2 D_4(V_3 + a_e V_{15})(cV_2 + V_{17}) - B_1 V_{10} + \nu_{ei}(V_9 - V_{11}) + \nu_{en}(V_{13} - V_{11})$$

$$C_{12} = \nu_{ni}(V_8 - V_{12}) + \nu_{ne}(V_{10} - V_{12})$$

$$C_{13} = \nu_{ni}(V_9 - V_{13}) + \nu_{ne}(V_{11} - V_{13})$$

$$C_{14} = c^2 \mu_o e\{D_2(V_3 + a_e V_{15})\ \exp[D_1(V_6 - a_i V_4)] - D_1(V_4 + a_i V_6)\ \exp[D_2(V_{15} - a_e V_3)]\}$$

$C_{15} = a_e C_3$

$C_{16} = -cC_1$

$C_{17} = -cC_2$

The diagonal matrix [D], written in terms of the elements of [U] is

$$[D] = \begin{bmatrix}
-c \\
& -c \\
& & u_e-a_e \\
& & & u_i-a_i & & & & & & & & & & \bigcirc \\
& & & & u_n-a_n \\
& & & & & u_i+a_i \\
& & & & & & u_n+a_n \\
& & & & & & & u_i' \\
& & & & & & & & u_i \\
& & & & & & & & & u_e \\
& & & & & & & & & & u_e \\
& & & & & & & & & & & u_n \\
& & & \bigcirc & & & & & & & & & u_n \\
& & & & & & & & & & & & & u_i+u_e \\
& & & & & & & & & & & & & & u_e+a_e \\
& & & & & & & & & & & & & & & c \\
& & & & & & & & & & & & & & & & c
\end{bmatrix}$$

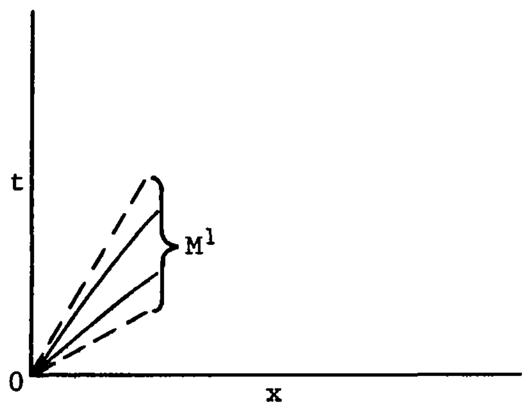
and

$$[U] = \begin{bmatrix} \zeta_1(x,t) \\ \zeta_2(x,t) \\ \zeta_3(x,t) \\ u_i(x,t) \\ u_e(x,t) \\ u_n(x,t) \\ v_i(x,t) \\ v_e(x,t) \\ v_n(x,t) \\ w_i(x,t) \\ w_e(x,t) \\ w_n(x,t) \\ E_1(x,t) \\ E_2(x,t) \\ E_3(x,t) \\ B_2(x,t) \\ B_3(x,t) \end{bmatrix} = \begin{bmatrix} D_1(V_6 - a_iV_4) \\ D_2(V_{15} - a_eV_3) \\ D_3(V_7 - a_nV_5) \\ D_1(V_4 + a_iV_6) \\ D_2(V_3 + a_eV_{15}) \\ D_3(V_5 + a_nV_7) \\ V_8 \\ V_{10} \\ V_{12} \\ V_9 \\ V_{11} \\ V_{13} \\ V_{14} \\ D_4(cV_{16} - V_1) \\ D_4(V_2 - cV_{17}) \\ D_4(cV_2 + V_{17}) \\ D_4(cV_1 + V_{16}) \end{bmatrix}$$

2. General Discussion of Boundary Conditions

Previously it was discussed how the physical problem of imposing a disturbance upon the flow becomes the mathematical problem of finding a solution of the governing equations given the initial values of the dependent variables and the values of some variables on the $x = 0$ boundary, which represents the disturbance. Of course, instead of equation (1.35), the problem now is to find a solution of the diagonal system (2.8) and then return to the original system through the transformation $[U] = [T][V]$. In connection with the boundary conditions it has also been discussed how the number of variables specified, or the number of relations such that these variables may be calculated, is not arbitrary.

The eigenvalues of [A] (also of [D]) are called characteristics and may be shown to be equal to dx/dt; thus, they may be sketched as curves on the x-t plane. Let M^1 be the number of positive eigenvalues of [A]. Therefore, there will be M^1 characteristic curves, corresponding to the positive eigenvalues of [A], which lie to the right of the origin of the x-t plane as shown below.



When the system was transformed into diagonal form the [T] matrix was arranged so that these M^1 characteristics are the last M^1 eigenvalues of [D]. Therefore, the following conditions must be imposed on [V] along the x = 0 boundary[7]:

(1) the values of $V_{17-(M^1-1)}, \cdots, V_{17}$ are prescribed along the x = 0 boundary as functions of time, or

(2) more generally, M^1 relations among $V_1, V_2, \cdots, V_{17}$ are given along x = 0 with the stipulation that it is possible to compute $V_{17-(M^1-1)}, \cdots, V_{17}$ from these relations.

3. Formulation of the Example Problem

From an inspection of the eigenvalues, or characteristics, as given on page 18, it is seen that if u_i, u_e, and u_n are considered to be negative at the origin, with $|u_i| > a_i$, $|u_n| > a_n$ and $|u_e| < a_e$, then only the last three characteristics lie to the right. Thus, for this particular case, one must specify V_{15}, V_{16}, and V_{17}, or three relations such that they can be calculated, on the x = 0 boundary as functions of time. If these particular restrictions are not forced upon u_i, u_e, and u_n, then one must specify more variables, depending upon the new restrictions.

Now if, on the x = 0 boundary, the assumption is made that

$$N_i(0,t) = N_e(0,t)$$

then one can show, from the expressions for the U's in terms of the V's given on page 27, that

$$V_{15}(0,t) = \frac{D_1}{D_2}\{V_6(0,t) - a_i V_4(0,t)\} + a_e V_3(0,t)$$

where $V_6(0,t)$, $V_4(0,t)$ and $V_3(0,t)$ will have been calculated in the solution technique by the time $V_{15}(0,t)$ is calculated. In addition, the assumption is made that the imposed disturbance is that of forcing the electric and magnetic fields in the y-direction to be certain functions of time at x = 0; i.e.,

$$E_2(0,t) = g_1(t)$$

$$B_2(0,t) = g_2(t)$$

where, as will be seen later, g_1 and g_2 must satisfy certain conditions in order for one to be assured of a unique solution existing. With the above, it can be shown from the expressions on page 203 that

$$V_{16}(0,t) = \frac{1}{c}\{\frac{1}{D_4}\, g_1(t) + V_1(0,t)\}$$

and

$$V_{17}(0,t) = \frac{1}{D_4}\, g_2(t) - cV_2(0,t)$$

where, as on bottom of previous page, $V_1(0,t)$ and $V_2(0,t)$ will have been calculated before $V_{16}(0,t)$ and $V_{17}(0,t)$. Instead of E_2 and B_2 one could have forced E_3 and B_3 or E_2 and E_3 or B_2 and B_3 at $x = 0$ and still have been able to solve for $V_{16}(0,t)$ and $V_{17}(0,t)$.

Initially it is required that the plasma flow be in equilibrium. Thus, remembering to satisfy the restrictions previously placed upon u_i, u_e, and u_n plus using number densities that might exist in an ionized gas at atmospheric pressure, the particular initial values assumed are

$$[U](x,0) = \begin{bmatrix} \zeta_1(x,0) \\ \zeta_2(x,0) \\ \zeta_3(x,0) \\ u_i(x,0) \\ u_e(x,0) \\ u_n(x,0) \\ v_i(x,0) \\ v_e(x,0) \\ v_n(x,0) \\ w_i(x,0) \\ w_e(x,0) \\ w_n(x,0) \\ E_1(x,0) \\ E_2(x,0) \\ E_3(x,0) \\ B_2(x,0) \\ B_3(x,0) \end{bmatrix} = \begin{bmatrix} \ln(10^{21}) \\ \ln(10^{21}) \\ \ln(3.62 \times 10^{24}) \\ -1500 \text{ m/sec} \\ -1500 \text{ m/sec} \\ -1500 \text{ m/sec} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to find the corresponding initial values of the diagonal system, the transformation

$$[V](x,0) = [T]^{-1}[U](x,0)$$

is used to yield

$$[V](x,0) = \begin{bmatrix} V_1\ (x,0) \\ V_2\ (x,0) \\ V_3\ (x,0) \\ V_4\ (x,0) \\ V_5\ (x,0) \\ V_6\ (x,0) \\ V_7\ (x,0) \\ V_8\ (x,0) \\ V_9\ (x,0) \\ V_{10}(x,0) \\ V_{11}(x,0) \\ V_{12}(x,0) \\ V_{13}(x,0) \\ V_{14}(x,0) \\ V_{15}(x,0) \\ V_{16}(x,0) \\ V_{17}(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ D_2\{-1500 - a_e \ln(10^{21})\} \\ D_1\{-1500 - a_i \ln(10^{21})\} \\ D_3\{-1500 - a_n \ln(3.62 \times 10^{24})\} \\ D_1\{\ln(10^{21}) - 1500\ a_i\} \\ D_3\{\ln(3.62 \times 10^{24}) - 1500\ a_n\} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ D_2\{\ln(10^{21}) - 1500\ a_e\} \\ 0 \\ 0 \end{bmatrix}$$

The temperature of each gas is taken to be $T_e = 10{,}000^oK$, and $T_i = T_n = 2000^oK$ which yields thermal velocities of $a_e = 3.9 \times 10^5$ m/sec, $a_i = 640$ m/sec, and $a_n = 640$ m/sec. Thus the -1500 m/sec assumed for the initial value of u_i, u_e, and u_n satisfies the restrictions discussed previously.

Values for the collision frequencies encountered in the species momentum equations are also needed. The effective frequencies can be calculated, as discussed in Section I to yield the values below. These calculations require values for the average collision frequencies which are obtained in Appendix B.

$$\nu_{ie} = 3.73 \times 10^5/\text{sec}, \ \nu_{in} = 2.95 \times 10^8/\text{sec}$$

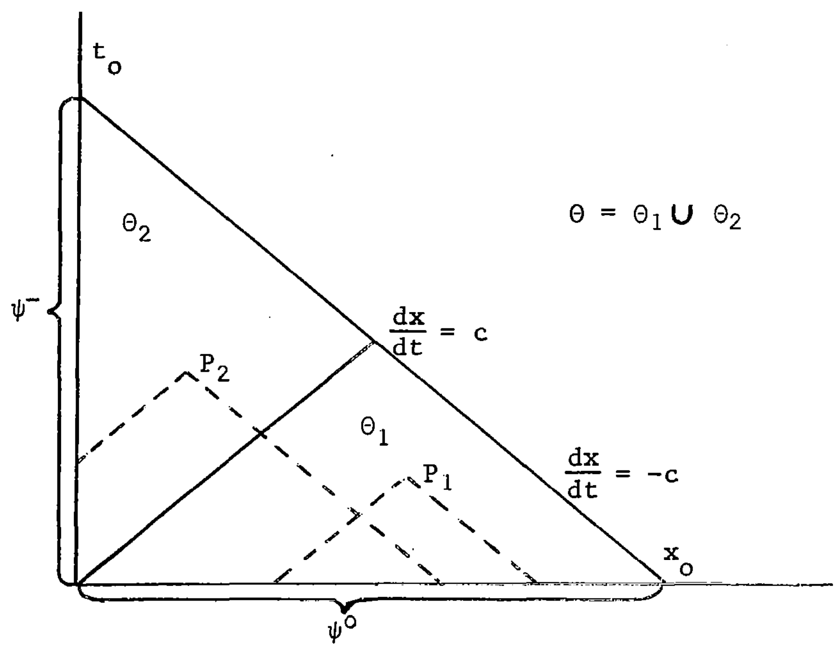
$$\nu_{ei} = 2.76 \times 10^{10}/\text{sec}, \; \nu_{en} = 1.44 \times 10^{11}/\text{sec}$$

$$\nu_{ni} = 0.81 \times 10^{5}/\text{sec}, \; \nu_{ne} = 5.36 \times 10^{2}/\text{sec}$$

In conclusion, particular initial values and boundary conditions (the imposed disturbance), plus values for the collision frequencies have been obtained, for which a solution of equation (2.8) is required. In addition, the linearized system of equations is also diagonalized and solved using the same values listed above.

4. Compatibility Conditions and the Existence Theorem

In this section the concept of a domain of dependence is discussed.



All points within the region bounded by the maximum characteristics ($\max |dx/dt| = c$) drawn from the end points of the initial interval ψ^o; i.e., Θ_1, are dependent upon only the initial values of the variables. For example, values at P_1 are only dependent upon that portion of the initial interval intersected by the maximum positive and negative characteristics drawn backward through P_1. This intersected portion of ψ^o is called the domain of dependence of P_1. All points of Θ_2 are dependent upon a portion of the initial interval as well as some of the x = 0 boundary. For example, values at P_2 are dependent upon that portion of the initial interval from x = 0 to where the maximum negative characteristic, drawn backwards through P_2, intersects it as well as that portion of the boundary up to where the maximum positive characteristic intersects the t-axis. Thus, if a solution exists it can only be within $\Theta(\Theta = \Theta_1 \cup \Theta_2)$since points outside this domain would depend upon initial values which are not given. Therefore, if the initial interval is ψ^o it makes no sense to specify boundary conditions for times exceeding t_o, where $t_o = \frac{x_o}{c}$.

Equation (2.8) may be written as

$$[V]_t - [D_1][V]_x = [C];\ [D_1] = -[D]$$

with

$$[V] = [f^o] \text{ on } \psi^o;\ [f^o] = [f^o](x) \tag{2.9}$$

and

$$[V^-] = [f^-] \text{ on } \psi^-;\ [f^-] = [f^-](t,[V^+]) \tag{2.10}$$

where

$$[D_1] = \begin{bmatrix} [D_1^+] & [0] \\ [0] & [D_1^-] \end{bmatrix};\ [D_1^+] > 0 \text{ and } [D_1^-] < 0$$

and $[V^+]$ and $[V^-]$ correspond to the partitioning of [D] shown above. ψ^o and ψ^- are as shown on the previous figure.

Assuming that the closed sets

$$\Omega = \{(x,t,[V])\,|\,0 \le x \le x_o,\ 0 \le t \le \delta_1,\ |[V] - [f^o]| \le \varepsilon\}$$

and

$$\Omega^{-} = \{(t,[V^{+}]) |\ 0 \le t \le \delta_1,\ |[V^{+}] - [f^{o^{+}}](0)| \le \varepsilon\}$$

exist, the following existence theorem taken from Thomée[10] can be stated.

Existence Theorem: If $[f^{o}]\varepsilon\xi^{2}$ on ψ^{o}, $[C]\varepsilon\xi^{2}$ on Ω, $[D_1]\varepsilon\xi^{2}$ on Ω, and $[f^{-}]\varepsilon\xi^{2}$ on Ω^{-}, plus if certain compatibility conditions are satisfied, then there exists a $\delta > 0(\delta < \delta_1)$ such that a solution $\varepsilon\xi^{1}$ exists in Θ^{δ}, where $\Theta^{\delta} = \Theta \cap \{0 \le t \le \delta\}$. In addition, the solution is unique.

The compatibility conditions which must be satisfied in the hypothesis of the existence theorem are:

(1) $[f^{-}](0,[f^{o^{+}}](0)) = [f^{o^{-}}](0)$

(2) $$\frac{\partial[f^{-}]}{\partial t}(0,[f^{o^{+}}](0)) + \frac{\partial[f^{-}]}{\partial[V^{+}]}(0,[f^{o^{+}}](0))\{[D_1^{+}](0,0,[f^{o}](0))\frac{d[f^{o^{+}}]}{dx}(0) + [C^{+}](0,0,[f^{o}](0))\} = [D_1^{-}](0,0,[f^{o}](0))\frac{d[f^{o^{-}}]}{dx}(0) + [C^{-}](0,0,[f^{o}](0))$$

where $\frac{\partial[f^{-}]}{\partial[V^{+}]}$ is the matrix with columns $\frac{\partial[f^{-}]}{\partial V_K^{+}}$, $K = 1,2,\cdots,14$.

Condition (1) expresses the continuity of [V] at the origin; whereas, condition (2) essentially expresses the fact that the differential equation applies at the origin. It is obtained in the following manner. With the partitioning of $[D_1]$, [V] and [C] previously illustrated, equation (2.8) may be partitioned as below.

$$[V^{+}]_t - [D_1^{+}][V^{+}]_x = [C^{+}] \tag{2.11}$$

and

$$[V^{-}]_t - [D_1^{-}][V^{-}]_x = [C^{-}] \tag{2.12}$$

Now using equation (2,9) it can be seen that applying the equation for

$[V^-]$ above at the origin yields

$$\frac{\partial[V^-]}{\partial t}(0,0) = [D_1^-](0,0,[f^o](0)\,)\,\frac{d[f^{o^-}]}{dx}(0) + [C^-](0,0,[f^o](0)\,) \tag{2.13}$$

However, equation (2.10) states that $[V^-]$ is specified on the x = 0 boundary as a function of time and $[V^+]$. Therefore the time derivative of the matrix of specified functions must be the same as the time derivative above. Therefore,

$$\frac{\partial[V^-]}{\partial t}(0,0) = \frac{\partial[f^-]}{\partial t}(0,0) = \frac{\partial[f^-]}{\partial t}(0,[f^{o^+}](0)\,) + \frac{\partial[f^-]}{\partial[V^+]}(0,[f^{o^+}](0)\,)\,\frac{\partial[V^+]}{\partial t}(0,0) \tag{2.14}$$

but,

$$\frac{\partial[V^+]}{\partial t}(0,0) = [D_1^+](0,0,[f^o](0)\,)\,\frac{d[f^{o^+}]}{dx}(0) + [C^+](0,0,[f^o](0)\,) \quad . \tag{2.15}$$

Substituting this into equation (2.14) and then the resulting expression into equation (2.13) yields the second compatibility condition.

From the previous discussion of the boundary conditions for the example problem

$$[f^-](t,[f^{o^+}]) = \begin{bmatrix} \frac{D_1}{D_2}\{V_6(x,0) - a_iV_4(x,0)\} + a_eV_3(x,0) \\ \frac{1}{c}\{\frac{1}{D_4}g_1(t) + V_1(x,0)\} \\ \frac{1}{D_4}g_2(t) - cV_2(x,0) \end{bmatrix} \tag{2.16}$$

whereas,

$$[f^{o^-}](x) = \begin{bmatrix} V_{15}(x,0) \\ V_{16}(x,0) \\ V_{17}(x,0) \end{bmatrix} . \tag{2.17}$$

From the assumed initial values given on page 207 it is seen that $V_1(x,0) = V_2(x,0) = V_{16}(x,0) = V_{17}(x,0) = 0$; thus, using this and applying condition (1) yields

$$\begin{bmatrix} \frac{D_1}{D_2}\{V_6(0,0) - a_i V_4(0,0)\} + a_e V_3(0,0) \\ \frac{1}{c}\frac{1}{D_4} g_1(0) \\ \frac{1}{D_4} g_2(0) \end{bmatrix} = \begin{bmatrix} V_{15}(0,0) \\ 0 \\ 0 \end{bmatrix} .$$

Making use of the relations between the V's and U's on page 203 it is seen that

$$N_i(0,0) = N_e(0,0)$$

which is satisfied. Also from the above

$$g_1(0) = 0 \text{ and } g_2(0) = 0 \quad . \tag{2.18}$$

Now, before applying the second condition observe that

$$\frac{\partial[f^-]}{\partial t}(t,[f^{o^+}]) = \begin{bmatrix} 0 \\ \frac{1}{c}\frac{1}{D_4}\frac{dg_1}{dt} \\ \frac{1}{D_4}\frac{dg_2}{dt} \end{bmatrix} .$$

Also, since $[f^{o^+}]$ consists of the first 14 elements of $[V](x,0)$ and $[f^{o^-}]$ the last three

$$\frac{d[f^{o^+}]}{dx} = [0] \text{ and } \frac{d[f^{o^-}]}{dx} = [0] \quad .$$

In addition, from an inspection of the elements of $[C]$ given on page 200

$$[C^+](0,0,[f^o](0)\ ) = [0]$$

and

$$[C^-](0,0,[f^o](0)\ ) = [0] \quad .$$

Thus, applying the second compatibility condition yields

$$\begin{bmatrix} 0 \\ \frac{1}{c}\frac{1}{D_4}\frac{dg_1}{dt}(0) \\ \frac{1}{D_4}\frac{dg_2}{dt}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

Therefore, at $t = 0$

$$\frac{dg_1}{dt} = 0 \tag{2.19}$$

$$\frac{dg_2}{dt} = 0 \quad . \tag{2.20}$$

Previously it was stated that $g_1(t)$ and $g_2(t)$ must satisfy certain conditions. These are given by (2.18), (2.19) and (2.20). The particular forms of g_1 and g_2 chosen are

$$g_1(t) = \tilde{E}_{02}(1 - \cos 2\pi\omega t) \tag{2.21}$$

$$g_2(t) = \tilde{B}_{02}(1 - \cos 2\pi\omega t) \tag{2.22}$$

which, as can easily be seen, satisfy the restrictions listed above. Thus the boundary conditions for the example, in terms of the transformed variables, are

$$V_{15}(0,t) = \frac{D_1}{D_2}\{V_6(0,t) - a_i V_4(0,t)\} + a_e V_3(0,t)$$

$$V_{16}(0,t) = \frac{1}{c}\{\frac{1}{D_4}\tilde{E}_{02}(1 - \cos 2\pi\omega t) + V_1(0,t)\}$$

$$V_{17}(0,t) = \frac{1}{D_4}\tilde{B}_{02}(1 - \cos 2\pi\omega t) - cV_2(0,t) \quad .$$

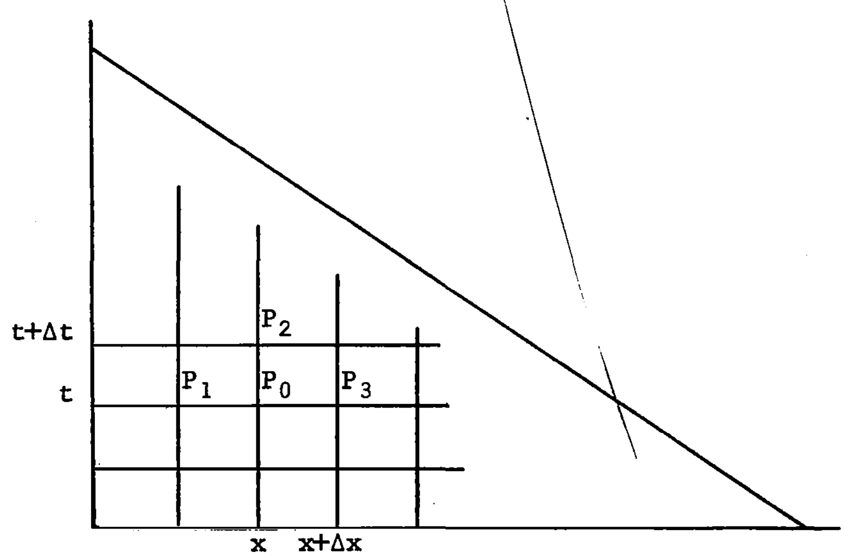
In conclusion, with the above forms for $g_1(t)$ and $g_2(t)$; i.e., the imposed disturbance, one is assured of satisfying the compatibility conditions. Thus, it is known from the existence theorem that a solution of equation (2.8) subject to the particular initial-values and boundary conditions assumed does exist in Θ^{δ}. Once this solution is found it is a simple matter to return to the original system.

SECTION III

In this section the finite-difference scheme used to form the difference equations from the differential equations is developed. Once the difference equations are obtained the consistency, stability, and convergence are analyzed.

1. Development of the Finite-Difference Scheme and the Difference Equations

When using finite-differences to find a numerical solution, values of the unknown variables are obtained at a discrete set of points called net points. A rectangular net of lines, to be superimposed on the (x,t) plane such that one family of lines is parallel to the x-axis and the other family of lines is parallel to the t-axis, are chosen. The lines are assumed equi-spaced with x interval Δx and t interval Δt, as illustrated below, where Δx and Δt are not necessarily equal.



If the function $W(x,t)$ is defined only at the net points $(k\Delta x, \ell\Delta t)$ in the (x,t) plane, for k,ℓ integers, then the values of $W(x,t)$ at the points P_i of the previous figure are:

$$P_0: \quad W(x,t)$$
$$P_1: \quad W(x - \Delta x,t)$$
$$P_2: \quad W(x,t + \Delta t)$$
$$P_3: \quad W(x + \Delta x,t)$$

The forward and backward space difference quotients at P_0 are written

$$\frac{1}{\Delta x} [W(x + \Delta x,t) - W(x,t)]$$

and

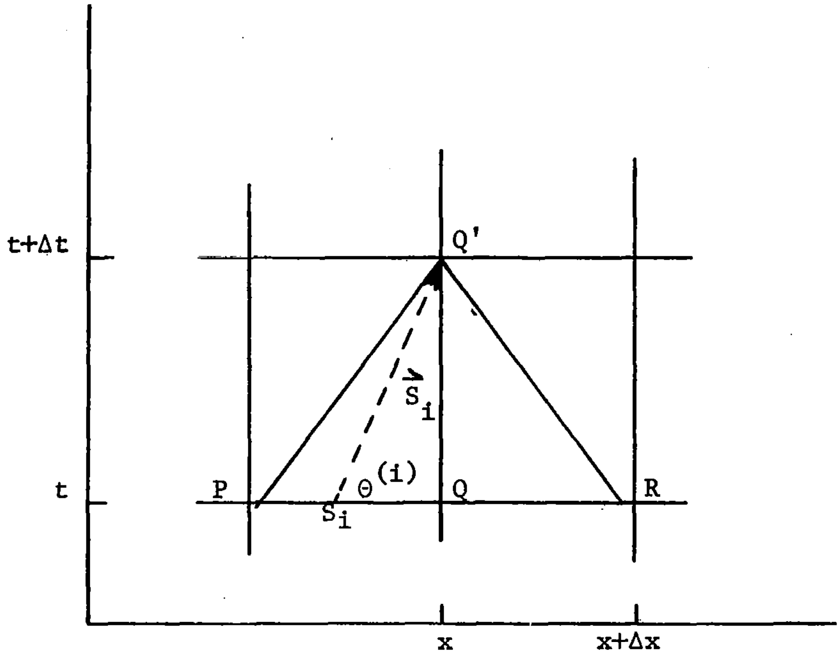
$$\frac{1}{\Delta x} [W(x,t) - W(x - \Delta x,t)]$$

respectively, and both approximate the partial derivative $\partial V/\partial x$ of the differentiable function $V(x,t)$ whose values coincide with those of $W(x,t)$ at each net point. Similarly, the forward time difference quotient at P_0 is written

$$W_t = \frac{1}{\Delta t} [W(x,t + \Delta t) - W(x,t)]$$

and approximates the partial derivative $\partial V/\partial t$.

