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DYNAMIC RESPONSE OF NONUNIFORM STRUCTURES TO CLASSES OF PRESSURE FIELDS

Milt G. Cottis
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16. Abstract A semi-analytical method is developed for the calculation of the response of nonuniform structures to deterministic and random excitation. The method is based on parametric representations of the impulse response and input functions. With these representations, a class of structures of specified geometry and a class of pressure fields of practical concern can be considered simultaneously in a single analytical calculation of structural response. In engineering applications, the parameters in the impulse response function can be fixed once the numerical solution of the associated eigenvalue problem is available; the input function parameters can be specified given a particular input function or pressure field data. This methodology is applied to nonuniform beams and circular cylindrical shells for which parametric response solutions are derived. The computerized version of these solutions is also presented. A sample engineering problem is solved and the results and efficiency of the present method are compared to the NASTRAN solution of the same problem.

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The computer codes presented in Section 3.0 were written by Mr. R. Janda of the Computing Group, KMS Technology Center. The NASTRAN calculation of the sample problem (Section 4.0) was carried out at Goddard Space Flight Center by Mr. G. Jones under the direction of Mr. J. P. Young, Technical Monitor for the study.

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LIST OF SYMBOLS

a_n	structural damping in the nth mode
$G(r, r_o; t-t_o)$	impulse response function
L	length of cylindrical shell
M_n	modal mass
m, n	mode numbers in circumferential and axial directions, respectively
$p(r, t)$	input function
$Q(r-r'; t-t')$	pressure cross-correlation function
r	spatial variable
R	nominal radius of cylindrical shell
t	temporal variable
$U(x)$	unit step function
$w(r, t)$	structural displacement response
$W(r, r', t-t')$	response cross-correlation function
X	$x-x'$
z	axial coordinate
Z	$z-z'$
$\delta(x)$	delta function
θ	circumferential angle
Θ	$\theta-\theta'$
τ	$t-t'$

LIST OF SYMBOLS
(continued)

$\psi_n(x)$	nth structural mode shape
ω	frequency variable (rad/sec)
ω_n	nth modal frequency
$*$	denotes complex conjugate
$\|x\|$	denotes absolute value of x
$\sum_{n,m}$	denotes double sum over n and m
$\hat{w}$	denotes Fourier transform of w

The following convention is adopted for one-dimensional Fourier transforms:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\ e^{-i\omega t} f(t)$$

$$f(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega\ e^{i\omega t} \hat{f}(\omega)$$

1.0 PROBLEM DEFINITION AND APPROACH

In view of the advances made in recent years in the development of aero and space vehicles, as well as naval vessels, an enormous amount of literature has been produced on the subject of structural dynamics (Reference 1). The behavior of structures when excited by the various pressure fields which are encountered in operation is crucial to the proper design of such vehicles and to the operation of on-board systems. By and large, available calculations of structural response follow one of two basic procedures: Analytical methods are employed in cases where the structure can be idealized to a sufficiently simple one and where the excitation can be represented by a relatively simple function; when the structure is complex (nonuniform), numerical techniques are used of which the most common involves the construction of a discrete structural model which approximates the original continuous structure (discrete element approach, Reference 2).

It is the purpose of the study reported herein to bridge the gap between those two approaches, that is, to use that portion of the discrete element approach (or, other numerical technique) which analytical methods cannot handle, in combination with an analytical calculation of the dynamic response. Inasmuch as the free vibrations of a complex structure cannot, in general, be established, analytically, this portion of the calculation is left to numerical techniques. Once results to the eigenvalue problem are available, however, analytical methods are used to complete the dynamic response calculation. This study employs the impulse response, or Green's function technique.

Two pieces of information are required in the application of the impulse response method: the impulse response function appropriate to the structure at hand and an appropriate input function which describes the spatial-temporal distribution of pressure over the structure. The impulse response function can always be written in the form of an eigenfunction expansion, so that, knowledge of the structural mode shapes and normal frequencies suffices (References 3, 4).* For complex structures of given geometry (i. e., describable in a given coordinate system), this study assumes parametric forms for the associated mode shapes, where the arbitrary parameters can be fixed once the numerical solution to the eigenvalue problem is available. These parametric forms are used in the expression for the impulse response function which, in turn, is used in the analytical calculation of the dynamic response. On the other hand, input functions are also parameterized in such a way that a whole class of functions can be considered simultaneously in the response calculation. This class represents most physically realizable pressure fields of practical concern. The result is a "bank" of response solutions where the arbitrary parameters can be fixed once (a) the numerical values for the mode shapes at a number of discrete points and values for the associated normal frequencies are available, and (b) a particular input function is given.

In any given application, this procedure constitutes, in effect, a hybrid calculation of structural response: the structure is discretized and the eigenvalue problem is solved by the use of appropriate computer codes; however, the calculation of the dynamic response does

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*In practical terms, the expansion of the Green's function in terms of structural mode shapes is generally valid if the damping is small. The assumption that such an expansion is possible is analogous to the assumption made for discrete systems with regard to the existence of a matrix which diagonalizes both $M^{-1}K$ and $M^{-1}C$, where M, K, C are the mass, spring and damping matrices, respectively (see Ref. 5).

not proceed numerically, but rather, the results of the eigenvalue routine are used to define values for the arbitrary parameters in the analytical response solutions. The above procedure is depicted in Figure 1. This study has applied it to one-dimensional structures (beams) and circular cylindrical shells in the case of either deterministic or random excitation.

Section 2.0 presents the fundamental equations employed by the impulse response method, the parametric forms for the structural mode shapes (impulse response function) and for the input function and the calculation of the response due to both deterministic and random pressure fields. Section 3.0 details the computer codes based on the analytical results and Section 4.0 applies the tools developed in this study to a specific engineering problem. The identical problem is also solved with the use of the NASTRAN code and the results of the two approaches are compared. NASTRAN is a general-purpose digital computer program designed to analyze the behavior of elastic structures under a range of loading conditions using a finite-element displacement method approach. A wide range of analytical capability is built into NASTRAN including modal analysis and the determination of the response of structures due to random loads, which were the two features of NASTRAN used in this investigation. General remarks with regard to the method and results of this study follow in Section 5.0.

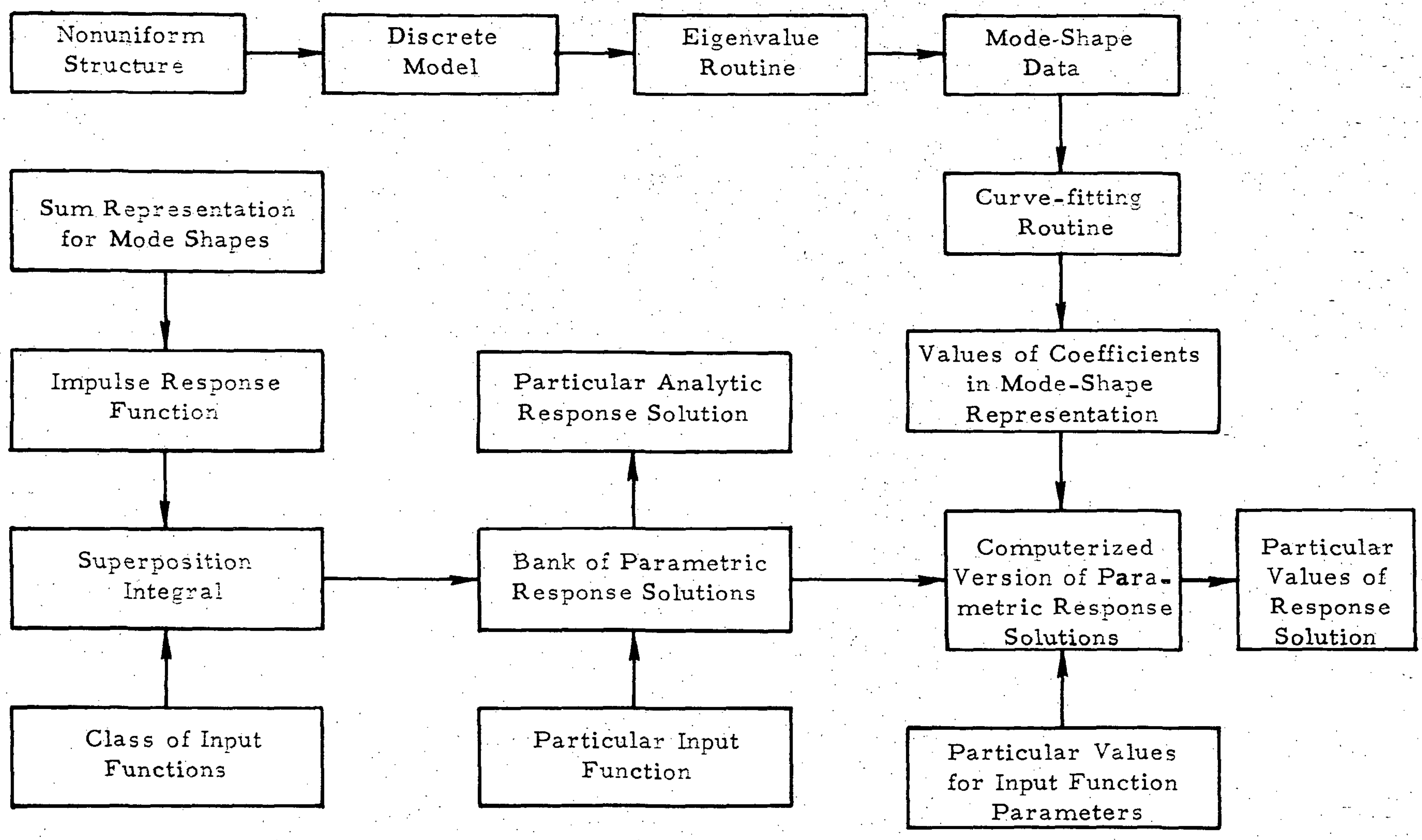

Figure 1. Approach in the calculation of the dynamic response of nonuniform structures.

2.0 ANALYSIS

2.1 Basic Equations

The linear displacement response of a structure obeys an equation of motion of the form

$$Q_{r,t}\, w(r,t) = p(r,t) \tag{1}$$

where $Q_{r,t}$ is a linear differential operator in the spatial and temporal variables r and t, $w(r,t)$ is the vibratory displacement response and $p(r,t)$ the forcing, or input function which describes the spatial-temporal distribution of pressure over the structure. Two special cases of Equation (1) arise when $p(r,t)$ is zero or an impulse:

$$Q_{r,t}\, w(r,t) = 0 \tag{2}$$

$$Q_{r,t}\, G(r, r_o; t-t_o) = \delta(r-r_o)\,\delta(t-t_o) \tag{3}$$

Equation (2) defines the free vibrations of the structure, that is, its solution yields the appropriate eigenfunctions (mode shapes) and eigenvalues (normal frequencies). Equation (3) defines the impulse response, or Green's function for the structure. It is the response at position r at time t due to an impulse at position r_o at time t_o; the impulse is represented by the product of delta functions.

Symbolically, the solution to Equation (1) is given by

$$w(r,t) = Q^{-1}_{r,t}\, p(r,t) \tag{4}$$

where $Q^{-1}_{r,t}$ is the inverse of the operator $Q_{r,t}$, that is, an integral operator. The kernel of this integral operator is the solution to Equation (3), $G(r, r_o; t-t_o)$. If initially the displacement and velocity responses are zero, Equation (4) has the explicit form

$$w(r, t) = \int_S dr_o \int_o^t dt_o \, G(r, r_o; t-t_o) \, p(r_o, t_o) \tag{5}$$

where it is assumed that the operator $Q_{r,t}$ is hermitian under the boundary conditions of the problem at hand, and where the spatial integration is over the surface of the structure, S. The Green's function is causal, that is, $G(r, r_o; t-t_o) = 0$ for $t < t_o$ and, therefore, the upper time limit in Equation (5) can be extended to $+\infty$. One can also extend the lower time limit to $-\infty$ by incorporating a unit step function in the definition of the input function, that is, $p(r, t) = p(r, t) \, U(t)$ where $U(t)$ is defined as

$$\begin{aligned} U(t) &= 0 \quad \text{for} \quad t < 0 \\ &= 1 \quad \text{for} \quad t > 0 \end{aligned}$$

Then Equation (5) reads

$$w(r, t) = \int_S dr_o \int_{-\infty}^{\infty} dt_o \, G(r, r_o; t-t_o) \, p(r_o, t_o) \tag{6}$$

The Fourier transform of Equation (6) with respect to the time t is

$$\hat{w}(r, \omega) = \int_S dr_o \, \hat{G}(r, r_o; \omega) \, \hat{p}(r_o, \omega) \tag{7}$$

where $\hat{w}$, $\hat{G}$, $\hat{p}$ are the Fourier transforms of w, G, p, respectively. For a one-dimensional structure (beam) of length L, Equation (7) reads

$$\hat{w}(x, \omega) = \int_{0}^{L} dx_{o} \; \hat{G}(x, x_{o}; \omega) \; \hat{p}(x_{o}, \omega) \tag{8}$$

while for a cylindrical shell of circular cross section and of length L,

$$\hat{w}(z, \theta, \omega) = R\int_{0}^{L} dz_{o} \int_{0}^{2\pi} d\theta_{o} \; \hat{G}(z, \theta, z_{o}, \theta_{o}; \omega) \; \hat{p}(z_{o}, \theta_{o}, \omega) \tag{9}$$

where R is the nominal radius and z, θ are the axial coordinate and circumferential angle, respectively.

