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PROPAGATION OF THE TRANSVERSE NORMAL STRESS IN A THICK PLATE DUE TO DISTRIBUTED LATERAL IMPULSIVE LOADINGS

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16. Abstract

A theoretical study of the elastic stresses produced in an infinite plate when struck by a high-speed object is presented. The solution is obtained by means of linear elasticity. Laplace transformation techniques are employed to solve the axisymmetric problem. The plate is loaded normal to its surface with a uniform load over a circular area. The normal stress at the wave front of the unreflected dilatation wave along the axis and its variation with the radius of loading are determined. Various facets of the problem are discussed.

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PROPAGATION OF THE TRANSVERSE NORMAL STRESS IN A THICK PLATE DUE TO DISTRIBUTED LATERAL IMPULSIVE LOADINGS

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SUMMARY

The purpose of this theoretical investigation is to determine the magnitude of the elastic stresses produced in an infinite elastic plate when an impulsive load is applied normal to the plate surface. The plate is assumed to be infinite and elastic with an axisymmetric normal load applied over a circular area. The first part of the analysis is devoted to establishing the equations of motion and the boundary conditions and transforming these equations by use of the Laplace transform. The normal stress distribution is determined in particular. The second part of the analysis deals with the inversion of the transformed equations and is accomplished by use of an expansion method of Cagniard which eliminates contour integration difficulties. At this point in the analysis the various wave fronts which contribute to the normal stress can be identified and are recognized.

The normal stress due to the initial, unreflected, irrotational wave is evaluated on the axis at the wave front, and the effect of varying the radius of the axisymmetric load is determined. The results compare favorably in the limit with the published results for normal stress due to a point load. It should be noted that only the normal stress at the wave front and on the axis is determined due to the mathematical complexities of the resulting inversion integrals. The solutions are obtained for both step and Dirac delta time variations.

INTRODUCTION

When a projectile traveling at relatively high velocity strikes a plate, elastic stress waves are usually initiated which propagate toward the back face of the plate. These elastic stress waves are strong compression waves which decay in magnitude as

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they propagate. When such waves arrive at the stress-free back surface, they are reflected as tensile stress waves, which, depending on the intensity of the wave and its duration, may cause fracture to occur. This process is termed spallation and can present a problem in some situations. One example of a problem area is the main wall of a double-walled meteoroid bumper system. The outer wall breaks up the impacting meteor into an expanding gas cloud which then impacts the main wall over a certain distributed area. Because of the intensity of loading and its duration, there is a possibility of spallation damage occurring.

Problems related to stress-wave reflection at a free boundary have been encountered by seismologists, with some pioneering work done by Cagniard in reference 1. A bibliography through 1960 of other authors having done work in this area is given by Miklowitz in reference 2. Several authors (Aliev in ref. 3, Kinslow in ref. 4, and Blake in ref. 5) have approached this problem by using a pressurized cavity as the source of a spherical disturbance along with an image system to preserve the stress-free boundary condition. Problems arise, however, in exactly matching these stress-free boundary conditions.

Another approach to the stress-wave propagation and reflection problem has been to consider a finite thickness plate with a normal force applied to one side. Some preliminary work for a distributive force on an elastic half space was done by Huth and Cole (ref. 6) and by Thiruvenkatachar (ref. 7). Further analysis by Broberg (ref. 8), Davids (ref. 9), Pytel and Davids (ref. 10), and Rae (ref. 11) included the effects of the stress-free surface. All of these plate analyses considered the load to be applied at a point. Distributive loading on a plate was formulated by Thiruvenkatachar (ref. 12) who obtained the Laplace transform solution without completely finding its inverse.

The present analysis considers a uniform normal load distributed over a circular area. The normal stress of the incident dilatational wave at the wave front under the load center is presented as a function of the radius of the loading. Numerical results are presented for various values of the radius of loading while maintaining a constant total load. A comparison is also made with the case of a point load to show the rapid decline in normal stress with increased loading area.

SYMBOLS

a radius of the load applied

c_1 dilatational wave speed, $\sqrt{\dfrac{\lambda + 2\mu}{\rho}}$

c_2 distortion wave speed, $\sqrt{\frac{\mu}{\rho}}$

F total force

$$F_1(\eta) = \frac{\eta\left(1 + 2q^2\eta^2\right)^2}{\left(1 + 2q^2\eta^2\right)^2 - 4q^3\eta^2\sqrt{1 + \eta^2}\sqrt{1 + q^2\eta^2}} \qquad \text{(eq. (21))}$$

h thickness of the plate

$I_1, I_2, \ldots, I_{12}$ integrals

$$N_1 = \frac{2\mu c_1^2}{p^2}\left[\xi^2 + \left(\frac{p^2}{2c_2^2}\right)\right]^2$$

$$N_2 = \left(\frac{2\mu c_1^2}{p^2}\right)\alpha\beta\xi^2$$

P intensity of applied impulsive pressure

p Laplace transform variable

$q = \frac{c_2}{c_1}$ (nomial value of 0.5 for aluminum)

r, θ, z cylindrical coordinates

t time

u_r, u_z displacements in radial and normal directions, respectively

$u = \cos\psi$

$y = -i\eta$

α, β constants

$$\beta = \sqrt{\xi^2 + \frac{p^2}{c_2^2}}$$

$\delta(t)$ Dirac delta

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \quad \text{(Dilatation)}$$

$$\eta = \frac{c_1 \xi}{p}$$

λ, μ Lamé constants

$$\xi = \sqrt{\alpha^2 - \frac{p^2}{c_1^2}}$$

ρ mass density

$\sigma_{zz}, \sigma_{rr}, \sigma_{rz}, \sigma_{\theta\theta}$ stress components

ψ = Dummy variable in Poisson integral representation of Bessel function (see eq. (19))

$$\Omega = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \quad \text{(Rotation)}$$

Subscript:

P dilatation wave

A dot over a symbol denotes differentiation with respect to time.

ANALYSIS

The problem considered is the determination of the stress response in a homogeneous, isotropic, infinite, elastic plate subjected to an impulsive pressure. The plate, of thickness h, is loaded with an impulsive pressure of intensity P. The pressure is uniformly distributed over a circular area of radius a on one face of the plate (see fig. 1). A system of cylindrical coordinates (r,θ,z) is defined with the origin at the

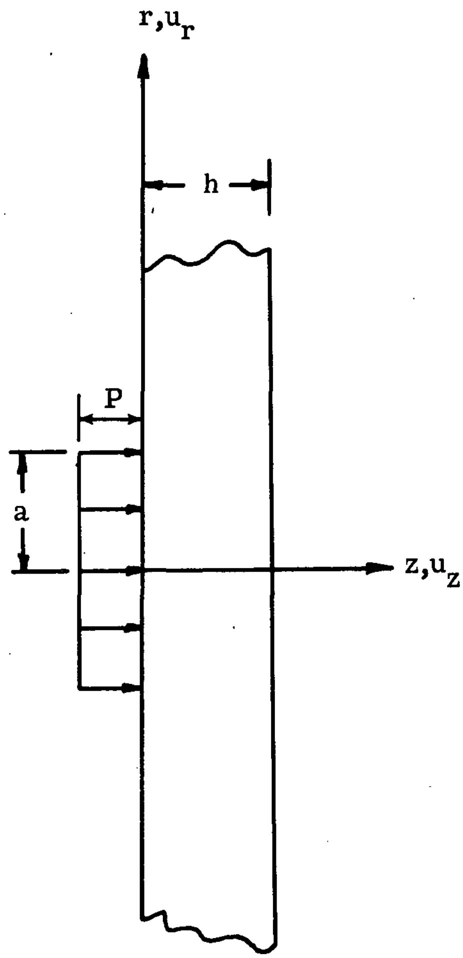


Figure 1.- Coordinates and loading.

center of the circular area of loading and the positive Z-axis normal to the plate and directed toward its other face. Due to symmetry of the loading, the problem is independent of θ. The displacements in the r- and z-directions are denoted as u_r and u_z, respectively.

Basic Equations

The problem is governed by the following set of basic equations:

Equations of motion (see ref. 13)

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial r} + \mu \frac{\partial \Omega}{\partial z} = \rho \frac{\partial^2 u_r}{\partial t^2} \tag{1}$$

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial z} - \frac{\mu}{r} \frac{\partial (r\Omega)}{\partial r} = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{2}$$

where

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \tag{3}$$

$$\Omega = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{4}$$

and λ, μ are Lamé constants and ρ is mass density.

Stress, displacement equations

$$\sigma_{rr} = \lambda\Delta + 2\mu \frac{\partial u_r}{\partial r} \tag{5}$$

$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu \frac{u_r}{r} \tag{6}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{7}$$

$$\sigma_{zz} = \lambda\Delta + 2\mu \frac{\partial u_z}{\partial z} \tag{8}$$

Boundary conditions

$$\sigma_{rz} = 0 \qquad (z = 0;\quad z = h;\quad t \geqq 0) \tag{9}$$

$$\sigma_{zz} = 0 \qquad (z = h;\quad t \geqq 0) \tag{10}$$

$$\sigma_{zz} = PH(t)\left[1 - H(r - a)\right] \qquad (z = 0 \text{ for step function loading}) \tag{11a}$$

$$\sigma_{zz} = P\delta(t)\left[1 - H(r - a)\right] \qquad (z = 0 \text{ for Dirac delta loading}) \tag{11b}$$

where $H(t)$ is the unit step function and $\delta(t)$ is the Dirac delta function.

