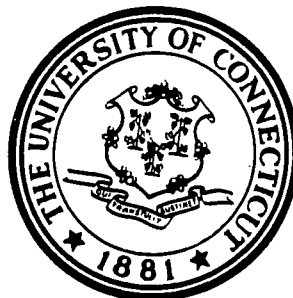


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SCHOOL OF ENGINEERING

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UTILIZING MISMATCHED STATE OBSERVERS

Bernard Goldstein

Technical Report 72-5

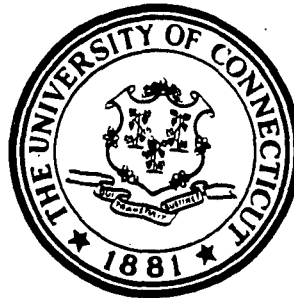
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July, 1972

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TABLE OF CONTENTS

	Page
I. Introduction and Outline of Research	1
1.1 Problem Statement	1
1.2 Background	2
1.3 Outline of the Report	7
II. Background In Observer Theory	9
2.1 Minimal Order State Observers for Time-Invariant Systems	9
2.2 Application of Observers in Optimal Control	12
2.3 Stability Properties of Closed Loop Systems	13
2.4 Minimal Order Observers With Parameter Uncertainties Present	14
III. Min-Max Design of Optimal Observer/Controller	19
3.1 Problem Formulation	19
3.2 Parameter Disturbance Formulation	20
3.3 Cost Functional Formulation	21
3.4 Solution for Disturbance γ	24
3.5 Modification of Cost Functional	26
3.6 Modification of Augmented State Equations	27
3.7 Initial Condition Transformation	28
3.8 Computation of the Gradient of the Cost Functional	31
3.9 Simplifying the Gradient of the Cost Functional	33
3.10 Additional Simplification of Gradient	34
3.11 Recursive Algorithm to Compute P^*	36
IV. Closed Loop Stability Considerations	38
4.1 Introduction	38
4.2 Lyapunov Function Formulation	38
V. Application of the Design	42
5.1 Introduction	42
5.2 Expansion of Eq. 3.64	42
5.3 Discussion of Simulation Results	43
5.3.1 Eigenvalue Placement	43
5.3.2 Uniqueness of The Minimum	45
5.3.3 Effects of Disturbance	45

Table of Contents - continued

VI. Conclusions and Recommendations	Page 63
6.1 Conclusions	63
6.2 Recommendations	65
VII. Bibliography	67

LIST OF TABLES

	Page
5.1 Effect of Q_{11} Variation on Closed Loop Dynamics	47
5.2 Effect of Q_{12} Variation on Closed Loop Dynamics	49
5.3 Effect of R Variation on Closed Loop Dynamics	51
5.4 Effect of Q_y Variation on Closed Loop Dynamics	53
5.5 Effect of Q_z Variation on Closed Loop Dynamics	55
5.6 Effect of Q_2 Variation on Closed Loop Dynamics	57
5.7 Region of Convergence for Various Initial Conditions	59
5.8 Effect of Disturbance on Closed Loop Dynamics	61

LIST OF ILLUSTRATIONS

	Page
3.1 Feedback Gain Matrix and State Observer	19
3.2 Controller and State Observer	22
5.1 Effect of Q_{11} Variation on Closed Loop Dynamics	48
5.2 Effect of Q_{12} Variation on Closed Loop Dynamics	50
5.3 Effect of R Variation on Closed Loop Dynamics	52
5.4 Effect of Q_y Variation on Closed Loop Dynamics	54
5.5 Effect of Q_z Variation on Closed Loop Dynamics	56
5.6 Effect of Q_2 Variation on Closed Loop Dynamics	58
5.7 Region of Convergence for Various Initial Conditions	60
5.8 Effect of Disturbance on Closed Loop Dynamics	62

NOTATION

$\bar{A}$	$n \times n$ plant matrix in non-canonic form
A	$n \times n$ plant matrix in canonic form
$\hat{A}$	$(2n-m) \times (2n-m)$ plant partition matrix
A_0	$n \times n$ nominal plant matrix
δA	$n \times n$ plant uncertainty matrix
$\hat{\delta A}$	$(2n-m) \times (2n-m)$ uncertainty partition matrix
A_L	$(q+m-m) \times 2n$ compatibility matrix
A^o	$(2n-m) \times (2n-m)$ closed loop system matrix
$\bar{B}$	$q \times n$ control matrix in non-canonic form
B	$q \times n$ control matrix in canonic form
$\hat{B}_0$	$q \times (2n-m)$ control partition matrix
B_0	$q \times n$ nominal control matrix
δB	$q \times n$ uncertainty control matrix
$\hat{B}$	$(q+n-m) \times (2n-m)$ control partition matrix
$\hat{\delta B}$	$(q+n-m) \times (2n-m)$ uncertainty partition matrix
$\hat{C}$	$(2n-m) \times 2n$ compatibility matrix
$\hat{D}$	$(2n-m) \times (2n-m)$ plant partition matrix
D	$m \times (n-m)$ observer input gain
E_1	$(2n-m) \times 2n$ compatibility matrix
E_2	$(q+n-m) \times (2n-m)$ compatibility matrix
E_3	$(q+n-m) \times (q+n-m)$ compatibility matrix
E_4	$(2n-m) \times 2n$ compatibility matrix
F	$(n-m) \times (n-m)$ observer dynamics
G	$q \times (n-m)$ observer control gain

Notation - continued

G_0	$qx(n-m)$ observer control gain
$\hat{G}$	$(2n-m) \times (2n-m)$ disturbance gain
$\bar{H}$	$n \times m$ output matrix in non-canonic form
H	$n \times m$ output matrix in canonic form
H_0	$n \times m$ nominal output matrix
H_a	Hamiltonian
I_1	$(q+n-m) \times q$ compatibility matrix
I_2	$(2n-m) \times 2n$ compatibility matrix
J	$n \times q$ feedback gain matrix
K	$m \times (n-m)$ observer free gain matrix
L_1	$m \times n$ observer output gain matrix
L_2	$(n-m) \times n$ observer output gain matrix
$\hat{L}_2$	$(2n-m) \times (2n-m)$ compatibility matrix
M	$(2n-m) \times (2n-m)$ transformation
$N(P)$	transpose of the gradient of the cost functional
P	$2n \times (q+n-m)$ partition matrix of gains J, K
P_1	$n \times n$ positive definite Riccati matrix
Q	$n \times n$ positive definite weighting matrix on state $\underline{x}$
Q_1	$n \times n$ positive definite weighting matrix on state $\underline{q}$
Q_2	$n \times n$ positive semi-definite weighting matrix on error vector $\underline{\epsilon}$
Q_y	$n \times n$ positive semi-definite weighting matrix on $L_1 \underline{y}$
Q_z	$(q+n-m) \times (q+n-m)$ positive semi-definite weighting matrix on $F \underline{z}$
$\hat{Q}_1$	$(2n-m) \times (2n-m)$ partition matrix for weight Q_1
$\hat{Q}_2$	$(2n-m) \times (2n-m)$ partition matrix for weight Q_2
$\hat{Q}_y$	$(2n-m) \times (2n-m)$ partition matrix for weight Q_y
$\hat{Q}_z$	$(q+n-m) \times (q+n-m)$ partition matrix for weight Q_z

Notation - continued

$\hat{Q}$	$(2n-m) \times (2n-m)$ overall weighting matrix
$\hat{Q}_A$	$(2n-m) \times (2n-m)$ partition matrix including constant weights of $\hat{Q}$
R	$q \times q$ positive definite weight on control $\underline{u}$
$\hat{R}$	$(q+n-m) \times (q+n-m)$ partition matrix for weight R
S	$(2n-m) \times (2n-m)$ positive definite symmetric Riccati matrix
T	$n \times (n-m)$ linear transformation between $\underline{z}$, $\underline{x}$
V_0	$(2n-m) \times (2n-m)$ positive semi-definite transformation initial condition covariance matrix on $\underline{v}$
W_0	$n \times n$ initial condition covariance matrix on $\underline{q}(0)$
X_0	$(2n-m) \times (2n-m)$ initial condition covariance matrix on $\underline{\omega}(0)$
Y	$(2n-m) \times (2n-m)$ positive definite weight on disturbance $\underline{y}$
δP	perturbation in gain P
δS	perturbation in gain S
$\Phi(\tau)$	$(2n-m) \times (2n-m)$ transition matrix
χ	$(2n-m) \times (2n-m)$ positive definite Riccati gain
$\underline{x}$	$1 \times n$ dimensional state vector in non-canonic space
$\hat{\underline{x}}$	$1 \times n$ dimensional estimated state vector in non-canonic space
$\underline{q}$	$1 \times n$ dimensional state vector in canonic space
$\hat{\underline{q}}$	$1 \times n$ dimensional estimated state vector in canonic space
$\underline{\epsilon}$	$1 \times (n-m)$ dimensional error vector in canonic space
$\underline{u}$	$1 \times q$ dimensional control vector in canonic space
$\underline{\omega}$	$1 \times (2n-m)$ dimensional state vector in canonic space
$\underline{v}(0)$	$1 \times (2n-m)$ dimensional transformed initial condition vector
$\underline{y}$	$1 \times m$ dimensional measurement vector
$\underline{z}$	$1 \times (n-m)$ dimensional observer output vector
$\underline{y}$	$1 \times (2n-m)$ dimensional disturbance vector

I. INTRODUCTION AND OUTLINE OF RESEARCH

1.1 Problem Statement

Control of linear dynamical systems has been the subject of wide investigation as evidenced by the numerous publications in this area [1,2,7,8,9,24]. The optimum control law is well known to be a linear function of the complete state vector, where the control gains are determined from the unique positive definite solution of a matrix Riccati equation. In actual practice, however, the entire state vector is rarely available for measurement. A control utilizing the optimal feedback gains may still be implemented, however, if an estimate of the total state vector is obtained through implementation of a state observer. In addition, feedback control techniques have been developed for cases in which the control is constrained to be a linear function of the measurable outputs [1,2,7,8].

The problem investigated in this report may be described as follows. Given the linear deterministic, continuous dynamical system described by the equation:

$$\dot{\underline{x}} = \bar{A}\underline{x} + \bar{B}\underline{u} \quad (1.1)$$

$$\underline{y} = \bar{H}\underline{x} \quad (1.2)$$

where

$\underline{x}$ is an n dimensional state vector

$\underline{y}$ is an m dimensional measurement vector

$\underline{u}$ is an r dimensional control vector

$\bar{A}$ is an $n \times n$ dimensional plant matrix

$\bar{B}$ is an $r \times n$ dimension input control matrix

$\bar{H}$ is an $n \times m$ dimension measurement matrix

It is desired to construct a full state feedback control law to regulate the dynamical system described by Eqs. 1.1, 1.2. It is assumed that exact knowledge of matrices $\bar{A}$ and $\bar{B}$ is not available, although the parameters are known to lie within closed bounds. In addition, the number of output measurements is, in general, less than the dimension of the state of the system.

1.2 Background

The problem of limited state feedback control may be classified as either fixed gain feedback or dynamical feedback control. Non-dynamical feedback control, while simpler to implement, suffers limitations since many inherently unstable systems may not easily be stabilized. The more general class of controllers are those which have dynamics. In these controllers, the compensator output may be directly utilized as the control, or the output of the dynamical controller may be combined in some appropriate manner with the measurable system output to form the required control signal.

Early investigation in the area of deterministic control of systems with limited accessible states may be attributed to Luenberger [15,16]. Although his primary emphasis was more in terms of state estimation, he did formulate a dynamical system whose structure plays an integral role in the design undertaken in this report. In Luenberger's work, it was shown that given a system described by Eqs. 1.1, 1.2 and complete knowledge of the system parameters, the state vector $\underline{x}$ may be constructed exactly with a stable linear system of order $n-m$ which was termed a minimal order state observer. The basic motivation in the development of this minimal order state observer is the notion that since there are " m " independent measurements available, it should be possible to

construct the entire n dimensional state vector of the system by generating only $n-m$ additional quantities and combining them appropriately with the " m " already existing outputs. Luenberger also demonstrated that a stable linear observer does not effect the stability of the closed loop system. However, the question of design in the presence of parameter uncertainty was left unanswered.

Dellon and Sarachik [3] studied the deterministic feedback optimal control problem with the standard quadratic cost function, from the standpoint of using a minimal order state observer in the feedback loop to reconstruct the state vector $\underline{x}$. Their analysis was generalized to time-varying systems, but did not consider the effect of parameter uncertainty in the design. They demonstrate that the resulting feedback control system is asymptotically stable even when the plant is originally unstable. Dellon and Sarachik's main contribution is the formulation of a transformation which provides an alternate, more tractable design approach to the structure of state observers and/or dynamic compensators. This transformation has been used extensively in additional research relating to observer design [13,14,25].

Neumann[4] investigated the effects upon the cost functional when an observer is used in a quadratic optimal control design for systems with limited accessible states. Similar to the previous authors, the systems considered are deterministic and do not account for parameter uncertainties. Neumann derives an expression for the increase in cost as a function of the initial condition error which exists between state observer output and some linear transformation of the true system state. He concludes that if no information is available about the initial conditions of some of the states of the system then perhaps it is more

advantageous to use a sub-optimal control [1] with feedback of those states which are measurable than a system and observer using the optimal feedback law.

Sarma and Jayaraj [5] also studied the problem of the effect upon the cost functional when a compatible dynamical observer is employed to construct the estimates of the inaccessible states. They extended the analysis performed in [4] to finite time optimal control systems. An expression is determined for the increase in cost as a function of the initial condition error. The authors demonstrate that under the condition of non-zero initial condition error, the smallest increase in cost in closed loop feedback control is achieved when the feedback control gain is obtained from the solution to the standard quadratic optimal control problem, assuming full state feedback.

Neumann [6] further investigated the utilization of dynamical controllers in limited state feedback control by considering the Luenberger design as a particular form of dynamical compensator. His analysis focuses on linear, time-invariant systems with complete parameter knowledge. The author has preserved the original Luenberger [15] form of the observer equations, and has not taken advantage of the observer canonical transformation derived in [3]. Neumann demonstrates in the analysis that in the computation of observer (controller) dynamics and feedback control gains, the feedback gain will in general differ from the optimal gain computed from the full state optimal control problem. The derived equations also show that the separation principle in linear optimal control design does not apply in this problem formulation, as compared to the case in which an n th order controller (observer), or Kalman filter, is used. Thus, although it is possible to obtain as

small an estimation error as desired by taking the magnitude of the eigenvalues of the dynamic controller (observer) large enough negative, a vanishing small increase in cost will not in general result for this specifically controlled regulator, unless the order of the dynamic compensator is equal to that of the controlled system. The approach does exhibit one weakness in that there generally results more unknowns than algebraic equations available for solution. Also, as in the case of previous authors, the question of design in the presence of system uncertainties was left unanswered.

This survey has thus far focused primarily on the derivation of dynamical controllers, particularly of the structure of the Luenberger observer, for limited state feedback control systems. Recent studies by Levine and Athans [7] have directed attention to the development of feedback control algorithms for time-varying multi-variable systems. Utilizing the minimum principle and assuming knowledge of the initial state statistics, an algorithm is derived which recursively estimates improved values of the fixed feedback gain parameters. A proof is also presented which guarantees monotonic convergence of the overall cost functional. It is stressed that since the solution is in general non-unique, a decrease in cost does not necessarily imply convergence to the optimum set of parameters.

An alternate approach to the work described in [7] was undertaken by Levine and Athans [8], constraining their design to linear, time-invariant systems. The design technique once again results in a set of matrix equations (in this case, algebraic) which must be solved recursively to determine the required feedback parameters. It is shown that if the output matrix $\bar{H}$ is non-singular, implying that all

states are measurable, then the control algorithm derived by this method is identical to the Kalman optimal controller. In both [7,8] the analysis assumes complete knowledge of all parameters.

Kosut [2] considered a broader class of fixed feedback controllers, limiting his analysis to linear, time-invariant deterministic systems. He classifies the feedback control laws as subject to either single control structure constraints, where the control vector $\underline{u}$ is constrained to be a linear combination of one set of measurable states, or multiple control structure constraints, where each element of the control vector, u_j , is a time-invariant, linear combination of different sets of measurable states. It may be noted that the approach taken by [2] yields results similar to [7,8] but with much less computational work. As in all the analysis previously discussed, [2] ignores the realistic problem of design in the presence of parameter uncertainties.

Athans and Johnson [9,10] extended the work of [8] assuming that the feedback control signal is determined as the output of a reduced order dynamical compensator. In their analysis, the gains of the dynamic compensator are designated as free design parameters. Following an optimization procedure analogous to [8], they derive a recursive algorithm for solution of the compensator design parameters. While the authors claim that the controller determined by this method may be of dynamic order less than the minimal order state observer, it is not totally clear how the designer selects the order of this feedback controller.

To review the previous paragraphs, the fundamental problem of control of linear dynamical systems when subject to limited accessible states has been introduced. The approaches taken by several researchers

to implement dynamical controllers, particularly of the structure of a Luenberger observer to control linear plants, has been discussed. It was also shown that [6], in particular, structured the problem similar to the research proposed in this investigation. However, as in all studies discussed in this background survey, the important question of observer/controller design in the presence of system parameter ignorance was largely left unanswered. In a complementary vein the investigation undertaken in this report represents a natural extension of the work discussed in [8,9]. The formulation of the control problem as a mini-max design permits a model of the effects of parameter uncertainty to play an integral role in the resulting design algorithm. Thus, it is believed that the approach taken in this report provides additional insight into the design of state observers (dynamic compensators) and represents a contribution to the theory of limited state feedback control.