If the excitation is random, the response is given by an appropriate statistical average of Equation (6). The most general average of practical concern is the response cross-correlation function between two space-time points (r, t) and (r', t'); it is given by

$$\langle w(r, t) \, w^{*}(r', t') \rangle = \int_{S} dr_{o} \int_{S} dr'_{o} \int_{-\infty}^{\infty} dt_{o} \int_{-\infty}^{\infty} dt'_{o} \; G(r, r_{o}; t-t_{o})$$

$$\cdot \, G^{*}(r', r'_{o}; t'-t'_{o}) \, \langle p(r_{o}, t_{o}) \, p^{*}(r'_{o}, t'_{o}) \rangle \tag{10}$$

where $(^{*})$ denotes complex conjugate and $\langle \cdots \rangle$ denotes the averaging process. The response autocorrelation, spatial correlation and mean square value result from Equation (10) by setting $r = r'$, $t = t'$ and the combination $r = r'$, $t = t'$, respectively. It is assumed that the pressure field is homogeneous in both space and time so that, the associated correlation function depends on the separation between the points (r, t) and (r', t'), that is

$$\langle p(r, t) \, p^{*}(r', t') \rangle = Q(r-r'; t-t') \tag{11}$$

Under these circumstances, the response correlation function is homogeneous in time, but not in space due to the finite extent of the structure, that is,

$$< w(r,t)\, w^*(r',t') > \;=\; W(r, r', t-t') \tag{12}$$

Using Equations (11), (12) in Equation (10) and taking the Fourier transform with respect to the variable $\tau = t-t'$, one obtains the response cross-spectral density:

$$\hat{W}(r, r'; \omega) = \int_S dr_o \int_S dr_o' \; \hat{G}(r, r_o; \omega)\, \hat{G}^*(r, r_o; \; \omega)\, \hat{Q}(r_o - r_o'; \omega) \tag{13}$$

where $\hat{W}, \hat{Q}$ are the Fourier transforms of W, Q, respectively. The power spectral density results from Equation (13) by setting $r = r'$. In one dimension, Equation (13) reduces to

$$\hat{W}(x, x'; \omega) = \int_o^L dx_o \int_o^L dx_o' \; \hat{G}(x, x_o; \omega)\; \hat{G}^*(x', x_o'; \; \omega) \hat{Q}(x_o - x_o'; \omega) \tag{14}$$

while, for a cylindrical shell

$$\hat{W}(z, \theta, z', \theta'; \omega) = R^2 \int_o^L dz_o \int_o^L dz_o' \int_o^{2\pi} d\theta_o \int_o^{2\pi} d\theta_o' \; \hat{G}(z, \theta, z_o, \theta_o; \omega)$$

$$\cdot\, \hat{G}^*(z', \theta', z_o', \theta_o'; \; \omega)\; \hat{Q}(z_o - z_o', \theta_o - \theta_o'; \omega) \tag{15}$$

The analytic evaluation of Equations (8), (9), (14) and (15) for a whole class of input functions $p(r,t)$ and $Q(r-r', t-t')$ is the purpose of this study. To do this, one must specify functional forms for the impulse response and input functions.

2.2 The Impulse Response Function

Under the assumption of small damping, the impulse response function can be written in the following form of an eigenfunction expansion (Ref. 3, 4).

$$G(r, r_o; t-t_o) = \sum_n (M_n \omega_n)^{-1} \psi_n(r) \psi_n^*(r_o) g_n(t-t_o) \tag{16}$$

and $$\hat{G}(r, r_o; \omega) = \sum_n (M_n \omega_n)^{-1} \psi_n(r) \psi_n^*(r_o) \hat{g}_n(\omega)$$

$$g_n(t-t_o) = \exp[-a_n(t-t_o)] \sin[\omega_n(t-t_o)] U(t-t_o) \tag{17}$$

where $\psi_n(r)$ are the structural mode shapes, n denotes the set of mode numbers required to specify a given mode, and M_n, ω_n, a_n are the modal masses, frequencies and damping, respectively. For uniform structures, the modal mass is a constant equal to the surface mass density and for most common structural geometries, the mode shapes $\psi_n(r)$ and frequencies ω_n are known from the analytic solution of the eigenvalue problem. In the case of nonuniform structures, the discrete element approach is usually employed to obtain values for M_n, ω_n and $\psi_n(r_i)$ at a discrete number of points on the structure r_i. This study assumes that such values are available in a given application which involves either a beam or a cylindrical shell. It is further assumed that trigonometric interpolation can be used to define a curve through the values of $\psi_n(r_i)$ at the spatial points r_i. The analytic forms of such trigonometric fits to mode-shape data are

$$\psi_n(x) = \sum_p A_p^n \sin k_p^n x + B_p^n \cos k_p^n x \tag{18}$$

$$\psi_{nm}(z,\theta) = \sum_p \sum_q \left\{ A_{pq}^{nm} \cos k_p^n z \sin k_q^m \theta + B_{pq}^{nm} \sin k_p^n z \cos k_q^m \theta + C_{pq}^{nm} \cos k_p^n z \cos k_q^m \theta + D_{pq}^{nm} \sin k_p^n z \sin k_q^m \theta \right\} \quad (19)$$

Values for the constant coefficients A, B, C, D and wave numbers k in the above expansions can be determined by an appropriate curve-fitting routine (Section 3.1). Knowledge of these values, however, is not required for the analytic calculation of the dynamic response. Note that Equation (19) does not assume separability in the variables z and θ, that is, $\psi_{nm}(z,\theta) \neq \psi_n(z)\ \psi_m(\theta)$. Use of the exponential forms for the trigonometric functions in Equation (19) results in the following form

$$\psi_{nm}(z,\theta) = \sum_p \sum_q \left\{ a_{pq}^{nm} \varphi_p(z) \varphi_q(\theta) + a_{pq}^{nm*} \varphi_p^*(z) \varphi_q^*(\theta) + b_{pq}^{nm*} \varphi_p(z) \varphi_q^*(\theta) + b_{pq}^{nm} \varphi_p^*(z) \varphi_q(\theta) \right\} \quad (20)$$

where $\varphi_p(z) = e^{ik_p^n z}$

$$\varphi_q(\theta) = e^{ik_q^m \theta}$$

$$a_{pq}^{nm} = \frac{1}{4i}\left(A_{pq}^{nm} + B_{pq}^{nm}\right) + \frac{1}{4}\left(C_{pq}^{nm} - D_{pq}^{nm}\right) \quad (21)$$

$$b_{pq}^{nm} = \frac{1}{4i}\left(A_{pq}^{nm} - B_{pq}^{nm}\right) + \frac{1}{4}\left(C_{pq}^{nm} + D_{pq}^{nm}\right)$$

The curve-fitting routine developed in this study (Section 3.1) makes use of the following ordering for the trigonometric expansion given by Equation (19):

$$\psi = a_1 + a_2 \cos 0\theta \cos Z + a_3 \cos 0\theta \cos 2Z + \ldots$$

$$+ a_n \cos 0\theta \cos [(n-1) Z] + a_{n+1} \cos \theta \cos 0Z$$

$$+ a_{n+2} \cos \theta \cos Z + \ldots + a_{2n} \cos \theta \cos [(n-1) Z]$$

$$+$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$+ a_{(m-1)n+1} \cos [(m-1) \theta] \cos 0Z + \ldots$$

$$+ a_{mn} \cos [(m-1) \theta] \cos [(n-1) Z]$$

$$+ a_{mn+1} \cos 0\theta \sin 0Z + \ldots + a_{mn+n} \cos 0\theta \sin [(n-1) Z]$$

$$+$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$+ a_{2mn-n+1} \cos [(m-1) \theta] \sin 0Z + \ldots$$

$$+ a_{2mn} \cos [(m-1) \theta] \sin [(n-1) Z]$$

$$+ a_{2mn+1} \sin 0\theta \cos 0Z + \ldots + a_{2mn+n} \sin 0\theta \cos [(n-1) Z]$$

$$+$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$+ a_{3mn-n+1} \sin[(m-1)\theta] \cos 0Z + \ldots$$

$$+ a_{3mn} \sin[(m-1)\theta] \cos[(n-1)Z]$$

$$+ a_{3mn+1} \sin 0\theta \sin 0Z + \ldots + a_{3mn+n} \sin 0\theta \sin[(n-1)Z]$$

$$+$$

$$\vdots$$

$$+ a_{4mn-n+1} \sin[(m-1)\theta] \sin 0Z + \ldots$$

$$+ a_{4mn} \sin[(m-1)\theta] \sin[(n-1)Z]$$

where $Z = \pi z/L$ and where the subscripts n, m bear no relationship to the mode numbers. Hence the indicies associated with a particular combination of trigonometric terms are given as follows (see Section 3.1 for notation):

$\cos i\theta \ \cos jZ \rightarrow NA$

$\cos i\theta \ \sin jZ \rightarrow NOFZ \cdot NTHA + NA$

$\sin i\theta \ \cos jZ \rightarrow 2 \cdot NOFZ \cdot NTHA + NA$

$\sin i\theta \ \cos jZ \rightarrow 3 \cdot NOFZ \cdot NTHA + NA$

where $NA = i \cdot NOFZ + j + 1$

Equation (16), with Equations (17), (18) and (20), defines the Green's function which is employed in Section 2.4 to derive analytic response solutions for beams and cylindrical shells. In a given application, all constants which appear in the above equations can easily be fixed once the numerical solution to the associated eigenvalue problem is available (Section 4.0).

Note that with proper choice of the coefficients A, B, C, D in Equations (18) and (19), Equation (16) reduces to the Green's function appropriate to a uniform beam or cylindrical shell (Refs. 3, 6). For instance, using Equation (18) with $p = 1$, $A_1 = (2/L)^{1/2}$, $B_1 = 0$, $k_1^n = n\pi/L$ in Equation (16) yields the impulse response function for a uniform beam. It follows that the response solutions of Section 2.4 are valid for both uniform and nonuniform beams and cylindrical shells.

2.3 The Input Function

It is assumed that the input functions in Equations (8), (9), (14) and (15) are separable in the variables, that is,

$$\hat{p}(x, \omega) = p_1(x)\, \hat{p}_3(\omega) \tag{22}$$

$$\hat{p}(z, \theta, \omega) = p_1(z)\, p_2(\theta)\, \hat{p}_3(\omega) \tag{23}$$

$$\hat{Q}(x-x'; \omega) = Q_1(x-x')\, \hat{Q}_3(\omega) \tag{24}$$

$$\hat{Q}(z-z', \theta-\theta'; \omega) = Q_1(z-z')\, Q_2(\theta-\theta')\, \hat{Q}_3(\omega) \tag{25}$$

The above assumption is standard in structural response calculations, though not fundamental to the methodology developed in this study. If one were to relax it, however, the mathematical manipulations would become essentially unmanageable.

We adopt a classification scheme for pressure fields, whereby input functions are classified according to the analyticity of their spectra*

*By analyticity of the input function spectrum, we mean the nature of the singularities of the corresponding Fourier transform in the complex plane.

(Refs. 7, 8). This property is sufficiently general to allow the grouping of different input functions into a class, and yet provides sufficient information for the analytic calculation of the response. In particular, we consider input functions whose Fourier transforms have any number of poles of arbitrary order.* In the case of deterministic excitation, the proper forms are

$$\hat{p}_1(k) = A_o\,\delta(k) + \sum_{j=1} B_j e^{-ikx_j} + \sum_{\alpha=1} \sum_{\beta=1} \frac{C_{\alpha,\beta}}{(k-\kappa_\alpha)^\beta} \tag{26}$$

$$\hat{p}_3(\omega) = H_o\,\delta(\omega) + \sum_{s=1} K_s e^{-i\omega t_s} + \sum_{\eta=1} \sum_{\sigma=1} \frac{N_{\eta,\sigma}}{(\omega-\nu_\eta)^\sigma} \tag{27}$$

where k is the Fourier conjugate of the coordinate x, or z and $\hat{p}_1(k)$ is the Fourier transform of $p_1(x)$, or $p_1(z)$. The first two terms in Equations (26), (27) have been included to account for the possibility of a discrete (delta function) and constant part of the spectrum. The last terms account for the singular portion of the spectrum in the complex k and ω planes. It is composed of an arbitrary number of complex poles κ_α, ν_η of arbitrary order. The poles are located in the upper half-plane only, that is, they have a positive imaginary part; e.g., $\nu_\eta = \nu'_\eta + i\nu''_\eta$, $\nu''_\eta > 0$.** The inverse Fourier transforms of Equations (26), (27) are given by

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*Such functions are termed meromorphic.

**Since $p_3(t)$ is causal, i.e., $p_3(t) = 0$ for $t < 0$, the poles ν_η must necessarily be in the upper ω half-plane. Also, one can associate a step function with $p_1(x)$ or $p_1(z)$ since the integrations in Equations (8) and (9) start at zero and thus treat them as "causal" functions. This requires the poles κ_α to be located in the upper k half-plane.

$$p_1(x) = A_o + \sum_{j=1} B_j \, \delta(x-x_j) + iU(x) \sum_{\alpha=1} \sum_{\beta=1} \frac{C_{\alpha,\beta}}{(\beta-1)!} e^{i\kappa_\alpha x} (ix)^{\beta-1} \quad (28)$$

$$p_3(t) = H_o + \sum_{s=1} K_s \, \delta(t-t_s) + iU(t) \sum_{\eta=1} \sum_{\sigma=1} \frac{N_{\eta,\sigma}}{(\eta-1)!} e^{i\nu_\eta t} (it)^{\sigma-1} \quad (29)$$

where the $(1/2\pi)$ factor in the inversion formula has been absorbed in the constant coefficients. $p_1(z)$ is given by Equation (28) with x replaced by z. In analogy to the above expressions, we assume the following form for the angular distribution of pressure.*

$$p_2(\theta) = D_o + \sum_{j'=1} E_{j'} \, \delta(\theta-\theta_{j'}) + i \sum_{\alpha'=1} \sum_{\beta'=1} \frac{F_{\alpha',\beta'}}{(\beta'-1)!} e^{i\kappa_{\alpha'}\theta} (i\theta)^{\beta'-1} \quad (30)$$

Thus, in the spatial-temporal variables, the class of input functions considered in this study includes a constant, a series of impulses and combinations of trigonometric functions, exponential functions and polynomials in the variables. This class of functions adequately represents physically realizable pressure fields of practical concern. In any given application, values for the constants in Equations (28) - (30) can easily be assigned (Section 2.4.1).