Initial conditions

$$u_r = u_z = 0 \qquad (t = 0, \quad r \geqq 0; \quad 0 \leqq z \leqq h) \tag{12a}$$

$$\dot{u}_r = \dot{u}_z = 0 \qquad (t = 0, \quad r \geqq 0; \quad 0 \leqq z \leqq h) \tag{12b}$$

Laplace Transformation of the Basic Equations

Equations (1) to (4) were transformed in time using Laplace transforms and were arranged to form two partial differential equations in terms of the Laplace transform of Δ and Ω. These equations are

$$\frac{\partial^2\overline{\Omega}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\Omega}}{\partial r} - \left(\frac{1}{r^2} + \frac{p^2}{c_2^{\,2}}\right)\overline{\Omega} + \frac{\partial^2\overline{\Omega}}{\partial z^2} = 0 \tag{13}$$

and

$$\frac{\partial^2\overline{\Delta}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\Delta}}{\partial r} - \frac{p^2}{c_1^{\,2}}\overline{\Delta} + \frac{\partial^2\overline{\Delta}}{\partial z^2} = 0 \tag{14}$$

where p is the transform variable. These equations were solved by the method of separation of variables and superposition (see appendix A and refs. 7 and 12). The expressions for the transform of the normal incident stress σ_{zz} can then be obtained by utilizing equations (3), (4), and (8).

Since the type of damage during spallation is caused by a reversal of the initial compression wave to a tensile wave at the rear free boundary of the plate, the normal stress, particularly along the Z-axis, is thought to be the major contributor to this damage. The transformed expression for the normal stress due to a Dirac delta loading function, is

$$\bar{\sigma}_{zz} = Pa\int_0^\infty \frac{M(z,\xi)\ J_1(a\xi)\ J_0(r\xi)\ d\xi}{G(\xi)} \tag{15}$$

The functional relationships $M(z,\xi)$ and $G(\xi)$ are defined in appendix A and $J_1(a\xi)$, $J_0(r\xi)$ are Bessel functions of the first kind. The transformed solution for a step loading is obtained from equation (15) by replacing P with P/p.

The solution as given by equation (15) can be expressed as a summation of terms, each of which can be identified as reflected and unreflected shear and dilatation waves. This expansion is given as

$$\bar{\sigma}_{zz} = aP\left[\left(\bar{Q}_0 + \bar{Q}_0{}^*\right) + \sum_{n=1}^{5} \left(\bar{Q}_n + \bar{Q}_n{}^*\right) + \ldots\right] \tag{16}$$

The values of $\bar{Q}_n$ and $\bar{Q}_n{}^*$ are given in appendix B, along with the details of the development of this equation. As can be shown, $\bar{Q}_0$ represents the initial dilatation wave, $\bar{Q}_0{}^*$ the initial shear wave, $\bar{Q}_1$ the dilatation wave after one reflection, and so forth. The result given by equation (16) has been previously obtained by Thiruvenkatachar in reference 12.

Inversion of Laplace Transforms

The present analysis considers the normal stress σ_{zz}, caused by the incident dilatation wave, and is represented by $\bar{Q}_0$ in equation (16). Therefore, the transformed stress which must be inverted is

$$\bar{\sigma}_{zz} = aP \int_0^\infty \left(\frac{N_1}{N_1 - N_2}\right) J_0(r\xi)\, J_1(a\xi)\, e^{-\alpha z}\, d\xi \tag{17}$$

where N_1 and N_2 are functional relationships found in appendix A. By introducing the variables $\eta = \frac{c_1 \xi}{p}$ and $q = \frac{c_2}{c_1}$ and by making use of the integral representation of the product of the Bessel functions (ref. 14), equation (17) becomes

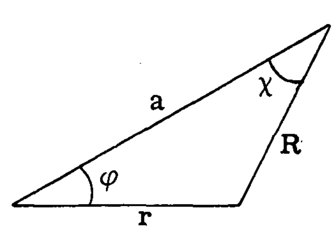
$$\bar{\sigma}_{zz} = \frac{aPp}{\pi c_1} \int_0^\infty \frac{\left(1 + 2q^2\eta^2\right)^2 e^{-\frac{pz}{c_1}\sqrt{1+\eta^2}}}{\left[\left(1 + 2q^2\eta^2\right)^2 - 4q^3\eta^2\sqrt{1+\eta^2}\sqrt{1+q^2\eta^2}\right]} \int_0^\pi J_1\left(\frac{p\eta}{c_1} R\right) \cos\chi\, d\varphi\, d\eta \tag{18}$$

where the trigonometric variables R, χ, and φ are shown in the following sketch and satisfy the following equations:

$$R^2 = a^2 + r^2 - 2\text{ arc cos }\varphi$$

$$R \sin\chi = r \sin\varphi$$

$$R \cos\chi = a - r \cos\varphi$$



In addition, the Bessel function in equation (18) can be replaced with its Poisson integral representation (ref. 15) to yield

$$\bar{\sigma}_{zz} = \frac{aPp^2}{\pi^2 c_1^2} \operatorname{Re} \int_0^{\pi} (a - r \cos \varphi) \int_0^{\pi} \sin^2\psi \int_0^{\infty} \frac{\eta\left(1 + 2q^2\eta^2\right)^2 e^{-\frac{p}{c_1}\left(z\sqrt{1+\eta^2} + i\eta R \cos \psi\right)}}{\left(1 + 2q^2\eta^2\right)^2 - 4q^3\eta^2\sqrt{1 + \eta^2}\sqrt{1 + q^2\eta^2}} \, d\eta \, d\psi \, d\varphi \quad (19)$$

Equation (19) can be recast into a more simplified representation by introducing the variable (see ref. 11)

$$c_1 t = z\sqrt{1 + \eta^2} + i\eta R \cos \psi \quad (20)$$

As η varies from zero to infinity, t is represented by a curve in the complex plane from z/c_1 to ∞ (shown in fig. 2). It is observed that $\eta = 0$ defines the wave front $z = c_1 t$. The path of integration can be transformed (see ref. 1) so that it lies along the real axis in the t-plane, and equation (19) becomes

$$\bar{\sigma}_{zz} = \frac{aPp^2}{\pi^2 c_1^2} \operatorname{Re} \int_0^{\infty} \int_0^{\pi} (a - r \cos \varphi) \, d\varphi \int_0^{\pi} \sin^2\psi \, d\psi \left[H\left(t - \frac{z}{c_1}\right) F_1(\eta) \frac{d\eta}{dt}\right] e^{-pt} \, dt \quad (21)$$

where

$$F_1(\eta) = \frac{\eta\left(1 + 2q^2\eta^2\right)^2}{\left(1 + 2q^2\eta^2\right)^2 - 4q^3\eta^2\sqrt{1 + \eta^2}\sqrt{1 + q^2\eta^2}}$$

and $H\left(t - \frac{z}{c_1}\right)$ is the unit step function.

The well-known Laplace transform of the second derivative of a function is given by

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = p^2\bar{f} - pf(0) - f'(0) \quad (22)$$

Therefore, noting that $\sigma_{zz}(0) = 0$, $0 \leq t < \frac{z}{c_1}$, and $\frac{d\sigma_{zz}}{dt}(0) = 0$, the inverse of equation (21) can be written as