1-3 Outline of the Report

Chapter 2 is a presentation of the development of the equations which describe the structure of an observer. The development is constrained to time-invariant systems, but includes the effects of uncertainties.

Chapter 3 presents the important contributions of this report. The optimal control problem is formulated, accounting for the uncertainty in the system design parameters. Because a constrained optimal control is prespecified, elimination of the initial condition state vector is required. The analysis clearly presents a method for eliminating this dependence at the expense of additional terms in the cost functional. The development is explained at each step such that the

final equations may be obtained without loss of continuity. Because some of the equations in the development are both long and cumbersome, the reader is referred to the appendices in the thesis where the particular equation is written in completeness.

The results of Chapter 3 are utilized in Chapter 4 to develop expressions for bounds on closed loop stability. These bounds are expressed as norms on the allowable variations in the A and B matrices

The theoretical developments of the previous chapters are applied to an example in Chapter 5, and a discussion of the pertinent developments of this research project and its implications are provided in Chapter 6.

II. BACKGROUND IN OBSERVER THEORY

The primary objective of this chapter is to review the important aspects in the development of minimal order state observers. This development is essential to its utilization in the optimal control formulation of Chapter 3.

The discussion is presented in two parts. The first section presents the observer development assuming complete knowledge of the system parameters. Also discussed is the implementation of a state observer in the standard optimal regulator control problem. The second part develops the overall system-observer equations when parameter uncertainties are present. We are also made aware of their effects on stability of the closed loop dynamical system.

2.1 Minimal Order State Observers For Time-Invariant Systems

The system under investigation is an n -dimensional, completely observable structure described by:

$$\dot{\underline{x}} = \bar{A}\underline{x} + \bar{B}\underline{u} \quad (2.1)$$

with m independent outputs

$$\underline{y} = \bar{H}\underline{x} \quad (2.2)$$

This system may be observed by an $n-m$ dimensional system

$$\dot{\underline{z}} = F\underline{z} + D\underline{y} + G\underline{u} \quad (2.3)$$

such that the output $\underline{z}$ is related to true $\underline{x}$ by:

$$\underline{z} = T\underline{x} + \underline{\varepsilon} \quad (2.4)$$

It may easily be shown that the differential equation in terms of the error $\underline{\varepsilon}$ is expressed as:

$$\dot{\underline{\epsilon}} = F\underline{\epsilon} \quad (2.5)$$

Since $\underline{\epsilon} = \underline{z} - T\underline{x}$, if the initial condition on the observer output $\underline{z}(0)$ is exactly equal to $T\underline{x}(0)$, then for all $t \geq 0$ the relationship $\underline{z} = T\underline{x}$ is valid, and it is possible to reconstruct the true state $\underline{x}$ from the output signals $\underline{y}$, $\underline{z}$. Letting:

$$\begin{pmatrix} \underline{y} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \bar{H} \\ T \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ \underline{\epsilon} \end{pmatrix}$$

we see that if

$$\begin{pmatrix} \bar{H} \\ T \end{pmatrix}^{-1}$$

exists, then $\underline{x}$ may be expressed as:

$$\underline{x} = L_1 \underline{y} + L_2 \underline{z} - L_2 \underline{\epsilon} \quad (2.6)$$

where

$$\begin{pmatrix} \bar{H} \\ T \end{pmatrix}^{-1} = (L_1, L_2) \quad (2.7)$$

Eq. 2.7 implies that the rows of T are linearly independent of the rows of $\bar{H}$. If we let

$$\hat{\underline{x}} = L_1 \underline{y} + L_2 \underline{z} \quad (2.8)$$

then Eq. 2.6 becomes

$$\underline{x} = \hat{\underline{x}} - L_2 \underline{\epsilon} \quad (2.9)$$

Thus we see that if $\underline{\epsilon}(0) = 0$ then $\hat{\underline{x}} = \underline{x}$ for $t \geq 0$. Otherwise, assuming F has all stable roots $\hat{\underline{x}}$ approaches $\underline{x}$ asymptotically.

In order for this dynamic system to be an estimator for the state $\underline{x}$, it is shown by Luenberger [15] that the following conditions are both necessary and sufficient:

$$T\bar{A} - FT = D\bar{H} \quad (2.10)$$

$$G = T\bar{B} \quad (2.11)$$

Furthermore, since it is required that Eq. 2.7 be valid, it may be determined from Eq. 2.10 that:

$$T\bar{A} = (D, F) \begin{pmatrix} \bar{H} \\ T \end{pmatrix}$$

or

$$D = T\bar{A}L_1 \quad (2.12)$$

$$F = T\bar{A}L_2 \quad (2.13)$$

$$I = L_1\bar{H} + L_2T \quad (2.13a)$$

Thus, the structure of the state observer is determined in terms of the transformation T and the parameters of the system to be observed.

Additional insight into the structure of observers is credited to Dellon and Sarachik [3], who introduced a transformation on state $\underline{x}$ which placed the system into an observer canonical form. If we recall that the measurement matrix was assumed to be of rank m , then the system described by Eq. 2.1 may always be transformed such that $\bar{H}$ is partitioned as:

$$\bar{H} = (\bar{H}_1 : \bar{H}_2)$$

where $\bar{H}_1$ is of rank m and $\bar{H}_2$ are the remaining columns. Introducing the linear transformation:

$$\underline{x} = \begin{bmatrix} \bar{H}_1^{-1} & -\bar{H}_1^{-1} \bar{H}_2 \\ 0 & I \end{bmatrix} \underline{q} \quad (2.14)$$

it may easily be shown that the new output matrix is:

$$H = (I \quad 0) \quad (2.15)$$

The importance of this transformation is that it allows one to insure that Eq. 2.7 exists if we assign:

$$T = \begin{pmatrix} \overset{m}{K} & \overset{n-m}{I} \end{pmatrix} \overset{n-m}{n-m} \quad (2.16)$$

By partitioning Eq. 2.13 appropriately, one can show that the dynamics of the state observer may be expressed as:

$$F = A_{22} + KA_{12}$$

The dual of Wonham's [26] result on controllability may then be invoked in [3] to state that if the pair (A_{22}, A_{12}) is completely observable, then the observers gain K may be adjusted to arbitrarily position the "n-m" eigenvalues of F .

2.2 Application of Observers in Optimal Control

A fundamental use of observers is in the design of feedback controllers for the linear regulator problem, where some of the states are inaccessible to measurement. It therefore becomes necessary to construct an observer to supply estimates of the unavailable states.

The standard optimal regulator control problem is described as follows. Determine a control $\underline{u}$ which minimizes

$$\min_{\underline{u}} I = \frac{1}{2} \int_0^{\infty} (\underline{q}^T Q \underline{q} + \underline{u}^T R \underline{u}) dt \quad (2.18)$$

subject to Eq. 2.1. Q is assumed positive semi-definite and R positive definite. The solution for control $\underline{u}$ is well known to be:

$$\underline{u} = -R^{-1} B^T P_1 \underline{q} \quad (2.19)$$

where P_1 is the unique positive definite solution to the matrix Riccati equation:

$$A^T P_1 + P_1 A - P_1 B R^{-1} B^T P_1 + Q = 0 \quad (2.20)$$

Since $\underline{q}$, by assumption, is not totally accessible to measurement, the optimal control cannot be implemented. If a minimal order state observer is constructed, then one may obtain an approximation to the optimal control as:

$$\hat{\underline{u}} = -R^{-1} B^T P_1 \hat{\underline{q}} \quad (2.21)$$

where

$$\hat{\underline{q}} = \underline{q} + L_2 \underline{\epsilon} \quad (2.22)$$

Then substituting into Eq. 2.1, we see that the approximation to the optimal control is expressed as:

$$\hat{\underline{u}} = -R^{-1} B^T P_1 \underline{q} - R^{-1} B^T P_1 L_2 e^{Ft} \underline{\epsilon}(0) \quad (2.23)$$

Thus, if the initial condition error is non-zero, an increase in cost generally results [14] over that obtained when only the optimal control is implemented.

2.3 Stability Properties of Closed Loop System

To determine the effect of an observer on the stability properties of a closed loop system, it is desired to control the system Eq. 2.1 by the control law of Eq. 2.19. Since the true state $\underline{q}$ is not available, we can express the dynamical Eq. 2.1 with control Eq. 2.19 as:

$$\begin{aligned} \dot{\underline{q}} &= A\underline{q} + B(-R^{-1} B^T P_1) \hat{\underline{q}} \\ &= A - (BR^{-1} B^T P_1)(L_1 \underline{y} + L_2 \underline{z}) \\ &= (A - BR^{-1} B^T P_1 L_1 H) \underline{q} - BR^{-1} B^T P_1 L_2 \underline{z} \end{aligned} \quad (2.24)$$

The observer equation may likewise be expressed as:

$$\begin{aligned} \dot{\underline{z}} &= F\underline{z} + D H \underline{q} + G(-R^{-1} B^T P_1)(L_1 \underline{y} + L_2 \underline{z}) \\ &= F\underline{z} + (D H - T B R^{-1} B^T P_1 L_1 H) \underline{q} - T B R^{-1} B^T P_1 L_2 \underline{z} \\ &= T(A - BR^{-1} B^T P_1) L_2 \underline{z} + T(A - BR^{-1} B^T P_1) L_1 H \underline{q} \end{aligned} \quad (2.25)$$

In matrix notation these equations may be expressed as:

$$\begin{bmatrix} \dot{\underline{q}} \\ \dot{\underline{z}} \end{bmatrix} = \begin{bmatrix} A - BR^{-1} B^T P_1 L_1 H & -BR^{-1} B^T P_1 L_2 \\ T(A - BR^{-1} B^T P_1) L_1 H & T(A - BR^{-1} B^T P_1) L_2 \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{z} \end{bmatrix} \quad (2.26)$$

The stability properties of this closed loop system become apparent when

the following coordinate transformation is applied:

$$\begin{bmatrix} \underline{q} \\ \underline{\epsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{T} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{z} \end{bmatrix} \quad (2.27)$$

This transformation, when applied to Eq. 2.26, yields the following representation for the closed loop dynamics:

$$\begin{bmatrix} \dot{\underline{q}} \\ \dot{\underline{\epsilon}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_1 & \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_1\mathbf{L}_2 \\ \mathbf{0} & \mathbf{TAL}_2 \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{\epsilon} \end{bmatrix} \quad (2.28)$$

It is readily apparent that the stability properties of this closed loop dynamical system are governed by the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_1$ and $\mathbf{TAL}_2$. If the system to be controlled is assumed controllable, then there exists a matrix $\mathbf{P}_1$ such that the closed loop system $\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_1$ is stable. In addition, one would always design the observer $\mathbf{F} = \mathbf{TAL}_2$ to contain stable roots.

In summary, then, the closed loop stability properties are unaltered when an observer is implemented. It merely adds dynamics of its own to the closed loop.

2.4 Minimal Order Observers With Parameter Uncertainty Present

When the system described by Eq. 2.1 contains uncertainty in the $\{\mathbf{A}, \mathbf{B}\}$ matrices, the system equations may be expressed as:

$$\begin{aligned} \dot{\underline{q}} &= (\mathbf{A}_0 + \delta\mathbf{A})\underline{q} + (\mathbf{B}_0 + \delta\mathbf{B})\underline{u} \\ \underline{y} &= \mathbf{H}_0\underline{q} \end{aligned} \quad (2.29)$$

where $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{H}_0$ are nominal values and $\delta\mathbf{A}$, $\delta\mathbf{B}$ represent the uncertainty. Our objective is to investigate the effects of this inherent error upon the state estimate $\hat{\underline{q}}$. Basing our design on the nominal system parameters and the observer design parameter $\mathbf{K}$, we may express

$$\hat{\underline{q}} = L_1 \underline{y} + L_2 \underline{z}$$

or

$$\dot{\hat{\underline{q}}} = L_1 \dot{\underline{y}} + L_2 \dot{\underline{z}}$$

Substituting for $\dot{\underline{y}}$ and $\dot{\underline{z}}$ from Eqs. 2.2, 2.3 and utilizing the relationships in Eqs. 2.10, 2.13a, we obtain an expression for the differential equation representing $\hat{\underline{q}}$ as:

$$\dot{\hat{\underline{q}}} = A_0 \hat{\underline{q}} + B_0 \underline{u} + L_2 F \underline{\epsilon} + L_1 H_0 (A \underline{q} + B \underline{u}) \quad (2.31)$$

Substituting for the true state $\underline{q}$ from eq. 2.22, Eq. 2.31 becomes:

$$\dot{\hat{\underline{q}}} = A_0 \hat{\underline{q}} + B_0 \underline{u} + (L_2 F - A_0 L_2) \underline{\epsilon} + L_1 H_0 (\delta A \underline{q} + \delta B \underline{u}) \quad (2.32)$$

As our interest lies in determining the existence of any steady state error, define:

$$\delta \underline{q} = \underline{q} - \hat{\underline{q}} \quad (2.33)$$

Then a differential equation in terms of this error may be expressed as:

$$\delta \dot{\underline{q}} = A \underline{q} + B_0 \underline{u} - A_0 \hat{\underline{q}} - B_0 \underline{u} - (L_2 F - A_0 L_2) \underline{\epsilon} - L_1 H_0 (\delta A \underline{q} + \delta B \underline{u}) \quad (2.34)$$

and substituting $\hat{\underline{q}} = \underline{q} - \delta \underline{q}$ yields:

$$\delta \dot{\underline{q}} = A_0 \delta \underline{q} - (L_2 F - A_0 L_2) \underline{\epsilon} + L_2 T (\delta A \underline{q} + \delta B \underline{u}) \quad (2.35)$$

or by making the substitution:

$$L_2 F - A_0 L_2 = -L_1 H_0 A_0 L_2$$

we obtain as the differential equation for state error as

$$\delta \dot{\underline{q}} = A_0 \delta \underline{q} + L_1 H_0 A_0 L_2 \underline{\epsilon} + L_2 T (\delta A \underline{q} + \delta B \underline{u}) \quad (2.36)$$

In a similar fashion, the equation which represents the inherent observer initial condition error may be shown to be expressed as:

$$\dot{\underline{\epsilon}} = F \underline{\epsilon} - T (\delta A \underline{q} + \delta B \underline{u}) \quad (2.37)$$

Then the overall dynamical system equations may be expressed as:

$$\begin{bmatrix} \delta \dot{\underline{q}} \\ \underline{\dot{\epsilon}} \end{bmatrix} = \begin{bmatrix} \underline{A}_0 & \underline{L}_1 \underline{H}_0 \underline{A}_0 \underline{L}_2 \\ 0 & \underline{T} \underline{A}_0 \underline{L}_2 \end{bmatrix} \begin{bmatrix} \delta \underline{q} \\ \underline{\epsilon} \end{bmatrix} + \begin{bmatrix} \underline{L}_2 \underline{T} \delta \underline{A} & \underline{L}_2 \underline{T} \delta \underline{B} \\ -\underline{T} \delta \underline{A} & -\underline{T} \delta \underline{B} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{u} \end{bmatrix} \quad (2.38)$$

It is apparent that state $\underline{q}$ and control $\underline{u}$ act as forcing functions as long as uncertainty in matrices $\underline{A}$ and/or $\underline{B}$ exist. It is of interest to know what effect this error has on the steady state operation of the overall system, and in particular, its effect on the system performance when a closed loop control is required.

Assume that in the open loop the input $\underline{u}$ is designated as a step of gain u . Then as $t \rightarrow \infty$, both $\underline{\dot{\epsilon}}$ and $\delta \dot{\underline{q}}$ approach zero, and thus we obtain from eqs. 2.37, 2.36:

$$\underline{F} \underline{\epsilon} - \underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) = 0 \quad (2.39)$$

$$\underline{A}_0 \delta \underline{q} + \underline{L}_1 \underline{H}_0 \underline{A}_0 \underline{L}_2 \underline{\epsilon} + \underline{L}_2 \underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) = 0 \quad (2.40)$$

Since $\underline{A}_0$ is assumed a stable matrix and the observer eigenvalues are by design stable, $\underline{A}_0^{-1}$ and $\underline{F}^{-1}$ exist, Hence:

$$\underline{\epsilon}_{ss} = \underline{F}^{-1} \underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) \quad (2.41)$$

$$\delta \underline{q}_{ss} = -\underline{A}_0^{-1} \underline{L}_1 \underline{H}_0 \underline{A}_0 \underline{L}_2 \underline{\epsilon}_{ss} - \underline{A}_0^{-1} \underline{L}_2 \underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) \quad (2.42)$$

Substituting Eq. 2.41 into Eq. 2.42 yields:

$$\delta \underline{q}_{ss} = -\underline{A}_0^{-1} (\underline{L}_1 \underline{H}_0 \underline{A}_0 \underline{L}_2 \underline{F}^{-1} + \underline{L}_2) \underline{T} (\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u})$$

and since from Eq. 2.1

$$\underline{q}_{ss} = -\underline{A}_0^{-1} \underline{B} \underline{u}$$

we obtain

$$\delta \underline{q}_{ss} = -\underline{A}_0^{-1} (\underline{L}_1 \underline{H}_0 \underline{A}_0 \underline{L}_2 \underline{F}^{-1} + \underline{L}_2) \underline{T} (\delta \underline{B} - \delta \underline{A} \cdot \underline{A}_0^{-1} \underline{B}) \underline{u} \quad (2.43)$$

It is apparent that so long as uncertainty exists in either matrices $\underline{A}$ and/or $\underline{B}$, the effect on design is transmitted as a steady state error in the estimate of true state $\underline{q}$.