In the case of random excitation, the correlation functions are required to be even functions of the variables. That is,

---

*$p_2(\theta)$ is defined in the interval 0 to 2π and, therefore, possesses a truncated Fourier transform which is analytic everywhere. The constants $\kappa_{\alpha'}$ are not associated with singularities of the spectrum.

$$Q_1(X) = Q_1(|X|)$$

$$Q_1(Z) = Q_1(|Z|)$$

$$Q_2(\Theta) = Q_2(|\Theta|) \text{ for } |\Theta| \leq \pi \qquad (31)$$

$$= Q_2(2\pi - |\Theta|) \text{ for } |\Theta| > \pi$$

$$Q_3(\tau) = Q_3(|\tau|)$$

where $X = x-x'$, $Z = z-z'$, $\Theta = \theta-\theta'$, $\tau = t-t'$. The condition on $Q_2(\Theta)$ also guarantees that the same value for the correlation is obtained at $\Theta = 0$ and $\Theta = 2\pi$. In terms of the spectra, the above conditions require that the singularities exist in both the upper and lower half-planes. Accordingly, the class of correlation functions analogous to the deterministic input functions given by Equations (26) - (30) is

$$\hat{Q}_1(k) = A_o \delta(k) + A_1 + \sum_{\alpha=1} \sum_{\beta=1} \frac{B_{\alpha,\beta}}{(k-\kappa_\alpha)^\beta} + \sum_{\gamma=1} \sum_{\epsilon=1} \frac{C_{\gamma,\epsilon}}{(k-\kappa_\gamma)^\epsilon} \qquad (32)$$

$$\hat{Q}_3(\omega) = H_o \delta(\omega) + H_1 + \sum_{\eta=1} \sum_{\sigma=1} \frac{N_{\eta,\sigma}}{(\omega-\nu_\eta)^\sigma} + \sum_{\xi=1} \sum_{\rho=1} \frac{M_{\xi,\rho}}{(\omega-\nu_\xi)^\rho} \qquad (33)$$

$$Q_1(X) = A_o + A_1 \delta(X) + iU(X) \sum_{\alpha=1} \sum_{\beta=1} \frac{B_{\alpha,\beta}}{(\beta-1)!} e^{i\kappa_\alpha X} (iX)^{\beta-1}$$

$$- iU(-X) \sum_{\gamma=1} \sum_{\epsilon=1} \frac{C_{\gamma,\epsilon}}{(\epsilon-1)!} e^{i\kappa_\gamma X} (iX)^{\epsilon-1} \qquad (34)$$

$$Q_3(\tau) = H_o + H_1 \delta(\tau) + iU(\tau) \sum_{\eta=1} \sum_{\sigma=1} \frac{N_{\eta,\sigma}}{(\sigma-1)!} e^{i\nu_\eta \tau} (i\tau)^{\sigma-1}$$

$$- iU(-\tau) \sum_{\xi=1} \sum_{\rho=1} \frac{M_{\xi,\rho}}{(\rho-1)!} e^{i\nu_\xi \tau} (i\tau)^{\rho-1} \qquad (35)$$

$$Q_2(\Theta) = D_o + D_1 \delta(\Theta) + \sum_{\eta=1} \sum_{\sigma=0} E_{\eta,\sigma} e^{i\kappa_\eta \Theta} \Theta^{\sigma} \qquad (36)$$

where k is the Fourier conjugate of X, or Z, ω is the Fourier conjugate of τ, and where the $(1/2\pi)$ factor in the inversion formula has again been absorbed in the constant coefficients. $Q_1(Z)$ is given by Equation (34) with X replaced by Z. The poles κ_α, ν_η are in the upper half-planes, that is, they have positive imaginary parts, while the poles κ_γ, ν_ξ are in the lower half-planes, that is, they have negative imaginary parts. Thus, in the spatial-temporal variables, the class of random pressure fields considered in this study is represented by correlation functions which may be constant (representing a perfectly correlated field), delta functions (representing a purely random field), or various combinations of trigonometric functions, exponential functions and polynomials in the variables. The arbitrary constants which appear in Equations (32) - (36) can easily be fixed in engineering applications (Section 4.0). A priori knowledge of values for these constants is not necessary, however, in order to proceed with the analytic evaluation of the response.

2.4 Response Calculation

2.4.1 Response to Deterministic Excitation

With Equations (16) and (22), Equation (8) reads

$$\hat{w}(x, \omega) = \hat{p}_3(\omega) \sum_n (M_n \omega_n)^{-1} \hat{g}_n(\omega)\ \psi_n(x) \int_0^L dx_o\ \psi_n^*(x_o)\ p_1(x_o) \qquad (37)$$

where $\hat{g}_n(\omega)$ is the Fourier transform of Equation (17) given by

$$\hat{g}_n(\omega) = \frac{-\omega_n}{(\omega-\Omega_1)\ (\omega-\Omega_2)} \quad ; \quad \left.\begin{matrix}\Omega_1 \\ \Omega_2\end{matrix}\right\} = \pm\ \omega_n + ia_n \qquad (38)$$

Using Equation (18) in exponential form,

$$\hat{w}(x, \omega) = \hat{p}_3(\omega) \sum_n (M_n \omega_n)^{-1} \hat{g}_n(\omega)\ \psi_n(x) \sum_p \left\{ a_p^n\, I_1(x_1, x_2) + a_p^{n*}\, I_2(x_1, x_2) \right\} \qquad (39)$$

where $$a_p^n = \frac{B_p^n}{2} + \frac{A_p^n}{2i}$$

$$I_1(x_1, x_2) = \int_{x_1}^{x_2} dx_o\ \varphi_p(x_o)\ p_1(x_o) \qquad (40)$$

$$I_2(x_1, x_2) = \int_{x_1}^{x_2} dx_o\ \varphi_p^*(x_o)\ p_1(x_o) \qquad (41)$$

where $\varphi_p(x)$ is given by Equation (21) with z replaced by x. The limits in Equations (40), (41) have been set at arbitrary values x_1, x_2

to allow for the possibility of truncation of $p_1(x)$. If the pressure field extends over the entire length of the beam, then $x_1 = 0$, $x_2 = L$; if it extends to a point $b < L$, then $x_1 = 0$, $x_2 = b$; if it extends from a point $a > 0$ to a point $b < L$, then $x_1 = a$, $x_2 = b$; and, finally, if the pressure is distributed from a point $a > 0$ to a point $b > L$, then $x_1 = a$, $x_2 = L$. With Equation (28), the evaluation of Equation (40) is straightforward yielding the result

$$I_1(x_1, x_2) = \frac{A_o}{ik_p}\left(e^{ik_p x_1} - e^{ik_p x_2}\right) + \sum_j B_j\, e^{ik_p x_j}$$

$$+ \sum_\alpha \sum_\beta \frac{(-1)^{\beta-1} C_{\alpha\beta}}{(k_p + \kappa_\alpha)^\beta} \sum_{r=0}^{\beta-1} \frac{1}{r!} \left[\left\{ - ix_2 (k_p + \kappa_\alpha) \right\}^r e^{i(k_p + \kappa_\alpha) x_2} \right.$$

$$\left. - \left\{ - ix_1 (k_p + \kappa_\alpha) \right\}^r e^{i(k_p + \kappa_\alpha) x_1} \right] \qquad (42)$$

where, for clarity in notation, we have dropped the superscript n from the wave numbers k_p. $I_2(x_1, x_2)$ is given by Equation (42) with k_p replaced by $- k_p$. Equation (39), with (38) and (42), gives the <u>beam solution</u> for the class of input functions considered in this study.

In the case of a cylindrical shell, Equation (9) with Equations (16), (20), (28) and (30) takes the form

$$\hat{w}(z,\theta,\omega) = \hat{p}_3(\omega)\ R \sum_{n,m} (M_{nm}\omega_{nm})^{-1}\ \hat{g}_{nm}(\omega)\ \psi_{nm}(z,\theta)$$

$$\cdot \sum_p \sum_q \Big\{ a_{pq}^{nm}\ I_1(z_1, z_2)\ I_1(\theta_1) + a_{pq}^{nm*}\ I_2(z_1, z_2)\ I_2(\theta_1)$$

$$+\ b_{pq}^{nm*}\ I_1(z_1, z_2)\ I_2(\theta_1) + b_{pq}^{nm}\ I_2(z_1, z_2)\ I_1(\theta) \Big\} \tag{43}$$

where $I_1(z_1, z_2)$, $I_2(z_1, z_2)$ are given by Equations (40), (41) and (42) with x_1, x_2 replaced by z_1, z_2, and where

$$I_1(\theta_1) = \int_0^{\theta_1} d\theta_o\ \varphi_q(\theta_o)\ p_2(\theta_o) \tag{44}$$

$$I_2(\theta_1) = \int_0^{\theta_1} d\theta_o\ \varphi_q^*(\theta_o)\ p_2(\theta_o) \tag{45}$$

The upper limit in Equations (44), (45) has been set at an arbitrary angle θ_1 to allow for the possibility of truncation of $p_2(\theta)$. If the pressure field is distributed over the entire shell circumference, then $\theta_1 = 2\pi$; if it is truncated at some angle $a < 2\pi$, then $\theta_1 = a$. Since the integrand in Equation (44) is identical

in form to that of Equation (40), it follows that $I_1(\theta_1)$ is given by Equation (42) under the following substitutions [see also Equations (28) and (30)]: $x_1 \to 0$, $x_2 \to \theta_1$, $k_p \to k_q$, $A_o \to D_o$, $B_j \to E_{j'}$, $x_j \to \theta_{j'}$, $C_{\alpha,\beta} \to F_{\alpha',\beta'}$, $\kappa_\alpha \to \kappa_{\alpha'}$. $I_2(\theta_1)$ is obtained from $I_1(\theta_1)$ by substituting k_q by $-k_q$. Equation (43) with (42) under the above substitutions gives the <u>cylindrical shell solution</u> for the class of pressure fields considered.

Equations (39) and (43) constitute a "bank" of deterministic response solutions for the class of input functions given by Equations (26) - (30). A particular response solution results once an input function belonging to the above class is specified. For instance, consider a beam loaded by a pressure field whose frequency spectrum is constant and whose spatial distribution over the entire length of the beam has the form

$$\hat{p}(x, \omega) = P_o e^{-0.5x} (x^2 + 1)$$

Comparison with Equation (27) yields

$$H_o = t_s = N_{\eta, r} = 0 , \quad \text{all } s, \eta, r$$

$$K_{s=1} = P_o$$

$$K_s = 0, \quad \text{all } s \neq 1$$

On the other hand, expanding the β sum in Equation (28),

$$p_1(x) = A_o + \sum_{j=1} B_j \, \delta(x-x_j) + U(x) \sum_{\alpha=1} e^{i\kappa_\alpha x} \left\{ i\, C_{\alpha, 1} + i\, C_{\alpha, 2}(ix) + \frac{i\, C_{\alpha, 3}}{2} (ix)^2 + \ldots \right\}$$

so that, by direct comparison we conclude

$$A_o = B_j = 0, \quad \text{all } j$$

$$\kappa_{\alpha=1} = 0.5\, i$$

$\kappa_{\alpha} = 0, \;\text{all}\; \alpha \neq 1$

$C_{1,1} = -i\;;\; C_{1,2} = 0\;;\; C_{1,3} = 2i$

$C_{\alpha,\beta} = 0, \;\text{all}\; \alpha \neq 1, \;\; \beta > 3\,.$

Substitution of the above values and $x_1 = 0$, $x_2 = L$ in Equation (39) yields the corresponding analytic response solution. In terms of the computerized version of Equation (39), the user simply inputs the above values (along with the mode shape parameters appropriate to the case at hand) to the code to obtain numerical values of $\hat{w}(x, \omega)$ at specified frequencies and positions (Section 3.2).

2.4.2 Response to Random Excitation

The response cross-spectral density for a beam is given by Equations (14), (16) and (24) as

$$\hat{W}(x, x'; \omega) = \hat{Q}_3(\omega) \sum_n \frac{\psi_n(x)\, \hat{g}_n(\omega)}{M_n\, \omega_n} \sum_{n'} \frac{\psi^*_{n'}(x')\, \hat{g}_{n'}(-\omega)}{M_{n'}\, \omega_{n'}}$$

$$\cdot \int_0^L dx_o \int_0^L dx_o'\; \psi^*_n(x_o)\; \psi_{n'}(x_o')\; Q_1(x_o - x_o')$$

Using Equation (18),

$$\hat{W}(x, x'; \omega) = \hat{Q}_3(\omega) \sum_n \frac{\psi_n(x)\, \hat{g}_n(\omega)}{M_n\, \omega_n} \sum_{n'} \frac{\psi^*_{n'}(x')\, \hat{g}_{n'}(-\omega)}{M_{n'}\, \omega_{n'}}$$

$$\cdot \sum_p \sum_r \left\{ h_1 I(k_p^n, k_r^{n'}) + h_1^* I(-k_p^n, -k_r^{n'}) + h_2 I(k_p^n, -k_r^{n'}) + h_2^* I(-k_p^n, k_r^{n'}) \right\} \quad (46)$$

where

$$h_1 = a_p^n a_r^{n'}$$

$$h_2 = a_p^n a_r^{n'*}$$

$$I(k_p^n, k_r^{n'}) = \int_o^L dx_o \int_o^L dx_o' \; \varphi_p(x_o) \; \varphi_r(x_o') \; Q_1(x_o - x_o') \tag{47}$$

The coefficients a_p^n, $a_r^{n'}$ are defined in Equation (39) and $\varphi_p(x)$ = $\exp(ik_p^n x)$, $\varphi_r(x') = \exp(ik_r^{n'} x')$.