Previously the domain of dependence concept was discussed; i.e., values at Q', of the following figure, depend upon that portion of the previous time step intersected by the maximum positive and negative characteristics drawn backwards through Q'. The time and spatial steps are selected such that the tangents to the characteristics at Q', when traced backwards, intersect the line through P and R at the points S_i; $i = 1,2,\cdots,17$, between P and R.



This assures one of keeping the domain of dependence intact and thus the finite-difference domain lies within the analytic domain. It is seen that the condition for the above to be satisfied is

$$\frac{\Delta t}{\Delta x} < \max \left|\frac{1}{\lambda^{(i)}}\right| \text{ where } \frac{dx^{(i)}}{dt} = \lambda^{(i)} \quad .$$

A typical equation of the diagonalized system given by equation (2.8) is

$$V_t^{(i)} + \lambda^{(i)} V_x^{(i)} = C^{(i)};\ i = 1,2,\cdots,17 \tag{3.1}$$

However, $(V_t^{(i)} + \lambda^{(i)} V_x^{(i)})$ can be considered to be a total time derivative along the curve $x^{(i)} = x^{(i)}(t)$ for which $dx^{(i)}/dt = \lambda^{(i)}$; thus, equation (3.1) can be written as

$$\frac{dV^{(i)}}{dt} = C^{(i)} \tag{3.2}$$

along

$$x^{(i)} = x^{(i)}(t), \text{ where } \frac{dx^{(i)}}{dt} = \lambda^{(i)} \quad .$$

Values of $V^{(i)}$ at Q' are written as $\overset{(i)}{V}(Q')$ and the discrete valued approximation is $\overset{(i)}{W}(Q')$. From the previous figure,

$$\vec{S}_i = S_{i_x}\hat{i} + S_{i_t}\hat{j}$$

where $\hat{i}$ is a unit vector in the x-direction and $\hat{j}$ is a unit vector along the t-axis. It is easily seen that

$$\tan\theta^{(i)} = \frac{S_{i_t}}{S_{i_x}} = \frac{dt}{dx^{(i)}} = \frac{1}{\lambda^{(i)}}$$

thus, setting $S_{i_t} = 1$ and $S_{i_x} = \lambda^{(i)}$ one obtains

$$\bar{S}_i = \lambda^{(i)}\hat{i} + \hat{j}$$

and

$$\hat{S}_i = \frac{\lambda^{(i)}}{\sqrt{1 + \lambda^{(i)2}}}\,\hat{i} + \frac{1}{\sqrt{1 + \lambda^{(i)2}}}\,\hat{j} \quad . \tag{3.3}$$

Now, the derivative of $V^{(i)}$ in the direction of $\hat{S}_i$ is

$$\hat{S}_i \cdot \nabla V^{(i)} = \frac{\lambda^{(i)}}{\sqrt{1 + \lambda^{(i)2}}}\,V_x^{(i)} + \frac{1}{\sqrt{1 + \lambda^{(i)2}}}\,V_t^{(i)}$$

or

$$\hat{S}_i \cdot \nabla V^{(i)} = \frac{1}{\sqrt{1 + \lambda^{(i)2}}}\,\frac{dV^{(i)}}{dt} \quad . \tag{3.4}$$

Now, let the finite-difference approximation to this directional derivative be

$$\vec{S}_i \cdot \nabla V^{(i)} \cong \frac{\overset{(i)}{W}(Q') - \overset{(i)}{W}(S_i)}{\overline{Q'S_i}} \quad . \tag{3.5}$$

However, from the previous figure it is seen that

$$\overline{QQ'} = \sin\theta^{(i)}\,\overline{S_iQ'}$$

where

$$\sin \theta^{(i)} = \frac{1}{\sqrt{1 + \lambda^{(i)2}}}$$

therefore,

$$\overline{S_iQ'} = \sqrt{1 + \lambda^{(i)2}} \quad \overline{QQ'} \quad .$$

Thus, equation (3.5) can be written as

$$\hat{S}_i \cdot \nabla V^{(i)} \cong \frac{1}{\sqrt{1 + \lambda^{(i)2}}} \quad \frac{W^{(i)}(Q') - W^{(i)}(S_i)}{\overline{QQ'}}$$

and thus, from equation (3.4)

$$\frac{dV^{(i)}}{dt} \cong \frac{W^{(i)}(Q') - W^{(i)}(S_i)}{\overline{QQ'}} \quad . \tag{3.6}$$

Then, from equation (3.2) one obtains

$$\frac{W^{(i)}(Q') - W^{(i)}(S_i)}{\overline{QQ'}} \cong C^{(i)}(Q') \quad . \tag{3.7}$$

The points S_i will not coincide with the net points (P,R) and thus $W^{(i)}(S_i)$ is undefined since $W^{(i)}$ is defined only at net points. This problem is solved by approximating $W^{(i)}(S_i)$ by linear interpolation between the values of $W^{(i)}$ at the adjacent net points. Clearly, if the gradient of $\overline{S_iQ'}$ is positive, S_i will lie between P and Q; whereas, if the gradient is negative, S_i will lie between Q and R.

If one assumes that S_i lies between P and Q, then using linear interpolation yields

$$W^{(i)}(S_i) = W^{(i)}(P) \frac{\overline{S_iQ}}{\overline{PQ}} + W^{(i)}(Q) \frac{\overline{S_iP}}{\overline{PQ}} \tag{3.8}$$

but,

$$\overline{S_iP} = \overline{PQ} - \overline{S_iQ}$$

therefore,

$$\overset{(i)}{W}(S_i) = \overset{(i)}{W}(P)\,\frac{\overline{S_iQ}}{\overline{PQ}} + \overset{(i)}{W}(Q)\,\frac{\overline{PQ} - \overline{S_iQ}}{\overline{PQ}} \tag{3.9}$$

Again considering the previous figure, one finds

$$\tan\,\theta^{(i)} = \frac{1}{\lambda^{(i)}} = \frac{\Delta t}{S_iQ} \quad ,$$

or

$$\overline{S_iQ} = \Delta t \lambda^{(i)} \quad .$$

Also

$$\overline{PQ} = \Delta x \quad ;$$

therefore,

$$\frac{\overline{S_iQ}}{\overline{PQ}} = \frac{\Delta t}{\Delta x}\,\overset{(i)}{\lambda}(Q') \quad .$$

Substituting the above in equation (3.9) and then using the resulting expression for $\overset{(i)}{W}(S_i)$ in equation (3.7) yields

$$\frac{\overset{(i)}{W}(Q') - \overset{(i)}{W}(Q)}{\Delta t} + \overset{(i)}{\Lambda}(Q')\,\frac{\overset{(i)}{W}(Q) - W(P)}{\Delta x} = \overset{(i)}{\tilde{C}}(Q')$$

where $\overset{(i)}{\Lambda}(Q')$ and $\overset{(i)}{\tilde{C}}(Q')$ involve $\overset{(i)}{W}(Q')$, which of course is not known. Therefore, one approximates $\overset{(i)}{\Lambda}(Q')$ and $\overset{(i)}{\tilde{C}}(Q')$ by their values at Q; i.e., the previous time step. One now concludes that if $\Lambda^{(i)}$ is positive the difference equations are

$$\frac{\overset{(i)}{W}(x,t+\Delta t) - \overset{(i)}{W}(x,t)}{\Delta t} + \overset{(i)}{\Lambda}(x,t)\,\frac{\overset{(i)}{W}(x,t) - \overset{(i)}{W}(x-\Delta x,t)}{\Delta x} = \overset{(i)}{\tilde{C}}(x,t) \tag{3.10}$$

whereas, if $\Lambda^{(i)}$ is negative the difference equations are obtained similarly as

$$\frac{\overset{(i)}{W}(x,t+\Delta t) - \overset{(i)}{W}(x,t)}{\Delta t} + \overset{(i)}{\Lambda}(x,t)\,\frac{\overset{(i)}{W}(x+\Delta x,t) - \overset{(i)}{W}(x,t)}{\Delta x} = \overset{(i)}{\tilde{C}}(x,t) \tag{3.11}$$

Thus, from the preceding equations one sees that for characteristics with a positive gradient, the spatial derivatives in the corresponding equations are determined by backward finite-difference quotients; whereas, for characteristics with a negative gradient, spatial derivatives are determined by forward finite-difference quotients.

In concluding this section, it should be noted that from an inspection of equation (3.10), it is seen that when applying the equation at $x = 0$, $\overset{(i)}{W}(-\Delta x,t)$ is required, which of course is not known. However, as has been previously stated, those variables associated with positive characteristics must be specified at $x = 0$ as a function of time. Thus, equation (3.10) is not applied at $x = 0$. Keeping this in mind, from the preceding difference equations, $\overset{(i)}{W}(x,t + \Delta t)$ can be found in terms of quantities calculated at the previous time step.

2. Consistency

When approximating the solution of differential equations by the solution of difference equations there are three primary considerations: consistency, stability, and convergence. In this section the consistency of the difference equations is analyzed, while the stability and convergence are considered in the following sections.

As previously stated, one can write the equations of the diagonal system as

$$\frac{\partial V^{(i)}}{\partial t} + \lambda^{(i)} \frac{\partial V^{(i)}}{\partial x} = C^{(i)}; \; i = 1,\cdots,17 \quad . \tag{3.12}$$

The difference equations corresponding to the differential equations above can be obtained, as illustrated in the previous section, by replacing the time derivative by a forward finite-difference quotient and the spatial derivative by a backward finite-difference quotient, assuming $\lambda^{(i)}$ is positive. Now, at the net point $(k\Delta x, \ell\Delta t)$ one can write

$$\left(\frac{\partial V^{(i)}}{\partial t}\right)_{k,\ell} = \frac{V^{(i)}_{k,\ell+1} - V^{(i)}_{k,\ell}}{\Delta t} + T_1^{(i)} \tag{3.13}$$

and

$$\left(\frac{\partial V^{(i)}}{\partial x}\right)_{k,\ell} = \frac{V_{k,\ell} - V_{k-1,\ell}}{\Delta x} + T_2^{(i)} \tag{3.14}$$

Therefore, substituting these expressions into equation (3.12) yields

$$\frac{V^{(i)}_{k,\ell+1} - V^{(i)}_{k,\ell}}{\Delta t} + \lambda^{(i)} \frac{V^{(i)}_{k,\ell} - V^{(i)}_{k-1,\ell}}{\Delta x} + T_1^{(i)} + \lambda^{(i)} T_2^{(i)} = C^{(i)} \quad . \tag{3.15}$$

Thus, if the truncation error

$$\tau^{(i)} = T_1^{(i)} + \lambda^{(i)} T_2^{(i)}$$

approaches zero as $\Delta t \to 0$ and $\Delta x \to 0$, the difference equations, obtained in the manner previously discussed, are said to be consistent.

Using Taylor's series with a remainder one can write

$$V^{(i)}_{k-1,\ell} = V^{(i)}_{k,\ell} - \Delta x\left(\frac{\partial V^{(i)}}{\partial x}\right)_{k,\ell} + \frac{(\Delta x)^2}{2} \frac{\partial^2 V^{(i)}_{(x',\ell)}}{\partial x^2} \tag{3.16}$$

where x' is some x between k and k-1. Similarly,

$$V^{(i)}_{k,\ell+1} = V^{(i)}_{k,\ell} + \Delta t\left(\frac{\partial V^{(i)}}{\partial t}\right)_{k,\ell} + \frac{(\Delta t)^2}{2} \frac{\partial^2 V^{(i)}_{(k,t')}}{\partial t^2} \tag{3.17}$$

where t' is some t between $\ell+1$ and ℓ. Now, using the above along with equations (3.13) and (3.14) it can be seen that

$$T_1^{(i)} = -\frac{\Delta t}{2} \frac{\partial^2 V^{(i)}_{(k,t')}}{\partial t^2}$$

and

$$T_2^{(i)} = \frac{\Delta x}{2} \frac{\partial^2 V^{(i)}_{(x',\ell)}}{\partial x^2}$$

Therefore, the total truncation error is

$$\tau^{(i)} = -\frac{\Delta t}{2}\frac{\partial^2 V^{(i)}_{(k,t')}}{\partial t^2} + \lambda^{(i)}\frac{\Delta x}{2}\frac{\partial^2 V^{(i)}_{(x',\ell)}}{\partial x^2}$$

which is $O(\Delta t,\Delta x)$ and thus approaches zero as Δt and $\Delta x \rightarrow 0$. Thus, the difference equations obtained by replacing the time derivative by forward differences and the spatial derivative by backward (or forward) differences are consistent with the differential equations.

3. Stability

In order to perform a stability analysis a typical difference equation is treated as being linear and the stability of that one equation is considered. By considering the worst possible case this then gives some indication of the stability of the nonlinear system.

A typical difference equation, using forward differences for the spatial derivative, is

$$f_{k,\ell+1} = f_{k,\ell} - \Lambda_{k,\ell}\frac{\Delta t}{\Delta x}(f_{k+1,\ell} - f_{k,\ell}) - b\Delta t f_{k,\ell} + \Delta t g_{k,\ell};\ k = 0,1,\cdots,(K-\ell) \tag{3.18}$$

whereas, if backward differences are used to replace the spatial derivatives, then

$$f_{k,\ell+1} = f_{k,\ell} - \Lambda_{k,\ell}\frac{\Delta t}{\Delta x}(f_{k,\ell} - f_{k-1,\ell}) - b\Delta t f_{k,\ell} + \Delta t g_{k,\ell};\ k = 1,2,\cdots,(K-\ell) \tag{3.19}$$

where K is the number of x-net points at $\ell = 0$, b consists of collision frequencies and g of course is merely an element of $[\tilde{C}]$ without the $(bf_{k,\ell})$ term. In the stability analysis, instead of $\Lambda_{k,\ell}$ one uses either the maximum or minimum value of $\Lambda_{k,\ell}$, depending on which yields the most severe restriction on the step sizes, in the preceding equation.

Now, at a particular time step $f_{k,\ell}$ takes on a value at each x-net point; thus, one can write all the values of $f_{k,\ell}$ at a particular time step as a column matrix. Therefore, equation (3.18), as well as equation (3.19), may be written in the following form:

$$[F]^{(\ell+1)} = [M][F]^{(\ell)} + [G]^{(\ell)} \quad . \tag{3.20}$$

Then the requirement for stability is that the eigenvalue of [M] having maximum absolute value must be less than 1.

It can be seen that the [M] associated with equation (3.18), where $\Lambda_{k,\ell}$ has been replaced by $-c$, has the form

$$m_{ij} = (1 - c\,\frac{\Delta t}{\Delta x} - b\Delta t)(\delta_{ij} - \delta_{i\zeta}) + c\,\frac{\Delta t}{\Delta x}\,\delta_{i+1,j}$$

where $\zeta = K-\ell+1$ and c is the speed of light. Therefore, it is seen that $f_{(K-\ell+1),\ell+1}$ has been taken to be zero although actually it cannot be calculated. Similarly the [M] associated with equation (3.19), where $\Lambda_{k,\ell}$ has been replaced by c, has the form

$$m_{ij} = (1 - c\,\frac{\Delta t}{\Delta x} - b\Delta t)(\delta_{ij} - \delta_{i1}) + c\,\frac{\Delta t}{\Delta x}\,\delta_{i-1,j} \quad .$$

Here the fact that $f_{0,\ell+1}$ is specified instead of calculated is taken into account. It is easily seen in both cases that [M] is a triangular matrix and thus its eigenvalues are merely its diagonal elements. In addition, the eigenvalues of both cases are the same; namely, 0 and $(1 - c\,\frac{\Delta t}{\Delta x} - b\Delta t)$. Therefore, for the difference equations to be stable

$$\left|1 - c\,\frac{\Delta t}{\Delta x} - b\Delta t\right| < 1$$

or

$$0 < (b\Delta t + \frac{\Delta t}{\Delta x}\,c) < 2 \quad .$$

Therefore,

$$\Delta t < \frac{2}{b + \frac{c}{\Delta x}} \tag{3.21}$$

must be satisfied for the difference equations to be stable. If one looks back at the governing equations, it is seen that the largest possible value for b is

$$b = \frac{a_e^2}{1 + a_e^2} (\nu_{ei} + \nu_{en}) \quad ;$$

thus,

$$\Delta t < \frac{2}{\frac{a_e^2}{1 + a_e^2} (\nu_{ei} + \nu_{en}) + \frac{c}{\Delta x}} \quad . \tag{3.22}$$

In conclusion, it should be remembered that equation (3.22) was not derived for the non-linear system, but it does provide some guide.

4. Convergence

When the difference between the finite difference solution and the actual solution of the differential equations approaches zero as a limit as the time step decreases, convergence is said to be satisfied. First define the error associated with each variable at the net point $(k\Delta x, \ell\Delta t)$ as

$$\omega_{k,\ell}^{(i)} = W_{i,\ell}^{(i)} - V_{k,\ell}^{(i)}; \; i = 1,2,\cdots,17 \quad .$$

Substituting this into equation (3.10) and rearranging yields

$$\begin{aligned}\omega_{k,\ell+1}^{(i)} = {} & (1 - \frac{\Delta t}{\Delta x} \Lambda_{k,\ell}^{(i)})\omega_{k,\ell}^{(i)} + \frac{\Delta t}{\Delta x} \Lambda_{k,\ell}^{(i)}\omega_{k-1,\ell}^{(i)} \\ & + (1 - \frac{\Delta t}{\Delta x} \Lambda_{k,\ell}^{(i)})V_{k,\ell}^{(i)} - V_{k,\ell+1}^{(i)} + \frac{\Delta t}{\Delta x} \Lambda_{k,\ell}^{(i)} \\ & V_{k-1,\ell}^{(i)} + \Delta t\tilde{C}_{k,\ell}^{(i)}, \; i = 1,2,\cdots,17\end{aligned}$$

Thus, letting $E_\ell = \max\limits_{(i,k)} |\omega_{k,\ell}^{(i)}|$ be a measure of the error, using the fact that $\frac{\Delta t}{\Delta x} \Lambda_{k,}^{(i)} < 1$ in a region where $|W_{k,\ell}^{(i)} - f^{o(i)}| < K_1$, and using the continuity of the solution, as given by the existence theorem, yields

$$E_{\ell+1} \leq 2E_\ell + \beta_1 \Delta t E_\ell + \beta_2 \Delta t$$

for $\ell \geq 0$, $E_0 = 0$ and where β_1 and β_2 are positive constants. The measure of the error from this inequality can always be made to satisfy that given in Ref. 9, for which it is shown that the measure does indeed approach zero as $\Delta t \to 0$. In addition, Courant, et al.[9] illustrates that there is a region contained within Θ^δ in which $|W_{k,\ell}^{(i)} - f^{o(i)}| < K_1$. Thus, one concludes that the finite difference solution converges to the solution of the differential equations as the time step approaches zero.

SECTION IV

When solving difference equations, using particular initial and boundary values, problems with stability which are not predicted in a linearized analysis such as in Section III may appear. In this section the investigation of the actual stability by computer experimentation is discussed and then a stable solution for the example problem is presented. The computer programs are listed in Appendix C.

1. Analysis of Computer Experimentation for the Example Problem

With the initial conditions previously assumed; i.e.,

$$[U](x,0) = \begin{bmatrix} \zeta_1(x,0) \\ \zeta_2(x,0) \\ \zeta_3(x,0) \\ u_i(x,0) \\ u_e(x,0) \\ u_n(x,0) \\ v_i(x,0) \\ v_e(x,0) \\ v_n(x,0) \\ w_i(x,0) \\ w_e(x,0) \\ w_n(x,0) \\ E_1(x,0) \\ E_2(x,0) \\ E_3(x,0) \\ B_2(x,0) \\ B_3(x,0) \end{bmatrix} = \begin{bmatrix} \ln(10^{21}) \\ \ln(10^{21}) \\ \ln(3.62 \times 10^{24}) \\ -1500 \text{ m/sec} \\ -1500 \text{ m/sec} \\ -1500 \text{ m/sec} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.1}$$

along with the forced boundary conditions at x = 0; i.e.,

$$N_i(0,t) = N_e(0,t)$$

$$E_2(0,t) = \tilde{E}_{02}(1 - \cos 2\pi\omega t)$$

$$B_2(0,t) = \tilde{B}_{02}(1 - \cos 2\pi\omega t) \qquad (4.2)$$

a solution of the difference equations for both the nonlinear and linear systems is desired. In addition, as given previously, $T_e = 10^4 {}^\circ K$, $T_i = T_n = 2000 {}^\circ K$ and the collision frequencies used are

$$\nu_{ie} = 3.73 \times 10^5 \text{ sec}^{-1}, \; \nu_{in} = 2.95 \times 10^8 \text{ sec}^{-1}$$

$$\nu_{ei} = 2.76 \times 10^{10} \text{ sec}^{-1}, \; \nu_{en} = 1.44 \times 10^{11} \text{ sec}^{-1}$$

$$\nu_{ni} = 0.81 \times 10^5 \text{ sec}^{-1}, \; \nu_{ne} = 5.36 \times 10^2 \text{ sec}^{-1} \quad . \qquad (4.3)$$

With the conditions above, there are still five parameters which must be assigned values before the difference equations can be solved: Δt, Δx, $\tilde{E}_{02}$, $\tilde{B}_{02}$ and ω. The discussion which follows is primarily for the nonlinear system since the linear equations were stable in all stable cases of the nonlinear equations.

Experimentation revealed that the x-component of the electron velocity, u_e, had the greatest tendency of all variables to be unstable. The solution for this variable was found to be very dependent upon the amplitude of the imposed magnetic field, $\tilde{B}_{02}$, where in general the larger $\tilde{B}_{02}$, the greater the tendency to instability. The amplitude of the imposed electric field had little influence upon the solution for u_e and thus the stability was not influenced significantly by this parameter.

With the amplitudes of the imposed magnetic and electric field disturbances, $\tilde{B}_{02}$ and $\tilde{E}_{02}$, equal to 0.025 Wb/m^2 and 100 V/m, respectively, at a frequency, ω, of 10^{10} sec^{-1}, the solution was unstable with a time step of 2×10^{-13} sec and a spatial step of 2×10^{-4} m (Fig. 1). This figure only shows the x-component of the electron velocity, at the first space step away from the location of the disturbance as a function of time, but all other variables were divergent from the outset also. Note that the solution is unstable even though the stability criteria developed in Section III are satisfied. In addition the linear system was also unstable under these conditions. However, it should be remembered that the linearized stability criteria were developed from a consideration of only one linearized equation, not the entire system of equations.

With the time step reduced to 5×10^{-14} sec and the spatial step to 2.5×10^{-5} m, the plot of u_e in Fig. 2 indicates a stable solution when compared to Fig. 1. After 600 time steps in Fig. 2 u_e is in stable oscillation; whereas, after an equivalent 150 time steps in Fig. 1, it has diverged. Figure 2 shows a small fluctuation in u_e, but it should be noted that the scale has been greatly expanded from that in Fig. 1. This fluctuation would be difficult to detect if u_e, for these conditions, were plotted with a scale such as that in Fig. 1.

It has been illustrated above that reducing the time and spatial steps yields a solution which becomes more stable. However, with the reduced time step, if one wishes to cover a larger portion of the forced oscillation cycle, ω must be increased. The results of increasing ω to 10^{11} sec^{-1} are presented in Fig. 3. Fifty time steps in Fig. 3 correspond to the same point in the forced oscillation cycle as do 500 time steps in Fig. 2. Therefore the plot of u_e in Fig. 3 is much less stable than that in Fig. 2. In Fig. 4 ω has been reduced to 4×10^{10} sec^{-1} and thus 125 time steps in Fig. 4 correspond to the same point in the forced oscillation cycle as do 500 time steps in Fig. 2 and 50 time steps in Fig. 3. Therefore, Fig. 4 is more stable than Fig. 3, but less stable than Fig. 2, as would be expected since ω lies between the values used in Fig. 2 and Fig. 3.

Previously it was stated that the amplitude of the imposed magnetic field influenced the stability, and from the above it is obvious that ω should be reduced below the value used in Fig. 4. Thus, with $\Delta t = 5 \times 10^{-14}$ sec and $\Delta x = 2.5 \times 10^{-5}$ m, it was decided to reduce ω and $\tilde{B}_{02}$ to values of 2×10^{10} sec^{-1} and 0.005 Wb/m^2, respectively, in the presentation of the plots of Figs. 5 and 6. Figure 5(a) illustrates that for these conditions the solution is stable; though, similar to Fig. 2, there is a slight fluctuation. This fluctuation is present to some extent in all the plots presented in Fig. 5.

The above discussion illustrates that the boundary values specified, as well as the time and spatial step sizes, influence the stability. It can be shown, by setting $T_i = T_n = 300^oK$ so that the equilibrium values of u_i, u_e, and u_n may be set equal to -500 m/sec, that the conditions under which Fig. 1 was obtained yield a stable solution. Thus, the particular initial values also have a great influence on the stability.