The effect on the open loop performance when uncertainty exists in the system is not extremely difficult to analyze. Ash [13] designed a sub-optimal observer which is capable of reducing the effects of errors such as described by Eq. 2.43. However, serious problems may occur when it is desired to regulate the system with a feedback control based on estimate $\hat{q}$. Under these circumstances, disturbances resulting from parameter uncertainties are no longer guaranteed to be bounded as in the previously discussed open loop situation.

The closed loop dynamics may be developed in a manner similar to Section 2.3. Letting the optimal control be determined from Eq. 2.19, the expression for the differential equation in state q becomes:

$$\dot{q} = (A_0 - B_0 R^{-1} B_0^T P_1 L_1 H_0) q - B_0 R^{-1} B_0^T P_1 L_2 z \quad (2.44)$$

$$+ (\delta A - \delta B R^{-1} B_0^T P_1 L_1 H_0) q - \delta B R^{-1} B_0^T P_1 L_2 z$$

Since the observer design is based on the nominal parameters, the differential equation for output z remains as Eq. 2.25. The closed loop dynamics now become:

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_0 - B_0 R^{-1} B_0^T P_1 L_1 H_0 + \delta A - \delta B R^{-1} B_0^T P_1 L_1 H_0 & -(B_0 + \delta B) R^{-1} B_0^T P_1 L_2 \\ T(A_0 - B_0 R^{-1} B_0^T P_1) L_1 H_0 & T(A_0 - B_0 R^{-1} B_0^T P_1) L_2 \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix} \quad (2.45)$$

Utilizing the transformation Eq. 2.27, we obtain:

$$\begin{bmatrix} \dot{q} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} A_0 - B_0 R^{-1} B_0^T P_1 + (\delta A - \delta B R^{-1} B_0^T P_1) & -(B_0 + \delta B) R^{-1} B_0^T P_1 L_2 \\ -T(\delta A - \delta B R^{-1} B_0^T P_1) & T A_0 L_2 + T \delta B R^{-1} B_0^T P_1 L_2 \end{bmatrix} \begin{bmatrix} q \\ \epsilon \end{bmatrix} \quad (2.46)$$

From Eq. 2.46, it is difficult to make any concrete statement regarding the stability of the closed loop system. This is due to the uncertainties existing in each of the four partitions of Eq. 2.46. It should be pointed out that [13] discussed the stability of the closed loop system under conditions of parameter ignorance. However, the uncertainty terms were incorrectly included in a bounded disturbance vector and did not actually appear in the stability analysis.

The problem of stability and closed loop observer performance under conditions of parameter uncertainty have been the prime source of motivation for this investigation. Unlike previous studies, the effects due to parameter uncertainty become an integral part of the design procedure. This study aims at control of linear, uncertain systems using the theory of Luenberger type observers in the development of a dynamical controller.

III. MIN-MAX DESIGN OF OPTIMAL OBSERVER/CONTROLLER

3.1 Problem Formulation

The primary objective of this formulation is to develop a minimal order dynamical controller, specifically of the Luenberger structure, to regulate an n -dimensional completely observable linear system. The regulation control is defined to be some linear function of the observer output $\hat{q}^1$. As will be shown in the analysis which follows, a minimax control design [28] will be utilized to yield a set of algebraic matrix equations which, when solved recursively, provide both a set of time-invariant feedback gains J and the free observer design parameters K . A diagram of the system under investigation is shown in Figure 3.1.

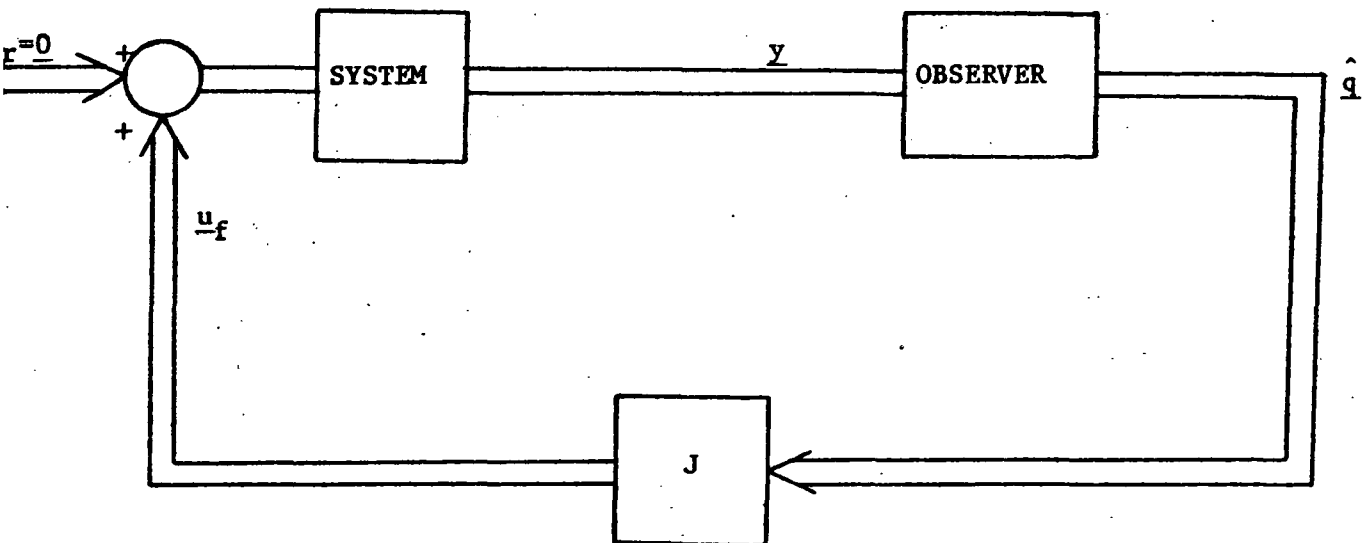


Figure 3.1 Feedback Gain Matrix & State Observer

¹ From this point on in the analysis, we assume that the system is in the observer canonic form.

3.2 Parameter Disturbance Formulation

Assuming the existence of parameter ignorance in the A and B matrices, the system equations have been shown in Eqs. 2.29, 2.37 to be:

$$\dot{\underline{q}} = \underline{A}_0 \underline{q} + \underline{B}_0 \underline{u} + (\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) \quad (3.1)$$

$$\dot{\underline{\epsilon}} = \underline{F} \underline{\epsilon} - \underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) \quad (3.2)$$

It has been shown in the analysis of Chapter 2 that little may be achieved by treating the parameter uncertainty explicitly. As a consequence, the design procedure is modified such that the uncertainty terms are treated as disturbance vectors $\underline{y}$. Accordingly, define:

$$\underline{G}_1 \underline{y}_1 = \delta \underline{A} \underline{q} + \delta \underline{B} \underline{u} \quad (3.3)$$

$$\underline{G}_2 \underline{y}_2 = -\underline{T}(\delta \underline{A} \underline{q} + \delta \underline{B} \underline{u}) \quad (3.4)$$

such that the state equations become

$$\dot{\underline{q}} = \underline{A}_0 \underline{q} + \underline{B}_0 \underline{u} + \underline{G}_1 \underline{y}_1 \quad (3.5)$$

$$\dot{\underline{\epsilon}} = \underline{F} \underline{\epsilon} + \underline{G}_2 \underline{y}_2 \quad (3.6)$$

Since $\underline{u} = \underline{u}_r + \underline{u}_f$, we may set $\underline{u}_r = 0$ as in the regulator problem and obtain:

$$\begin{bmatrix} \dot{\underline{q}} \\ \dot{\underline{\epsilon}} \end{bmatrix} = \begin{bmatrix} \underline{A}_0 & 0 \\ 0 & \underline{F} \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{\epsilon} \end{bmatrix} + \begin{bmatrix} \underline{B}_0 \\ 0 \end{bmatrix} \underline{u}_f + \begin{bmatrix} \underline{G}_1 & 0 \\ 0 & \underline{G}_2 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \quad (3.7)$$

Define $\underline{\omega}^T$ as the augmented vector $(\underline{q}^T, \underline{\epsilon}^T)$. Then

$$\dot{\underline{\omega}} = \hat{\underline{D}} \underline{\omega} + \hat{\underline{B}}_0 \underline{u}_f + \hat{\underline{G}} \underline{y} \quad (3.8)$$

where $\hat{\underline{D}}$, $\hat{\underline{B}}_0$ and $\hat{\underline{G}}$ are approximately defined partition matrices.

Since Eq. 3.8 includes the disturbance component $\hat{\underline{G}} \underline{y}$, it is necessary to utilize this term in the control design proposed in this investigation.

A convenient means of accounting for the disturbance y is to formulate the control problem as worst case or minimax design [28] as to be shown in the following section.

3.3 Cost Functional Formulation

As in any optimal control problem, it is necessary to formulate a meaningful cost functional. The detailed system diagram shown in Figure 3.2 provides insight into the selection of the cost functional to be utilized.

Because the regulation control problem is considered, it is necessary to place constraints on both the state q and the control u_f (herein designated as u). In addition, it is reasonable to place some constraints on the observer design parameters, in order that these gains remain bounded. In the approach taken by Athans and Johnson [9] quadratic weighting is placed on the dynamic compensator design parameters. However, because the dynamic compensator to be developed is specifically of the structure of a Luenberger observer, it may easily be shown that the gains $\{D, F, G_0, L_1\}^1$ are uniquely related through a single design matrix K . Thus, by constraining any one of the four observer parameters, the remaining three are implicitly constrained. With the above assumptions in mind, let the following cost functional be defined:

$$\begin{aligned} \min_{J,K} \max_{Y} I[J,K;Y] = \frac{1}{2} \int_{t_0}^t & (q^T Q_1 q + \epsilon^T L_2^T Q_2 L_2 \epsilon + u^T R u \\ & + y^T L_1^T Q_y L_1 y + z^T F^T Q_z F z - y^T Y y) dt \end{aligned} \quad (3.9)$$

¹ L_2 is not a function of any design parameters.

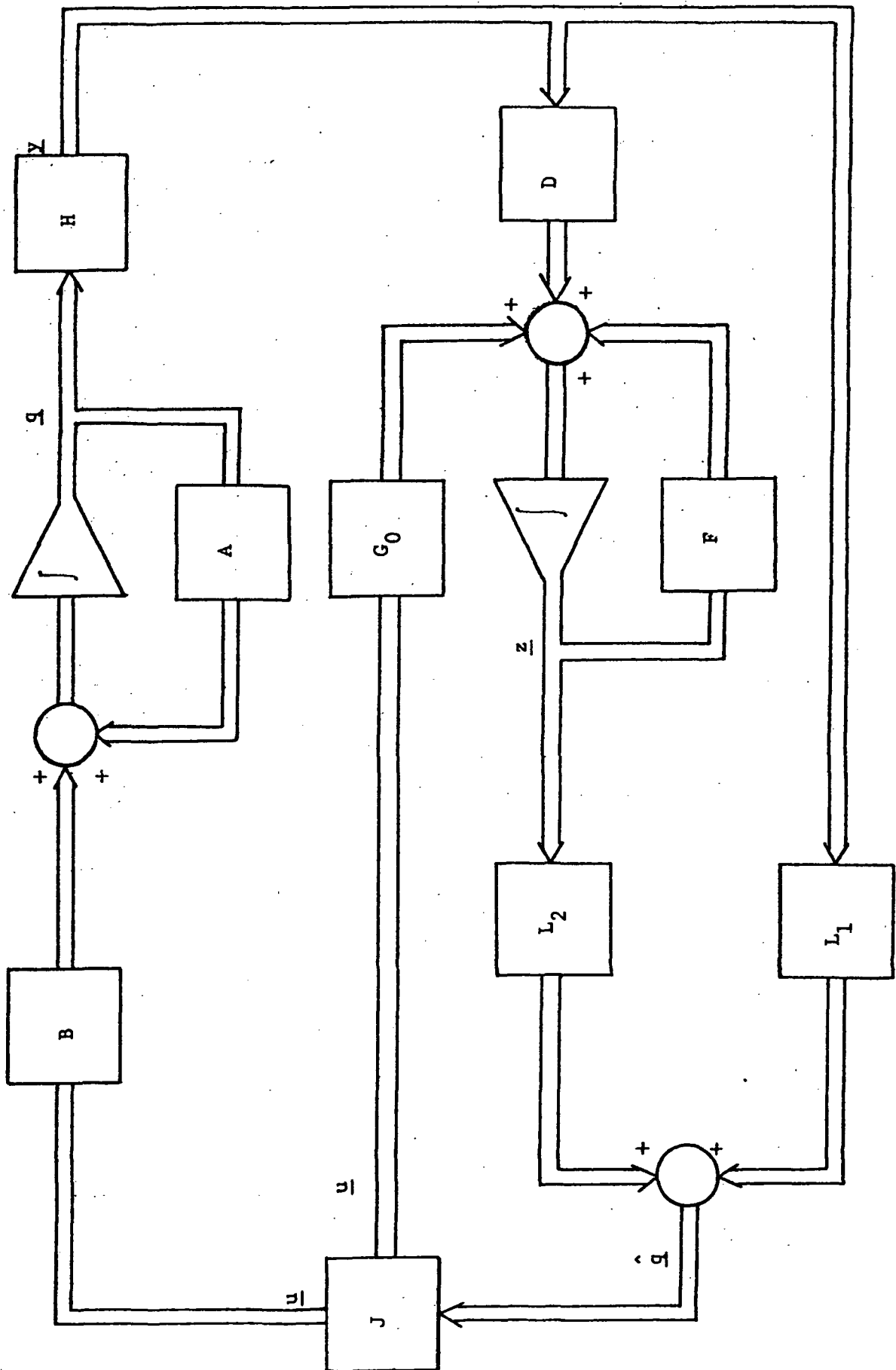


Figure 3.2 Controller and State Observer

Essentially, we desire to achieve some trade-off between state estimation and control. The first component of the cost functional with positive definite weighting Q_1 requires that $\underline{q}$ be regulated as close to zero as possible. The second term, however, weights the degree to which estimate $\hat{\underline{q}}$ tracks true state $\underline{q}$, since $\underline{q}$, $\hat{\underline{q}}$ are related through Eq. 2.22. It should be made clear that because the system design is performed in the observer canonic space, only the last "n-m" components of $\underline{q}-\hat{\underline{q}}$ influence the overall cost. Hence, this component places a penalty on state estimation, attempting to keep the inherent error (due to parameter uncertainties and incorrect initial conditions) minimal.

The quadratic weighting on control $\underline{u}$ actually places a penalty on the elements of $\underline{J}$ since the control has been constrained to be a function of $\hat{\underline{q}}$.

It has previously been stated that constraining any one of the four observer gains implicitly constrains the remaining three. To generalize the analysis, quadratic weighting has been placed on both the partition $\underline{L}_1\underline{y}$ of estimate $\hat{\underline{q}}$ and on the observer feedback $\underline{F}\underline{z}$. The justification for this generalization is as follows.

Both quadratic terms place a constraint on the positioning of the observer poles. As shown in Appendix A of [29], however, the constraint $\underline{L}_1\underline{y}$ is quadratic in gain $\underline{K}$ while constraints on any of the remaining three observer design parameters result in terms fourth order in $\underline{K}$. It appears reasonable to determine whether any design trade-off exists through variation of weights Q_y , Q_z . Hence, the analysis is formulated utilizing both observer parameters in the cost functional, Eq. 3.9.

The final component, $\underline{\gamma}$, acts as a disturbance vector to account for any parameter uncertainties in the system. This disturbance, to be

determined by maximizing Eq. 3.9 with respect to $\underline{\gamma}$, will be shown to act upon the system from a worst case design point of view. In light of the above assumptions, the design approach to be taken is as follows:

Determine a function $\underline{\gamma} = \underline{\gamma}^*$ which maximizes functional Eq. 3.9, subject to a constrained feedback control $\underline{u} = \underline{J}\hat{\underline{q}}$. Having determined $\underline{\gamma}^*$ (shown to be obtained from the solution to a steady state algebraic Riccati equation, which is a function of $\underline{J}, \underline{K}$), determine a parameter set $\{\underline{J}, \underline{K}\}$, assuming such a set exists, which minimizes the overall cost $I[\underline{J}, \underline{K}; \underline{\gamma}^*]$.

Once the optimization yields a set $\{\underline{J}^*, \underline{K}^*; \underline{\gamma}^*\}$, the stability of the closed loop system may be examined utilizing the optimum gains in a manner to be discussed in Chapter 4.

3.4 Solution For Disturbance $\underline{\gamma}$

The resulting equations which are necessary to compute the parameters $\{\underline{J}, \underline{K}\}$ are determined by first expanding each component of the cost functional, Eq. 3.9. In terms of $(\underline{q}^T, \underline{\varepsilon}^T) = \underline{\omega}^T$, one obtains the following augmented weight $\hat{\underline{Q}}$ as a function of gains $\{\underline{J}, \underline{K}\}$ ¹.

