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Finite-β Perpendicular Fast Shocks

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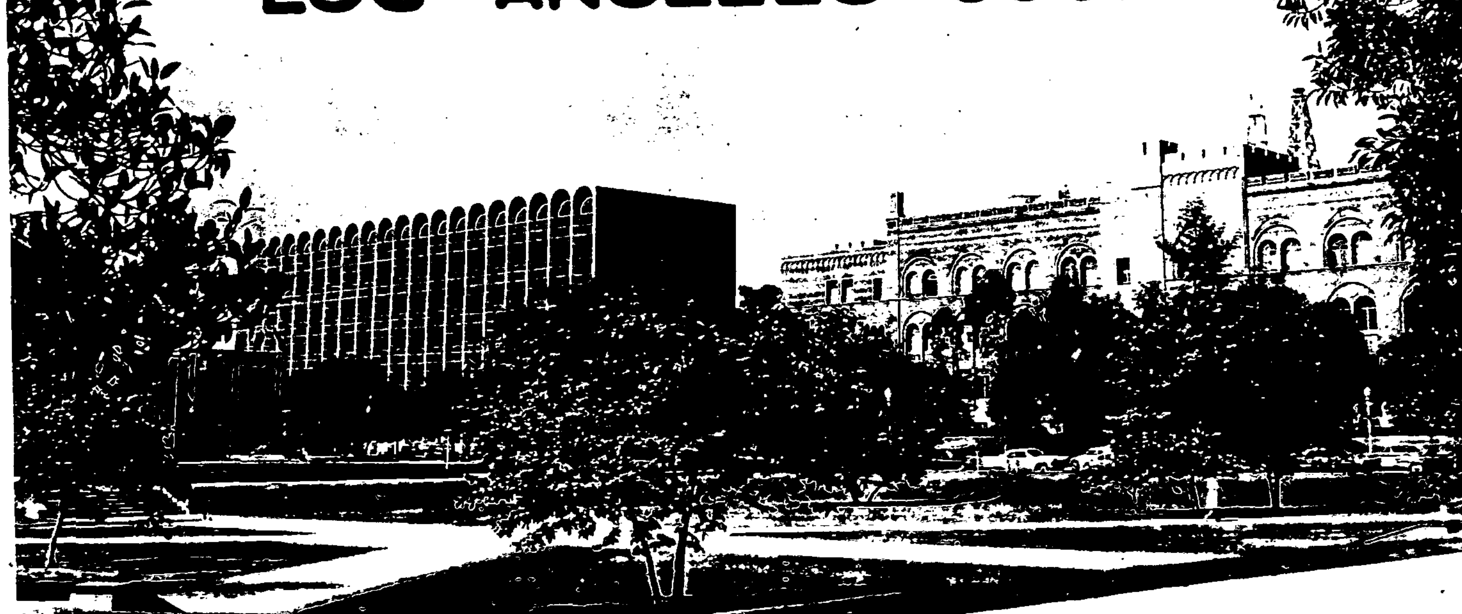

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ABSTRACT

In a finite-β plasma ion cyclotron radius dispersion forms a trailing wave train for a perpendicular fast shock. Collisionless dissipation is provided by the three wave decay of the wave train into very oblique fast and parallel Alfven waves. Particle thermalization results from Landau damping of oblique fast wave turbulence. The shock damping length to three wave decay is many ion cyclotron radii. Undamped Alfven turbulence should persist far downstream from the shock.

1.0 INTRODUCTION

The high-β ($\beta^{\pm} = 8\pi NT^{\pm}/B^2$), truly collision-free plasma which permeates the interplanetary medium and most of the magnetosphere has stimulated considerable theoretical and experimental research on the structure of high-β collisionless shocks [Kennel and Sagdeev, 1967a, b; Tidman, 1967; Kennel and Petschek, 1968; Fredricks et al., 1970]. Theoretical interest centers on determining the macroscopic or fluid behavior of the plasma flowing through a shock and specifying the collisionless plasma turbulence which provides microscopic dissipation. Two distinct approaches to this problem have evolved in the shock literature: a fully turbulent shock model in which collisionless dissipation dominates the structure [Fishman et al., 1960; Tidman, 1967; Kennel and Sagdeev, 1967a], and fluid shock models in which plasma dynamics are governed by dispersive hydromagnetic equations and weak collisionless dissipation [Sagdeev, 1966]. The large parameter range and high variability of space plasmas permits ample room for application of both methods.

In this paper the fluid approach is employed to investigate a fast shock propagating strictly perpendicular to the magnetic field in a high-β plasma. The shock structure is a nonlinear wave train with ion cyclotron radius (ICR) oscillation lengths which trails in the downstream flow. This nonlinear wave train decays by the three wave mode coupling process into oblique fast and parallel Alfven waves, thus relinquishing ordered shock oscillation energy to incoherent wave turbulence. The irreversible energy exchange between waves is the collisionless dissipation needed to spatially damp wave train oscillations and complete the shock transition.

Kennel and Sagdeev [1967b] first pointed out that finite-β perpendicular shocks should possess ion cyclotron radius wave train structure. As recognized by MacMahon [1968] and by Fredricks and Kennel [1968], however, Kennel and Sagdeev used an incomplete set of ICR fluid equations, and hence obtained the incorrect sign of ICR dispersion for the fast mode. (They found the ICR wave train leading in the upstream flow.) Consequently the three wave decay of the wave train into perpendicular magnetosonic turbulence proposed for the shock dissipation is invalid. However, the physical discussion of wave train formation, the three wave decay instability, as well as the basic analytical techniques employed to investigate wave trains and turbulent dissipation presented in Kennel and Sagdeev [1967b] are essentially correct and are recommended to the interested reader. Therefore in this paper there is little need to dwell extensively on the physics of wave trains or to expound on calculational details.

Section 2.1 presents and briefly comments on the set of fluid equations, Chew-Goldberger-Low hydromagnetics with first order ion cyclotron radius corrections, used in the analysis. The linear dispersion relation for the perpendicular fast wave is derived. For wavelengths comparable to the ICR, the wave propagates dispersively with phase speeds less than the long wavelength hydromagnetic fast speed.