2. Presentation of the Solution of the Example Model

As stated above, the plots presented in Figs. 5 and 6 were obtained for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\omega = 2 \times 10^{10}$ sec^{-1}, $\tilde{B}_{02} = 0.005$ Wb/m^2, and $\tilde{E}_{02} = 100$ V/m. The initial and forced boundary conditions are given in equations (4.1), (4.2) and (4.3).

Figures 5(a) - 5(k) are plots of those variables with significant changes from their equilibrium values as functions of time after one spatial step; whereas, Figs. 6(a) - 6(j) are plots of the variables versus spatial distance after 100 time steps. Note that for these latter plots the portion after about 17 spatial steps is shown enlarged. Figure 5(l) is a plot of u_e at the origin versus time and Figs. 6(k) and 6(l) are plots uf u_e and E_1 versus spatial distance after 150 time steps. As can be seen from plots of the variables versus spatial distance, all variables rapidly approach their equilibrium values after only a few spatial steps. This is the reason for presenting plots of the variables versus time at or near the origin.

Note that both the linear and non-linear solutions are presented on each plot. Figures 5(a) and 5(g) illustrate that in the linear case neither u_e nor E_1, when plotted versus time, ever change from their respective initial or equilibrium value. However, from the same figures this is not the case in the nonlinear solutions since here u_e and E_1 have a time dependence similar to that illustrated by the other variables. In a similar manner Figs. 6(a) and 6(f) illustrate that when plotted versus the spatial coordinate, once again the linear solutions of u_e and E_1 show no change; whereas, the corresponding nonlinear solutions do. Note that since the nonlinear solutions approach their equilibrium values after only a few spatial steps the two cases quickly become identical. The remaining plots in

Figs. 5 and 6 indicate that the linear and non-linear solutions of all other variables are essentially the same.

If one inspects the linear equations it is obvious that with an initial equilibrium state the equations which involve the x-components of the vector quantities are uncoupled from those involving the y and z components. Thus, when forcing boundary values of only variables in these directions, the variables in the x-direction can never change from their equilibrium values. It is readily seen that this uncoupling does not occur in the nonlinear equations. Thus, one would expect the linear and nonlinear solutions of the x-components of the electron velocity and electric field, u_e and E_1, to differ to some extent.

CONCLUSIONS

The effect of forcing a disturbance upon a flowing three-fluid plasma initially in equilibrium has been investigated. Thus it was necessary to solve the nonlinear equations governing the effect of forcing a disturbance at a point, as a function of time, upon a three-fluid plasma initially in equilibrium. For an example problem the solution was obtained using first the nonlinear and then the linearized equations.

A diagonalization of the system of equations was required in order to have them in a workable form. This resulted in the assumption that the temperature of each fluid is a constant in order to prevent extra derivatives from appearing. In order to be assured of the existence of a unique solution, the boundary values of only certain variables could be prescribed as the disturbance. These boundary values, which are functions of time, had to satisfy certain compatibility conditions at the origin of the x-t plane. The time and spatial steps used in the difference equations were restricted to be extremely small by the speed of light and the magnitude of the collision frequencies. This was required in order for a stable solution which was a good approximation to the solution of the differential equations to exist. The number of field points which could be stored in the computer then determined the region in which the finite difference solution could be obtained. This storage problem is a very real one for a system which contains 17 field variables, even though values were stored for at most two time steps.

A particular example for which a unique solution was shown to exist was formulated. A numerical solution of this example, for both the linear and nonlinear cases, which converged to the actual solution of the differential equations was obtained from the consistent set of difference equations developed. Computer experimentation revealed that though the stability criteria developed for one linearized equation were satisfied, a stable solution was not necessarily obtained. In addition to the reduction of the step sizes, care had to be taken in the specification of the frequency and amplitude of the forced oscillation of the magnetic field in the y-direction at $x = 0$. From the solution of this example, it was found that all variables approach their equilibrium values in a very few spatial steps upstream of what is considered to be the disturbance. Absolutely nothing about the downstream section can be said. In fact, initial conditions cannot even be specified there since if they were, variables could not be prescribed as functions of time at the point considered to be $x = 0$.

A goal of this study was to compare the solutions corresponding to the linear and nonlinear equations for the example formulated. With this example it was found that the x-components of the electron velocity and the electric field had different values for the linear and nonlinear solutions. However, even for these variables the solutions became the same within a very few spatial steps. Thus, a short distance from the disturbance the linearized equations gave solutions as accurate as those obtained using the nonlinear difference equations.

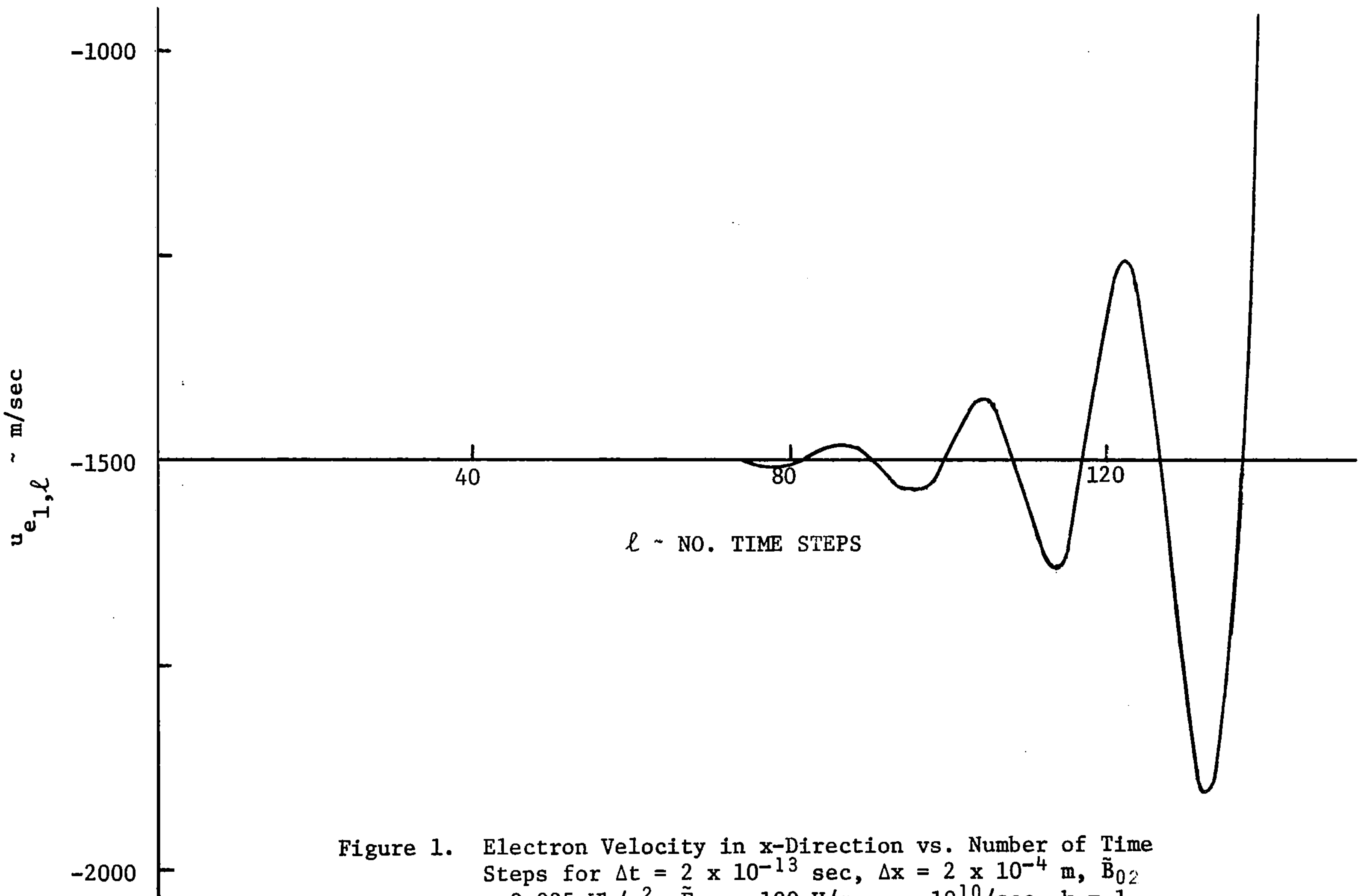


Figure 1. Electron Velocity in x-Direction vs. Number of Time Steps for $\Delta t = 2 \times 10^{-13}$ sec, $\Delta x = 2 \times 10^{-4}$ m, $\tilde{B}_{02} = 0.025$ Wb/m^2, $\tilde{E}_{02} = 100$ V/m, $\omega = 10^{10}$/sec, k = 1.

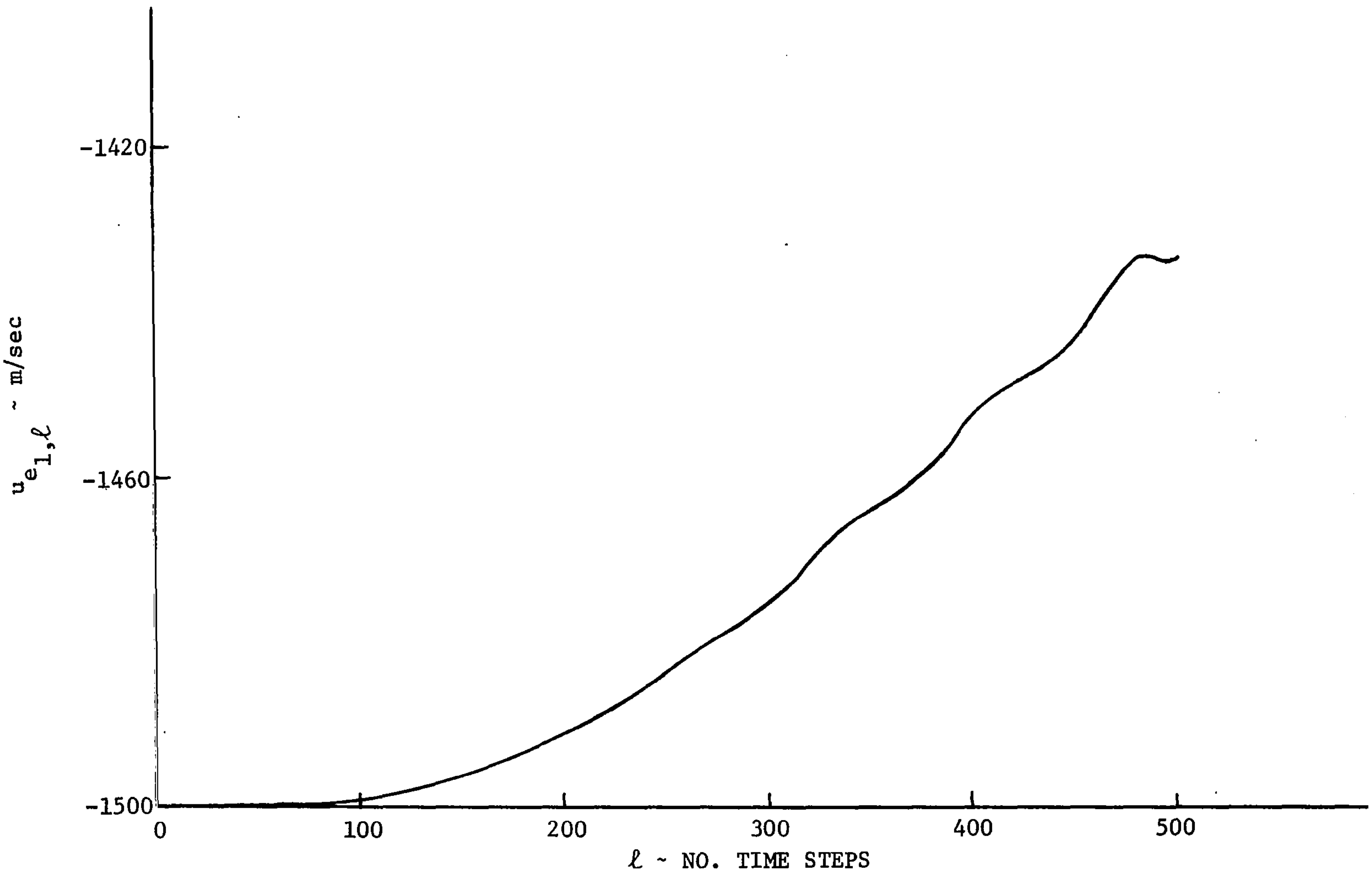


Figure 2. Electron Velocity in x-Direction vs. Number of Time Steps for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\tilde{B}_{02} = 0.025$ Wb/m^2, $\tilde{E}_{02} = 100$ V/m, $\omega = 10^{10}$/sec, k = 1.

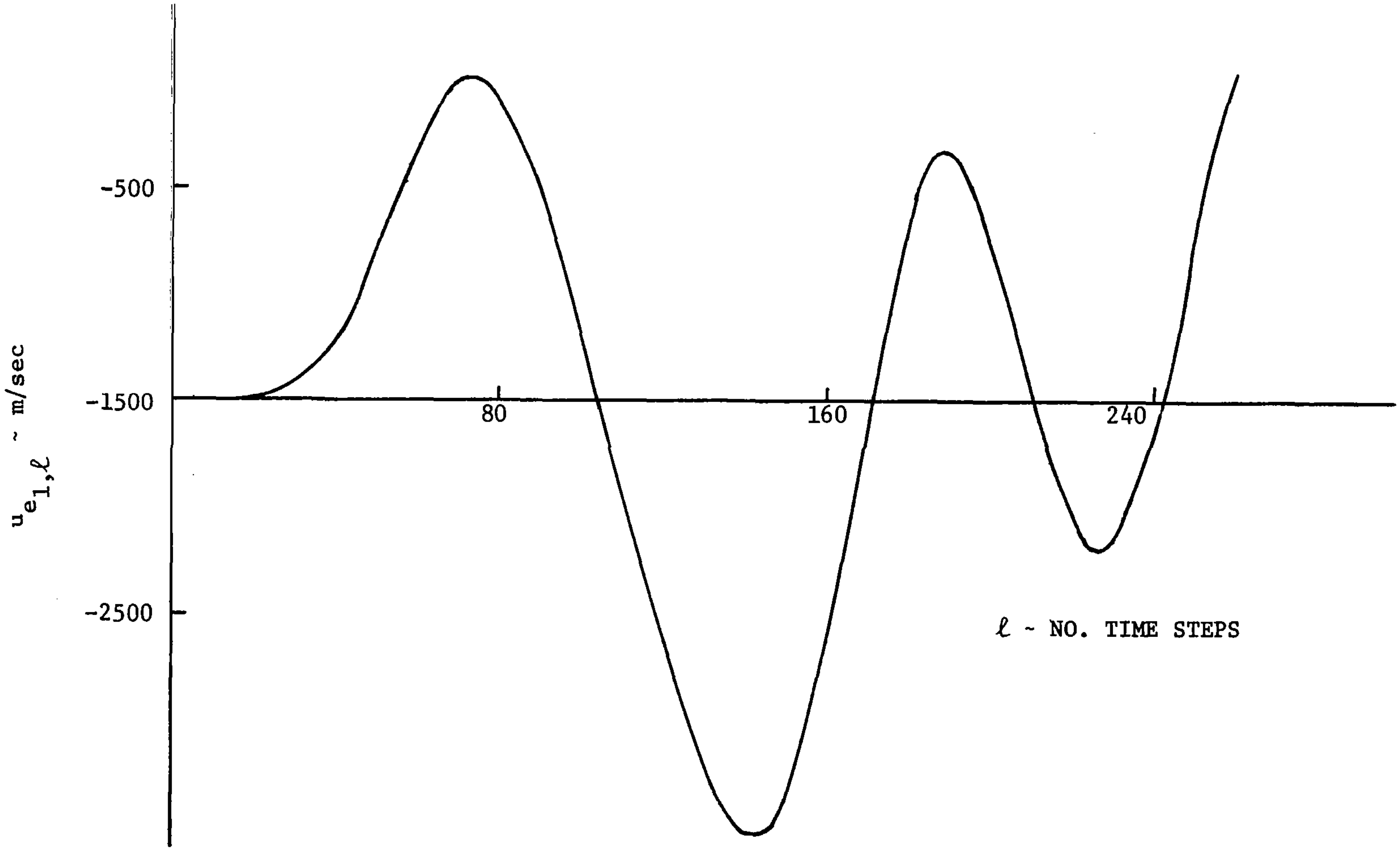


Figure 3. Electron Velocity in x-Direction vs. Number of Time Steps for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\tilde{B}_{02} = 0.025$ Wb/m^2, $\tilde{E}_{02} = 100$ V/m, $\omega = 10^{11}$/sec, $k = 1$.

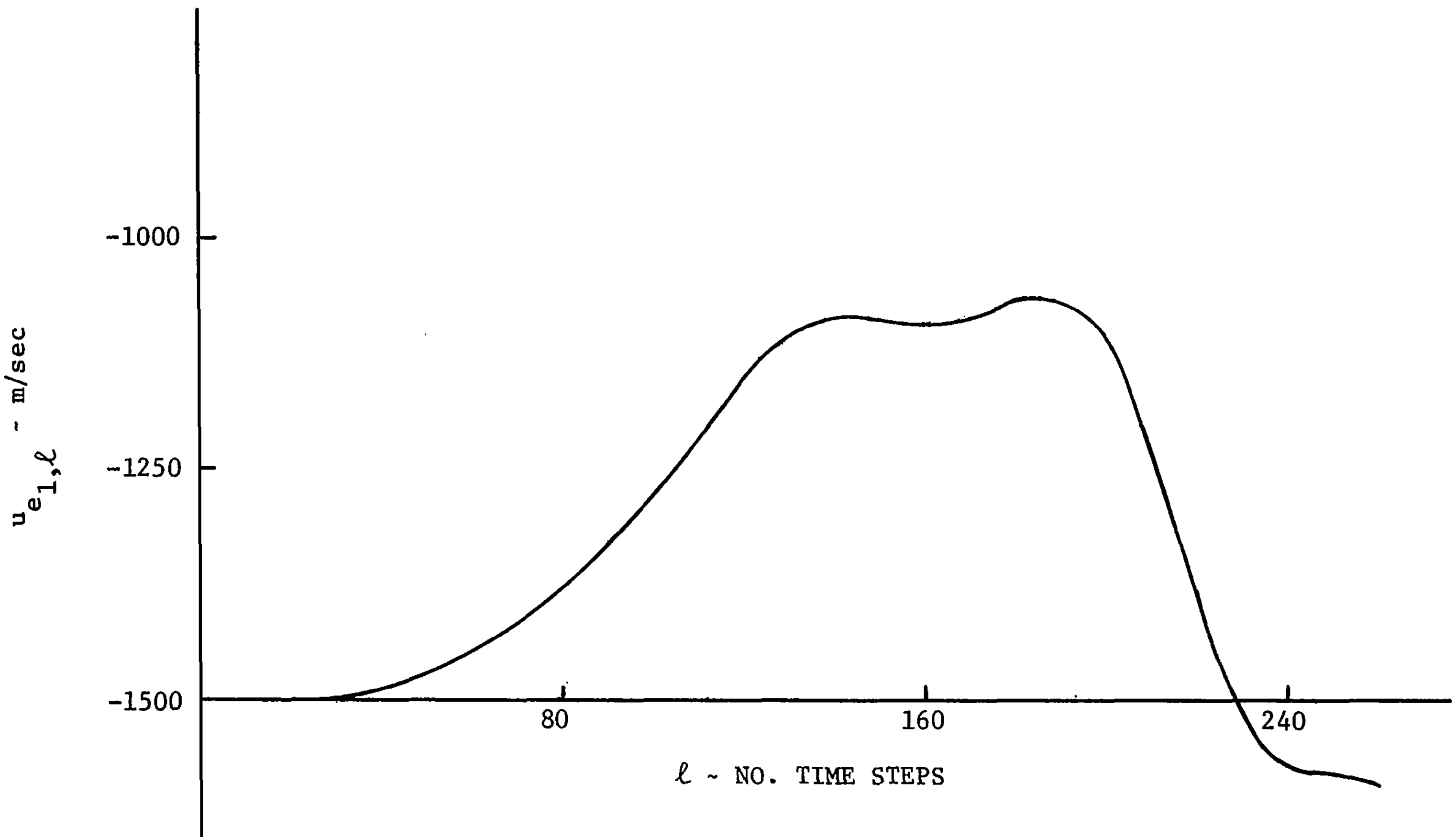


Figure 4. Electron Velocity in x-Direction vs. Number of Time Steps for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\tilde{B}_{02} = 0.025$ Wb/m^2, $\tilde{E}_{02} = 100$ V/m, $\omega = 4 \times 10^{10}$/sec, $k = 1$.

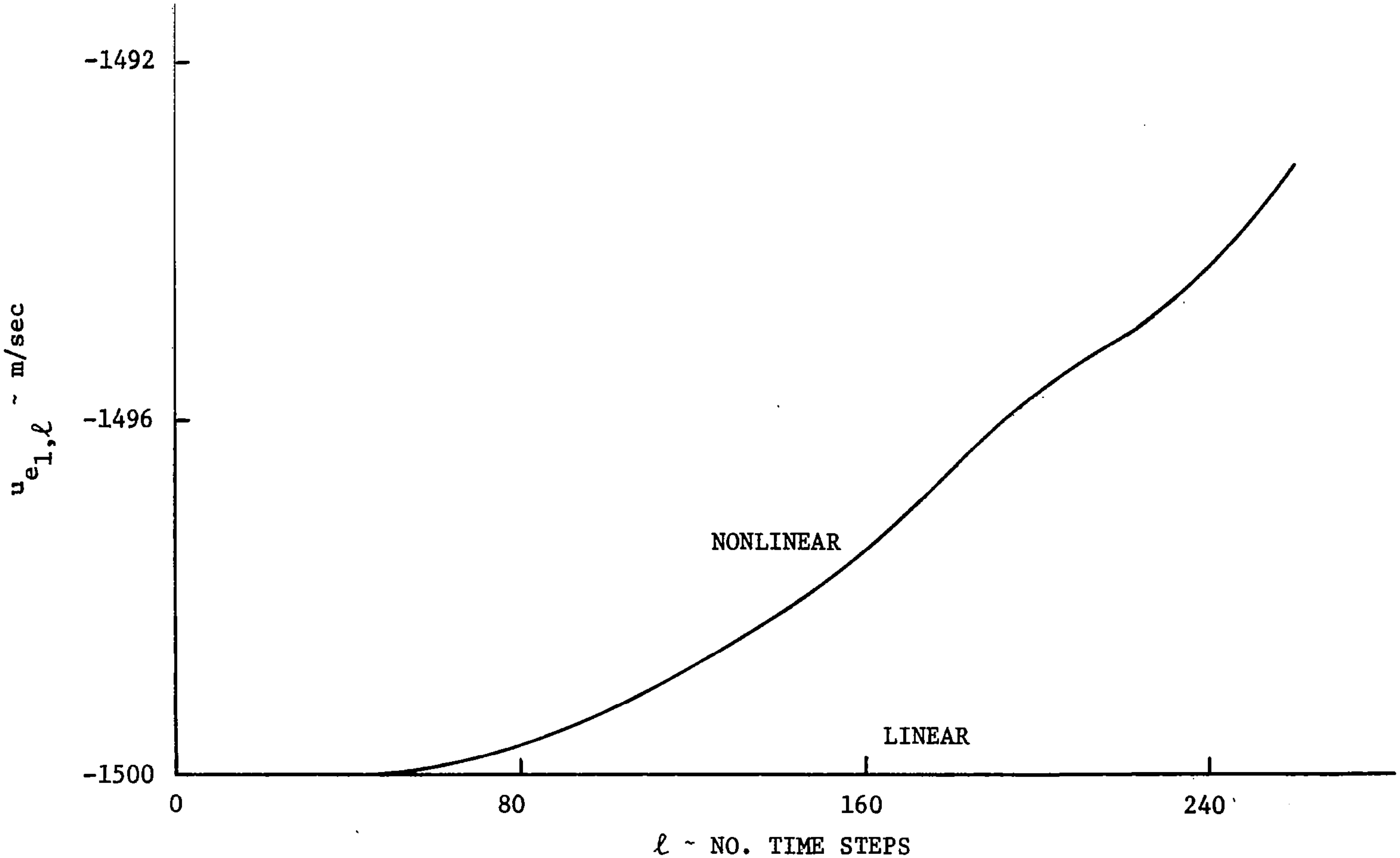


Figure 5(a). Electron Velocity in x-Direction vs. Number of Time Steps for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\tilde{B}_{02} = 0.005$ Wb/m^2, $\tilde{E}_{02} = 100$ V/m, $\omega = 2 \times 10^{10}$ sec, k = 1.

