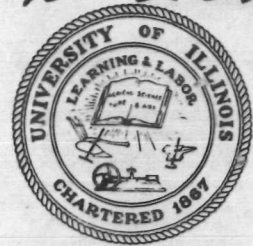


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A STUDY OF NOZZLE AND EJECTOR FLOW PROBLEMS BY THE METHOD OF INTEGRAL RELATIONS

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by

L. D. HOWLETT and W. L. CHOW

Prepared Under

Research Grant NASA NGL 14-005-140

FLUID DYNAMIC AND HEAT TRANSFER PROBLEMS
ASSOCIATED WITH AIR BREATHING PROPULSIVE SYSTEMS

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Engineering Experiment Station
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FOREWORD

This research was carried out under Research Grant NASA NGL 14-005-140 entitled "Fluid Dynamic and Heat Transfer Problems Associated with Air Breathing Propulsive Systems." This report presents a method dealing with flow within nozzles or ejector systems based on the method of integral relations.

ABSTRACT

The application of the method of integral relations to nozzle and ejector flow problems has been examined. For nozzle flow problems, the formulation is general that the approaching flow may be rotational. Particular attention has been given to the phenomenon of choking under non-uniform flow conditions. Numerical integration of the governing ordinary differential equations has also been investigated. This scheme of analysis has also been applied to study the interacting flow field within an ejector system.

NOMENCLATURE

a,b,c	polynomial coefficients
A	nozzle flow area
A,B,C, D,E,F	coefficients of ordinary differential equations
I,J	indices
N	coordinate conversion factor ($N = 0$ for two-dimensional plane flow, $N = 1$ for axisymmetric)
$\dot{m}$	dimensionless mass flux
M	Mach number
M*	u/c^* , c^* , speed of sound at $M = 1$
P	pressure
q	dimensionless velocity magnitude, $q = (u^2 + v^2)^{1/2} / V_m$
R	throat wall radius of curvature, normalized by nozzle throat wall half height y_t , ($y_t = 1$)
$\bar{R}$	gas constant
T_o	stagnation temperature
u	velocity in x-direction
v	velocity in y-direction
V_m	theoretical maximum velocity, $V_m = [(2\gamma\bar{R} T_o)/(\gamma - 1)]^{1/2}$
x	axial coordinate, normalized by the nozzle throat height
y	normal (radial) coordinate, normalized by the nozzle throat height
α	Mach angle
γ	ratio of specific heats, $\gamma = 1.4$ for air
η	dimensionless normal (radial) coordinate
θ	streamline angle
λ	limiting value of coefficients of ordinary differential equations evaluated at the nozzle throat
ρ	density

Subscripts

c	evaluated at the nozzle axis or inner streamline
o	stagnation conditions
st	evaluated along a streamline
t	evaluated at the nozzle throat
w	evaluated at the nozzle outer wall

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1. INTRODUCTION

Since flow through nozzles have wide industrial applications, a large amount of work has been directed toward the mathematical solution of nozzle flow fields. Experimental and theoretical investigations have been made about the behavior of nozzle flows for more than one hundred years. Even with this vast amount of work already applied toward solving this problem, no mathematical solution of the nonlinear governing equations has yet been found for the complete nozzle flow field.

Several solutions for the two-dimensional nozzle flow problem have been published. The basic procedure employed in most of these solutions is first to simplify the governing equations by assuming an inviscid, irrotational flow field and then express the velocity distribution as a power series. For a given wall shape, the unknown coefficients of the series expression for the velocities may be obtained. Alternatively, for a given velocity distribution (say along the nozzle centerline) a corresponding bounding streamline may be determined.

The approach presented in this study is to develop the method of integral relations into a form useful for solving two-dimensional (or axisymmetric) nozzle problems. The method of integral relations reduces the governing nonlinear partial differential equations into a system of ordinary differential equations. These ordinary differential equations may then be solved by numerical integration procedures provided proper initial values of the unknown variables are known.

In application of the method of integral relations to investigate supersonic flow past a blunt body, the unknown dependent variable is the

shock stand off distance. A proper value of the initial shock stand off distance produces physically realistic velocities near the location of the body sonic point. Since this scheme of analysis has been successful in predicting both shock shapes and velocity distributions for such flow problems, it is discussed in more detail in the following paragraph.

The method of integral relations as applied to find the solution of nonlinear fluid dynamics differential equations was introduced by Dorodnitsyn and presented at the Ninth International Congress of Applied Mechanics at Brussels in 1957. Belotserkovskii developed this general method to solve the supersonic flow of a perfect, inviscid gas over a blunt body. The basic approach of this method as presented by Hayes and Probstein [1]*, and others, is to approximate the variation of the dependent variables (u, v) by interpolating polynomials. Upon suitably arranging the governing equations, and integrating along the normal coordinate, the nonlinear partial differential equations reduce to a set of nonlinear, first order ordinary differential equations in the stream-wise direction. These equations can be conveniently programmed on a computer, and may be solved basically as the classical initial value problem. However, the solution still behaves as a two-point boundary value problem, since initial values must be chosen and iterated upon to produce meaningful results at locations where the velocity becomes sonic. This scheme of analysis thus illustrates the elliptic character of the flow field.

The integral-relation technique is an approximate solution as are all integral methods since details of the flow field are not obtained, but rather overall characteristics are determined with the method. It

*Numbers in brackets refer to entries in REFERENCES.

is also an approximation in the sense that variations of flow properties in the body oriented normal direction are specified by interpolating polynomials. Higher order approximations may be obtained by using more terms of the polynomials or by dividing the flow field into multi-strips. As would be expected, each method of higher approximation adds complications to the solution.

Traugott [2] presented the solution for flow over a blunt body using a one strip approximation with first a linear variation, then a cubic variation of flow properties in the normal direction. His first approximation yielded good values of the shock shape but resulting in poor prediction of the pressure coefficient near areas where the body exhibited sharp changes of curvature. His second approximation produced better results than the linear approximation for both pressure coefficient and shock shape. In a second paper [3], he described some of the phenomena of blunt body flows and showed that the integral method failed when a shock was present in the flow field and when the flow became over expanded.

In a critical review of the blunt body method, Xerikos and Anderson [4] pointed out the complexities which arise in the numerical integration scheme. From calculations of their own, they found that the initial estimation of the shock stand-off distance must be made accurate to more than the eighth decimal place in order to carry the solution past the sonic singular point into the supersonic flow region. They questioned the applicability of the integral method in the supersonic region, however, since the integral method essentially propagated wall disturbances along the normal body oriented coordinate instead of along characteristic lines of the flow field.

By applying the integral method to the flow region between an oblique shock and a pointed body of revolution, South and Newman [5] have shown that the integral-relation method is self-compensating for this incorrect zone of influence since their calculations for shock curvature and pressure coefficient agree well with exact results predicted by the method of characteristics. For a more blunted body, their two strip solution showed some disagreement with the method of characteristics.

A large amount of effort has been directed toward solving the nozzle problem. As noted in a 1964 survey paper by Hall and Sutton [6], the first significant attempts to understand the steady flow of a compressible fluid through a duct or a nozzle were published by Navier in 1829 and by St. Venant and Wantzel in 1831. Since that time, there has been a considerable amount of effort direct toward understanding the flow of compressible fluids through ducts. Theoretical solutions have been developed for both the low subsonic and the supersonic flow regimes which compare well with experimental results. The transonic regime of the nozzle flow field remains the most difficult region for a theoretical solution due to the basic nature of the governing equations. It also is the most important region of the flow field since the choking flow rate is determined by the transonic throat conditions.

The behavior of the one-dimensional solution has been examined in depth by Shapiro and Hawthorne [7]. Their well known work includes the effects of area variation, friction, heat exchange, and chemical reaction on the flow field behavior. Although the one-dimensional theory shows the general characteristics of the nozzle flow field, it is

inadequate for regions where the area changes rapidly such as the geometric throat.

Even before this detailed investigation of the one-dimensional theory, Meyer published the first indirect two-dimensional solution to the nozzle problem. By specifying the velocity distribution along the axis, he was able to calculate velocity profiles and stream line shapes. In 1930, Taylor published the first direct solution to the nozzle problem. After expanding the velocity potential into a double series, he evaluated the series coefficients in terms of the nozzle wall shape. Oswatitsch and Rothstein [8] published a less complex solution to the nozzle problem by considering the two-dimensional solution as a perturbation of the one-dimensional solution.

Hall [9] improved the direct solution technique by solving the perturbed one-dimensional problem in a stretched coordinate system.

Since 1964, there have been additional improvements of the two-dimensional solution techniques. Most of the work has been aimed at obtaining a direct solution since these are more valuable for design work. Due to the need of practical applications, these studies were more concerned with nozzles with small radii of curvature at the throat.

Hopkins and Hill [10] developed an indirect solution for nozzles with small throat radii of curvature. They expanded the coordinates and the velocity magnitude and direction as fourth order polynomials in terms of the stream function. After specifying the centerline velocity distribution, they were able to calculate both normal velocity distributions and streamline shapes. Their analytical results agreed well with data by Bach, Massier, and Gier [11] taken for conical

nozzles with small throat radius of curvature.

Shelton [12] utilized Hopkins and Hill's method to obtain a direct solution to the nozzle problem. He assumed the centerline velocity to be in polynomial form, and determined the polynomial coefficients by an iteration process in which he compared the predicted nozzle wall shape with the specified wall shape. Kliegel and Quan [13] developed a direct solution to the nozzle problem using a procedure similar to that of Hall. By choosing a different normalizing factor for their coordinate transformation, their results were valid for all (subsonic, transonic, and supersonic) flow regimes.

Thompson [14,15] developed a solution for transonic flow in a curved channel. He made use of the integral continuity relation to evaluate the choking conditions of the flow.

The first attempt to apply the method of integral relations to the nozzle problem was made by Holt [16]. He did not use the natural boundary conditions of symmetry about the nozzle centerline.

Liddle and Archer [17] in 1971 solved the nozzle problem by the method of integral relations. They used the conditions of irrotational flow and have made use of the symmetry conditions in the nozzle flow field. In light with the true integral relations method, they developed a multiple strip solution. By comparison with experimental data, they concluded that the single strip solution yields valid results for nozzles with large radii of curvature, and the two-strip solution produces better results for nozzles with small throat radii of curvature.

There are several different ways of determining the choking conditions for the previously mentioned nozzle solution techniques. First, for the one-dimensional solution, the choking conditions are set at the

location where the velocity becomes sonic. Since the mass flux per unit area and hence total nozzle mass flux is maximum at this condition, the choking condition is also known as the condition of maximum mass flux. For the indirect solutions, such as those developed by Meyer [6] and Hall [9], the choking case arises naturally from the solution for a constantly increasing centerline velocity distribution. There are several methods used to specify choking for the direct solutions to the nozzle problem. Oswatitsch and Rothstein's [8] solution to the nozzle problem defined the choking condition in terms of behavior of certain integral curves. The integral curve which allowed a continuous finite acceleration through the throat region corresponded to the choked condition. They showed that this curve was dependent upon a proper choice for the initial velocity value.

Kliegel and Quan [13] defined choked conditions from equations which became singular at the geometric throat. Thompson [14,15] used the integral continuity relation. He noted that there are two methods for specifying the choking condition. One method may be derived from the mathematical singular condition at the throat and the other which he termed "the physical condition" was obtained by specifying choking as the maximum mass flow condition. The two results are nearly the same, however. Liddle and Archer [17] specified the physical or maximum flow rate as the choking condition.

The present investigation presents a solution to the nozzle problem developed with the method of integral relations. This solution differs from that presented by Liddle and Archer in several ways.

In the derivation of the governing relations, the present work will not be restricted to irrotational flow; thus, the analysis may

readily be applied to flow cases such as by pass jet engine nozzle flows, where the stagnation conditions of the flow are generally not uniform. Furthermore, the main emphasis of this part of the work is directed toward understanding the choking condition. Since in all practical flow situations nozzle entrance section does not have a smooth profile, the starting of calculations within converging section of the nozzle must also be investigated.

One of the early motivations for this investigation was to develop an improved method over the one-dimensional simplified description of the secondary stream within an ejector flow field. Thus, this scheme of analysis by the method of integral relations was subsequently applied to study this interaction problem. The solution was also evaluated and compared against previous analytical results and experimental data.

2. THE NOZZLE PROBLEM

2.1 DEVELOPMENT OF THE INTEGRAL RELATION

Consider the two-dimensional or axisymmetric inviscid compressible flow within a nozzle with arbitrary wall shape $y_w(x)$. The approaching flow may have non-uniform stagnation pressures and temperatures but constant ratio of specific heat γ and gas constant R . The conservation principles may be written as

$$\frac{\partial \rho u y^N}{\partial x} + \frac{\partial \rho v y^N}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial p}{\partial y} + \frac{N}{y} \rho v^2 = 0 \quad (2)$$

where $N = 0$ for two-dimensional flow and $N = 1$ for axisymmetric flow.

It is understood that the flow may be non-homentropic but is isentropic along streamlines.

We now integrate Eqs. (1) and (2) between two streamlines $y_w(x)$ and $y_c(x)$ as shown in Fig. 1 and obtain

$$\int_{y_c}^{y_w} \frac{\partial \rho u y^N}{\partial x} dy + \int_{y_c}^{y_w} \frac{\partial \rho v y^N}{\partial y} dy = 0 \quad (3)$$

$$\int_{y_c}^{y_w} \left[\frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{N}{y} \rho v^2 + \frac{\partial p}{\partial y} \right] dy = 0 \quad (4)$$

Using Leinetz's rule for differentiating under the integral sign given by

$$\begin{aligned} \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} dy &= \frac{d}{dx} \int_{a(x)}^{b(x)} f(x,y) dy - f(x,b) \frac{db}{dx} \\ &\quad + f(x,a) \frac{da}{dx} \end{aligned} \quad (5)$$

and carrying out the integration, we obtain

$$\begin{aligned} \frac{d}{dx} \int_{y_c}^{y_w} \rho u y^N dy - (\rho u y^N)_{y_w} y'_w + (\rho u y^N)_{y_c} y'_c \\ + \rho v y^N \Big|_{y_c}^{y_w} = 0 \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{d}{dx} \int_{y_c}^{y_w} \rho u v dy - (\rho u v)_{y_w} y'_w + (\rho u v)_{y_c} y'_c + \int_{y_c}^{y_w} \frac{N}{y} \rho v^2 dy \\ + \rho v^2 \Big|_{y_c}^{y_w} + p \Big|_{y_c}^{y_w} = 0 \end{aligned} \quad (7)$$

where the (') represents differentiation with respect to x .

Along a streamline we have

$$\frac{v}{u} = y' \Big|_{\text{streamline}} \quad (8)$$

Equations (6) and (7) may then be reduced to

$$\frac{d}{dx} \int_{y_c}^{y_w} \rho u y^N dy = 0 \quad (9)$$

$$\frac{d}{dx} \int_{y_c}^{y_w} \rho u v dy + N \int_{y_c}^{y_w} \frac{\rho v^2}{y} dy + P_w - P_c = 0 \quad (10)$$

It is also convenient to integrate Eq. (9) to obtain an expression for the mass flux

$$\dot{m} = \text{const} = (2\pi)^N \int_{y_c}^{y_w} \rho u y^N dy \quad (11)$$

Further study of the problem requires information about the integrands. We shall assume the integrands to be represented by polynomial functions of the dimensionless normal coordinate η , with

$$\eta = \frac{y - y_c}{y_w - y_c} \quad y_c < y < y_w \quad (12)$$

These polynomial expressions may usually be written as

$$\begin{aligned} \rho u &= a_0(x) + a_1(x) \eta + a_2(x) \eta^2 + \dots + a_n \eta^n \\ \rho u v &= b_0(x) + b_1(x) \eta + b_2(x) \eta^2 + \dots + a_n \eta^n \\ \rho v^2 &= c_0(x) + c_1(x) \eta + c_2(x) \eta^2 + \dots + c_n \eta^n \end{aligned} \quad (13)$$

The coefficient functions within these polynomials can be determined in terms of the nozzle wall shape and the local velocity values on the two streamlines. The order of the polynomials is restricted by the number of available boundary conditions. For the flow problem bounded by the streamline $y_c(x) \neq 0$, terms up to first order may be evaluated. For the nozzle flow problem, we set $y_c(x) \equiv 0$ and determine the second order polynomial coefficients from the boundary conditions including the condition of symmetry along the nozzle centerline.

Thus, for the nozzle problem, we may assume

$$\rho u = (\rho u)_c + [(\rho u)_w - (\rho u)_c] \eta^2 \quad (14)$$

$$\rho u v = (\rho u)_c v_w \eta + [(\rho u)_w - (\rho u)_c] v_w \eta^3 \quad (15)$$

$$\rho v^2 = (\rho v^2)_w \eta^2 \quad (16)$$

with the understanding that

$$(\rho u)_c = \rho_{oc} v_{mc} (1 - q_c^2)^{1/\gamma-1} q_c \quad (17)$$

$$(\rho u)_w = \rho_{ow} v_{mw} (1 - q_w^2)^{1/\gamma-1} q_w \frac{1}{(1 + y_w'^2)^{1/2}} \quad (18)$$

$$V_w = V_{mw} q_w \frac{y'_w}{(1 + y'^2_w)^{1/2}} \quad (19)$$

$$(\rho V^2)_w = \rho_{ow} V_{mw}^2 q_w^2 (1 - q_w^2)^{1/\gamma-1} \frac{y'^2_w}{1 + y'^2_w} \quad (20)$$

where q_c and q_w are the fluid speed normalized by the corresponding maximum possible speeds V_{mc} and V_{mw} . The relations

$$u_w = V_{mw} q_w \frac{1}{(1 + y'^2_w)^{1/2}} \quad (21)$$

$$v_w = V_{mw} q_w \frac{y'_w}{(1 + y'^2_w)^{1/2}} \quad (22)$$

have also been employed. Here both x - and y -coordinates have also been normalized by the nozzle throat y_t . Upon substituting the expressions given by Eqs. (14), (15), and (16), performing the integrations and considerable manipulation, Eqs. (9) and (10) become

$$A q'_w + B q'_c = C \quad (23)$$

$$D q'_w + E q'_c = F_1 + F_2 \quad (24)$$

where

$$A = \left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) \frac{(1 - q_w^2)^{1/\gamma-1} \left[1 - \frac{\gamma+1}{\gamma-1} q_w^2 \right]}{(1 + y'^2_w)^{1/2} (1 - q_w^2)} \quad (25)$$

$$B = \frac{2}{N+1} \frac{(1 - q_c^2)^{1/\gamma-1}}{(1 - q_c^2)} \left(1 - \frac{\gamma+1}{\gamma-1} q_c^2 \right) \quad (26)$$

$$C = y'_w \left\{ \left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) \frac{(1 - q_w^2)^{1/\gamma-1} q_w}{(1 + y'^2_w)^{3/2}} y'_w - \frac{N+1}{y_w} \left[\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right. \right. \\ \left. \left. \cdot \frac{(1 - q_w^2)^{1/\gamma-1} q_w}{(1 + y'^2_w)^{1/2}} + \frac{2}{N+1} (1 - q_c^2)^{1/\gamma-1} q_c \right] \right\} \quad (27)$$

$$D = y'_w \left[2 \left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) q_w (1 - q_w^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma}{\gamma-1} q_w^2 \right) \right. \\ \left. + q_c (1 - q_c^2)^{1/\gamma-1} (1 + y'^2_w)^{1/2} \right] \quad (28)$$

$$E = y'_w q_w (1 - q_c^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma+1}{\gamma-1} q_c^2 \right) \\ \cdot (1 + y'^2_w)^{1/2} \quad (29)$$

$$F_1 = - \frac{y'^2_w}{y_w} \left[(2N+1) \left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) q_w^2 (1 - q_w^2)^{1/\gamma-1} \right. \\ \left. + (1 + y'^2_w)^{1/2} q_w q_c (1 - q_c^2)^{1/\gamma-1} \right] \quad (30)$$

$$F_2 = \frac{4}{y_w} (1 + y'^2_w) \sqrt{\frac{T_{oc}}{T_{ow}}} \frac{\gamma-1}{2\gamma} \left[(1 - q_c^2)^{\gamma/\gamma-1} - (1 - q_w^2)^{\gamma/\gamma-1} \frac{p_{ow}}{p_{oc}} \right] \\ - y'_w \left[\frac{(1 - y'^2_w)}{(1 + y'^2_w)} \left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) q_w^2 (1 - q_w^2)^{1/\gamma-1} \right. \\ \left. + \frac{q_w q_c (1 - q_c^2)^{1/\gamma-1}}{(1 + y'^2_w)^{1/2}} \right] \quad (31)$$

The dimensionless mass flux $\dot{m}$ may be found by carrying out the integration in Eq. (11)

$$\dot{m} = (y_w)^N \frac{y_w}{N+3} \left[\left(\frac{p_{ow}}{p_{oc}} \sqrt{\frac{T_{oc}}{T_{ow}}} \right) \frac{q_w (1 - q_w^2)^{1/\gamma-1}}{(1 + y_w'^2)^{1/2}} + \frac{2}{N+1} q_c (1 - q_c^2)^{1/\gamma-1} \right] \quad (32)$$

These simultaneous differential equations (Eqs. (23) and (24)) may be integrated to establish velocity distributions for a given nozzle shape and mass flux. Well established numerical integration techniques such as the Euler predictor corrector method or the Runge Kutta method are available for carrying out the numerical integration.

Before we carry out the calculations and establish the characteristics of the numerical solution, it is important to examine the approximations made in the mass flux profiles. It will also be valuable to examine the characteristics of the coefficients A, B, C, etc., and the resulting behavior of the velocity gradients q'_w and q'_c .

2.2 DISCUSSION OF THE VELOCITY AND MASS FLUX PROFILE APPROXIMATIONS

The final justification for making any assumptions in the mathematical solution of a physical problem must be obtained by comparing the calculated results to experimental results. The purpose of this section is to present some justification for making the approximations employed in this analysis. We shall also point out a discrepancy between the approximate mass flux profile and the true physical behavior.

As previously discussed, the integral relations method was developed as a direct solution to the supersonic blunt body problem. The basic approach was to approximate the velocity or mass flux profiles by

interpolating polynomials in terms of the normal coordinate. The coefficients were evaluated in terms of conditions along the body and at the outer shock. Higher order approximate solutions were obtained by dividing the flow region bounded by the body and the shock into strips and the properties between strips were again approximated by interpolating polynomials. Increasing the number of strips results in an increased number of ordinary differential equations which must be solved simultaneously at each step of the integration.

Hayes and Probstein [1], in summarizing the development of the integral relations solution technique, showed that the two strip solution to the blunt body problem developed by Belotserkovskii agrees well with experimental data for the shock and sonic line shape. Based on these results and the fact that the nozzle problem is similar to the blunt body problem, it is attempted to adapt this solution technique to the nozzle problem.

The boundary conditions for the nozzle problem are the velocity values at the nozzle wall and along the centerline. By using the condition of symmetry about the nozzle centerline, velocity and mass flux profiles can be approximated as a second order polynomial for the axial velocity component and as a first order polynomial for the normal velocity.

There is a discrepancy in the second order mass flux approximation, however. In the transonic flow regime, the velocity magnitude varies from a subsonic value at the nozzle centerline to a supersonic value at the nozzle wall. The mass flux must then have a maximum value somewhere between the centerline and the wall. At the nozzle throat, where the vertical velocity component is zero, the mass flux is represented by ρu .

Our second order approximation does not allow a point of maximum flux density occurring within the field thus underestimating the mass flux. Since this deficiency is more pronounced when the throat radius of curvature is small, it is expected that this solution will be valid only for nozzles with large throat radius of curvature.

It is also recognized that in the expression given in Eq. (16), which will be needed for axisymmetric flows, the density has been approximately estimated by that at the wall.

2.3 ANALYSIS OF EQUATIONS

Before initiating a numerical solution to the pair of ordinary differential equations (Eqs. (23) and (24)), we can determine some important characteristics of the solution by examining the behavior of the coefficients of these equations.

We now specialize to flow cases with uniform stagnation state for the approaching flow. The flow is thus irrotational and the ratios p_{ow}/p_{oc} and T_{ow}/T_{oc} are unity.