¹ $\underline{K}$ is one component of the transformation $\underline{T} = (\underline{K}:\underline{I})$

$$\underline{\omega}^T \hat{Q} \underline{\omega} = \underline{\omega}^T \begin{bmatrix} Q_1 + J^T R J + Q_y & J^T R J L_2 \\ -Q_y L_2^T - (L_2^T)^T Q_y & + \\ + (L_2^T)^T Q_y (L_2^T) & (T A_0 L_2^T)^T Q_z (T A_0 L_2) \\ + (T A_0 L_2^T)^T Q_z (T A_0 L_2 T) & \\ \hline L_2^T J^T R J + & L_2^T Q_z L_2 \\ (T A_0 L_2)^T Q_z (T A_0 L_2 T) & + L_2^T J^T R J L_2 \\ & + (T A_0 L_2)^T Q_z (T A_0 L_2) \end{bmatrix} \underline{\omega} \quad (3.10)$$

Substituting into the cost functional we have:

$$\min_{J,K} \max_{\underline{Y}} I[J,K;\underline{Y}] = \frac{1}{2} \int_{t_0}^t (\underline{\omega}^T \hat{Q} \underline{\omega} - \underline{Y}^T \underline{Y} \underline{Y}) dt \quad (3.11)$$

To determine the form of the disturbance $\underline{Y}$, define a Hamiltonian, H_a , as follows:

$$H_a = \underline{\lambda}^T \underline{f} + L = \underline{\lambda}^T [\hat{D} \underline{\omega} + \hat{B}_0 \underline{u} + \hat{G} \underline{Y}] + \frac{1}{2} (\underline{\omega}^T \hat{Q} \underline{\omega} - \underline{Y}^T \underline{Y} \underline{Y}) \quad (3.12)$$

Since the feedback control $\underline{u}$ has been specified as $\underline{u} = J \hat{q}$, substitute this control into Eq. 3.12 after modifying $\underline{u}$ to a more appropriate form.

As one wishes to determine the gains $\{J,K\}$, where K is a partition of observer transformation T , define a matrix P as:

$$P = \begin{bmatrix} J & 0 \\ 0 & T \end{bmatrix} = \begin{matrix} n & m & n-m \\ q \begin{bmatrix} J & 0 & 0 \\ 0 & K & I \end{bmatrix} \end{matrix} \quad (3.13)$$

The matrix P includes in one augmented form, all the necessary gains to be determined. Thus the control may be expressed as:

$$\underline{u} = \hat{J}\underline{q} = J(\underline{q} + L_2\underline{\epsilon}) = \begin{bmatrix} J & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{\epsilon} \end{bmatrix} = I_1 P I_2 \underline{\omega} \quad (3.14)$$

Inserting this control into the Hamiltonian, H_a becomes:

$$H_a = \underline{\lambda}^T [(\hat{D} + \hat{B}_0 I_1 P I_2) \underline{\omega} + \hat{G} \underline{Y}] + \frac{1}{2} (\underline{\omega}^T \hat{Q} \underline{\omega} - \underline{Y}^T Y \underline{Y})$$

and extremising H_a with respect to disturbance $\underline{Y}$ yields:

$$\underline{Y} = Y^{-1} \hat{G}^T \underline{\lambda} \quad (3.15)$$

By defining a costate $\underline{\lambda} = S \underline{\omega}$, it may easily be shown [29] that matrix S is obtained from the solution to the following equation:

$$(\hat{D} + \hat{B}_0 I_1 P I_2)^T S + S(\hat{D} + \hat{B}_0 I_1 P I_2) + \hat{Q} + S \hat{G} Y^{-1} \hat{G}^T S = -\dot{S} \quad (3.16)$$

and if $t \rightarrow \infty$, one obtains a steady state algebraic matrix equation:

$$(\hat{D} + \hat{B}_0 I_1 P I_2)^T S + S(\hat{D} + \hat{B}_0 I_1 P I_2) + \hat{Q} + S \hat{G} Y^{-1} \hat{G}^T S = 0 \quad (3.17)$$

Eq. 3.17 reveals that in order to obtain an explicit set of values for Riccati gain S , it is necessary to have complete knowledge of the elements of P , which have yet to be determined. This is a well known characteristic of the minimax design technique. In the optimization design procedure to follow, the variation in S , resulting from variations in matrix P , is accounted for by obtaining an explicit relation for variation in S , due to P , to first order.

3.5. Modification of Cost Functional

Having formulated an approach to determine the worst case disturbance vector $\underline{Y}$, we are in a position to proceed with the minimization of $I(P, \underline{Y})$ with respect of P . Inserting Eq. 3.15 into Eq. 3.11, the functional becomes:

$$\min_P I(P, \underline{Y}) = \frac{1}{2} \int_{t_0}^{\infty} \underline{\omega}^T (\hat{Q} - S \hat{G} Y^{-1} \hat{G}^T S) \underline{\omega} \, dt \quad (3.18)$$

At this stage, it is advantageous to modify all terms of $\hat{Q}$ to an appropriate augmented matrix form. Utilizing a set of partition matrices defined in Appendix B of [29], the overall weighting function may be shown to be expressed as:

$$\begin{aligned}
 \hat{Q} = & \hat{Q}_A + E_1^T P^T R P E_1 + E_1^T P^T R P E_1 L_2 - L_2^T E_1^T P^T R P E_1 + L_2^T E_1^T P^T R P E_1 L_2 \\
 & + \hat{Q}_y L_2 E_2 P E_4 + E_4^T P^T E_2^T L_2^T \hat{Q}_y + E_4^T P^T E_2^T L_2^T \hat{Q}_y L_2 E_2 P E_4 \\
 & + E_4^T P^T A_L^T P^T E_3^T Q_2 E_3 P A_L P E_4 + E_4^T P^T A_L^T P^T E_3^T Q_2 E_3 P A_L E_2^T \\
 & + E_2 A_L^T P^T E_3^T Q_2 E_3 P A_L P E_4 + E_2 A_L^T P^T E_3^T Q_2 E_3 P A_L E_2^T
 \end{aligned} \quad (3.19)$$

where

$$\hat{Q}_A = \hat{Q}_1 + \hat{Q}_y + L_2^T Q_2 L_2 \quad (3.20)$$

and E_1, E_2, E_3, E_4 and A_L are constant partition matrices defined in [29].

3.6 Modification of Augmented State Equation

State equation 3.7 must be partitioned in a fashion similar to that utilized in determining $\hat{Q}$. Setting $\underline{u} = J\hat{q}$ we have:

$$\begin{bmatrix} \dot{\hat{q}} \\ \underline{\dot{\epsilon}} \end{bmatrix} = \begin{bmatrix} A_0 + B_0 J & B_0 J L_2 \\ 0 & T A_0 L_2 \end{bmatrix} \begin{bmatrix} \hat{q} \\ \underline{\epsilon} \end{bmatrix} + \hat{G} Y^{-1} \hat{G}^T S \underline{\omega} \quad (3.21)$$

Since the component " $\hat{G} Y^{-1} \hat{G}^T S$ " is already expressed as a $(2n-m) \times (2n-m)$ matrix, it is necessary only to concentrate on the first component. Accordingly, define the following partition matrices:

$$\hat{A} = \begin{bmatrix} \overset{n}{A_0} & \overset{n-m}{0} \\ 0 & 0 \end{bmatrix} \begin{matrix} n \\ n-m \end{matrix} \quad \hat{B} = \begin{bmatrix} \overset{q}{B_0} & \overset{n-m}{0} \\ 0 & I \end{bmatrix} \begin{matrix} n \\ n-m \end{matrix} \quad \hat{C} = \begin{bmatrix} \overset{n}{I} & \overset{n-m}{L_2} \\ 0 & A_0 L_2 \end{bmatrix} \begin{matrix} n \\ n \end{matrix}$$

then:

$$\hat{A} + \hat{B}P\hat{C} = \begin{bmatrix} A_0 + B_0^J & B_0^{JL_2} \\ 0 & TA_0^{L_2} \end{bmatrix}$$

and thus the overall closed loop state equation may be expressed as:

$$\dot{\underline{\omega}} = (\hat{A} + \hat{B}P\hat{C} + \hat{G}Y^{-1}\hat{G}^T S)\underline{\omega} \quad (3.22)$$

It is well known that the solution to Eq. 3.22 is:

$$\underline{\omega} = e^{(\hat{A} + \hat{B}P\hat{C} + \hat{G}Y^{-1}\hat{G}^T S)t} \underline{\omega}(t_0) = e^{At} \underline{\omega}(t_0)$$

where:

$$\underline{\omega} = \phi(t) \underline{\omega}(t_0) \quad (3.23)$$

and $\phi(t)$ is the transition matrix satisfying

$$\dot{\phi}(t) = (\hat{A} + \hat{B}P\hat{C} + \hat{G}Y^{-1}\hat{G}^T S) \phi(t) \quad (3.24)$$

Having obtained a closed form expression for state $\underline{\omega}$ the cost functional becomes:

$$\min_P I(P, \underline{\gamma}) = \frac{1}{2} \int_{t_0}^{\infty} \underline{\omega}^T(t_0) \phi^T(t) [\hat{Q} - S\hat{G}Y^{-1}\hat{G}^T S] \phi(t) \underline{\omega}(t_0) dt \quad (3.25)$$

3.7 Initial Condition Transformation

In order to operate on the cost functional expressed by Eq. 3.25, it is necessary to eliminate the dependence, upon the cost $I(P, \underline{\gamma})$, of the initial condition vector $\underline{\omega}(t_0)$. One such approach, similar to that taken in [7,8] is to assume that the initial state $\underline{q}(t_0)$ is a random variable, uniformly distributed on a unit sphere with zero mean and known covariance W_0 . Then, from the constraint equations for an observer, the initial state $\underline{z}(t_0)^1$ becomes:

$$\underline{z}(0) = T\underline{q}(0) + \underline{\varepsilon}(0)$$

¹ Let $t_0 = 0$ from hereon.

Since the designer is free to choose the initial condition vector $\underline{z}(0)$, set $\underline{z}(0)$ as:

$$\underline{z}(0) = T^T E[\underline{q}(0)] = 0$$

Then the initial error between state and compensator output is:

$$\underline{\varepsilon}(0) = -T\underline{q}(0)$$

and the initial condition error covariance is found as:

$$\overline{\underline{\varepsilon}(0)\underline{\varepsilon}(0)^T} = \overline{T(\underline{q}(0)\underline{q}(0)^T)T^T} = TW_0T^T \quad (3.26a)$$

The cross terms $\overline{\underline{\varepsilon}(0)\underline{q}(0)^T}$ and $\overline{\underline{q}(0)\underline{\varepsilon}(0)^T}$ are found to be

$$\overline{\underline{\varepsilon}(0)\underline{q}(0)^T} = -T \overline{\underline{q}(0)\underline{q}(0)^T} = -TW_0$$

and

$$\overline{\underline{q}(0)\underline{\varepsilon}(0)^T} = -W_0T^T \quad (3.26b)$$

Thus the overall initial condition covariance matrix X_0 is found to be:

$$X_0 = \begin{bmatrix} W_0 & -W_0T^T \\ -TW_0 & TW_0T^T \end{bmatrix} \quad (3.27)$$

In its present form, the covariance matrix X_0 is not very useful. It would be advantageous, however, if X_0 is in strictly diagonal form, the reasoning behind this assumption becoming clear in the following section. To obtain such an expression, define a transformation M such that

$$\underline{v}(0) = M^T \underline{\omega}(0)$$

where

$$\overline{\underline{v}(0)\underline{v}(0)^T} = M^T \overline{\underline{\omega}(0)\underline{\omega}(0)^T} M = M^T X_0 M = V_0$$

and we require that V_0 be a diagonal matrix. Thus, we must seek out a

transformation such that the diagonality of V_0 is assured. To obtain such a transformation, proceed as follows:

Pre and post multiply X_0 by H_1^T and H_1 respectively, where

$$H_1 = \begin{bmatrix} I & T^T \\ 0 & I \end{bmatrix}$$

thus:

$$H_1^T X_0 H_1 = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} W_0 & -W_0 T^T \\ -T W_0 & T W_0 T^T \end{bmatrix} \begin{bmatrix} I & T^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} W_0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the transformation $M=H_1$ so that one has

$$M = \begin{bmatrix} I & T^T \\ 0 & I \end{bmatrix}; \quad M^{-1} = \begin{bmatrix} I & -T^T \\ 0 & I \end{bmatrix} \quad (3.28)$$

To implement the transformation we must obtain matrix M in terms of the gain matrix P . Consider M^{-T} and partition as follows:

$$M^{-T} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} = I - E_2 P E_4 \quad (3.29a)$$

$$\text{and} \quad M^{-1} = (I - E_2 P E_4)^T \quad (3.29b)$$

With explicit expressions determined for M^{-1} in terms of the augmented gain matrix P , the cost functional becomes:

$$\min_P I(P, \gamma) = \frac{1}{2} \int_0^\infty \underline{v}^T(0) (I - E_2 P E_4)^T \phi^T(t) [\hat{Q} - S \hat{G} Y^{-1} \hat{G}^T S] \phi(t) (I - E_2 P E_4) \underline{v}(0) dt \quad (3.30)$$

A convenient means of eliminating the dependence of the cost $I(P, \gamma)$ upon the initial conditions is to minimize the expected value of $I(P, \gamma)$. Note that the initial condition covariance matrix X_0 has been diagonalized through

the transformation Eq. 3.28. Taking expectations of Eq. 3.30 yields:

$$E[I(P, \underline{y})] = \hat{I}(P, \underline{y}) = \frac{1}{2} \text{TR} \int_0^{\infty} (I - E_2 P E_4)^T \Phi^T(t) [\hat{Q} - S \hat{G} Y^{-1} \hat{G}^T S] \Phi(t) (I - E_2 P E_4) V_0 dt \quad (3.31)$$

The function $I(P, \underline{y})$ is a real function where P contains $2n \times (q+n-m)$ variables. However, the values we wish to determine are $\{J, K\}$, since the remaining elements are either zero or one. Thus the actual number of elements to be computed is:

$$P = \begin{bmatrix} \overset{n}{J} & \overset{m}{0} & \overset{n-m}{0} \\ 0 & K & I \end{bmatrix} \begin{matrix} q \\ n-m \end{matrix} = q \times n + [(n-m) \times m]$$

There does exist an additional complication, however. The fact that Riccati gain S , which determines the extremum in disturbance, is a function of P , implies that a variation in $\hat{I}(P, \underline{y})$ with respect to P results in a variation in S . The functional dependence of S upon P is accounted for by determining an expression which relates the dependence of S to that of P .

3.8 Computation of $\partial \hat{I}(P, \underline{y}) / \partial P$

A necessary condition that there exists a P^* which minimizes $\hat{I}(P, \underline{y})$ is:

$$\left. \frac{\partial \hat{I}(P, \underline{y})}{\partial P} \right|_{P=P^*} = 0 \quad (3.32)$$

Let the closed loop plant matrix be expressed as:

$$A^0 = \hat{A} + \hat{B} P \hat{C} + \hat{G} Y^{-1} \hat{G}^T S \quad (3.33)$$

If we allow a small perturbation in P such that $P \rightarrow P + \epsilon \delta P$, then this implies that $S \rightarrow S + \epsilon \delta S$. Replacing $\hat{Q}$ by Eq. 3.19, substituting for the transition matrix in Eq. 3.31, and finally perturbing the cost functional by the above variations yields:

$$\begin{aligned}
\hat{I}(P + \epsilon \delta P, S + \epsilon \delta S) &= \frac{1}{2} \text{TR} \int_0^\infty [I - E_2(P + \epsilon \delta P) E_4]^T \\
&\quad e^{[A^0 + \epsilon(\hat{B}\delta P\hat{C} + \hat{G}Y^{-1}\hat{G}^T\delta S)]^T t} \\
&\quad [\hat{Q}_A + E_1^T(P + \epsilon \delta P)^T \hat{R}(P + \epsilon \delta P) E_1 - E_1^T(P + \epsilon \delta P)^T \hat{R}(P + \epsilon \delta P) E_1 \hat{L}_2 \\
&\quad - \hat{L}_2^T E_1^T(P + \epsilon \delta P)^T \hat{R}(P + \epsilon \delta P) E_1 + \hat{L}_2^T E_1^T(P + \epsilon \delta P)^T \hat{R}(P + \epsilon \delta P) E_1 \hat{L}_2 \\
&\quad + \hat{Q}_y \hat{L}_2^T E_2(P + \epsilon \delta P) E_4 + E_4^T(P + \epsilon \delta P)^T E_2^T \hat{L}_2^T \hat{Q}_y \\
&\quad + E_4^T(P + \epsilon \delta P)^T E_2^T \hat{L}_2^T \hat{Q}_y \hat{L}_2^T E_2(P + \epsilon \delta P) E_4 \\
&\quad + E_4^T(P + \epsilon \delta P)^T A_L^T(P + \epsilon \delta P)^T E_3^T \hat{Q}_z E_3(P + \epsilon \delta P) A_L(P + \epsilon \delta P) E_4 \\
&\quad + E_4^T(P + \epsilon \delta P)^T A_L^T(P + \epsilon \delta P)^T E_3^T \hat{Q}_z E_3(P + \epsilon \delta P) A_L E_2^T \\
&\quad + E_2 A_L^T(P + \epsilon \delta P)^T E_3^T \hat{Q}_z E_3(P + \epsilon \delta P) A_L(P + \epsilon \delta P) E_4 \\
&\quad + E_2 A_L^T(P + \epsilon \delta P)^T E_3^T \hat{Q}_z E_3(P + \epsilon \delta P) A_L E_2^T \\
&\quad - (S + \epsilon \delta S) \hat{G}Y^{-1} \hat{G}^T (S + \epsilon \delta S)] e^{[A^0 + \epsilon(\hat{B}\delta P\hat{C} + \hat{G}Y^{-1}\hat{G}^T\delta S)]^T t} \\
&\quad [I - E_2(P + \epsilon \delta P) E_4] V_0 \quad dt \tag{3.34}
\end{aligned}$$

Equation 3.34 reveals some complexities which must be circumvented before proceeding with the analysis. It will first be necessary to obtain an appropriate representation for the term

$$e^{[A^0 + \epsilon(\hat{B}\delta P\hat{C} + \hat{G}Y^{-1}\hat{G}^T\delta S)]^T t}$$

In addition, the variation in δS must be determined as a function of δP . Appendices D and E of [29] present first order approximations to handle both of the above difficulties.