In the shock interior the fluid transition between upstream and downstream flow states is organized into a train of coherent or laminar oscillations which are weakly damped by collisionless dissipation. These wave trains consist of dispersively propagating waves whose phase speed matches the local fluid velocity; the relevant standing waves must be on the same branch of the dispersion relation as the waves which form the shock. Since ICR dispersion slows the fast wave speed, the wave train with ICR oscillation lengths trails in the downstream shock flow. In Section 2.2 a differential equation for the flow velocity through the shock is derived. Here ICR dispersion balances the nonlinear hydromagnetic steepening of the shock and establishes a steady shock flow. The scale length of the leading velocity gradient in the shock and the downstream oscillation wavelength is determined from the differential equation in Section 2.3. The ICR wave train is found to be a valid description of the shock structure only if the shock is weak, i.e., if the Mach number is much less than two. Hence the present analysis does not apply to the strong bow shock interaction near the subsolar region of the magnetosphere. ICR fast shock structure might occur in the weak shock region on the magnetosphere flanks or for shocks propagating in the interplanetary medium.

Section 3.0 analyzes the three wave decay of the nonlinear perpendicular wave train into oblique fast and parallel Alfven waves. The matrix elements coupling the three waves are derived in Section 3.1, following the standard techniques [see Galeev and Karpman, 1963; Kadomtsev, 1965; Sagdeev, 1966; Sagdeev and Galeev, 1969]. The growth rate of the decay instability is estimated in terms of the wave train oscillation amplitude in Section 3.3. The Landau or transit time damping rate of the oblique fast waves is shown to be less than the decay growth rate provided that β does not greatly exceed unity. In Section 3.4 the wave train spatial damping provided by three wave decay turbulence is found to be many ICR scale lengths. Since Alfven waves are not Landau damped, Alfven turbulence generated by the decay instability should persist far downstream from the shock.

2.0 ION CYCLOTRON RADIUS WAVE TRAIN FOR PERPENDICULAR FAST SHOCKS

This section briefly reviews previous work on the ion cyclotron radius (ICR) perpendicular fast shock wave train [see Kinsinger and Auer, 1969; Goldberg, 1970], and makes specific the problem at hand. The time dependent Chew-Goldberger-Low (CGL) hydromagnetic equations correct to first order in the small ICR expansion are presented in Section 2.1; the perpendicular fast mode dispersion relation is determined. Following standard analytical techniques, a differential equation for the time independent wave train structure is derived in Section 2.2; the wave train properties are examined in Section 2.3.

2.1 THE CGL-ICR FLUID SET

For time scales long compared to the ion cyclotron period and space scales longer than the ICR, plasma dynamics are often reasonably well approximated by moments of the Vlasov equation. Of course, phenomena which depend on details of the particles' distribution functions, such as Landau and cyclotron resonance, and heat flow, cannot be treated by fluid equations. By systematically expanding the Vlasov moments a set of fluid equations with first order ICR corrections can be obtained to describe motion both parallel and perpendicular to the magnetic field. The formulation employed here is due to MacMahon [1965], and the interested reader is referred to his paper for details. A common difficulty with equations derived from a moment hierarchy is that they fail to close, i.e., lower order moments depend on higher order ones. This difficulty is avoided for the degenerate case of shock propagation perpendicular to the magnetic field [MacMahon, 1965] and for small amplitude linear waves. The considerations of this paper are restricted to these cases and, therefore, the fluid equations presented below are a truncation of the full set.

The equation for the continuity of mass flow is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0 \tag{2.1}$$

where ρ is the mass density and $\underline{v}$ is the fluid velocity. The dynamical force equation is

$$\rho\,\frac{d\underline{v}}{dt} = -\,\nabla\cdot\underset{\approx}{P}^{(1)} + \frac{\underline{J}\times\underline{B}}{c} \tag{2.2}$$

where $\underset{\approx}{P}^{(1)}$ is the pressure tensor summed for ions and electrons, and the superscript (1) denotes retention of first order ICR contributions. $\underline{J}$ is the current density, $\underline{B}$ the magnetic field, and c is the velocity of light. Gaussian units are used. The relation between $\underline{J}$ and $\underline{B}$ is specified by Ampere's equation

$$\nabla\times\underline{B} = \frac{4\pi}{c}\underline{J} \quad . \tag{2.3}$$

The electric field $\underline{E}$ is governed by the particularly simple Ohm's law

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0 \quad . \tag{2.4}$$

The neglect of contributions to Ohm's law from finite ion and electron inertia, and resistivity is a further restriction on the calculations performed here. When $\beta^{\pm} = 8\pi P^{\pm}/B^2 > 1$, inertial dispersion is a smaller effect than ICR dispersion. Faraday's law

$$\nabla\times\underline{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{2.5}$$

completes the Maxwell set.

For shock investigations an equation expressing the conservation of energy is required

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + p_{\perp}^{(1)} + \frac{1}{2}\,p_{\parallel}^{(1)} + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\rho\underline{v}\left(\frac{v^2}{2} + \frac{p_{\perp}^{(1)}}{\rho} + \frac{p_{\parallel}^{(1)}}{2\rho}\right)\right.$$

$$\left. + \underline{v}\cdot\underset{\approx}{p}^{(1)} + \frac{\underline{B}\times(\underline{v}\times\underline{B})}{4\pi} + \underline{q}_{\perp}^{\perp(1)} + \underline{q}_{\perp}^{\parallel(1)}\right] = 0 \qquad (2.6)$$

$p_{\perp}^{(1)}$ and $p_{\parallel}^{(1)}$ are pressures perpendicular and parallel to the magnetic field. Subsidiary equations for $p_{\perp}^{(1)}$ and $p_{\parallel}^{(1)}$ are needed below, and will be written here in a form appropriate only for perpendicular shock and linear wave propagation

$$\frac{dp_{\perp}^{(1)}}{dt} + p_{\perp}^{(1)}\,(\nabla\cdot\underline{v} + \nabla_{\perp}\cdot\underline{v}) + \nabla\cdot\underline{q}_{\perp}^{\perp(1)} = 0 \qquad (2.7)$$

$$\frac{1}{2}\,\frac{dp_{\parallel}^{(1)}}{dt} + p_{\parallel}^{(1)}\left(\frac{3}{2}\,\nabla\cdot\underline{v} - \nabla_{\perp}\cdot\underline{v}\right) + \nabla\cdot\underline{q}_{\perp}^{\parallel(1)} = 0 \qquad (2.8)$$

where $\nabla_{\perp}$ is the perpendicular gradient operator. Contributions to (2.6), (2.7), and (2.8) from zero order heat flow along $\underline{B}$ have been neglected; these heat flow moments depend on the exact shape of the distribution function and are not determined by the moment equations. Their neglect represents the closure of the moment hierarchy.