The dimensionless variables are limited according to physical considerations to the following range of values:

$$0 \leq q_w \leq 1$$

$$0 \leq q_c \leq 1$$

$$1 \leq y_w \text{ finite}$$

$$y'_w \text{ finite}$$

$$y''_w \text{ finite} \tag{33}$$

Within this range of values for the variables, the coefficients are found to have zeros at:

$$A = 0 \text{ at } q_w^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$B = 0 \text{ at } q_c^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$C = 0 \text{ at } y'_w = 0$$

$$D = 0 \text{ at } y'_w = 0$$

$$E = 0 \text{ at } q_c^2 = \frac{\gamma - 1}{\gamma + 1}, y'_w = 0$$

$$F_1 = 0 \text{ at } y'_w = 0 \quad (34)$$

Examination of the momentum equation (Eq. (24)) shows that F_2 must be equal to zero at $y'_w = 0$. From the above considerations and the expressions for the velocity gradients

$$q'_w = \frac{CE - BD}{AE - BD} \quad (35)$$

$$q'_c = \frac{AF - DC}{AE - BD} \quad (36)$$

where

$$F = F_1 + F_2 \quad (37)$$

we find three possible critical points in the solution. These occur at

- a. The geometric throat, $y'_w = 0$,
- b. The location of the axis sonic point, $q_c^2 = \frac{\gamma - 1}{\gamma + 1}$, and
- c. The location where $AE - BD = 0$ and neither conditions (a) or (b) are satisfied.

Numerical calculations for several values of the wall parameters while varying the velocities throughout their physically meaningful range, indicated that this third possible critical point does not occur. The first two critical points reveal some interesting characteristics

about the nozzle solution, however,

2.3.1 The Geometric Throat

As mentioned previously, evaluating the momentum equation at the geometric throat shows that the expression F_2 must vanish at this location

$$F_2 \Big|_t = \frac{2(\gamma-1)}{y_w \gamma} \left[(1 - q_c^2)^{\gamma/\gamma-1} - (1 - q_w^2)^{\gamma/\gamma-1} \right] - y_w'' \left[q_w^2 (1 - q_w^2)^{1/\gamma-1} + q_w q_c (1 - q_c^2)^{1/\gamma-1} \right] = 0 \quad (38)$$

Thus, for a given nozzle shape and a given ratio of specific heats, there is a definite relation between the throat wall and centerline velocities. This relation is plotted in Fig. 2 for several values of the throat radius of curvature. It is interesting to note that the curves appear to be linear although this behavior would not be expected from an examination of Eq. (38).

Liddle and Archer's analysis [17] yielded throat conditions in a much simpler form given by

$$q_c = q_w \left(1 - \frac{y_w y_w''}{2} \right) \quad (39)$$

This relation between the throat centerline and wall velocities is approximately the same as that found from Eq. (38) (see Fig. 3). The difference in their analytical form is due to application of the irrotationality condition, given by

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad (40)$$

The dimensionless mass flux can be calculated from Eq. (32) for a given nozzle shape and a pair of proper throat velocity values

determined from Eq. (38). Figure 4 shows this variation in mass flux as a function of the throat wall velocity. This curve reaches a maximum mass flux value for a specific value of the nozzle throat wall velocity. Thompson [15] has termed this condition as the "physical" choking condition.

It is pertinent to point out that this maximum mass flux has nothing to do with choking. The original meaning of choking refers to the phenomenon that the downstream disturbance cannot propagate upstream that the upstream flow no longer responds to additional reduction in downstream back-pressure. For one-dimensional analysis, it is coincident that both maximum mass flux condition and choking occur together. Although, during choking, the nozzle does deliver a physical maximum amount of mass rate of flow, it would be incorrect to label a condition of maximum mass rate of flow obtained from analysis as choking.

The condition of choking actually derives from the physical requirement that the velocity gradient at the throat of the nozzle must be finite so that the flow may accelerate continuously through the transonic flow regime. We may investigate this condition by examining the behavior of the governing equations. We will refer to this last choking condition as the continuous acceleration criterion for choking.

For the pair of ordinary differential equations,

$$A q'_w + B q'_c = C \quad (41)$$

$$D q'_w + E q'_c = F_1 + F_2 \quad (42)$$

Since the coefficients C , D , E , F_1 , and F_2 approach zero as the throat is approached, one realizes that one of the two possible situations may occur:

1. $q'_w = q'_c = 0$ at $y'_w = 0$
2. $q'_w/q'_c = -B/A$ at $y'_w = 0$

(43)

We now determine the criterion for the second of these conditions to exist, namely, to determine the required conditions for a nonzero, finite value for the velocity gradients at the throat. Dividing Eq. (24) by y'_w and lettering y'_w approach zero, we get

$$\lim_{y'_w \rightarrow 0} \left[\frac{D}{y'_w} q'_w + \frac{E}{y'_w} q'_c = \frac{F_1}{y'_w} + \frac{F_2}{y'_w} \right] \quad (44)$$

where

$$\frac{D}{y'_w} = \bar{D} \neq 0 \text{ at } y'_w = 0$$

$$\frac{E}{y'_w} = \bar{E} \neq 0 \text{ at } y'_w = 0$$

$$\frac{F_1}{y'_w} = 0 \quad \text{at } y'_w = 0$$

and

$$\frac{F_2}{y'_w} \rightarrow 0/0 \quad \text{at } y'_w \rightarrow 0 \quad (45)$$

Equation (44) becomes

$$\bar{D} q'_{wt} + \bar{E} q'_{ct} = \lim_{y'_w \rightarrow 0} \left[\frac{F_2}{y'_w} \right] \quad (46)$$

Applying L'Hospital's rule to find the limiting value of

$$\lim_{y'_w \rightarrow 0} \left[\frac{F_2}{y'_w} \right]$$

and rearranging terms, we find that Eq. (46) can be expressed as:

$$\lambda_1 q'_{w_t} + \lambda_2 q'_{c_t} = \lambda_3 y'''_{w_t} \quad (47)$$

where

$$\begin{aligned} \lambda_1 &= y'''_{w_t} \left[q_{w_t} (1 - q_{w_t}^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma}{\gamma-1} q_{w_t}^2 \right) \right. \\ &\quad \left. + \frac{q_{c_t}}{2} (1 - q_{c_t}^2)^{1/\gamma-1} \right] - q_{w_t} (1 - q_{w_t}^2)^{1/\gamma-1} \\ \lambda_2 &= \frac{y'''_{w_t}}{2} q_{w_t} (1 - q_{c_t}^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma+1}{\gamma-1} q_{c_t}^2 \right) \\ &\quad + q_{c_t} (1 - q_{c_t}^2)^{1/\gamma-1} \\ \lambda_3 &= -\frac{q_{w_t}}{4} \left[q_{w_t} (1 - q_{w_t}^2)^{1/\gamma-1} + q_{c_t} (1 - q_{c_t}^2)^{1/\gamma-1} \right] \end{aligned} \quad (48)$$

The resulting governing equations at the geometric throat are

$$A q'_{w_t} + B q'_{c_t} = 0 \quad (41)$$

$$\lambda_1 q'_{w_t} + \lambda_2 q'_{c_t} = \lambda_3 y'''_{w_t} \quad (47)$$

Since there is a difference in the analysis between a symmetric nozzle (defined here as $y'''_w = 0$) and a nonsymmetric nozzle (defined as $y'''_w \neq 0$), we discuss each case separately.

2.3.1.1 A Symmetric Nozzle ($y'''_w = 0$)

For a symmetric nozzle, Eqs. (41) and (47) be-

come:

$$A q'_{w_t} + B q'_{c_t} = 0 \quad (41)$$

$$\lambda_1 q'_{w_t} + \lambda_2 q'_{c_t} = 0 \quad (49)$$

The two possible solutions to these equations are:

$$q'_{w_t} = q'_{c_t} = 0$$

at the throat, and

$$\begin{vmatrix} A & B \\ \lambda_1 & \lambda_2 \end{vmatrix} = 0 \quad \text{with} \quad q'_{w_t} \quad q'_{c_t} \neq 0 \quad (50)$$

For the second case, evaluation of the determinant, Eq. (50), yields a specific pair of throat velocities for the given wall shape and γ .

This condition is defined as the true choking condition. As the numerical results will demonstrate, this specific pair of throat velocity values is the only pair which allow finite, nonzero velocity gradients at the geometric throat.

While it is possible to calculate the proper throat velocity pairs, the corresponding choking mass flux, and the ratio of throat velocity gradients at the choked condition, we cannot calculate the magnitude of these velocity gradients. Further attempts in applying L'Hospital's rule to the governing equations at the geometric throat introduce unknown higher velocity gradients with each new equation.

The results of this analysis of the choking conditions show that a two-dimensional flat nozzle has higher throat velocities at the choked condition than does an axially symmetric nozzle with the same throat radius of curvature (see Fig. 5). We can also note that the mass flow per unit area at choking is approximately the same for the two nozzle shapes with the same throat radius of curvature as plotted in Fig. 6.

Although the difference in flow coefficient due to different definition of choking is small (as shown in Fig. 7 for a two-dimensional flat nozzle), it is expected that the difference may not be negligible

for nozzles with small radii of curvature at the throat.

2.3.1.2 A Nonsymmetric Nozzle ($y'''_t \neq 0$)

The governing equations at the geometric throat for a nonsymmetric nozzle are

$$A q'_{w_t} + B q'_{c_t} = 0 \quad (41)$$

$$\lambda_1 q'_{w_t} + \lambda_2 q'_{c_t} = \lambda_3 y'''_t \quad (47)$$

Using Cramer's rule to solve for the velocity gradients, we have

$$q'_{w_t} = \frac{-B\lambda_3 y'''_t}{\begin{vmatrix} A & B \\ \lambda_1 & \lambda_2 \end{vmatrix}}, \quad q'_{c_t} = \frac{A\lambda_3 y'''_t}{\begin{vmatrix} A & B \\ \lambda_1 & \lambda_2 \end{vmatrix}} \quad (51)$$

Examination of the above terms shows that

$$A = 0 \text{ at } q_{w_t}^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$B = 0 \text{ at } q_{c_t}^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$\lambda_3 \neq 0 \quad (52)$$

$$\begin{vmatrix} A & B \\ \lambda_1 & \lambda_2 \end{vmatrix} \text{ is always finite.}$$

Therefore, for a nonsymmetric nozzle, the velocity gradients may be nonzero at the geometric throat. The ratio of the velocity gradients is plotted in Fig. 8 versus the throat wall velocity. The fact that this curve becomes undefined at the condition where the throat wall velocity is sonic, indicates only that the centerline velocity gradient is equal to zero while the wall velocity gradient remains finite.

Further reasoning along this line yields the conclusion that the

choking condition for a nonsymmetric nozzle shape must occur at a lower mass flux rate than for a symmetric nozzle with the same throat height and radius of curvature. We deduce this result by noting that the determinant

$$\begin{vmatrix} A & B \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

approaches zero as the throat velocity increases. Since at least one of the numerators in Eq. (51) remain nonzero, a minimum of one of the velocity gradients must become undefined at the condition for choking for a symmetric nozzle if the choking of the nonsymmetric nozzle has not occurred.

Therefore, the choking condition in a nonsymmetric nozzle must occur at a lower throat velocity than that for a symmetric nozzle. The graph of mass flux versus the wall throat velocity in Fig. 4 shows that for a throat wall velocity lower than the true choking value, the mass flux must also be lower. In addition, we note that the difference in the choking conditions between a symmetric and a nonsymmetric nozzle is proportional to the quantity y'''_{w_t} .

For the nonsymmetric nozzle, we can determine the throat velocity gradients for any given mass flux and nozzle shape. We have also shown that the choking mass flux must be less than the value corresponding to the vanishing determinant. The precise values of q_{c_t} and q_{w_t} at the choking condition cannot be made from this analysis, however. We will later see that the choking condition is determined by calculations near the throat.

2.3.2 Axis Sonic Point

The second critical point occurs at the location where the

centerline velocity becomes sonic. From conditions at this critical point, we will be able to determine the spatial location as a function of the choking mass flow rate and the nozzle shape.

The expression for the velocity gradients at any point in the nozzle can be written as

$$q'_w = \frac{CE - BF}{AE - BF}, \quad q'_c = \frac{AF - DC}{AE - BF} \quad (53)$$

If we factor out the term

$$1 - \left[\frac{\gamma + 1}{\gamma - 1} \right] q_c^2$$

from coefficients B and E, we find that the wall gradient remains defined at the axis sonic point

$$q_c^2 = \frac{\gamma - 1}{\gamma + 1}$$

The axis velocity gradient becomes undefined unless the quantity $AF - DC$ approaches zero as the axis velocity approaches sonic since the quantity $AE - BF$ goes to zero.

Once the choking mass is specified, the location of the axis sonic point can be determined by solving

$$AF - DC = 0 \quad (54)$$

subject to the conditions

$$q_c^2 = \frac{\gamma - 1}{\gamma + 1}$$

$$q_w = f(\text{mass flux, wall shape, } q_c)$$

with the wall shape specified as a function of axial location.

The solution curves for Eq. (54) are plotted in Figs. 9 and 10 for a two-dimensional nozzle and an axisymmetric nozzle, respectively. The proper solution for the location of the axis sonic point in the nozzle problem is determined from the choking flow rate. Both of these

graphs show that the axis sonic point is located farther downstream at larger values of the choking mass flux.

For the nozzle presented in these two figures, the choking dimensionless mass flux was calculated to be 0.81056 for the axisymmetric nozzle and 0.258018 for the two-dimensional flat nozzle. We also find that the axis sonic point is located closer to the geometric throat for a two-dimensional flat nozzle than for an axisymmetric nozzle, which is agreeable with the early findings of Sauer [18].

2.4 ONE-DIMENSIONAL SOLUTION CHARACTERISTICS

Although the one-dimensional solution to compressible flow problems has been discussed in detail in the literature by Shapiro and Hawthorne [7], it will be helpful to re-examine the choking criterion for the nozzle problem. Our objective is to compare the previously mentioned choking criterion in terms of the one-dimensional solution. In making this comparison, we will introduce the one-dimensional characteristic curves in terms of the velocity gradient at constant mass flux versus the axial location in a particular nozzle.

Before proceeding with this discussion, we note that our analysis of the solution characteristics is for inviscid steady compressible flow through a converging-diverging nozzle such as shown in Fig. 1. With flow from left to right in this nozzle, the normal operating conditions produce subsonic flow in the converging portion of the nozzle upstream of the throat. Downstream from the throat the velocity may be either subsonic or supersonic depending on the downstream back pressure.

The concept of choking flow in a compressible nozzle is discussed in textbooks on fluid mechanics. A usual approach to this material is to examine the pressure distribution in a converging-diverging nozzle for given stagnation conditions as the nozzle back pressure is lowered. A critical value of the back pressure is found below which the back pressure has no influence on the pressure distribution upstream of the throat. If we also consider the mass flux through the nozzle, we find that the mass flux increases as the back pressure is decreased until the critical value of back pressure is reached. After this, further reductions in the back pressure do not change the mass flow rate.

Shapiro [19] examines the flow through a nozzle in terms of the characteristic curves which are plots of the velocity variation through the nozzle for a specific value of the mass flux. Figure 11 shows these characteristic curves for flow through a hyperbolic nozzle. As indicated in the graph, the curves of velocity distribution in the nozzle at mass fluxes greater than choking do not exist at the geometric throat.

Another way to look at the characteristics of flow through a nozzle is to plot the velocity gradients for a specific mass flux as a function of the axial location in the nozzle. These curves are shown in Fig. 12. It can be readily seen that these curves contain the same information shown in Fig. 11. The advantage to this type of graph is that we can illustrate physically possible flow conditions as those which result in continuous, finite velocity gradients throughout the nozzle. Also, these velocity gradient curves show the characteristics of the nozzle solution more vividly than do the velocity curves. The difference

between the two types of curves becomes more pronounced as we examine their behavior for mass flows very near choking in the vicinity of the throat.

It is obvious that during choking, the velocity gradient at the throat is positive (for nozzle flows) and finite. It is given by

$$\left(\frac{dM^*}{dx}\right)_t = \left[\frac{1}{\gamma + 1} \left(\frac{1}{A} \frac{d^2 A}{dx^2} \right)_t \right]^{1/2} \quad (55)$$

which can be found when the limiting process is applied to the well known Hugoniot expression

$$\frac{1}{V} \frac{dV}{dx} = \frac{dA}{A} \frac{1}{1 - M^2}$$

2.5 NUMERICAL INTEGRATION TECHNIQUE

A second order implicit predictor-corrector numerical technique was used to integrate this pair of ordinary differential equations (Eqs. (35) and (36)). At each step of the integration, an iteration on the predicted velocity values was performed until the predicted values at the new station repredicted themselves with an error of less than 0.001 percent. As the solution approached the critical points, the step size was reduced to maintain the 0.001 percent predictor accuracy. A block diagram of this process is shown in Fig. 13.

The numerical results obtained with this second order method were checked against results with smaller step size. Also, they were compared with both a first order solution and a fourth order Runge-Kutta integration scheme. All solution techniques produced the same results except the first order solution differed in the region near the throat.

One solution to a nozzle flow field from the subsonic upstream

portion to the axis sonic point costs approximately \$3.00 on the University of Illinois IBM 360-75 computer. This cost includes the \$1.00 cover charge assessed to each program and the 360-75 processor time of approximately 15 seconds. This computation includes the iterations required to initiate the calculations in the subsonic flow field and in the transonic flow field downstream from the throat.

This second order implicit calculation scheme was used to integrate the equations from some initial location, where both the wall and the centerline velocities were known, to a location where the calculation scheme failed to continue. There were several regions where this restriction occurred. For mass flows less than choking, the equations could be numerically integrated in the direction toward the throat in both the upstream converging section and the downstream diverging section. For mass flows above the choking flow rate, the velocity gradients became undefined near the throat. When choking occurs the calculation could not be continued downstream from the axis sonic point. This behavior of the numerical solution was unchanged when the equations were integrated by the fourth order Runge-Kutta numerical scheme.

Since the schemes for choosing the starting conditions for the numerical solution are implicitly associated with the behavior of the governing equations, we will discuss both of these conditions together. Furthermore, we will break down the discussion into the different regimes of the nozzle solution. As we will show in the following discussion, the solution is quite directional. The numerical iteration diverges if the solution is attempted in the wrong direction.

2.5.1 Upstream of the Throat--Accelerating Flow

The nozzle solution was started by assuming values for the

nozzle mass flux and the wall velocity at a specified axial location for a given nozzle shape. Normally, the one-dimensional velocity value for the wall velocity was chosen at a location far upstream of the geometric throat. From these two initial values, the centerline velocity, and hence the velocity gradients, were calculated from Eqs. (35) and (36) to initiate the step by step integration.

The solution curves were initially very sensitive to the chosen value of the initial wall velocity for a given mass flux. As shown in Fig. 14, a 2 percent change in the initial value for the velocity causes a corresponding 50 percent change in the initial value for the velocity gradient.

This behavior is unchanged if the numerical iteration step size is reduced. Physically, the different initial values in the wall and centerline velocities for the same mass flux represent different nozzle shapes upstream from the starting location. However, all curves of the same mass rate of flow coalesce quickly with each other as shown in Fig. 14. The upstream nozzle shape has little effect on the downstream subsonic flow field. Figure 15 shows the velocity gradient at a certain downstream section for different starting conditions. This behavior was also noted by Oswatitsch and Rothstein [8].

Although not plotted here, the curves for the centerline velocity gradients illustrate the same behavior. We could also show these characteristics by plotting the curves of velocity distribution. However, as noted above, the curves of velocity gradients present an amplified picture of the solution characteristics.

If the nozzle solution is initiated at a section where the wall slope is equal to zero, a unique starting velocity pair exists for each specified

value of mass flux. This starting velocity pair may be determined for each mass flux value by solving Eqs. (32) and (38) simultaneously for the initial velocities (q_w and q_c). Both starting methods yield the same downstream results for a given nozzle shape and mass flux. The method which initiates the solution at a location where the wall slope is not zero is better for determining the effects of inlet conditions, however.

2.5.2 Downstream from the Nozzle Throat

The solution curves for the region downstream from the nozzle throat were obtained by a marching integration which began somewhere downstream and continued upstream toward the throat. These solution curves can be classified into two categories termed the accelerating branch and the de-accelerating branch. The accelerating branch contains the solution curves for continuous acceleration from subsonic to supersonic velocities in the two-dimensional nozzle. The de-accelerating branch includes the solution curves for nozzle flow fields which correspond to continuous change from an accelerating to a de-accelerating flow in the vicinity of the geometric throat.

To further clarify the two categories of solution curves, we shall examine the one-dimensional velocity gradient curves shown in Fig. 12. There is only one proper solution curve that belongs to the accelerating branch: This proper curve is determined by an isentropic expansion from subsonic to supersonic velocities in a converging-diverging nozzle. It is also important to note the behavior of improper solution curves. Here, we see that a mass flux greater than the choking value produces a solution curve with undefined velocity gradients in the vicinity of the nozzle throat while a mass flux less than the choking value has velocity

gradients which go to zero at the geometric throat.

There are an infinite number of proper solution curves which belong to the de-accelerating branch. If we classify these solution curves in terms of the nozzle mass flux, the proper solution curves are bounded by the choking mass flux and the zero mass flux condition. All of these solution curves have a zero velocity gradient at the geometric throat except the choking case. For the choking condition, the velocity increases to the sonic value at the throat, then begins to decrease downstream from the throat. The velocity gradient for this case is double valued at the nozzle throat.

2.5.2.1 Accelerating Branch

The curves presented here were calculated in initiating the solution in the supersonic region of the flow field and then marching back to the nozzle throat with the previously described numerical calculation method. The solution curve for a given mass flux was initiated by searching for a supersonic velocity pair (q_w, q_c) at the starting location which produced the specified mass flux according to Eq. (32), and yielded positive values for the velocity gradients calculated from Eqs. (23) and (24). The correct pair could not be determined accurately; however, different curves with the same mass rate of flow converge quickly into a single curve, and there is no difficulty in arriving at the correct pair of velocities at a certain downstream section.

The solution curves for a hyperbolic nozzle are presented in Fig. 16. Their behavior is similar to that described for the one-dimensional solution, in that the velocity gradients become undefined for mass flows greater than the choking value and the velocity gradients go to zero for mass flows less than the choking value. Although the axis velocity

gradients are not plotted here, they have the same general characteristics as the wall velocity gradients.

Attempts to continue the solution downstream from the throat in the transonic flow regime were not successful due to difficulties in determining the precise starting conditions. Also, numerical difficulties prevented continuation of the solution from the supersonic section upstream across the axis sonic point location. In the supersonic region, the solution was directionally stable only when the marching iteration proceeded toward the throat.

2.5.2.2 De-Accelerating Branch

Again, the proper direction for the marching solution in the downstream de-accelerating branch is toward the nozzle throat. Attempts to continue the de-accelerating solution away from the throat did not produce meaningful results (as shown in Fig. 17). These same directional properties can be observed in the upstream subsonic flow characteristics. In attempts to march away from the nozzle throat with a forward numerical integration scheme, small numerical errors cause the numerical integration to follow one of the diverging solution curves. The numerical integration is quite insensitive to small numerical errors when marching in the direction toward the nozzle throat, however. The downstream de-accelerating solution curves were, therefore, calculated by the same method described for the upstream subsonic solution.

2.6 DISCUSSION OF RESULTS

The preceding discussion has developed an integral solution to the two-dimensional nozzle problem into a form which can readily be solved

by well known numerical marching integration schemes. Although engineering applications normally require only knowledge of the velocity or pressure distributions in a nozzle, the main emphasis of this work has been directed toward determining and understanding the velocity gradient distributions. As demonstrated earlier, the velocity gradients present an amplified picture of the solution behavior.

The wall velocity gradients for a hyperbolic nozzle are shown in Fig. 18. This nozzle shape is also termed a symmetric nozzle since the wall height is symmetric about the nozzle throat plane. These curves indicate that extremely small changes in the mass flux near the choking value cause significant differences in the calculated velocity gradients. The choking condition, defined as the mass flux which causes a smooth continuous acceleration through the transonic flow regime, must occur at a mass flux between 0.258017 and 0.258019 as seen from the velocity gradient curves. The actual choking condition as calculated from the throat singular conditions for this two-dimensional flat nozzle occurs at a mass flux of approximately 0.258018.

The exact velocity gradient distribution for the choked flow condition was not calculated due to numerical accuracy problems. Although the marching integration scheme used for these calculations maintained the mass flux constant to six decimal place accuracy would be required to determine the velocity gradients through the complete transonic flow regime. Fortunately, such accuracy is not required for physical applications where just velocity or pressure distributions are desired. Near the choking condition, an error in the mass flux of 0.1 percent causes a corresponding error of 1 percent in the calculated velocity values.