Substitution into Eq. 3.34 of the results derived in Appendices D, E of [29] yields an expression which is too involved and cumbersome for

this report. Rather, the mechanics of the optimization procedure are merely outlined, continually referring the reader to [29] for a comprehensive explanation of the details.

The modified Eq. 3.34 (due to insertion of D. E) must be expanded in order to disregard terms of order $\epsilon > 1$. This operation is shown in detail in Appendices F and G of [29]. The resulting equation, a set of integral expressions linear in perturbation δP , is given by Eq. 3.48 in [29].

The following section presents a lemma which manifests the reasoning for the approach taken in this report.

3.9 Simplifying the Gradient of the Cost Functional

The motivation for formulating the problem in the manner described is to take advantage of the following lemma proposed by Kleinman [27]:

Lemma: Let $\hat{I}(P, \underline{Y})$ be a trace function. If one can express:

$$\hat{I}(P + \epsilon \delta P, \underline{Y}) - \hat{I}(P, \underline{Y}) = \epsilon \text{TR}[N(P) \delta P]^1$$

Then as $\epsilon \rightarrow 0$

$$\partial \hat{I}(P, \underline{Y}) / \partial P = N^T(P) \quad (3.37)$$

To obtain an expression in the form of Eq. 3.37 we make use of the following properties of trace matrices.

$$1. \text{TR}[XYZ] = \text{TR}[ZXY] = \text{TR}[YZX]$$

$$2. \text{TR}[\underline{X}] = \text{TR}[\underline{X}^T]$$

Because of the length of Eq. 3.48 in [29] it is more feasible to consider each term individually. To preserve the continuity of this report, the expansion of Eq. 3.48 is shown in Appendix I. Each term has been manipulated in such fashion that the Lemma of Section 3.9 may be directly

¹ Recall that the variation in cost is with respect to certain defined elements of P ; i.e. $P[J, K]$.

applied. Collecting all terms from Appendix I, the final expression for the gradient, $N^T(P)$, is shown in Appendix J.

By considering the infinite time case, the integral expressions of Appendix J may be replaced by algebraic matrix equivalencies.

3.10 Additional Simplification of Gradient

In its present form, the equations of Appendix J in [29] are not extremely useful. By applying integral properties which relate to the interchange of order of integration, the equations may be reformulated into a more appealing format. Once again, because of length considerations, the reader is referred to Section 3.10 of [29] for the details. We merely express the resulting algorithm as the following set of equations:

$$\begin{aligned}
 & \hat{B}^T \chi \hat{C}^T - E_2^T (I - E_2 P E_4) V_0 E_4^T + \hat{R} P E_1 \psi E_1^T - \hat{R} P E_1 \psi (E_1 \hat{L}_2)^T - \hat{R} P E_1 \hat{L}_2 \psi E_1^T \\
 & + \hat{R} P E_1 \hat{L}_2 \psi (E_1 \hat{L}_2)^T + (\hat{Q}_y \hat{L}_2 E_2)^T \psi E_4^T + (E_2^T \hat{L}_2^T \hat{Q}_y \hat{L}_2 E_2 P E_4) \psi E_4^T \\
 & + (A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L P E_4) \psi E_4^T + (E_3^T \hat{Q}_z E_3 P A_L P E_4) \psi (A_L P E_4)^T \\
 & + (E_3^T \hat{Q}_z E_3 P A_L P E_4) \psi (E_2 A_L^T) + (E_3^T \hat{Q}_z E_3 P A_L E_2^T) \psi (A_L P E_4)^T \\
 & + (A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L E_3^T) \psi E_4^T + (E_3^T \hat{Q}_z E_3 P A_L E_2^T) \psi (E_2 A_L^T) = 0
 \end{aligned} \tag{3.38}$$

subject to:

$$A^0 \psi + \psi A^{0T} + (I - E_2 P E_4) V_0 (I - E_2 P E_4)^T = 0 \tag{3.39}$$

$$\begin{aligned}
& (A + BPC)^T \chi + \chi(A + BPC) + Q_A + E_1^T P^T R P E_1 \\
& - E_1^T P^T \hat{R} P E_1 \hat{L}_2 - \hat{L}_2^T E_1^T P^T \hat{R} P E_1 + \hat{L}_2^T E_1^T P^T \hat{R} P E_1 \hat{L}_2 \\
& + \hat{Q}_y \hat{L}_2^T E_2 P E_4 + E_4^T P^T E_2 \hat{L}_2^T \hat{Q}_y + E_4^T P^T E_2 \hat{L}_2^T \hat{Q}_y \hat{L}_2 E_2 P E_4 \\
& + E_4^T P^T A_L^T P^T E_3 \hat{Q}_z E_3 P A_L P E_4 + E_4^T P^T A_L^T P^T E_3 \hat{Q}_z E_3 P A_L E_2^T \\
& + E_2^T A_L^T P^T E_3 \hat{Q}_z E_3 P A_L P E_4 + E_2^T A_L^T P^T E_3 \hat{Q}_z E_3 P A_L E_2^T + \chi \hat{G} Y^{-1} \hat{G}^T \chi = 0
\end{aligned} \tag{3.40}$$

As explained in [29], the χ, ψ are Riccati and Lyapunov type matrices, being that they are the solutions to Eqs. 3.52, 3.53 of [29], respectively. Note that if parameter ignorance is non-existent, and no disturbances are present, $Y^{-1} \rightarrow 0$, and the quadratic term χ in Eq. 3.38 drops out.

Equations 3.38-3.40 may be solved recursively to obtain the desired gain P^* . In addition to P , the recursive equations also provide a positive definite symmetric χ matrix. This function is utilized in Chapter 4 to derive a stability bound on closed loop performance.

The discussion which follows presents the required recursive algorithm. Note that from Eq. 3.40 an approximation has been made such that the quadratic term in χ is represented at the n th iteration while the linear term in χ is at the $n+1$ st iteration. This approximation is justified as it simplifies the computational algorithm by reducing Eq. 3.40 to a linear equation in terms of χ_{n+1} .

3.11 Recursive Algorithm to Compute Parameter P*

$$\begin{aligned}
(1) \quad & \hat{B}^T \chi_{n+1} \psi_{n+1} \hat{C}^T - E_2^T \chi_{n+1} (I - E_2 P_{n+1} E_4) V_0 E_4^T + \hat{R} P_{n+1} E_1 \psi_{n+1} E_1^T \\
& - \hat{R} P_{n+1} E_1 \psi_{n+1} (E_1 \hat{L}_2)^T \\
& - \hat{R} P_{n+1} E_1 \hat{L}_2 \psi_{n+1} E_1^T + \hat{R} P_{n+1} E_1 \hat{L}_2 \psi_{n+1} (E_1 \hat{L}_2)^T + (\hat{Q}_y \hat{L}_2 E_2)^T \psi_{n+1} E_4^T \\
& + (E_2^T \hat{L}_2 \hat{Q}_y \hat{L}_2 E_2 P_{n+1} E_4) \psi_{n+1} E_4^T + (A_{L,n+1}^T P_{n+1}^T E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_4) \psi_{n+1} E_4^T \\
& + (E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_4) \psi_{n+1} (A_{L,n+1} E_4)^T + (E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_4) \psi_{n+1} (E_2 A_L^T) \\
& + (E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_2^T) \psi_{n+1} (A_{L,n+1} E_4)^T + A_{L,n+1}^T P_{n+1}^T E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_2^T) \psi_{n+1} E_4^T \\
& + (E_3^T \hat{Q}_z E_3 P_{n+1} A_{L,n+1} E_2^T) \psi_{n+1} (E_2 A_L^T) = 0 \tag{3.41}
\end{aligned}$$

$$\begin{aligned}
(2) \quad & (\hat{A} + \hat{B} P_{n+1} \hat{C} + \hat{G} Y^{-1} \hat{G}^T \chi_{n+1}) \psi_{n+1} + \psi_{n+1} (\hat{A} + \hat{B} P_{n+1} \hat{C} + \hat{G} Y^{-1} \hat{G}^T \chi_{n+1})^T \\
& + (I - E_2 P_{n+1} E_4) V_0 (I - E_2 P_{n+1} E_4)^T = 0 \tag{3.42}
\end{aligned}$$

$$\begin{aligned}
(3) \quad & (\hat{A} + \hat{B} P_n \hat{C})^T \chi_{n+1} + \chi_{n+1} (\hat{A} + \hat{B} P_n \hat{C}) + \hat{Q}_A + E_1^T P_n^T \hat{R} P_n E_1 \\
& - E_1^T P_n^T \hat{R} P_n E_1 \hat{L}_2 - (P_n E_1 \hat{L}_2)^T \hat{R} P_n E_1 + (P_n E_1 \hat{L}_2)^T \hat{R} P_n E_1 \hat{L}_2 \\
& + \hat{Q}_y \hat{L}_2 E_2 P_n E_4 + (\hat{L}_2 E_2 P_n E_4)^T \hat{Q}_y + (\hat{L}_2 E_2 P_n E_4)^T \hat{Q}_y \hat{L}_2 E_2 P_n E_4 \\
& + (E_3 P_n A_{L,n} E_4)^T \hat{Q}_z E_3 P_n A_{L,n} E_4 + (E_3 P_n A_{L,n} E_4)^T \hat{Q}_z E_3 P_n A_{L,n} E_2^T \\
& + (E_3 P_n A_{L,n} E_2)^T \hat{Q}_z E_3 P_n A_{L,n} E_4 + (E_3 P_n A_{L,n} E_2)^T \hat{Q}_z E_3 P_n A_{L,n} E_2^T \\
& + \chi_n \hat{G} Y^{-1} \hat{G}^T \chi_n = 0 \tag{3.43}
\end{aligned}$$

- i. Select P_0 , χ_0 such that $\hat{A} + \hat{B}P_0\hat{C}$ is stable and χ_0 is positive definite.
- ii. Compute χ_1 from (3).
- iii. Substitute χ_1 into (1), (2). Solve simultaneous non-linear equations for P_1 , ψ_1 .

This completes one iteration. Repeat until some prespecified criterion on convergence is satisfied.

IV. CLOSED LOOP STABILITY CONSIDERATIONS

4.1 Introduction

Formulation of the problem investigated in this report in terms of a minimax control design accounts to some degree for the parameter uncertainty effects inherent in the closed loop system. It has been pointed out that in addition to the solution for matrix P, the recursive optimization procedure also computes a positive definite symmetric $(2n-m) \times (2n-m)$ matrix gain, designated as χ . This chapter presents an approach toward the determination of bounds on allowable parameter variations to insure stability of the closed loop system.

4.2 Lyapunov Function Formulation

Assume that the design procedure converges to a solution P^* , χ^* . If the variation in output matrix H is zero, as assumed in this analysis, the closed loop dynamics are:

$$\begin{bmatrix} \dot{\underline{q}} \\ \dot{\underline{e}} \end{bmatrix} = \begin{bmatrix} A_0 + B_0 J^* + \delta A + \delta B J^* & (B_0 + \delta B) J^* L_2 \\ -T^*(\delta A + \delta B J^*) & T^* A_0 L_2 - T^* \delta B J^* L_2 \end{bmatrix} \begin{bmatrix} \underline{q} \\ \underline{e} \end{bmatrix}^1 \quad (4.1)$$

In an attempt to determine the allowable variations on δA , δB to insure stability, begin by defining partition matrices as follows:

¹ from here-on we drop the asterisk notation on $\{J, K\}$

$$\hat{\delta A} = \begin{bmatrix} \delta A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} n \\ n-m \end{matrix} \quad (4.2a)$$

$$\hat{\delta B} = \begin{bmatrix} \delta B & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} n \\ n-m \end{matrix} \quad (4.2b)$$

Utilizing these partition matrices it may be shown that the system matrix of Eq. 4.1 can be expressed as:

$$\dot{\underline{\omega}} = [(\hat{A} + \hat{B}\hat{P}\hat{C}) + (I - E_2 P E_4) (\delta \hat{A} + \delta \hat{B}\hat{P}\hat{C})] \underline{\omega} \quad (4.3)$$

and we observe that the nominal and uncertain terms have been separated.

In order to formulate a stability criterion, define a function:

$$V(\underline{\omega}) = \frac{1}{2} \underline{\omega}^T \chi \underline{\omega} \quad (4.4)$$

This function is positive definite since χ , the solution to Eq. 3.63 is based on an assumed stable (nominal) closed loop system and positive definite weighting matrix $\hat{Q}_A$.

Taking the derivative of Eq. 4.5 yields:

$$\begin{aligned} \dot{V}(\underline{\omega}) = & \frac{1}{2} \underline{\omega}^T [(\hat{A} + \hat{B}\hat{P}\hat{C})^T \chi + \chi (\hat{A} + \hat{B}\hat{P}\hat{C}) \\ & + ((I - E_2 P E_4) \delta \hat{A})^T \chi + \chi ((I - E_2 P E_4) \delta \hat{A}) \\ & + ((I - E_2 P E_4) \delta \hat{B}\hat{P}\hat{C})^T \chi + \chi ((I - E_2 P E_4) \delta \hat{B}\hat{P}\hat{C})] \underline{\omega} \end{aligned} \quad (4.5)$$

From Eq. 3.63 we have:

$$(\hat{A} + \hat{B}\hat{P}\hat{C})^T \chi + \chi (\hat{A} + \hat{B}\hat{P}\hat{C}) = [\hat{Q} + \chi \hat{G} Y^{-1} \hat{G}^T \chi]$$

Substituting Eq. 4.6 into Eq. 4.5 and removing the negative sign

$\dot{V}(\underline{\omega})$ becomes

$$\begin{aligned} \dot{V}(\underline{\omega}) = & -\frac{1}{2} \underline{\omega}^T [\hat{Q} + \chi \hat{G} Y^{-1} \hat{G}^T \chi \\ & - ((I - E_2 P E_4) \delta \hat{A})^T \chi - \chi ((I - E_2 P E_4) \delta \hat{A}) \\ & - ((I - E_2 P E_4) \delta \hat{B}\hat{P}\hat{C})^T \chi - \chi ((I - E_2 P E_4) \delta \hat{B}\hat{P}\hat{C})] \underline{\omega} \end{aligned} \quad (4.7)$$

Conditions on allowable variations in $\delta\hat{A}$, $\delta\hat{B}$ to insure that $\dot{V}(\underline{\omega})$ is negative definite must be investigated. Replacing $\hat{Q}$ by Eq. 3.19, assume that Eq. 4.7 can be expressed as follows:¹

$$\begin{aligned} \dot{V}(\underline{\omega}) = & -\frac{1}{2}\underline{\omega}^T \left\{ [\hat{Q}_1 + E_1^T P^T \hat{R} P E_1 - E_1^T P^T \hat{R} P E_1 \hat{L}_2 - \hat{L}_2^T E_1^T P^T \hat{R} P E_1 \right. \\ & + \hat{L}_2^T E_1^T P^T \hat{R} P E_1 \hat{L}_2 + \chi \hat{G} Y^{-1} \hat{G}^T \chi - ((I - E_2 P E_4) \delta\hat{A})^T \chi \\ & - \chi((I - E_2 P E_4) \delta\hat{A})] \\ & + [\hat{L}_2^T \hat{Q}_2 \hat{L}_2 + \hat{Q}_y + \hat{Q}_y \hat{L}_2 E_2 P E_4 + E_u^T P^T E_2^T \hat{L}_2 \hat{Q}_y \\ & + E_4^T P^T E_2^T \hat{L}_2 \hat{Q}_y \hat{L}_2 E_2 P E_4 + E_4^T P^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L P E_4] \\ & + E_4^T P^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L E_2^T + E_2 A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L P E_4 \\ & + E_2 A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L E_2^T - ((I - E_2 P E_4) \delta\hat{B} \hat{P} \hat{C})^T \chi \\ & \left. - \chi((I - E_2 P E_4) \delta\hat{B} \hat{P} \hat{C}) \right\} \underline{\omega} \end{aligned} \quad (4.8)$$

One approach to determine stability bounds is to derive the allowable variations in the norms of $\delta\hat{A}$, $\delta\hat{B}$. In this light, let the augmented vector $\underline{\omega}$ be defined on a normal space [12] as:

$$||\underline{\omega}|| = [\underline{\omega}^T \underline{\omega}]^{\frac{1}{2}}$$

and define a matrix Z such that

$$||Z|| = \sup ||Z\underline{\omega}||$$

$$||\underline{\omega}|| = 1$$

so that

$$||Z|| = \lambda_{\max}^{\frac{1}{2}} \{Z^T Z\}$$

¹ As there are essentially five weighting coefficients which determine the resulting bound, several alternative combinations are possible.