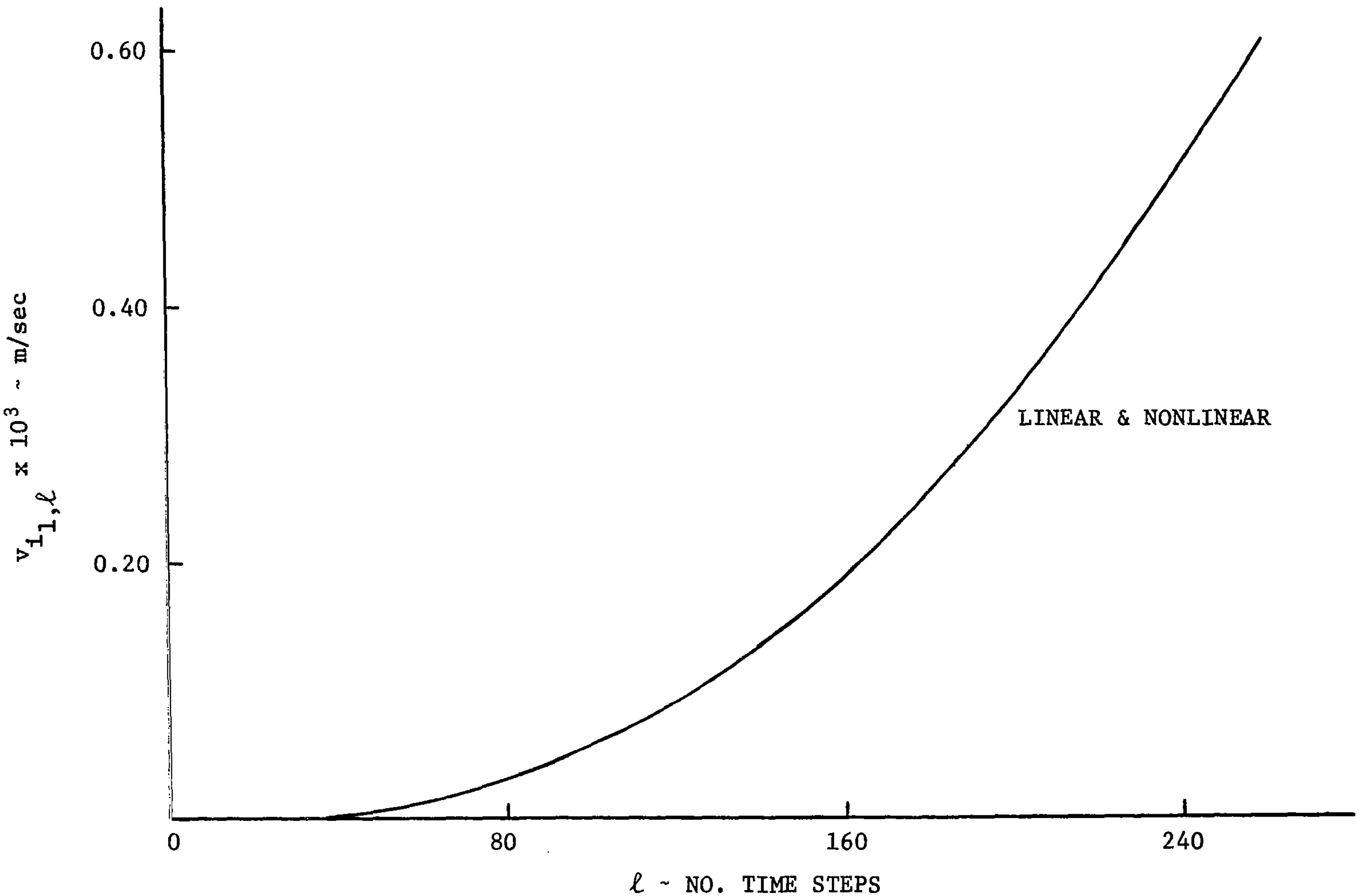


Figure 5(b). Ion Velocity in y-Direction vs. Number of Time Steps.

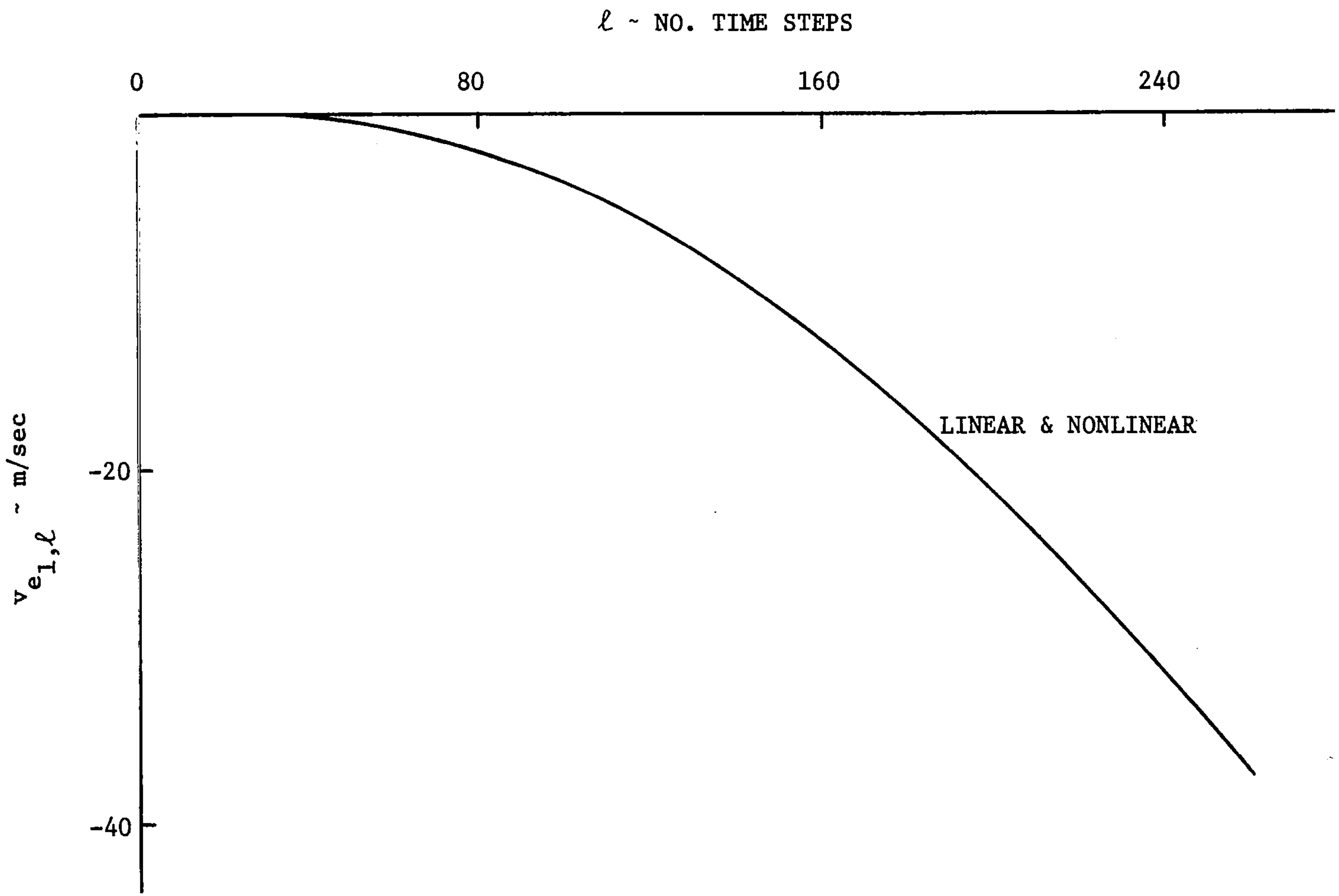


Figure 5(c). Electron Velocity in y-Direction vs. Number of Time Steps.

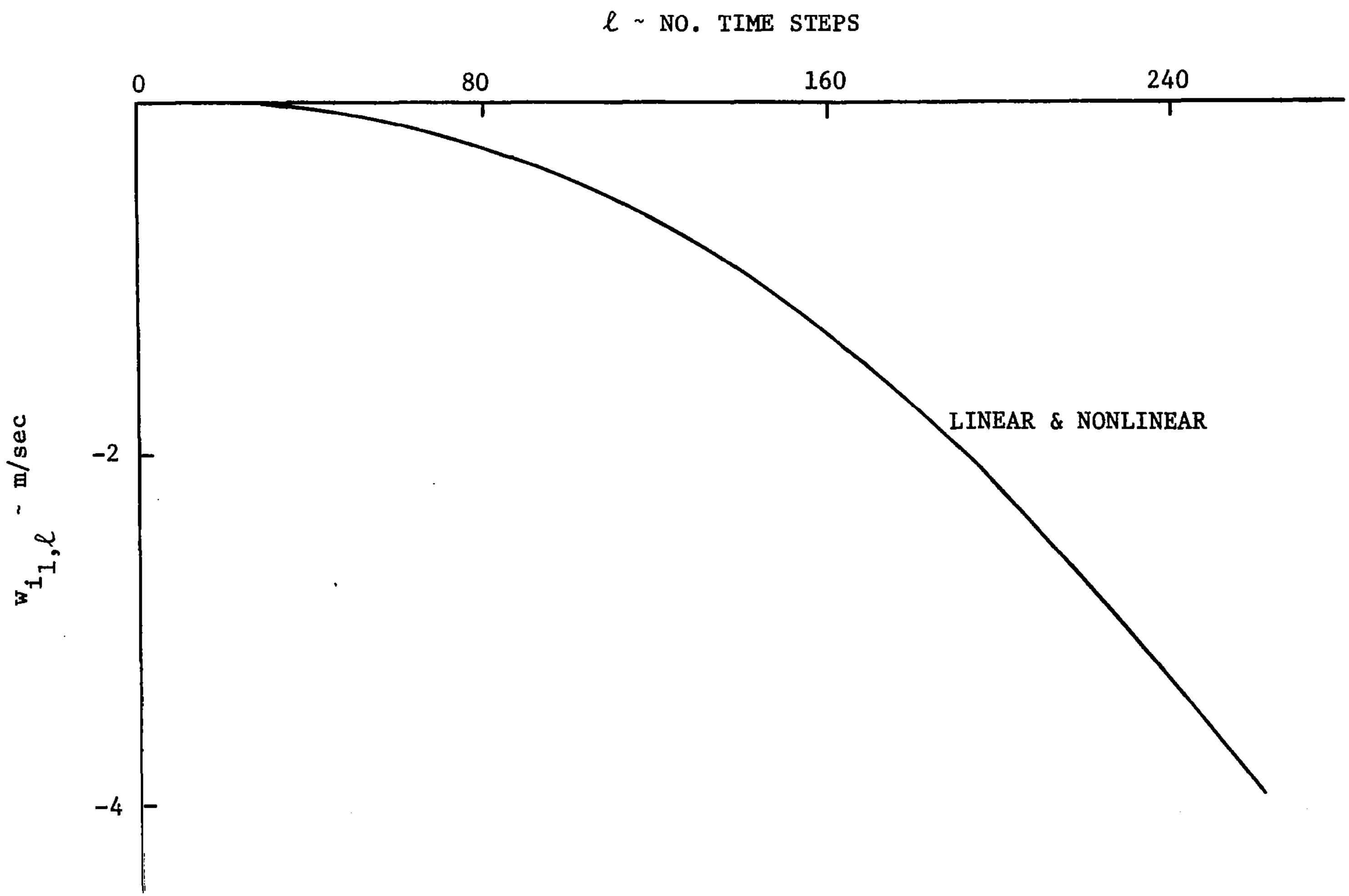


Figure 5(d). Ion Velocity in z-Direction vs. Number of Time Steps.

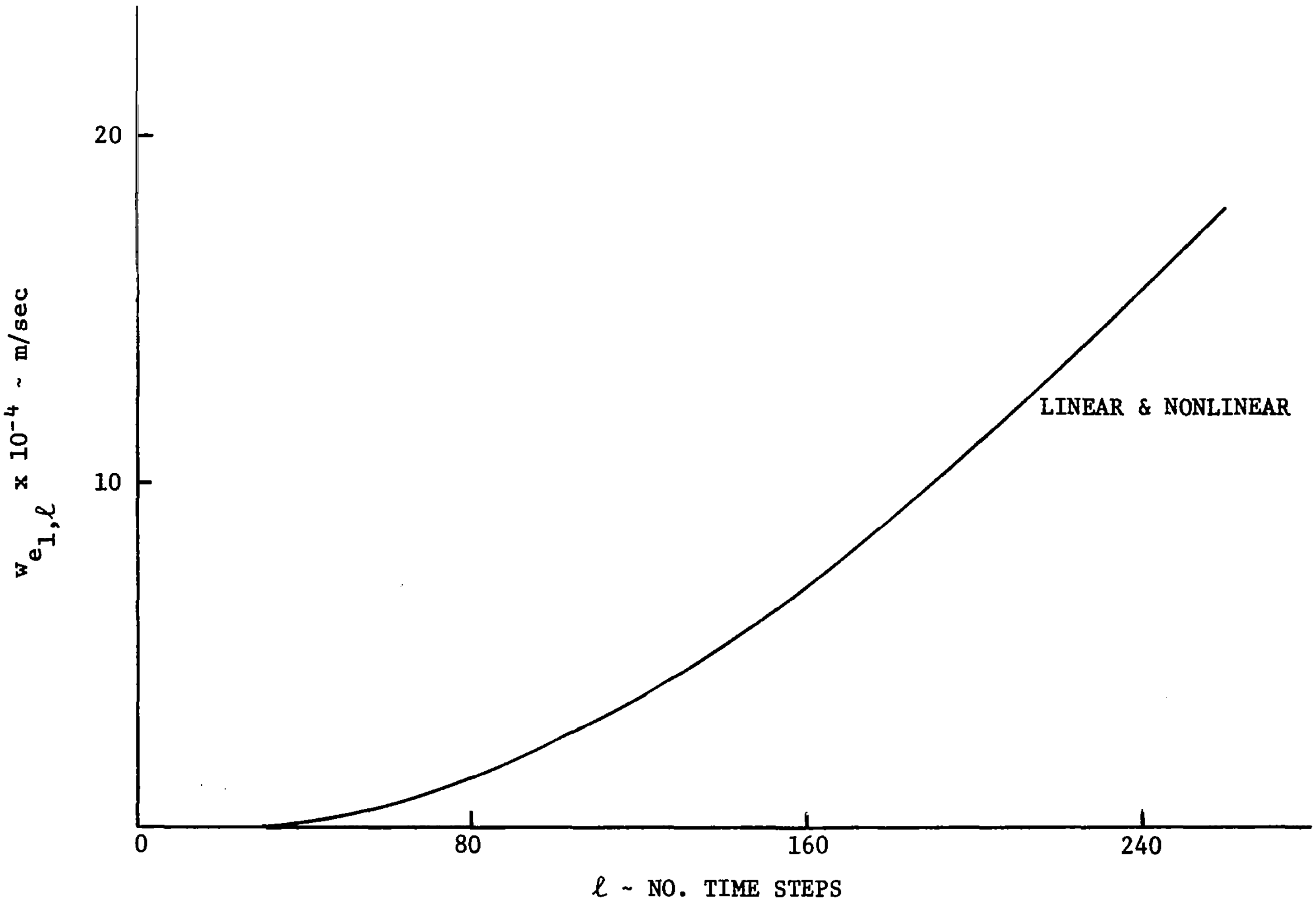


Figure 5(e). Electron Velocity in z-Direction vs. Number of Time Steps.

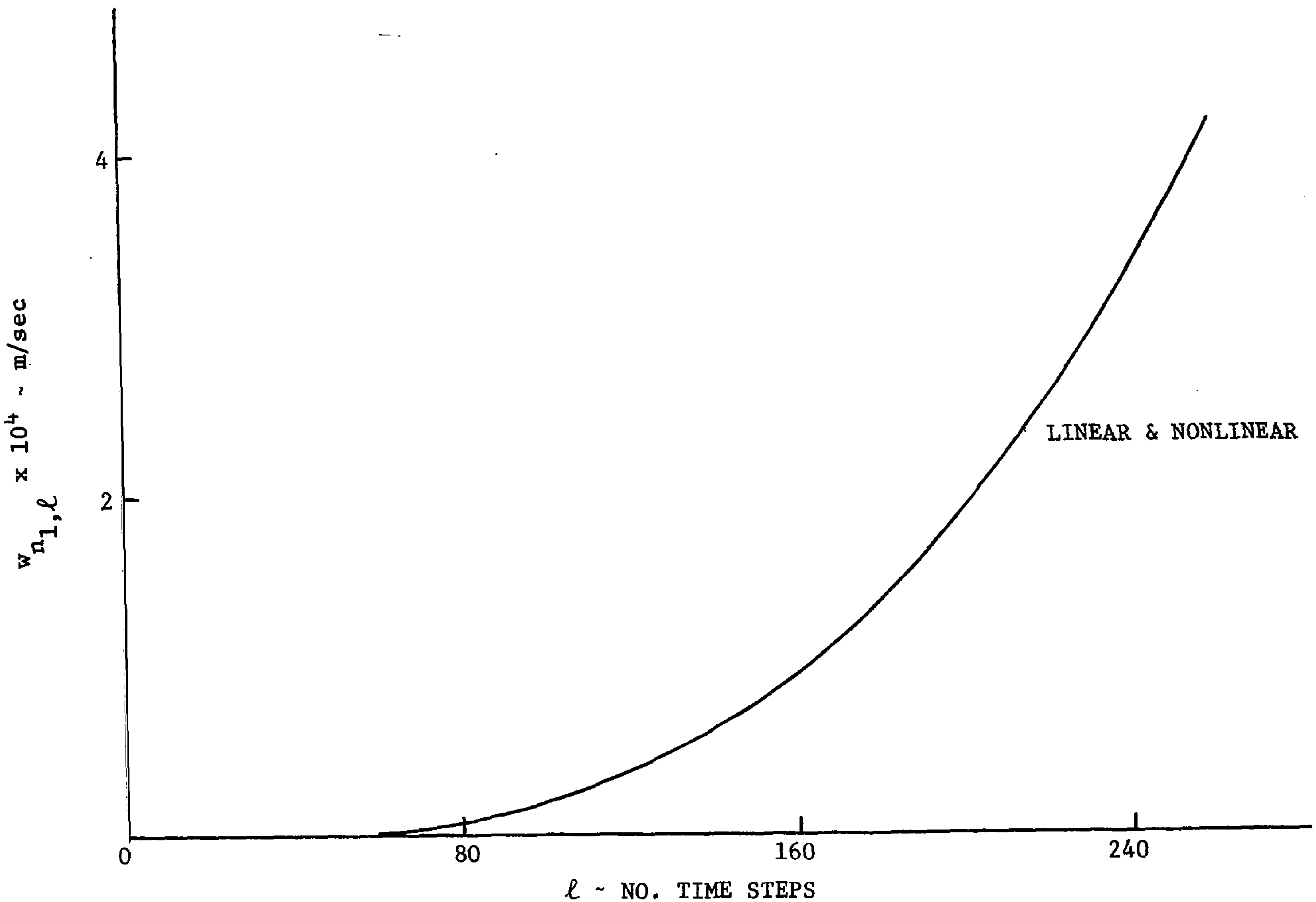


Figure 5(f). Neutral Species Velocity in z-Direction vs. Number of Time Steps.

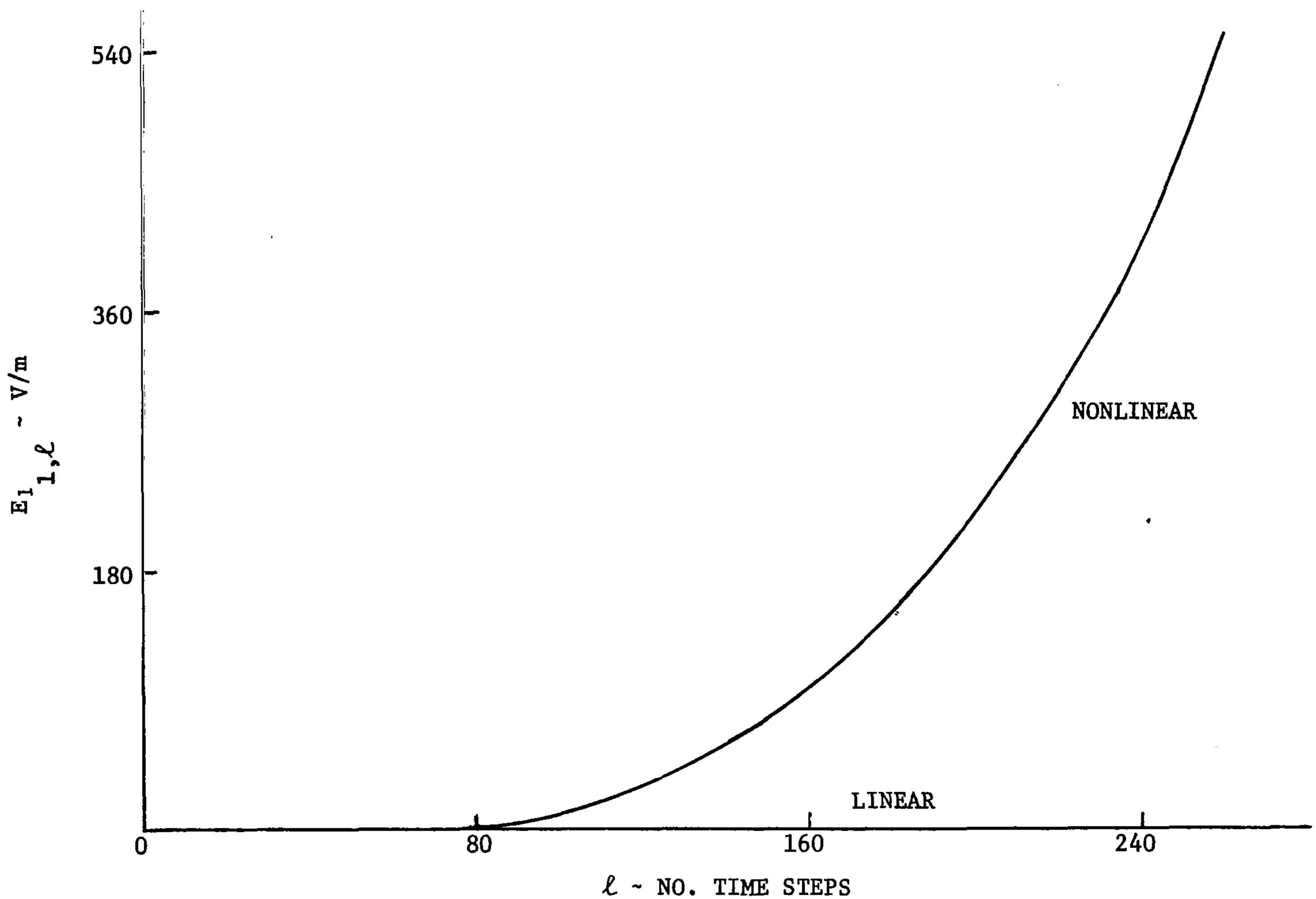


Figure 5(g). Electric Field in x-Direction vs. Number of Time Steps.

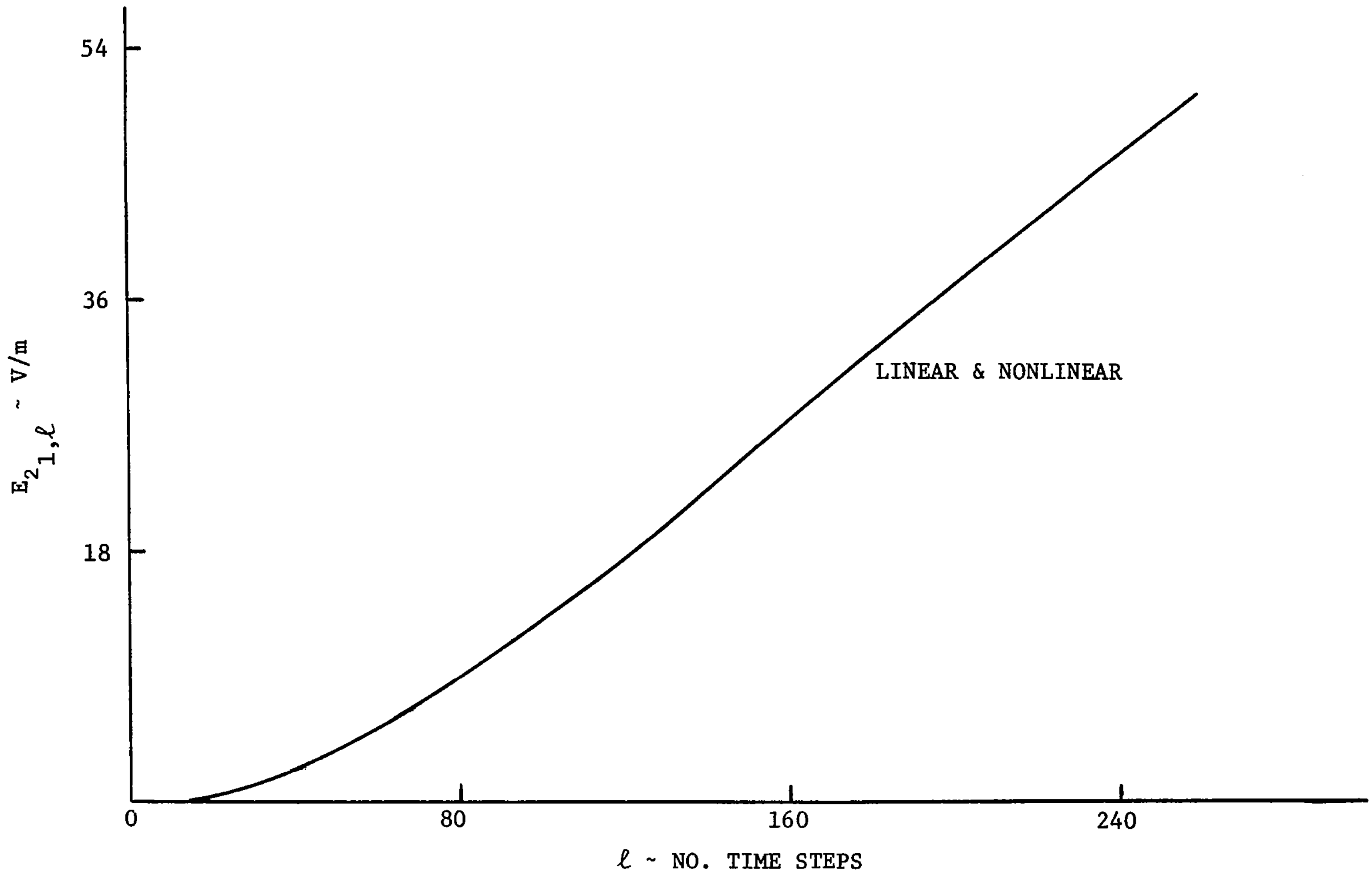


Figure 5(h). Electric Field in y-Direction vs. Number of Time Steps.