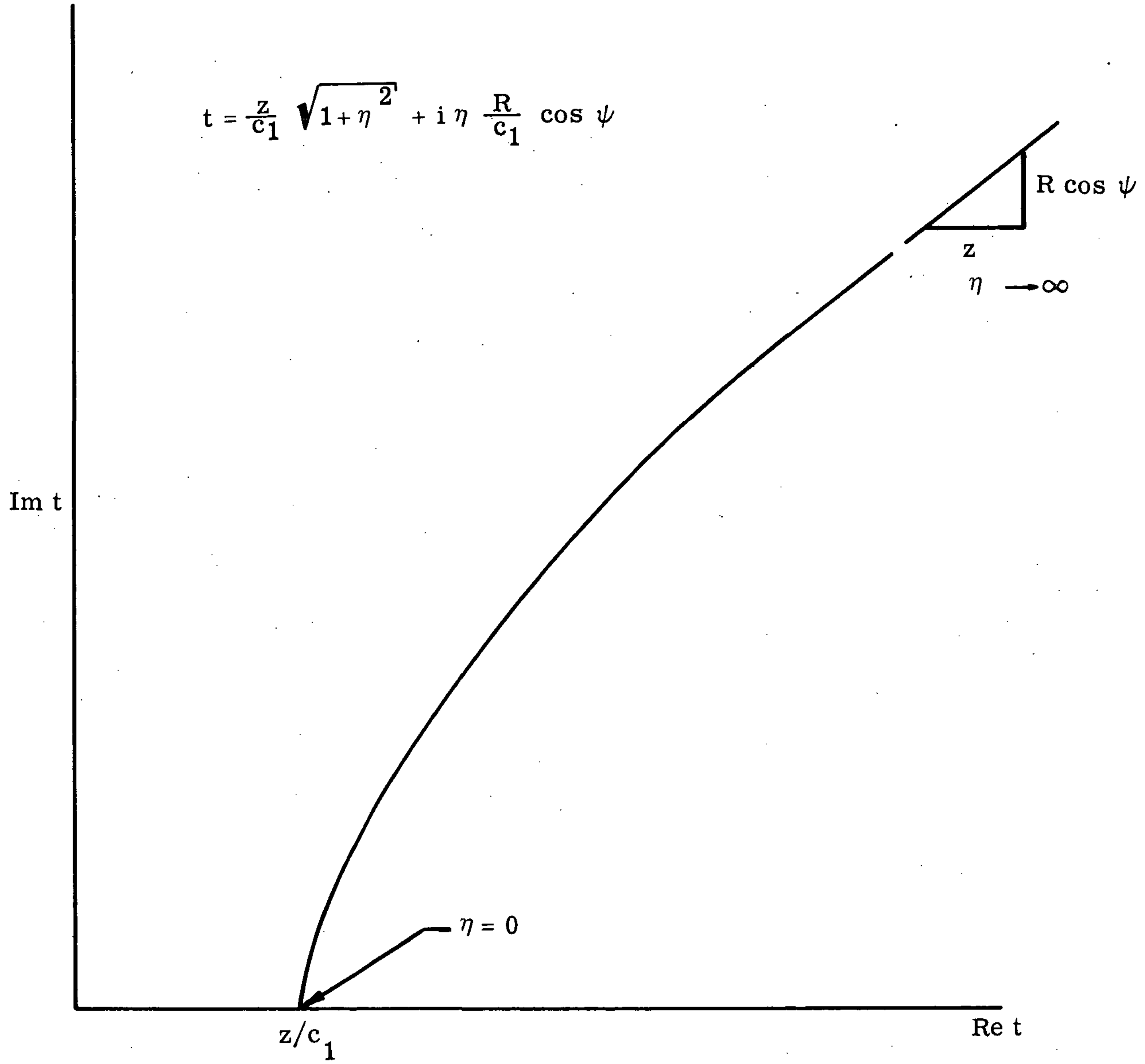


Figure 2.- Variation of η with t in the complex plane.

$$\sigma_{zz} = \frac{aP}{\pi^2 {c_1}^2} H\left(t - \frac{z}{c_1}\right) \mathrm{Re}\left\{\frac{d^2}{dt^2}\left[\int_0^{\pi} (a - r\cos\varphi)\, d\varphi \int_0^{\pi} F_1(\eta)\, \frac{d\eta}{dt} \sin^2\psi\, d\psi\right]\right\} \tag{23}$$

Inasmuch as equation (23) represents a stress wave propagating in the positive z-direction with velocity c_1, it is the incident dilatation wave. For a step loading, equation (23) would contain only one time derivative.

Stress at Wave Front

If a fracture occurs during the reflection process, it is most likely to occur on the axis where $r = 0$. Setting $r = 0$ in equation (23) yields

$$\left(\sigma_{zz}\right)_P = \frac{a^2 P}{\pi c_1^2} \operatorname{Re}\left[\frac{d^2}{dt^2} \int_0^{\pi} F_1(\eta) \frac{d\eta}{dt} \sin^2\psi \, d\psi\right] \tag{24}$$

where the subscript P denotes dilatation wave. For this condition $R = a$, equation (20) becomes

$$c_1 t = z\sqrt{1 + \eta^2} + i\eta a \cos\psi \tag{25}$$

It will be observed from equation (25) that as long as the value of a is nonzero, η is a complex function of ψ, which makes $F_1(\eta)$ and $d\eta/dt$ in equation (24) also functions of ψ. These functional relationships must be determined before the integral can be evaluated. The value of a is assumed to be a positive number and $a = 0$ is treated as a special case. Solving for η from equation (25) yields

$$\eta = \frac{-iac_1 t \cos\psi \pm z\sqrt{c_1^2 t^2 - \left(z^2 + a^2\cos^2\psi\right)}}{z^2 + a^2\cos^2\psi} \tag{26}$$

In this expression the positive sign must be chosen since $\eta = 0$ defines the wave front $z = c_1 t$. From equation (26) it can be seen that η is, in general, a complicated function of ψ which makes the integration shown in equation (24) a formidable task. In order to make the problem somewhat tractable, the solution was sought at the wave front.

Case of a Point Load Where $a = 0$

For the special case of $a = 0$, η is real and independent of ψ and can be taken outside the integral in equation (24). Equation (24) can then be written as

$$\left(\sigma_{zz}\right)_P = \frac{F}{2\pi c_1^2} \frac{d^2}{dt^2}\left[\frac{-\eta\dot{\eta}\left(1 - 2q^2\eta^2\right)^2}{\left(1 - 2q^2\eta^2\right)^2 + 4q^3\eta^2\sqrt{1 - \eta^2}\,\sqrt{1 - q^2\eta^2}}\right] \tag{27}$$

where $\pi a^2 P$ is replaced by F in the limit.

Step load at a point.- The normal stress at the wave front $z = c_1 t$ for a step load is obtained from equation (27) by replacing the operator d^2/dt^2 by d/dt. The resulting normal stress σ_{zz} becomes

$$\left(\sigma_{zz}\right)_P = \frac{F}{2\pi z^2}\left(1 + 8q^3\right) \tag{28}$$

where F is the total force. This result is identical to that obtained by Rae in reference 11.

Dirac delta at a point.- From equation (27) it is possible to obtain the incident normal stress at the wave front due to a Dirac delta loading at a point which is written as

$$\left(\sigma_{zz}\right)_P = \frac{4\tilde{F}q^3c_1}{\pi z^3}\left(5 + 2q^2 + 16q^3\right) \tag{29}$$

where $\tilde{F}$ is the impulse of the load.

Case of a distributed load where $a \neq 0$.- For the case of a distributed load, the solution near the wave front $z \approx c_1t$ will always satisfy the relation

$$c_1t \leqq \sqrt{z^2 + a^2\cos^2\psi} \tag{30}$$

This makes the term under the radical in equation (26) negative and makes η pure imaginary. (In order to consider the stress at any point other than at the wave front, η would, in general, have to be considered a complex number, dependent upon z, a, and ψ.) Therefore, by letting $\eta = iy$ and $\dot{\eta} = -i\dot{y}$ where

$$y = \frac{c_1ta\cos\psi + z\sqrt{z^2 + a^2\cos^2\psi - c_1{}^2t^2}}{z^2 + a^2\cos^2\psi} \tag{31}$$

equation (24) becomes

$$\left(\sigma_{zz}\right)_P = \frac{a^2P}{\pi c_1{}^2}\frac{d^2}{dt^2}\int_0^\pi \frac{-y\dot{y}\left(1 - 2q^2y^2\right)^2 \sin^2\psi\, d\psi}{\left(1 - 2q^2y^2\right)^2 + 4q^3y^2\sqrt{1 - y^2}\,\sqrt{1 - q^2y^2}} \tag{32}$$

Near the wave front, $z \approx c_1t$ and $y << 1$. Using this knowledge and the fact that $q < 1$ allows equation (32) to be expanded, using the binomial expansion, to yield

$$\left(\sigma_{zz}\right)_P = \frac{a^2P}{\pi c_1{}^2}\frac{d^2}{dt^2}\int_0^\pi \frac{-y\dot{y}\left(1 - 4q^2y^2 + 4q^4y^4\right)\sin^2\psi\, d\psi}{\left(1 - 4q^2y^2 + 4q^4y^4\right) + 4q^3y^2\left(1 - \frac{q^2y^2}{2} - \frac{q^4y^4}{8} + \ldots\right)\left(1 - \frac{y^2}{2} - \frac{y^4}{8} + \ldots\right)} \tag{33}$$

The expansion was truncated at powers of y^2 to yield

$$\left(\sigma_{zz}\right)_P = \frac{a^2 P}{\pi c_1^2} \frac{d^2}{dt^2} \int_0^{\pi} y\dot{y}\left(4q^3y^2 - 1\right) \sin^2\psi \, d\psi \tag{34}$$

It can be shown from equation (34) that the step load for $a = 0$ given by equation (28) can still be obtained, but the Dirac delta load for $a = 0$ becomes

$$\left(\sigma_{zz}\right)_P = \frac{12\tilde{F}q^3 c_1}{\pi z^3} \tag{35}$$

which is less than that determined in equation (29). Since the series expansion has been truncated, the solution presented in equation (35) is an approximation. Consequently, the results of the investigation for the Dirac delta loading is limited to the case where $a > 0$.

Substituting for y and $\dot{y}$ from equation (31) into equation (34) and making the substitution $u = \cos\psi$ results in an integrand with 12 identifiable terms. Equation (34) can be written as

$$\left(\sigma_{zz}\right)_P = \frac{a^2 P}{\pi c_1^2} \frac{d^2}{dt^2} \sum_{n=1}^{12} I_n \tag{36}$$

where I_n are the integrals of these 12 terms. A representative example of one of these integrals is I_1, written as

$$I_1 = \int_{-1}^{1} \frac{c_1^2 t\left(z^2 - a^2u^2\right)\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^2} \tag{37}$$

and the remaining 11 integrals are presented in appendix C.