Similarly, Equations (15), (16), (20) and (25) yield the response cross-spectral density for a cylindrical shell in the form

$$\hat{W}(z, \theta, z', \theta'; \omega) = \hat{Q}_3(\omega) \, R^2 \sum_{n,m} \frac{\psi_{nm}(z,\theta) \, \hat{g}_{nm}(\omega)}{M_{nm} \, \omega_{nm}} \sum_{n',m'} \frac{\psi^*_{n'm'}(z',\theta') \, \hat{g}_{n'm'}(-\omega)}{M_{n'm'} \, \omega_{n'm'}}$$

$$\sum_{p,q} \sum_{r,s} \Big\{ H_1 I(k_p^n, k_r^{n'}, k_q^m, k_s^{m'}) + H_1^* I(-k_p^n, -k_r^{n'}, -k_q^m, -k_s^{m'})$$

$$+ H_2 I(k_p^n, -k_r^{n'}, k_q^m, -k_s^{m'}) + H_2^* I(-k_p^n, k_r^{n'}, -k_q^m, k_s^{m'}) + H_3 I(k_p^n, -k_r^{n'}, k_q^m, k_s^{m'})$$

$$+ H_3^* I(-k_p^n, k_r^{n'}, -k_q^m, -k_s^{m'}) + H_4 I(k_p^n, k_r^{n'}, k_q^m, -k_s^{m'}) + H_4^* I(-k_p^n, -k_r^{n'}, -k_q^m, k_s^{m'})$$

$$+ J_1 I(-k_p^n, -k_r^{n'}, k_q^m, k_s^{m'}) + J_1^* I(k_p^n, k_r^{n'}, -k_q^m, -k_s^{m'}) + J_2 I(-k_p^n, k_r^{n'}, k_q^m, -k_s^{m'})$$

$$+ J_2^* I(k_p^n, -k_r^{n'}, -k_q^m, k_s^{m'}) + J_3 I(-k_p^n, k_r^{n'}, k_q^m, k_s^{m'}) + J_3^* I(k_p^n, -k_r^{n'}, -k_q^m, -k_s^{m'})$$

$$+ J_4 I(-k_p^n, -k_r^{n'}, k_q^m, -k_s^{m'}) + J_4^* I(k_p^n, k_r^{n'}, -k_q^m, k_s^{m'}) \Big\} \tag{48}$$

where

$$H_1 = a_{pq}^{nm} a_{rs}^{n'm'} \qquad J_1 = b_{pq}^{nm} b_{rs}^{n'm'}$$

$$H_2 = a_{pq}^{nm} a_{rs}^{n'm'*} \qquad J_2 = b_{pq}^{nm} b_{rs}^{n'm'*}$$

$$H_3 = a_{pq}^{nm} b_{rs}^{n'm'} \qquad J_3 = b_{pq}^{nm} a_{rs}^{n'm'}$$

$$H_4 = a_{pq}^{nm} b_{rs}^{n'm'*} \qquad J_4 = b_{pq}^{nm} a_{rs}^{n'm'*} \tag{49}$$

$$I(k_p^n, k_r^{n'}, k_q^m, k_s^{m'}) = P_{pr}(L)\, S_{qs}(2\pi) \tag{50}$$

$$P_{pr}(L) = \int_0^L dz_o \int_0^L dz_o'\, \varphi_p(z_o)\, \varphi_r(z_o')\, Q_1(z_o - z_o') \tag{51}$$

$$S_{qs}(2\pi) = \int_0^{2\pi} d\theta_o \int_0^{2\pi} d\theta_o'\, \varphi_q(\theta_o)\, \varphi_s(\theta_o')\, Q_2(\theta_o - \theta_o') \tag{52}$$

The constants in Equation (49) are defined in Equation (21) and $\varphi_p(z) = \exp(ik_p^n z)$, $\varphi_r(z') = \exp(ik_r^{n'} z)$, $\varphi_q(\theta) = \exp(ik_q^m \theta)$, and $\varphi_s(\theta) = \exp(ik_s^{m'} \theta)$. The summation symbols in Equation (48) denote a double summation, e.g., $\sum_{p,q}$ stands for $\sum_p \sum_q$. Noting that Equations (47) and (51) are identical [i.e., $I(k_p^n, k_r^{n'}) = P_{pr}(L)$], it follows that the evaluation of both Equations (46) and (48) rests on performing the integrations in Equations (51) and (52) for the class of input functions given by Equations (34) and (36). This calculation is presented in detail in the Appendix. The results are

$$
\begin{aligned}
P_{pr}(L) = {} & -A_o \frac{\left(e^{ik_p L}-1\right)\left(e^{ik_r L}-1\right)}{k_p k_r} + A_1 \frac{\left[e^{i(k_p+k_r)L}-1\right]}{i(k_p+k_r)} \\
& + \sum_{\alpha,\beta} \frac{B_{\alpha,\beta}}{(\beta-1)!}\, i^{\beta} \left\{ \frac{-1}{i(k_r-\kappa_\alpha)} \left(e^{i(k_p+\kappa_\alpha)L} \left[\frac{L^{\beta-1}}{i(k_p+\kappa_\alpha)} + \sum_{b=1}^{\beta-1} \frac{(-1)^b (\beta-1)(\beta-2)\ldots(\beta-b)}{i^{b+1}(k_p+\kappa_\alpha)^{b+1}} L^{\beta-b-1} \right] - \frac{(-1)^{\beta-1}(\beta-1)!}{i^{\beta}(k_p+\kappa_\alpha)^{\beta}} \right) \right. \\
& + \sum_{a=1}^{\beta-1} \frac{(-1)^a (\beta-1)(\beta-2)\ldots(\beta-a)}{(-i)^{a+1}(k_r-\kappa_\alpha)^{a+1}} \left(e^{i(k_p+\kappa_\alpha)L} \left[\frac{L^{\beta-a-1}}{i(k_p+\kappa_\alpha)} + \sum_{b=1}^{\beta-a-1} \frac{(-1)^b (\beta-a-1)(\beta-a-2)\ldots(\beta-a-b)}{i^{b+1}(k_p+\kappa_\alpha)^{b+1}} L^{\beta-a-b-1} \right] \right. \\
& \left.\left. - \frac{(-1)^{\beta-a-1}(\beta-a-1)!}{i^{\beta-a}(k_p+\kappa_\alpha)^{\beta-a}} \right) - \frac{(-1)^{\beta-1}(\beta-1)!}{(-i)^{\beta}(k_r-\kappa_a)^{\beta}} \frac{\left[e^{i(k_p+k_r)L}-1\right]}{i(k_p+k_r)} \right\} \\
& - \sum_{\gamma,\epsilon} \frac{C_{\gamma,\epsilon}}{(\epsilon-1)!}\, i^{\epsilon} \left\{ \frac{-1}{i(k_r-\kappa_\gamma)} \left(e^{i(k_r-\kappa_\gamma)L} \left[\frac{(-L)^{\epsilon-1}}{i(k_p+\kappa_\gamma)} + \sum_{b=1}^{\epsilon-1} \frac{(-1)^b (\epsilon-1)(\epsilon-2)\ldots(\epsilon-b)}{i^{b+1}(k_p+\kappa_\gamma)^{b+1}} (-L)^{\epsilon-b-1} \right] - \frac{(-1)^{\epsilon-1}(\epsilon-1)!}{i^{\epsilon}(k_p+\kappa_\gamma)^{\epsilon}} \right) \right.
\end{aligned}
$$

$$+\sum_{a=1}^{\epsilon-1} \frac{(-1)^a (\epsilon-1)(\epsilon-2)\ldots(\epsilon-a)}{(-i)^{a+1}(k_r-\kappa_\gamma)^{a+1}} \left(e^{i(k_r-\kappa_\gamma)L} \left[\frac{(-L)^{\epsilon-a-1}}{i(k_p+\kappa_\gamma)} + \sum_{b=1}^{\epsilon-a-1} \frac{(-1)^b (\epsilon-a-1)(\epsilon-a-2)\ldots(\epsilon-a-b)}{i^{b+1}(k_p+\kappa_\gamma)^{b+1}} (-L)^{\epsilon-a-b-1} \right] \right.$$

$$\left. - \frac{(-1)^{\epsilon-a}(\epsilon-a-1)!}{i^{\epsilon-a}(k_p+\kappa_\gamma)^{\epsilon-a}} e^{i(k_p+k_r)L} \right) + \frac{(-1)^{\epsilon-1}(\epsilon-1)!}{(-i)^\epsilon (k_r-\kappa_\gamma)^\epsilon} \frac{\left[e^{i(k_p+k_r)L}-1\right]}{i(k_p+k_r)} \Bigg\} = I(k_p^n, k_r^{n'}) \qquad (53)$$

$$S_{qs} = - D_o \frac{\left(e^{2\pi i k_q} - 1\right)\left(e^{2\pi i k_s} - 1\right)}{k_q k_s} + D_1 \frac{\left[e^{i(k_q + k_s)2\pi} - 1\right]}{i(k_q + k_s)} + S_1 + S_2 + S_3 + S_4 \tag{54}$$

where

$$S_1 = \sum_{\eta,\sigma} E_{\eta,\sigma} \Bigg[\frac{-1}{i(k_s - \kappa_\eta)} \Bigg\{ e^{i(k_q + \kappa_\eta)\pi} \left(\frac{\pi^\sigma}{i(k_q + \kappa_\eta)} + \sum_{b=1}^{\sigma} \frac{(-1)^b \sigma!}{(\sigma-b)!} \frac{\pi^{\sigma-b}}{i^{b+1}(k_q + \kappa_\eta)^{b+1}} \right) - \frac{(-1)^\sigma \sigma!}{i^{\sigma+1}(k_q + \kappa_\eta)^{\sigma+1}} \Bigg\}$$

$$+ \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{1}{(-i)^{a+1}(k_s - \kappa_\eta)^{a+1}} \Bigg\{ e^{i(k_q + \kappa_\eta)\pi} \left(\frac{\pi^{\sigma-a}}{i(k_q + \kappa_\eta)} + \sum_{b=1}^{\sigma-a} \frac{(-1)^b (\sigma-a)!}{(\sigma-a-b)!} \frac{\pi^{\sigma-a-b}}{i^{b+1}(k_q + \kappa_\eta)^{b+1}} \right)$$

$$- \frac{(-1)^{\sigma-a}(\sigma-a)!}{i^{\sigma-a+1}(k_q + \kappa_\eta)^{\sigma-a+1}} \Bigg\}$$

$$- \frac{(-1)^\sigma \sigma!}{(-i)^{\sigma+1}(k_s - \kappa_\eta)^{\sigma+1}} \frac{\left[e^{i(k_q + k_s)\pi} - 1\right]}{i(k_q + k_s)} + \left\{ \frac{e^{i(k_q + k_s)2\pi} - e^{i(k_q + k_s)\pi}}{i(k_q + k_s)} \right\}$$

$$\cdot \Bigg\{ e^{-i(k_s - \kappa_\eta)\pi} \left(\frac{\pi^\sigma}{-i(k_s - \kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{(-i)^{a+1}(k_s - \kappa_\eta)^{a+1}} \right) - \frac{(-1)^\sigma \sigma!}{(-i)^{\sigma+1}(k_s - \kappa_\eta)^{\sigma+1}} \Bigg\} \Bigg] \tag{55}$$

$$
\begin{aligned}
S_2 = \sum_{\eta,\sigma} E_{\eta,\sigma} \Bigg[& \left\{ \frac{e^{i(k_q+k_s)\pi} - 1}{i(k_q+k_s)} \right\} \cdot \left\{ e^{i(k_s+\kappa_\eta)\pi} \left(\frac{\pi^\sigma}{i(k_s+\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{i^{a+1}(k_s+\kappa_\eta)^{a+1}} \right) - \frac{(-1)^\sigma \sigma!}{i^{\sigma+1}(k_s+\kappa_\eta)^{\sigma+1}} \right\} \\
& + \frac{e^{i(k_s+k_q)2\pi}}{i(k_s+\kappa_\eta)} \left\{ e^{-i(k_q-\kappa_\eta)\pi} \left(-\frac{\pi^\sigma}{i(k_q-\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^\sigma \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{(-i)^{a+1}(k_q-\kappa_\eta)^{a+1}} \right) - \frac{(-1)^\sigma \sigma!}{(-i)^{\sigma+1}(k_q-\kappa_\eta)^{\sigma+1}} \right\} \\
& + e^{i(k_s+k_q)2\pi} \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{1}{i^{a+1}(k_s+\kappa_\eta)^{a+1}} \left\{ e^{-i(k_q-\kappa_\eta)\pi} \left(-\frac{\pi^{\sigma-a}}{i(k_q-\kappa_\eta)} + \sum_{b=1}^{\sigma-a} \frac{(-1)^b(\sigma-a)!}{(\sigma-a-b)!} \frac{\pi^{\sigma-a-b}}{(-i)^{b+1}(k_q-\kappa_\eta)^{b+1}} \right) \right. \\
& \left. \qquad - \frac{(-1)^{\sigma-a}(\sigma-a)!}{(-i)^{\sigma-a+1}(k_q-\kappa_\eta)^{\sigma-a+1}} \right\} \\
& - \frac{(-1)^\sigma \sigma!}{i^{\sigma+1}(k_s+\kappa_\eta)^{\sigma+1}} \left\{ \frac{e^{i(k_q+k_s)2\pi} - e^{i(k_q+k_s)\pi}}{i(k_q+k_s)} \right\} \Bigg]
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
S_3 = \sum_{\eta,\sigma} E_{\eta,\sigma}\, e^{-2\pi i k_s} \Bigg[& \left\{ \frac{e^{i(k_q+k_s)2\pi} - e^{i(k_q+k_s)\pi}}{i(k_q+k_s)} \right\} \cdot \left\{ e^{i(k_s+\kappa_\eta)\pi} \left(\frac{\pi^{\sigma}}{i(k_s+\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{i^{a+1}(k_s+\kappa_\eta)^{a+1}} \right) \right\} \\
& - \frac{e^{i(k_q+k_s)2\pi}}{i(k_s+\kappa_\eta)} \left\{ e^{-i(k_q-\kappa_\eta)\pi} \left(-\frac{\pi^{\sigma}}{i(k_q-\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{(-i)^{a+1}(k_q-\kappa_\eta)^{a+1}} \right) - \frac{(-1)^{\sigma} \sigma!}{(-i)^{\sigma+1}(k_q-\kappa_\eta)^{\sigma+1}} \right\} \\
& - e^{i(k_q+k_s)2\pi} \sum_{a=1}^{\sigma} \frac{(-1)^{\sigma} \sigma!}{(\sigma-a)!} \frac{1}{i^{a+1}(k_s+\kappa_\eta)^{a+1}} \cdot \left\{ e^{-i(k_q-\kappa_\eta)\pi} \left(\frac{-\pi^{\sigma-a}}{i(k_q-\kappa_\eta)} + \sum_{b=1}^{\sigma-a} \frac{(-1)^b (\sigma-a)!}{(\sigma-a-b)!} \frac{\pi^{\sigma-a-b}}{(-i)^{b+1}(k_q-\kappa_\eta)^{b+1}} \right) \right. \\
& \left. - \frac{(-1)^{\sigma-a}(\sigma-a)!}{(-i)^{\sigma-a+1}(k_q-\kappa_\eta)^{\sigma-a+1}} \right\} \Bigg] \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
S_4 = \sum_{\eta,\sigma} E_{\eta,\sigma}\, e^{2\pi i k_s} \Bigg[& \left\{ \frac{e^{i(k_q+k_s)\pi} - 1}{i(k_q+k_s)} \right\} \cdot \left\{ e^{-i(k_s-\kappa_\eta)\pi} \left(- \frac{\pi^\sigma}{i(k_s-\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{(-i)^{a+1}(k_s-\kappa_\eta)^{a+1}} \right) \right\} \\
& + \frac{1}{i(k_s-\kappa_\eta)} \left\{ e^{i(k_q+\kappa_\eta)\pi} \left(\frac{\pi^\sigma}{i(k_q+\kappa_\eta)} + \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{\pi^{\sigma-a}}{i^{a+1}(k_q+\kappa_\eta)^{a+1}} \right) - \frac{(-1)^\sigma \sigma!}{i^{\sigma+1}(k_q+\kappa_\eta)^{\sigma+1}} \right\} \\
& - \sum_{a=1}^{\sigma} \frac{(-1)^a \sigma!}{(\sigma-a)!} \frac{1}{(-i)^{a+1}(k_s-\kappa_\eta)^{a+1}} \left\{ e^{i(k_q+\kappa_\eta)\pi} \left(\frac{\pi^{\sigma-a}}{i(k_q+\kappa_\eta)} + \sum_{b=1}^{\sigma-a} \frac{(-1)^b (\sigma-a)!}{(\sigma-a-b)!} \frac{\pi^{\sigma-a-b}}{i^{b+1}(k_q+\kappa_\eta)^{b+1}} \right) \right. \\
& \left. - \frac{(-1)^{\sigma-a}(\sigma-a)!}{i^{\sigma-a+1}(k_q+\kappa_\eta)^{\sigma-a+1}} \right\} \Bigg] \qquad (58)
\end{aligned}
$$