The quantities $\underline{q}_{\perp}^{\perp(1)}$ and $\underline{q}_{\perp}^{\parallel(1)}$ are the first order ICR heat flows perpendicular to $\underline{B}$. Only $\underline{q}_{\perp}^{\perp(1)}$ is needed, and subject to the same restrictions appropriate to (2.7) and (2.8), takes the form

$$\underline{q}_{\perp}^{\perp(1)} = -\frac{2\underline{B}\times}{\Omega_{+}|\underline{B}|}\left[\frac{p_{\perp}^{(0)+}}{\rho}\,\nabla\cdot\underset{\approx}{p}^{(1)}\right] \qquad (2.9)$$

$P_\perp^{(0)+}$ is the zeroth order ion pressure; $\Omega_+ = eB/M_+c$ is the ion gyrofrequency; e is the electronic charge and M_+ the ion mass. In (2.9) terms involving heat flow and fourth order moments have been neglected; these terms can be significant for plasmas whose distribution functions deviate greatly from Maxwellian [MacMahon, 1965, 1968].

Since the factor $1/\Omega_+$ appears in (2.9), $q_\perp^{\perp(1)}$ is formally of first order; hence, formally, $\nabla\cdot\underset{\approx}{P}^{(1)}$ could be replaced by $\nabla\cdot\underset{\approx}{P}^{(0)}$, whereupon it is easily shown that $q_\perp^{\perp(1)}$ vanishes. ICR dispersion, however, couples different degrees of freedom, and the lowest order ICR corrections to long wavelength hydromagnetics are products of two ICR terms. Therefore in (2.9) $\nabla\cdot\underset{\approx}{P}^{(1)}$ must be retained since $q_\perp^{\perp(1)}$ contributes to, and is actually the dominant contribution to, ICR dispersion [MacMahon, 1968].

If $\underline{B}$ is assumed to be in the z direction, the only components of $\underset{\approx}{P}^{(1)}$ required below are $P_{xx}^{(1)}$ and $P_{xy}^{(1)}$; these are

$$P_{xx}^{(1)} = P_\perp^{(1)} - \frac{P_\perp^{(0)+}}{2\Omega_+}\frac{dv_y}{dx} \tag{2.10}$$

$$P_{xy}^{(1)} = \frac{P_\perp^{(0)+}}{2\Omega_+}\frac{du}{dx} \tag{2.11}$$

u and v_y are the fluid velocities in the x and y direction, respectively.

Before passing to the shock wave train, a useful exercise is to determine the linear dispersion relation for the perpendicular fast wave. Linearizing about a stationary, uniform equilibrium of infinite spatial extent and assuming harmonic perturbations of the form $\delta u(xyz,t) = \delta u \exp[ikx-\omega t]$, (2.1) - (2.11) reduce to the dispersion relation

$$\frac{\omega^2}{k^2} = C_F{}^2 \quad (1 - \frac{3}{4} \frac{P_\perp^{(0)+}}{\rho C_F{}^2} k_\perp{}^2 R_+{}^2) \tag{2.12}$$

C_F is the hydromagnetic fast speed, $C_F = [(B^2/4\pi\rho) + (2P_\perp^{(1)}/\rho)]^{1/2}$ and $R_+ = (P_\perp^{(0)+}/\rho\Omega_+{}^2)^{1/2}$ is the ion cyclotron radius based on the zeroth order ion pressure. The small ICR expansion of the fluid equations requires that $k_\perp R_+ < 1$; shorter wavelengths can only be treated correctly by the kinetic theory [Fredricks, 1968].

As found by MacMahon [1968] and by Fredricks and Kennel [1968], ICR dispersion, $k_\perp{}^2 R_+{}^2$ term in (2.12), decreases the phase velocity below the hydromagnetic fast speed. In the context of shock wave trains, therefore, the dispersively propagating fast wave can phase stand, $\omega/k = u$ = flow velocity, only in the downstream shock flow. (The downstream flow velocity must be less than C_F by the shock evolutionary conditions [Kantrowitz and Petschek, 1966].) Hence linear wave theory predicts that the $\beta > 1$ perpendicular fast shock wave train trails behind the shock leading edge.

2.2 DERIVATION OF THE SHOCK WAVE TRAIN

When dissipative processes are weak, the collisionless shock transition from upstream to downstream stationary states takes the form of a nonlinear dispersively propagating wave whose phase velocity matches the local flow speed. This wave train is described by a differential equation which balances dispersion and weak dissipation against nonlinear shock steepening. Since the analytical techniques for obtaining the differential equation are standard [see Sagdeev, 1966; Cavaliere and Englemann, 1967; Kennel and Sagdeev, 1967b], only the highlights are presented. A small viscosity is included in the analysis to simulate weak collisionless dissipation, a specific model of which is the subject of Section 3.0.

Wave train solutions to (2.1) - (2.11) are sought which are time independent in the co-moving shock frame. The shock normal is assumed to be in the x-direction, and is perpendicular to the magnetic field in the z-direction; plasma quantities are assumed to be functions of x, only. Equations (2.1) - (2.6) now possess a first integral with respect to x; in regions where the flow is spatially uniform, these integrals are the familiar shock Rankine-Hugoniot relations which connect the upstream and downstream states. For (2.1) - (2.6), however, only five independent relations between the six dynamical variables are obtained; hence an additional conservation law is needed.

Goldberg [1970] has pointed out that for perpendicular propagation, the equation for $P_{\parallel}^{(1)}$, (2.8), provides the required conservation law. In the shock frame (2.8) becomes

$$\frac{d}{dx} \left(P_{\parallel}^{(1)} u + 2\underline{q}_{\perp}^{\parallel(1)} \cdot \hat{x} \right) = 0 \quad . \tag{2.13}$$

Since $\underline{q}_{\perp}^{\parallel(1)} \cdot \hat{x}$ vanishes for a spatially uniform flow, (2.13) yields $P_{\parallel}^{(1)} u \propto P_{\parallel}^{(1)}/\rho = T_{\parallel} = \text{const.}$; for perpendicular shocks the parallel temperature is unchanged.

Since viscous dissipation is assumed small, second order terms in the coefficient of viscosity, μ , are dropped in the derivation; in addition, terms of order $\mu(R_+/L)$, where L is a characteristic shock scale length, can be neglected. With these approximations, (2.1) - (2.6) and (2.13) reduce to the following wave train differential equation:

$$\frac{3}{4}\ \frac{\Omega_{+}u}{\rho_1 u_1}\ R_{+}^{2}\ \frac{d}{dx}\left[\frac{p_{\perp}^{(0)+}}{\Omega_{+}}\ \frac{d\ \Delta u}{dx}\right] + \frac{\mu u}{\rho_1 u_1}\ \frac{d\ \Delta u}{dx} =$$

$$= u_1^{\ 2}\left[\frac{3}{2}\ (\Delta u)^2 + \Delta u\ \frac{M_{F_1}^2 - 1}{M_{F_1}^2}\right] \tag{2.14}$$

u_1 denotes the upstream flow speed, and $M_{F_1} = u_1/C_{F_1}$ is the magnetosonic Mach Number; $\Delta u = (u-u_1)/u_1$.