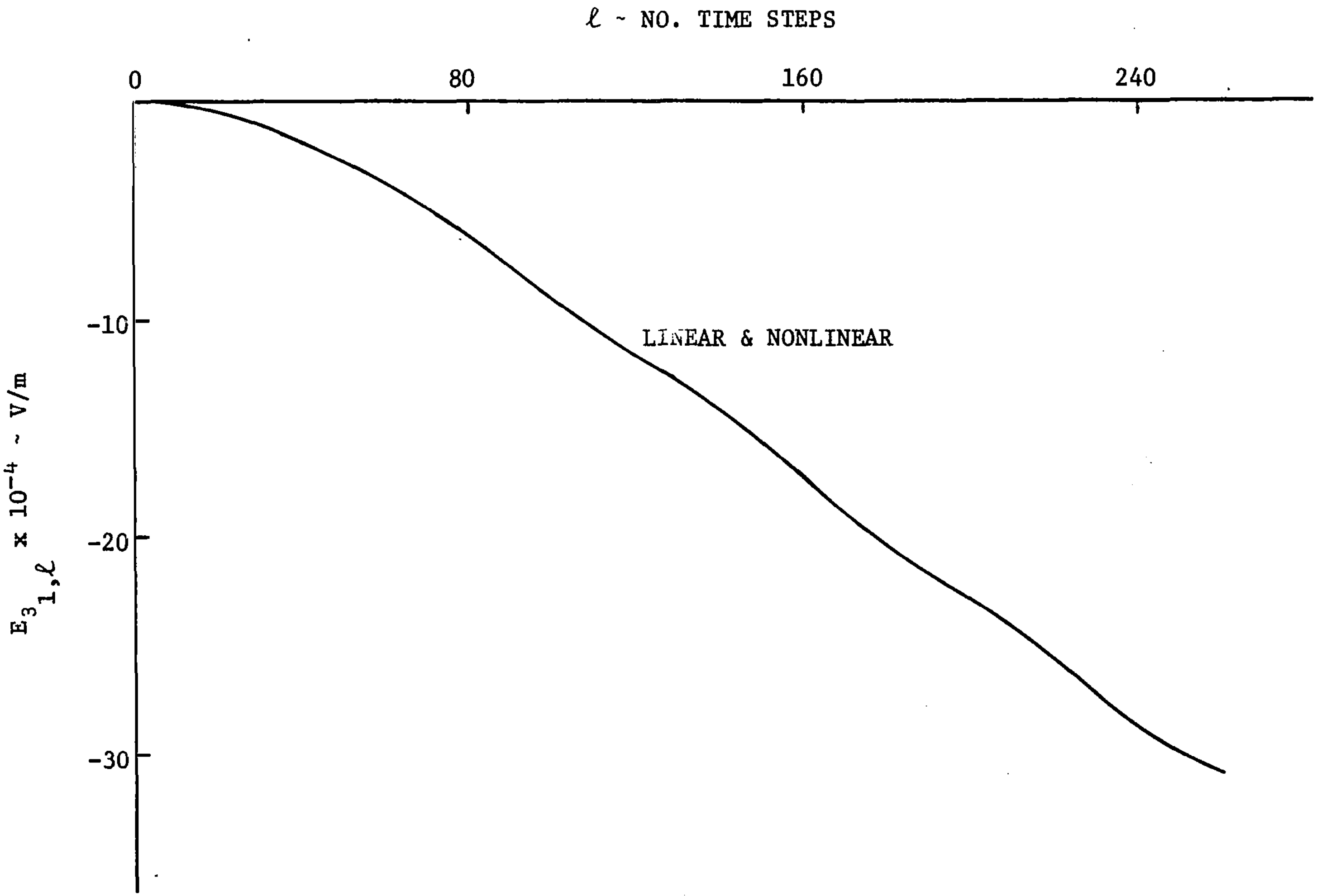


Figure 5(i). Electric Field in z-Direction vs. Number of Time Steps.

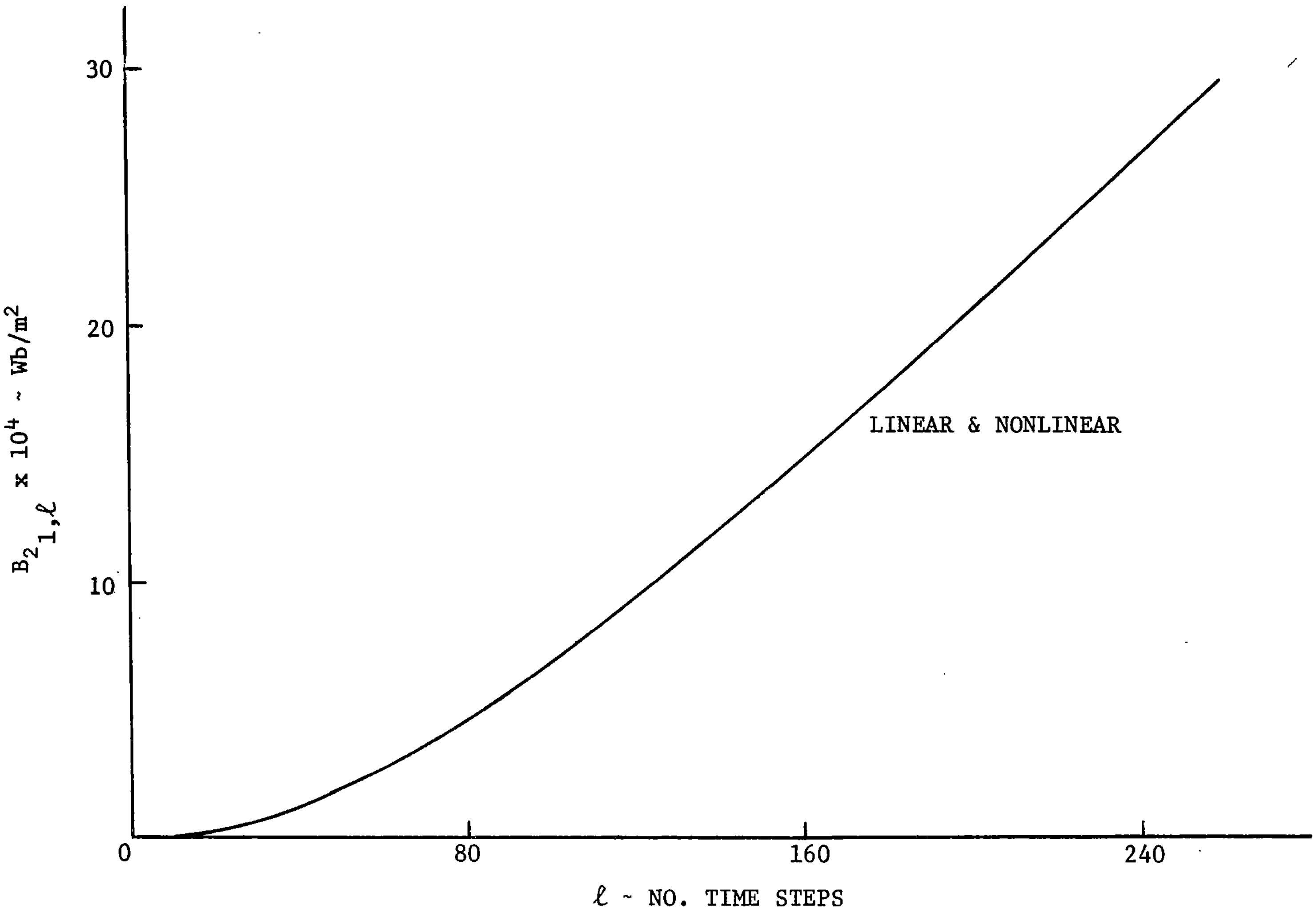


Figure 5(j). Magnetic Field in y-Direction vs. Number of Time Steps.

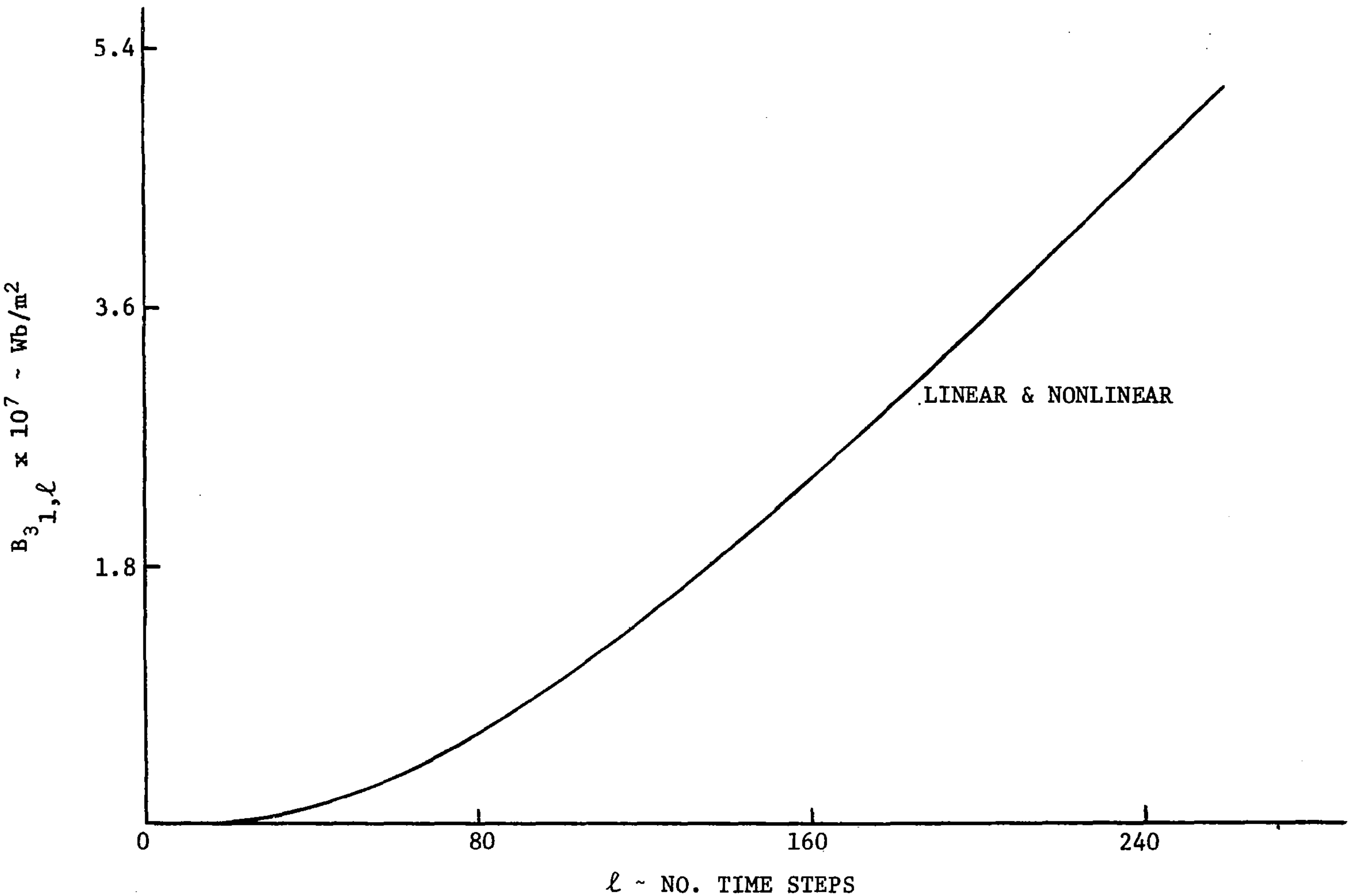


Figure 5(k). Magnetic Field in z-Direction vs. Number of Time Steps.

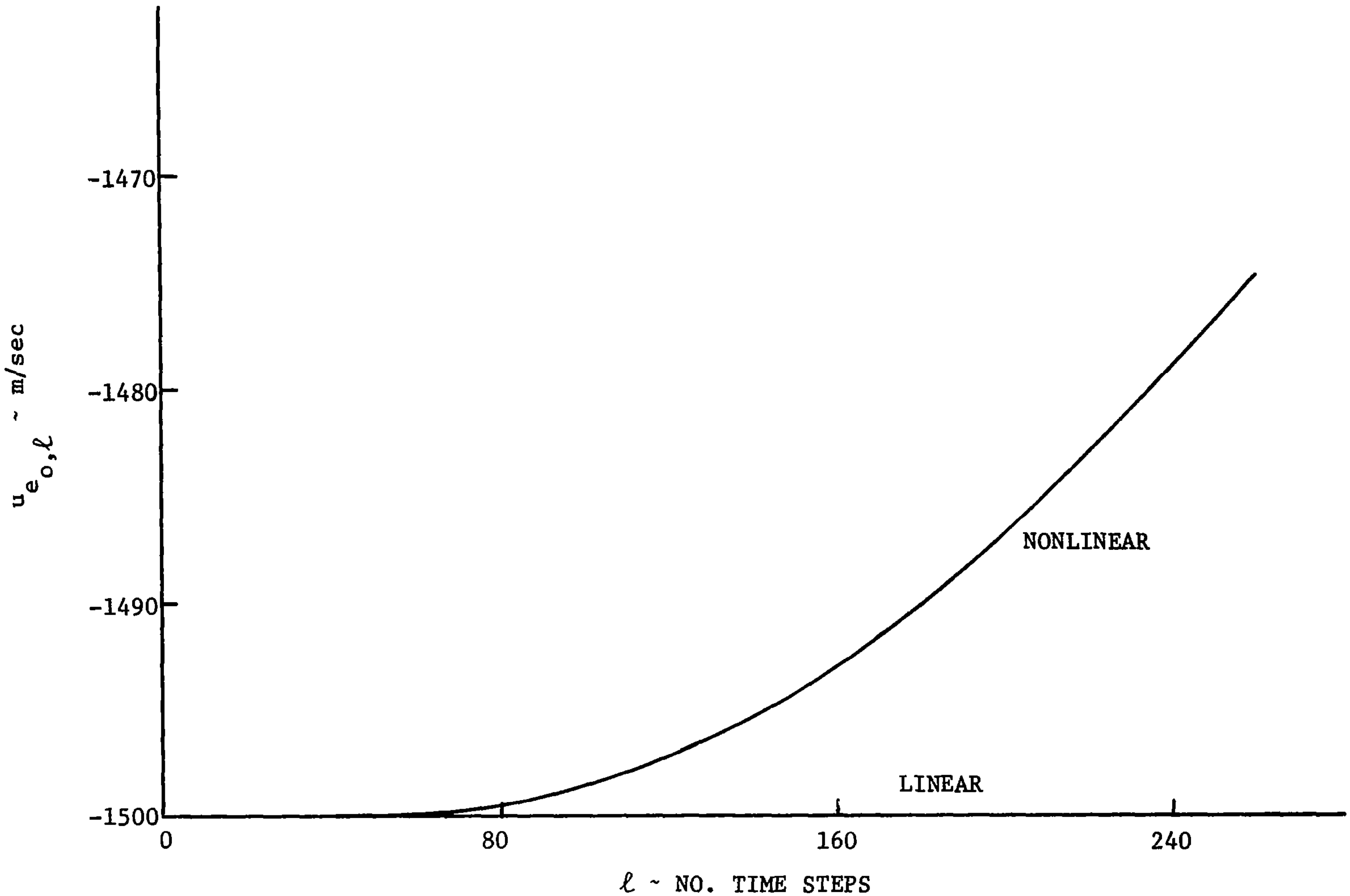


Figure 5(1). Electron Velocity in x-Direction vs. Number of Time Steps at k = 0.

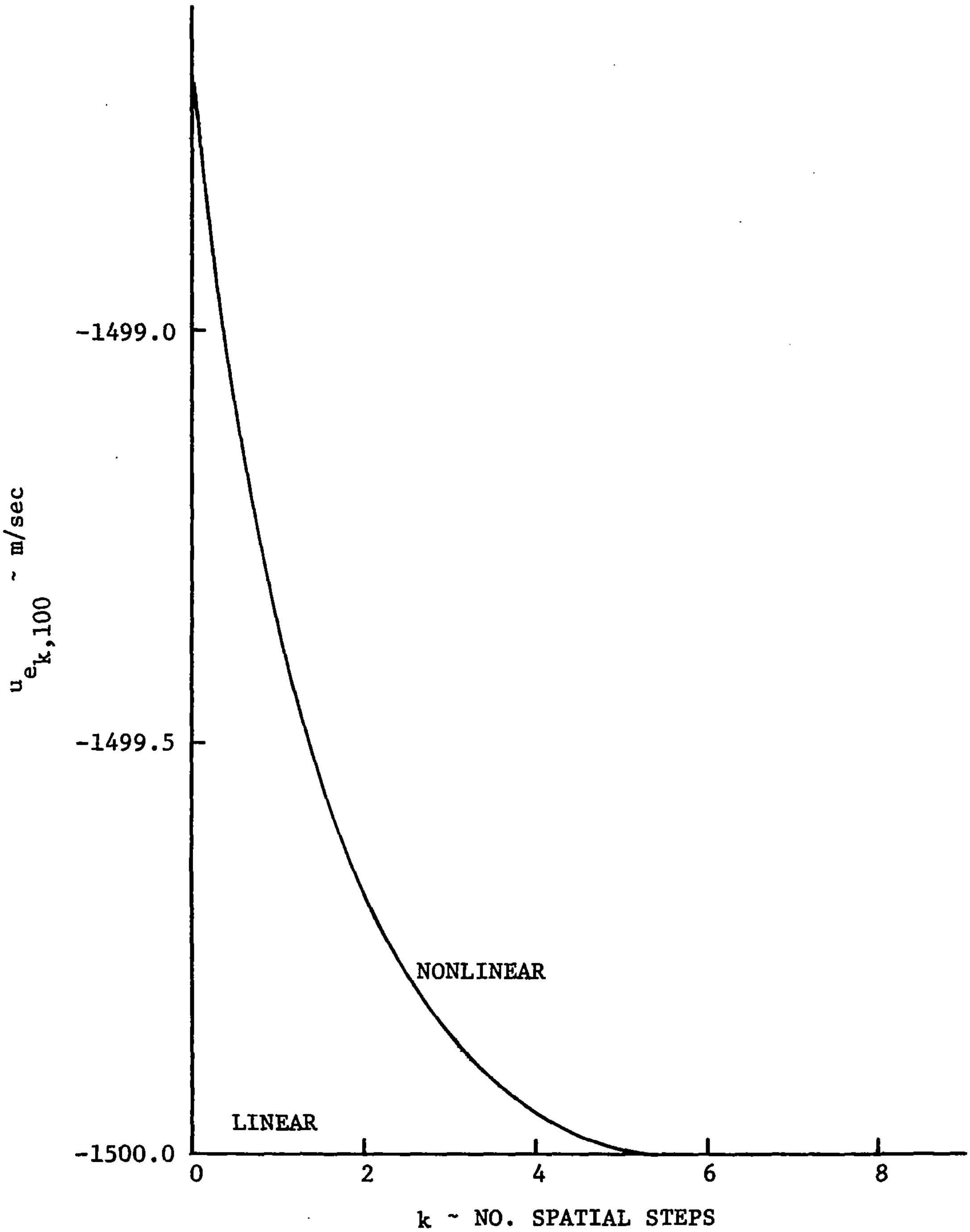


Figure 6(a). Electron Velocity in x-Direction vs. Number of Spatial Steps for $\Delta t = 5 \times 10^{-14}$ sec, $\Delta x = 2.5 \times 10^{-5}$ m, $\tilde{B}_{02} = 0.005$ Wb/m^2, $\tilde{E}_{02} =$ 100 V/m, $\omega = 2 \times 10^{10}$/sec, $\ell = 100$.

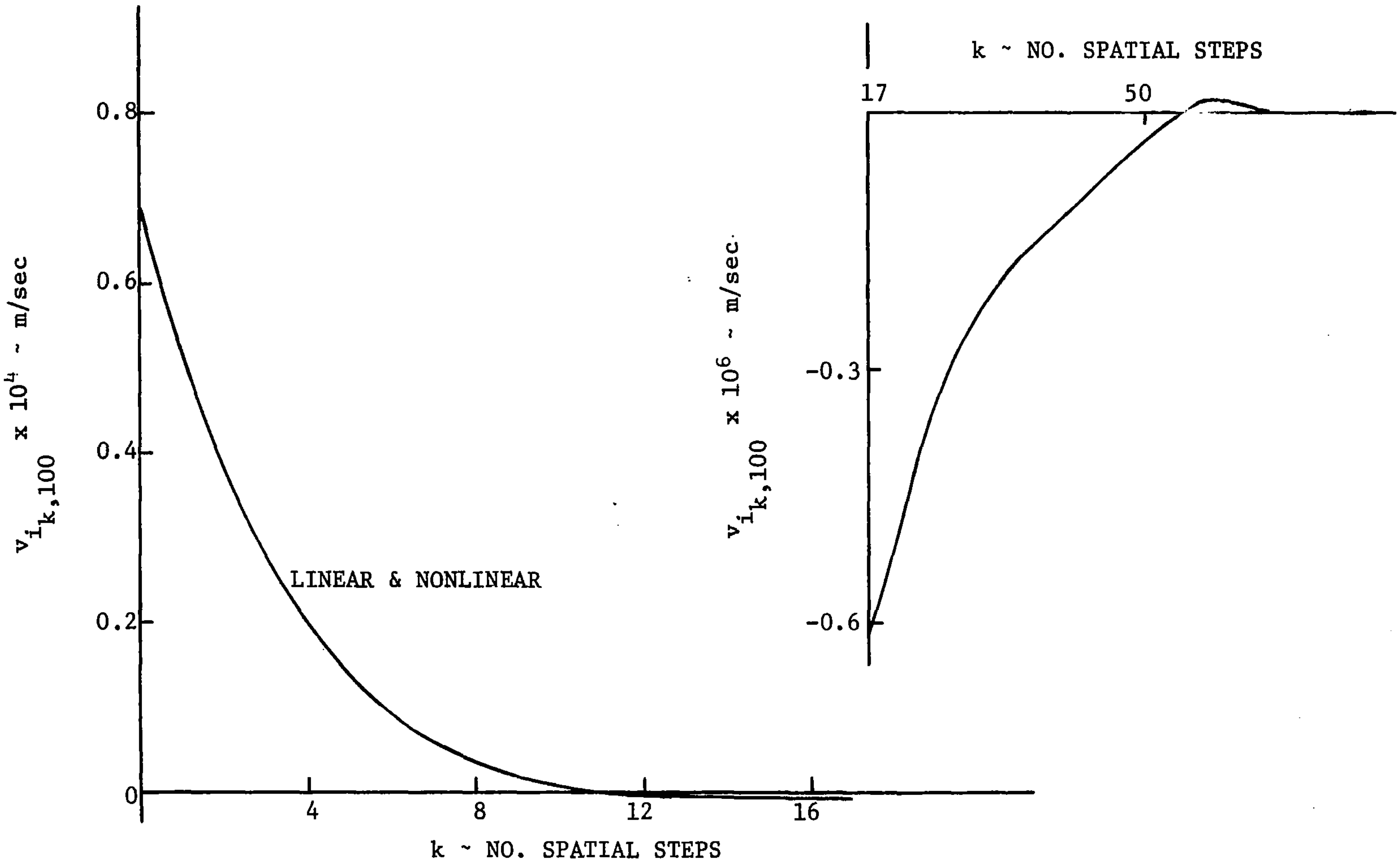


Figure 6(b). Ion Velocity in y-Direction vs. Number of Spatial Steps.

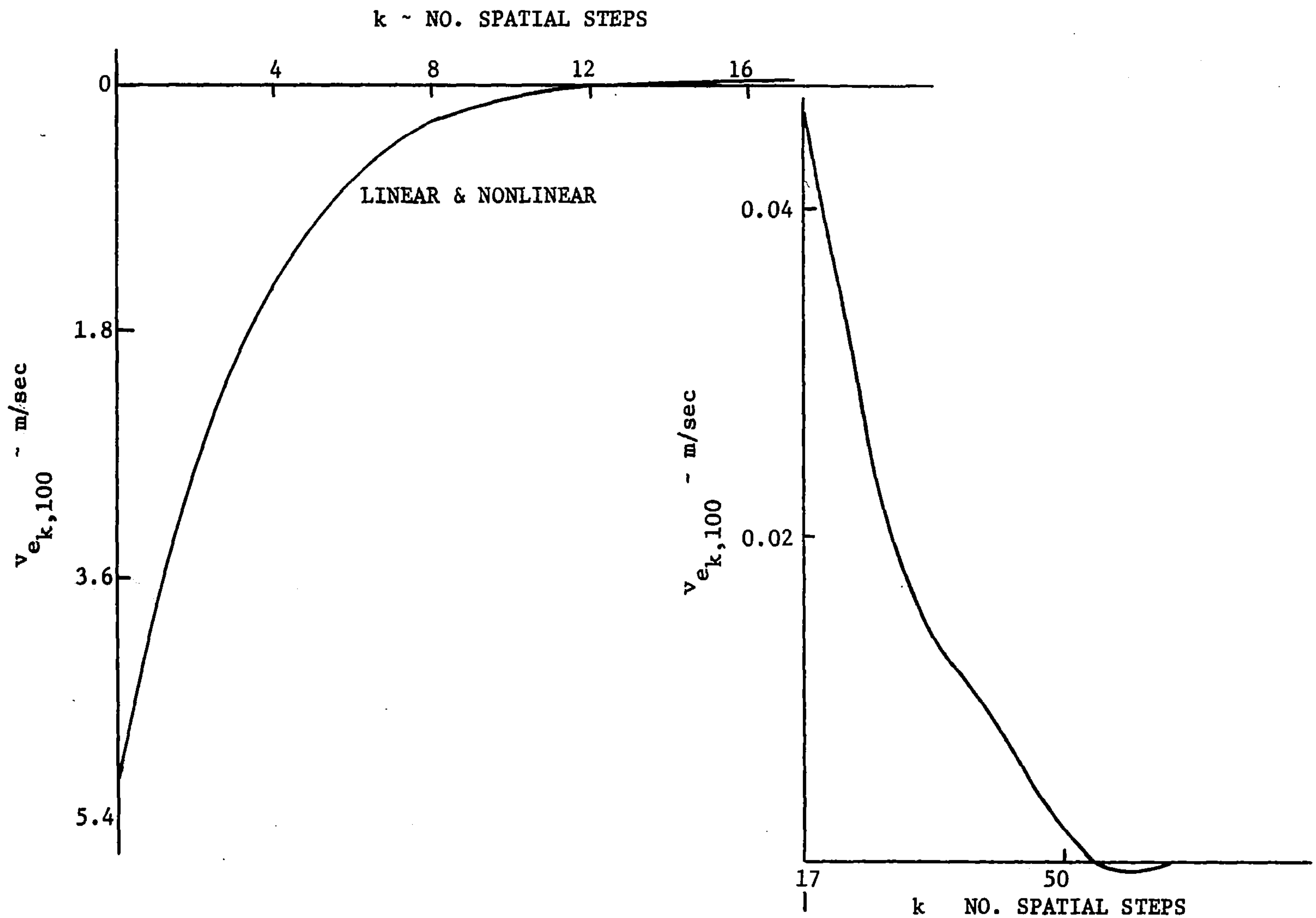


Figure 6(c). Electron Velocity in y-Direction vs. Number of Spatial Steps.