In order to evaluate these integrals, an expansion was utilized. Since $|u| \leq 1$ over the open interval $-1 < u < 1$, then

$$\left(1 - u^2\right)^{1/2} = 1 - \frac{u^2}{2} - \frac{u^4}{8} - \ldots \tag{38}$$

This expansion diverges at the end points. However, the end points of the interval in the 12 integrals may be excluded since their integrands at the end points remain finite. For ease of computation, this series expansion was truncated such that the numerator of the integrals always contained terms of u to the fourth power. The expressions for the normal stress $(\sigma_{zz})_P$ at the wave front for the step load and Dirac delta load are shown in appendix D.

RESULTS

Normal impulsive loading of an elastic plate by a uniform and point load was investigated. Results were obtained for the incident normal dilatation stress wave along the axis at the wave front. The time-dependent step and Dirac delta loadings were considered.

Figure 3 shows the typical results of the incident normal stress at the wave front for a step load applied uniformly over a circular area as a function of the radius of loading a. The total load applied in each case was a constant 27.95 N (2π lb), and the distance traveled was $z = 1.27$ cm (0.5 in.). The plate was assumed to be aluminum. It will be noted that the solution for $a \neq 0$, as given by equation (D1), limits to the solution for an $a = 0$, as given by equation (28). The decrease in normal stress for increasing a is rapid when a is small.

Figure 4 shows corresponding results for a Dirac delta load uniformly applied over a circular area. The load was determined from $\tilde{F} = 44.718t_0$ N-s $\left(\tilde{F} = \frac{16t_0\pi}{5} \text{ lb-s}\right)$, where t_0 is chosen as 2.45×10^{-6} s from reference 8 (according to the Hertz theory of impact). Again as a is increased, the stress level at the wave front decreases. It will again be noted that the solution for $a \neq 0$, as given by equation (D8), is consistent with the solution for $a = 0$, as given by equation (29).

Figure 5 shows the nondimensional ratio of normal stress at the dilatation wave front to applied pressure, as a function of a/z for an applied step load. This was obtained from equation (D1). The curve in figure 5 shows (as expected) that either as a gets large (plane wave) or as z gets small (the pressure application point) the ratio of normal stress to applied pressure approaches unity. Another observation that can be made is that, for a given z, the plane-wave case (i.e., where no attenuation at the wave front is experienced) is approached very rapidly for values of $a > z$.

Figure 6 shows the progression of the wave front as it propagates into the plate for a step input. The total load applied is again a constant 27.95 N (2π lb) but the intensity of the pressure P is varied. At $z = 0$ the normal stress σ_{zz} equals the applied pressure for a particular radius a. As a increases, the rate of decay decreases, and the solution rapidly approaches that of a plane wave.

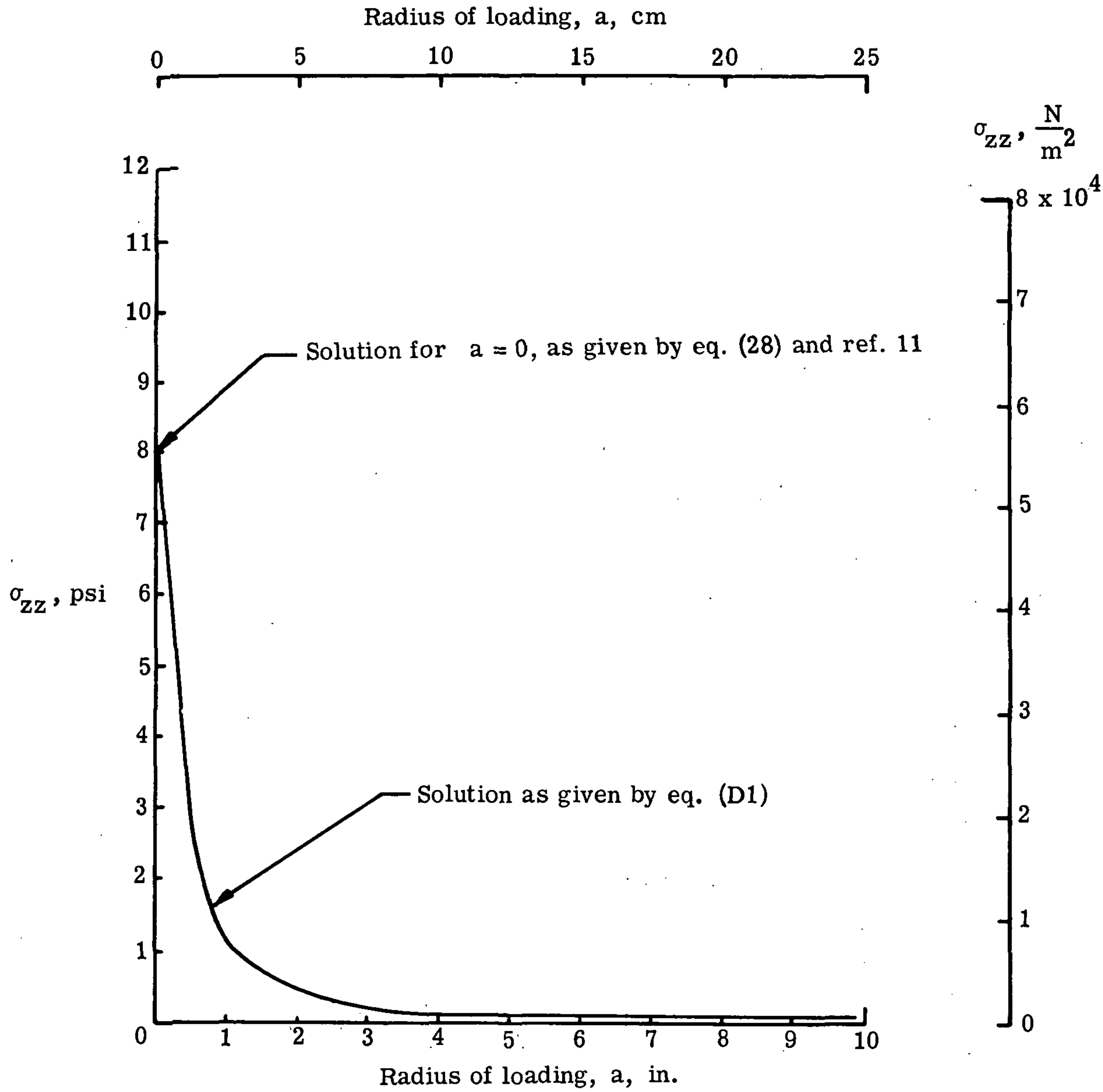


Figure 3.- Incident normal stress along the axis at the wave front for a step input. $F = 27.95$ N (2π lb); $z = 1.27$ cm (0.5 in.).