For simplicity in notation, the superscripts have been suppressed in the wave numbers k, that is

$$k_p \equiv k_p^n, \; k_r \equiv k_r^{n'}, \; k_q \equiv k_q^m \text{ and } k_s \equiv k_s^{m'}$$

Equation (53) in (46) yields the response cross-spectral density in parametric form for a beam excited by the class of input functions given by Equations (33), (34), while Equation (48), with Equations (53) - (58), gives the corresponding solutions for a cylindrical shell when the input function belongs to the class given by Equations (33), (34) and (36). The response power spectral density at position x on the beam, or position z, θ on the shell, results by setting $x = x'$ in (46) and $z = z'$, $\theta = \theta'$ in (48). The mean-square response can be obtained by further integrating the above equations over all frequencies.

If the analyst is given an input function which belongs to the class considered, values for the pressure field parameters can easily be specified and the particular analytic response solutions obtained from the above equations. On the other hand, the computerized version of the solutions provides means for obtaining numerical response values. The section which follows describes the codes based on Equations (39), (43), (46) and (48).

3.0 COMPUTER CODE - USER'S GUIDE

3.1 Curve-Fitting Routine for Structural Mode Shapes (FITMSC)

3.1.1 Program Description

The FITMSC program fits the normal component of the structural mode shape as computed by a numerical code, such as NASTRAN. The fits are in the form of products of trigonometric functions. The code uses stepwise multiple regression to evaluate the coefficients of the fit.

3.1.2 Input

Section 3.1.9 presents the necessary input to be read into the code.

3.1.3 Governing Equations

The governing equations are given in Section 2.2.

3.1.4 Method of Solution

The calculation procedure is described in Section 2.0.

3.1.5 Restart

The code is set up to skip past as many sets of modal data as necessary in order to refit a particular mode shape.

3.1.6 Termination

The code will terminate when the number of modes to be fitted have been fitted. The termination occurs because of lack of additional input data.

3.1.7 Computer Conversion

The program was written in FORTRAN V. It was used extensively on the UNIVAC 1108 (EXEC 8).

To adapt this code to CDC 6600, or IBM 7090 or 360, the following changes must be made:

1. All control cards at the beginning of each subroutine must be eliminated and/or replaced with appropriate cards.

3.1.8 Equipment Requirements

The code uses logical units (15) from which the mode shapes are obtained. No other storage units are needed. The present form of the code needs approximately 32,000 cells to execute, most of which is used for the data itself. Tapes 5 and 6 are used for reading and writing, respectively. Two or three cards will be punched for each fit.

3.1.9 Program Input Requirements

A tape with the mode shape data must be mounted on logical unit 15. See Sec. 3.1.15 for the listing of the code to generate this data tape from NASTRAN card output.

3.1.10 Card Formats

NOTE: All integer variables must be right adjusted. Floating point fields must contain a decimal which may be arbitrarily located in the field width.

CARD GROUP 1

Columns	Description
1-5	ITEST, case number. (Integer)
6-10	LIMIT, maximum number of steps in the regression analysis. (Integer)
11-15	NZS, number of constant Z for which the modal data is given.
16-20	NTH, number of angles for which the modal data is given.
21-25	NPOLI, K_p, number of terms in the fits in the spatial variable. (Integer)
26-30	NPOLK, K_q, number of terms in the fits in the azimuthal variable.
31-35	ISKIP, number of modes on the data tape to be skipped. If ISKIP is zero, no modal data will be skipped.

CARD GROUP 2

Columns	Description
1-10	EFIN, the F* value for entering a variable into the regression. (Floating point)
11-20	EFOUT, the F value for removing a variable from the regression. (Floating point)
21-30	THNOT, θ_0, the modal data is tabulated for various angles starting at θ_0. (Floating point)
31-40	DTH, $\Delta\theta$, angular increments (in degrees) for the modal data is tabulated. (Floating point)
41-50	DELZ, ΔZ, spatial increments in Z (in inches) for the tabulation of the modal data. (Floating point)
51-60	ZNOT, Z_0, length of the cylinder in inches. (Floating point)

* F - F test is a significance test.

The above card groups are repeated for each mode shape to be fitted.

3.1.11 Sample Input

The following is a listing of the input data used to fit all of the mode shapes for model 1 (Section 4.0). Sample 2 is input to fit modes 3, 4, and 8 only.

INPUT SAMPLE 1

1	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.
2	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.
3	11	7	10	6	2	0		
1.05		1.05		0.		36.	22.	132.
4	11	7	10	6	2	0		
1.05		1.05		0.		36.	22.	132.
5	11	7	10	6	2	0		
1.05		1.05		0.		36.	22.	132.
6	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.
7	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.
8	11	7	10	6	3	0		
1.05		1.05		0.		36.	22.	132.
9	11	7	10	6	3	0		
1.05		1.05		0.		36.	22.	132.
10	11	7	10	7	3	0		
1.05		1.05		0.		36.	22.	132.
11	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.
12	11	7	10	4	4	0		
1.05		1.05		0.		36.	22.	132.

INPUT SAMPLE 2

```
    3    11     7    10     6     2     2
 1.05         1.05         0.          36.         22.          132.
    4    11     7    10     6     2     0
 1.05         1.05         0.          36.         22.          132.
    8    11     7    10     6     3     3
 1.05         1.05         0.          36.         22.          132.
```

3.1.12 Description of Output

Variable	Line 1
ITEST	
LIMIT	
NZS	
NTH	Input data, see Section 3.1.10
NPOLI	
NPOLK	
ISKIP	

Variable	Line 2
EFIN	
EFOUT	
THNOT	
DTH	Input data, see Section 3.1.10
DELZ	
ZNOT	

Following the input is a printout of the pertinent information at each step of the regression. The last set of variable numbers and the values of the coefficients are punched on cards for later code runs.

A comparison is made and printed between the actual data and the predicted data (data calculated from the last set of coefficients). These columns are self-explanatory.

3.1.13 Code Expansion

Following is a listing of minimum storage requirements. NOTE: MAXN and MAXNP1 must be altered to the dimension sizes of X and A, respectively.

Minimum storage requirements:

a) (MAXN, MAXNP1) cells per variable X

b) MAXN cells per variable W, XBAR, SIG, IVAR, B, SB

c) MAXNP1 cells per variable R

d) NTH cells per variable THA

e) NZS cells per variable ZHA

f) (MAXNP1, MAXNP1) cells per variable A

NOTE: 1) MAXN $\geq$ Number of modal data points.
2) MAXNP1 $\geq 4 \cdot$ NPOLI $\cdot$ NPOLK + 1