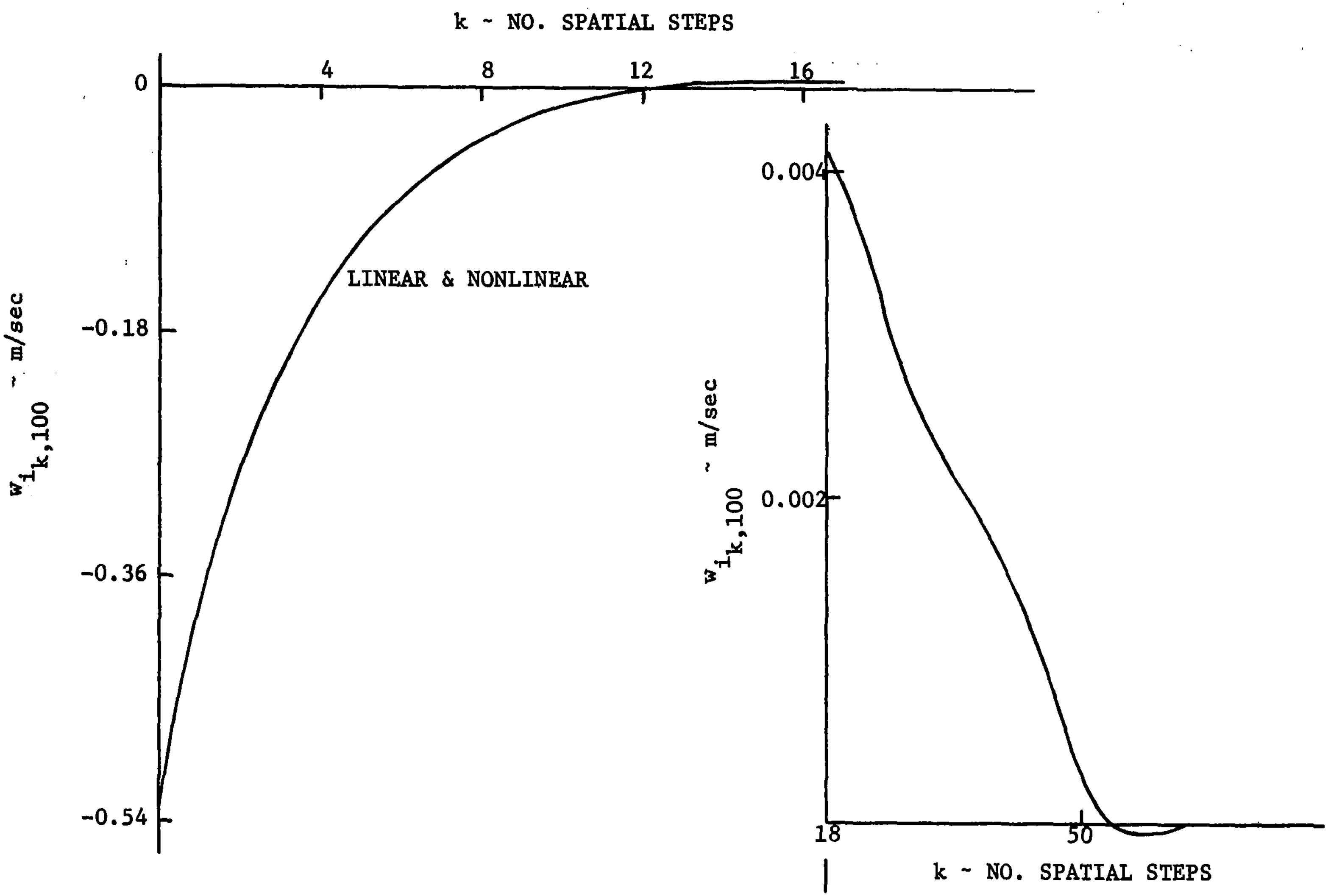


Figure 6(d). Ion Velocity in z-Direction vs. Number of Spatial Steps.

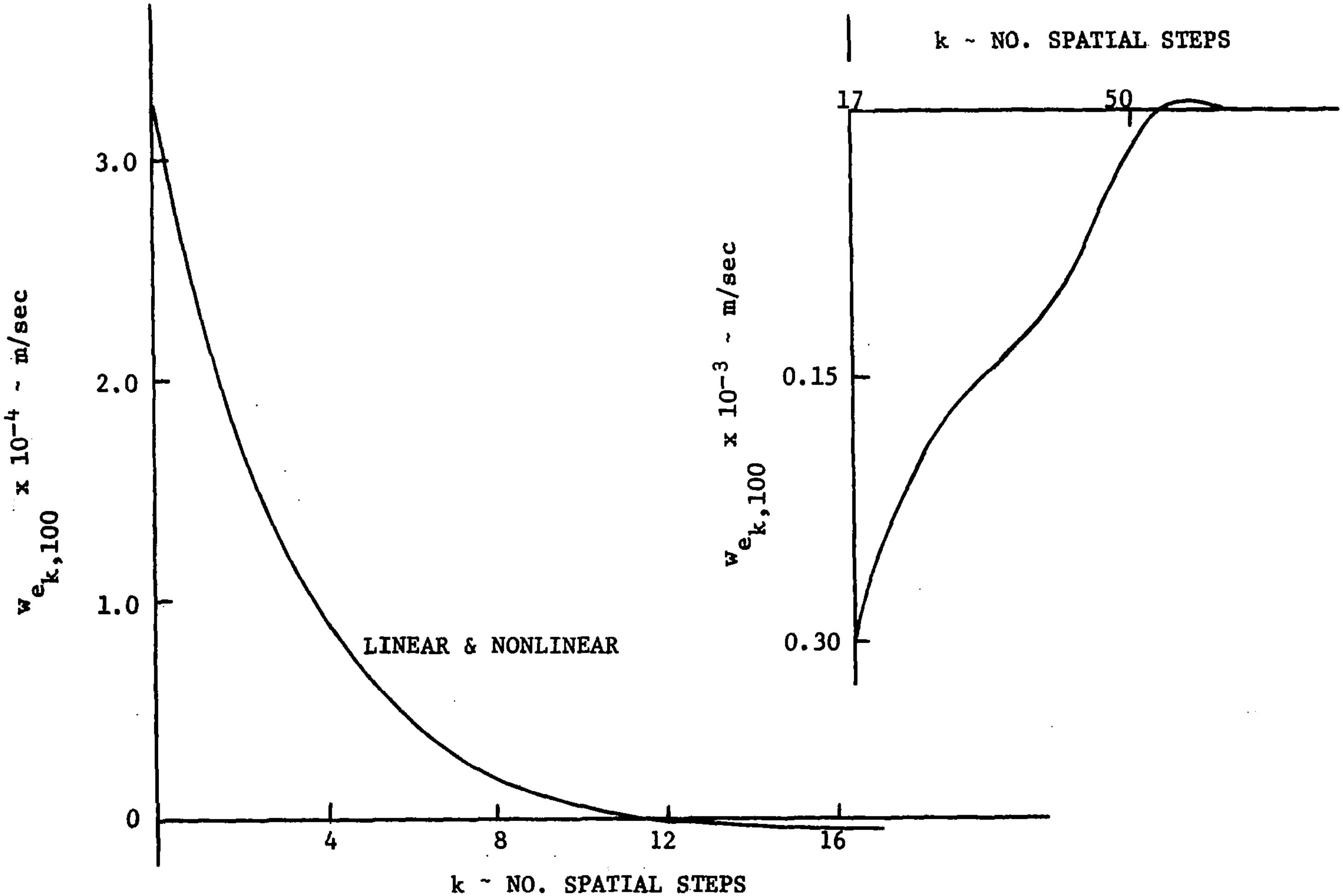


Figure 6(e). Electron Velocity in z-Direction vs. Number of Spatial Steps.

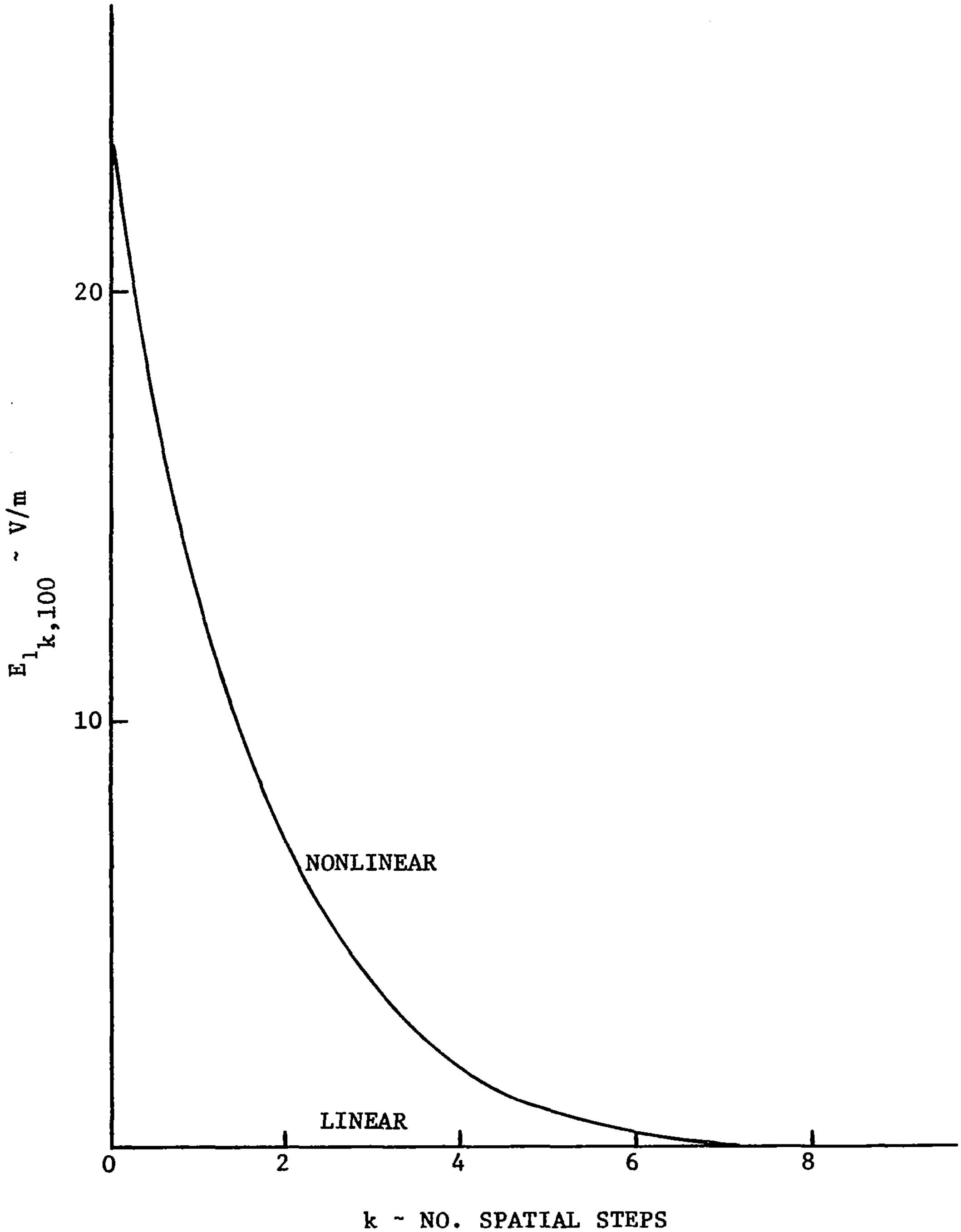


Figure 6(f). Electric Field in x-Direction vs. Number of Spatial Steps.

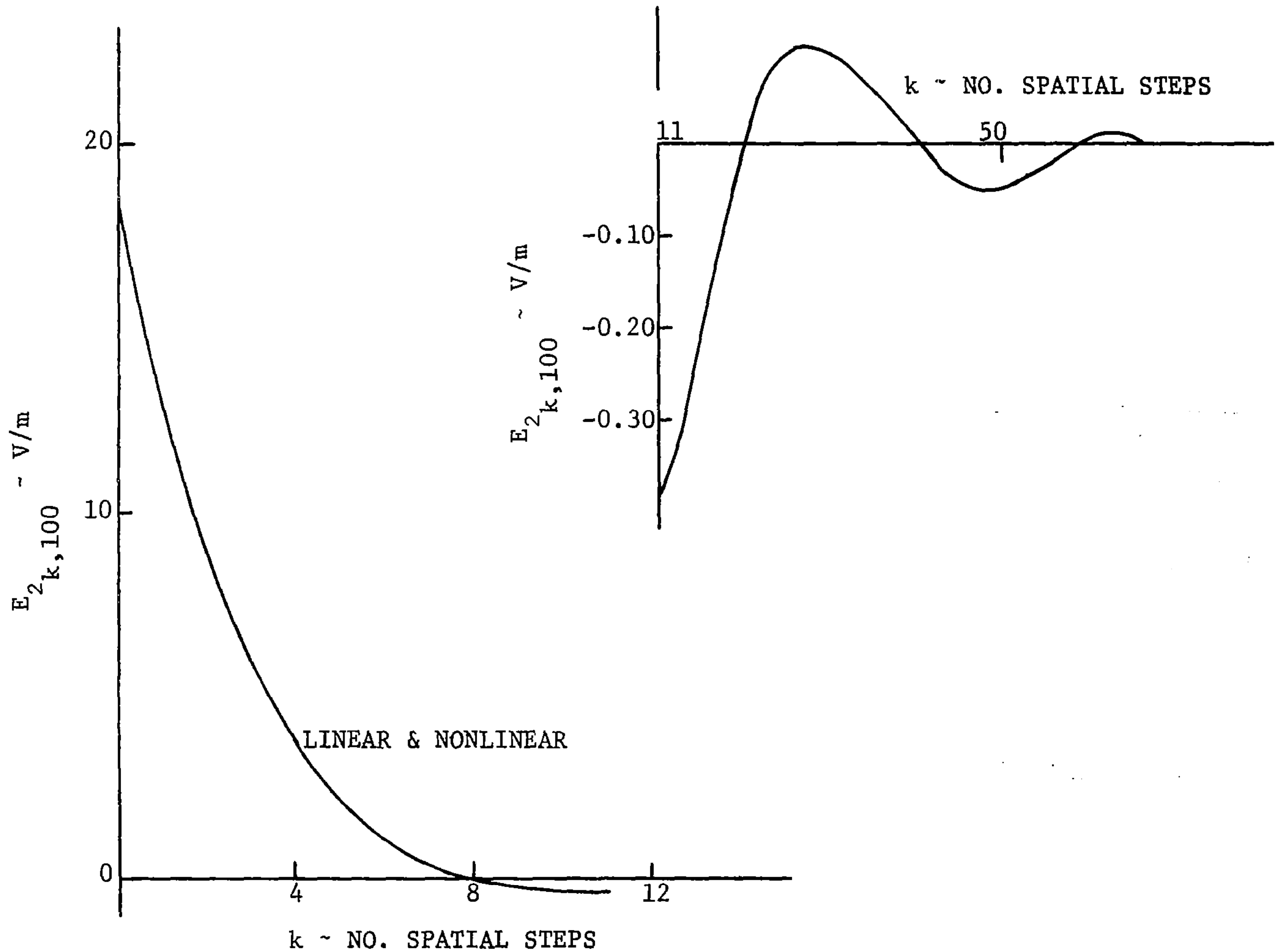


Figure 6(g). Electric Field in y-Direction vs. Number of Spatial Steps.

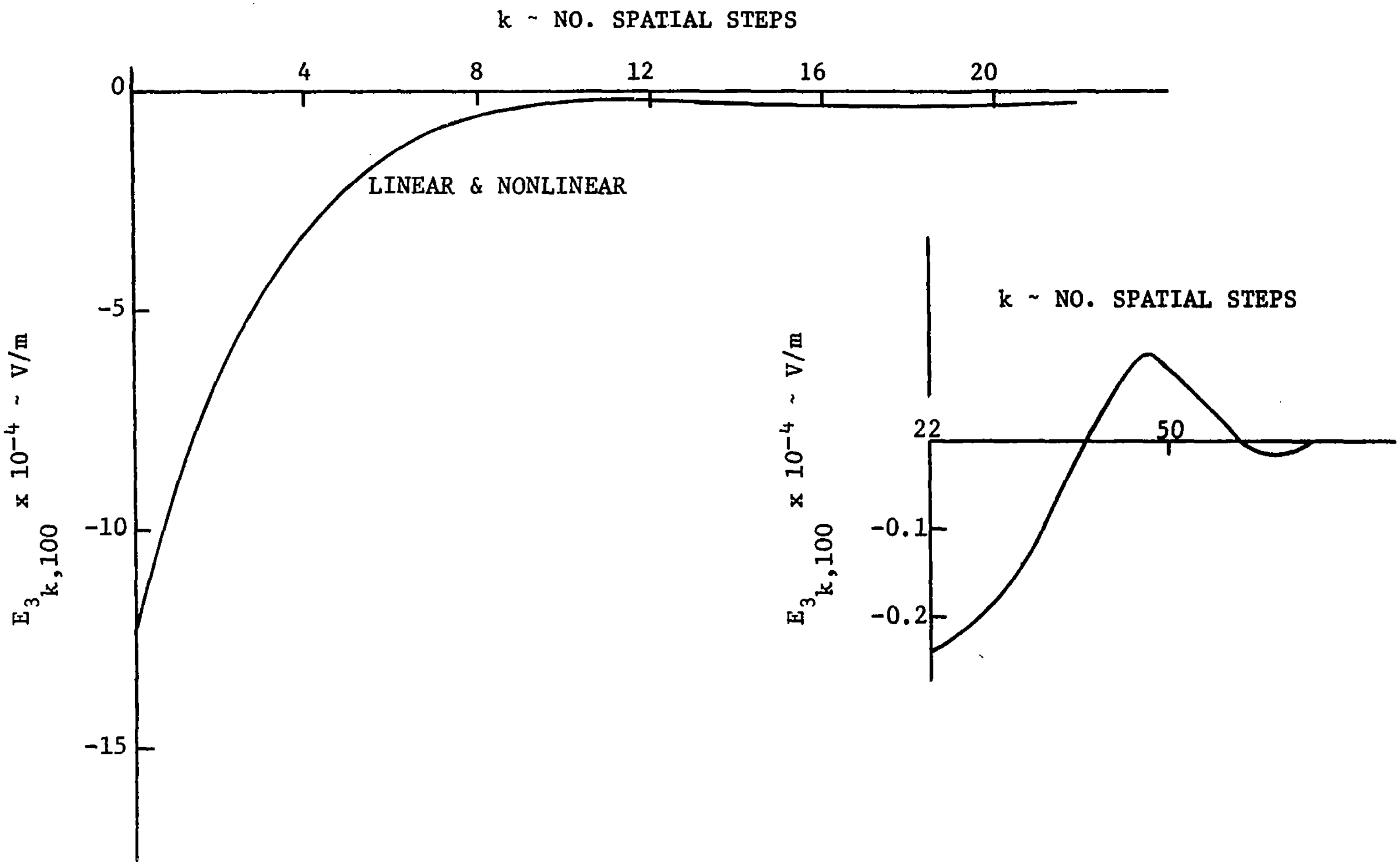


Figure 6(h). Electric Field in z-Direction vs. Number of Spatial Steps.

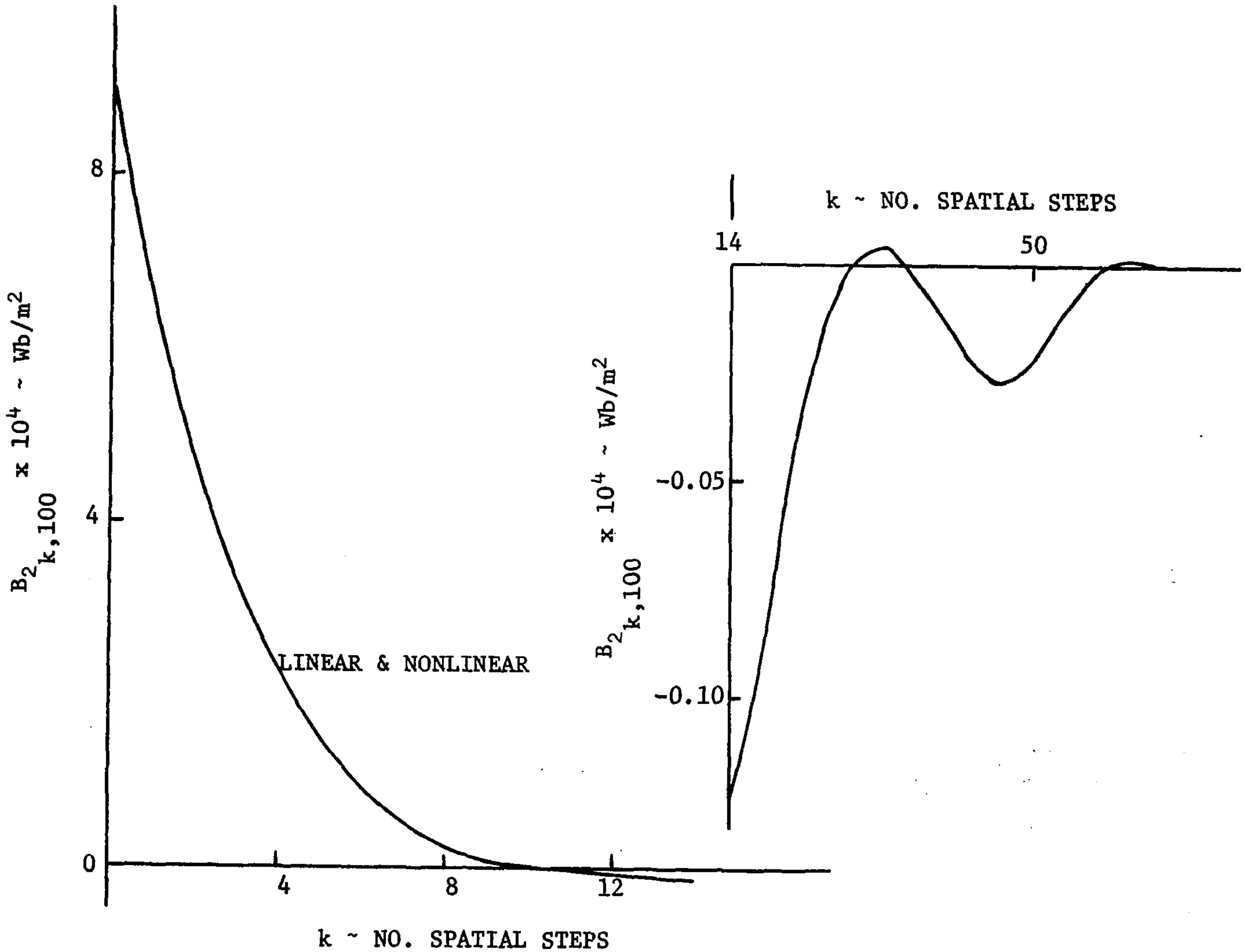


Figure 6(i). Magnetic Field in y-Direction vs. Number of Spatial Steps.