The axis and wall velocity distributions for the two-dimensional

choked condition are shown in Fig. 19 along with the one-dimensional values. For comparison purposes, the results from Oswatitsch [8] for the same nozzle are also shown. These two-dimensional velocity curves were calculated by stepping the marching solution across the throat singular point. That is, velocity gradients at location $x = -0.012$ were used to predict the velocities at location $x = 0.005$. This stepping procedure allowed the solution to continue smoothly across the throat singular point to the axis sonic point. The supersonic values were determined from calculations which initiated downstream and marched upstream to the axis sonic point.

For mass flows less than choking, the velocity gradients go to zero at the nozzle throat. Both the axis and the wall velocity gradients are shown in Fig. 20 for a subcritical value of the mass flux. As expected, the axis velocity gradients have a smaller magnitude than the wall velocity gradient.

The characteristic curves for a nonsymmetric nozzle are shown in Fig. 21. These curves have the same general shape as those for a symmetric nozzle except that the velocity gradients for subcritical values of the mass flux may be nonzero at the nozzle throat. Thus, a nonsymmetric nozzle (determined by $y_w''' \neq 0$) may have continued acceleration for a short distance downstream from the throat at subcritical values of mass flux.

For a two-dimensional flow in a nozzle, sonic velocities in the flow field may be attained at subcritical values of mass flux. Figure 22 shows the location of the wall sonic upstream from the throat at mass flows less than or equal to the choking value. For this hyperbolic nozzle, the mass flux must be within 0.25 percent of the choking value to obtain sonic velocities anywhere in the nozzle. For nozzles with a

smaller throat radius of curvature, the sonic condition would be reached at lower values of mass flux relative to the choking value.

The differences in mass flux and throat velocities between the maximum possible mass flux condition and the choked flow condition are plotted against the inverse throat radius of curvature in Figs. 23 and 24 for a two-dimensional flat nozzle. Figures 25 and 26 contain similar curves for an axisymmetric nozzle. The differences between calculated throat velocities and mass flux are very small for nozzles with large throat radius of curvature. However, as previously discussed, these small differences in the mass flux or velocities are magnified in the considerations of the velocity gradient curves.

The nozzle discharge coefficient defined as the ratio of the inviscid two-dimensional choking mass flux to the one-dimensional value is plotted in Fig. 27 against the inverse throat radius of curvature. The discharge coefficient predicted by this method is approximately 2 percent lower than that predicted by Kliegel and Levine [20] at a throat radius of curvature of $R = 2$. Kliegel and Levine's calculated discharge coefficient agrees very well with experimental results for a nozzle with much smaller throat radius of curvature.

Perhaps it is appropriate to mention that in the study of subsonic flow upstream of the throat, the problem is of the elliptic type, while the scheme of analysis by integral relations follows a procedure usually adopted for parabolic problems. The elliptic character of the flow field is exhibited in the iteration of the initially selected parameters such that the flow field satisfies certain conditions at the throat. On the other hand, for the supersonic flow downstream of the throat, where the problem is of the hyperbolic type, calculations have to be carried out

toward the upstream throat. Although the correct information may be produced from repeated calculations, this scheme of calculation does not illustrate the hyperbolic character of the problem. Since the method of characteristics is powerful and useful in dealing with hyperbolic type of problems, we are inclined to favor the application of the method of characteristics for the supersonic flow within nozzles. However, the present analysis may be suitable to consider flow with shock within the divergent nozzle. In this respect, we are not sure that the present method is inferior to other methods of analysis. Many interesting mathematical questions must be answered before any conclusions can be reached for the comparison of different methods.

Finally, it is interesting to point out that Eqs. (23) and (24) relate three functions, y_w , q_w , and q_c . With any one of them specified, the other two unknown functions may be determined by integrating this pair of equations. Thus, for the direct or indirect solution of nozzle flow problems, same system of equations may be employed.

3. INVISCID INTERACTION BETWEEN TWO STREAMS

The present study of nozzle flow problems was originally directed toward developing a solution method for multiple stream interaction problems such as that of a supersonic ejector system. Ejectors have historically been used to pump a low energy secondary fluid by using a high velocity primary fluid. They have been widely applied for removing noncondensable gases from a stream condensor. More recently, the ejector principle has been applied to jet propulsion systems for improving the performance and the propulsion efficiency of the engine.

The flow configuration for a supersonic ejector is shown in Fig. 28. Initially, the primary stream is supersonic and the secondary outer stream is subsonic. The secondary shroud is axisymmetric and is not restricted to a constant area. A theoretical analysis for this problem has been presented by Addy [21] and by Chow and Addy [22]. They calculated the supersonic inner stream by the method of characteristics and the subsonic outer stream by a quasi one-dimensional method in which the one-dimensional secondary stream area was specified to lie in the plane normal to the dividing streamline. The viscous effect was also taken into account. They showed that the inviscid interaction predominantly determines the flow field except at low values of secondary mass flow rates.

The purpose of this analysis is to develop a solution method for the two-stream ejector problem using two-dimensional flow considerations for the outer stream. Boundary conditions for the outer stream flow field limit the stream density to a linear variation across the stream. The inner stream is calculated by the method of characteristics and an iteration scheme is used to calculate the two-stream ejector flow field.

3.1 OUTER STREAM ANALYSIS

For the approaching flow with uniform stagnation state, the integral forms of the continuity and the momentum equations as derived in Section 2.1 for axisymmetric flow fields are:

$$\frac{d}{dx} \int_{y_c}^{y_w} \frac{\rho u y}{\rho_o V_m} dy = 0 \quad (11)$$

$$\frac{d}{dx} \int_{y_c}^{y_w} \frac{\rho u v}{\rho_o V_m^2} dy + \int_{y_c}^{y_w} \frac{1}{y} \frac{\rho v^2}{\rho_o V_m^2} dy + \frac{P_w - P_c}{\rho_o V_m^2} = 0 \quad (12)$$

The integrands are now approximated by polynomials in the form of

$$\text{Integrand} = a_o + a_1 \eta$$

where

$$\eta = \frac{y}{y_w}, \eta_c \leq \eta \leq 1, \text{ and } \eta_c \geq 0 \quad (56)$$

and the coefficients a_o, a_1 are evaluated in terms of the property values along the inner and outer streamline boundaries (y_c, y_w).

These boundary conditions are

$$y = y_w, \eta = 1, q = q_w, \rho = \rho_w \quad (57)$$

$$y = y_c, \eta = \eta_c, q = q_c, \rho_u = \rho_c$$

The integrands become

$$\frac{\rho u}{\rho_o V_m} = \frac{F_{1c} - \eta_c F_{1w}}{1 - \eta_c} + \frac{F_{1w} - F_{1c}}{1 - \eta_c} \eta \quad (58)$$

$$\frac{\rho u v}{\rho_o V_m^2} = \frac{F_{2c} - \eta_c F_{2w}}{1 - \eta_c} + \frac{F_{2w} - F_{2c}}{1 - \eta_c} \eta \quad (59)$$

$$\frac{1}{y} \frac{\rho}{\rho_o} \frac{v^2}{v_m^2} = \frac{1}{y} \frac{\rho v^2}{\rho_o v_m^2} = \frac{F_{3c} - \eta_c F_{3w}}{1 - \eta_c} + \frac{F_{3w} - F_{3c} \eta}{1 - \eta_c} \quad (60)$$

where

$$\begin{aligned} F_{1c} &= \frac{(1 - q_c^2)^{1/\gamma-1} q_c}{(1 + y_c'^2)^{1/2}}, \quad F_{1w} = \frac{(1 - q_w^2)^{1/\gamma-1} q_w}{(1 + y_w'^2)^{1/2}} \\ F_{2c} &= \frac{(1 - q_c^2)^{1/\gamma-1} q_c^2 y_c'}{(1 + y_c'^2)}, \quad F_{2w} = \frac{(1 - q_w^2)^{1/\gamma-1} q_w^2 y_w'}{(1 + y_w'^2)} \\ F_{3c} &= \frac{(1 - q_c^2)^{1/\gamma-1} q_c^2 y_c'^2}{(1 + y_c'^2) y_c}, \quad F_{3w} = \frac{(1 - q_w^2)^{1/\gamma-1} q_w^2 y_w'^2}{(1 + y_w'^2) y_w} \end{aligned}$$

After carrying out the indicated integration, differentiation, and the required algebraic manipulation, the governing equations become two ordinary nonlinear differential equations.

$$A \frac{dq_w}{dx} + B \frac{dq_c}{dx} = C \quad (61)$$

$$D \frac{dq_w}{dx} + E \frac{dq_c}{dx} = F \quad (62)$$

where

$$A = \frac{y_w^2}{6} (2 - \eta_c - \eta_c^2) \frac{(1 - q_w^2)^{[(2-\gamma)/(\gamma-1)]}}{(1 + y_w'^2)^{1/2}} \left(1 - \frac{\gamma+1}{\gamma-1} q_w^2 \right) \quad (63)$$

$$B = \frac{y_w^2}{6} (1 + \eta_c - 2\eta_c^2) \frac{(1 - q_c^2)^{[(2-\gamma)/(\gamma-1)]}}{(1 + y_c'^2)^{1/2}} \left(1 - \frac{\gamma+1}{\gamma-1} q_c^2 \right) \quad (64)$$

$$\begin{aligned}
C = & \frac{y_w^2}{6} \left[(2 - \eta_c - \eta_c^2) \frac{(1 - q_w^2)^{1/\gamma-1} q_w}{(1 + y_w'^2)^{3/2}} y_w' y_w'' \right. \\
& + (1 + \eta_c - 2\eta_c^2) \frac{(1 - q_c^2)^{1/\gamma-1} q_c}{(1 + y_c'^2)^{3/2}} y_c' y_c'' \left. \right] \\
& - \frac{y_w y_w'}{3} \left[(2 - \eta_c - \eta_c^2) \frac{(1 - q_w^2)^{1/\gamma-1} q_w}{(1 + y_w'^2)^{1/2}} \right. \\
& + (1 + \eta_c - 2\eta_c^2) \frac{(1 - q_c^2)^{1/\gamma-1} q_c}{(1 + y_c'^2)^{1/2}} \left. \right] \\
& + \frac{y_w^2}{6} \frac{q_w (1 - q_w^2)^{1/\gamma-1}}{(1 + y_w'^2)^{1/2}} (1 + 2\eta_c) \eta_c' \\
& - \frac{y_w^2}{6} \frac{q_c (1 - q_c^2)^{1/\gamma-1}}{(1 + y_c'^2)^{1/2}} (1 - 4\eta_c) \eta_c' \tag{65}
\end{aligned}$$

$$D = \frac{2y_w'}{1 + y_w'^2} q_w (1 - q_w^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma}{\gamma-1} q_w^2 \right) \tag{66}$$

$$E = \frac{2y_c'}{1 + y_c'^2} q_c (1 - q_c^2)^{[(2-\gamma)/(\gamma-1)]} \left(1 - \frac{\gamma}{\gamma-1} q_c^2 \right) \tag{67}$$

$$\begin{aligned}
F = & -(1 - q_w^2)^{1/\gamma-1} q_w^2 \frac{(1 - y_w'^2)}{(1 + y_w'^2)^2 y_w'} \\
& - (1 - q_c^2)^{1/\gamma-1} q_c^2 \frac{(1 - y_c'^2)}{(1 + y_c'^2)^2 y_c'} \\
& - \left[\frac{(1 - q_w^2)^{1/\gamma-1} q_w^2 y_w'}{(1 + y_w'^2)} + \frac{(1 - q_c^2)^{1/\gamma-1} q_c^2 y_c'}{(1 + y_c'^2)} \right] \\
& \cdot \left[\frac{y_w'}{y_w} - \frac{\eta_c'}{1 - \eta_c} \right] - \frac{(1 - q_w^2)^{1/\gamma-1} q_w^2 y_w'^2}{y_w (1 + y_w'^2)} \\
& - \frac{(1 - q_c^2)^{1/\gamma-1} q_c^2 y_c'^2}{y_c (1 + y_c'^2)} - \frac{\gamma - 1}{\gamma} \frac{1}{y_w (1 - \eta_c)} \\
& \cdot \left[(1 - q_w^2)^{\gamma/\gamma-1} - (1 - q_c^2)^{\gamma/\gamma-1} \right] \tag{68}
\end{aligned}$$

And the expression for the mass flux can be written as

$$\begin{aligned}
\dot{m} = & \frac{\pi y_w^2}{3} \left[\frac{(1 - q_w^2)^{1/\gamma-1} q_w^2}{(1 + y_w'^2)^{1/2}} (2 - \eta_c - \eta_c^2) \right. \\
& \left. + \frac{(1 - q_c^2)^{1/\gamma-1} q_c^2}{(1 + y_c'^2)^{1/2}} (1 + \eta_c - 2\eta_c^2) \right] \tag{69}
\end{aligned}$$

Analysis of the coefficients in Eqs. (61) and (62) show that there are critical points in the flow field when the expression $\begin{vmatrix} A & B \\ D & E \end{vmatrix}$ goes to zero. Expanding this determinant and equating it to zero, we get

$$\begin{aligned}
& (2 - \eta_c - \eta_c^2) \left(1 - \frac{\gamma + 1}{\gamma - 1} q_w^2 \right) \frac{y'_c}{(1 + y'^2_c)^{1/2}} q_c \\
& \cdot \left(1 - \frac{\gamma}{\gamma - 1} q_c^2 \right) - (1 + \eta_c - 2\eta_c^2) \left(1 - \frac{\gamma + 1}{\gamma - 1} q_c^2 \right) \\
& \cdot \frac{y'_w}{(1 + y'^2_w)^{1/2}} q_w \left(1 - \frac{\gamma}{\gamma - 1} q_w^2 \right) = 0 \tag{70}
\end{aligned}$$

For an annulus in which the inner streamline slope is identically zero ($y'_c = 0$ and $y_c = \text{constant}$), these critical points occur at:

- a. $y'_w = 0$ minimum area
- b. $q_c = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$ center streamline sonic point
- c. $q_w = \sqrt{\frac{\gamma - 1}{\gamma}}$ wall streamline velocity
 $M_w^* = 1.31$ for $\gamma = 1.4$ (71)

An annulus in which the outer streamline slope is identically zero ($y'_w = 0$ and $y_w = \text{constant}$) will have critical points at:

- a. $y'_c = 0$ minimum area
- b. $q_w = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$ wall streamline sonic point
- c. $q_c = \sqrt{\frac{\gamma - 1}{\gamma}}$ center streamline velocity
 $M_c^* = 1.31$ for $\gamma = 1.4$ (72)

The location of these critical points as listed above in terms of axial position or velocity values is readily established by examination of Eq. (70) with either y'_w or y'_c equal to zero.

Location of critical points for the more general case where both the inner (y_c) and the outer (y_w) streamlines vary with axial location is much more difficult. As seen by examining Eq. (70) for zeros, each

critical point for this general annulus shape is a function of the inner and outer streamline shape and the velocity values along these streamlines. Numerical calculations for the annulus bounded by

$$y_w = 1.2 (1 + 0.2 x^2)^{1/2} \quad (73)$$

and

$$y_c = 0.2 [1 + 0.05 (x + 0.1)^2]^{1/2} \quad (74)$$

are shown in Fig. 29. These indicate that the first critical point occurs in the vicinity of the nozzle minimum area. The annulus shape is plotted in Fig. 30.

For a given annulus shape, further analysis of the velocity gradients in the vicinity of the critical points such as followed for the nozzle problem could be used to predict the choking condition and the axial location of critical velocities in the annulus flow field. In the two-stream interaction problem, the annulus shape is not known *a priori*; therefore, the velocity gradient behavior in the vicinity of the minimum area critical point is used to determine the choked condition. Although we would expect additional critical points downstream from this location, their existence is not important for the two-stream supersonic ejector solution presented here.

3.2 INNER SUPERSONIC STREAM

The inner supersonic stream is assumed to be an axisymmetric, steady, isentropic, irrotational supersonic flow field. At the primary nozzle exit plane, the supersonic flow field is furthermore assumed to be uniform. The well developed method of characteristics was used for calculating the supersonic velocity field downstream from the primary

nozzle exit plane.

Initial conditions for the method of characteristics were established by dividing the Mach line originating at the nozzle lip into 19 equally spaced locations and by a Prandtl-Meyer expansion about the nozzle lip from the primary stream pressure in the exit plane to the secondary stream pressure. This Prandtl-Meyer expansion was subdivided into approximately one degree increments.

The calculations for the axisymmetric supersonic flow field can be classified as one of three basic processes:

- a. Field point calculation,
- b. Axis point calculation, or
- c. Boundary point calculation.

The first two these unit processes have been described in detail by Shapiro (Vol. II) [19], Addy [21], and Brown [23]. These will be presented here for completeness. The boundary point calculation process used here must be compatible with the outer stream solution by integral relations along the common boundary (inviscid interaction). It differs from the boundary point calculations used by Addy [21] and by Brown [23], and will therefore be discussed in more detail.

For the known field points (points 1 and 2 in Fig. 31a) in the primary flow field, the unknown values at field point 3 can be found with an iteration process using the finite difference equations:

$$(y_3 - y_1) = (x_3 - x_1) \tan (\theta - \alpha)_{13} \quad (75)$$

$$(y_3 - y_2) = (x_3 - c_2) \tan (\theta + \alpha)_{23} \quad (76)$$

where θ and α are the streamline and Mach angles, respectively. The double subscript indicates that average values are used in the calculation. The values of M_3^* and θ_3 can be found by simultaneous solution

of the following equations

$$(\theta_3 - \theta_1) + \frac{M_3^* - M_1^*}{(M^* \tan \alpha)_{13}} - \left[\frac{1}{y} \frac{\tan \theta \tan \alpha}{\tan \theta - \tan \alpha} \right]_{13} (y_3 - y_1) = 0 \quad (77)$$

$$(\theta_3 - \theta_2) - \frac{M_3^* - M_2^*}{(M^* \tan \alpha)_{23}} + \left[\frac{1}{y} \frac{\tan \theta \tan \alpha}{\tan \theta + \tan \alpha} \right]_{23} (y_3 - y_2) = 0 \quad (78)$$

There are two types of axis point calculations. For the first type (Fig. 31b), the property values are known along the axis. The equations for the unknown field point 3 in terms of the known axis point 2 and field point 1 are identical with the just described field point equations. Care must be taken in the iteration, however, so that the average value y_{23} is never set equal to zero.

The second type of axis point calculation (Fig. 31c) locates an unknown point 3 along the axis when the values at field point 1 are known. For this case, simultaneous solution of Eqs. (75) and (77) yields the axial location (x_3) and the dimensionless velocity (M_3^*) at point 3 since $y_3 = 0$ and $\theta_3 = 0$.

The boundary point calculation is unique to the two-stream interaction problem. Along the streamline boundary between the primary and secondary streams, both the pressures and the flow angles must match. The approach used here to determine the properties at a boundary point required an iteration between the primary and secondary stream solutions.

a. At some axial location I (see Fig. 32), the primary stream boundary point (0) is known and the conditions of the secondary stream are known (q_{c_I}, q_{w_I}).

b. The pressure gradient along the dividing line is estimated from the outer stream velocity gradient. Since the primary and secondary stream pressure gradients must be equal along the dividing

streamline, the primary stream velocity gradient is related to the secondary stream velocity gradient along this dividing streamline, by

$$\frac{dM_p^*}{dx} = \frac{p_{os}}{p_{op}} \frac{M_s^*}{M_p^*} \frac{\left[1 - \frac{\gamma_s - 1}{\gamma_s + 1} M_s^{*2}\right]^{1/\gamma_s - 1}}{\left[1 - \frac{\gamma_p - 1}{\gamma_p + 1} M_p^{*2}\right]^{1/\gamma_p - 1}} \left[\frac{\frac{\gamma_s}{\gamma_s + 1}}{\frac{\gamma_p}{\gamma_p + 1}} \right] \frac{dM_s^*}{dx} \quad (79)$$

where p_{op} and p_{os} are defined as the primary and secondary stream stagnation pressure values along this dividing streamline.

c. The properties at the primary stream boundary point are then calculated from the known primary stream velocity gradient along the dividing streamline in conjunction with a characteristic curve of family II originating from the point 2 (see Fig. 32). We may write

$$M_3^* = \frac{dM_p^*}{dx} \bigg|_I (x_3 - x_1) + M_{pI}^* \quad (80)$$

$$y_3 = y_o + (x_3 - x_o) [\tan \theta]_{o3} \quad (81)$$

along the boundary, and

$$(y_3 - y_2) = (x_3 - x_2) \tan (\theta + \alpha)_{23} \quad (76)$$

$$(\theta_3 - \theta_2) - \frac{M_3^* - M_2^*}{(M^* \tan \alpha)_{23}} + \left[\frac{1}{y} \frac{\tan \theta \tan \alpha}{\tan \theta + \tan \alpha} \right]_{23}$$

$$\cdot (y_3 - y_2) = 0 \quad (78)$$

along the family II characteristic. Simultaneous solution yields the properties at the unknown boundary point 3.

3.3 TWO STREAM SOLUTION METHOD AND RESULTS

As previously discussed, the method of solution for the secondary stream by integral relations has been developed to calculate the flow

field behavior when the initial velocity values and the inner and outer streamline shapes are given. The primary stream calculation method has been developed to determine the streamline location given the pressure gradient along the dividing streamline. Solution of the two-stream problem is accomplished through an iteration process which utilizes the characteristics of the two-solution methods. The proper location of the dividing streamline satisfies the basic requirement that the pressure must be continuous across the dividing streamline.

The two-stream calculation was initiated at the primary nozzle exit plate from the known initial values of the primary and secondary stream stagnation conditions and the primary stream exit velocity. The secondary stream initial velocity profile (assumed as one-dimensional) was then determined from the calculated secondary stream initial pressure ratio. After calculating the initial secondary stream velocity gradients from Eqs. (61) and (62) and initiating the method of characteristics solution for the primary stream, the two-stream solution was continued by an iteration calculation.

Referring to Fig. 22, the iteration method for calculating the two-stream solution is initiated at some axial location (I) where all variables are known for both the primary and the secondary streams. The iteration procedure is then:

- a. The method of characteristics solution is carried out to evaluate the variables at the field point 2.
- b. The pressure gradient along the dividing streamline is estimated from the known properties at section I.
- c. The location of the boundary point 3 is calculated from the dividing streamline pressure gradient and the properties

along the family II characteristics running through the field point 2.

- d. The geometric location of boundary point 3 and the flow direction at this point are used to estimate the streamline shape between points 0 and 3.
- e. The outer stream integral relations solution is then calculated from known values at section I to section J bounded by the dividing streamline shape estimated in step d above.
- f. At section J, the calculated pressure along the dividing streamline from the secondary stream solution is compared with that calculated from the primary stream solution. If the two pressures agree within a specified percentage (0.01 percent), the solution is continued downstream. If the two pressures do not agree, the pressure gradient between sections I and J is re-estimated and the calculations for boundary point 3 are repeated.

The wall pressure distribution for a variable area supersonic ejector is shown in Fig. 33. These results agree with the experimental and theoretical results published by Chow and Yeh [24]. We also note the behavior of the integral relations solution in this figure. The calculated wall pressure distribution shows typical saddle-point behavior as the secondary stream approaches the "choking condition." Very small changes in the secondary stream initial conditions cause large variations in the calculated pressure distributions in this region.

4. SUMMARY AND CONCLUSIONS

The preceding study has developed an integral solution to the two-dimensional nozzle problem which can be solved by conventional numerical marching integration techniques. This characteristic of the solution makes it readily compatible with another direct solution procedure (the method of characteristics). Thus, multiple stream interaction problems involving subsonic and supersonic flow fields may be solved with a marching integration scheme from given initial conditions and a specified wall geometry. The solution method for a two-stream supersonic ejector problem has been developed to illustrate this procedure.

In the development of the solution method a discrepancy in the two methods of defining the choking condition used in the literature was noted. Although the actual difference in physical results such as velocity and mass flux distributions obtained from the two considerations is negligibly small, a proper understanding of the nozzle problem requires that this discrepancy be resolved.

Following this reasoning, it was found that the velocity gradient distribution curves were extremely sensitive to the nozzle mass flux, particularly near the choking value. From analysis of the velocity gradient behavior, it was concluded that the proper choking condition is determined by the finite velocity gradient condition at the nozzle throat.

Further analysis of the velocity gradients and the mathematical characteristics of the solution indicate that a symmetric nozzle (determined by $y''' = 0$ at the throat) has a larger value of choking mass flux than a non-symmetric nozzle (defined by $y''' \neq 0$ at the throat)

of the same throat area and radius of curvature. The non-symmetric nozzle may also have finite non-zero values of acceleration at the nozzle throat for subcritical values of mass flux.