The first term in (2.14) represents ICR dispersion and will yield an oscillatory trailing wave train. The first derivative term represents weak viscous dissipation. The right hand side of this second order equation can be interpreted as a nonlinear driving force; setting this term equal to zero yields the Rankine-Hugoniot relations for u . An analogy between (2.14) and weakly damped particle motion in an anharmonic potential well is presented in Sagdeev [1966] and in Kennel and Sagdeev [1967b]. The next section discusses an alternative method for obtaining the wave train properties.

2.3 STRUCTURE OF THE ICR WAVE TRAIN

The wave train differential equation is nonlinear since both the coefficients of the differential operators and the driving force are nonlinear functionals of the velocity. Equation (2.14), however, does possess two stationary points located at the upstream and downstream solutions of the Rankine-Hugoniot relations [zeros of the right hand side of (2.14)]. A general technique for determining the qualitative features of the nonlinear equation is to linearize about the stationary points and examine the stability of the linearized solutions. For wave trains shock transition requires that the perturbations be unstable (stable) upstream (downstream) [see Sagdeev, 1966; Cavaliere and Englemann, 1967]. The linearized solutions determine whether the wave train oscillations occur in the upstream or downstream flow and provide estimates of the shock scale lengths.

After some manipulation, the linearized version of (2.14) for the perturbed velocity δu becomes

$$\frac{3}{4} R_+^2 \frac{d^2\delta u}{dx^2} + \frac{\mu u}{P_\perp^{(0)+}} \frac{d\delta u}{dx} - \frac{u^2 - C_F^2}{P_\perp^{(0)+}/\rho} \delta u = 0 \tag{2.15}$$

where the coefficients of δu are now to be evaluated about either the upstream or downstream flow. If terms of orders μ^2 are neglected, solutions to (2.15) in the form $\delta u \propto \exp(\lambda x)$ are

$$\lambda = \frac{-\mu u}{\frac{3}{2} P_\perp^{(0)+} R_+^2} \pm \frac{1}{R_+} \left[\frac{\rho\, C_F^2 (M_F^2-1)}{\frac{3}{4} P_\perp^{(0)+}} \right]^{1/2} \tag{2.16}$$

Since $M_{F_1}^2-1 > 0$ upstream, (2.16) yields u decreasing exponentially from u_1 with a scale height given by

$$L \sim \left[\frac{3}{4} \frac{P_\perp^{(0)+}}{\rho C_F^2} \right]^{1/2} \frac{R_+}{|M_F^2-1|^{1/2}} \tag{2.17}$$

Hence ICR dispersion determines the thickness of the leading edge. Downstream $M_F^2-1 < 0$, and solutions of (2.16) are damped oscillations with wavelengths given by (2.17) evaluated about the downstream flow and damping length $\sim \frac{3}{2} P_\perp^{(0)+} R_+^2/\mu u^2$. Therefore as predicted by the linear wave dispersion relation [Macmahon, 1968; Fredricks and Kennel, 1968] ICR dispersion forms a trailing shock wave train.

The first order ICR-CGL fluid equations provide an accurate description of the shock structure if, and only if, the shock thickness $L \ll R_+$. Therefore from (2.17) the upstream Mach number is limited to values $M_{F_1}^2 - 1 \ll \frac{3}{4} P_\perp^{(0)+}/\rho C_F{}^2 < 1$, so that ICR dispersive shocks must be weak. Coroniti [1970] has shown that stronger high-β fast shocks steepen until thicknesses of order the electron intertia length C/ω_{p_-} $(\omega_{p_-}^2 = 4\pi Ne^2/M_-)$, are reached; for solar wind plasmas, $C/\omega_{p_-} \ll R_+$. The shock structure is now described by a trailing C/ω_{p_-} length wave train.

In summary, the ICR-CGL hydromagnetic equations describe plasma behavior over scale lengths long compared to R_+. A closed set of Rankine-Hugoniot relations for perpendicular shocks exists with $T_\parallel$ constant across the shock. When ICR dispersion is balanced against nonlinear steepening, a trailing wave train with R_+ scale lengths resolves the shock structure. This shock solution is valid only for weak shocks, $M_F^2 - 1 \ll 1$. In the above analysis weak collisionless dissipation was assumed to provide an irreversible shock transition. The next section considers a specific model, the nonlinear three wave decay process, for this dissipation.

3.0 THREE WAVE DECAY INSTABILITY OF A NONLINEAR PERPENDICULAR FAST WAVE

3.1 INTRODUCTION

A nonlinear wave, such as a wave train, does not always constitute an equilibrium state since the oscillation amplitude is far from the thermal level. Two perturbation waves, either thermal or nonthermal fluctuations, can couple to or scatter off the nonlinear wave spatial gradients [Galeev and Karpman, 1963; Kadomtsev, 1965]. If these mode couplings resonantly extract energy from the nonlinear wave, thus amplifying the perturbation waves, the nonlinear wave amplitude can decay faster than the rate set by particle-particle collisions. A nonlinear wave train, therefore, can parametrically excite incoherent wave turbulence which, in general, will not phase stand in the shock flow, and which provides a form of irreversible collisionless dissipation that damps the wave train amplitude. Particle distributions adjust by damping the turbulent waves, eventually relaxing to a uniform downstream state.

The three wave decay instability is possible only if the wave frequencies and wave vectors of the interacting modes satisfy the following resonance conditions [Sagdeev and Galeev, 1969]

$$\omega_1 = \omega_0 + \omega_2 \tag{3.1}$$

$$\underline{k}_1 = \underline{k}_0 + \underline{k}_2 \tag{3.2}$$

ω_0, $\underline{k}_0$ refer to the nonlinear wave, and ω_1, $\underline{k}_1$ and ω_2, $\underline{k}_2$ are the two perturbing waves. In analogy with quantum mechanics, (3.1) and

(3.2) express the conservation of wave energy and wave momentum in the interaction. In addition to (3.1) and (3.2) the matrix elements coupling the three waves must be non-zero and have the appropriate sign for instability.