$\lambda_{\max} \{ \}$ indicates the maximum eigenvalue of a real, symmetric and positive semi-definite matrix. Based on Eq. 4.8, $\dot{V}(\underline{w})$ is assumed negative definite, and hence the system is asymptotically stable, if the following conditions are satisfied:

$$||\delta\hat{A}|| < \frac{\lambda_{\min} \left[\begin{array}{l} \hat{Q}_1 + E_1^T P^T \hat{R} P E_1 - E_1^T P^T \hat{R} P E_1 \hat{L}_2 - \hat{L}_2^T E_1^T P^T \hat{R} P E_1 \\ + \hat{L}_2^T E_1^T P^T \hat{R} P E_1 + \chi \hat{G} Y^{-1} \hat{G}^T \chi \end{array} \right]}{2 ||\chi||} \quad (4.9)$$

$$||\delta\hat{B}|| < \frac{\lambda_{\min} \left[\begin{array}{l} \hat{L}_2^T \hat{Q}_2 \hat{L}_2 + \hat{Q}_y + \hat{Q}_y \hat{L}_2 E_2 P E_4 + E_4^T P^T E_2^T \hat{L}_2^T \hat{Q}_y \\ + E_4^T P^T E_2^T \hat{L}_2^T \hat{Q}_y L_2 E_2 P E_4 + E_4^T P^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L P E_4 \\ + E_4^T P^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L E_2^T + E_2^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L P E_4 \\ + E_2^T A_L^T P^T E_3^T \hat{Q}_z E_3 P A_L E_2^T \end{array} \right]}{2 ||\chi|| \quad ||\hat{P}\hat{C}||} \quad (4.10)$$

where the norm of $(I - E_2 P E_4)$ is unity since

$$||I - E_2 P E_4|| = \left\| \begin{bmatrix} I & 0 \\ -T & I \end{bmatrix} \right\| = 1$$

We remark that the above bounds are based on an arbitrary separation of the weighting functions of Eq. 4.6. It is conceivable that rearrangement of the weights may result in more or less conservative bounds.

V. APPLICATION OF THE DESIGN TECHNIQUE

5.1 Introduction

The design algorithm expressed by Eqs.3.41-3.43 is applied to a system described by the following equations. Note that the system is assumed to already be in the observer canonical form.

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2$$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + b u$$

$$y = x_1$$

Since the system has a single output, we require an observer design with one pole. From Eq. 2.7

$$\begin{bmatrix} H \\ T \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \Rightarrow L_1 = \begin{bmatrix} 1 \\ -k \end{bmatrix} ; L_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

5.2 Objectives of The Simulation

The primary objective of the simulation study is to obtain some measure of both the versatility and flexibility of the design algorithm. Such questions under consideration are as follows:

1. What degree of freedom does the designer have in optimally positioning the eigenvalues of the closed loop dynamical system?
2. What flexibility exists in positioning the observer/compensator eigenvalues?
3. What gain is achieved by modeling system parameter uncertainties as extremum disturbance vectors?

The simulation results presented in the following section provide partial answers to these questions. A series of plots, depicting variation in eigenvalue positioning with respect to the various weighting functions in Eq. 3.9 are presented. In addition, it is shown that a closed bounded region of initial conditions P_0 exists for which convergence to a unique minimum occurs.

5.3 Discussion of Simulation Results

5.3.1 Eigenvalue Placement

Figures 5.1 - 5.3 display the pole migration of the overall dynamical system subject to variations in weights Q_1 and R . Subject to observability, controllability constraints these graphs demonstrate that the eigenvalues of $A+BJ$ may arbitrarily be positioned. However, the data presented in Tables 5.1 - 5.4 also indicate that excessively long convergence times are required. As an example, for $q_{11} = 5 \times 10^3$ (Table 5.1), 15 minutes CPU time was required to position the root from

$$\lambda_s^{ic} = 1 \pm j\sqrt{2} ; \quad \lambda_0^{ic} = -2$$

to

$$\lambda_s = -7.60 \pm j 0.533 \quad \lambda_0 = -10.127$$

Yet the bound $\epsilon_n \approx 9.5 \times 10^{-7}$ and the cost is still decreasing. In theory, then, the algorithm permits arbitrary placement of closed loop eigenvalues, providing the required computation time is available.

The convergence time for those data points whose functional value $L_{n+1} < 10^{-19}$ was generally less than 4 minutes. Data points for which convergence was not achieved in this time period are so indicated. Hence, if data run "7" in Fig. 5.3 would have converged to $L_{n+1} \leq 10^{-19}$ in the allotted computation period, its true position would be approximately

$$\lambda_s = -8. \pm j3.25.$$

The following two graphs, Figs. 5.4, 5.5, provide insight into the pole migration when Q_y and Q_z are varied. In both cases, the closed loop system poles become more oscillatory as the observer gain is penalized more heavily thru Q_y , Q_z . Migration of the observer pole to the right as Q_y , Q_z is increased verifies that for variation in cost $\hat{I}(P, \underline{y})$ to remain relatively invariant, an increase in the observer penalties implies a decrease in $||k||$. Since, for this example, $\lambda_0 = 2(1+k)$, it is apparent that as $||k|| \rightarrow 1.0$ (k negative) the pole moves toward the origin. It may also be observed that pole positioning in both graphs is relatively equivalent. However, since Q_z penalizes the observer gain to fourth order, less weighting Q_z (for $Q_z > 1.0$) is required for approximately equivalent pole placement.

The final penalty function relating to eigenvalue positioning is shown in Fig. 5.6. It was hypothesized that an increase in weight Q_2 requires the inherent observer error $\underline{\epsilon}$ to decay more rapidly. In the simulation studies performed, Q_2 was varied from 10^{-2} to 10^3 . The resulting variation in λ_0 , though much smaller than anticipated, is shown in the figure.

Some preliminary conclusions may be arrived at based on the simulation results discussed. It is apparent that the designer has more control over the placement of system poles than observer eigenvalues. It should be recalled, however, that since both observer and feedback gains are computed simultaneously, the separation principle [24] applied by previous authors [13, 14] is not fully applicable. This restriction in design is due to the high degree of coupling which exists between the state equation and initial condition error equation (Eqs. 3.1, 3.2) when $\underline{u} = J\hat{\underline{q}}$.

5.3.2 Uniqueness of The Minimum

It has been previously mentioned that whether or not a unique minimum exists is a question of considerable design importance. The simulation studies provide evidence that a unique minimum does indeed exist. Figure 5.7 displays a number of initial parameter guesses, P_0 , to initiate the optimization procedure. For the penalty weights given in Table 5.7, the optimum $P=P^*$ is clearly shown in the figure. Observe that initial conditions 1-8, 12-14 as indicated in Table 5.7, all converge to P^* . These points (at least for control gain J_0) form a closed region around P^* . A sampling of initial conditions, P_0 , outside this region (9-11 for example) however, fail to converge to P^* . In fact, these points fail to converge to any location.

Thus, it is concluded that there exists a closed, bounded region R_0 of initial conditions P_0 such that convergence to a unique $P=P^*$ exists. For all P_0 outside R_0 , convergence to any $P \neq P^*$ does not exist.

5.3.3 Effects of Disturbance

The primary objective of the investigation was to determine whether any significant gain in design is achieved by modeling existing system parameter uncertainties as extremum disturbance vectors. Simulation results, shown in Fig. 5.8, display the effects of added disturbance gain $\hat{G}$ (Eq. 3.8) to the optimization computation. In this example the penalty Y was set to unity and the disturbance was varied utilizing $\hat{G}$. The results indicate that for convergence to a unique solution P^* to occur, the optimization procedure may tolerate only extremely small magnitudes of $\hat{G}$ (see Table 5.8). For those cases in which convergence did occur, a slight stabilizing effect in the closed loop pole positioning is observable. However, increased

disturbance drives the observer/controller pole to the right. For disturbance gains $\hat{G} > G_{\max}$, where $G_{\max}$ is determined experimentally, divergence occurs. Table 5.8 indicates the allowable range of $\hat{G}$ to insure convergence. Also presented is the estimated range of allowable parameter variations as determined from Eqs. 4.9, 4.10. Little significant difference is observed between the disturbance free cases and those simulations for which disturbance was included. Since the weights Q_y , Q_z are relatively small in comparison to Q_1 , Q_2 and R , it is not unreasonable that the resulting bound $||\delta\hat{B}||$ is computed to be approximately zero.

TABLE 5.1
EFFECT OF Q_{11} VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1		Q_2	Q_y	Q_z	R	P^*			λ^{*CLS}	λ^*_0	J	Time	ϵ
	1	10					j_1	j_2	k					
1	1	10	1.0	0.1	0.01	1.0	-1.1	-6.1	-1.7	-2.57+ j 1.31	-1.38	19.34	1.05	10^{-19}
2	10						-1.4	-6.4	-1.8	-2.69+ j 1.99	-1.69	25.73	1.19	10^{-19}
3	50						-4.3	-7.3	-2.0	-3.14+ j 2.82	-2.13	41.85	2.07	10^{-19}
4	100						-6.5	-8.0	-2.1	-3.51+ j 3.27	-2.33	55.47	2.47	10^{-19}
5	250						-11.1	-9.4	-2.3	-4.21+ j 3.99	-2.56	85.09	4.00	$\approx 10^{-17}$
6	500						-15.3	-10.8	-2.5	-4.93+ j 4.39	-2.98	121.93	4.00	$\approx 10^{-12}$
7	1000						-15.2	-12.3	-3.5	-5.67+ j 3.54	-4.93	180.08	4.00	$\approx 10^{-9}$
8	5000						-19.9	-16.2	-6.1	-7.60+ j 0.53	-10.12	480.73	15.00	$\approx 10^{-7}$
9	10000						3.7	-17.9	-13.9	-7.7, -16.17	-25.85	1032.11	4.00	$\approx 10^{-3}$

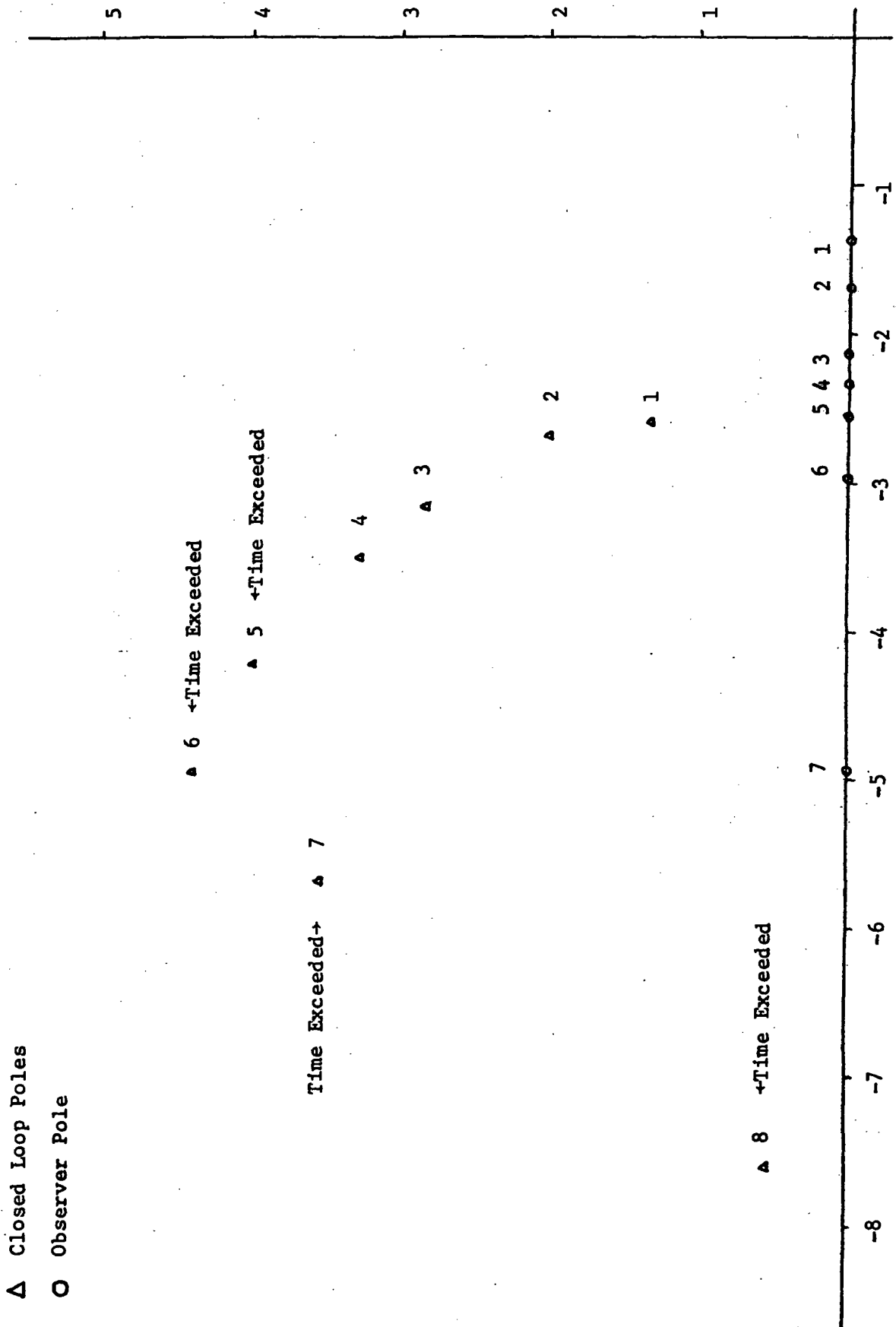


Figure 5.1 Effect of Q_{11} Variation On Closed Loop Dynamics

TABLE 5.2
EFFECT OF Q_{12} VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1	Q_2	Q_y	Q_z	R	P^* $j_1 \quad j_2$	λ^*_{CLS}	λ^*_0	J	Time	ϵ
1	100	0.1	1.0	0.1	1.0	-6.6 -7.2 -2.2	-3.09 $^{+1}_{-1}$ 13.57	-2.36	46.82	2.01	10^{-19}
2		1.0				-6.6 -7.3 -2.2	-3.14 $^{+1}_{-1}$ 13.54	-2.36	47.63	2.02	10^{-19}
3		10.0				-6.5 -8.0 -2.2	-3.52 $^{+1}_{-1}$ 13.27	-2.33	55.47	2.27	10^{-19}
4		20.0				-6.5 -8.8 -2.1	-3.90 $^{+1}_{-1}$ 12.93	-2.31	63.87	2.32	10^{-19}
5		35.0				-6.4 -9.8 -2.1	-4.42 $^{+1}_{-1}$ 12.28	-2.29	76.05	3.11	10^{-19}
6		50.0				-6.2 -10.7 -2.2	-4.88 $^{+1}_{-1}$ 11.19	-2.31 ¹	87.86	3.52	10^{-19}
7		100.0				-4.4 -13.1 -2.3	-2.49, -9.62	-2.66 ¹	121.62	4.00	$\approx 10^{-15}$
8		500.0				0.7 -22.3 -2.7	-1.13, -20.18	-3.36 ¹	401.70	4.00	$\approx 10^{-8}$