Additional work that may be undertaken to improve and develop this solution method are

- a. Improve the polynomial approximations for the normal velocity distributions by using multiple strip integral relations.
- b. Investigate the solution characteristics for a non-symmetric nozzle shape.
- c. Explore the condition of flow with shocks within the divergent nozzle by the integral relations.

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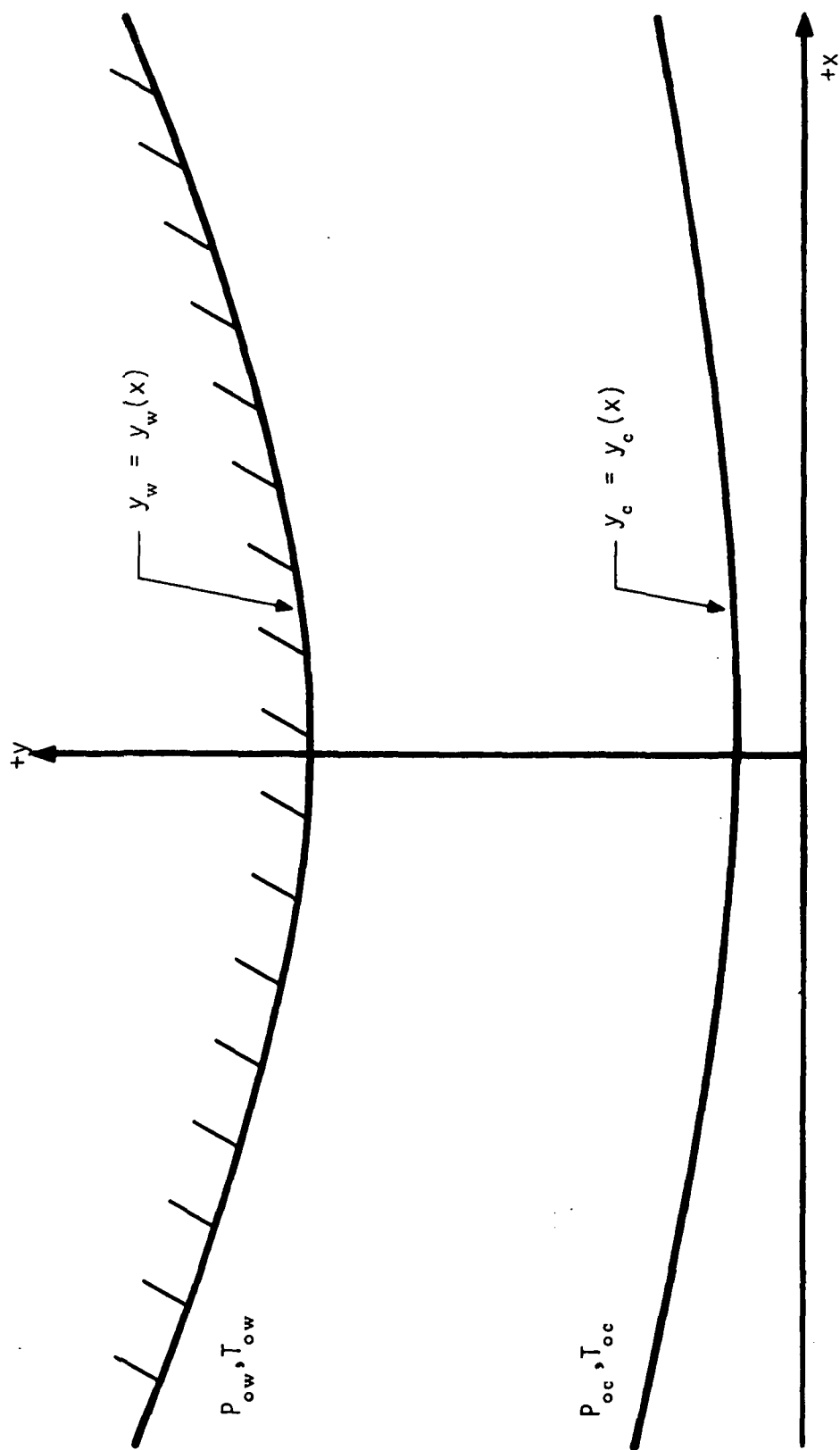


Figure 1 Flow Field Geometry ($y_c = 0$ for Nozzle Problem)

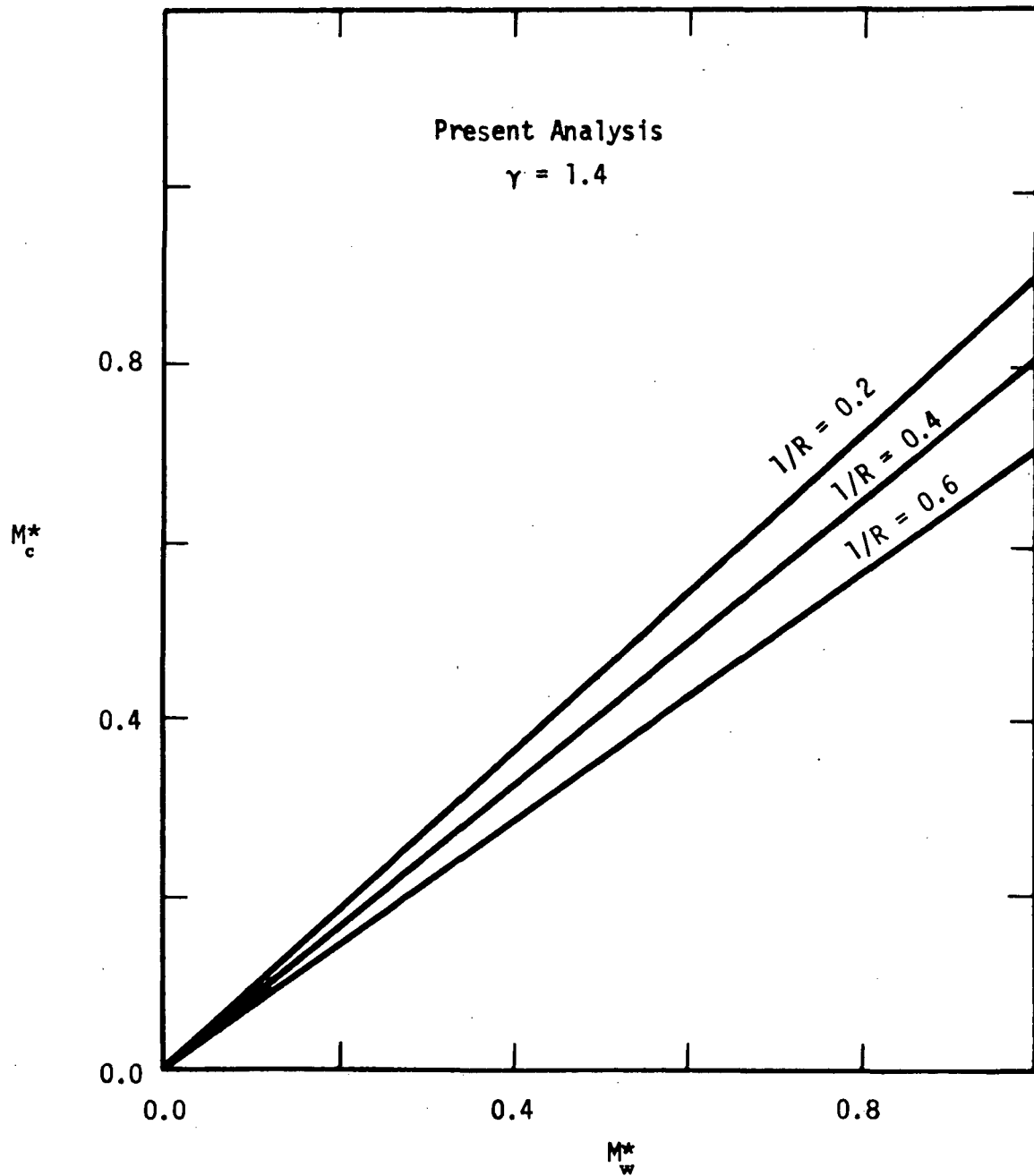


Figure 2 Proper Velocity Pairs at the Geometric Throat (Present Analysis)

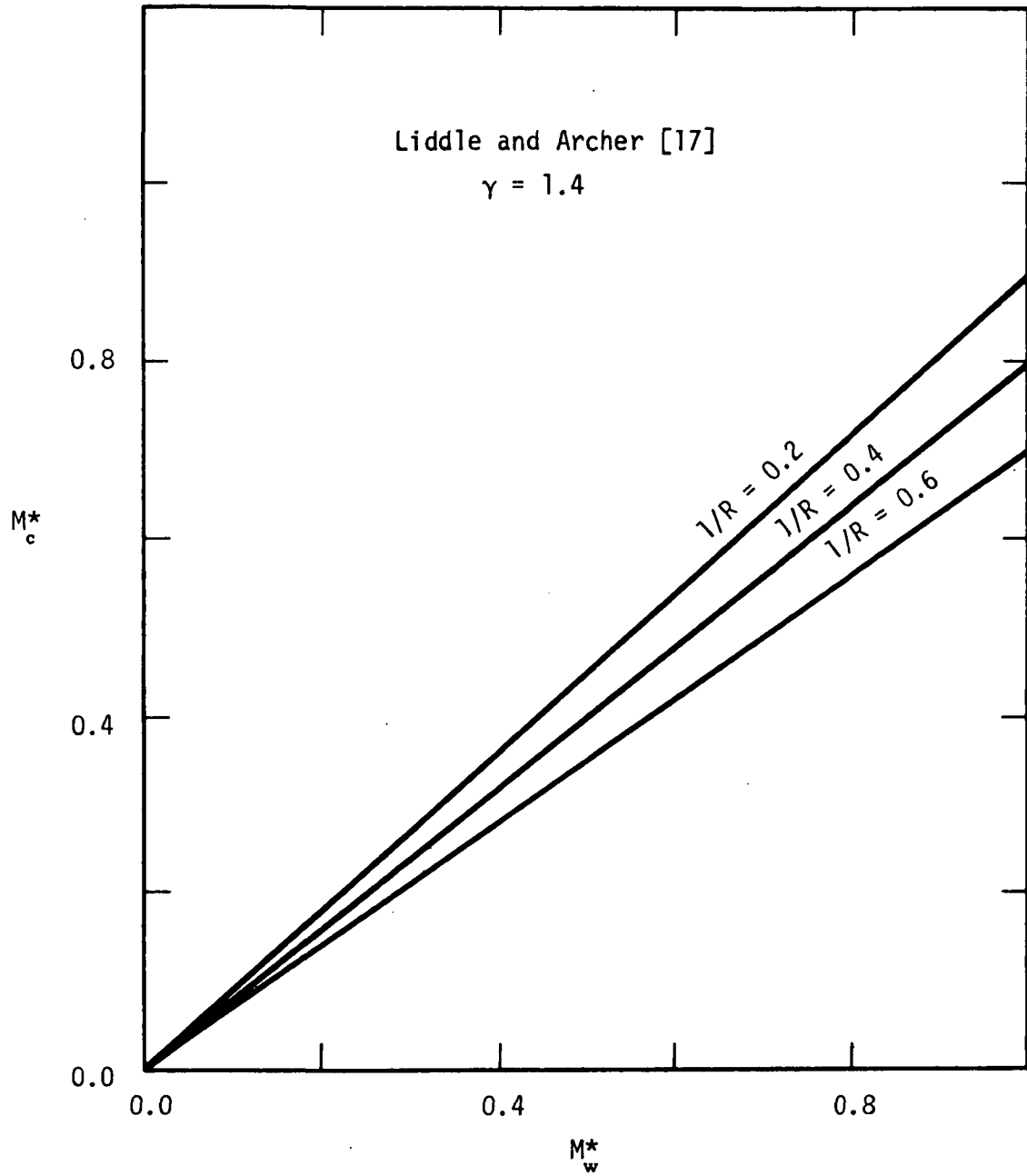


Figure 3 Proper Velocity Pairs at the Geometric Throat
(Liddle and Archer [17])

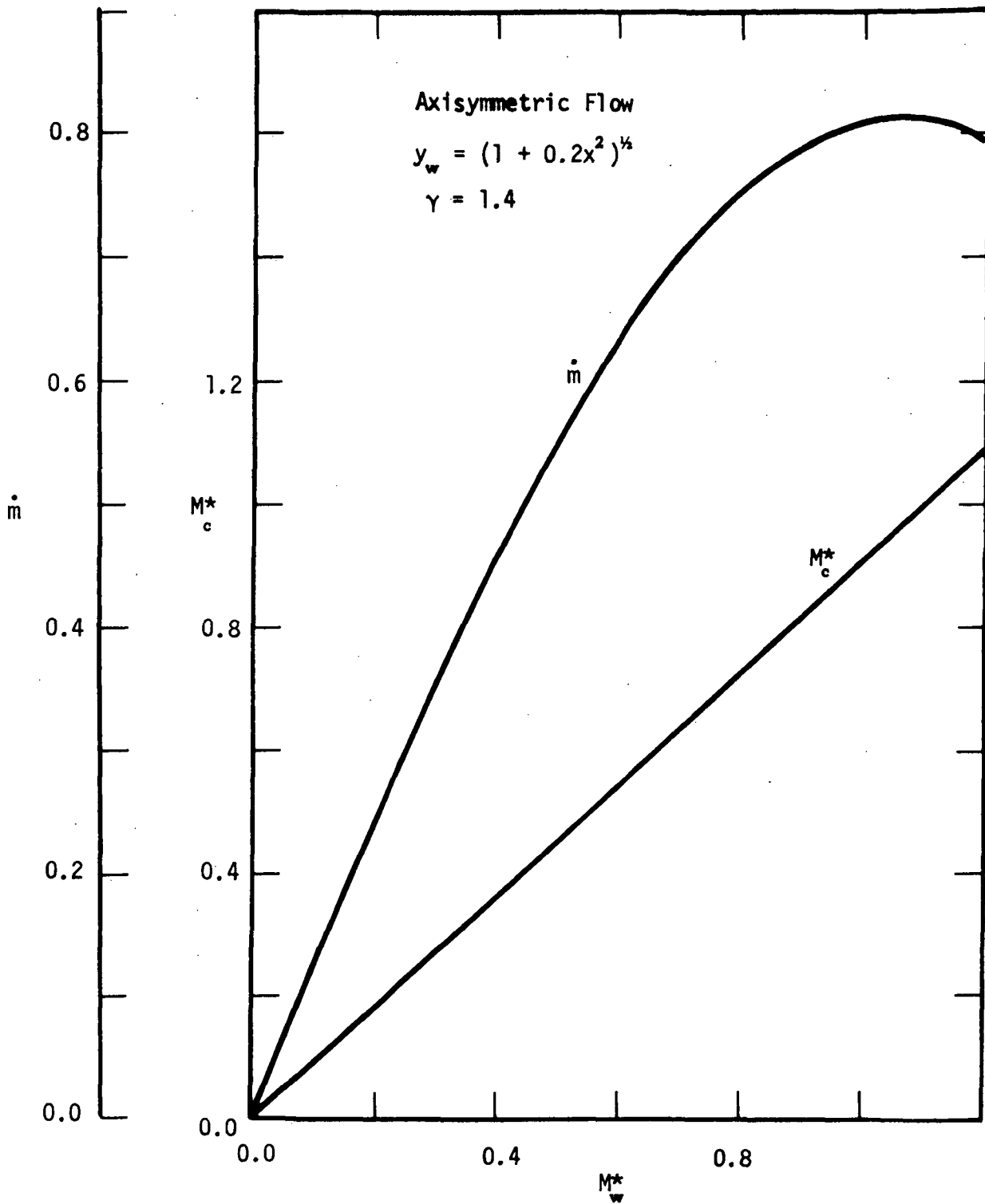


Figure 4 Conditions at the Nozzle Throat

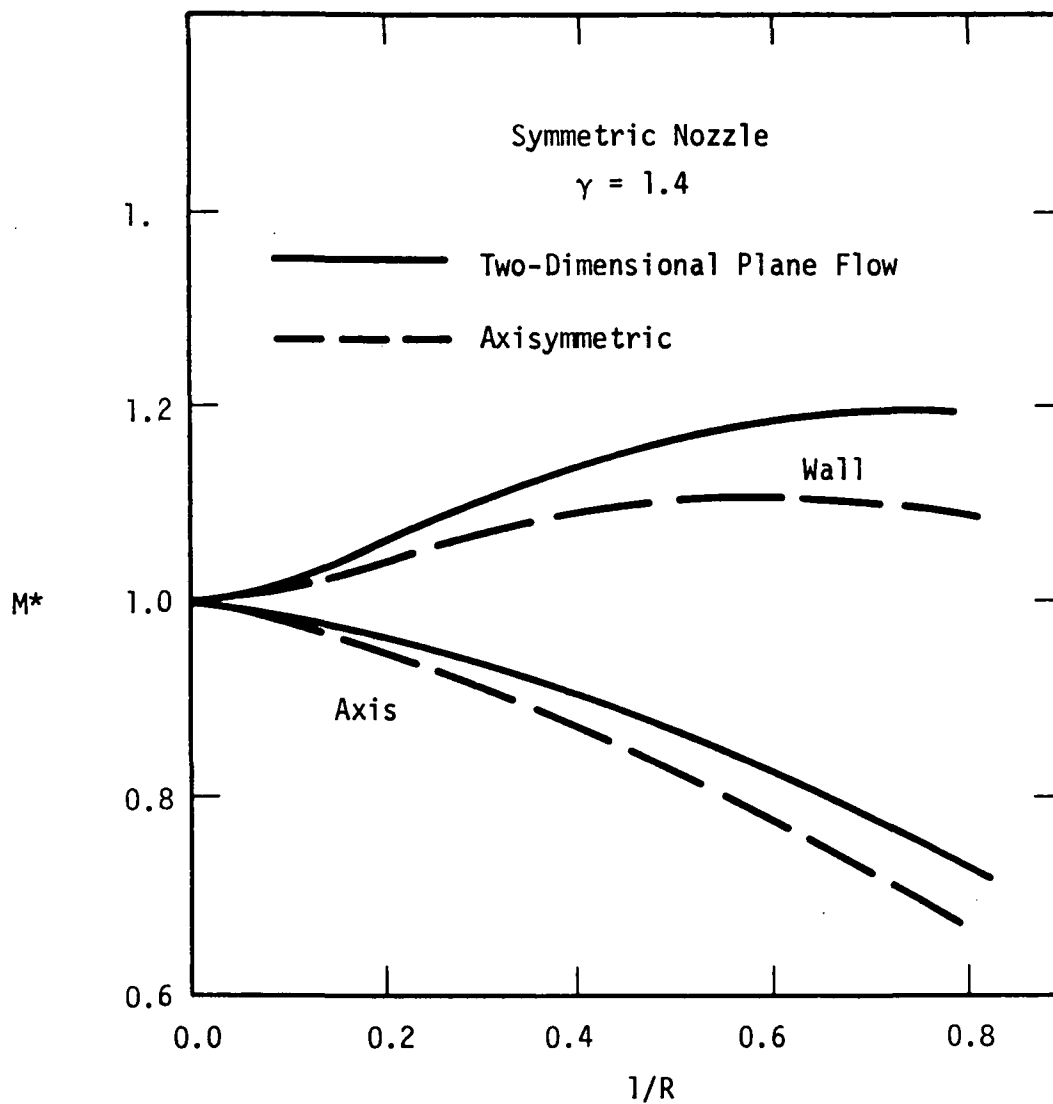


Figure 5 Comparison between Two-Dimensional and Axisymmetric Throat Velocities at the Choking Condition