A general theorem governing possible decays is that three waves on the same branch of the dispersion relation can mode couple only if ω increases with increasing $\underline{k}$ (see Sagdeev and Galeev [1969] or Kennel and Sagdeev [1967b] for proof). Therefore, from (2.12), decays between three perpendicular high-β fast waves are disallowed [MacMahon, 1968]. The fast mode, however, can unstably interact with waves on other branches of the dispersion relation provided that the decay is not forbidden by polarization restrictions.

This section considers the decay of the perpendicular ICR wave train into slightly oblique fast waves and Alfven waves propagating parallel to the magnetic field. The motivation for treating this particular case is that Alfven waves are undamped in high-β plasmas [Barnes, 1967], and therefore could possibly be observed in the downstream flow by spacecraft. Other decay modes are not considered but could contribute to the total shock dissipation. The matrix elements for the interaction are derived in Section 3.2. The growth rate for the decay instability is determined in Section 3.3. Section 3.4 estimates the wave train damping length.

3.2 THE MODE COUPLING MATRIX ELEMENTS

Since the ICR shock must be weak, $M_F^2 - 1 \ll 1$, the downstream wave train oscillation amplitude is small. Therefore locally the wave train should approximately obey the linear wave dispersion relation, and the wave polarizations can be interrelated by linear theory. For the perpendicular ICR shock wave train, the polarizations are δV_x, δV_y, $\delta\rho$, δB_z, and $\delta \underset{\approx}{P}^{(1)}$; the temporal and spatial dependence is of harmonic form $\exp[ik_{\perp_0} x - i\omega_0 t]$. The slightly oblique fast mode is denoted by $\underline{V}_1$, $\underline{b}_1$, ρ_1, $\underset{\approx}{P}_1^{(1)}$, and oscillates as $\exp[i(k_{\perp_1} x + k_{\parallel_1} z) - i\omega_1 t]$. Similarly the Alfven wave polarizations are V_{y_2}, b_{y_2}, which vary as $\exp[ik_{\parallel_2} z - i\omega_2 t]$.

From (3.2), the wave vectors of the three interacting modes must satisfy $k_{\perp_0} = k_{\perp_1} \equiv k_\perp$ and $k_{\parallel_1} = k_{\parallel_2} \equiv k_\parallel$. For slightly oblique propagation of wave (1), $k_\perp >> k_\parallel$, an assumption which permits considerable simplification in determining the matrix elements coupling the three waves.

The calculation follows the method of Sagdeev and Galeev [1969]. The dispersion relation for each mode is determined from Eqs. (2.1) - (2.9) retaining the resonant nonlinear coupling of the perturbation waves to the large amplitude wave; the resonant couplings are of the form $\delta V_x V_{x_1}$ for the Alfven wave and $\delta V_x^* V_{y_2}$ for the oblique fast mode (* denotes complex conjugate). Other nonlinear terms such as $V_{x_1}^2$ or $V_{x_1} V_{y_2}$ are neglected. The linear wave polarizations are then used to eliminate all the variables in terms of V_{x_1}, V_{y_2}, and δV_x. The wave polarizations are nonlinearly coupled by matrix elements, the relative signs of which determine the stability of the interaction. Since the analytical procedures are standard, the calculation is outlined only for the Alfven wave; the result for the fast mode is simply stated. Furthermore assuming $k_\perp >> k_\parallel$, ICR terms of order $k_\parallel R_+ << 1$ contribute negligibly compared to $k_\perp R_+ < 1$ terms and are dropped.

Substitution of the above polarizations into the y-component of Ohm's law, (2.4) and (2.5), and use of (3.1) and (3.2) yields the following equation for b_{y_2}

$$b_{y_2}\exp[ik_{\parallel}z-\omega_2 t] = \left[-\frac{k_{\parallel}V_{y_2}B_0}{\omega_2}-\frac{k_{\parallel}}{\omega_2}(\delta V_y^* b_{z_1}+V_{y_1}\delta B_z^*)\right]\exp[ik_{\parallel}z-i\omega_2 t]$$

$$+\ \text{nonresonant terms} \qquad (3.3)$$

B_o is the equilibrium magnetic field strength. The nonresonant terms vanish after the phase averaging. The lowest order linear polarizations for δB_z, δV_y, b_z, and V_{y_1} are given by

$$\delta B_z = \frac{k_{\perp}\delta V_x B_0}{\omega_0} \qquad (3.4)$$

$$\delta V_y = i\,\frac{k_{\perp}{}^2 R_+{}^2\Omega_+\delta V_x}{\omega_0} \qquad (3.5)$$

$$b_{z_1} = \frac{k_{\perp}V_{x_1}B_0}{\omega_1} \qquad (3.6)$$

$$V_{y_1} = i\,\frac{k_{\perp}{}^2 R_+{}^2\Omega_+ V_{x_1}}{2[\omega_1-(k_{\parallel}{}^2C_A{}^2/\omega_1)]} \qquad (3.7)$$

where $C_A = (B_0{}^2/4\pi\rho)^{1/2}$, the Alfven speed. Substitution of (3.4) - (3.7) into (3.3) yields

$$b_{y_2} = -\frac{k_{\parallel}V_{y_2}B_0}{\omega_2} + i\,\frac{k_{\parallel}k_{\perp}\Omega_+B_0k_{\perp}{}^2R_+{}^2\delta V_x^*V_{x_1}}{2\omega_0\omega_1\omega_2}\ \frac{k_{\parallel}{}^2C_A{}^2}{\omega_1{}^2[1-(k_{\parallel}{}^2C_A{}^2/\omega_1{}^2)]} \qquad (3.8)$$

Note that since the oblique fast mode has $\omega_1 \sim k_\perp C_F$, $k_\parallel^2 C_A^2/\omega_1^2 \sim k_\parallel^2/k_\perp^2 << 1$, and the nonlinear contribution to (3.8) is small.

Performing an identical analysis on the y momentum equation yields for V_{y_2}

$$-i\omega_2 V_{y_2} = \frac{ik_\parallel B_0 b_{y_2}}{4\pi\rho_0} + i\,\frac{k_\parallel B_0 b_{y_1}}{4\pi\rho_0}\left(\frac{\delta B_z^*}{B_0} - \frac{\delta\rho^*}{\rho_0}\right) + ik_\perp(V_{x_1}\delta V_y^* - V_{y_1}\delta V_x^*) \tag{3.9}$$

The last two nonlinear terms arise from $\underline{V}\cdot\nabla\underline{V}$. The perpendicular fast wave polarizations satisfy $\delta(B_z/\rho) = 0$; hence the nonlinear term proportional to b_{y_1} vanishes identically. Substituting (3.5), (3.7), and (3.8) into (3.9) and neglecting terms of order $k_\parallel^2 C_A^2/\omega_1^2 << 1$, yields the following simple result

$$-i\left(\omega_2 - \frac{k_\parallel^2 C_A^2}{\omega_2}\right) V_{y_2} = \frac{k_\perp\Omega_+ k_\perp^2 R_+^2}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_0}\right)\delta V_x^* V_{x_1} \tag{3.10}$$

The coefficient of V_{y_2} is just the linear Alfven wave dispersion relation. Note that the nonlinear terms are proportional to $k_\perp^2 R_+^2$.