¹ still decreasing

Δ Closed Loop Poles
 O Observer Pole

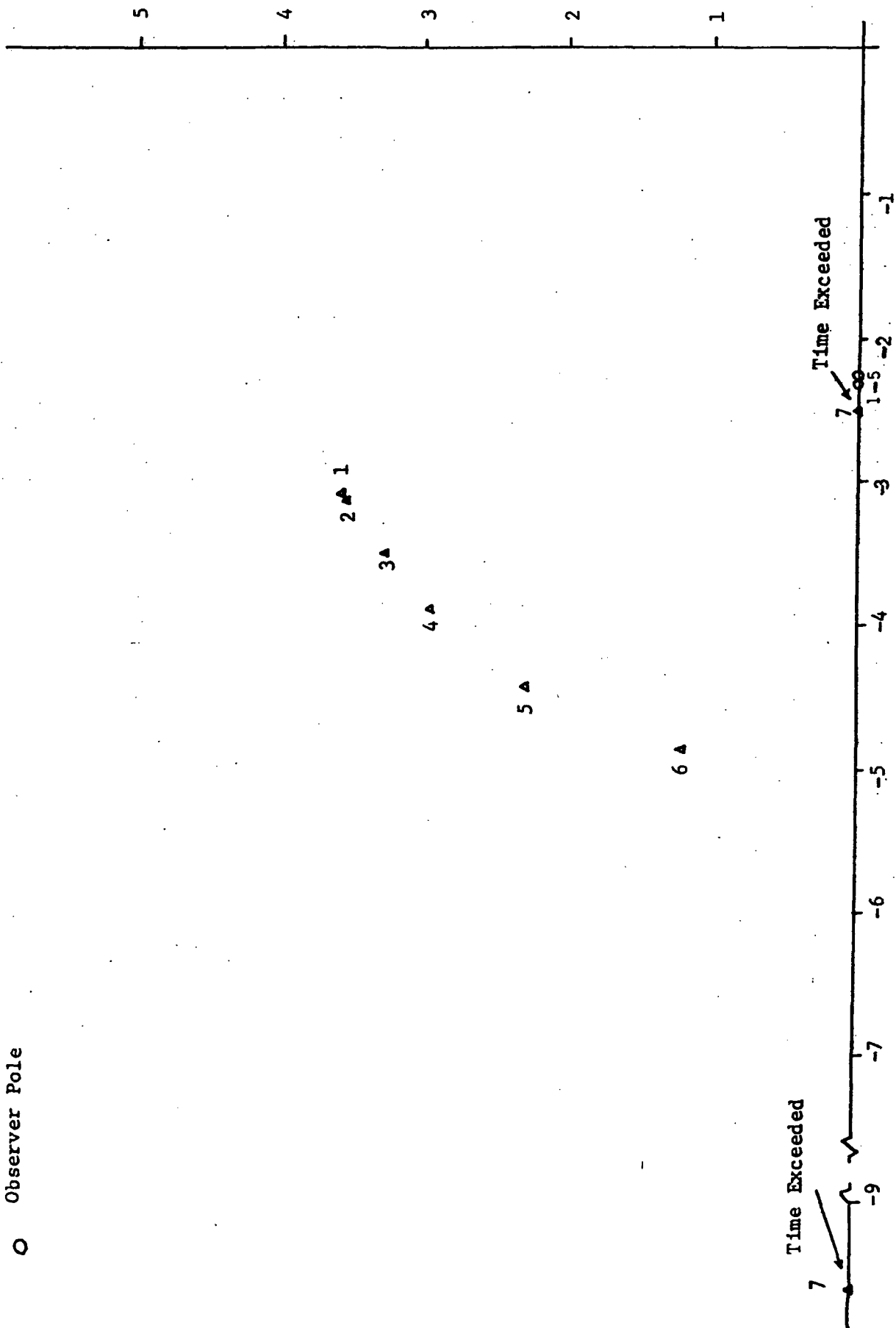


Figure 5.2 Effect of Q_{12} Variation On Closed Loop Dynamics

TABLE 5.3
EFFECT OF R VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1	Q_2	Q_y	Q_z	R	P^* $j_1 \quad j_2 \quad k$	λ^*_{CLS}	λ^*_0	J	Time	ϵ
1	150	25	1.0	0.1	0.01	10.0	-1.0 -5.2 -2.2	-2.09+j2.18	206.12	2.08	10^{-19}
2						5.0	-2.3 -6.2 -2.2	-2.59+j2.47	142.93	2.31	10^{-19}
3						1.0	-8.1 -9.6 -2.2	-4.30+j3.07	79.14	3.40	10^{-19}
4						0.8	-9.4 -10.3 -2.2	-4.63+j3.10	73.53	3.13	10^{-19}
5						0.6	-11.2 -11.2 -2.2	-5.89+j2.90	67.25	3.33	10^{-19}
6						0.4	-14.2 -12.8 -2.2	-7.63+j0.63	59.89	4.00	$\approx 10^{-16}$
7						0.2	-20.2 -16.3 -2.3	-4.74, -15.5	50.45	4.00	$\approx 10^{-13}$
8						0.1	-25.1 -21.2 -2.6	-1.27, -40.1	43.93	4.00	$\approx 10^{-10}$
9						0.01		-9.68	38.75	1.20	$\approx 10^{-3}$

Δ Closed Loop Poles

O Observer Pole

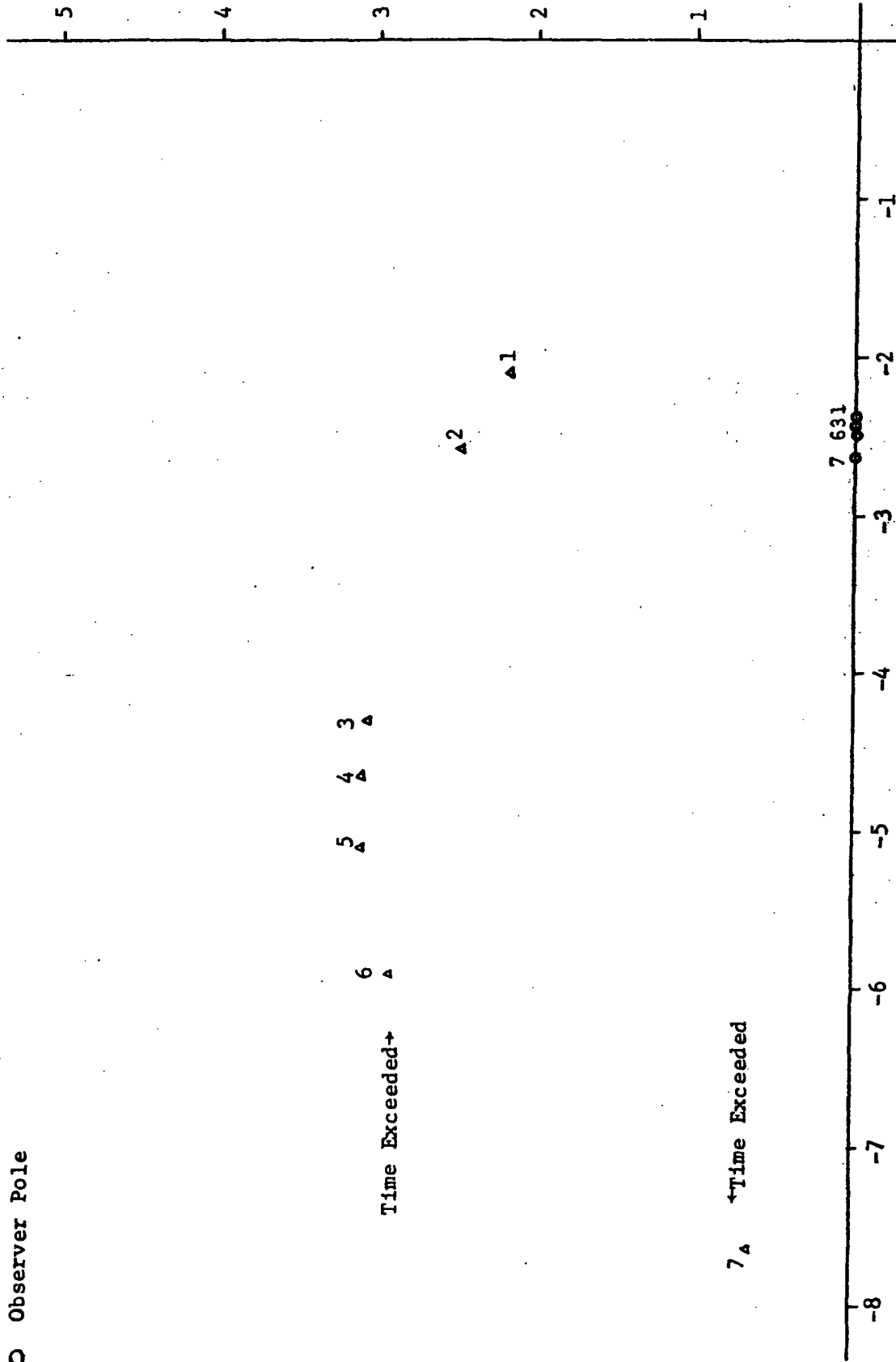


Figure 5.3 Effect of R Variation On Closed Loop Dynamics

TABLE 5.4
EFFECT OF Q_y VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1	Q_2	Q_y	Q_z	R	P^* $j_1 \quad j_2 \quad k$			λ_{CLS}^*	λ_0^*	J	Time	ϵ
1	150	25	1.0	.01	1.0	-7.7	-9.6	-2.3	-4.30±12.94	-2.53	79.04	3.55	10^{-19}
2			.1			-8.1	-9.6	-2.2	-4.30±13.07	-2.42	79.14	3.40	10^{-19}
3			.5			-9.1	-9.6	-2.1	-4.30±13.37	-2.17	79.51	3.06	10^{-19}
4			1.0			-9.8	-9.6	-2.0	-4.28±13.60	-2.00	79.94	2.31	10^{-19}
5			1.5			-10.3	-9.5	-1.9	-4.26±13.75	-1.89	80.34	2.25	10^{-19}
6			2.0			-10.7	-9.5	-1.9	-4.24±13.87	-1.80	80.73	1.56	10^{-19}
7			5.0			-12.3	-9.3	-1.8	-4.15±14.31	-1.52	82.76	1.16	10^{-19}
8			10.0			-13.6	-9.1	-1.6	-4.05±14.68	-1.30	85.67	1.22	10^{-19}
9			25.0			-15.8	-8.8	-1.5	-3.89±15.23	-1.02	92.95	1.44	10^{-19}

▲ Closed Loop Poles
○ Observer Pole

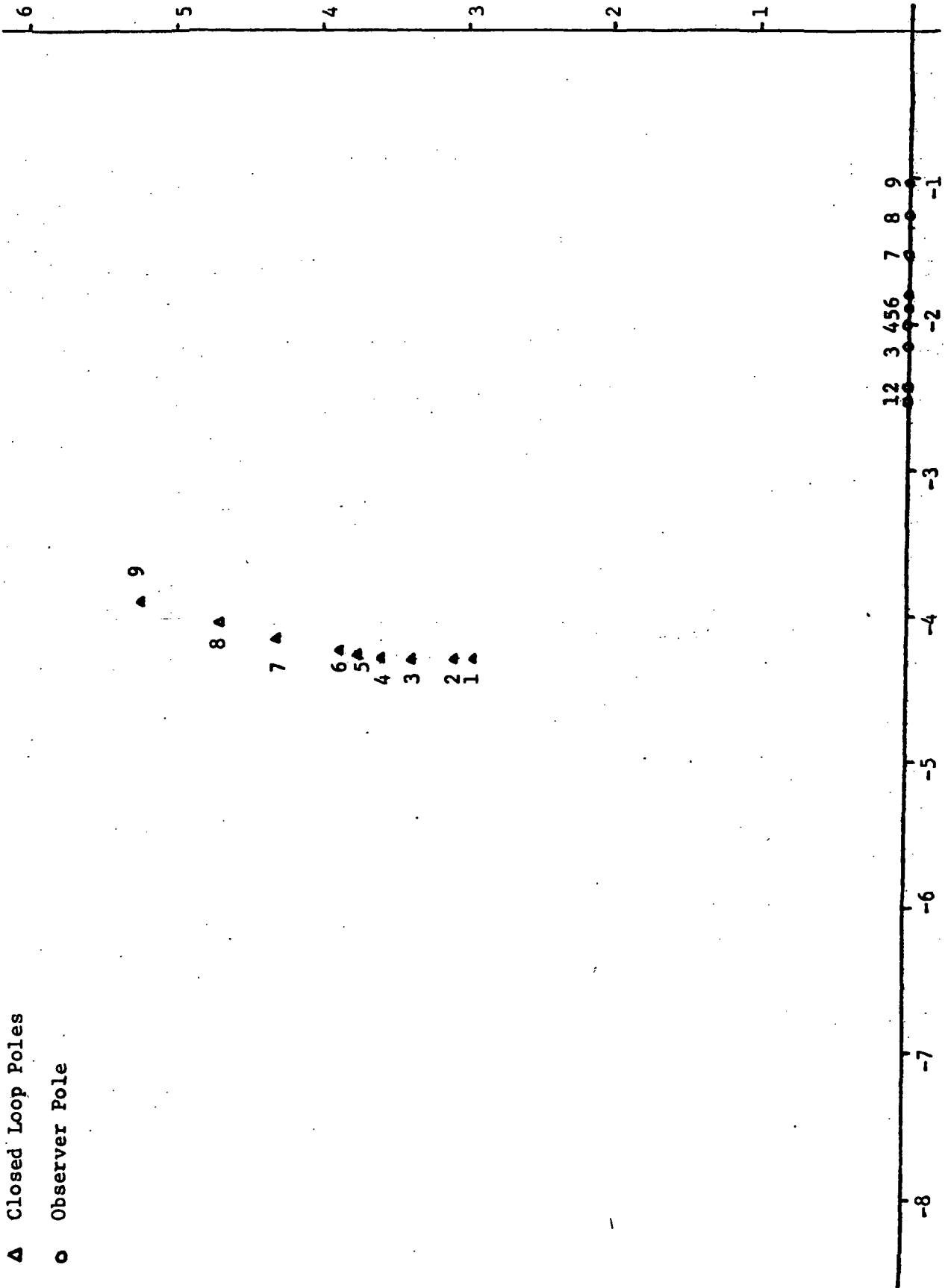


Figure 5.4 Effect Of Q_y Variation On Closed Loop Dynamics

TABLE 5.5
EFFECT OF Q_z VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1		Q_2	Q_y	Q_z	R	P^* $j_1 \quad j_2$		λ^{*}_{CLS}	λ^*_0	J	Time	ϵ	
1	150	25	1.0	0.1	0.001	1.0	-6.2	-9.4	-2.5	-4.21±12.48	-3.03	79.08	3.34	10^{-19}
2					0.01		-8.1	-9.6	-2.2	-4.30±13.07	-2.42	79.14	3.40	10^{-19}
3					0.1		-10.1	-9.5	-1.9	-4.26±13.67	-1.92	79.71	2.33	10^{-19}
4					1.0		-12.8	-9.0	-1.6	-4.01±14.54	-1.26	82.60	2.25	10^{-19}
5					2.0		-13.7	-8.7	-1.5	-3.86±14.81	-1.05	84.63	2.20	10^{-19}
6					2.5		-13.9	-8.6	-1.5	-3.81±14.91	-0.99	85.47	3.00	10^{-10}

Δ Closed Loop Poles
 O Observer Pole

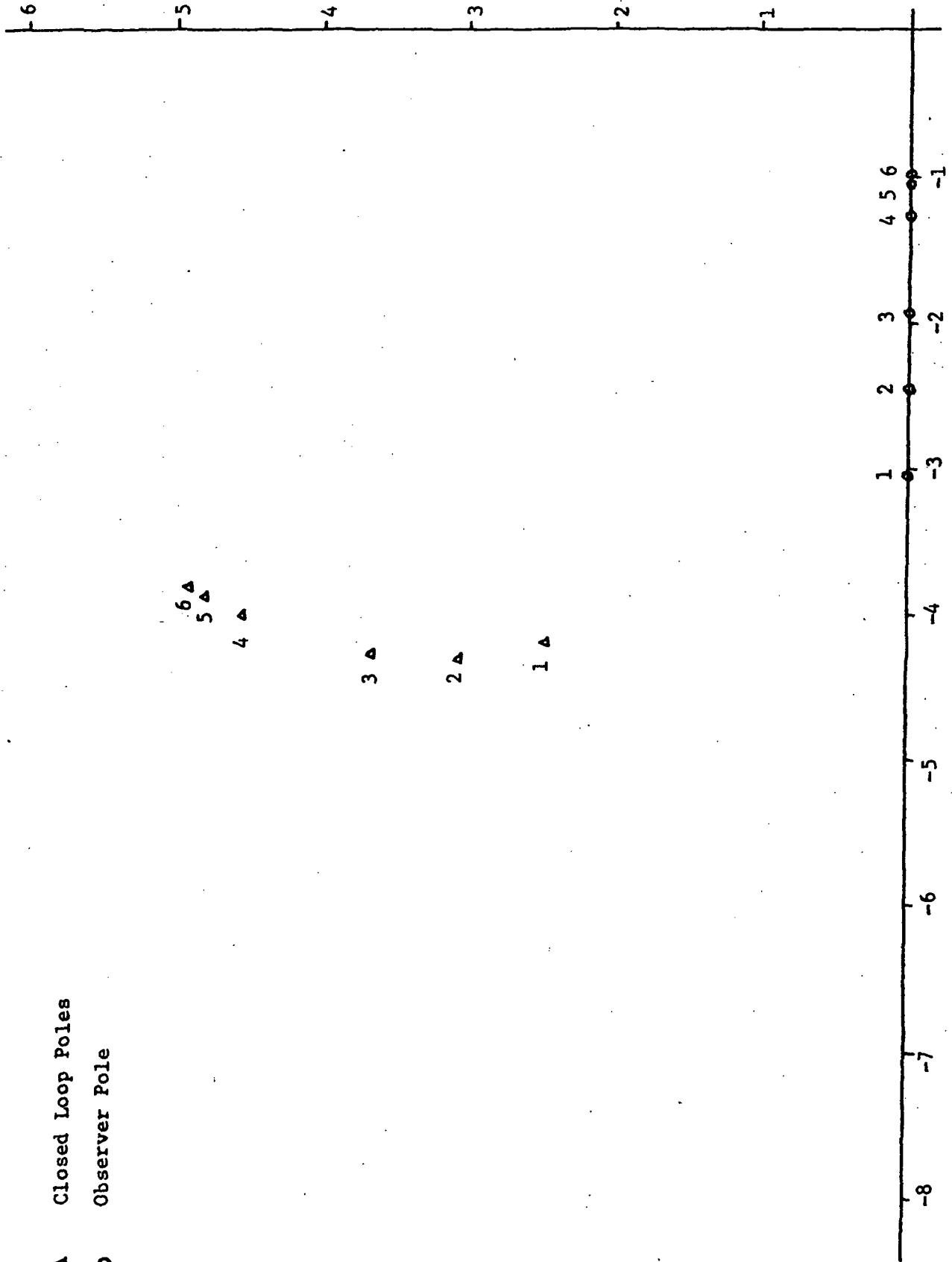


Figure 5.5 Effect Of Q_z Variation On Closed Loop Dynamics

TABLE 5.6
EFFECT OF Q_2 VARIATION ON CLOSED LOOP DYNAMICS

Run #	Q_1	Q_2	Q_y	Q_z	R	P*			λ_{CLS}^*	λ_0^*	J	Time	ϵ
1	150 25	.01	0.1	0.01	1.0	-8.3	-9.6	-2.2	-4.30±j3.11	-2.38	78.53	3.47	10^{-19}
2		.1				-8.3	-9.6	-2.2	-4.30±j3.10	-2.39	78.59	3.37	10^{-19}
3		1.0				-8.2	-9.6	-2.2	-4.30±j3.07	-2.42	79.14	3.40	10^{-19}
4		10.0				-7.5	-9.6	-2.3	-4.28±j2.86	-2.62	84.59	3.11	10^{-19}
5		50.0				-7.0	-9.5	-2.4	-4.26±j2.71	-2.76	108.74	1.21	10^{-19}
6		100.0				-6.9	-9.5	-2.4	-4.26±j2.68	-2.80	138.92	1.05	10^{-19}
7		1000.				-6.8	-9.5	-2.4	-4.25±j 2.66	-2.85	682.12	1.02	10^{-19}