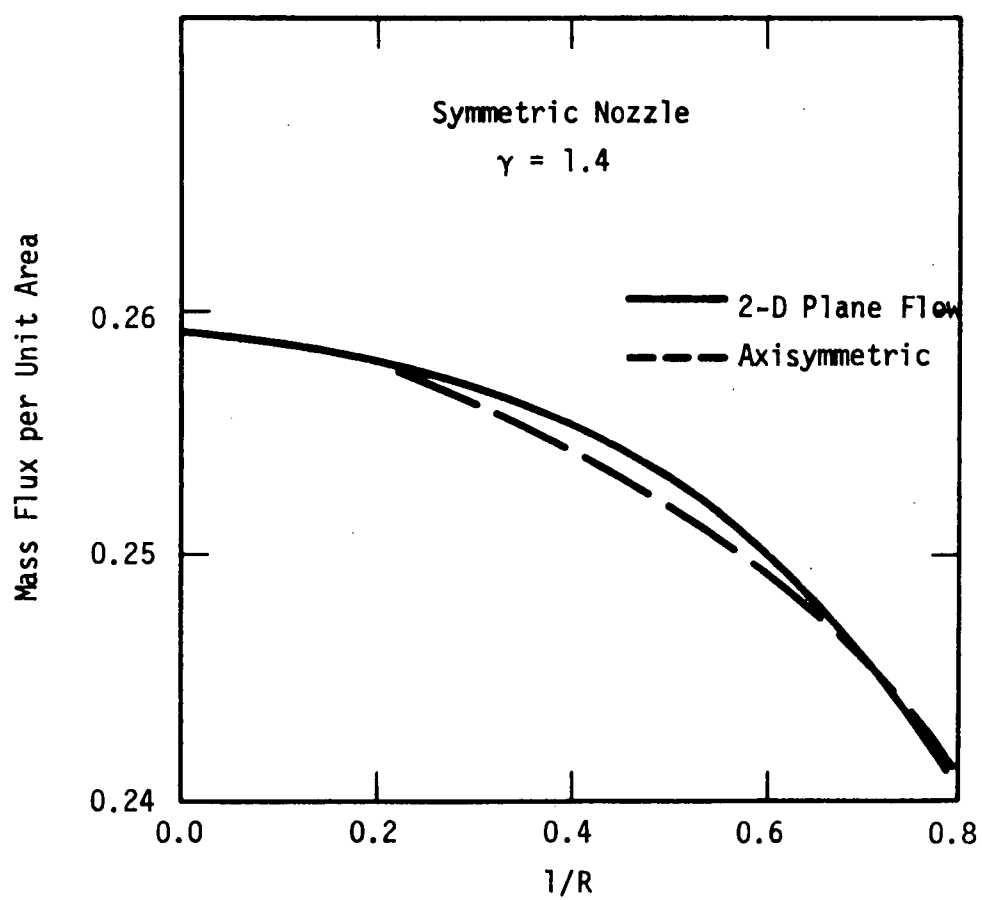


Figure 6 Dimensionless Choking Mass Flux per Unit Area at the Geometric Throat

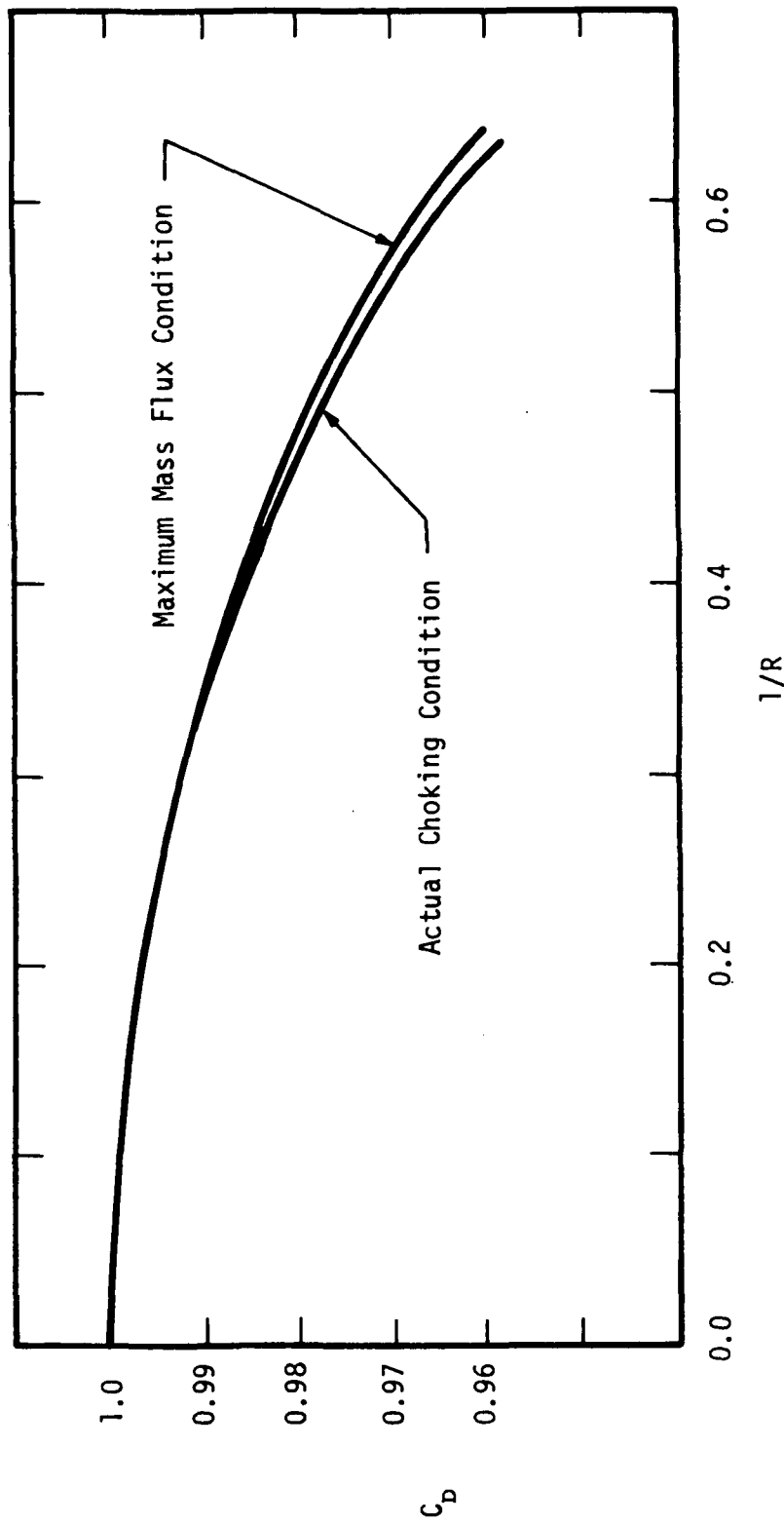


Figure 7 Flow Coefficient vs. the Inverse Throat Radius of Curvature

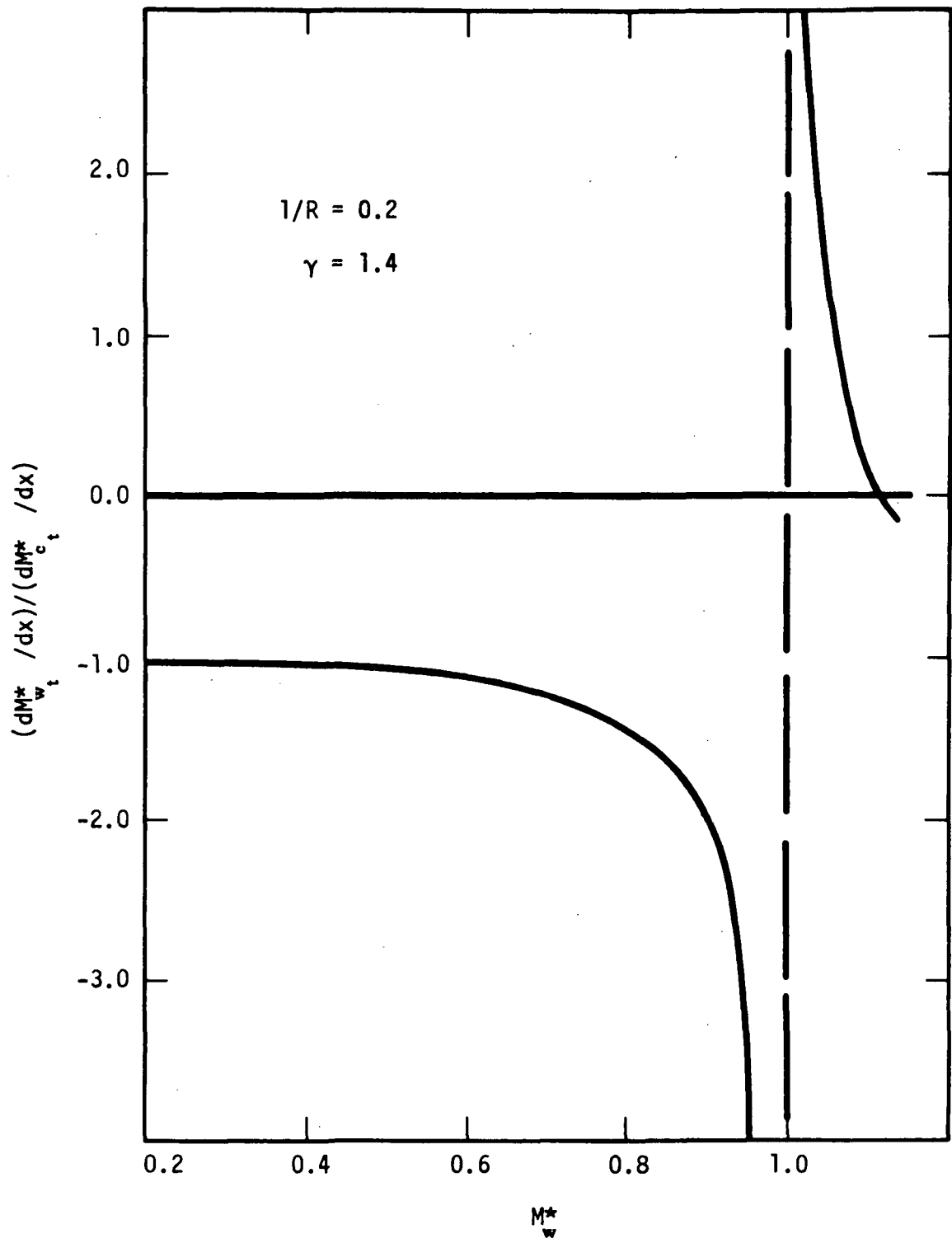


Figure 8 Ratio of Velocity Gradients at the Geometric Throat

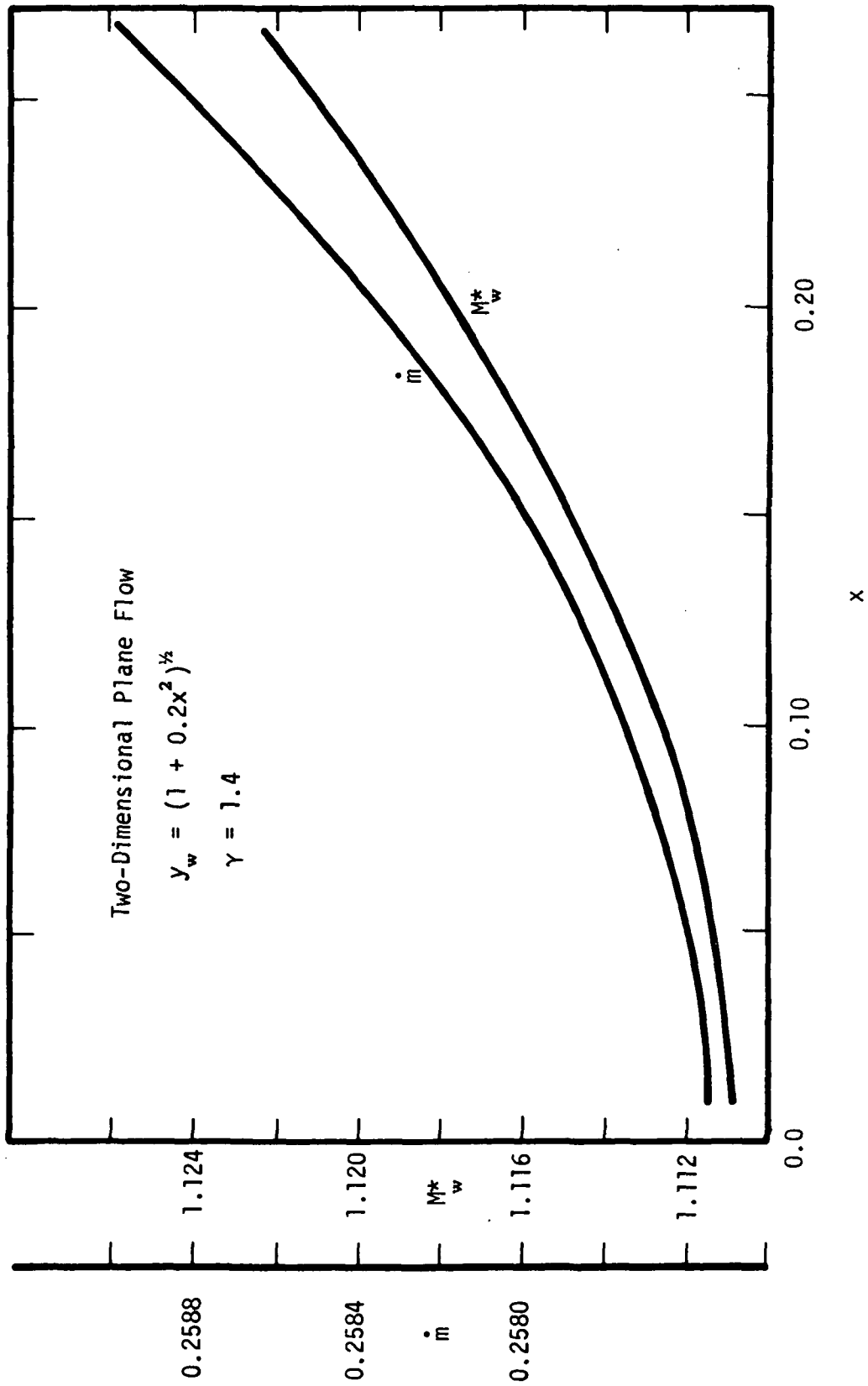


Figure 9 Conditions for Location of Axis Sonic Point
(Two-Dimensional Plane Flow)

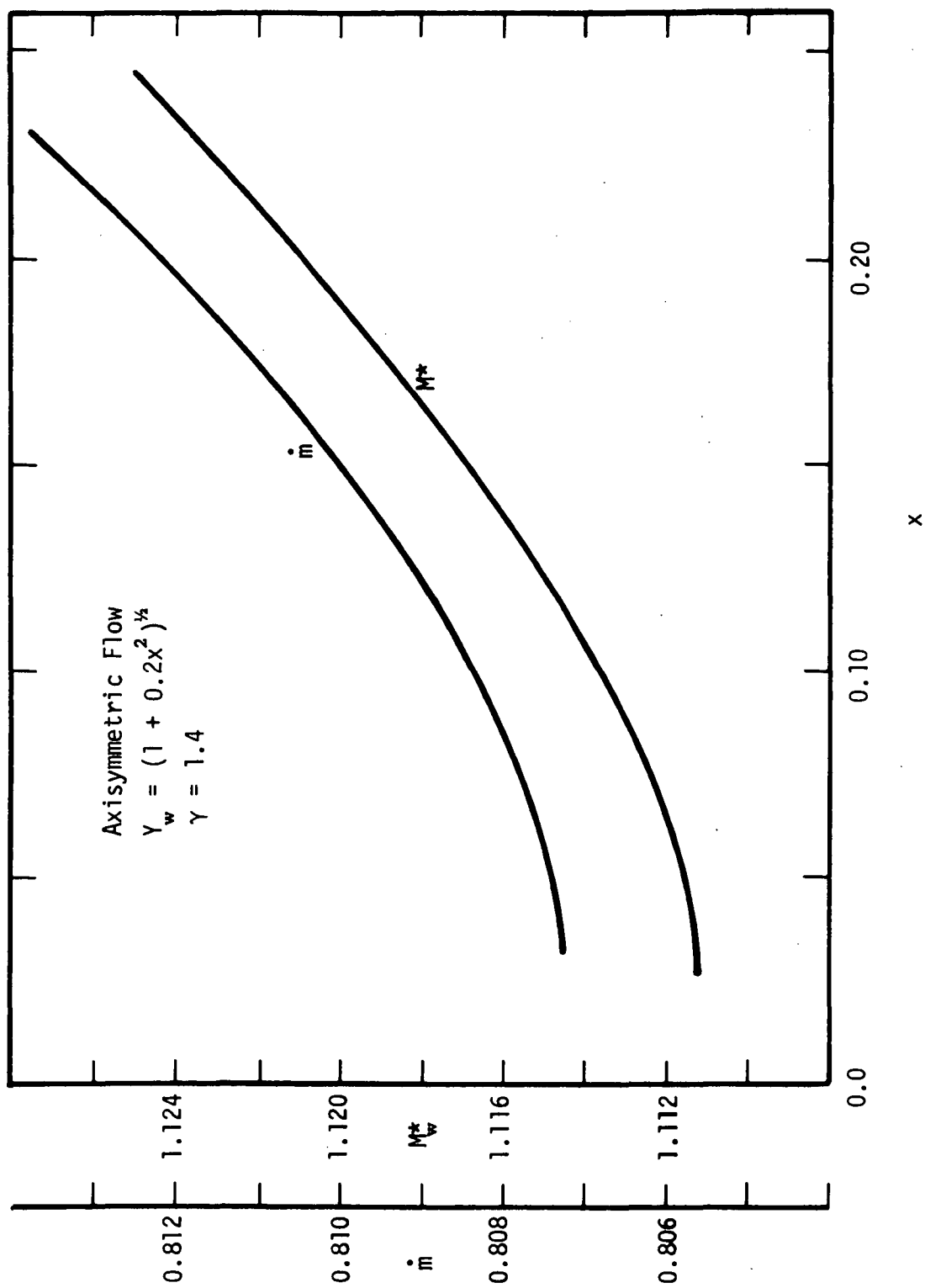


Figure 10 Conditions for Location of Axis Sonic Point (Axisymmetric Flow)