A more convenient and transparent form of (3.10) is obtained by reformulating the above analysis in a Hamiltonian representation [Sagdeev and Galeev, 1969]. The time dependence of the wave amplitude is now assumed to have the form $V_{y_2} = V_{y_2}(t)\exp[ik_\parallel z - i\omega_2 t]$ where $V_{y_2}(t)$ is considered as slowly varying. Use of the linear dispersion relation for ω_2 then permits (3.10) to be written as

$$\frac{\partial V_{y_2}(t)}{\partial t} = \frac{k_\perp\Omega_+ k_\perp^2 R_+^2}{2\omega_0\omega_1\omega_2}\left[2 - \frac{\omega_2}{\omega_1}\right]\omega_2\omega_1\;\delta V_x^* V_{x_1} \tag{3.11}$$

where $\omega_0 = \omega_1 - \omega_2$ was used.

An identical analysis for the oblique high-β fast mode yields for $V_{x_1}(t)$

$$\frac{\partial V_{x_1}(t)}{\partial t} = - \frac{k_{\parallel}^2 C_A^2 k_{\perp} \Omega_+ k_{\perp}^2 R_+^2}{2\omega_0 \omega_1 \omega_2} \delta V_x V_{y_2} \tag{3.12}$$

where terms of order $k_{\parallel}^2 C_A^2/\omega_1^2 << 1$ and $k_{\parallel} R_+ << 1$ have been neglected. A similar equation for the time rate of change of δV_x could be obtained.

3.3 DECAY INSTABILITY GROWTH RATE

Now consider the simple case where the wave train amplitude greatly exceeds both perturbation amplitudes, $\delta V_x >> V_{x_1}, V_{y_2}$. If the growth time for the instability is long compared to a flow time across a wave train scale length, the time dependence of δV_x in (3.11) and (3.12) can be neglected in the determination of the initial decay instability growth rate. Of course after many e-folding times of the instability, δV_x will be reduced to the perturbation amplitude level; however, by this time the shock transition will be completed. Note that the nonlinear limit cycle of the decay instability, i.e., when all three modes share equal amplitudes [Sagdeev and Galeev, 1969], need not be considered since the perturbed waves are convected downstream out of the shock region.

If $V_{x_1}(t)$ and $V_{y_2}(t)$ vary as $\exp[\gamma t]$, (3.11) and (3.12) reduce to

$$\gamma^2 = - \frac{k_{\parallel}^2 C_A^2 k_{\perp}^4 (P_{\perp}^{(0)+}/\rho_0) k_{\perp}^2 R_+^2}{4|\omega_0 \omega_1 \omega_2|^2} \left[2 - \frac{\omega_2}{\omega_1} \right] \omega_1 \omega_2 \; |\delta V_x|^2 \tag{3.13}$$

Instability results only if sign $(\omega_1\omega_2) < 0$, or from (3.1) if $\omega_0 > |\omega_1| + |\omega_2|$. Hence the initial large amplitude wave decays to waves of lower frequency or, by the quantum analogy, the decay occurs only to lower energy states.

To further determine the growth rate, $k_\parallel$, ω_1, and ω_2 must be estimated. Setting $\omega_2 = -k_\parallel C_A$, $k_\parallel > 0$, $\omega_0 = M_{F_2} k_\perp C_F$, from (3.1) and (2.12) ω_1 becomes $\omega_1 = M_{F_2} k_\perp C_F - k_\parallel C_A \approx k_\perp C_F\,[1 - (3/8\rho)(P_\perp^{(0)+}/C_F^{\,2})k_\perp^{\,2}R_+^{\,2}]$ where $k_\perp^{\,2}R_+^{\,2} < 1$ was assumed. Note that $\omega_1/k_\perp < u_2$; hence the perturbed waves are convected downstream. From the Rankine-Hugoniot relations for a perpendicular shock [Anderson, 1963], M_{F_2} is given by

$$M_{F_2} = \left[\frac{(M_{F_1}^2 - 1) + 3}{4(M_{F_1}^2 - 1) + 3}\right]^{1/2} \approx \frac{3 - M_{F_1}^2}{2} \tag{3.14}$$

if $M_{F_1}^2 - 1 \ll 1$. Eliminating M_{F_2} then yields

$$\frac{2k_\parallel C_A}{k_\perp C_F} \approx -(M_{F_1}^2 - 1) + \frac{3}{4}\frac{P_\perp^{(0)+}}{\rho_0 C_F^{\,2}} k_\perp^{\,2}R_+^{\,2} \approx \mathcal{O}\left(\frac{3}{4}\;\frac{P_\perp^{(0)+}}{\rho_0 C_F^{\,2}} k_\perp^{\,2}R_+^{\,2}\right)$$

$$\approx \mathcal{O}\,(|2\omega_2/\omega_1|) \tag{3.15}$$

as an order of magnitude estimate for $k_\parallel$ and $|\omega_2/\omega_1|$. The wave train amplitude is estimated as $\delta V_x \sim (\delta\rho/\rho_0)(\omega_0/k_\perp) \sim M_{F_2} C_F(M_{F_1}^2 - 1) \sim C_F(M_{F_1}^2 - 1)^2$. Substitution for ω_0, δV_x, and $|2\omega_2/\omega_1|$ into (3.13) yields as a very rough estimate

$$\gamma \sim \frac{\sqrt{3}}{4} \frac{k_\perp p_\perp^{(0)+}}{\rho_0 C_F} k_\perp^2 R_+^2 (M_{F_1}^2 - 1) \quad (3.16)$$