3.1.14 Code Listing

(following)

```
@ASG,T 15,T,RJ3
@HDG,P       POLREG USES UNIVAC MULTIPLE REGRESSION TO FIT A 2-D POLY FOR COTTI
@FOR,IS POLREG
      COMMON X(200,86),W(200),XBAR(86),A(86,86),SIG(100),IVAR(66),B(86),
     1 SB(86),R(200) ,THA(200),ZHA(200)
      REWIND 15
      MAXN=200
      MAXNP1=86
    1 READ (5,30) ITEST,LIMIT,NZS,NTH,NPOLI,NPOLK,ISKIP
      WRITE(6,35) ITEST,LIMIT,NZS,NTH,NPOLI,NPOLK,ISKIP
      NP1=4*(NPOLI*NPOLK)+1
      NPL1=NP1-1
      IF(ISKIP.EQ.0) GO TO 210
      DO 211 I=1,ISKIP
  211 READ (15) EI
  210 READ (15) EI,MODE,NPTS,(X(J,NP1),DUM1,DUM2,DUM3,DUM4,DUM5,J=1,
     1NPTS)
      READ (5,31) EFIN,EFOUT,THNOT,DTH,      DELZ,ZNOT
      WRITE(6,36) EFIN,EFOUT,THNOT,DTH,      DELZ,ZNOT
      IW=0
      THNOT=THNOT*.01745
      DTH=DTH*.01745
      PI=3.14159265
      DELZ=DELZ/ZNOT*PI
      ZNOT=PI
      TH=THNOT
      DO 2 I=1,NTH
      THA(I)=TH
    2 TH=TH+DTH
      ZZ=ZNOT
      DO 3 I=1,NZS
      ZHA(I)=ZZ
    3 ZZ=ZZ-DELZ
      J=0
      DO 200 I=1,NTH
      TH=THA(I)
      DO 200 L=1,NZS
      ZZ=ZHA(L)
      MO=0
      J=J+1
      NO=NPOLI*NPOLK
      NOP=2*NO
      NOR=3*NO
      DO 200 M=1,NPOLI
      TS=FLOAT(M-1)*TH
       TT=COS(TS)
      TL=SIN(TS)
      DO 200 K=1,NPOLK
      ARG=FLOAT(K-1)*ZZ
      TP=COS(ARG)
      TR=SIN(ARG)
      NOP=NOP+1
      NOR=NOR+1
```

```
      MO=MO+1
      NO=NO+1
      X(J,MO)=TT*TP
      X(J,NO)=TT*TR
      X(J,NOP)=TL*TP
  200 X(J,NOR)=TL*TR
      IF(IW) 5,5,6
    6 READ (5,31) (W(I),I=1,NPTS)
      WRITE(6,34) (W(I),I=1,NPTS)
    5 IND=0
      ISTEP=-1
    7 CALL RESTEM(X,NPTS,NP1,MAXN,MAXNP1,W,IW,EFIN,EFOUT,XBAR,A,SIG,CONS
     1T,NVAR,FLEVEL,SY,NOIN,IVAR,B,SB,R,IND)
      IF(IND)13,12,13
   13 ISTEP=ISTEP+1
   14 WRITE(6,39) ISTEP
      IF(NVAR)18,19,19
   18 NVAR=-NVAR
      WRITE(6,130) NVAR
      GO TO 10
   19 WRITE(6,131) NVAR
   10 WRITE(6,132) FLEVEL,SY,CONST,(IVAR(I),B(I),SB(I),I=1,NOIN)
      IF(ISTEP-LIMIT)7,24,24
   24 IND=-1
      GO TO 7
   12 WRITE(6,133)
      IST=1
      MCO=NOIN
      IF(IVAR(1)-1) 1210,1210,1211
 1210 CONST=B(1)+CONST
      IST=2
      MCO=MCO-1
 1211 PUNCH 30,NPOLI,NPOLK,MCO,(IVAR(I),I=IST,NOIN)
      PUNCH 136,CONST,(B(I),I=IST,NOIN)
      DO 29 I=1,NPTS
      DEV=X(I,NP1)-R(I)
   29 WRITE(6,134) I,X(I,NP1),R(I),DEV
      GO TO 1
   30 FORMAT(16I5)
   31 FORMAT(8E10.3)
   32 FORMAT(10H0 RAW DATA)
   33 FORMAT(1X,10E12.5)
   34 FORMAT(19H0 WEIGHTING FACTORS/(1X,10E12.5))
   35 FORMAT(9H1ITEST = I5,11H,  LIMIT = I5,9H,  NZS = I5, 9H,  NTH = I5
     1,11H,  NPOLI = I5,11H,  NPOLK = I5,11H,  ISKIP = I5)
   36 FORMAT(8H0EFIN = E13.6,11H,  EFOUT = E13.6,11H,  THNOT = E13.6,
     19H,  DTH = E13.6/8H0DELZ = E13.6,10H,  ZNOT = E13.6)
   39 FORMAT(11H0 STEP NO. I3)
  130 FORMAT(5X,17H VARIABLE REMOVED I3)
  131 FORMAT(5X,19H VARIABLE ENTERING I3)
  132 FORMAT(5X,7H FLEVEL,E13.6/5X,20H STANDARD ERROR OF Y, E13.6/5X,
     19HCONSTANT ,E13.6//15X,46H VARIABLE      COEFFICIENT   STD ERROR OF
     1 COEFF,/(17X,3H X I2,E16.5,E18.5))
  133 FORMAT(30H1 PREDICTED VS  ACTUAL RESULTS/8H OB. NO.,8X,7H ACTUAL,
```

```
     110X,10H PREDICTED, 9X,10H DEVIATION)
  134 FORMAT(I5,E18.6,E18.6,E19.6)
  136 FORMAT(5E15.8)
      END
@FOR,IS RESTEM
      SUBROUTINE RESTEM (X,N,NP1,MAXN,MAXNP1,W,IW,EFIN,EFOUT,XBAR,A,SIG,RESTEM
     1 CONST,NVAR,FLEVEL,SY,NOIN,IVAR,B,SB,R,IND)                       RESTEM
C-----------------------------------------------------------------------
C     MULTIPLE REGRESSION PROGRAM                                       RESTEM
C-----------------------------------------------------------------------
      DIMENSION X(MAXN,1),W(1),XBAR(1),A(MAXNP1,1),SIG(1),IVAR(1),B(1), RESTEM
     1 SB(1),R(1)                                                       RESTEM
C-----------------------------------------------------------------------
C     IND=0 UPON FIRST CALL TO SUBRT.                                   RESTEM
C-----------------------------------------------------------------------
      IF (IND) 175,100,160                                              RESTEM
  100 IND=1                                                             RESTEM
C-----------------------------------------------------------------------
C     TEST IF WEIGHTS ARE INPUT. IF NOT, SET W(J)=1.0...WEIGHT SUM=N    RESTEM
C     IF INPUT, NORMALIZE WEIGHTS SO THAT AVERAGE OF WEIGHTS IS 1.0     RESTEM
C-----------------------------------------------------------------------
      IF (IW) 101,101,102                                               RESTEM
C-----------------------------------------------------------------------
C     NOT INPUT                                                         RESTEM
C-----------------------------------------------------------------------
  101 DO 103 I=1,N                                                      RESTEM
  103 W(I)=1.0                                                          RESTEM
      GO TO 104                                                         RESTEM
C     INPUT                                                             RESTEM
  102 TEMP=0.0                                                          RESTEM
      DO 105 I=1,N                                                      RESTEM
  105 TEMP=TEMP+W(I)                                                    RESTEM
      TEMP=TEMP/N                                                       RESTEM
      DO 106 I=1,N                                                      RESTEM
  106 W(I)=W(I)/TEMP                                                    RESTEM
C-----------------------------------------------------------------------
C     COMPUTE MEAN OF EACH VARIABLE=XBAR                                RESTEM
C-----------------------------------------------------------------------
  104 DO 114 J=1,NP1                                                    RESTEM
      XBAR(J)=0.0                                                       RESTEM
      DO 115 I=1,N                                                      RESTEM
  115 XBAR(J)=XBAR(J)+W(I)*X(I,J)                                       RESTEM
  114 XBAR(J)=XBAR(J)/N                                                 RESTEM
C-----------------------------------------------------------------------
C     COMPUTE WEIGHTED RESIDUAL SUMS OF SQUARES AND CROSS PRODUCTS      RESTEM
C-----------------------------------------------------------------------
      DO 117 I=1,NP1                                                    RESTEM
      DO 117 J=I,NP1                                                    RESTEM
      A(I,J)=0.0                                                        RESTEM
      DO 116 K=1,N                                                      RESTEM
  116 A(I,J)=A(I,J)+W(K)*(X(K,I)-XBAR(I))*(X(K,J)-XBAR(J))              RESTEM
  117 CONTINUE                                                          RESTST
C-----------------------------------------------------------------------
C     COMPUTE STANDARD DEVIATION.  SET DIAGONALS OF CORRELATION MTRX=1. RESTEM
```

```
C     NORMALIZE, THEN EXPAND UPPER TRIANGULAR MATRIX TO FULL                RESTEM
C-----------------------------------------------------------------------
      DO 120 I=1,NP1                                                        RESTEM
      SIG(I)=SQRT(A(I,I))                                                   RESTEM
  120 A(I,I)=1.0                                                            RESTEM
      NP=NP1-1                                                              RESTEM
      DO 121 I=1,NP                                                         RESTEM
      II=I+1                                                                RESTEM
      DO 121 J=II,NP1                                                       RESTEM
      A(I,J)=A(I,J)/(SIG(I)*SIG(J))                                         RESTEM
  121 A(J,I)=A(I,J)                                                         RESTEM
C-----------------------------------------------------------------------
C     COMPUTE DEGREES OF FREEDOM                                            RESTEM
C-----------------------------------------------------------------------
      PHI=N-1.0                                                             RESTEM
C-----------------------------------------------------------------------
C     INITIALIZATION PROCEDURE FOR DETERMINING MOST SIGNIFICANT VARIABLERESTEM
C     TO BE ADDED  TO REGRESSION                                            RESTEM
C     COMPUTE STANDARD ERROR OF DEPENDENT VARIABLE                          RESTEM
C-----------------------------------------------------------------------
  125 SY=SIG(NP1)*SQRT(A(NP1,NP1)/PHI)                                      RESTEM
C-----------------------------------------------------------------------
C     ZERO COEFFICIENTS ARRAY                                               RESTEM
C-----------------------------------------------------------------------
      DO 131 I=1,NP                                                         RESTEM
  131 B(I)=0.0                                                              RESTEM
      VMIN = -10.E34                                                        RESTEM
      VMAX=0.0                                                              RESTEM
      NMIN=0                                                                RESTEM
      NMAX=0                                                                RESTEM
      NOIN=0                                                                RESTEM
      DO 150 I=1,NP                                                         RESTEM
C     GO TO 142
      IF (A(I,I)-.00001) 150,150,141                                        RESTEM
C-----------------------------------------------------------------------
C     COMPUTE VARIANCE                                                      RESTEM
C-----------------------------------------------------------------------
  141 V=A(I,NP1)*A(NP1,I)/A(I,I)                                            RESTEM
      IF (V) 142,150,143                                                    RESTEM
  143 IF (V-VMAX) 150,150,144
  144 VMAX=V
      NMAX=I                                                                RESTEM
      GO TO 150                                                             RESTEM
C-----------------------------------------------------------------------
C     X(I) IS IN REGRESSION...COMPUTE COEFFICIENT B AND STAND. ERROR OF RESTEM
C     COEFF.                                                                RESTEM
C-----------------------------------------------------------------------
  142 NOIN=NOIN+1                                                           RESTEM
      IVAR(NOIN)=I                                                          RESTEM
      B(NOIN)=A(I,NP1)*SIG(NP1)/SIG(I)                                      RESTEM
      SB(NOIN)=SY*SQRT(A(I,I))/SIG(I)                                       RESTEM
      IF (V-VMIN) 150,150,145
  145 VMIN=V
      NMIN=I                                                                RESTEM
```

```
   150 CONTINUE                                                         RESTEM
C-----------------------------------------------------------------------
C      COMPUTE CONSTANT                                                 RESTEM
C-----------------------------------------------------------------------
       TEMP=0.0                                                         RESTEM
       DO 151 I=1,NOIN                                                  RESTEM
       II=IVAR(I)                                                       RESTEM
   151 TEMP=TEMP+B(I )*XBAR(II)                                         RESTEM
       CONST=XBAR(NP1)-TEMP
       RETURN
C-----------------------------------------------------------------------
C      COMPUTE FLEVEL                                                   RESTEM
C-----------------------------------------------------------------------
   160 FLEVEL=VMIN*PHI/A(NP1,NP1)                                       RESTEM
C-----------------------------------------------------------------------
C      COMPARE F LEVELS...                                              RESTEM
C-----------------------------------------------------------------------
       IF (EFOUT+FLEVEL) 153,153,152
   152 K=NMIN
       PHI=PHI-1.0                                                      RESTEM
       NVAR=-K                                                          RESTEM
       GO TO 200                                                        RESTEM
   153 FLEVEL=VMAX*(PHI-1.)/(A(NP1,NP1)-VMAX)                           RESTEM
       IF (EFIN-FLEVEL) 154,175,175                                     RESTEM
   154 K=NMAX                                                           RESTEM
       PHI=PHI-1.0                                                      RESTEM
       NVAR=K                                                           RESTEM
C-----------------------------------------------------------------------
C      CALCULATE NEW MATRIX
C-----------------------------------------------------------------------
   200 DO 210 I=1,NP1
       IF (I-K) 230,210,230
   230 DO 240 J=1,NP1                                                   RESTEM
       IF (J-K) 260,240,260                                             RESTEM
   260 A(I,J)=A(I,J) - A(I,K)*A(K,J)/A(K,K)                             RESTEM
   240 CONTINUE                                                         RESTEM
   210 CONTINUE                                                         RESTEM
       DO 280 I=1,NP1                                                   RESTEM
       IF (I-K) 300,280,300                                             RESTEM
   300 A(I,K)=-A(I,K)/A(K,K)                                            RESTEM
   280 CONTINUE                                                         RESTEM
       DO 320 J=1,NP1                                                   RESTEM
       IF (J-K) 340,320,340                                             RESTEM
   340 A(K,J)=A(K,J)/A(K,K)                                             RESTEM
   320 CONTINUE                                                         RESTEM
   350 A(K,K)=1.0/A(K,K)                                                RESTEM
       GO TO 125                                                        RESTEM
   175 DO 178 J=1,N                                                     RESTEM
       TEMP=0.0                                                         RESTEM
       DO 177 I=1,NOIN                                                  RESTEM
       II=IVAR(I)                                                       RESTEM
   177 TEMP=TEMP + B(I)*X(J,II)                                         RESTEM
   178 R(J)=CONST+TEMP                                                  RESTEM
C-----------------------------------------------------------------------
```

```
C     SET INDICT. TO INDICATE END OF COMPUTATION HAS BEEN REACHED        RESTEM
C-----------------------------------------------------------------------
      IND=0                                                              RESTEM
      RETURN                                                             RESTEM
      END                                                                RESTEM
@MAP,IS ,ABS
  IN POLREG
@XQT  ABS
```

3.1.15 Suitable Data to Tape for FILMSC

(following)

```
@ASG,T 15,T,PJ5
@FOR,IS CDTP
C   TAKES GODDARD CARD DATA AND PUTS IT ON TAPE FOR COTTIS
        DIMENSION DUM(10),T1(400),T2(400),T3(400),R1(400),R2(400),R3(400)
      1 READ (5,30) NPTS,NMODE
     30 FORMAT(16I5)
        DO 10 L=1,NMODE
        READ (5,31) (DUM(I),I=1,9)
     31 FORMAT(A1)
        READ (5,32) EI,MODE
     32 FORMAT(12X,E15.4,9X,I5)
        DO 9 J=1,NPTS
      9 READ (5,33) T1(J),T2(J),T3(J),R1(J),R2(J),R3(J)
     33 FORMAT(18X,3E18.5)
     10 WRITE(15) EI,MODE,NPTS,(T1(J),T2(J),T3(J),R1(J),R2(J),R3(J),J=1,
       1NPTS)
        END FILE 15
        GO TO 1
        END
@MAP,IS ,ABS
  IN CDTP
@XQT ABS
   140    20
```

3.2 Response to Deterministic Excitation (DEXCYL)

3.2.1 Program Description

The DEXCYL program computes the dynamic response of a cylinder or beam for a class of deterministic pressure fields. The structural mode shapes in the form of combinations of trigonometric functions with arbitrary constant coefficients are assumed. These coefficients are obtained from the computer code "FITMSC."

3.2.2 Input

Section 3.1.9 presents the necessary input to be read into the code.

3.2.3 Governing Equations

The governing equations are given in Section 2.4.1.

3.2.4 Method of Solution

The computational procedure is described in Section 2.0.

3.2.5 Restart

The code computes the response at prescribed positions and frequencies. Should a restart be necessary, it may be done without having to recalculate any of the previous data points. Each point is computed independently of previous points.

3.2.6 Termination

The code will terminate when the input data has been exhausted.

3.2.7 Computer Conversion

The program was written in FORTRAN V. It has been used extensively on the UNIVAC 1108 (EXEC 8). The only obvious changes necessary to adapt the code to CDC 6600 or IBM 7090 are the control cards. All control cards at the beginning of each subroutine must be eliminated and/or replaced with appropriate control cards.

3.2.8 Equipment Requirements

The code uses no tape units other than 5 and 6 for input and output, respectively. Approximately 13,000 cells are needed to execute in the present form.

3.2.9 Program Input Requirements

NOTE: Certain real variable arrays were equivalenced to their complex counterparts to avoid the problem of format variation from machine to machine for complex variables.

3.2.10 Card Formats

NOTE: All integer variables must be right adjusted. Floating point fields must contain a decimal which may be arbitrarily located in the field width.

CARD GROUP 1

Columns	Description
1-5	NTHA, the response is computed for a number of angular positions, θ, and a number of positions on the cylinder Z, NTHA is the number of θ's. (Integer)
6-10	NOFZ, number of Z's for which the response is calculated. (Integer)
11-15	NFREQ, number of frequencies for which the response is computed. (Integer)
16-20	NMODES, number of mode shapes for which fit data is available. (Integer)
21-25	NBETA, number of β terms in $p_1(Z)$. (Integer)
26-30	NBETAP, number of β' terms in $p_2(\theta)$. (Integer)
31-35	NALP, number of α terms in $p_1(Z)$. (Integer)
36-40	NALPP, number of α' terms in $p_2(\theta)$. (Integer)
41-45	NJS, number of B_j's in $p_1(Z)$. (Integer)
46-50	NJPS, number of E_j''s in $p_2(\theta)$. (Integer)
51-55	NSS, number of K_s's in $\hat{p}_3(\omega)$. (Integer)
56-60	NETA, number of ν_η's in $\hat{p}_3(\omega)$. (Integer)
61-65	NSIG, number of σ's in $\hat{p}_3(\omega)$. (Integer)

CARD GROUP 2

Columns	Description
1-10	ZNOT, length of the cylinder in inches. (Floating point)
11-20	RCYL, radius of the cylinder. (Floating point)
21-30	Z1 } Z1-Z2 is the extent of the pressure field.
31-40	Z2 } (Floating point)
41-50, 51-60 61-70, 71-80	Z, positions along the cylinder for which the response will be computed. (Floating point, NOFZ numbers in all)

CARD GROUP 3

Columns	Description
1-10	THNOT, θ, extent of the pressure field in the azimuthal direction. (Floating point)
11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	TTH, θ, angular values for which the response is calculated. (Floating point, NTHA numbers in all).