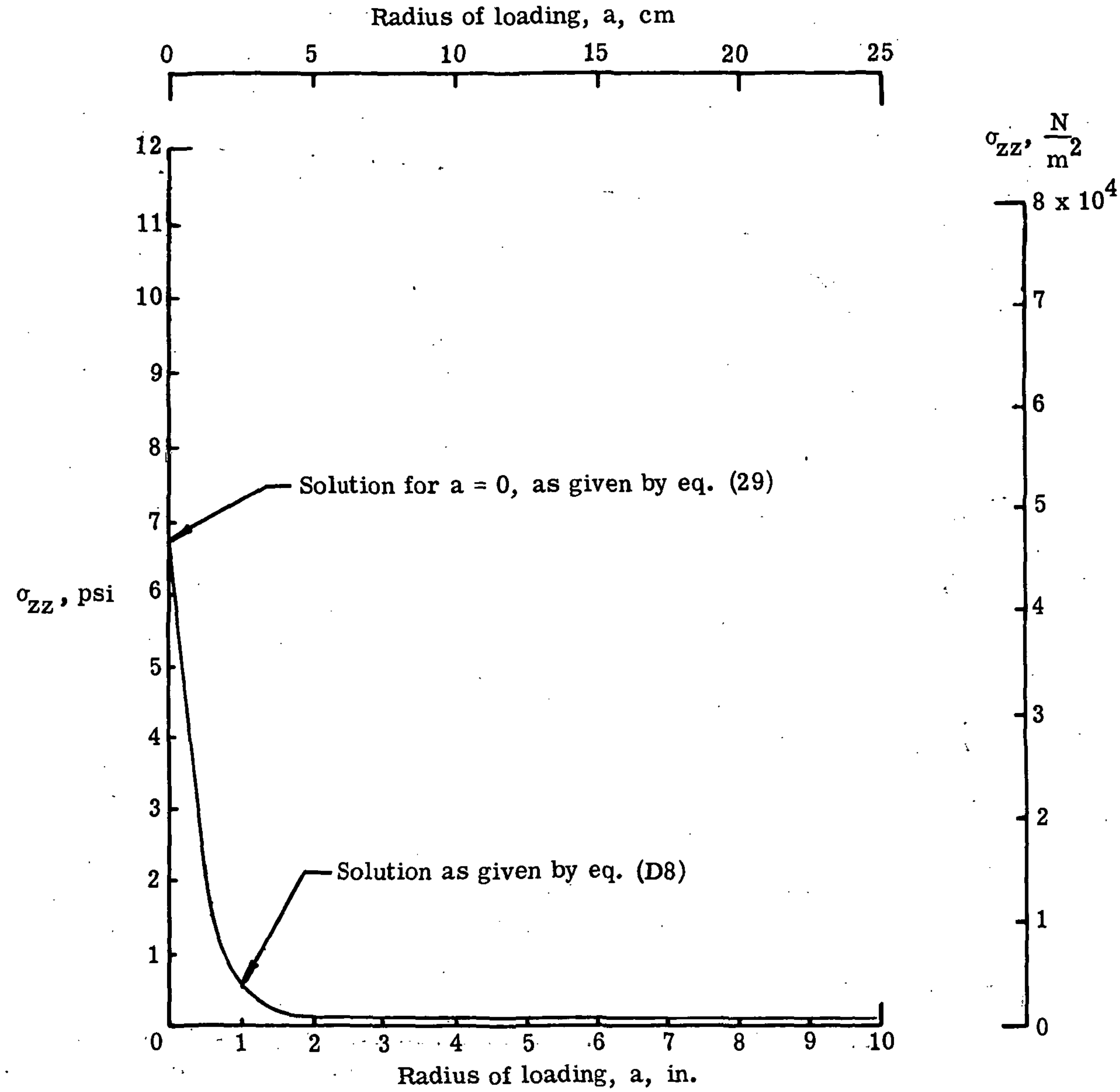


Figure 4.- Incident normal stress along the axis at the wave front for a Dirac delta input. $\tilde{F} = 44.718t_0$ N-s $\left(\frac{16t_0\pi}{5}\ \text{lb-s}\right)$; z = 2.54 cm (1.0 in.); $t_0 = 2.45 \times 10^{-6}$ s.

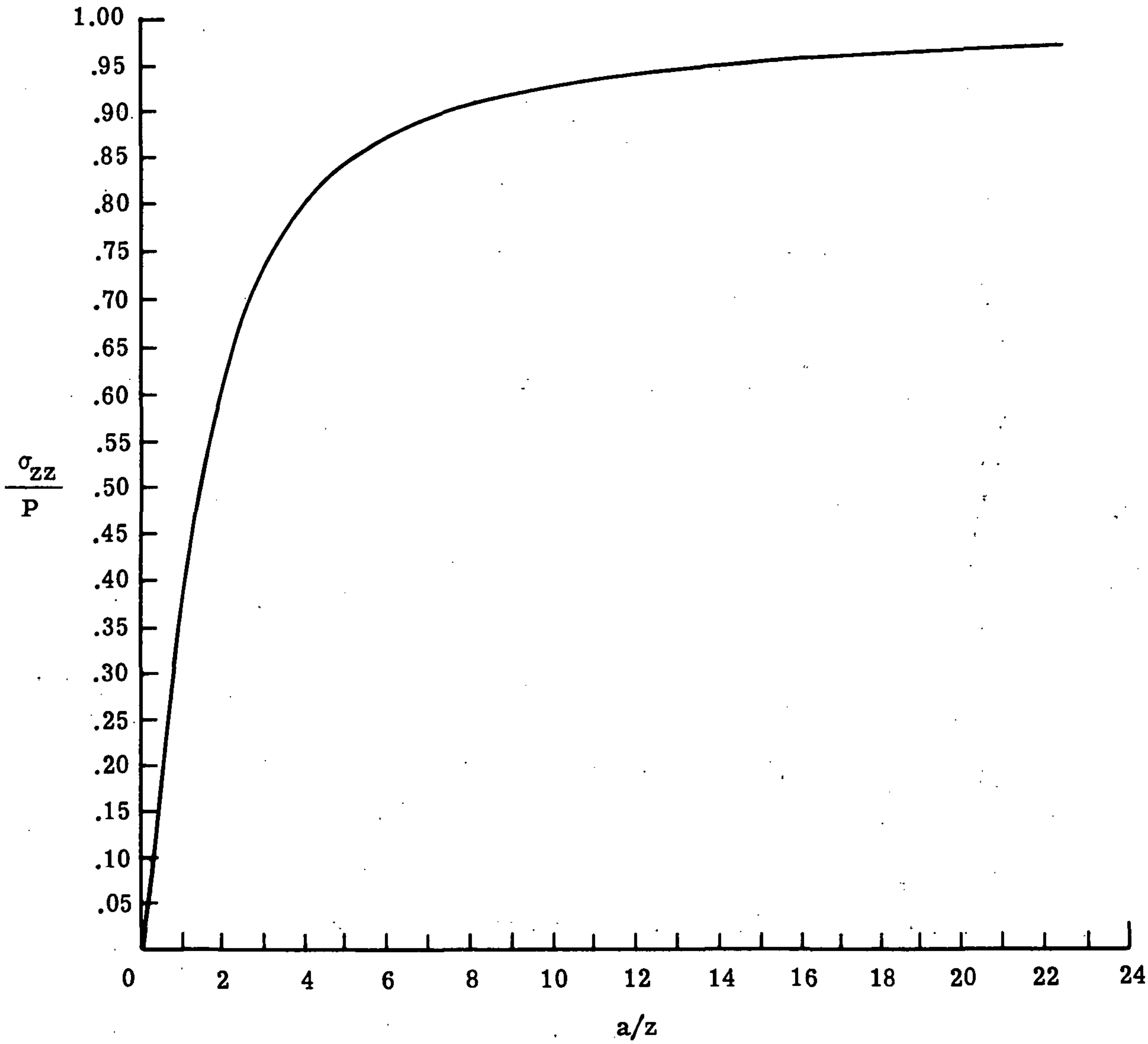


Figure 5.- Normal stress as a function of a/z for a constant pressure amplitude P.

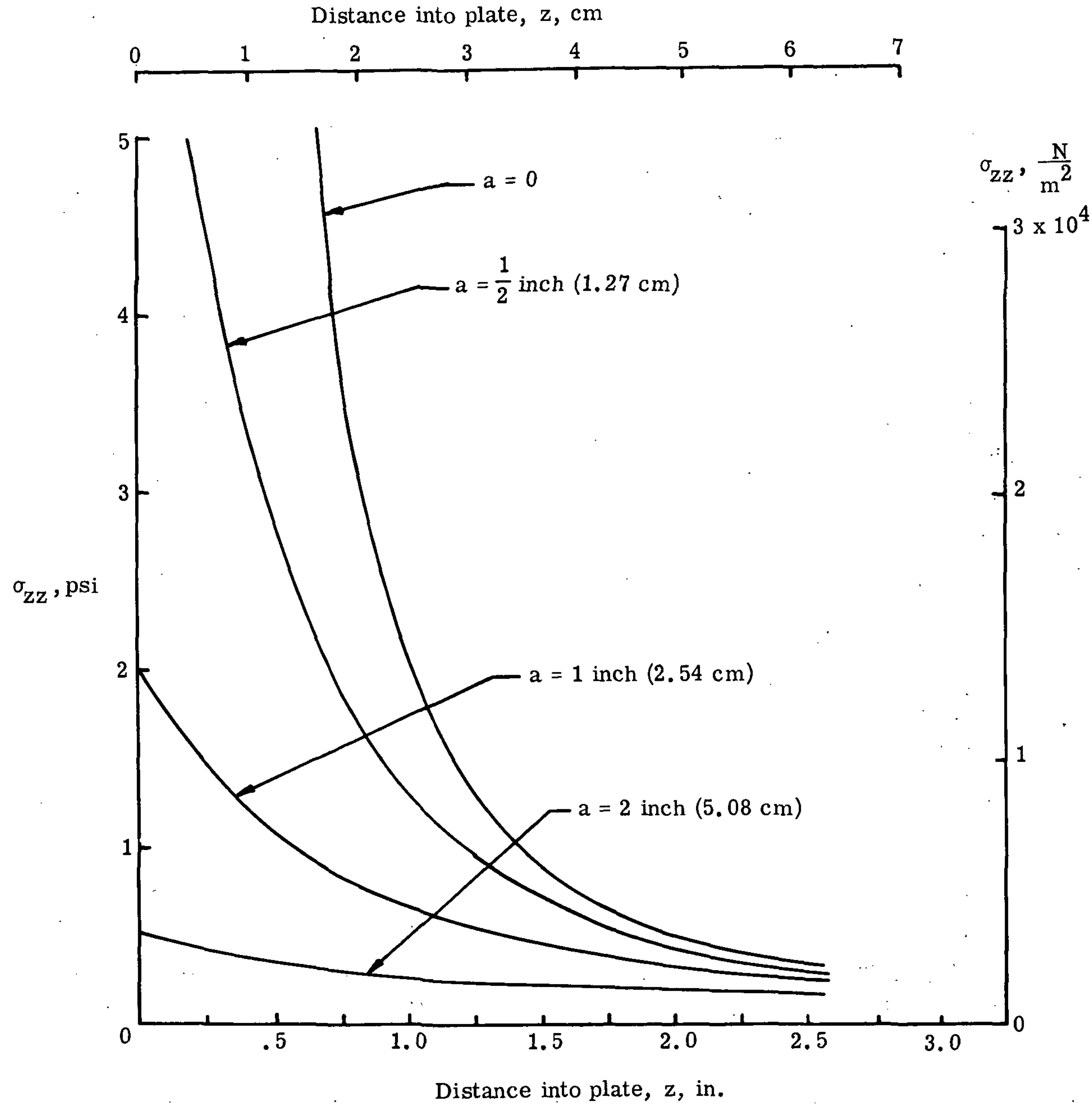


Figure 6.- Variation of normal stress with depth z for a step input of constant total load (variable pressure amplitude).