The above calculation assumed that oblique fast and Alfven waves were present in the noise fluctuation spectrum and could be parametrically amplified by three wave decay. Oblique fast waves, but not Alfven waves, have resonant Landau or transit time damping interactions with thermal ions and electrons [Stix, 1962; Barnes, 1967]. If the nonlinear growth rate (3.16) is less than the Landau damping decrement, then little or no energy can be coupled into fast waves by nonlinear mode coupling, and the three wave decay instability probably does not provide the shock dissipation. The oblique fast wave Landau damping rate is approximately [Stix, 1962]

$$|\gamma_L^{\pm}| \simeq \frac{\sqrt{\pi}}{2} \beta^{\pm} \frac{\omega^2}{k_\parallel C_\pm} \exp[-\omega^2/k_\parallel^2 C_\pm^2] \quad (3.17)$$

where $C_\pm^2 = 2T^\pm/M_\pm$, the specie thermal speed. Substitution of (3.15) for $k_\parallel$, use of the downstream phase standing condition $\omega_0 = M_{F_2} k_\perp C_F = k_\perp C_F [1 - \frac{3}{4} (P_\perp^{(0)+}/\rho C_F^2) k_\perp^2 R_+^2]^{1/2}$ to estimate $\omega \sim \omega_0$ and $k_\perp^2 R_+^2$ in terms of M_{F_2}, and elimination of M_{F_2} by (3.14), yields for $|\gamma_L^{\pm}|$

$$|\gamma_L^{\pm}| \simeq \sqrt{\pi} \beta^{\pm} \frac{C_A}{C_\pm} \frac{k_\perp C_F}{(M_{F_1}^2 - 1)} \exp\left[- \frac{4C_A^2}{C_\pm^2 (M_{F_1}^2 - 1)^2} \right] \quad (3.18)$$

The ratio of the nonlinear growth to the linear damping rate then becomes

$$\frac{\gamma}{|\gamma_L^{\pm}|} \sim \frac{P_\perp^{(0)+}}{4\rho_0 C_F^2} \frac{C_\pm}{C_A \beta^{\pm}} k_\perp^2 R_+^2 (M_{F_1}^2 - 1)^2 \exp\left[\frac{4C_A^2}{C_\pm^2 (M_{F_1}^2 - 1)^2} \right] \quad (3.19)$$

Since $C_-/C_A\beta^- \sim \sqrt{M_+/M_-}\ 1/\sqrt{\beta^-} >> 1$ unless $\beta_- \sim M_+/M_-$, electron Landau damping does not suppress the nonlinear decay instability. Ion Landau damping is also small provided the exponential factor is large, or $M_{F_1}^2 - 1 << 2/\sqrt{\beta^+}$. Hence three wave decay is an effective dissipation process for the perpendicular fast shock if $\beta^\pm \sim 1$, but becomes considerably weaker for higher $\beta^\pm$.

3.4 ESTIMATE OF THE WAVE TRAIN DAMPING LENGTH

The three wave decay instability extracts energy from the ordered oscillations of the wave train and generates incoherent wave turbulence in the downstream flow. Hence the wave train should persist for a length $L_D \sim M_{F_2} C_F/\gamma$ before being damped to the fluctuation level. Using (3.16) L_D can be estimated as

$$L_D \sim \frac{M_{F_2} C_F}{\gamma} \sim \frac{4}{\sqrt{3}} \frac{\rho_0 C_F^2}{P_\perp^{(0)+}} \frac{R_+}{k_\perp^3 R_+^3} \quad . \tag{3.20}$$

From the downstream phase standing condition, $\omega_0 = M_{F_2} k_\perp C_F$ and (2.12) and (3.14), $k_\perp$ is estimated as $k_\perp^2 R_+^2 \gtrsim \frac{4}{3} (\rho C_F^2/P_\perp^{(0)+})(M_{F_1}^2 - 1)$; L_D then becomes of order

$$L_D \sim \frac{3}{2} \left(\frac{P_\perp^{(0)+}}{\rho_0 C_F^2} \right)^{1/2} \frac{R_+}{(M_{F_1}^2 - 1)^{3/2}} \quad . \tag{3.21}$$

For $M_{F_1}^2 - 1 << 1$, the wave train should exist many ion gyro-radii in the downstream flow.

If no other particle dissipation processes, such as current instabilities, are involved in the shock structure, shock heating of the particle distributions occurs primarily by Landau damping of oblique fast waves. The particle thermalization length should be of order

$L_T^{\pm} \sim M_{F_2} C_F / |\gamma_L^{\pm}|$, or substituting (3.18)

$$L_T^{\pm} \sim \frac{1}{2} \left(\frac{3P_{\perp}^{(0)+}}{\pi\rho C_F^2} \right)^{1/2} \frac{C_{\pm}}{\beta^{\pm} C_A} (M_{F_1}^2 - 1)^{3/2} R_+ \exp\left[\frac{4C_A^2}{C_{\pm}^2 (M_F^2 - 1)^2} \right] \tag{3.22}$$

Note that by (3.19) $L_T^{\pm} > L_D$; hence particle dissipation persists further downstream than does wave train dissipation.

Since Alfven waves are undamped, after many L_D or $L_T^{\pm}$ lengths downstream the wave spectrum will be dominated by transverse magnetic oscillations.

3.5 SUMMARY

The parametric amplification of oblique fast and parallel Alfven waves at the expense of the wave train amplitude produces the shock dissipation required by the Rankine-Hugoniot relations. In the downstream flow coherent wave train oscillations gradually become disordered and finally damp to the fluctuation level. Oblique fast wave turbulence is Landau damped by ions and electrons, thus producing downstream particle thermalization, until only Alfven turbulence remains. The relaxation lengths for the wave train and particles are many times R_+ ; hence the complete shock transition region should be much broader than the initial gradient of the magnetic field at the leading edge. Landau damping of oblique fast waves probably restricts the particular decay instability dissipation considered here to shock flows in which $\beta^{\pm}$ does not greatly exceed unity. The spatial structure of the shock is sketched in Fig. 1.

4.0 DISCUSSION

The three wave decay model of collisionless dissipation probably greatly oversimplifies the turbulent relaxation of the downstream shock flow. Clearly decays into other modes of hydromagnetic turbulence are possible and would also contribute to wave train dissipation. Hence the damping length determined from a single decay instability only represents an upper limit to the shock length.