Two-Dimensional Flow

$$y_w = (1 + 0.2x^2)^{1/2}$$

$$\gamma = 1.4$$

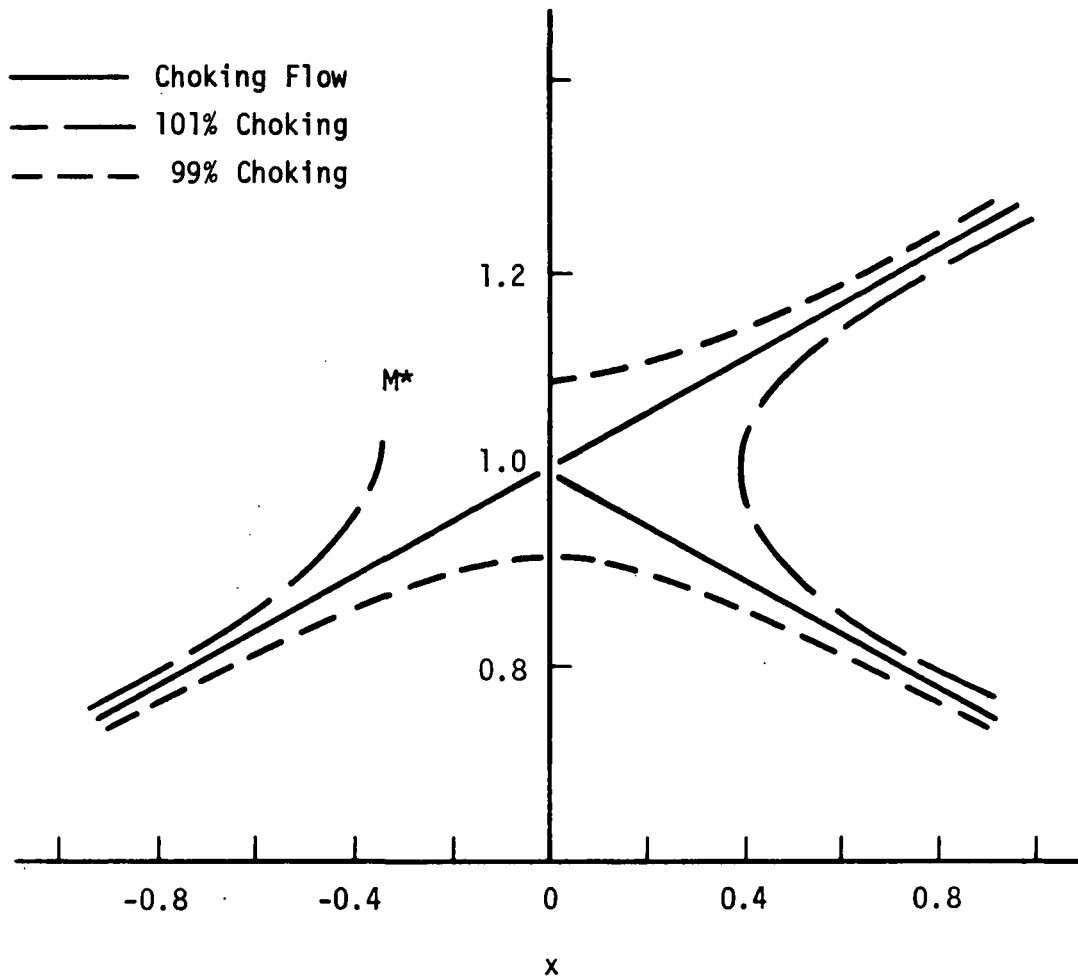


Figure 11 One-Dimensional Velocity Distribution

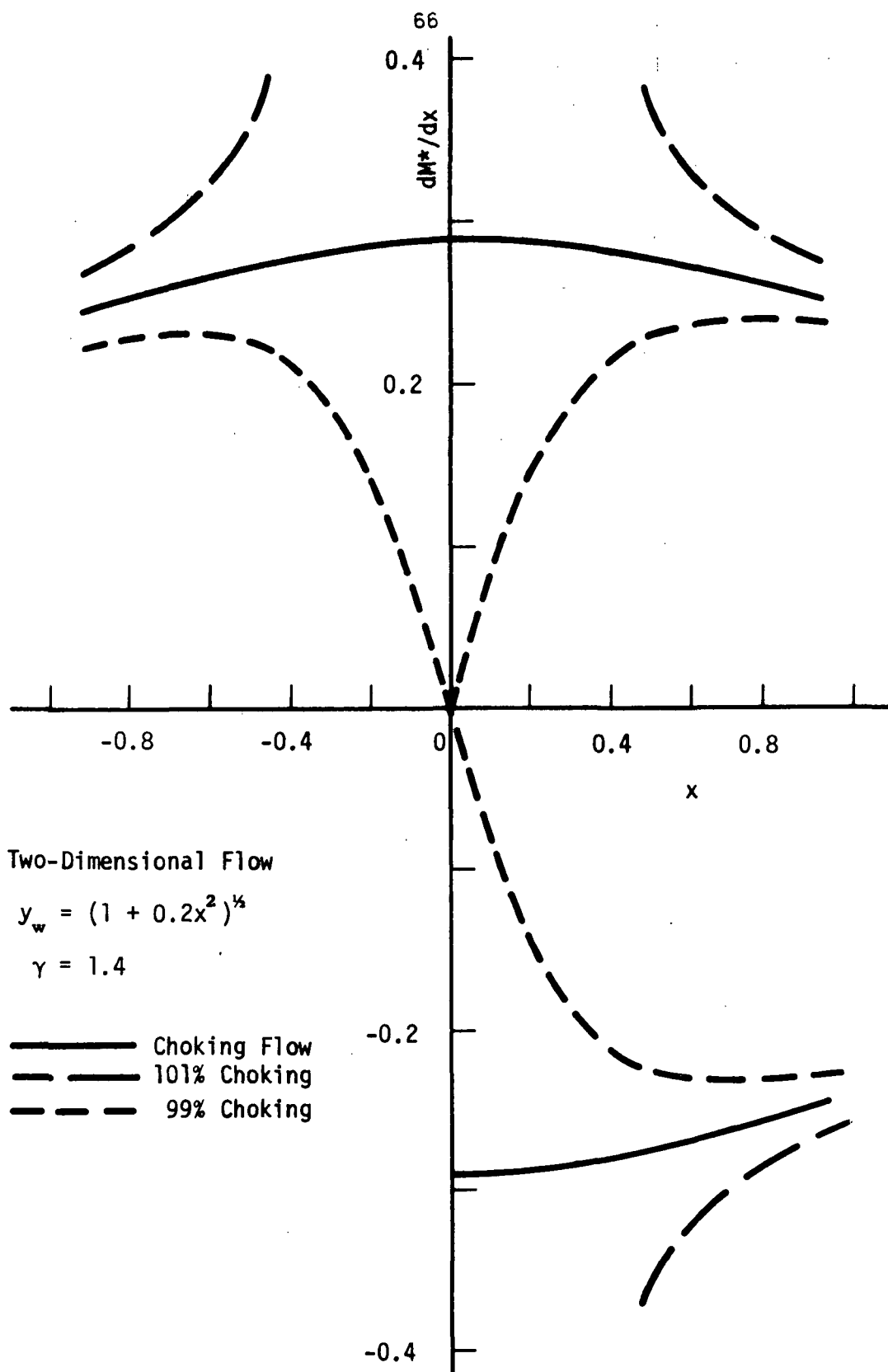


Figure 12 One-Dimensional Velocity Gradient Distribution

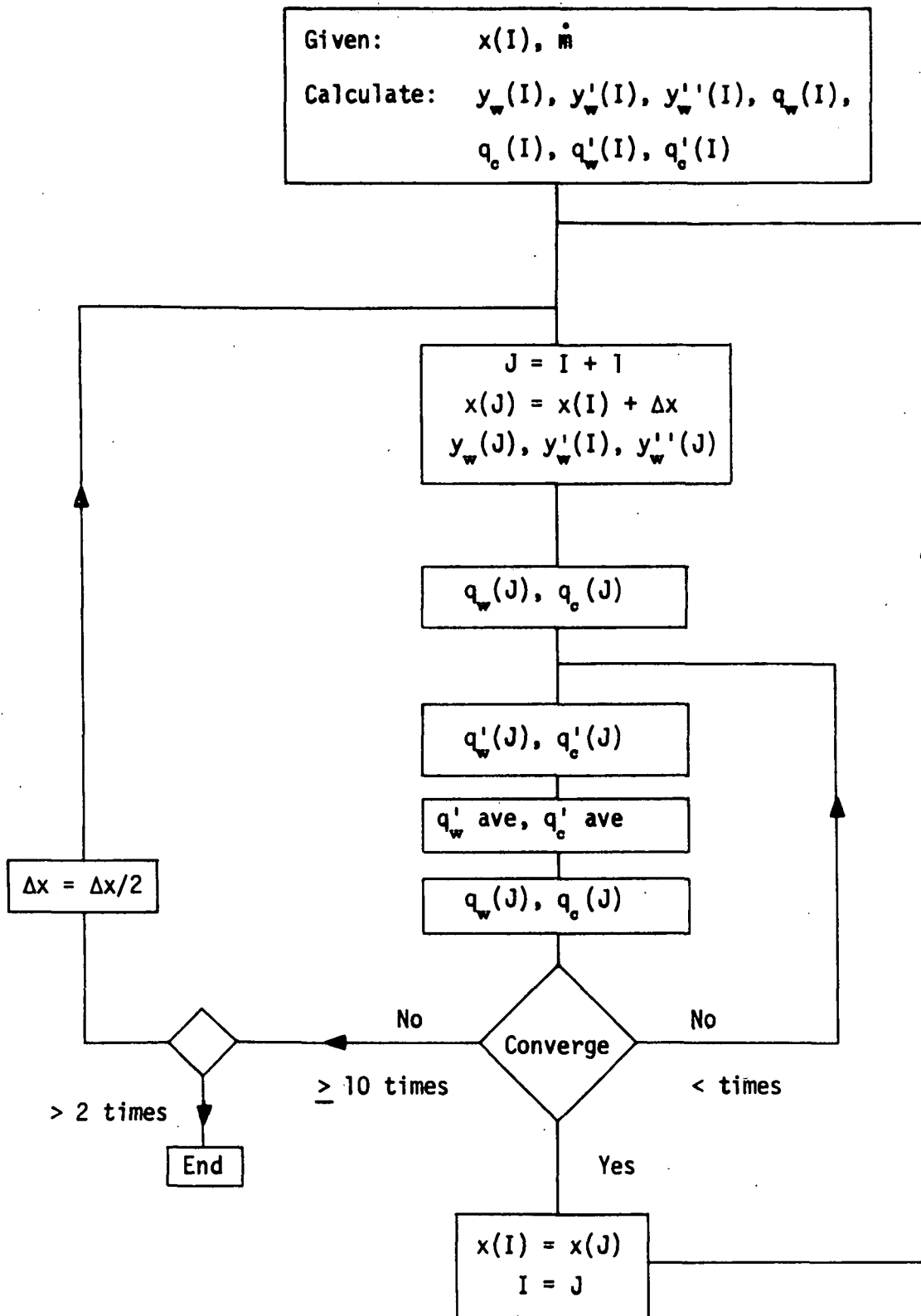


Figure 13 Numerical Integration Method

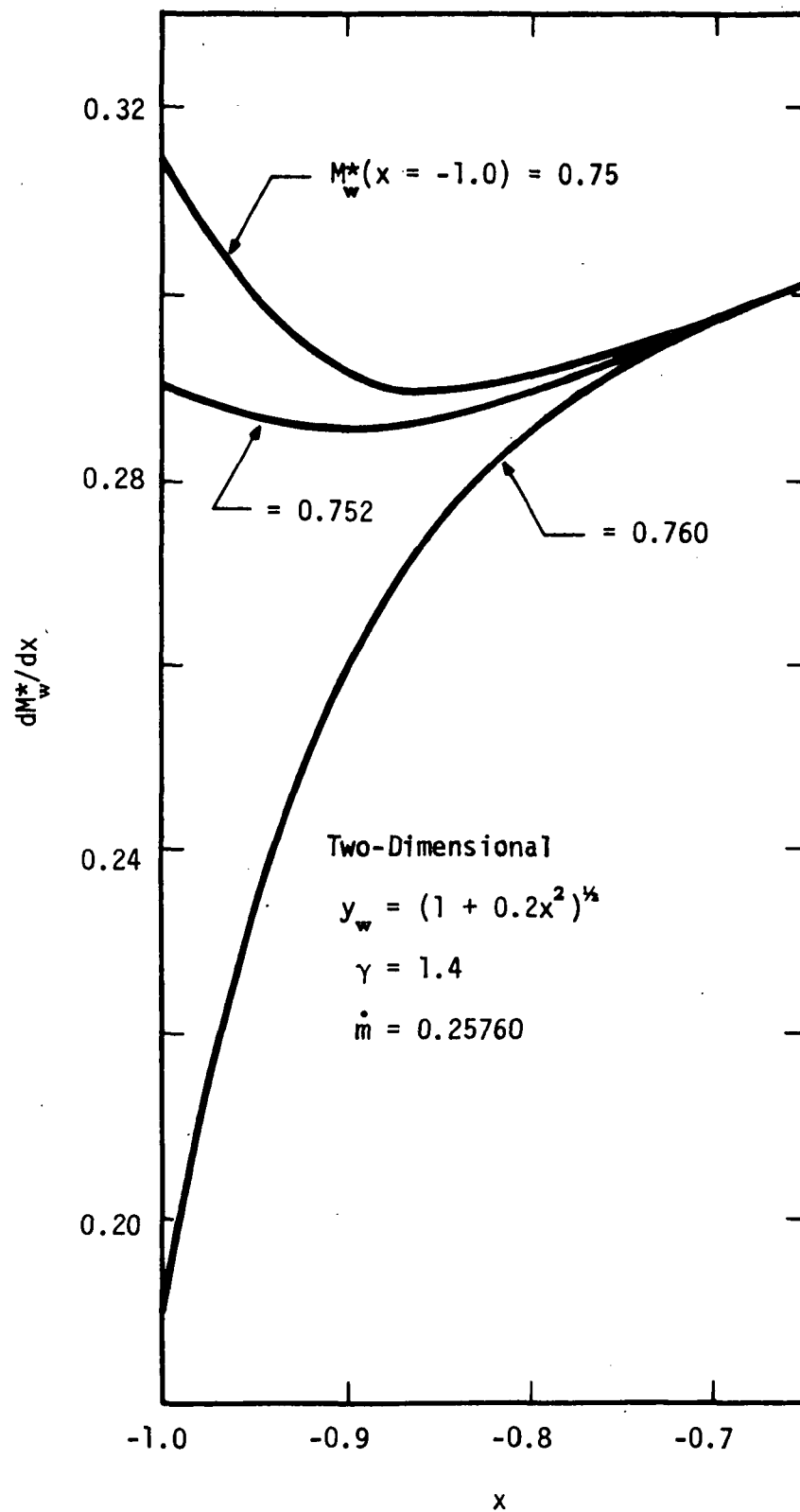


Figure 14 Influence of Initial Velocity Value on the Velocity Gradient Distribution

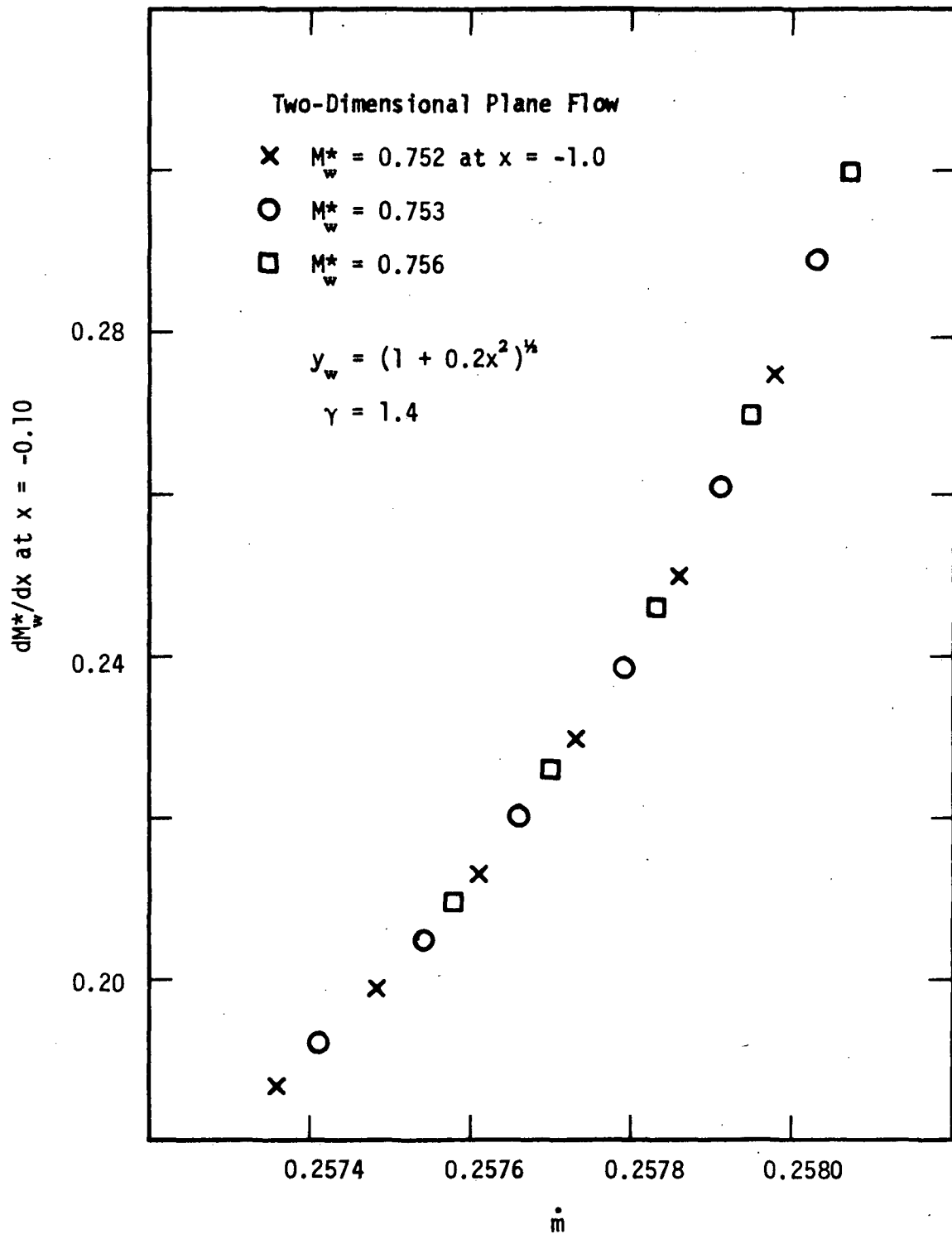


Figure 15 Independence of Downstream Velocity Gradients from Initial Velocities for a Specified Mass Flux