CONCLUDING REMARKS

The normal stress due to the incident dilatational wave produced in an infinite plate when loaded normal to its surface has been evaluated. The loading was distributed uniformly over a circular area. Solutions have been obtained on the axis at the wave front for step and Dirac delta time variations.

Although the solutions presented are limited, they do show the effect of increasing the area of loading. In a similar manner, other contributions to the normal stress, such as the incident shear and reflected shear and dilatational waves, could be obtained. However, the superposition of one wave upon another cannot be determined by this solution since the different wave fronts pass the same point at different times.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., June 20, 1972.

APPENDIX A

DEVELOPMENT OF TRANSFORMED NORMAL STRESS

By substituting equations (3) and (4) into equations (1) and (2) and taking the Laplace transform, two partial differential equations result, one of which (eq. (14)) is

$$\frac{\partial^2 \bar{\Delta}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Delta}}{\partial r} - \frac{p^2}{c_1^2}\bar{\Delta} + \frac{\partial^2 \bar{\Delta}}{\partial z^2} = 0 \tag{A1}$$

where p is the Laplace transform variable. Separation of variables by assuming

$$\bar{\Delta} = R(r)\, Z(z)$$

yields

$$\bar{\Delta} = \int_0^\infty \left[A(\xi)\, J_0(r\xi) \sinh \alpha z + B(\xi)\, J_0(r\xi) \cosh \alpha z\right] d\xi \tag{A2}$$

where $\xi^2 = \alpha^2 - \frac{p^2}{c_1^2}$, α is a constant, and $A(\xi)$ and $B(\xi)$ are arbitrary functions.

Similarly, the other partial differential equation (eq. (13)), when solved, results in

$$\bar{\Omega} = \int_0^\infty \left[C(\xi)\, J_1(r\xi) \sinh \beta z + D(\xi)\, J_1(r\xi) \cosh \beta z\right] d\xi \tag{A3}$$

where $\beta^2 = \xi^2 + \frac{p^2}{c_2^2}$, β is a constant, and $C(\xi)$ and $D(\xi)$ are arbitrary functions.

Substitution of the values for $\bar{\Delta}$ and $\bar{\Omega}$ into the Laplace transform of the differential equations (1) and (2) yields

$$\begin{aligned} p^2 u_r = \int_0^\infty \Big\{ & c_1^2\left[-A(\xi) \sinh \alpha z - B(\xi) \cosh \alpha z\right] \\ & + c_2^2 \beta \xi^{-1}\left[C(\xi) \cosh \beta z + D(\xi) \sinh \beta z\right]\Big\}\, J_1(r\xi)\, d\xi \end{aligned} \tag{A4}$$

and

$$p^2 u_z = \int_0^\infty \left\{ \alpha c_1{}^2 \xi^{-1} \left[A(\xi) \cosh \alpha z + B(\xi) \sinh \alpha z \right] \right.$$
$$\left. - c_2{}^2 \left[C(\xi) \sinh \beta z + D(\xi) \cosh \beta z \right] \right\} J_0(r\xi)\, d\xi \tag{A5}$$

From equations (7) and (8) the transformed stresses are found to be

$$\bar{\sigma}_{zz} = \lambda \int_0^\infty \left[A(\xi) \sinh \alpha z + B(\xi) \cosh \alpha z \right] J_0(r\xi)\, d\xi$$
$$+ \frac{2\mu}{p^2} \int_0^\infty \left\{ \alpha^2 c_1{}^2 \xi^{-1} \left[A(\xi) \sinh \alpha z + B(\xi) \cosh \alpha z \right] \right.$$
$$\left. - \beta c_2{}^2 \left[C(\xi) \cosh \beta z + D(\xi) \sinh \beta z \right] \right\} J_0(r\xi)\, \xi\, d\xi \tag{A6}$$

and

$$\bar{\sigma}_{rz} = \frac{2\mu}{p^2} \int_0^\infty \left\{ -c_1{}^2 \alpha \left[A(\xi) \cosh \alpha z + B(\xi) \sinh \alpha z \right] \right.$$
$$\left. + c_2{}^2 \xi^{-1} \left(\xi^2 + \frac{p^2}{2c_2{}^2} \right) \left[C(\xi) \sinh \beta z + D(\xi) \cosh \beta z \right] \right\} J_1(r\xi)\, \xi\, d\xi \tag{A7}$$

By applying the transformed boundary conditions, equations (9), (10), and (11b), to equations (A6) and (A7) the following equations are obtained:

$$-2c_1{}^2 \alpha A(\xi) + 2c_2{}^2 \xi^{-1} \left(\xi^2 + \frac{p^2}{2c_2{}^2} \right) D(\xi) = 0 \tag{A8}$$

$$-2c_1{}^2 \alpha \left[A(\xi) \cosh \alpha h + B(\xi) \sinh \alpha z \right]$$
$$+ 2c_2{}^2 \xi^{-1} \left(\xi^2 + \frac{p^2}{2c_2{}^2} \right) \left[C(\xi) \sinh \beta h + D(\xi) \cosh \beta h \right] = 0 \tag{A9}$$

$$\left(\lambda + \frac{2\mu}{p^2} \alpha^2 c_1{}^2 \right) B(\xi) - \frac{2\mu}{p^2} \beta \xi c_2{}^2 C(\xi) = PaJ_1(a\xi) \tag{A10}$$

$$\left(\lambda + \frac{2\mu}{p^2}\alpha^2 c_1^{\ 2}\right)\xi^{-1}\left[A(\xi)\sinh\alpha h + B(\xi)\cosh\alpha h\right]$$

$$-\frac{2\mu}{p^2}\beta c_2^{\ 2}\left[C(\xi)\cosh\beta h + D(\xi)\sinh\beta h\right] = 0 \tag{A11}$$

Equation (A10) is obtained with the help of the Hankel inversion theorem (ref. 16) and an expression from reference 14. Equations (A8) to (A11) are solved simultaneously for $A(\xi)$, $B(\xi)$, $C(\xi)$, and $D(\xi)$ which are then substituted into the expression for the normal stress, equation (A6). This gives the relation for the transformed normal stress as

$$\bar{\sigma}_{zz} = Pa\int_0^{\infty}\frac{M(z,\xi)\ J_1(a\xi)\ J_0(r\xi)\ d\xi}{G(\xi)} \tag{A12}$$

where

$$M(z,\xi) = \left(\cosh\alpha z + \cosh\beta z\right)\left[N_1N_2\left(1 - \cosh\alpha h\cosh\beta h\right)\right]$$

$$+ \left(N_1^{\ 2}\cosh\alpha z + N_2^{\ 2}\cosh\beta z\right)\left(\sinh\alpha h\sinh\beta h\right)$$

$$+ \left(N_1\sinh\alpha z - N_2\sinh\beta z\right)\left(N_2\sinh\alpha h\cosh\beta h - N_1\cosh\alpha h\sinh\beta h\right) \tag{A13}$$

$$G(\xi) = \left(N_1^{\ 2} + N_2^{\ 2}\right)\left(\sinh\alpha h\sinh\beta h\right) + 2N_1N_2\left(1 - \cosh\alpha h\cosh\beta h\right) \tag{A14}$$

and

$$N_1 = \frac{2\mu c_1^{\ 2}}{p^2}\left[\xi^2 + \left(\frac{p^2}{2c_2^{\ 2}}\right)\right]^2 \tag{A15}$$

$$N_2 = \left(\frac{2\mu c_1^{\ 2}}{p^2}\right)\alpha\beta\xi^2 \tag{A16}$$

APPENDIX B

TRANSFORMED STRESS, SHOWING WAVE COMPONENTS

The transformed normal stress is given by equation (15) as

$$\bar{\sigma}_{zz} = Pa \int_0^\infty \frac{M(z,\xi)\, J_1(a\xi)\, J_0(r\xi)\, d\xi}{G(\xi)} \tag{B1}$$

Let

$$\left.\begin{aligned} t_1 &= \tanh \frac{\alpha h}{2} \\ t_2 &= \tanh \frac{\beta h}{2} \\ \chi &= \frac{t_1}{t_2} \end{aligned}\right\} \tag{B2}$$

then

$$\left.\begin{aligned} \cosh \alpha h &= \frac{1 + t_1^2}{1 - t_1^2} \\ \sinh \alpha h &= \frac{2t_1}{1 - t_1^2} \end{aligned}\right\} \tag{B3}$$

and

$$\begin{aligned} \frac{M(z,\xi)}{G(\xi)} = \frac{1}{2}\left[\frac{1}{\left(N_1^2 + N_2^2\right)\chi - N_1N_2\left(1 + \chi^2\right)}\right]\Bigg\{ & N_1 \cosh \alpha z\left[-N_2\left(1 + \chi^2\right) + 2\chi N_1\right] \\ & + N_2 \cosh \beta z\left[2\chi N_2 - N_1\left(1 + \chi^2\right)\right] + \Big[N_1 \sinh \alpha z \\ & - N_2 \sinh \beta z\Big]\left[\frac{\chi}{t_2} N_2\left(1 + t_2^2\right) - \frac{N_1}{t_2}\left(1 + t_1^2\right)\right]\Bigg\} \end{aligned} \tag{B4}$$