NOTE: In the following three groups it is assumed the number of terms for each variable is one. If there are more terms the columns will be shifted appropriately.

CARD GROUP 4

Columns	Description
1-10, 11-20	AA, A_0, constant pressure field term in $p_1(Z)$, real part followed by imaginary part. (Floating point)
21-30, 31-40	BB, B_j, real component followed by the imaginary component of B_j. (Floating point) - there will be NJS successive paris of B_j. If NJS = 0 two zero fields must be present.
41-50, 51-60	CC, $C_{\alpha, \beta}$, real and imaginary components of $C_{\alpha, \beta}$. There will be NALP pairs of numbers for $\beta = 1$, followed by NALP pairs of numbers for $\beta = 2$, etc. If NALP = 0 there must be two fields containing zero. (Floating point)
61-70, 71-80	ZKALP, K_α, real and imaginary components of K_α. There must be NALP pairs of numbers. At least two fields (which may be zero) must be present. (Floating point)
New card, 1-10, 11-20	ZJP, Z_j', real and imaginary components of Z_j'. There must be at least two fields and up to NJPS pairs of numbers in all. (Floating point)

CARD GROUP 5

Columns	Description
1-10, 11-20	DD, D_0, constant pressure field term in $p_2(\theta)$, real part followed by the imaginary part. (Floating point)
21-30, 31-40	EE, E_j', real component followed by the imaginary component of E_j'. (Floating point - there will be NJBS successive pairs of β_j. If NJBS = 0 two zero fields must be present)
41-50, 51-60	FF, $F_{\alpha', \beta'}$, real and imaginary components of $F_{\alpha', \beta'}$. There will be NALPP pairs of numbers for $\beta' = 1$, followed by NALPP pairs of numbers for $\beta' = 2$ etc. If NALPP = 0 two fields containing zero must be present. (Floating point)
61-70, 71-80	ZKALPP, $K_{\alpha'}$, real and imaginary components of $K_{\alpha'}$. There must be NALPP pairs of numbers. At least two fields (which may be zero) must be present. (Floating point)
New card, 1-10 11-20	THP, θ_j', real and imaginary components of θ_j'. There must be at least two fields and up to NJPS pairs of numbers in all. (Floating point)

CARD GROUP 6

Columns	Description
1-10	HCAP, H_0, constant pressure field term in $\hat{p}_3(\omega)$. (Floating point)
11-20, 21-30	ZKS, K_s, real component followed by the imaginary component of K_s. (Floating point - there will be NSS successive pairs of K_s. If NSS = 0 two zero fields must be present.)
31-40	TS, t_s appearing in $\hat{p}_3(\omega)$. There must be at least one field (which may be zero) and up to NSS numbers. (Floating point)
41-50, 51-60	ZNN, $N_{\eta,\sigma}$, real and imaginary components of $N_{\eta,\sigma}$. There will be NSIG pairs of numbers for $\eta = 1$ followed by NSIG pairs of numbers for $\eta = 2$ etc. until the list of η is complete. If NSIG = 0 two fields containing zero must be present. (Floating point)
61-70, 71-80	ZNU, ν_η, real and imaginary components of ν_η. There must be at least two fields and up to NETA pairs of numbers in all. (Floating point)

CARD GROUP 7

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	AMN1, α_{nm}, damping constants. (Floating point - NMODES numbers in all)

CARD GROUP 8

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	WMN, ω_{nm}, modal frequency in cycles per second. (Floating point - eight numbers per card, NMODES in all.)

CARD GROUP 9

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	OM, ω frequencies for which the responses are to be calculated. (Floating point - eight numbers per card NFREQ in all.)

CARD GROUP 10

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	ZMAS, M_{nm}, modal mass. (Floating point - eight numbers per card NMODES numbers in all.)

CARD GROUP 11

Columns	Description
1-5	NTH, number of θ terms in the fitting of the modal data. (Integer)
6-10	NKZ, number of Z terms in the fitting of the modal data. (Integer)
11-15	NCOE, number of non-zero terms in the fit, excluding the constant term.
16-20, 21-25, 26-30, 31-35, 36-40, 41-45, 46-50, 51-55, 56-60, 61-65, 66-70, 71-75, 76-80	MCO, index tagging the fit coefficients with the appropriate trigonometric term. (Integer - NCOE numbers in all)

CARD GROUP 12

Columns	Description
1-15	CON, constant term in the fits to the mode shapes. (Floating point)
15-30, 31-45, 46-60, 61-75	COE, fit coefficients to the mode shapes. (Floating point - NCOE numbers in all).

Card groups 11 and 12 are repeated once for each mode number. These two card groups are the direct output of the FITMSC computer code.

NOTE: To run the deterministic beam problem set:

1) $R = RCYL = 1$

2) $E_{j'} = EE(1) + iEE(2) = 1+0i$

3) $NJPS = 1$

4) $F_{\alpha', \beta'} = FF(1) + iFF(2) = 0$

5) $D_0 = DD(1) + iDD(2) = 0$

6) NTH = 1 for all modes

3.2.11 Sample Input

The following is a listing of the input data used to obtain the sample output. It was the model 1 data (Section 4.0) as computed from FITMSC.

SAMPLE INPUT

```
   19     1     1    12     1     1     1     1     0     0     0     0     0
 132.            0.         132.        66.
 3.14159265  50.         0.          20.         40.         60.         80.         100.
 120.        140.        160.        180.        200.        220.        240.        260.
 280.        300.        320.        340.        360.
 0.          0.          0.          0.          1.          0.          -.1         .025
 0.          0.
 0.          0.          0.          0.          1.          0.          0.          .05
 0.          0.
 .016        0.          0.          0.          0.          0.          0.          0.
 .02         .02         .02         .02         .02         .02         .02         .02
 .02         .02         .02         .02
 149.8409    149.8413    154.4314    154.4314    170.1672    187.4919    187.4922    258.4653
 258.4653    259.1931    280.3604    280.3606
 149.8409
 .5878779    .6263621    .5698155    .5654379    1.089890    .7388390    .6730367    .7642241
 .7751001    1.479049    1.021387    1.006094
    4     4     4    30    32    62    64
    .64891715-05  -.75502460+00   .25676082-01  -.61556394+00   .20933483-01
    4     4     4    30    32    62    64
    .11708251-03   .63559545+00  -.21614622-01  -.77966249+00   .26513895-01
    6     2     3     9    22    46
    .18283015-04  -.19143746-01   .98810329+00   .16254598+00
    6     2     3    22    33    46
    .12709915-03   .16226757+00   .19080996-01  -.98501851+00
    6     2     2    11    24
   -.54830642-06  -.20595947-01   .93730116+00
    4     4     4    26    28    58    60
    .16967466-05   .51843181+00  -.95938868-02   .93018614+00  -.12878921-01
    4     4     5    22    26    28    58    60
   -.10586593-03  -.49546157-03  -.79255699+00   .12295176-01   .59039415+00
   -.91589419-02
    6     3     3    14    33    59
    .15926782-09   .88414990-02  -.10017767+01   .53869285+00
    6     3     3    33    50    69
   -.14583651-08   .54168539+00  -.89123033-02   .10092856+01
    7     3     2    17    39
   -.19943995-09  -.10770447-01   .11265665+01
    4     4    11     7    18    22    24    30    37    39    54    56    58    62
   -.22611810-06   .71489229-04  -.14072712-03  -.48495815+00  -.20647036-02
   -.76995498-04   .22655226-01  -.22655292-01   .86073080+00   .11291609-01
   -.20228178-03  -.90219221-04
    4     4    11     5     7    18    22    24    26    30    39    54    56    58
    .22150605-06   .22537380-01  -.22537106-01   .13347271-03   .85426307+00
    .11224375-01  -.17934819-03  -.11332811-03  -.70947396-04   .48112981+00
 .20484      -02  -.11030      -03
```

3.2.12 Description of Output

The input is printed immediately after it is read. No labels are printed. The order is the same as listed for the input cards.

Variable	Line 1
FREQUENCY	ω, frequency of the calculation.
THETA	θ — Coordinates of the point at which the dynamic
Z	Z — response is being computed.
RESPONSE	W, real and imaginary components of the dynamic response at θ, Z, and ω.

Variable	Line 2
MAGNITUDE	$\|W\|$, magnitude of the dynamic response at θ, Z, and ω.
PHASE	φ, phase of the dynamic response at θ, Z, and ω.

3.2.13 Sample Output

Following is a partial listing of a case run earlier. This is not the case run with the sample input. It corresponds to the sample case with $a_{mn} = .02/\omega_{mn}$.

```
***INPUT DATA***
 19     1     1    12     1     1     1     1     0     0     0     0     0
   .13200+03     .00000       .13200+03    .66000+02
   .31416+01     .50000+02    .00000       .20000+02    .40000+02    .60000+02
   .16000+03     .18000+03    .20000+03    .22000+03    .24000+03    .26000+03
   .36000+03
   .00000        .00000       .00000       .00000       .10000+01    .00000
   .00000        .00000       .00000       .00000       .10000+01    .00000
   .16000-01     .00000       .00000       .00000       .00000       .00000
   .20000-01     .20000-01    .20000-01    .20000-01    .20000-01    .20000-01
   .20000-01     .20000-01
   .14984+03     .14984+03    .15443+03    .15443+03    .17017+03    .18749+03
   .28036+03     .28036+03
   .14984+03
   .58788+00     .62636+00    .56982+00    .56544+00    .10899+01    .73884+00
   .10214+01     .10061+01
    4     4     4    30    32    62    64
  .64891715-05  -.75502460+00   .25676082-01  -.61556394+00   .20933483-01
    4     4     4    30    32    62    64
  .11708251-03   .63559545+00  -.21614622-01  -.77966249+00   .26513896-01
    6     2     3     9    22    46
  .18283015-04  -.19143746-01   .98810329+00   .16254598+00
    6     2     3    22    33    46
  .12709915-03   .16226757+00   .19080996-01  -.98501851+00
    6     2     2    11    24
 -.54830642-06  -.20595947-01   .98730116+00
    4     4     4    26    28    58    60
  .16967466-05   .61843181+00  -.95938868-02   .83018614+00  -.12878921-01
    4     4     5    22    26    28    58    60
 -.10586593-03  -.49546157-03  -.79255699+00   .12295176-01   .59039415+00
 -.91589418-02
    6     3     3    14    33    69
  .15926782-09   .88414990-02  -.10017767+01   .53869285+00
    6     3     3    33    50    69
 -.14583651-08   .54168539+00  -.89123033-02   .10092856+01
    7     3     2    17    39
 -.17943995-09  -.10770447-01   .11265665+01
    4     4    11     7    18    22    24    30    37    39    54    56    58    62
 -.22611810-06   .71489229-04  -.14072712-03  -.48495815+00  -.20647036-02
 -.76995498-04   .22655226-01  -.22655292-01   .86073080+00   .11291609-01
 -.20228178-03  -.90219221-04
    4     4    11     5     7    18    22    24    26    30    39    54    56    58
  .22150605-06   .22537380-01  -.22537106-01   .13347271-03   .85426307+00
  .11224375-01  -.17934819-03  -.11332811-03  -.70947395-04   .48112981+00
  .20484000-02  -.11030000-03

***END OF INPUT DATA***


FREQUENCY =    .149841+03,   THETA =    .000000     ,   Z =    .660000+02,   RE
                                                                             MA

FREQUENCY =    .149841+03,   THETA =    .199962+02,   Z =    .660000+02,   RE
```

```
                                                                              MAGNITUDE =     .509408-01,  PHASE =    .:
ICY =   .149841+03,  THETA =   .399925+02,  Z =   .660000+02,  RESPONSE =    .251220-01   -.419690-01
                                                                              MAGNITUDE =     .489133-01,  PHASE =    .:
!CY =   .149841+03,  THETA =   .599887+02,  Z =   .660000+02,  RESPONSE =    .209086-02    .345552-02
                                                                              MAGNITUDE =     .403885-02,  PHASE =    .!
!CY =   .149841+03,  THETA =   .799849+02,  Z =   .660000+02,  RESPONSE =   -.230303-01    .454314-01
                                                                              MAGNITUDE =     .509353-01,  PHASE =    .;
!CY =   .149841+03,  THETA =   .999811+02,  Z =   .660000+02,  RESPONSE =   -.251293-01    .419970-01
                                                                              MAGNITUDE =     .489411-01,  PHASE =    .1
!CY =   .149841+03,  THETA =   .119977+03,  Z =   .660000+02,  RESPONSE =   -.211031-02   -.341499-02
                                                                              MAGNITUDE =     .401442-02,  PHASE =    .2
!CY =   .149841+03,  THETA =   .139974+03,  Z =   .660000+02,  RESPONSE =    .230148-01   -.454083-01
                                                                              MAGNITUDE =     .509077-01,  PHASE =    .2
!CY =   .149841+03,  THETA =   .159970+03,  Z =   .660000+02,  RESPONSE =    .251314-01   -.420036-01
                                                                              MAGNITUDE =     .489478-01,  PHASE =    .3
NCY =   .149841+03,  THETA =   .179966+03,  Z =   .660000+02,  RESPONSE =    .212444-02    .339550-02
                                                                              MAGNITUDE =     .400533-02,  PHASE =    .5
NCY =   .149841+03,  THETA =   .199962+03,  Z =   .660000+02,  RESPONSE =   -.230089-01    .454046-01
                                                                              MAGNITUDE =     .509018-01,  PHASE =    .1
NCY =   .149841+03,  THETA =   .219958+03,  Z =   .660000+02,  RESPONSE =   -.251461-01    .420285-01
                                                                              MAGNITUDE =     .489767-01,  PHASE =    .1
NCY =   .149841+03,  THETA =   .239955+03,  Z =   .660000+02,  RESPONSE =   -.214973-02   -.335797-02
                                                                              MAGNITUDE =     .398714-02,  PHASE =    .2
NCY =   .149841+03,  THETA =   .259951+03,  Z =   .660000+02,  RESPONSE =    .229936-01   -.453835-01
                                                                              MAGNITUDE =     .508760-01,  PHASE =    .2
NCY =   .149841+03,  THETA =   .279947+03,  Z =   .660000+02,  RESPONSE =    .251486-01   -.420374-01
                                                                              MAGNITUDE =     .489857-01,  PHASE =    .3
NCY =   .149841+03,  THETA =   .299943+03,  Z =   .660000+02,  RESPONSE =    .215853-02    .333465-02
                                                                              MAGNITUDE =     .397229-02,  PHASE =    .57
```

3.2.14 Code Expansion

Following is a list of minimum storage requirements.