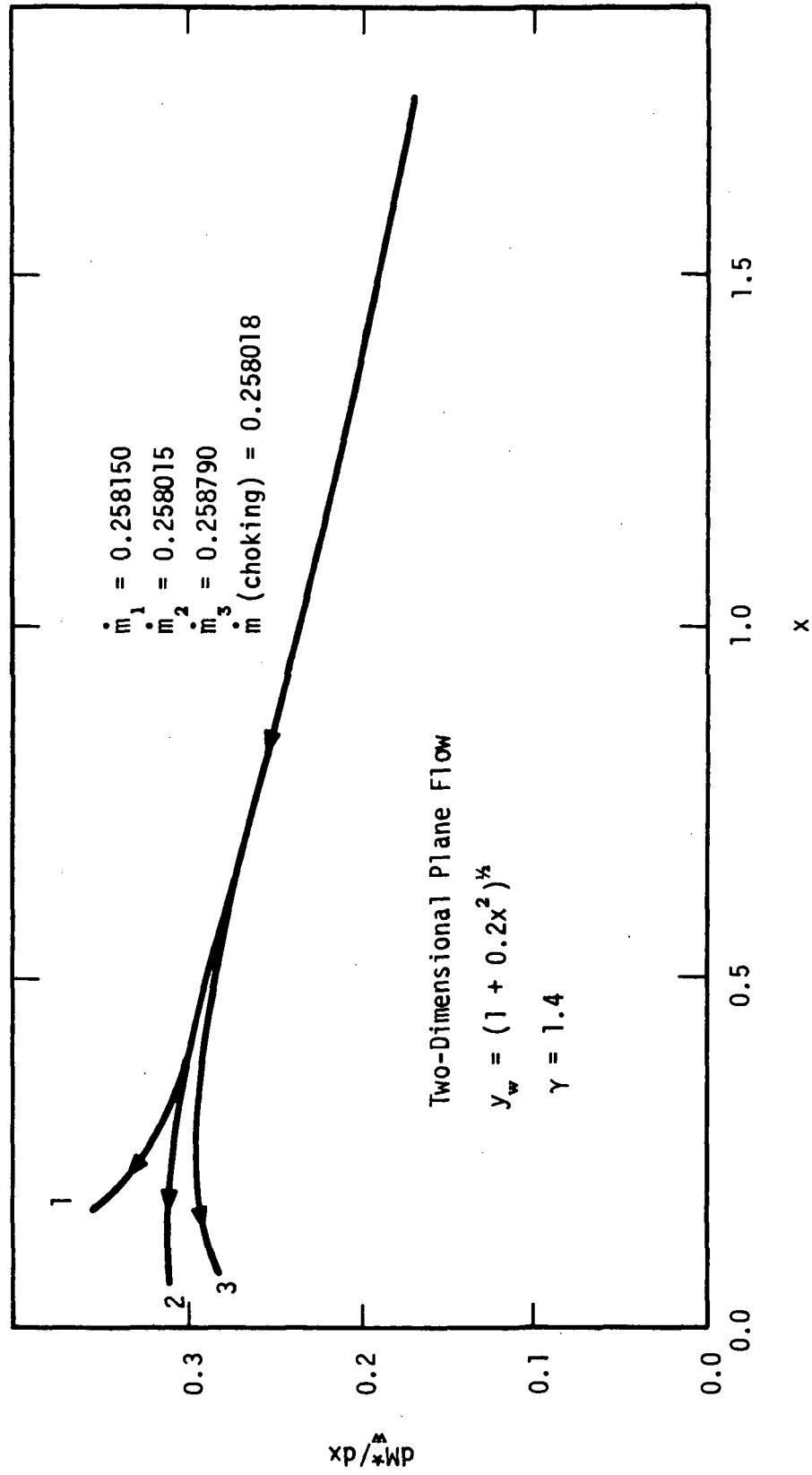


Figure 16 Two-Dimensional Velocity Gradient vs. Axial Location

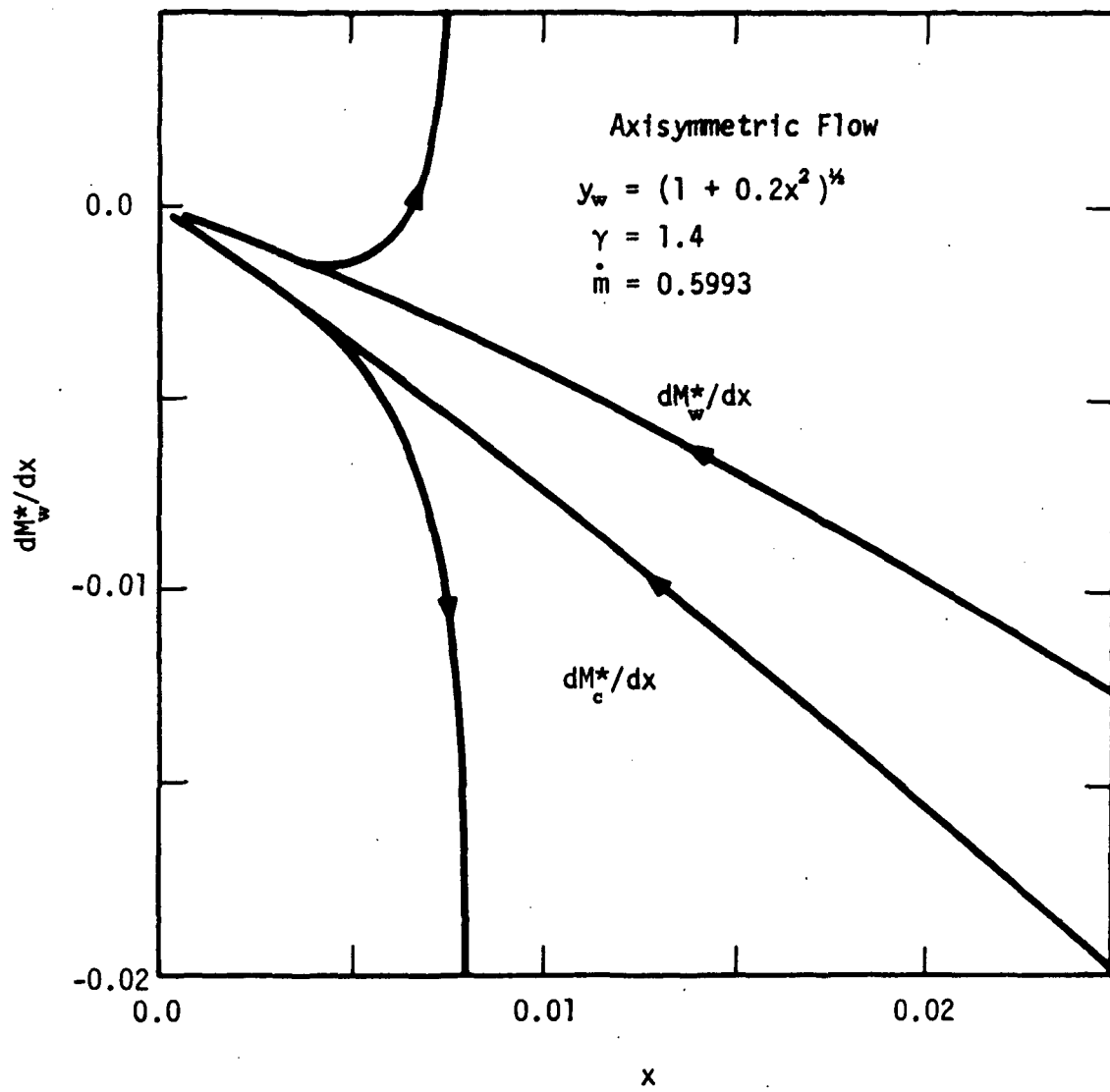


Figure 17 Directional Properties of Downstream De-accelerating Flow

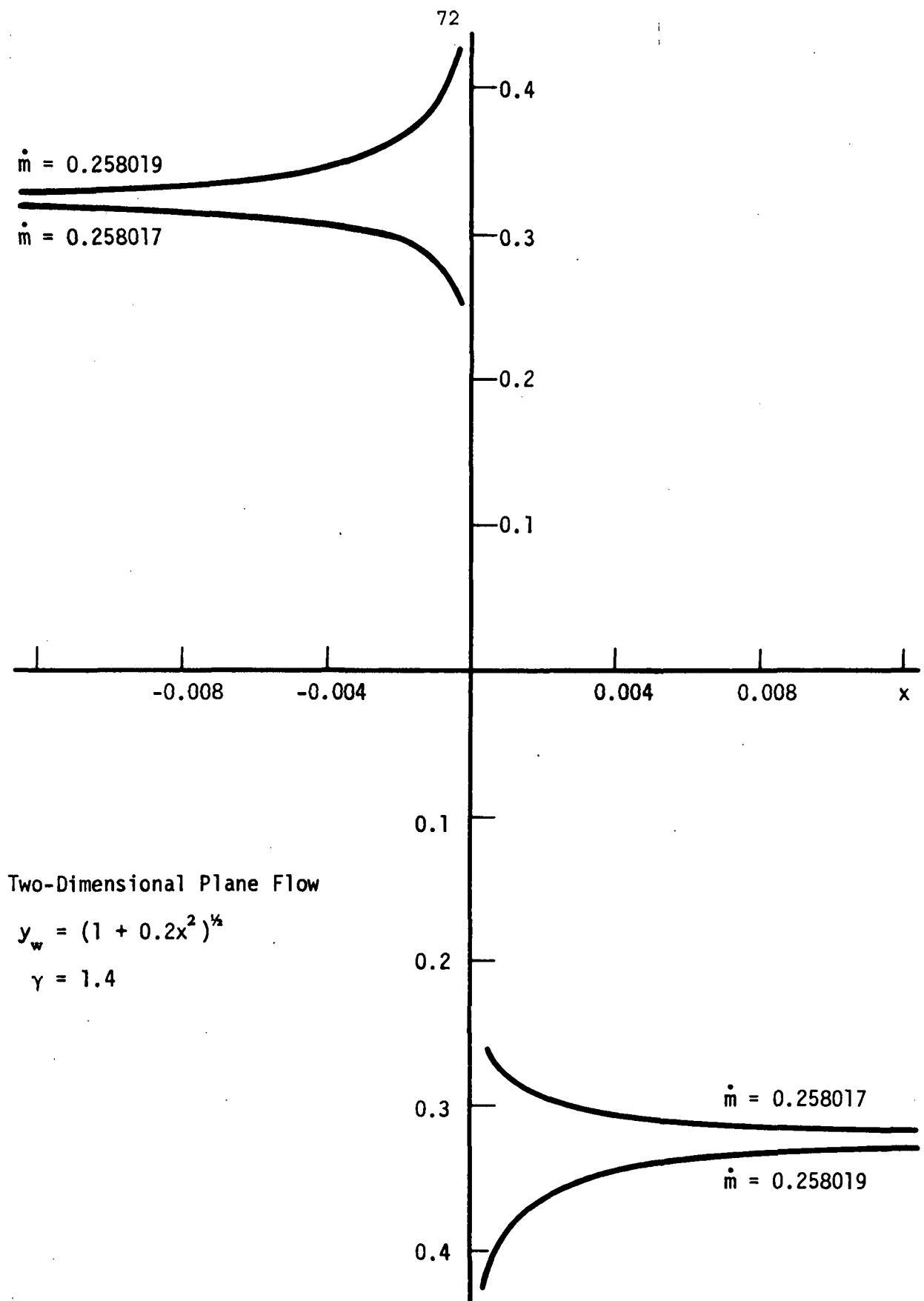


Figure 18 Wall Velocity Gradients Behavior at Mass Flows Near Choking

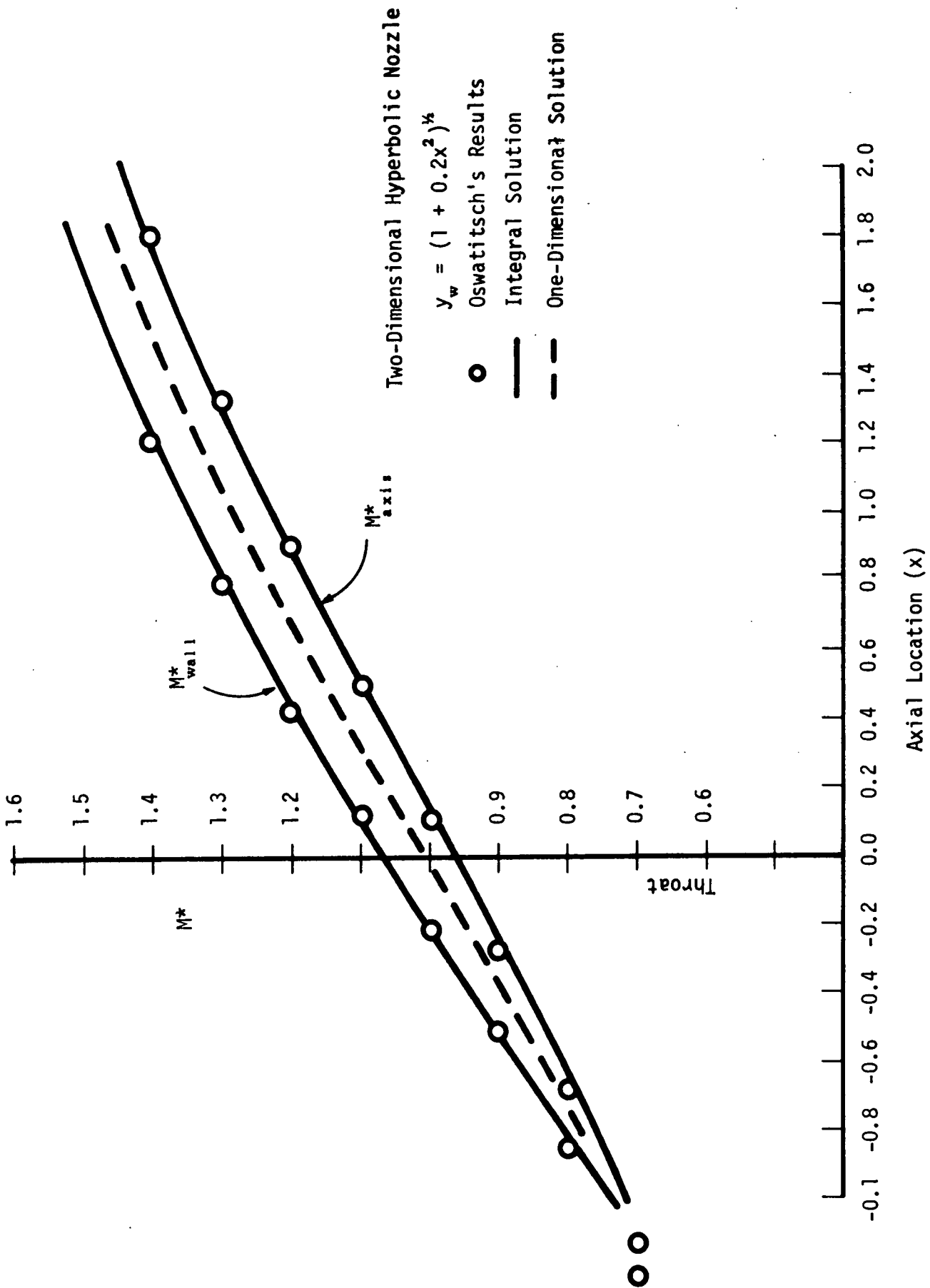


Figure 19 Calculated Velocity Distributions

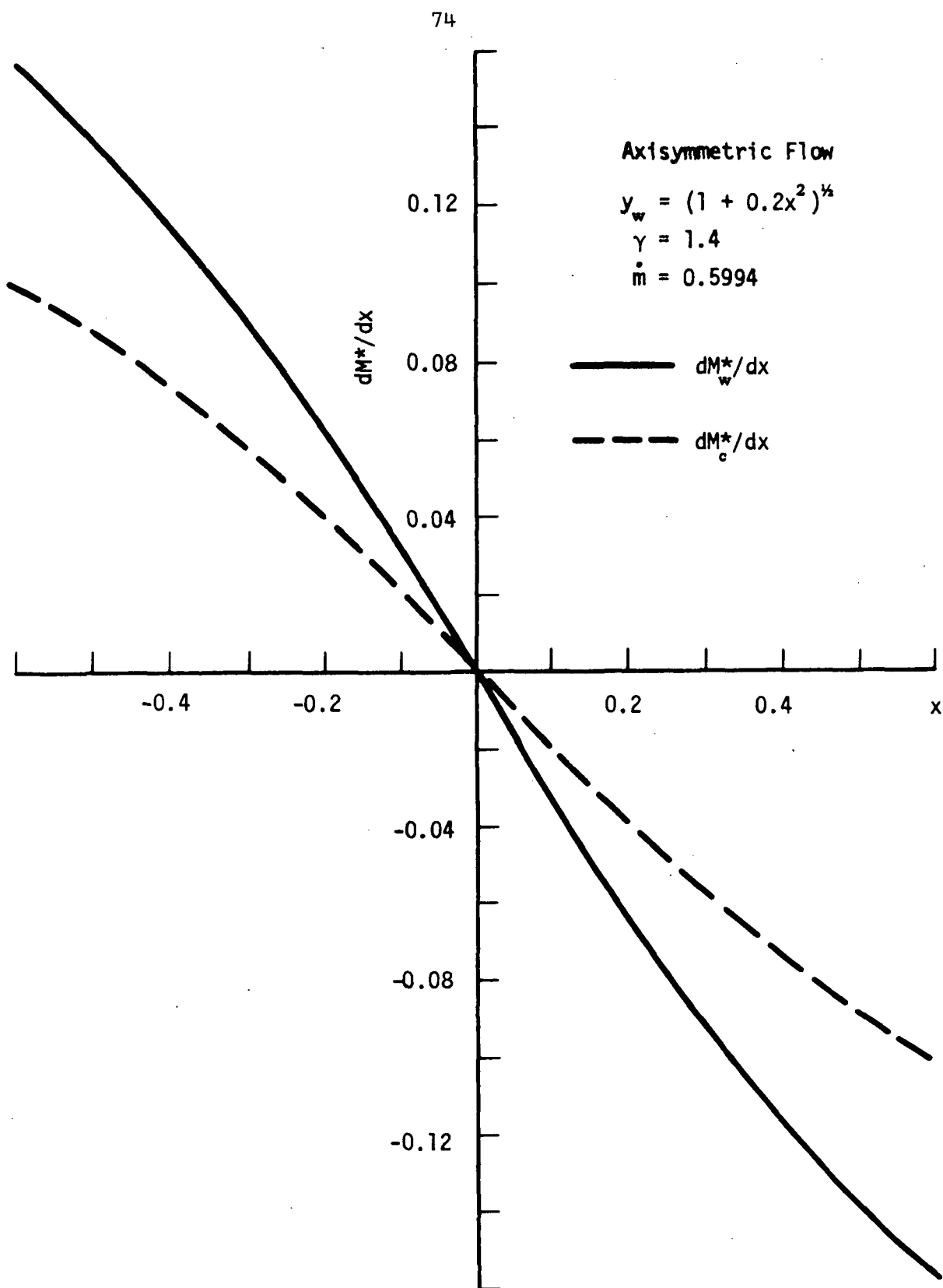


Figure 20 Velocity Gradient Distribution for Mass Flux Less than Choking

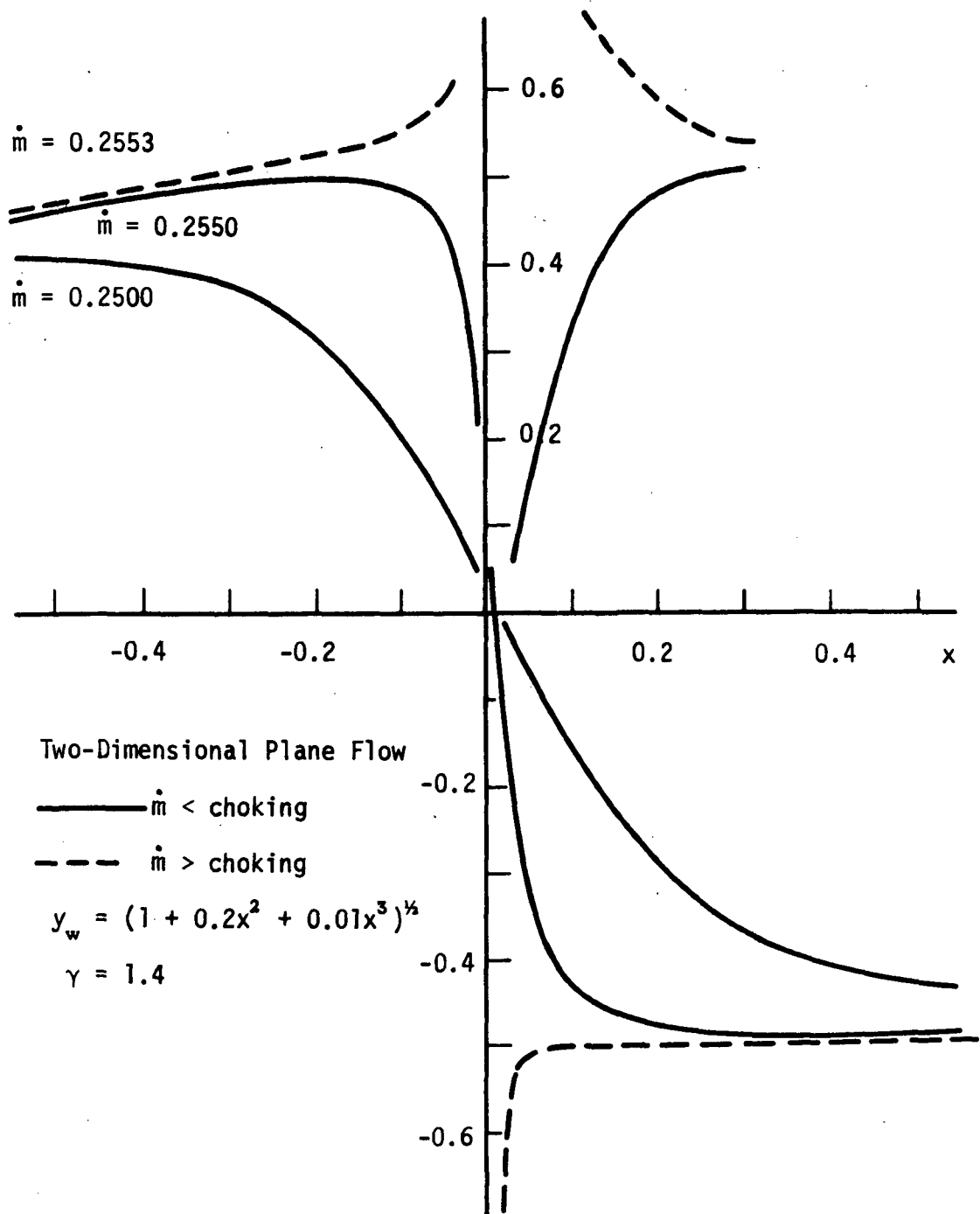


Figure 21 Velocity Gradient Distribution for a Non-symmetric Nozzle ($y_w''' \neq 0$)

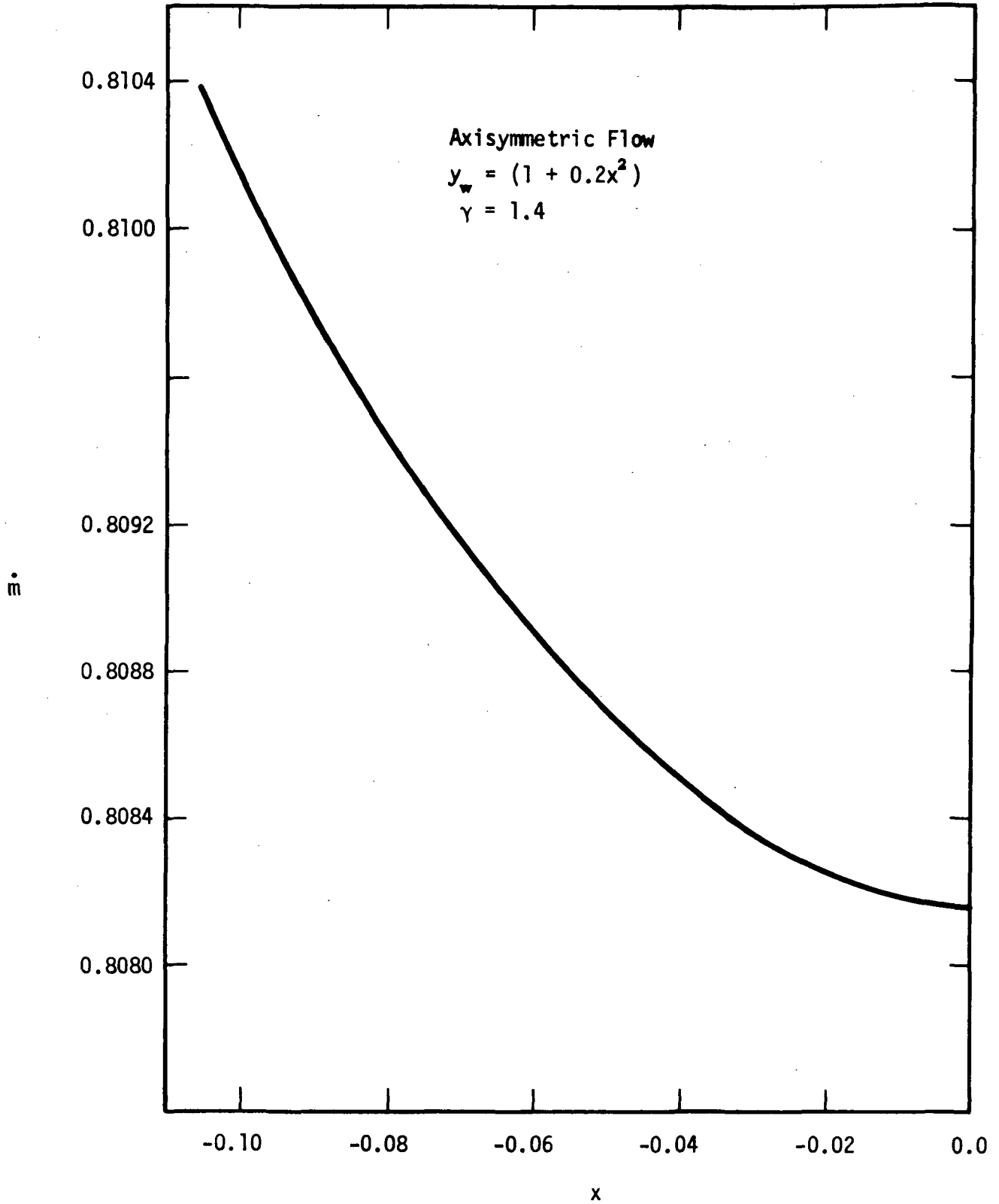


Figure 22 Location of Wall Sonic Point Upstream from Nozzle Throat at Subcritical Values of Mass Flux

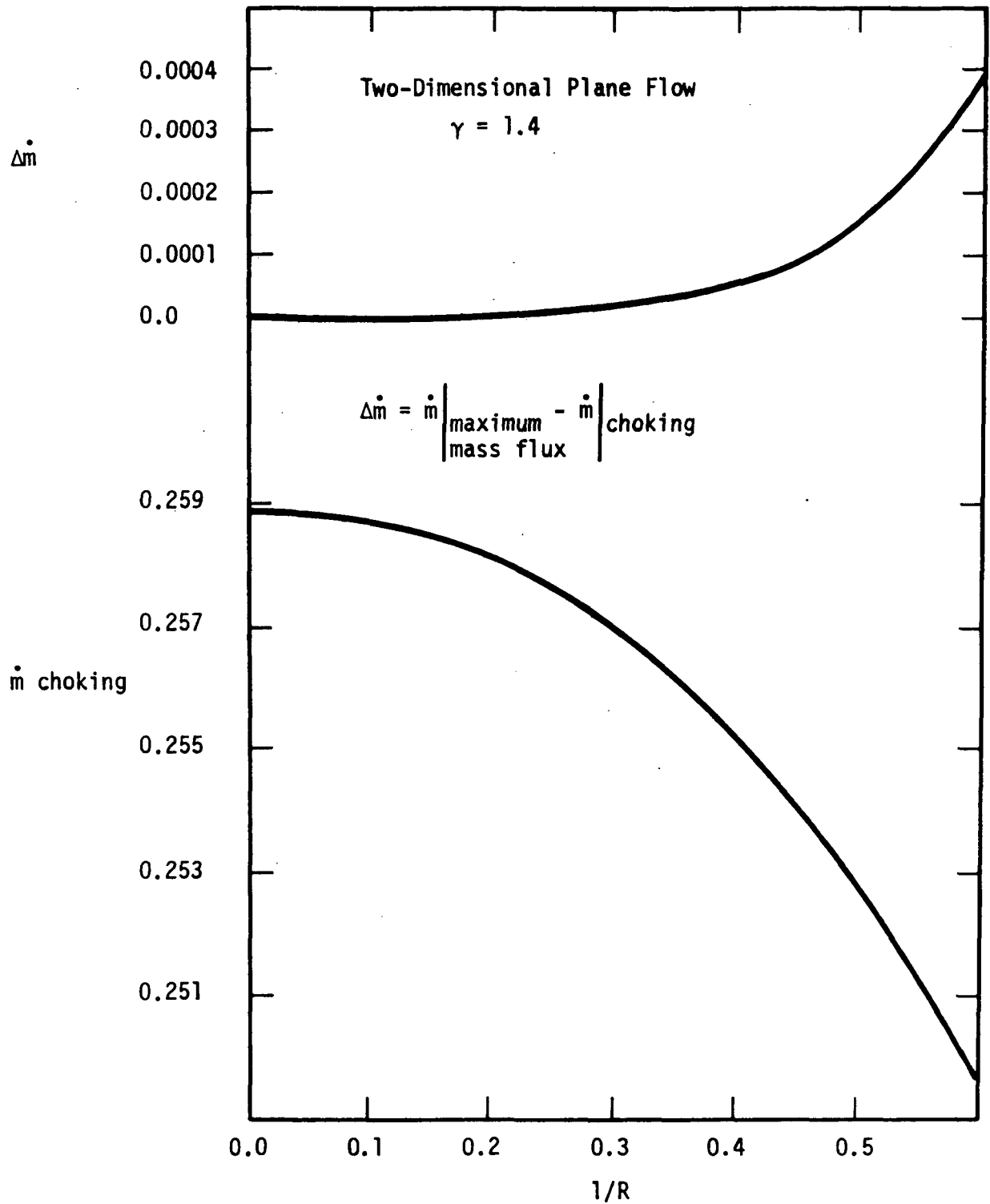


Figure 23 Choking Mass Flux vs. Inverse Throat Radius of Curvature (Two-Dimensional Plane Flow)

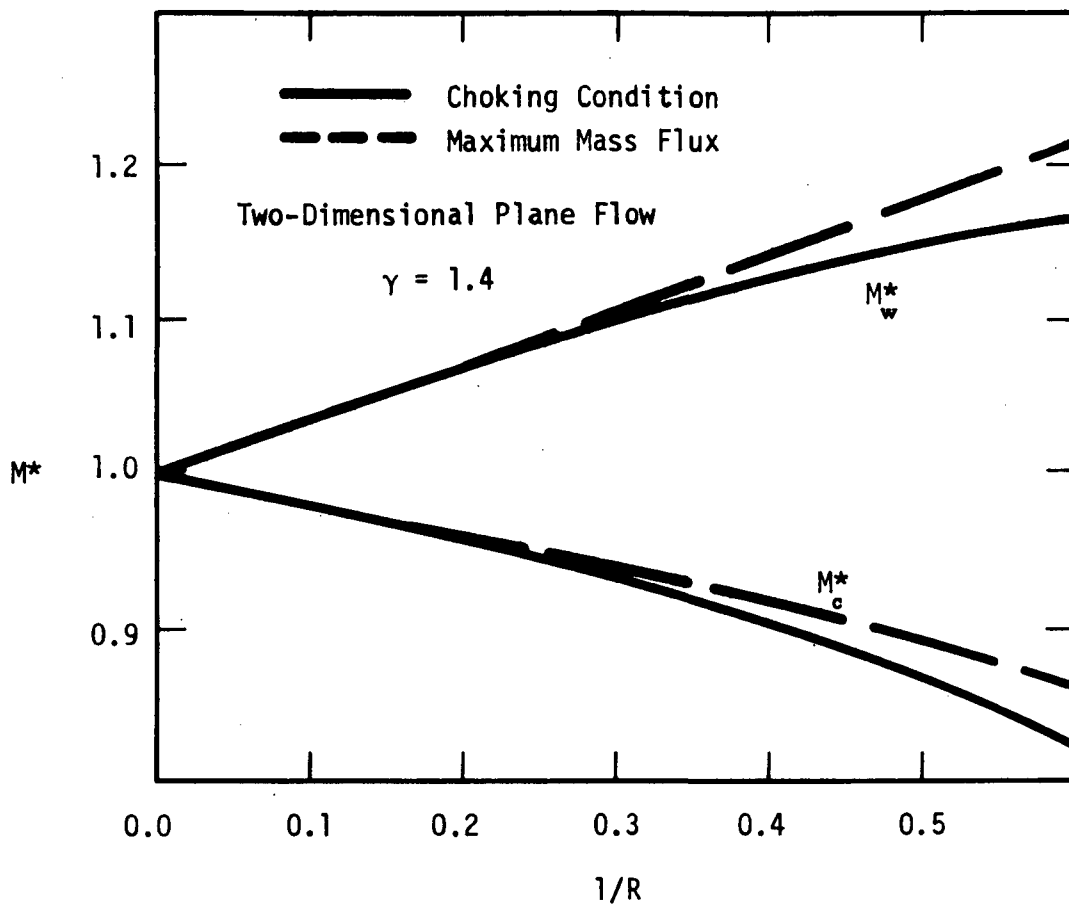


Figure 24 Calculated Throat Velocities vs. Inverse Throat Radius of Curvature (Two-Dimensional Plane Flow)

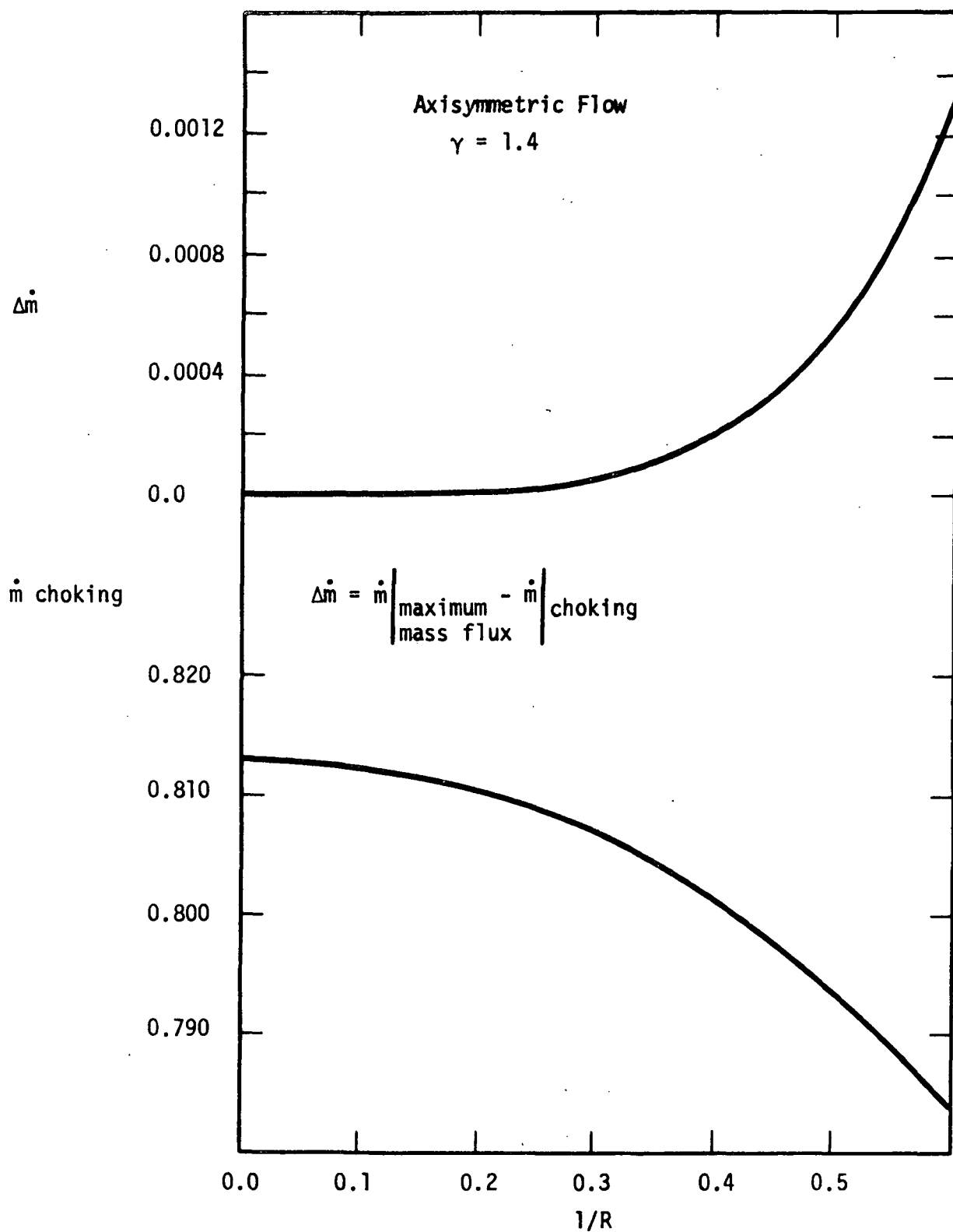


Figure 25 Choking Mass Flux vs. Inverse Throat Radius of Curvature (Axisymmetric)

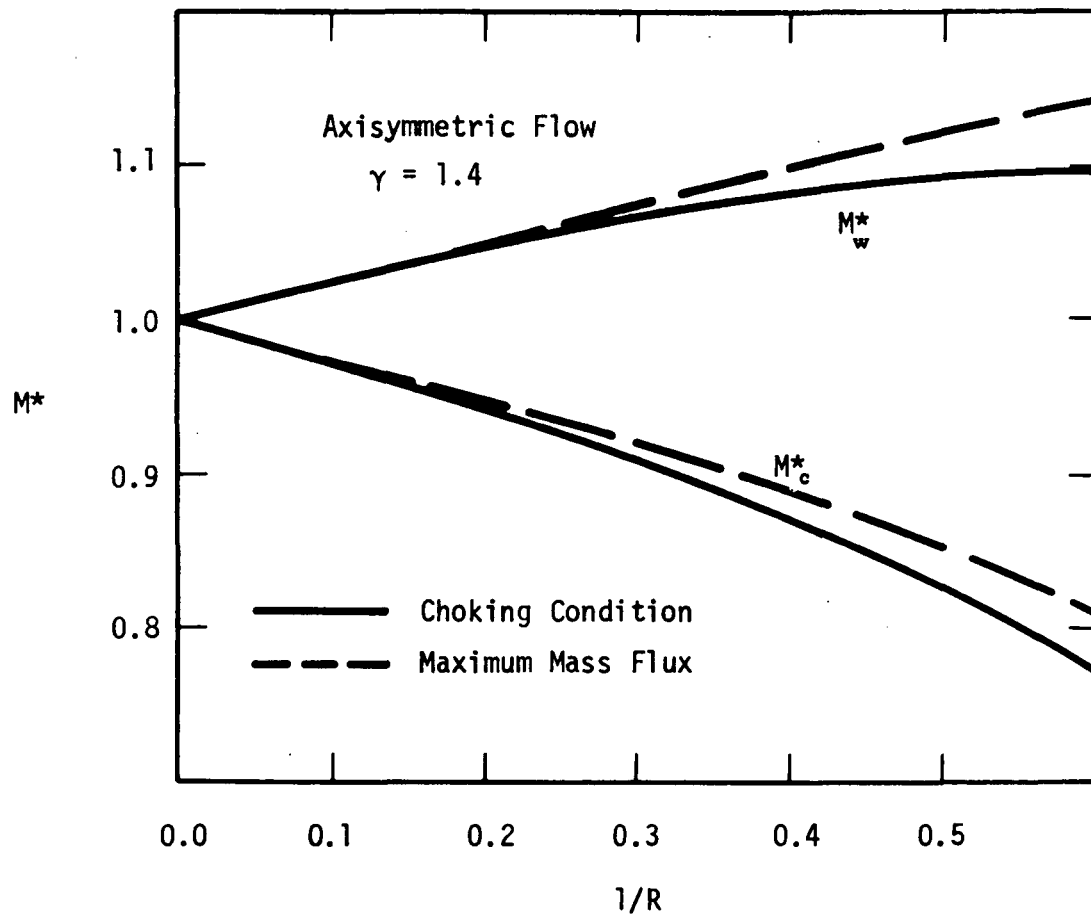


Figure 26 Calculated Throat Velocities vs. Inverse Throat Radius of Curvature (Axisymmetric Flow)