Substituting equation (B4) into equation (B1) gives

$$\bar{\sigma}_{zz} = \frac{aP}{2}\int_0^{\infty} \left[F_1 N_1 \cosh \alpha z + F_2 N_2 \cosh \beta z + F_3\left(N_1 \sinh \alpha z - N_2 \sinh \beta z\right)\right] J_0(r\xi)\, J_1(a\xi)\, d\xi \tag{B5}$$

where

$$F_1 = \frac{1}{N_1} + \frac{N_2}{N_1\left(N_1\chi - N_2\right)} + \frac{1}{N_1 - \chi N_2} \tag{B6}$$

$$F_2 = \frac{1}{N_2} + \frac{N_1}{N_2\left(N_2\chi - N_1\right)} + \frac{1}{N_2 - \chi N_1} \tag{B7}$$

$$F_3 = \frac{1}{t_2\left(N_2 - \chi N_1\right)} + \frac{t_1}{\chi N_2 - N_1} \tag{B8}$$

Let $X = e^{-\alpha h}$ and $Y = e^{-\beta h}$, then

$$\left.\begin{aligned} t_1 &= \frac{1 - X}{1 + X} \\ t_2 &= \frac{1 - Y}{1 + Y} \\ \chi &= \left(\frac{1 - X}{1 + X}\right)\left(\frac{1 + Y}{1 - Y}\right) \end{aligned}\right\} \tag{B9}$$

Substituting equation (B9) into equation (B6), after some algebraic manipulation, results in

$$F_1 = \frac{1}{N_1}\left[1 + \frac{1}{1 - \frac{N_2}{N_1}\left(\frac{1 - X}{1 + X}\right)\left(\frac{1 + Y}{1 - Y}\right)} - \frac{1}{1 - \frac{N_1}{N_2}\left(\frac{1 - X}{1 + X}\right)\left(\frac{1 + Y}{1 - Y}\right)}\right] \tag{B10}$$

which, after using the binomial theorem for $\frac{1}{1 + X}$ and $\frac{1}{1 - Y}$, becomes

$$F_1 = \frac{1}{N_1}\left\{1 + \sum_{n=0}^{\infty}\left[\left(\frac{N_2}{N_1}\right)^n - \left(\frac{N_1}{N_2}\right)^n\right]\left[\left(\frac{1 - X}{1 + X}\right)\left(\frac{1 + Y}{1 - Y}\right)\right]^n\right\} \tag{B11}$$

Similarly

$$F_2 = \frac{1}{N_2}\left\{1 + \sum_{n=0}^{\infty}\left[\left(\frac{N_1}{N_2}\right)^n - \left(\frac{N_2}{N_1}\right)^n\right]\left[\left(\frac{1 - X}{1 + X}\right)\left(\frac{1 + Y}{1 - Y}\right)\right]^n\right\} \tag{B12}$$

and

$$F_3 = \sum_{n=0}^{\infty}\left[N_1(1 - X)\right]^n\left(\frac{1 + Y}{N_2}\right)^{n+1}\left(\frac{1}{1 + X}\right)^n\left(\frac{1}{1 - Y}\right)^{n+1}$$
$$- \sum_{n=0}^{\infty}\left[N_2(1 + Y)\right]^n\left(\frac{1 - X}{N_1}\right)^{n+1}\left(\frac{1}{1 + X}\right)^{n+1}\left(\frac{1}{1 - Y}\right)^n \tag{B13}$$

Equation (B5) can now be rewritten as

$$\bar{\sigma}_{zz} = \frac{aP}{4}\int_0^{\infty}\left[\left(F_1N_1 + F_3N_1\right)e^{\alpha z} + \left(F_1N_1 - F_3N_1\right)e^{-\alpha z}\right.$$
$$\left. + \left(F_2N_2 - F_3N_2\right)e^{\beta z} + \left(F_2N_2 + F_3N_2\right)e^{-\beta z}\right] J_0(r\xi)\, J_1(a\xi)\, d\xi \tag{B14}$$

The coefficient of $e^{\alpha z}$, can be expressed as

$$F_1N_1 + F_3N_1 = 2X\left[1 + \frac{N_2}{N_1} + \frac{N_1}{N_2} + \left(\frac{N_1}{N_2}\right)^2 + \left(\frac{N_2}{N_1}\right)^2 + \ldots\right]$$
$$+ 2X^2\left[-1 - 3\frac{N_2}{N_1} - \frac{N_1}{N_2} - 3\left(\frac{N_1}{N_2}\right)^2 - 5\left(\frac{N_2}{N_1}\right)^2 - \ldots\right]$$
$$+ 4XY\left[\frac{N_2}{N_1} + \frac{N_1}{N_2} + 2\left(\frac{N_1}{N_2}\right)^2 + 2\left(\frac{N_2}{N_1}\right)^2 + \ldots\right] + \ldots$$

or

$$F_1N_1 + F_3N_1 = -4X^2\left[\frac{N_1(N_1 + N_2)}{(N_1 - N_2)^2}\right] + 4XY\left[\frac{2N_1N_2}{(N_1 - N_2)^2}\right] + \ldots \tag{B15}$$

Similarly the coefficient of $e^{-\alpha z}$ is found to be

$$F_1N_1 - F_3N_1 = \frac{4N_1}{N_1 - N_2} + \frac{4X^2N_1(N_1 + N_2)^2}{(N_1 - N_2)^3} + \frac{8Y^2N_1N_2(N_1 + N_2)}{(N_1 - N_2)^3}$$

$$- \frac{8XYN_1N_2(3N_1 + N_2)}{(N_1 - N_2)^3} + \ldots \tag{B16}$$

Thus, from equations (B14), (B15), and (B16) it follows that the stress can be written as

$$\tilde{\sigma}_{zz} = aP\left[\bar{Q}_0 + \bar{Q}_0^* + \sum_{n=1}^{5}\left(\bar{Q}_n + \bar{Q}_n^*\right) + \ldots\right] \tag{B17}$$

where

$$\bar{Q}_0 = \int_0^\infty \left(\frac{N_1}{N_1 - N_2}\right) J_0(r\xi)\, J_1(a\xi)\, e^{-\alpha z}\, d\xi \tag{B18}$$

$$\bar{Q}_1 = -\int_0^\infty \left[\frac{N_1(N_1 + N_2)}{(N_1 - N_2)^2}\right] J_0(r\xi)\, J_1(a\xi)\, e^{-\alpha(2h-z)}\, d\xi \tag{B19}$$

$$\bar{Q}_2 = \int_0^\infty \left[\frac{2N_1N_2}{(N_1 - N_2)^2}\right] J_0(r\xi)\, J_1(a\xi)\, e^{-h(\alpha+\beta)+\alpha z}\, d\xi \tag{B20}$$

$$\bar{Q}_3 = \int_0^\infty \left[\frac{N_1(N_1 + N_2)^2}{(N_1 - N_2)^3}\right] J_0(r\xi)\, J_1(a\xi)\, e^{-\alpha(2h+z)}\, d\xi \tag{B21}$$

$$\bar{Q}_4 = -\int_0^\infty \left[\frac{2N_1\left(N_2^{\,2} + 3N_1N_2\right)}{(N_1 - N_2)^3}\right] J_0(r\xi)\, J_1(a\xi)\, e^{-h(\alpha+\beta)-\alpha z}\, d\xi \tag{B22}$$

$$\bar{Q}_5 = \int_0^\infty \left[\frac{2N_1N_2\left(N_1 + N_2\right)}{\left(N_1 - N_2\right)^3}\right] J_0(r\xi)\, J_1(a\xi)\, e^{-2\beta h-\alpha z}\, d\xi \tag{B23}$$

The values of $\bar{Q}_n^*$ $(n = 0,1,2,3,4,5)$ are obtained from $\bar{Q}_n$ by interchanging N_1 with N_2 and α with β. The results given by equations (B17) to (B23) are identical to those presented by Thiruvenkatachar in reference 12.