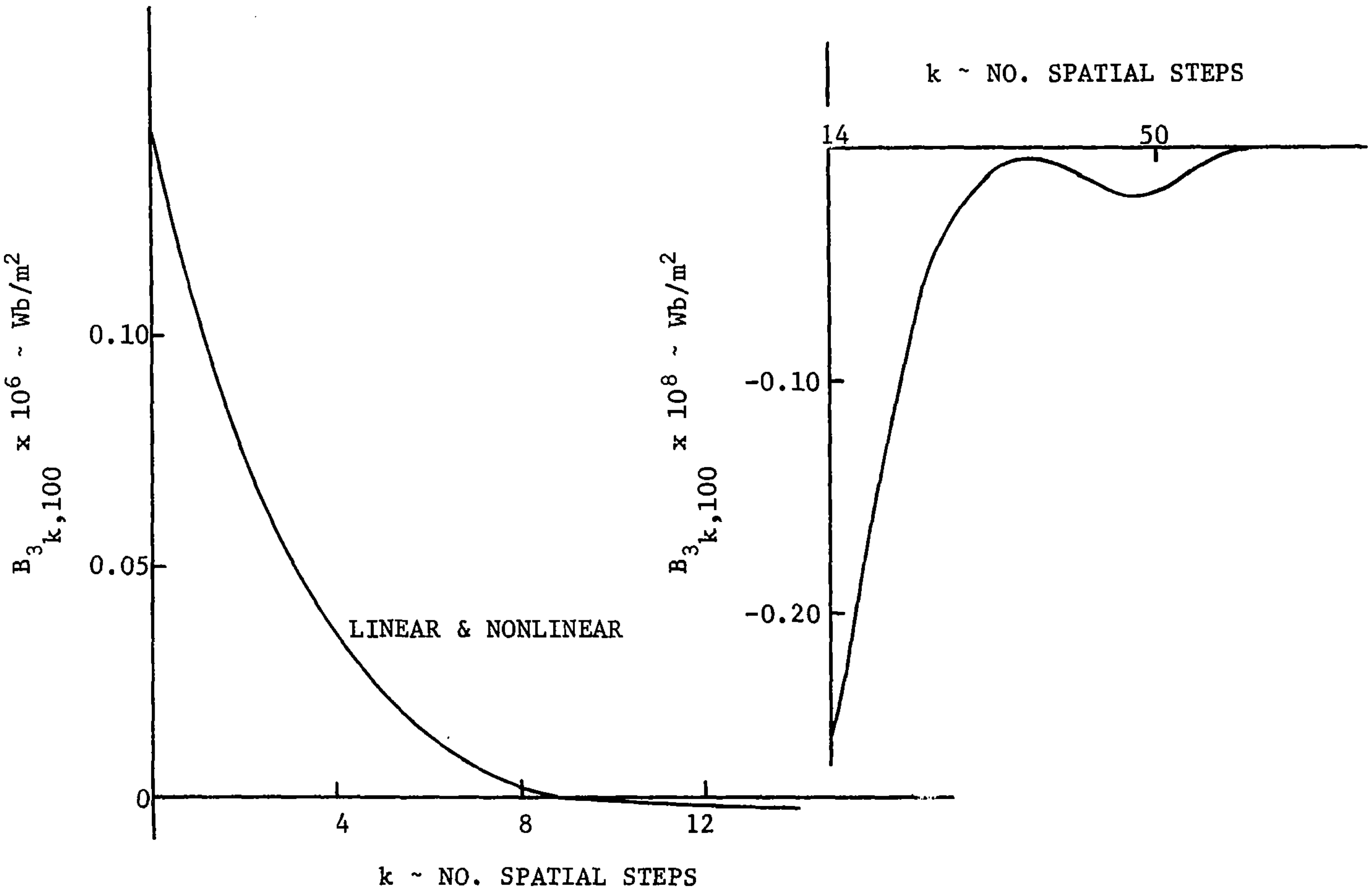


Figure 6(j). Magnetic Field in z-Direction vs. Number of Spatial Steps.

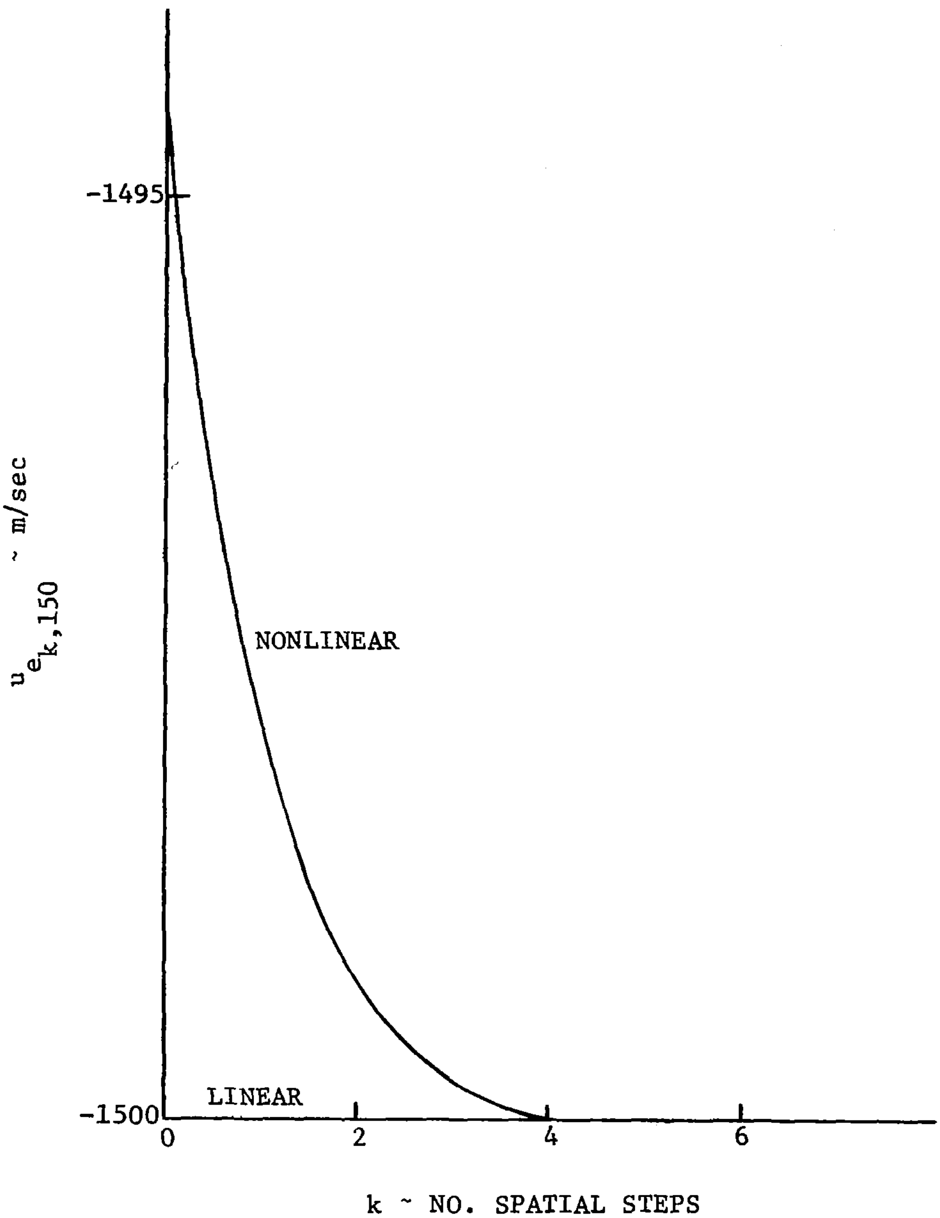


Figure 6(k). Electron Velocity in x-Direction vs. Number of Spatial Steps at ℓ = 150.

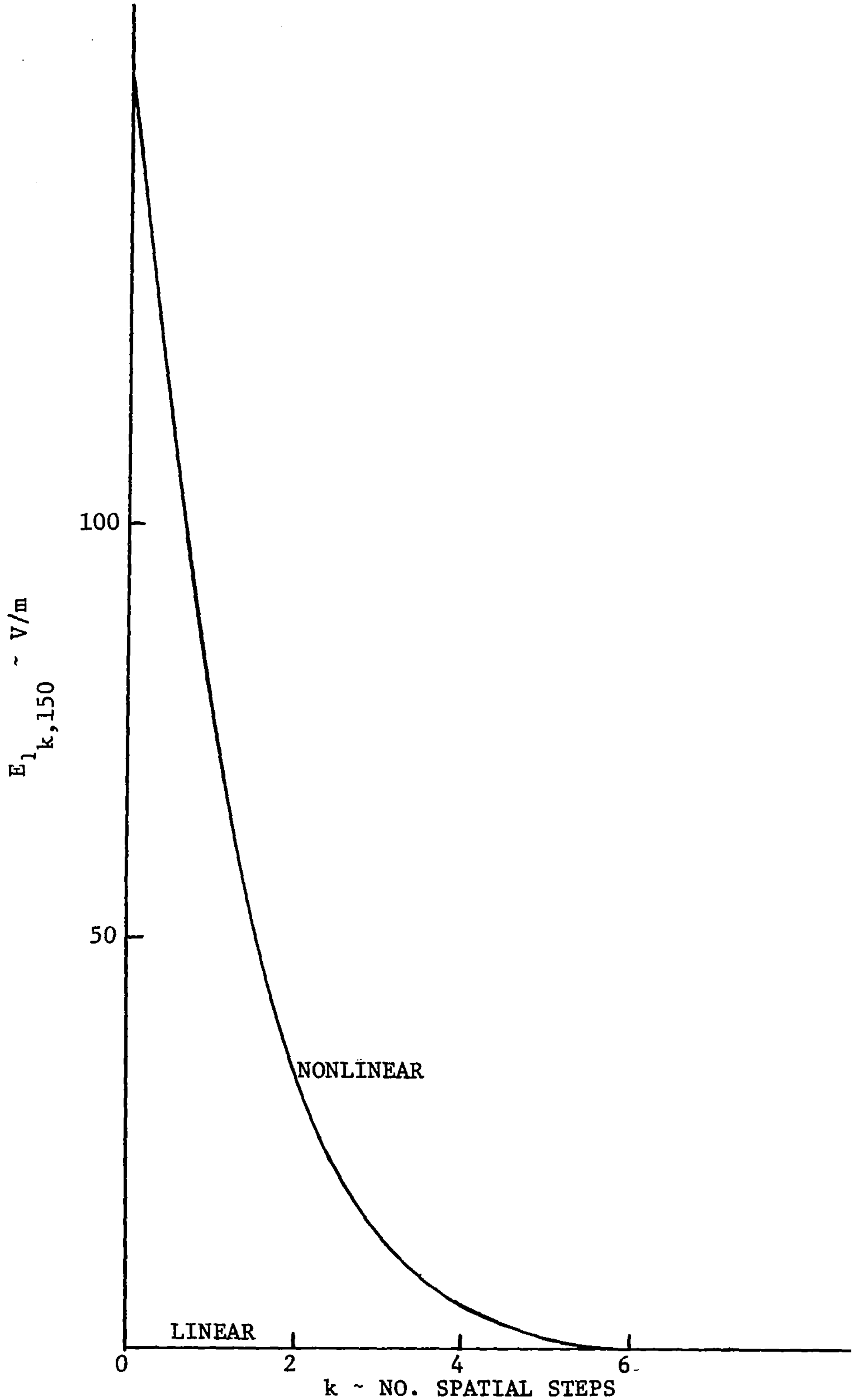


Figure 6(1). Electric Field in x-Direction vs. Number of Spatial Steps at ℓ = 150.

APPENDIX A: GRAM-SCHMIDT ORTHOGONALIZATION PROCEDURE

Given the set of 17 linearly independent eigenvectors of [A], it is desired to find a corresponding set of unit orthogonal eigenvectors. The Gram-Schmidt orthogonalization procedure[12] is used for this purpose. First, let

$$\hat{e}_1 = \frac{\overline{X}_1}{\ell(\overline{X}_1)} \quad \text{where} \quad \ell(\overline{X}_1) = (\overline{X}_1,\overline{X}_1)^{\frac{1}{2}} \quad .$$

Then, let

$$\overline{\xi}_2 = \overline{X}_2 - c_1\hat{e}_1$$

and thus,

$$(\hat{e}_1,\overline{\xi}_2) = (\hat{e}_1,\overline{X}_2) - c_1(\hat{e}_1,\hat{e}_1) \quad .$$

Now, since an orthogonal set is desired, $(e_1,\overline{\xi}_2) = 0$. Therefore, from the above, $c_1 = (\hat{e}_1,\overline{X}_2)$ and thus

$$\overline{\xi}_2 = \overline{X}_2 - (\hat{e}_1,\overline{X}_2)\hat{e}_1$$

and

$$\hat{e}_2 = \frac{\overline{\xi}_2}{\ell(\overline{\xi}_2)} \quad .$$

If this procedure is continued, it is seen that in general

$$\overline{\xi}_j = \overline{X}_j - \sum_{i=1}^{j-1}(\hat{e}_i,\overline{X}_j)\hat{e}_i \; ; \; j = 2,3,\cdots,17$$

and then

$$\hat{e}_j = \frac{\overline{\xi}_j}{\ell(\overline{\xi}_j)} \; ; \; j = 2,3,\cdots,17$$

yields the unit orthogonal set required.

APPENDIX B: CALCULATION OF AVERAGE COLLISION FREQUENCIES

The expressions for the average collision frequencies below are taken from Murphree and Yamada.[13] Thus,

$$\nu_{ei} = \frac{4}{3}\,\frac{1}{(4\pi K_o)^2}\,\left(\frac{2\pi}{m_e}\right)^{\frac{1}{2}}\,\frac{e^4 N_e}{(KT_e)^{3/2}}\,\ln\left(\frac{\lambda_d}{r_c}\right)$$

where

$$\lambda_d = \left(\frac{K_o KT_e}{N_e e^2}\right)^{\frac{1}{2}}$$

and

$$r_c = \frac{e^2}{6\pi K_o KT_e}$$

Using $N_e = 10^{15}/cm^3$ and $T_e = 10^4\ {}^oK$, ν_{ei} is calculated to be 2.76 x 10^{10}/sec. Now

$$\nu_{en} = c_e N_n Q_{en}$$

where

$$c_e = \left(\frac{8KT_e}{\pi m_e}\right)^{\frac{1}{2}}$$

and

$$Q_{en} = \frac{\pi d^2}{4} \quad .$$

Thus, using d = 2.87 x 10^{-8} cm as the diameter of an argon atom and $N_n = 3.62 \times 10^{18}/cm^3$, ν_{en} is calculated to be 1.44 x 10^{11}/sec. Also,

$$\nu_{in} = c_i N_n Q_{in}$$

where

$$c_i = \left(\frac{8KT_i}{m_i}\right)^{\frac{1}{2}} \quad .$$

Using $T_i = 2000^oK$, ν_{in} is calculated to be 5.9 x 10^8/sec. With these average collision frequencies and the expressions from Ref. 1, the effective collision frequencies given in Section II may be obtained.

APPENDIX C: COMPUTER PROGRAMS

```
      DOUBLE PRECISION ZETA1(301,2),ZETA2(301,2),ZETA3(301,
     12),V1(301,2),V2(301,2),V3(301,2),V4(301,2),V5(301,2)
     2,V6(301,2),V7(301,2),V8(301,2),V9(301,2),V10(301,2),
     3V11(301,2),V12(301,2),V13(301,2),V14(301,2),V15(301,
     42),V16(301,2),V17(301,2),D1,D2,D3,D4,NI(301,2),NE(301
     5,2),NN(301,2),UI(301,2),UE(301,2),UN(301,2),VI(301,2)
     6VE(301,2),VN(301,2),WI(301,2),WE(301,2),WN(301,2)
      DOUBLE PRECISION E1(301,2),E2(301,2),E3(301,2),B2(301
     1,2),B3(301,2),B1,C,A1,A2,A3,C1,C2,C3,C4,C5,C6,C7,C8,
     2C9,C10,C11,C12,C13,C14,C15,C16,C17,E,MI,ME,MN,X,T,CF12
     3,CF13,CF21,CF23,CF31,CF32,MU,DSQRT,NIO,NEO,NNO,DCOS,
     4E02,B02,OMEGA,G1(2),G2(2)
C*****************************************************************
C     GIVEN THE INITIAL VALUES OF THE ELECTROMAGNETIC AND
C     FLUID FIELD VARIABLES, PLUS BOUNDARY VALUES FOR SOME
C     VARIABLES, THIS PROGRAM USES AN EXPLICIT FINITE -
C     DIFFERENCE SCHEME TO SOLVE THE NONLINEAR EQUATIONS.
C     THE VARIABLES ARE PRINTED AT A PARTICULAR TIME STEP
C     VERSUS THE SPACE STEPS.
C     THE FOLLOWING DEFINES THE SYMBOLS USED.
C     NI,NE,NN=NUMBER DENSITIES OF IONS,ELECTRONS, AND
C     NEUTRALS RESPECTIVELY.
C     ZETA1,ZETA2,ZETA3=NATURAL LOG OF THE ION,ELECTRON,AND
C     NEUTRAL NUMBER DENSITIES RESPECTIVELY.
C     UI,UE,UN=VELOCITY IN X DIRECTION OF IONS,ELECTRONS,
C     AND NEUTRALS
C     VI,VE,VN=VELOCITY IN Y DIRECTION OF IONS,ELECTRONS,AND
C     NEUTRALS
C     WI,WE,WN=VELOCITY IN Z DIRECTION OF IONS,ELECTRONS,
C     AND NEUTRALS
C     E1,E2,E3=ELECTRIC FIELD COMPONENTS IN THE X,Y,Z
C     DIRECTIONS
C     B1,B2,B3=MAGNETIC FIELD COMPONENTS IN THE X,Y,Z
C     DIRECTIONS
C     A1,A2,A3=THERMAL VELOCITIES OF THE IONS,ELECTRONS,AND
C     NEUTRALS ASSUMING EACH FLUID IS ISOTHERMAL
C     MI,ME,MN=MASS OF AN ION,ELECTRON,OR NEUTRAL
C     X,T=THE STEP SIZES IN THE SPATIAL AND TIME COORDINATES
C     C=SPEED OF LIGHT
C     MU=PERMEABILITY OF A VACUUM
C     KO=PERMITTIVITY OF A VACUUM
C     T1,T2,T3=TEMPERATURE OF THE IONS,ELECTRONS,AND NEUTRAL
C     BC=BOLTZMANN'S CONSTANT
C     E=ELECTRONIC CHARGE
C     V1,V2,...,V17=THE ELEMENTS OF THE COLLUM MATRIX V , IN
C     THE TRANSFORMATION U=TV.
C  ****************** STATEMENT OF THE PROBLEM **********
C     THE PROBLEM IS SUCH THAT AT X=0 WE FORCE E2 AND B2 TO
C     BE CERTAIN FUNCTIONS OF TIME, PLUS, WE ALSO SPECIFY
C     THAT NE=NI AT X=0. THEN IF WE ASSUME THAT UI,UE,AND UN
C     ARE ALL IN THE NEGATIVE X DIRECTION AND THAT UI .GT.
C     A1, UE .LT. A2, AND UN .GT. A3 WE MUST CALCULATE ALL
C     OTHER VARIABLES AT X=0. OF COURSE WE ALSO CALCULATE
C     ALL OTHER VARIABLES AT LATER TIMES AND DIFFERENT
```

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C      SPATIAL LOCATIONS. INSTEAD OF E2 AND B2 WE COULD HAVE
C      SPECIFIED B2 AND B3 OR E2 AND E3 OR E3 AND B3.
C*************************************************************
       NIO=1.0D21
       NEO=1.0D21
       NNO=3.62D24
       X=2.5D-5
       PI=3.1415927D0
       E=1.60D-19
       B1=1.0D-4
       C=3.0D8
       MI=0.673D-25
       ME=9.11D-31
       MN=MI
       MU=12.56D-7
       BC=1.38D-23
       T1=2000.0D0
       T2=1.0D4
       T3=2000.0D0
       CF12=3.73D5
       CF13=2.95D8
       CF21=2.76D10
       CF23=1.44D11
       CF31=0.815D5
       CF32=5.36D2
       E02=100.0D0
       B02=0.005D0
       OMEGA=2.0D10
       A1=DSQRT(BC*T1/MI)
       A2=DSQRT(BC*T2/ME)
       A3=DSQRT(BC*T3/MN)
       D1=1.0D0/DSQRT(1.0D0+A1**2)
       D2=1.0D0/DSQRT(1.0D0+A2**2)
       D3=1.0D0/DSQRT(1.0D0+A3**2)
       D4=1.0D0/DSQRT(1.0D0+C**2)
C      THIS READS IN THE INITIAL VALUES.THE ORIGIN IS TAKEN
C      TO BE (1,1)
       DO 80 K=1,301
       L=1
       NI(K,L)=NIO
       NE(K,L)=NEO
       NN(K,L)=NNO
       UI(K,L)=-1500.0D0
       UE(K,L)=-1500.0D0
       UN(K,L)=-1500.0D0
       VI(K,L)=0.0D0
       VE(K,L)=0.0D0
       VN(K,L)=0.0D0
       WI(K,L)=0.0D0
       WE(K,L)=0.0D0
       WN(K,L)=0.0D0
       E1(K,L)=0.0D0
       E2(K,L)=0.0D0
       E3(K,L)=0.0D0
```

```
      B2(K,L)=0.0D0
      B3(K,L)=0.0D0
      ZETA1(K,L)=DLOG(NI(K,L))
      ZETA2(K,L)=DLOG(NE(K,L))
      ZETA3(K,L)=DLOG(NN(K,L))
      V1(K,L)=D4*(C*B3(K,L)-E2(K,L))
      V2(K,L)=D4*(E3(K,L)+C*B2(K,L))
      V3(K,L)=D2*(UE(K,L)-A2*ZETA2(K,L))
      V4(K,L)=D1*(UI(K,L)-A1*ZETA1(K,L))
      V5(K,L)=D3*(UN(K,L)-A3*ZETA3(K,L))
      V6(K,L)=D1*(ZETA1(K,L)+A1*UI(K,L))
      V7(K,L)=D3*(ZETA3(K,L)+A3*UN(K,L))
      V8(K,L)=VI(K,L)
      V9(K,L)=WI(K,L)
      V10(K,L)=VE(K,L)
      V11(K,L)=WE(K,L)
      V12(K,L)=VN(K,L)
      V13(K,L)=WN(K,L)
      V14(K,L)=E1(K,L)
      V15(K,L)=D2*(ZETA2(K,L)+A2*UE(K,L))
      V16(K,L)=D4*(C*E2(K,L)+B3(K,L))
      V17(K,L)=D4*(B2(K,L)-C*E3(K,L))
   80 CONTINUE
C     THE INITIAL VALUES HAVE NOW BEEN READ IN AND HAVE
C     UNDER GONE THE TRANSFORMATION V=(T INVERSE)(U)
      M1=1
   22 J=301-M1
      T=5.0D-14
      M2=100
      DO 54 K=1,J
      C1=-D4*C**2*MU*E*(V10(K,L)*DEXP(D2*(V15(K,L)-A2*
     1V3(K,L)))-V8(K,L)*DEXP(D1*(V6(K,L)-A1*V4(K,L))))
      C2=D4*C**2*MU*E*(V11(K,L)*DEXP(D2*(V15(K,L)-A2*V3
     1(K,L)))-V9(K,L)*DEXP(D1*(V6(K,L)-A1*V4(K,L))))
      C3=D2*(-(E/ME)*(V14(K,L)+V10(K,L)*D4*(C*V1(K,L)+V16
     1(K,L))-V11(K,L)*D4(C*V2(K,L)+V17(K,L)))+CF21*(D1*
     2(V4(K,L)+A1*V6(K,L))-D2*(V3(K,L)+A2*V15(K,L)))+CF23*
     3(D3*(V5(K,L)+A3*V7(K,L))-D2*(V3(K,L)+A2*V15(K,L))))
      C4=D1*((E/MI)*(V14(K,L)+V8(K,L)*D4*(C*V1(K,L)+V16
     1(K,L))-V9(K,L)*D4*(C*V2(K,L)+V17(K,L)))+CF12*(D2*
     2(V3(K,L)+A2*V15(K,L))-D1*(V4(K,L)+A1*V6(K,L)))+CF13*
     3(D3*(V5(K,L)+A3*V7(K,L))-D1*(V4(K,L)+A1*V6(K,L))))
      C5=D3*(CF31*(D1*(V4(K,L)+A1*V6(K,L))-D3*(V5(K,L)+A3*
     1V7(K,L)))+CF32*(D2*(V3(K,L)+A2*V15(K,L))-D3*(V5(K,L)+
     2A3*V7(K,L))))
      C6=A1*C4
      C7=A3*C5
      C8=(E/MI)*(D4*(C*V16(K,L)-V1(K,L))+B1*V9(K,L)-D1*D4*
     1(V4(K,L)+A1*V6(K,L))*(C*V1(K,L)+V16(K,L)))+CF12*
     2(V10(K,L)-V8(K,L))+CF13*(V12(K,L)-V8(K,L))
      C9=(E/MI)*(D4*(V2(K,L)-C*V17(K,L))+D1*D4*(V4(K,L)+A1*
     1V6(K,L))*(C*V2(K,L)+V17(K,L))-B1*V8(K,L))+CF12*(V11
     2(K,L)-V9(K,L))+CF13*(V13(K,L)-V9(K,L))
      C10=(-E/ME)*(D4*(C*V16(K,L)-V1(K,L))+B1*V11(K,L)-D2*
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      1D4*(V3(K,L)+A2*V15(K,L))*(C*V1(K,L)+V16(K,L)))+CF21*
      2(V8(K,L)-V10(K,L))+CF23*(V12(K,L)-V10(K,L))
       C11=-(E/ME)*(D4*(V2(K,L)-C*V17(K,L))+D2*D4*(V3(K,L)+
      1A2*V15(K,L))*(C*V2(K,L)+V17(K,L))-B1*V10(K,L))+CF21*
      2(V9(K,L)-V11(K,L))+CF23*(V13(K,L)-V11(K,L))
       C12=CF31*(V8(K,L)-V12(K,L))+CF32*(V10(K,L)-V12(K,L))
       C13=CF31*(V9(K,L)-V13(K,L))+CF32*(V11(K,L)-V13(K,L))
       C14=C**2*MU*E*(D2*(V3(K,L)+A2*V15(K,L))*DEXP(D1*(V6
      1(K,L)-A1*V4(K,L)))-D1*(V4(K,L)+A1*V6(K,L))*DEXP(D2*
      2(V15(K,L)-A2*V3(K,L))))
       C15=A2*C3
       C16=-C*C1
       C17=-C*C2
       V1(K,L+1)=V1(K,L)+(T/X)*C*(V1(K+1,L)-V1(K,L))+T*C1
       V2(K,L+1)=V2(K,L)+(T/X)*C*(V2(K+1,L)-V2(K,L))+T*C2
       V3(K,L+1)=V3(K,L)-(T/X)*(D2*(V3(K,L)+A2*V15(K,L))-A2)*
      1(V3(K+1,L)-V3(K,L))+T*C3
       V4(K,L+1)=V4(K,L)-(T/X)*(D1*(V4(K,L)+A1*V6(K,L))-A1)*
      1(V4(K+1,6)-V4(K,L))+T*C4
       V5(K,L+1)=V5(K,L)-(T/X)*(D3*(V5(K,L)+A3*V7(K,L))-A3)*
      1(V5(K+1,L)-V5(K,L))+T*C5
       V6(K,L+1)=V6(K,L)-(T/X)*(D1*(V4(K,L)+A1*V6(K,L))+A1)*
      1(V6(K+1,L)-V6(K,L))+T*C6
       V7(K,L+1)=V7(K,L)-(T/X)*(D3*(V5(K,L)+A3*V7(K,L))+A3)*
      1(V7(K+1,L)-V7(K,L))+T*C7
       V8(K,L+1)=V8(K,L)-(T/X)*(D1*(V4(K,L)+A1*V6(K,L)))*
      1(V8(K+1,L)-V8(K,L))+T*C8
       V9(K,L+1)=V9(K,L)-(T/X)*(D1*(V4(K,L)+A1*V6(K,L)))*
      1(V9(K+1,L)-V9(K,L))+T*C9
       V10(K,L+1)=V10(K,L)-(T/X)*(D2*(V3(K,L)+A2*V15(K,L)))*
      1(V10(K+1,L)-V10(K,L))+T*C10
       V11(K,L+1)=V11(K,L)-(T/X)*(D2*(V3(K,L)+A2*V15(K,L)))*
      1(V11(K+1,L)-V11(K,L))+T*C11
       V12(K,L+1)=V12(K,L)-(T/X)*(D3*(V5(K,L)+A3*V7(K,L)))*
      1(V12(K+1,L)-V12(K,L))+T*C12
       V13(K,L+1)=V13(K,L)-(T/X)*(D3*(V5(K,L)+A3*V7(K,L)))*
      1(V13(K+1,L)-V13(K,L))+T*C13
       V14(K,L+1)=V14(K,L)-(T/X)*(D1*(V4(K,L)+A1*V6(K,L))+D2*
      1(V3(K,L)+A2*V15(K,L)))*(V14(K+1,L)-V14(K,L))+T*C14
       IF(K.EQ.1)GO TO 53
       V15(K,L+1)=V15(K,L)-(T/X)*(D2*(V3(K,L)+A2*V15(K,L))+
      1A2)*(V15(K,L)-V15(K-1,L))+T*C15
       V16(K,L+1)=V16(K,L)-(T/X)*C*(V16(K,L)-V16(K-1,L))+T*
      1C16
       V17(K,L+1)=V17(K,L)-(T/X)*C*(V17(K,L)-V17(K-1,L))+T*
      1C17
       GO TO 18
C*************************************************************
C      G1 AND G2 ARE THE VALUES OF E2 AND B2 AT X=0 AS A
C      FUNCTION OF TIME
   53  G1(L+1)=E02*(1.0D0-DCOS(2.0D0*PI*OMEGA*M1*T))
       G2(L+1)=B02*(1.0D0-DCOS(2.0D0*PI*OMEGA*M1*T))
       V15(1,L+1)=(D1/D2)*(V6(1,L+1)-A1*V4(1,L+1))+A2*V3(1,L+
      11)
```