Δ Closed Loop Poles
 $\circ$ Observer Poles

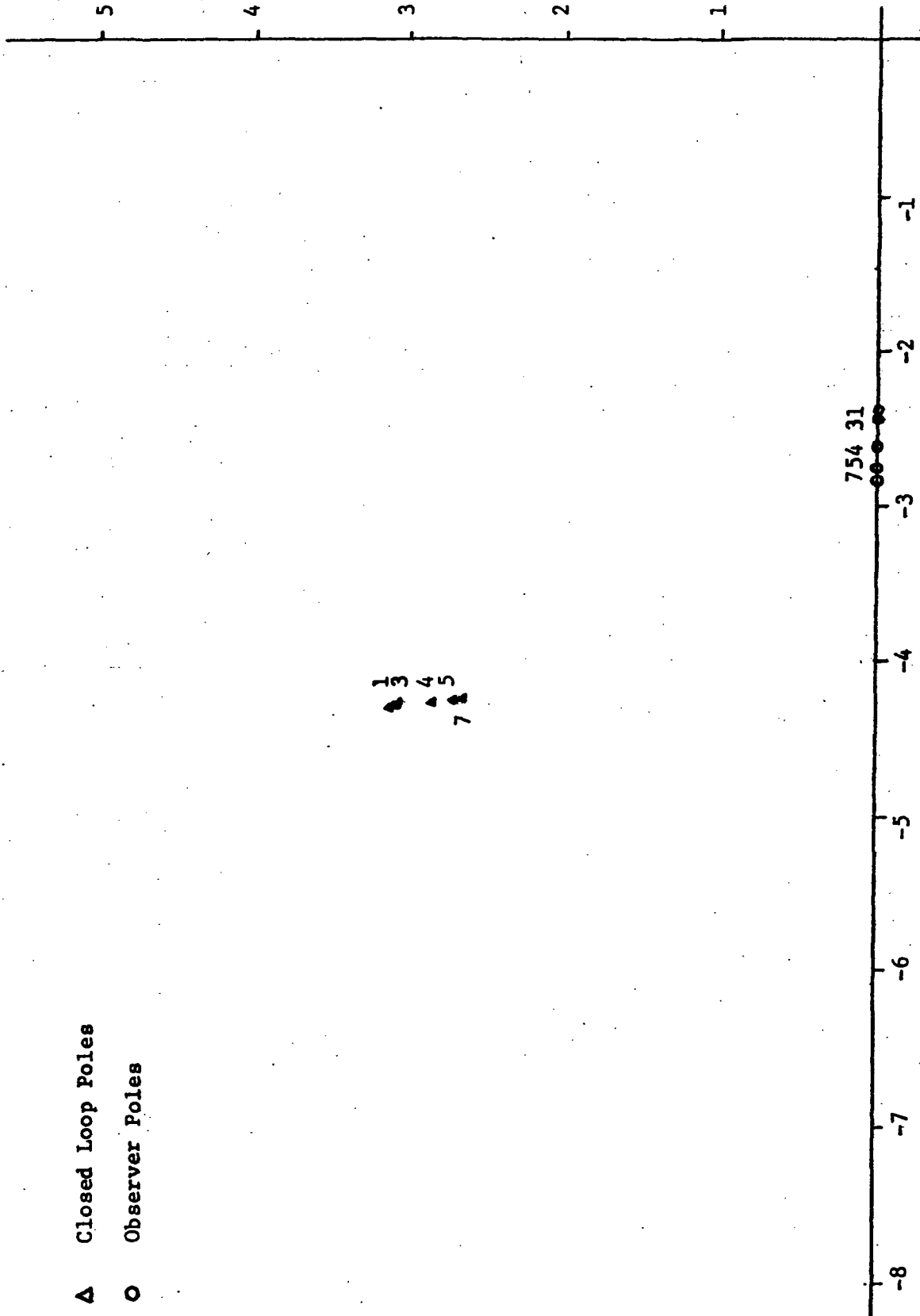


Figure 5.6 Effect Of Q_2 Variation On Closed Loop Dynamics

TABLE 5.7
REGION OF CONVERGENCE FOR VARIOUS INITIAL CONDITIONS P_0

Run #	Q ₁		Q ₂	Q _y	Q _z	R	Initial Conditions j ₁ j ₂ k			λ_s^{ic}	λ_0^{ic}	λ_{CLS}^*	λ_0^*	J	Time	ϵ	
1	150	25	1.0	0.1	0.01	1.0	1.0	-3.0	-2.0	-1.*j1.4	-2.0	-4.30* ~~j~~ 3.1	-2.42	79.14	3.40	10 ⁻¹⁹	
2							1.5	-2.5	-2.0	-7.*j.96	-2.0	-4.30* ~~j~~ 3.1	-2.42	79.14	3.22	10 ⁻¹⁹	
3							1.5	-2.0	-2.5	-5.* ~~j~~ .86	-3.0	-4.30* ~~j~~ 3.1	-2.42	79.14	3.28	10 ⁻¹⁹	
4							1.0	-2.5	-2.5	-7.* ~~j~~ 1.4	-3.0	-4.30* ~~j~~ 3.1	-2.42	79.14	3.23	10 ⁻¹⁹	
5							0.0	-4.0	-2.0	-1.5* ~~j~~ 1.9	-2.0	-4.30* ~~j~~ 3.1	-2.42	79.14	2.51	10 ⁻¹⁹	
6							-4.0	-5.0	-2.0	-2.* ~~j~~ 3.3	-2.0	-4.30* ~~j~~ 3.1	-2.40	79.14	3.07	10 ⁻¹⁹	
7							-3.0	-9.0	-2.0	-4.* ~~j~~ 1.0	-2.0	-4.30* ~~j~~ 3.1	-2.41	79.14	2.51	10 ⁻¹⁹	
8							-40.0	-21.0	-2.0	-10.* ~~j~~ 1.7	-2.0	-4.30* ~~j~~ 3.1	-2.40	79.14	3.07	10 ⁻¹⁹	
9							-10.0	-3.0	-2.5	-1.* ~~j~~ 4.9	-3.0	Fails to Converge					
10							-23.0	-3.0	-2.0	-1.* ~~j~~ 7.1	-2.0	Fails to Converge					
11							-64.0	-19.0	-2.0	-9.* ~~j~~ 8.2	-2.0	Fails to Converge					
12							-12.0	-9.0	-3.0	-4.* ~~j~~ 5.6	-4.0	-4.30* ~~j~~ 3.1	-2.40	79.14	2.48	10 ⁻¹⁹	
13							-26.0	-15.0	-2.0	-7.* ~~j~~ 4.5	-2.0	-4.30* ~~j~~ 3.1	-2.40	79.14	2.37	10 ⁻¹⁹	
14							-16.7	-15.0	-2.0	-7.* ~~j~~ 1.2	-2.0	-4.30* ~~j~~ 3.1	-2.40	79.14	2.46	10 ⁻¹⁹	

Fails to Converge
▲ 11

Fails to Converge
▲ 10

Fails to Converge
▲ 9

▲ Closed Loop Poles

○ Observer Pole

Optimum Pole X
Location

Optimum Compensator
Location

1, 2, 5, 6, 7, 8, 13, 14

Figure 5.7

Region Of Convergence For Various Initial Conditions

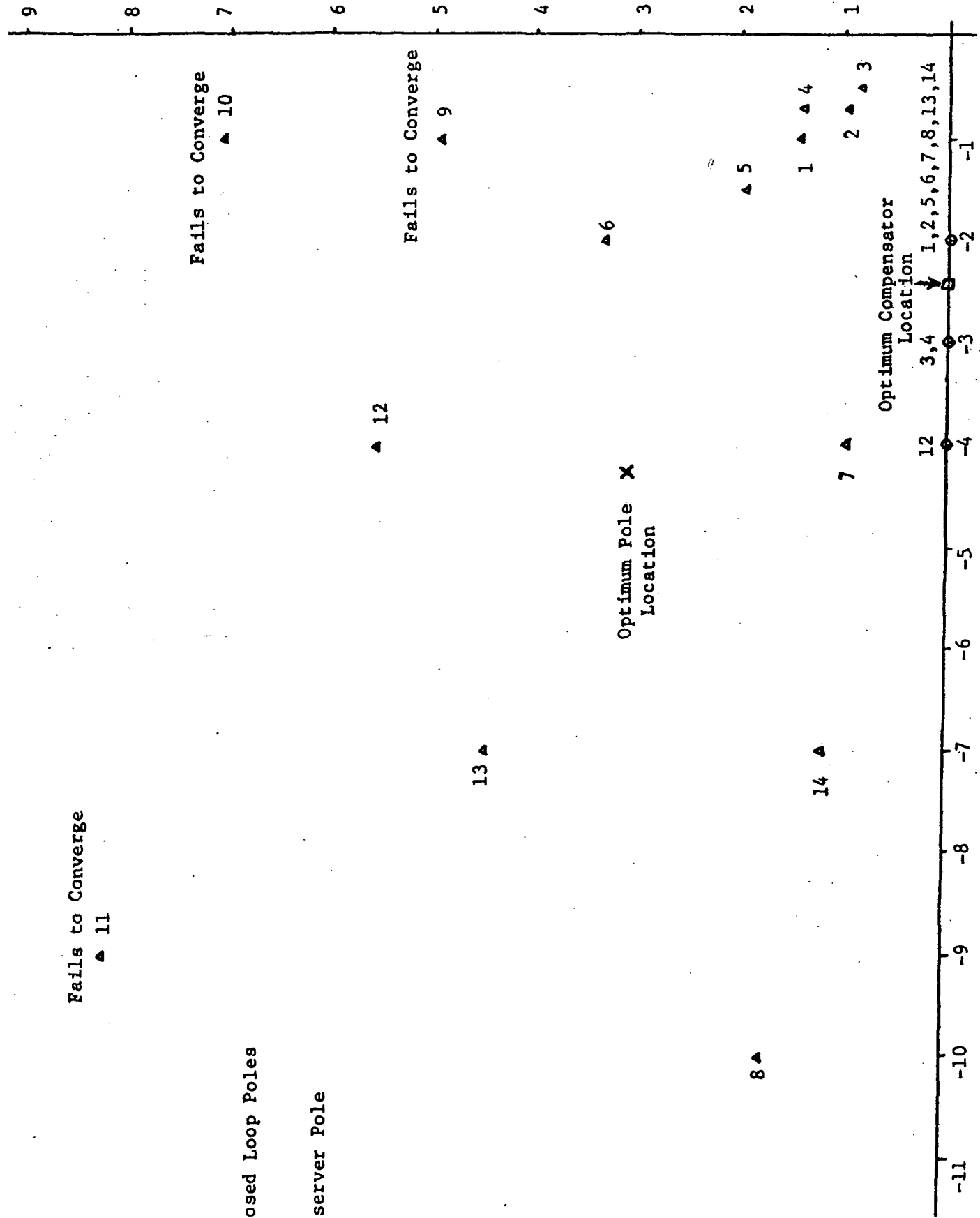


TABLE 5.8
EFFECT OF DISTURBANCE ON CLOSED LOOP DYNAMICS

Run #	Q_1	Q_2	Q_y	Q_z	R	Noise $q_1 \quad q_2 \quad q_3$			λ^*_{CLS}	λ_0	J	ϵ	$\% \ \delta A\ $
1	100	10	1.0	.05	.001	2.0	.0	.0	.0	-2.81	12.65	10^{-19}	7.66
2							.05	.05	.05	-2.86	12.78	10^{-19}	8.37
3							.10	.05	.05	-2.95	13.04	10^{-19}	7.91
4							.1375	.05	.05	-3.05	13.21	10^{-19}	7.41
5							.15	.05	.05	Diverges			
6							.1375	.075	.075	Diverges			

△ Closed Loop Poles

○ Observer Pole

△⁴
△³
△²
△¹

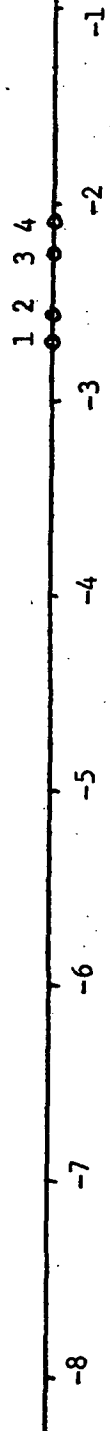


Figure 5.8 Effect Of Disturbance On Closed Loop Dynamics

VI. CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

A new algorithm has been developed for limited state feedback control system design. The recursive computation procedure presented in this report simultaneously computes a full state time-invariant feedback control law and the dynamics of a minimal order state observer. In contrast to previous work, this investigation has attempted to extend existing design techniques to account for system parameter ignorances.

Chapter II presents the background necessary to formulate a design approach. It was also shown that in circumstances where the effects of parameter uncertainties cannot be ignored, it is not possible to draw any accurate conclusions regarding the stability properties of the closed loop dynamical system.

Chapter III presents the major development of this investigation. To account for the possible parameter ignorances inherent in the system, the problem is formulated as a minimax control design. The theoretical development is shown to result in a set of algebraic matrix equations. A recursive computation procedure is proposed to obtain a solution to the resulting matrix equations.

The results of Chapter III are utilized in Chapter IV to derive a region of guaranteed asymptotic stability for the overall closed loop dynamical performance. The derived bounds are shown to be limits on the allowable variation in the norms of matrices $\{A, B\}$.

To obtain some measure of the merits of the design, the theoretical results were applied to a problem discussed in Chapter V. Several preliminary conclusions were drawn from the computer simulation studies, the major result being that the roots of the closed loop dynamical system, $A_0 + B_0 J$, may be arbitrarily positioned through adjustment of the penalty functions in the original optimal control cost functional. Such freedom of pole placement in observer design is not nearly as evident. In general, the dynamics of the observer (compensator) root locations appear dependent on the placement of the closed loop system poles, one explanation for this being the high degree of coupling in the state and observer equations,

A third major result demonstrated in the simulation studies is the fact that a unique minimum exists. It was observed that initial conditions P_0 which did not result in convergence to the optimal solution did not converge to any solution. Thus it was concluded that a closed bounded region R_0 of initial conditions P_0 exist such that convergence to a unique $P = P^*$ occurs.

The major objective of this study was to determine whether any advantage was gained by modeling the parameter uncertainty inherent in the system by extremum disturbances. Simulation results provide evidence that very little uncertainty may be tolerated before divergence occurs in the recursive computation algorithm. For those magnitudes of uncertainty for which convergence did result, some slight stabilizing effect in the closed loop pole placement was observed. The dynamics of the observer, however, showed a shift to the right. Finally, the stability bound derived in Chapter IV yields extremely conservative estimates of regions of allowable parameter variation.

The fact that the latter results suggest relatively little gain by modeling uncertainties as extremum disturbances does not place severe restrictions on the resulting design algorithm. Rather the results indicate that alternate methods of solutions to Eqs. 3.38-3.40 should be investigated in the expectation of permitting higher disturbance levels. This recommendation is deferred to the following section.

6.2 Recommendations

The treatment presented in this report is by no means complete. Based on both the simulation results of Chapter V and the theoretical results of the previous chapters, several areas of investigations are suggested for possible research studies:

- a) The recursive algorithm described by Eqs. 3.41-3.43 has been shown to converge monotonically to the optimum set of feedback and compensator parameters P^* . However, the convergence times were shown to be excessively long. Techniques should be investigated such that convergence times may be significantly reduced.
- b) The addition of disturbance gain into the algorithmic solution has been demonstrated that only small magnitudes of disturbance are tolerable for convergence to result. It is suggested that alternate methods of solution to Eqs. 3.41-3.43 be investigated to determine whether the disturbance magnitude may be increased. One possible approach is to solve the entire set of non-linear equations, Eqs. 3.41-3.43, simultaneously as:

$$\min L_{n+1} = \frac{1}{2} \sum_{L=1}^{(qxn)+(n)x(n-m) + (n+m)x(n+m+1)} f_i(P, X)$$

When $L_{n+1} < \epsilon_{n+1}$, set $\epsilon_{n+2} = \frac{\epsilon_{n+1}}{2}$ and repeat the iterative process until the desired value of L_{n+j} is achieved.

c) Simulation of a more complex example, such as a 3rd or 4th order system, should provide further insight into the applicability of the design technique. One primary consideration should be to consider the design of multiple observers.

d) The design technique proposed in this investigation has been limited to linear, time-invariant, deterministic systems. It is suggested that the design be extended to account for random disturbances including state and measurement noises. Due to the nature of the state equation formulation, random measurement noise would generally be modeled as a Gauss-Markov process which leads to an argumentation in the state equations. However, this should not present any additional complexities other than a redefinition of the compatibility portion matrices.

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