APPENDIX C

TWELVE INTEGRALS USED IN EQUATION (36)

In equation (36) the solution of the normal stress was shown to be

$$\left(\sigma_{zz}\right)_P = \frac{a^2 P}{\pi c_1^2} \frac{d^2}{dt^2} \sum_{n=1}^{12} I_n \tag{C1}$$

The twelve integrals of this equation are as follows:

$$I_1 = \int_{-1}^{1} \frac{c_1^2 t\left(z^2 - a^2u^2\right)\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^2} \tag{C2}$$

$$I_2 = \int_{-1}^{1} \frac{c_1^3 z a t^2 u\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^2\left(z^2 - c_1^2t^2 + a^2u^2\right)^{1/2}} \tag{C3}$$

$$I_3 = -\int_{-1}^{1} \frac{c_1 z a u\left(1 - u^2\right)^{1/2}\left(z^2 - c_1^2t^2 + a^2u^2\right)^{1/2}}{\left(z^2 + a^2u^2\right)^2} \tag{C4}$$

$$I_4 = \int_{-1}^{1} \frac{4q^3 c_1^4 t^3 a^2 u^2\left(a^2u^2 - z^2\right)\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^4} \tag{C5}$$

$$I_5 = \int_{-1}^{1} \frac{4q^3 z^2 c_1^2 t\left(a^2u^2 - z^2\right)\left(z^2 - c_1^2t^2 + a^2u^2\right)\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^4} \tag{C6}$$

$$I_6 = -\int_{-1}^{1} \frac{8q^3 c_1^3 t^2 z a u\left(z^2 - a^2u^2\right)\left(z^2 - c_1^2t^2 + a^2u^2\right)^{1/2}\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^4} \tag{C7}$$

$$I_7 = \int_{-1}^{1} \frac{-4q^3 c_1^5 t^4 a^3 z u^3\left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2u^2\right)^4\left(z^2 - c_1^2t^2 + a^2u^2\right)^{1/2}} \tag{C8}$$

$$I_8 = \int_{-1}^{1} \frac{4q^3 c_1{}^3 t^2 z a^3 u^3 \left(1 - u^2\right)^{1/2} \left(z^2 - c_1{}^2 t^2 + a^2 u^2\right)^{1/2} du}{\left(z^2 + a^2 u^2\right)^4} \tag{C9}$$

$$I_9 = \int_{-1}^{1} \frac{-4q^3 c_1{}^3 z^3 t^2 a u \left(1 - u^2\right)^{1/2} \left(z^2 - c_1{}^2 t^2 + a^2 u^2\right)^{1/2} du}{\left(z^2 + a^2 u^2\right)^4} \tag{C10}$$

$$I_{10} = \int_{-1}^{1} \frac{4q^3 c_1{}^3 z^3 a u \left(1 - u^2\right)^{1/2} \left(z^2 - c_1{}^2 t^2 + a^2 u^2\right)^{3/2} du}{\left(z^2 + a^2 u^2\right)^4} \tag{C11}$$

$$I_{11} = -\int_{-1}^{1} \frac{8q^3 z^2 c_1{}^4 t^3 a^2 u^2 \left(1 - u^2\right)^{1/2} du}{\left(z^2 + a^2 u^2\right)^4} \tag{C12}$$

$$I_{12} = \int_{-1}^{1} \frac{8q^3 z^2 c_1{}^2 t a^2 u^2 \left(1 - u^2\right)^{1/2} \left(z^2 - c_1{}^2 t^2 + a^2 u^2\right) du}{\left(z^2 + a^2 u^2\right)^4} \tag{C13}$$

APPENDIX D

EXPRESSIONS FOR THE NORMAL STRESS

The normal stress on the axis at the wave front $z = c_1 t$ for a step load is given by

$$\left(\sigma_{zz}\right)_P = \frac{a^2 P}{\pi c_1^2} \frac{d}{dt} \sum_{n=1}^{12} I_n \tag{D1}$$

where, at $z = c_1 t$

$$\frac{dI_1}{dt} = \frac{c_1^2}{8a^4}\left\{\frac{\left(6a^2 - z^2\right)\left(4a^2 + 3z^2\right)}{z^2 + a^2} + \frac{\tan^{-1}(a/z)}{az}\left[z^2\left(3z^2 - 16a^2\right)\right]\right\} \tag{D2}$$

$$\frac{dI_4}{dt} = \frac{6q^3 c_1^2 z^2}{a^2}\left[\frac{z^2\left(z^2 + 4a^2\right)}{3\left(z^2 + a^2\right)^3} - \frac{\left(7z^2 + 16a^2\right)}{12\left(z^2 + a^2\right)^2} + \frac{1}{8\left(z^2 + a^2\right)} + \frac{\tan^{-1}(a/z)}{8az}\right] \tag{D3}$$

$$\begin{aligned}
\frac{dI_5}{dt} = \frac{q^3 z^2 c_1^2}{2a^4}\Bigg\{ & \frac{1}{3\left(z^2 + a^2\right)^3}\left[3z^2\left(16a^4 + 8a^2z^2 - z^4\right) + z^4\left(z^2 - 4a^2\right)\right] \\
& + \frac{1}{12z^2\left(z^2 + a^2\right)^2}\left[3z^2\left(32a^4 - 32a^2z^2 + 7z^4\right) + z^2\left(-7z^4 + 4a^2z^2 - 96a^4\right)\right] \\
& + \frac{1}{8z^4\left(z^2 + a^2\right)}\left[3z^2\left(32a^4 - z^4\right) + z^2\left(z^2 + 8a^2\right)\left(z^2 - 4a^2\right)\right] \\
& + \frac{\tan^{-1}(a/z)}{8az^5}\left[3z^2\left(32a^4 - z^4\right) + z^2\left(z^2 + 8a^2\right)\left(z^2 - 4a^2\right)\right]\Bigg\}
\end{aligned} \tag{D4}$$

$$\frac{dI_{11}}{dt} = \frac{-12q^3 c_1^2 z^4}{a^2}\left[-\frac{2a^2 + z^2}{3\left(z^2 + a^2\right)^3} + \frac{2a^2 + 7z^2}{12z^2\left(z^2 + a^2\right)^2} + \frac{2a^2 - z^2}{8z^4\left(z^2 + a^2\right)} + \frac{\tan^{-1}(a/z)}{8az^5}\left(2a^2 - z^2\right)\right] \tag{D5}$$

$$\frac{dI_{12}}{dt} = \frac{4q^3z^2c_1{}^2}{a^2}\left\{\frac{1}{3\left(z^2 + a^2\right)^3}\left[-z^4 + 3z^2\left(z^2 + 2a^2\right)\right]\right.$$

$$+ \frac{1}{12z^2\left(z^2 + a^2\right)^2}\left[z^2\left(7z^2 - 12a^2\right) - 3z^2\left(7z^2 + 2a^2\right)\right]$$

$$+ \frac{1}{8z^4\left(z^2 + a^2\right)}\left[z^2\left(4a^2 - z^2\right) + 3z^2\left(z^2 - 2a^2\right)\right]$$

$$\left. + \frac{\tan^{-1}(a/z)}{8az^5}\left[z^2\left(4a^2 - z^2\right) + 3z^2\left(z^2 - 2a^2\right)\right]\right\} \tag{D6}$$

$$\frac{dI_2}{dt} = \frac{dI_3}{dt} = \frac{dI_6}{dt} = \frac{dI_7}{dt} = \frac{dI_8}{dt} = \frac{dI_9}{dt} = \frac{dI_{10}}{dt} = 0 \tag{D7}$$

The normal stress at the wave front $z = c_1t$, for a Dirac delta loading is given by

$$\left(\sigma_{zz}\right)_P = \frac{a^2P}{\pi c_1{}^2}\frac{d^2}{dt^2}\sum_{n=1}^{12} I_n \tag{D8}$$

where, at $z = c_1t$,

$$\frac{d^2}{dt^2} I_1 = \frac{d^2I_2}{dt^2} = \frac{d^2I_3}{dt^2} = \frac{d^2I_6}{dt^2} = \frac{d^2I_7}{dt^2} = \frac{d^2I_8}{dt^2} = \frac{d^2I_9}{dt^2} = \frac{d^2I_{10}}{dt^2} = 0 \tag{D9}$$

$$\frac{d^2I_4}{dt^2} = \frac{12q^3c_1{}^3z}{a^2}\left[\frac{z^2\left(z^2 + 4a^2\right)}{3\left(z^2 + a^2\right)^3} - \frac{7z^2 + 16a^2}{12\left(z^2 + a^2\right)^2} + \frac{1}{8\left(z^2 + a^2\right)} + \frac{\tan^{-1}(a/z)}{8az}\right] \tag{D10}$$

$$\frac{d^2I_5}{dt^2} = \frac{3q^3c_1{}^3z^3}{a^4}\left[\frac{1}{3\left(z^2 + a^2\right)^3}\left(16a^4 + 8a^2z^2 - z^4\right) + \frac{1}{12z^2\left(z^2 + a^2\right)^2}\left(32a^4 - 32a^2z^2 + 7z^4\right)\right.$$

$$\left. + \frac{1}{8z^4\left(z^2 + a^2\right)}\left(32a^4 - z^4\right) + \frac{\tan^{-1}(a/z)}{8az^5}\left(32a^4 - z^4\right)\right] \tag{D11}$$

$$\frac{d^2I_{11}}{dt^2} = -\frac{24q^3{c_1}^3z^3}{a^2}\left[-\frac{2a^2 + z^2}{3\left(z^2 + a^2\right)^3} + \frac{2a^2 + 7z^2}{12z^2\left(z^2 + a^2\right)^2} + \frac{2a^2 - z^2}{8z^4\left(z^2 + a^2\right)} + \frac{\tan^{-1}(a/z)}{8az^5}\left(2a^2 - z^2\right)\right] \tag{D12}$$

$$\frac{d^2I_{12}}{dt^2} = \frac{24q^3{c_1}^3z^3}{a^2}\left[\frac{z^2 + 2a^2}{3\left(z^2 + a^2\right)^3} - \frac{7z^2 + 2a^2}{12z^2\left(z^2 + a^2\right)^2} + \frac{z^2 - 2a^2}{8z^4\left(z^2 + a^2\right)} + \frac{\tan^{-1}(a/z)}{8az^5}\left(-2a^2 + z^2\right)\right] \tag{D13}$$

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