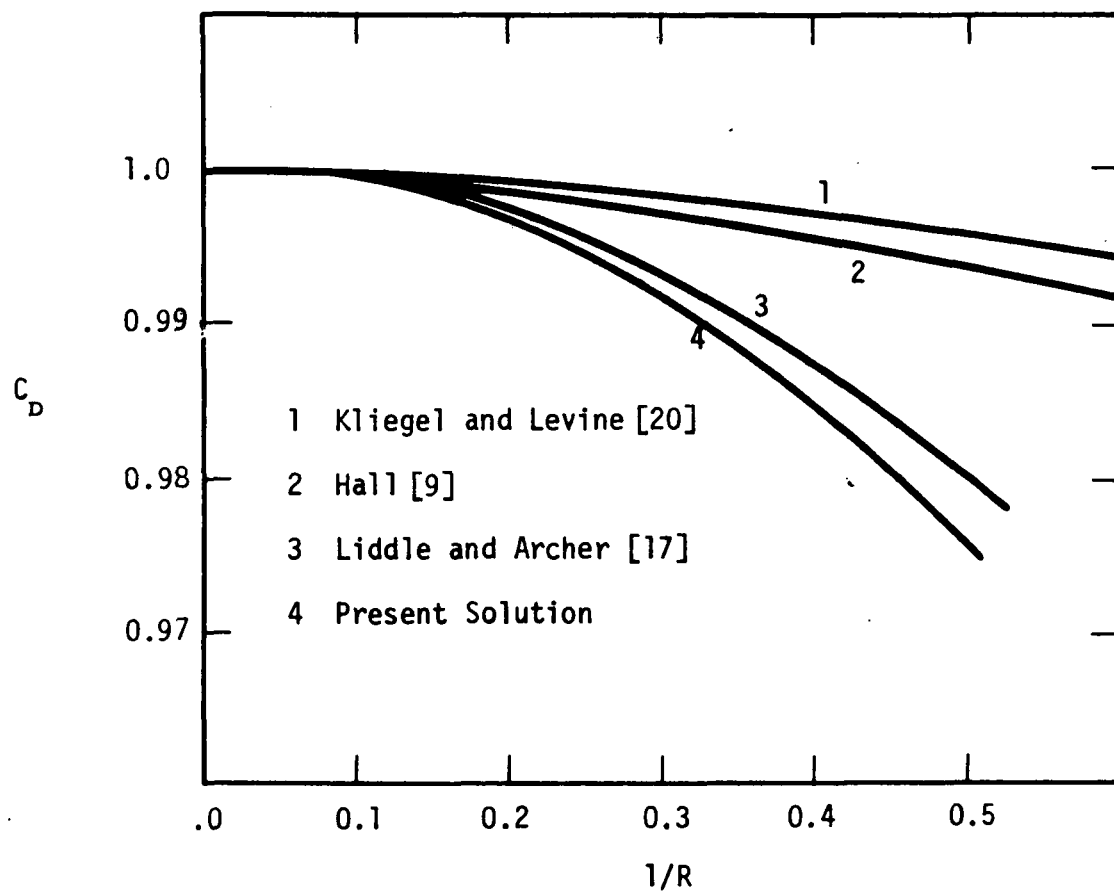


Figure 27 Discharge Coefficient vs. Inverse Throat Radius of Curvature

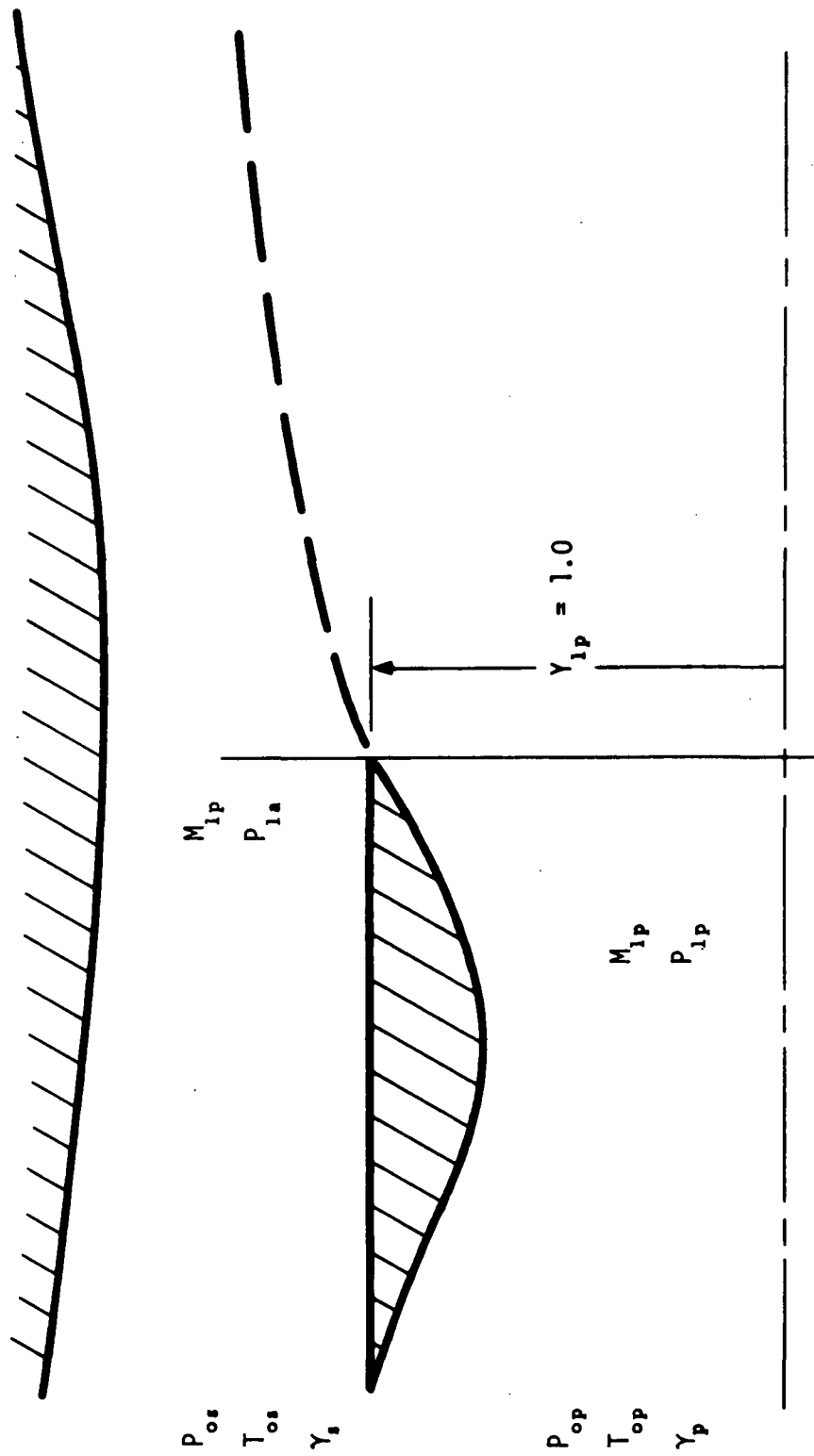


Figure 28 Supersonic Ejector Configuration

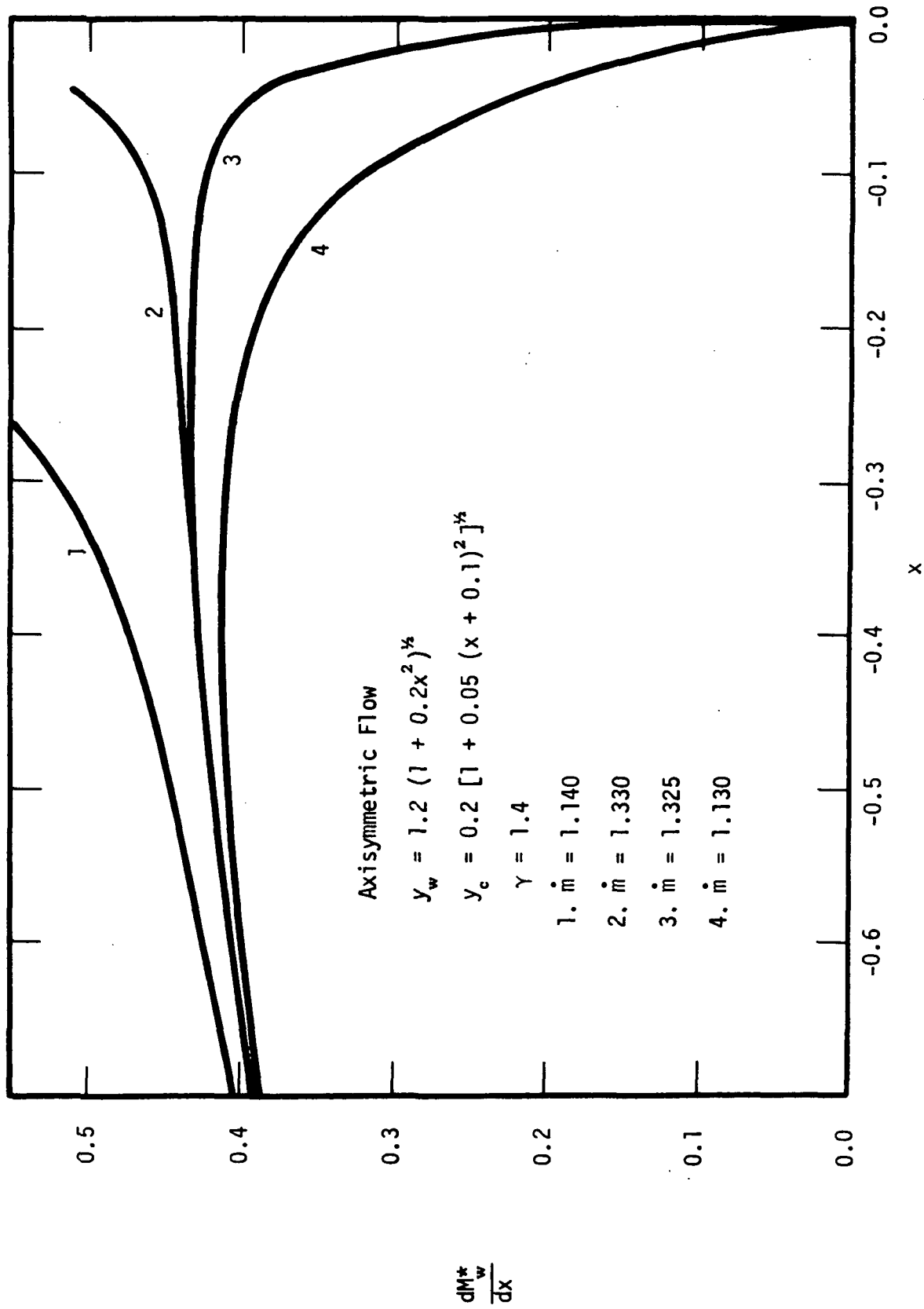


Figure 29 Velocity Gradient Distribution (Linear Velocity Profile Approximations)

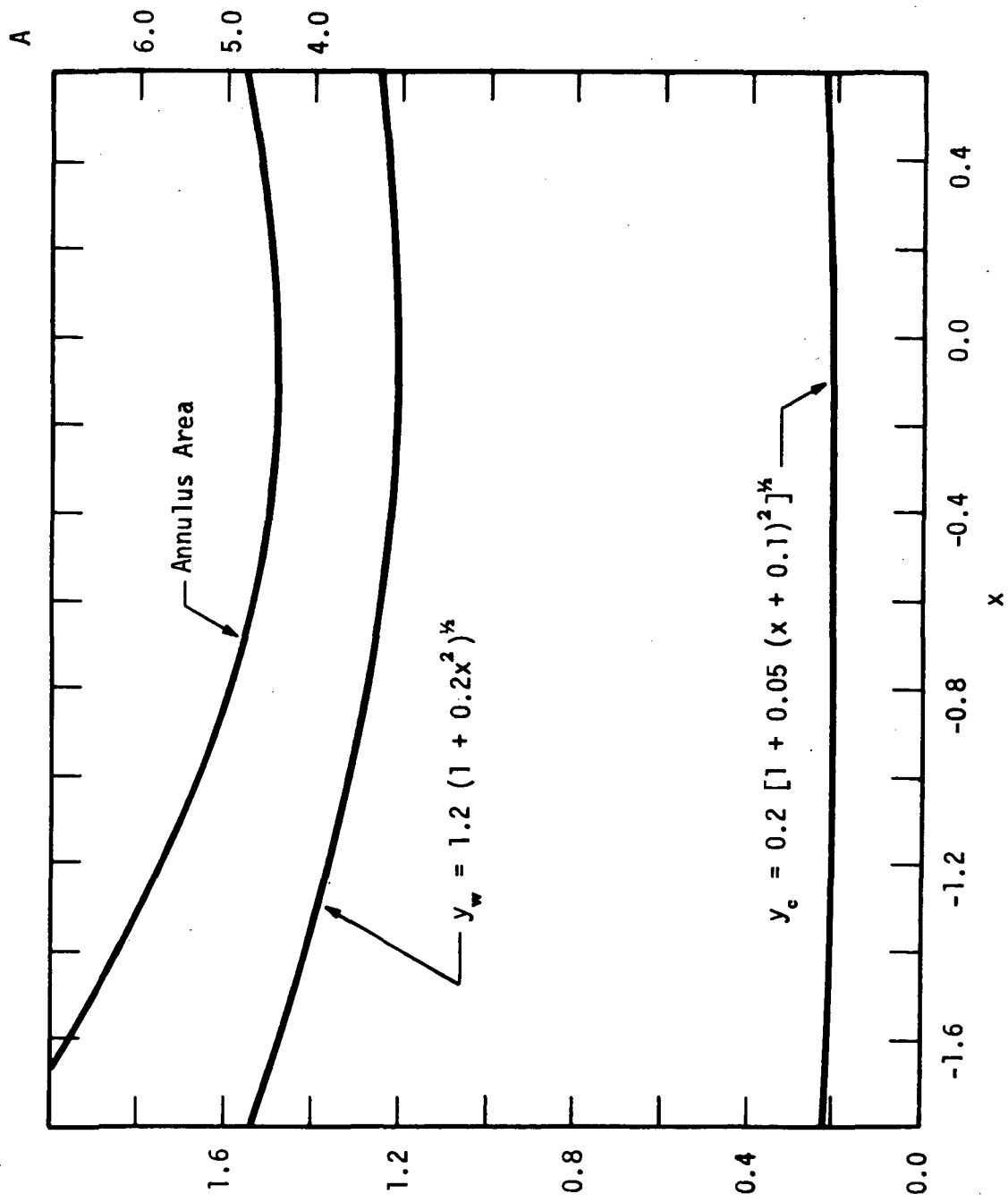
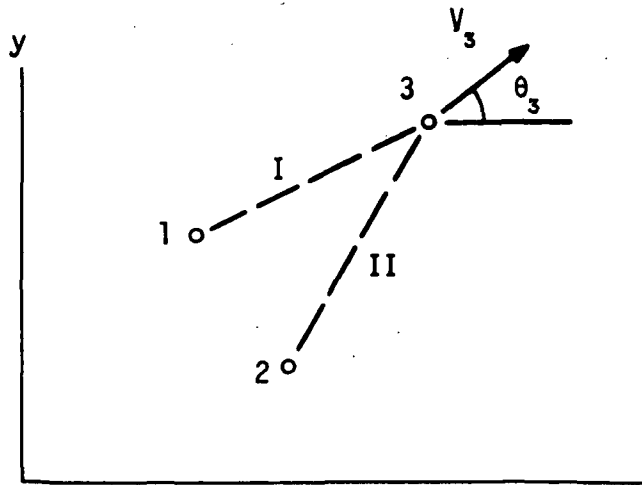
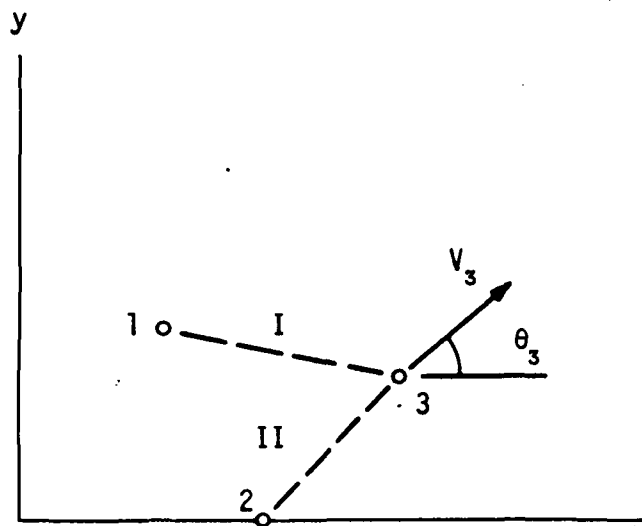


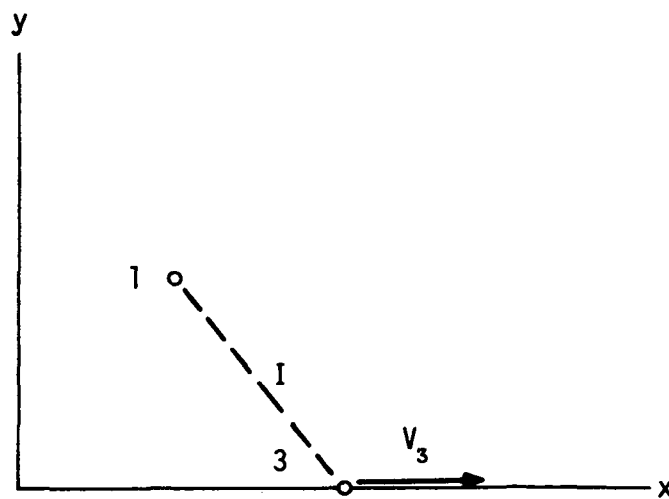
Figure 30 Annulus Shape



a Field Point



b Known Point on Axis



c Unknown Point on Axis

Figure 31 Method of Characteristics (Unit Processes)

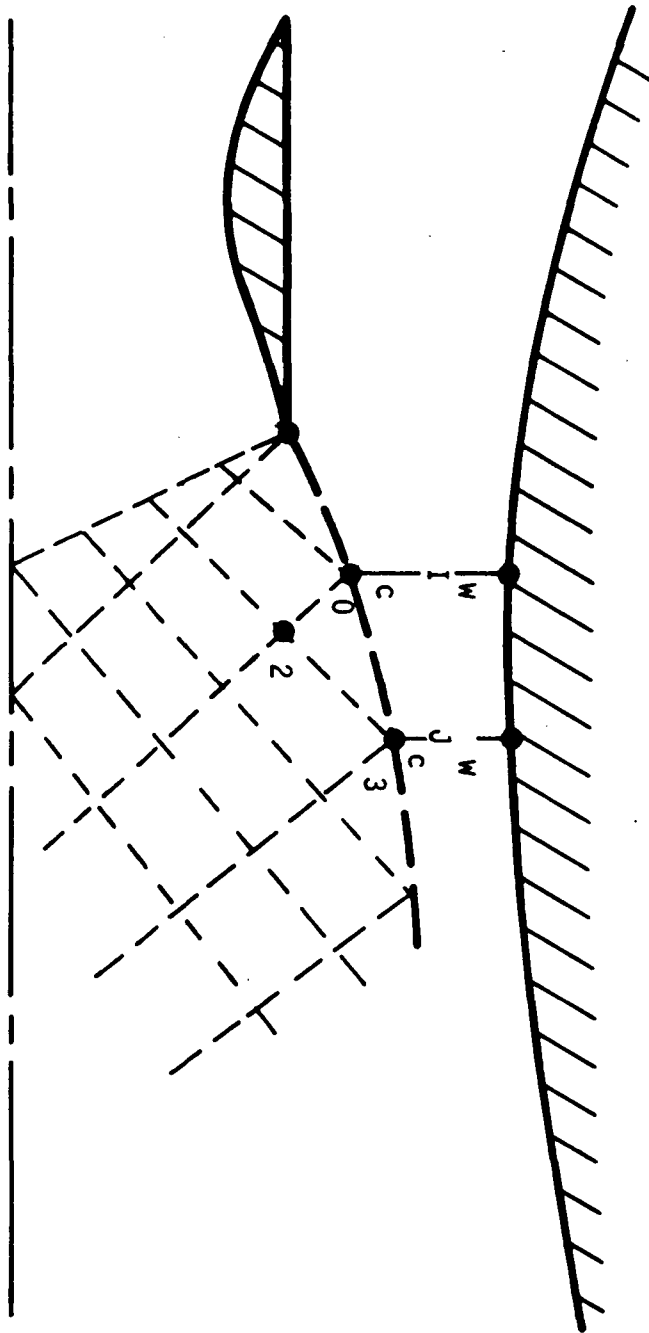


Figure 32 Two Stream Boundary Point Calculation

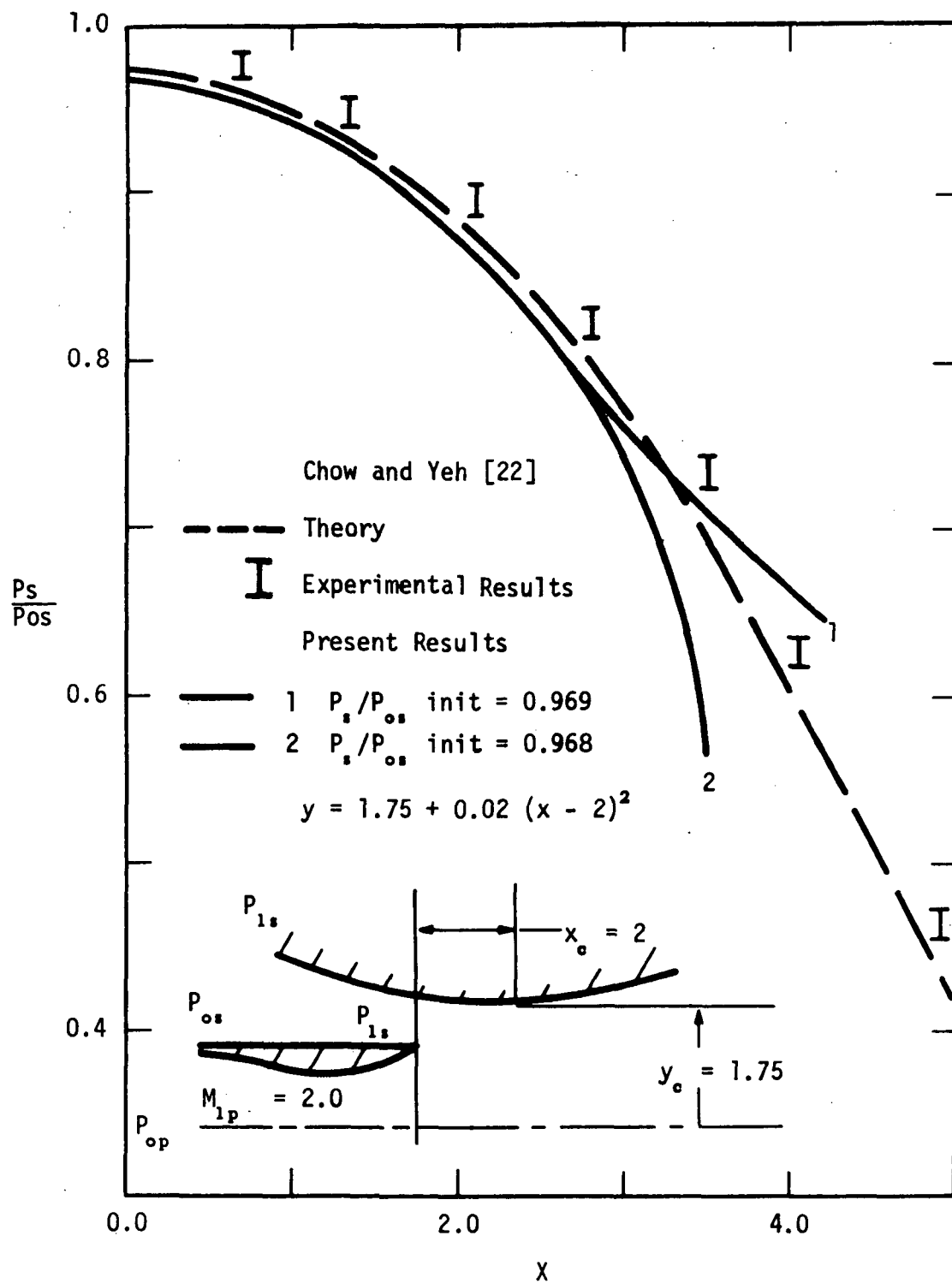


Figure 33 Supersonic Ejector Results