Heat transport in the downstream flow might also be quite complicated. If particle thermalization proceeds by the linear Landau damping of oblique fast wave turbulence, the quasi-linear theory predicts that the resonant diffusion is entirely in parallel energy [Kennel and Englemann, 1966]. The Rankine-Hugoniot relations, however, require that $T_{\parallel}$ remain unchanged across the shock; hence additional turbulent dissipation is needed to convert acquired parallel energy into perpendicular energy. For $T_{\parallel}^{\pm} > T_{\perp}^{\pm}$ and $\beta^{\pm}$ large, parallel propagating fast waves are unstable to the nonresonant fire-hose and to a resonant ion instability [Kennel and Sagdeev, 1967a; Kennel and Petschek, 1966]. Kennel and Scarf [1968] have shown that the ion resonant growth rates are exponentially enhanced if the electrons are also $T_{\parallel}^{-} > T_{\perp}^{-}$ anisotropic; the instability now can affect the main bulk of the ion distribution, thus greatly increasing the turbulent dissipation. The downstream wave spectrum, therefore, is likely to exhibit a variety of turbulent modes each attempting to drive the plasma toward a stationary configuration.

The decay instability considered here predicts that Alfven wave turbulence should persist far downstream from the shock. Detection of these waves, however, does not itself constitute a verification of the proposed dissipation model since the fully turbulent shock theories of Kennel and Sagdeev [1967a] and of Camac et al. [1962] also predict downstream Alfven turbulence. Direct observations of the internal shock structure are required to differentiate between the various theories.

Typical solar wind flows have moderate to high Mach numbers $M_F \sim 4$ to 10. Therefore in the spatial region surrounding the bow of the magnetosphere where the shock is strong, trailing ICR wave trains of the type discussed here will not generally be part of the shock structure. Fredricks et al. [1970] have reported that near the sub-solar region the bow shock is characterized by electron inertia lengths, C/ω_{p_-}, which are much less than $\dot{R}_+$ when $\beta^{\pm} \sim 1$. Far out on the flanks of the magnetosphere, however, the shock strength is weak. Also shocks propagating in the interplanetary medium might be weak. Hence the trailing ICR fast shock structure could be detected in these regions by high telemetry rate satellites.

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REFERENCES

Anderson, J. E., Magnetohydrodynamic Shock Waves, M.I.T. Press, 1963.

Barnes, A., Stochastic electron heating and hydromagnetic wave damping, Phys. Fluids, 10, 2427, 1967.

Camac, M., A. R. Kantrowitz, M. M. Litvak, R. M. Patrick, and H. E. Petschek, Shock waves in collision-free plasmas, Nucl. Fusion Suppl., part 2, 423, 1962.

Cavaliere, A., and F. Englemann, Laminar hydromagnetic waves of finite amplitude in plasmas of low temperature, Nucl. Fusion, 7, 137, 1967.

Coroniti, F. V., Dispersion discontinuities of strong collisionless shocks, submitted to J. Plasma Physics, 1970.

Fishman, F. J., A. R. Kantrowitz, and H. E. Petschek, Magnetohydrodynamic shock wave in a collision-free plasma, Rev. Mod. Phys. 32, 959, 1960.

Fredricks, R. W., Structure of generalized ion Bernstein modes from the full electromagnetic dispersion relation, J. Plasma Phys., 2, 365, 1968.

Fredricks, R. W., and C. F. Kennel, Magnetosonic wave propagating across a magnetic field in a warm collisionless plasma, J. Geophys. Res., 73, 7429, 1968.

Fredricks, R. W., G. M. Crook, C. F. Kennel, I. M. Green, F. L. Scarf, P. J. Coleman, and C. T. Russel, Electrostatic turbulence in bow shock magnetic structures: OGO-5 observations, J. Geophys. Res., 1970.

Galeev, A. A., and V. I. Karpman, Turbulence theory of a weakly nonequilibrium low density plasma and structure of shock waves, Soviet Phys. JETP, 17, 403, 1963.

Goldberg, P., The structure of collisionless shock waves in a high-β plasma, Phys. Fluids, 1970.

Kadomtsev, B. B., Plasma Turbulence, Academic Press, London and New York, 1965.

Kantrowitz, A. R., and H. E. Petschek, MHD characteristics and shock waves, in Plasma Physics in Theory and Application, edited by W. B. Kunkel, 148, McGraw-Hill Book Co., New York, 1966.

Kennel, C. F., and H. E. Petschek, Limit on stably trapped particle fluxes, J. Geophys. Res., 71, 1, 1966.

Kennel, C. F., and F. E. Englemann, Velocity space diffusion from weak plasma turbulence in a magnetic field, Phys. Fluids, 9, 2377, 1966.

Kennel, C. F., and R. Z. Sagdeev, Collisionless shock waves in high-β plasmas, 1 and 2, J. Geophys. Res., 72, 3303 and 3327, 1967a, b.

Kennel, C. F., and H. E. Petschek, Magnetic turbulence in shocks, in Physics of the Magnetosphere, edited by R. L. Carovillano, J. F. McClay, and H. R. Radoski, 485, D. Reidel Publ. Co., Dordrecht-Holland, 1968.

Kennel, C. F., and F. L. Scarf, Thermal anisotropies and electromagnetic instabilities in the solar wind, J. Geophys. Res., 73, 6149, 1968.

Kinsinger, R. E., and P. L. Auer, Finite Larmor radius effects in the magnetosonic pulse, Phys. Fluids, 12, 2580, 1969.

MacMahon, A., Finite gyro-radius corrections to the hydromagnetic equations for a Vlasov plasma, Phys. Fluids, 8, 1840, 1965.

MacMahon, A., Discussion of paper by C. F. Kennel and R. Z. Sagdeev, 'Collisionless shock waves in high-β plasmas, 2,' J. Geophys. Res., 73, 7538, 1968.

Sagdeev, R. Z., Cooperative phenomena and shock waves in collisionless plasmas, Rev. Plasma Phys. 4, 23, Consultants Bureau, New York, 1966.

Sagdeev, R. Z., and A. A. Galeev, Nonlinear Plasma Theory, W. A. Benjamin, Inc., 1969.

Stix, T. H., The Theory of Plasma Waves, McGraw-Hill, New York, 1962.

Tidman, D. A., Turbulent shock waves in plasmas, Phys. Fluids, 10, 547, 1967.

FIGURE CAPTION

Fig. 1. The spatial structure of the magnetic field in the ICR fast shock is sketched against distance through the shock. Upstream is to the left, downstream to the right. At the leading edge B increases exponentially with a scale height $L \sim R_+$. Wave train oscillations, also with R_+ scale lengths, trail in the downstream flow. The wave train decays into oblique fast and parallel Alfven waves by three wave mode couplings, thus providing irreversible collisionless dissipation. Wave train oscillations are damped over a length $L_D >> R_+$. Ion and electron thermalization proceeds by the Landau damping of oblique fast waves. Undamped Alfven waves should be observed far downstream.

Structure of ion cyclotron radius perpendicular fast shock