```
      V16(1,L+1)=(1.0D0/C)*((1.0D0/D4)*G1(L+1)+V1(1,L+1))
      V17(1,L+1)=(1.0D0/D4)*G2(L+1)-C*V2(1,L+1)
   18 IF(M1.EQ.M2)GO TO 17
      GO TO 54
   17 ZETA1(K,2)=D1*(V6(K,2)-A1*V4(K,2))
      ZETA2(K,2)=D2*(V15(K,2)-A2*V3(K,2))
      ZETA3(K,2)=D3*(V7(K,2)-A3*V5(K,2))
      NI(K,2)=DEXP(ZETA1(K,2))
      NE(K,2)=DEXP(ZETA2(K,2))
      NN(K,2)=DEXP(ZETA3(K,2))
      UI(K,2)=D1*(V4(K,2)+A1*V6(K,2))
      UE(K,2)=D2*(V3(K,2)+A2*V15(K,2))
      UN(K,2)=D3*(V5(K,2)+A3*V7(K,2))
      VI(K,2)=V8(K,2)
      VE(K,2)=V10(K,2)
      VN(K,2)=V12(K,2)
      WI(K,2)=V9(K,2)
      WE(K,2)=V11(K,2)
      WN(K,2)=V13(K,2)
      E1(K,2)=V14(K,2)
      E2(K,2)=D4*(C*V16(K,2)-V1(K,2))
      E3(K,2)=D4*(V2(K,2)-C*V17(K,2))
      B2(K,2)=D4*(C*V2(K,2)+V17(K,2))
      B3(K,2)=D4*(C*V1(K,2)+V16(K,2))
      WRITE(6,62) K,M1,NI(K,2),K,M1,NE(K,2),K,M1,NN(K,2)
      WRITE(6,64) K,M1,UI(K,2),K,M1,UE(K,2),K,M1,UN(K,2)
      WRITE(6,66) K,M1,VI(K,2),K,M1,VE(K,2),K,M1,VN(K,2)
      WRITE(6,68) K,M1,WI(K,2),K,M1,WE(K,2),K,M1,WN(K,2)
      WRITE(6,70) K,M1,E1(K,2),K,M1,E2(K,2),K,M1,E3(K,2)
      WRITE(6,72) K,M1,B2(K,2),K,M1,B3(K,2)
   54 CONTINUE
C     IN ORDER TO NOT OVERLOAD THE STORAGE CAPACITY OF THE
C     COMPUTER I AM RENAMING THE VARIABLES AFTER EVERY
C     TIME STEP
      DO 82 K=1,J
      V1(K,1)=V1(K,2)
      V2(K,1)=V2(K,2)
      V3(K,1)=V3(K,2)
      V4(K,1)=V4(K,2)
      V5(K,1)=V5(K,2)
      V6(K,1)=V6(K,2)
      V7(K,1)=V7(K,2)
      V8(K,1)=V8(K,2)
      V9(K,1)=V9(K,2)
      V10(K,1)=V10(K,2)
      V11(K,1)=V11(K,2)
      V12(K,1)=V12(K,2)
      V13(K,1)=V13(K,2)
      V14(K,1)=V14(K,2)
      V15(K,1)=V15(K,2)
      V16(K,1)=V16(K,2)
      V17(K,1)=V17(K,2)
   82 CONTINUE
      M1=M1+1
```

```
      IF(M1.LE.M2)GO TO 22
70    FORMAT('0','E1(',I3,',',I3,') =',D22.16,5X,'E2(',I3,'
     1,',I3,') =',D22.16,5X,'E3(',I3,',',I3,') =',D22.16)
62    FORMAT('0','NI(',I3,',',I3,') =',D22.16,5X,'NE(',I3,',
     1',I3,') =',D22.16,5X,'NN(',I3,',',I3,') =',D22.16)
64    FORMAT('0','UI(',I3,',',I3,') =',D22.16,5X,'UE(',I3,'
     1,',I3,') =',D22.16,5X,'UN(',I3,',',I3,') =',D22.16)
66    FORMAT('0','VI(',I3,',',I3,') =',D22.16,5X,'VE(',I3,'
     1,',I3,') =',D22.16,5X,'VN(',I3,',',I3,') =',D22.16)
68    FORMAT('0','WI(',I3,',',I3,') =',D22.16,5X,'WE(',I3,'
     1,',I3,') =',D22.16,5X,'WN(',I3,',',I3,') =',D22.16)
72    FORMAT('0','B2(',I3,',',I3,') =',D22.16,5X,'B3(',I3,'
     1,',I3,') =',D22.16)
      STOP
      END
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      DOUBLE PRECISION ZETA1(301,2),ZETA2(301,2),ZETA3(301,
     12),V1(301,2),V2(301,2),V3(301,2),V4(301,2),V5(301,2)
     2,V6(301,2),V7(301,2),V8(301,2),V9(301,2),V10(301,2),
     3V11(301,2),V12(301,2),V13(301,2),V14(301,2),V15(301,
     42),V16(301,2),V17(301,2),D1,D2,D3,D4,NI(301,2),NE(301
     5,2),NN(301,2),UI(301,2),UE(301,2),UN(301,2),VI(301,2)
     6VE(301,2),VN(301,2),WI(301,2),WE(301,2),WN(301,2)
      DOUBLE PRECISION E1(301,2),E2(301,2),E3(301,2),B2(301
     1,2),B3(301,2),B1,C,A1,A2,A3,C1,C2,C3,C4,C5,C6,C7,C8,
     2C9,C10,C11,C12,C13,C14,C15,C16,C17,E,MI,ME,MN,X,T,CF12
     3,CF13,CF21,CF23,CF31,CF32,MU,DSQRT,NIO,NEO,NNO,DCOS,
     4E02,B02,OMEGA,G1(2),G2(2),UIO,UEO,UNO
C*********************************************************
C     GIVEN THE INITIAL VALUES OF THE ELECTROMAGNETIC AND
C     FLUID FIELD VARIABLES, PLUS BOUNDARY VALUES FOR SOME
C     VARIABLES, THIS PROGRAM USES AN EXPLICIT FINITE -
C     DIFFERENCE SCHEME TO SOLVE THE LINEARIZED EQUATIONS.
C     THE VARIABLES ARE PRINTED AT A PARTICULAR SPACE STEP
C     VERSUS THE TIME STEPS
C     THE FOLLOWING DEFINES THE SYMBOLS USED.
C     NI,NE,NN=NUMBER DENSITIES OF IONS,ELECTRONS, AND
C     NEUTRALS RESPECTIVELY.
C     ZETA1,ZETA2,ZETA3=RATIO OF FLUCTUATION IN NUMBER
C     DENSITY TO THE EQUILIBRIUM VALUE FOR IONS,ELECTRONS
C     AND NEUTRALS
C     UI,UE,UN=VELOCITY IN X DIRECTION OF IONS,ELECTRONS,
C     AND NEUTRALS
C     VI,VE,VN=VELOCITY IN Y DIRECTION OF IONS,ELECTRONS,AND
C     NEUTRALS
C     WI,WE,WN=VELOCITY IN Z DIRECTION OF IONS,ELECTRONS,
C     AND NEUTRALS
C     E1,E2,E3=ELECTRIC FIELD COMPONENTS IN THE X,Y,Z
C     DIRECTIONS
C     B1,B2,B3=MAGNETIC FIELD COMPONENTS IN THE X,Y,Z
C     DIRECTIONS
C     A1,A2,A3=THERMAL VELOCITIES OF THE IONS,ELECTRONS,AND
C     NEUTRALS ASSUMING EACH FLUID IS ISOTHERMAL
C     MI,ME,MN=MASS OF AN ION,ELECTRON,OR NEUTRAL
C     X,T=THE STEP SIZES IN THE SPATIAL AND TIME COORDINATES
C     C=SPEED OF LIGHT
C     MU=PERMEABILITY OF A VACUUM
C     KO=PERMITTIVITY OF A VACUUM
C     T1,T2,T3=TEMPERATURE OF THE IONS,ELECTRONS,AND NEUTRAL
C     BC=BOLTZMANN'S CONSTANT
C     E=ELECTRONIC CHARGE
C     V1,V2,...,V17=THE ELEMENTS OF THE COLLUM MATRIX V , IN
C     THE TRANSFORMATION U=TV.
C  ****************** STATEMENT OF THE PROBLEM ***********
C     THE PROBLEM IS SUCH THAT AT X=0 WE FORCE E2 AND B2 TO
C     BE CERTAIN FUNCTIONS OF TIME, PLUS, WE ALSO SPECIFY
C     THAT NE=NI AT X=0. THEN IF WE ASSUME THAT UI,UE,AND UN
C     ARE ALL IN THE NEGATIVE X DIRECTION AND THAT UI .GT.
C     A1, UE .LT. A2, AND UN .GT. A3 WE MUST CALCULATE ALL
C     OTHER VARIABLES AT X=0. OF COURSE WE ALSO CALCULATE
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C     ALL OTHER VARIABLES AT LATER TIMES AND DIFFERENT
C     SPATIAL LOCATIONS. INSTEAD OF E2 AND B2 WE COULD HAVE
C     SPECIFIED B2 AND B3 OR E2 AND E3 OR E3 AND B3.
C***********************************************************
      NIO=1.0D21
      NEO=1.0D21
      NNO=3.62D24
      UIO=-1500.0D0
      UEO=-1500.0D0
      UNO=-1500.0D0
      X=2.5D-5
      PI=3.1415927D0
      E=1.60D-19
      B1=1.0D-4
      C=3.0D8
      MI=0.673D-25
      ME=9.11D-31
      MN=MI
      MU=12.56D-7
      BC=1.38D-23
      T1=2000.0D0
      T2=1.0D4
      T3=2000.0D0
      CF12=3.73D5
      CF13=2.95D8
      CF21=2.76D10
      CF23=1.44D11
      CF31=0.815D5
      CF32=5.36D2
      E02=100.0D0
      OMEGA=2.0D10
      B02=0.005D0
      A1=DSQRT(BC*T1/MI)
      A2=DSQRT(BC*T2/ME)
      A3=DSQRT(BC*T3/MN)
      D1=1.0D0/DSQRT(1.0D0+A1**2)
      D2=1.0D0/DSQRT(1.0D0+A2**2)
      D3=1.0D0/DSQRT(1.0D0+A3**2)
      D4=1.0D0/DSQRT(1.0D0+C**2)
C     THIS READS IN THE INITIAL VALUES.THE ORIGIN IS TAKEN
C     TO BE (1,1)
      DO 80 K=1,301
      L=1
      NI(K,L)=0.0D0
      NE(K,L)=0.0D0
      NN(K,L)=0.0D0
      UI(K,L)=0.0D0
      UE(K,L)=0.0D0
      UN(K,L)=0.0D0
      VI(K,L)=0.0D0
      VE(K,L)=0.0D0
      VN(K,L)=0.0D0
      WI(K,L)=0.0D0
      WE(K,L)=0.0D0
```

```
      WN(K,L)=0.0D0
      E1(K,L)=0.0D0
      E2(K,L)=0.0D0
      E3(K,L)=0.0D0
      B2(K,L)=0.0D0
      B3(K,L)=0.0D0
      ZETA1(K,L)=NI(K,L)/NIO
      ZETA2(K,L)=NE(K,L)/NEO
      ZETA3(K,L)=NN(K,L)/NNO
      V1(K,L)=D4*(C*B3(K,L)-E2(K,L))
      V2(K,L)=D4*(E3(K,L)+C*B2(K,L))
      V3(K,L)=D2*(UE(K,L)-A2*ZETA2(K,L))
      V4(K,L)=D1*(UI(K,L)-A1*ZETA1(K,L))
      V5(K,L)=D3*(UN(K,L)-A3*ZETA3(K,L))
      V6(K,L)=D1*(ZETA1(K,L)+A1*UI(K,L))
      V7(K,L)=D3*(ZETA3(K,L)+A3*UN(K,L))
      V8(K,L)=VI(K,L)
      V9(K,L)=WI(K,L)
      V10(K,L)=VE(K,L)
      V11(K,L)=WE(K,L)
      V12(K,L)=VN(K,L)
      V13(K,L)=WN(K,L)
      V14(K,L)=E1(K,L)
      V15(K,L)=D2*(ZETA2(K,L)+A2*UE(K,L))
      V16(K,L)=D4*(C*E2(K,L)+B3(K,L))
      V17(K,L)=D4*(B2(K,L)-C*E3(K,L))
   80 CONTINUE
C     THE INITIAL VALUES HAVE NOW BEEN READ IN AND HAVE
C     UNDER GONE THE TRANSFORMATION V=(T INVERSE)(U)
      M1=1
   22 J=301-M1
      T=5.0D-14
      DO 54 K=1,J
      C1=-D4*C**2*MU*E*(NEO*V10(K,L)-NIO*V8(K,L))
      C2=D4*C**2*MU*E*(NEO*V11(K,L)-NIO*V9(K,L))
      C3=D2*(-(E/ME)*V14(K,L)+CF21*(D1*(V4(K,L)+A1*V6(K,L))-
     1D2*(V3(K,L)+A2*V15(K,L)))+CF23*(D3*(V5(K,L)+A3*V7
     2(K,L))-D2*(V3(K,L)+A2*V15(K,L))))
      C4=D1*((E/MI)*V14(K,L)+CF12*(D2*(V3(K,L)+A2*V15(K,L))-
     1D1*(V4(K,L)+A1*V6(K,L)))+CF13*(D3*(V5(K,L)+A3*V7
     2(K,L))-D1*(V4(K,L)+A1*V6(K,L))))
      C5=D3*(CF31*(D1*(V4(K,L)+A1*V6(K,L))-D3*(V5(K,L)+A3*
     1V7(K,L)))+CF32*(D2*(V3(K,L)+A2*V15(K,L))-D3*(V5(K,L)+
     2A3*V7(K,L))))
      C6=A1*C4
      C7=A3*C5
      C8=(E/MI)*(D4*(C*V16(K,L)-V1(K,L))+B1*V9(K,L)-UIO*D4*
     1(C*V1(K,L)+V16(K,L)))+CF12*(V10(K,L)-V8(K,L))+CF13*
     2(V12(K,L)-V8(K,L))
      C9=(E/MI)*(D4*(V2(K,L)-C*V17(K,L))+UIO*D4*(C*V2(K,L)+
     1V17(K,L))-B1*V8(K,L))+CF12*(V11(K,L)-V9(K,L))+CF13*
     2(V13(K,L)-V9(K,L))
      C10=-(E/ME)*(D4*(C*V16(K,L)-V1(K,L))+B1*V11(K,L)-UEO*
     1D4*(C*V1(K,L)+V16(K,L)))+CF21*(V8(K,L)-V10(K,L))+CF23*
```

```
     2(V12(K,L)-V10(K,L))
      C11=-(E/ME)*(D4*(V2(K,L)-C*V17(K,L))+UEO*D4*(C*
     1V2(K,L)+V17(K,L))-B1*V10(K,L))+CF21*(V9(K,L)-V11(K,LE
     2)+CF23*(V13(K,L)-V11(K,L))
      C12=CF31*(V8(K,L)-V12(K,L))+CF32*(V10(K,L)-V12(K,L))
      C13=CF31*(V9(K,L)-V13(K,L))+CF32*(V11(K,L)-V13(K,L))
      C14=C**2*MU*E*(UEO*NIO*ZETA1(K,L)-UIO*NEO*ZETA2(K,L)+
     1NIO*D2*(V3(K,L)+A2*V15(K,L))-NEO*D1*(V4(K,L)+A1*
     2V6(K,L)))
      C15=A2*C3
      C16=-C*C1
      C17=-C*C2
      V1(K,L+1)=V1(K,L)+(T/X)*C*(V1(K+1,L)-V1(K,L))+T*C1
      V2(K,L+1)=V2(K,L)+(T/X)*C*(V2(K+1,L)-V2(K,L))+T*C2
      V3(K,L+1)=V3(K,L)-(T/X)*(UEO-A2)*(V3(K+1,L)-V3(K,L))+
     1T*C3
      V4(K,L+1)=V4(K,L)-(T/X)*(UIO-A1)*(V4(K+1,L)-V4(K,L))+
     1T*C4
      V5(K,L+1)=V5(K,L)-(T/X)*(UNO-A3)*(V5(K+1,L)-V5(K,L))+
     1T*C5
      V6(K,L+1)=V6(K,L)-(T/X)*(UIO+A1)*(V6(K+1,L)-V6(K,L))+
     1T*C6
      V7(K,L+1)=V7(K,L)-(T/X)*(UNO+A3)*(V7(K+1,L)-V7(K,L))+
     1T*C7
      V8(K,L+1)=V8(K,L)-(T/X)*UIO*(V8(K+1,L)-V8(K,L))+T*C8
      V9(K,L+1)=V9(K,L)-(T/X)*UIO*(V9(K+1,L)-V9(K,L))+T*C9
      V10(K,L+1)=V10(K,L)-(T/X)*UEO*(V10(K+1,L)-V10(K,L))+T*
     1C10
      V11(K,L+1)=V11(K,L)-(T/X)*UEO*(V11(K+1,L)-V11(K,L))+T*
     1C11
      V12(K,L+1)=V12(K,L)-(T/X)*UNO*(V12(K+1,L)-V12(K,L))+T*
     1C12
      V13(K,L+1)=V13(K,L)-(T/X)*UNO*(V13(K+1,L)-V13(K,L))+T*
     1C13
      V14(K,L+1)=V14(K,L)-(T/X)*(UIO+UEO)*(V14(K+1,L)-
     1V14(K,L))+T*C14
      IF(K.EQ.1)GO TO 53
      V15(K,L+1)=V15(K,L)-(T/X)*(UEO+A2)*(V15(K,L)-V15(K-
     11,L))+T*C15
      V16(K,L+1)=V16(K,L)-(T/X)*C*(V16(K,L)-V16(K-1,L))+T*
     1C16
      V17(K,L+1)=V17(K,L)-(T/X)*C*(V17(K,L)-V17(K-1,L))+T*
     1C17
      GO TO 54
C**************************************************************
C     G1 AND G2 ARE THE VALUES OF E2 AND B2 AT X=0 AS A
C     FUNCTION OF TIME
C**************************************************************
  53  G1(L+1)=E02*(1.0D0-DCOS(2.0D0*PI*OMEGA*M1*T))
      G2(L+1)=B02*(1.0D0-DCOS(2.0D0*PI*OMEGA*M1*T))
      V15(1,L+1)=(D1/D2)*(V6(1,L+1)-A1*V4(1,L+1))+A2*V3(1,L+
     11)
      V16(1,L+1)=(1.0D0/C)*((1.0D0/D4)*G1(L+1)+V1(1,L+1))
      V17(1,L+1)=(1.0D0/D4)*G2(L+1)-C*V2(1,L+1)
```

```
54  CONTINUE
    M2=2
    ZETA1(M2,2)=D1*(V6(M2,2)-A1*V4(M2,2))
    ZETA2(M2,2)=D2*(V15(M2,2)-A2*V3(M2,2))
    ZETA3(M2,2)=D3*(V7(M2,2)-A3*V5(M2,2))
    NI(M2,2)=ZETA1(M2,2)*NIO
    NE(M2,2)=ZETA2(M2,2)*NEO
    NN(M2,2)=ZETA3(M2,2)*NNO
    UI(M2,2)=D1*(V4(M2,2)+A1*V6(M2,2))
    UE(M2,2)=D2*(V3(M2,2)+A2*V15(M2,2))
    UN(M2,2)=D3*(V5(M2,2)+A3*V7(M2,2))
    VI(M2,2)=V8(M2,2)
    VE(M2,2)=V10(M2,2)
    VN(M2,2)=V12(M2,2)
    WI(M2,2)=V9(M2,2)
    WE(M2,2)=V11(M2,2)
    WN(M2,2)=V13(M2,2)
    E1(M2,2)=V14(M2,2)
    E2(M2,2)=D4*(C*V16(M2,2)-V1(M2,2))
    E3(M2,2)=D4*(V2(M2,2)-C*V17(M2,2))
    B2(M2,2)=D4*(C*V2(M2,2)+V17(M2,2))
    B3(M2,2)=D4*(C*V1(M2,2)+V16(M2,2))
    NI(M2,2)=NI(M2,2)+NIO
    NE(M2,2)=NE(M2,2)+NEO
    NN(M2,2)=NN(M2,2)+NNO
    UI(M2,2)=UI(M2,2)+UIO
    UE(M2,2)=UE(M2,2)+UEO
    UN(M2,2)=UN(M2,2)+UNO
    WRITE(6,62) M2,M1,NI(M2,2),M2,M1,NE(M2,2),M2,M1,
   1NN(M2,2)
    WRITE(6,64) M2,M1,UI(M2,2),M2,M1,UE(M2,2),M2,M1,
   1UN(M2,2)
    WRITE(6,66) M2,M1,VI(M2,2),M2,M1,VE(M2,2),M2,M1,
   1VN(M2,2)
    WRITE(6,68) M2,M1,WI(M2,2),M2,M1,WE(M2,2),M2,M1,
   1WN(M2,2)
    WRITE(6,70) M2,M1,E1(M2,2),M2,M1,E2(M2,2),M2,M1,
   1E3(M2,2)
    WRITE(6,72) M2,M1,B2(M2,2),M2,M1,B3(M2,2)
C   IN ORDER TO NOT OVERLOAD THE STORAGE CAPACITY OF THE
C   COMPUTER I AM RENAMING THE VARIABLES AFTER EVERY TIME
C   STEP
    DO 82 K=1,J
    V1(K,1)=V1(K,2)
    V2(K,1)=V2(K,2)
    V3(K,1)=V3(K,2)
    V4(K,1)=V4(K,2)
    V5(K,1)=V5(K,2)
    V6(K,1)=V6(K,2)
    V7(K,1)=V7(K,2)
    V8(K,1)=V8(K,2)
    V9(K,1)=V9(K,2)
    V10(K,1)=V10(K,2)
    V11(K,1)=V11(K,2)
```

```
      V12(K,1)=V12(K,2)
      V13(K,1)=V13(K,2)
      V14(K,1)=V14(K,2)
      V15(K,1)=V15(K,2)
      V16(K,1)=V16(K,2)
      V17(K,1)=V17(K,2)
 82   CONTINUE
      M1=M1+1
      IF(M1.LE.270)GO TO 22
 70   FORMAT('0','E1(',I3,',',I3,') =',D22.16,5X,'E2(',I3,'
     1,',I3,') =',D22.16,5X,'E3(',I3,',',I3,') =',D22.16)
 62   FORMAT('0','NI(',I3,',',I3,') =',D22.16,5X,'NE(',I3,',
     1',I3,') =',D22.16,5X,'NN(',I3,',',I3,') =',D22.16)
 64   FORMAT('0','UI(',I3,',',I3,') =',D22.16,5X,'UE(',I3,'
     1,',I3,') =',D22.16,5X,'UN(',I3,',',I3,') =',D22.16)
 66   FORMAT('0','VI(',I3,',',I3,') =',D22.16,5X,'VE(',I3,'
     1,',I3,') =',D22.16,5X,'VN(',I3,',',I3,') =',D22.16)
 68   FORMAT('0','WI(',I3,',',I3,') =',D22.16,5X,'WE(',I3,'
     1,',I3,') =',D22.16,5X,'WN(',I3,',',I3,') =',D22.16)
 72   FORMAT('0','B2(',I3,',',I3,') =',D22.16,5X,'B3(',I3,'
     1,',I3,') =',D22.16)
      STOP
      END
```

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