	AA	(2)
	AMN1	(2 • NMODES)
	AMN2	(2 • NMODES)
	ASM	(NTH, NKZ)
	ASMST	(NTH, NKZ)
	BB	(2 • NJS)
	BCAP	(NJS)
	BSM	(NTH, NKZ)
	BSMST	(NTH, NKZ)
	CC	(2 • NALP, NBETA)
	CCAP	(NALP, NBETA)
*	CCAP	(NALP, NBETA) or CCAP (NALPP, NBETAP) whichever is larger.
	COE	(NCOE, NMODES)
*	COE	(NALP, NBETA) or COE (NALPP, NBETAP) whichever is larger.
*	COEZ	(NALP) or COEZ (NALPP) whichever is larger.
*	COKP	(NALP, NBETA) or COKP (NALPP, NBETAP) whichever is larger.
	CON	(NMODES)
	DD	(2)
	ECAP	(NJPS)
	EE	(2 • NJPS)
*	FACT	(NBETA) or FACT (NBETAP) whichever is larger.
	FCAP	(NALPP, NBETAP)
	FF	(2 • NALPP, NBETAP)
	GHAT	(NFREQ, NMODES)
	MCO	(NCOE, NMODES)
	NCOEE	(NMODES)
	NKZ	(NMODES)
	NTH	(NMODES)
	OM	(NFREQ)
	OMEG1	(NMODES)
	OMEG2	(NMODES)
	PHI	(NMODES)
	SRE	(NMODES)
	THP	(NJPS)
	TI1	(NTH)
	TI2	(NTH)
	TS	(NSS)
	TTH	(NTHA)
	WMN	(NMODES)

	Z	(NOFZ)
*	Z11	(NBETA) or FACT (NBETAP) whichever is larger.
*	Z22	(NBETA) or FACT (NBETAP) whichever is larger.
	ZI1	(NKZ)
	ZI2	(NKZ)
	ZJP	(NJS)
	ZKA	(NALP)
	ZKALP	(2 • NALP)
	ZKALPP	(2 • NALPP)
	ZKAP	(NALPP)
	ZKKS	(NSS)
	ZKP	(NKZ)
	ZKQ	(NTH)
	ZKS	(2 • NSS)
	ZMAS	(NMODES)
	ZNCAP	(NSIG, NETA)
	ZNN	(2 • NSIG, NETA)
	ZNU	(2 • NETA)
	ZNUCAP	(NETA)

NOTE: The following variables must be dimensioned the same to avoid errors in subroutine FINDI: (CCAP and FCAP), (FF and CC). These arrays must be based on the largest values of (NALP, NALPP) and (NBETA, NBETAP). The present form of the code allows up to ten values for NALP, NALPP, NBETA, and NBETAP.

Variables tagged by an (*) are located in subroutine FINDI.

3.2.15 Code Listing

(following)

```
@FOR,IS DEXCYL
      COMPLEX ACAP,BCAP,CCAP,DCAP,ECAP,FCAP, RESPS,       ZKA, ZKAP,
     1 ZNCAP, ZNUCAP, OMEG1, OMEG2, PHAT, GHAT, CN, CM
     2,ASM, BSM, ASMST, BSMST, ZI1, ZI2, TI1, TI2, SMB, SM, CB1, CB2
     3,ZKKS,SRF
      DIMENSION AA(2), BCAP(20), BB(40), CCAP(10,10), CC(20,10), DD(2),
     1 ECAP(20), EE(40), FCAP(10,10), FF(20,10),            ZKA(20),
     2 ZKALP(40), ZKAP(20), ZKALPP(40),          TS(40), ZNCAP(10,10),
     3 ZNN(20,10), ZNUCAP(20), ZNU(40), OMEG1(50), AMN1(100), OMEG2(50),
     4 AMN2(100), ZKKS(20), ZKS(40), Z(100), TTH(100), WMN(50), OM(50)
     5,ZMAS(50),ASM(10,10),BSM(10,10),MCO(15,20),NTH(50),NKZ(50),CON(50)
     6, COF(15,30), NCOFE(30), ZKP(100), ZKQ(100), BSMST(10,10)
     7, ASMST(10,10), ZI1(50), ZI2(50), TI1(50), TI2(50), SRF(50),
     8 PHI(50),GHAT(25,30),ZJP(20),THP(30)
      EQUIVALENCE (ACAP,AA), (BCAP,BB), (CCAP,CC), (DCAP,DD), (ECAP,EE),
     1 (FCAP,FF), (ZKKS,ZKS),(ZKA,ZKALP), (ZKAP,ZKALPP),
     2 (ZNCAP,ZNN), (ZNUCAP,ZNU), (OMEG1,AMN1), (OMEG2,AMN2)
      THCON=57.29578
      CM=(0.,1.)
      PI2=6.2831853O
      PI=3.14159265
    1 READ (5,31) NTHA,NOFZ,NFREQ,NMODES,NBETA,NBETAP,NALP,NALPP,NJS,
     1NJPS,NSS,NETA,NSIG
      WRITE(6,38)
      WRITE(6,31) NTHA,NOFZ,NFREQ,NMODES,NBETA,NBETAP,NALP,NALPP,NJS,
     1NJPS,NSS,NETA,NSIG
      LJ=NJS*2
      LA=NALP*2
      LB=NBETA
      IF(LJ.LT.2) LJ=2
      IF(LA.LT.2) LA=2
      IF(LB.LT.1) LB=1
      READ (5,32) ZNOT,Z1,Z2,(Z(I),I=1,NOFZ)
      WRITE(6,34) ZNOT,Z1,Z2,(Z(I),I=1,NOFZ)
      READ (5,32) THNOT,RCYL,(TTH(I),I=1,NTHA)
      WRITE(6,34) THNOT,RCYL,(TTH(I),I=1,NTHA)
      READ (5,32) AA,(BB(I),I=1,LJ),((CC(L,J),L=1,LA),J=1,LB),(ZKALP(K)
     1,K=1,LA),(ZJP(KK),KK=1,NJS)
      WRITE(6,34) AA,(BB(I),I=1,LJ),((CC(L,J),L=1,LA),J=1,LB),(ZKALP(K)
     1,K=1,LA),(ZJP(KK),KK=1,NJS)
      LJ=NJPS*2
      LA=NALPP*2
      LB=NBETAP
      IF(LJ.LT.2) LJ=2
      IF(LA.LT.2) LA=2
      IF(LB.LT.1) LB=1
      READ (5,32) DD,(EE(I),I=1,LJ),((FF(L,J),L=1,LA),J=1,LB),(ZKALPP(K)
     1,K=1,LA),(THP(KK),KK=1,NJPS)
      WRITE(6,34) DD,(EE(I),I=1,LJ),((FF(L,J),L=1,LA),J=1,LB),(ZKALPP(K)
     1,K=1,LA),(THP(KK),KK=1,NJPS)
      LJ=NSS
      LA=NETA
      LB=NSIG*2
      IF(LJ.LT.1) LJ=1
```

```
      LK=2*LJ
      IF(LA.LT.1) LA=1
      LC=2*LA
      IF(LB.LT.2) LB=2
      READ (5,32) HCAP,(ZKS(I),I=1,LK),(TS(J),J=1,LJ),((ZNN(L,K),L=1,LB)
     1,K=1,LA),(ZNU(M),M=1,LC)
      WRITE(6,34) HCAP,(ZKS(I),I=1,LK),(TS(J),J=1,LJ),((ZNN(L,K),L=1,LB)
     1,K=1,LA),(ZNU(M),M=1,LC)
      LJ=2*NMODES
      READ (5,32) (AMN1(I),I=2,LJ,2)
      WRITE(6,34) (AMN1(I),I=2,LJ,2)
      READ (5,32) (WMN(I),I=1,NMODES)
      WRITE(6,34) (WMN(I),I=1,NMODES)
      READ (5,32) (OM(I),I=1,NFREQ)
      WRITE(6,34) (OM(I),I=1,NFREQ)
      DO 14 I=1,NFREQ
   14 OM(I)=PI2*OM(I)
      J=0
      DO 15 I=1,NMODES
      WMN(I)=WMN(I)*PI2
      J=J+2
   15 AMN1(J)=AMN1(J)*WMN(I)
      CO=PI/ZNOT
      DO 42 I=1,NTHA
   42 TTH(I)=TTH(I)*.0174532925
      READ (5,32) (ZMAS(I),I=1,NMODES)
      WRITE(6,34) (ZMAS(I),I=1,NMODES)
      L=-1
      DO 20 I=1,NMODES
      L=L+2
      AMN1(L)=WMN(I)
      AMN2(L)=-WMN(I)
   20 AMN2(L+1)=AMN1(L+1)
      DO 43 I=1,NFREQ
      DO 43 NM=1,NMODES
   43 GHAT(I,NM)=-1./((OM(I)-OMEG1(NM))*(OM(I)-OMEG2(NM))*ZMAS(NM))
      DO 100 NM=1,NMODES
      READ (5,31) NTH(NM),NKZ(NM),NCOE,(MCO(L,NM),L=1,NCOE)
      WRITE(6,31) NTH(NM),NKZ(NM),NCOE,(MCO(L,NM),L=1,NCOE)
      READ (5,30) CON(NM),(COE(L,NM),L=1,NCOE)
      WRITE(6,30) CON(NM),(COE(L,NM),L=1,NCOE)
      CO=.25/SQRT(ZMAS(NM))
      NCOEF(NM)=NCOE
      NZS=NKZ(NM)
      NTHS=NTH(NM)
      DO 12 I=1,NZS
   12 ZKP(I)=-2.
      DO 13 J=1,NTHS
   13 ZKQ(J)=-2.
      DO 110 I=1,NZS
      DO 110 J=1,NTHS
      ASMST(I,J)=(0.,0.)
      BSMST(I,J)=(0.,0.)
      BSM  (I,J)=(0.,0.)
```

```
  110 ASM  (I,J)=(0.,0.)
      ZKP(1)=0.
      ZKQ(1)=0.
      CA=CON(NM)*CO
      ASM(1,1)=CA
      BSM(1,1)=CA
      DO 6 LL=1,NCOF
      L=(MCO(LL,NM)-1)/NKZ(NM)+1
      M=(L-1)/NTH(NM)+1
      L1=MOD(L-1,NTH(NM))
      L2=MOD(MCO(LL,NM)-1,NKZ(NM))
      D1=L1
      D2=L2
      L3=L2+1
      L4=L1+1
      CB=COF(LL,NM)
C     WRITE(6,31) NM,L1,L2,L3,L4,L,M,LL
      GO TO (8,9,10,11),M
    8 ASM(L3,L4)=ASM(L3,L4)+CB
      BSM(L3,L4)=BSM(L3,L4)+CB
      ASMST(L3,L4)=ASMST(L3,L4)+CB
      BSMST(L3,L4)=BSMST(L3,L4)+CB
      GO TO 7
    9 CN=CM*CB
      ASM(L3,L4)=ASM(L3,L4)-CN
      BSM(L3,L4)=BSM(L3,L4)+CN
      ASMST(L3,L4)=ASMST(L3,L4)+CN
      BSMST(L3,L4)=BSMST(L3,L4)-CN
      GO TO 7
   10 CN=CM*CB
      ASM(L3,L4)=ASM(L3,L4)-CN
      BSM(L3,L4)=BSM(L3,L4)-CN
      ASMST(L3,L4)=ASMST(L3,L4)+CN
      BSMST(L3,L4)=BSMST(L3,L4)+CN
      GO TO 7
   11 ASM(L3,L4)=ASM(L3,L4)-CB
      BSM(L3,L4)=BSM(L3,L4)+CB
      ASMST(L3,L4)=ASMST(L3,L4)-CB
      BSMST(L3,L4)=BSMST(L3,L4)+CB
    7 ZKP(L3)=D2*CD
      ZKQ(L4)=D1
C     WRITE(6,34) CB,ZKP(L3),ZKQ(L4),ASM(L3,L4),BSM(L3,L4),CN
    6 CONTINUE
      DO 18 IP=1,NZS1
      L3=IP+1
      DO 18 IT=1,NTHS1
      L4=IT+1
      ASM(L3,L4)=ASM(L3,L4)*CO
      BSM(L3,L4)=BSM(L3,L4)*CO
      ASMST(L3,L4)=ASMST(L3,L4)*CO
   18 BSMST(L3,L4)=BSMST(L3,L4)*CO
      ASMST(1,1)=ASM(1,1)
      BSMST(1,1)=BSM(1,1)
C     DO 402 IP=1,NZS
```

```
C      WRITE(6,34) (ASMST(IP,IT),IT=1,NTHS)
C 402  WRITE(6,34) (BSMST(IP,IT),IT=1,NTHS)
       DO 90 IP=1,NZS
       ZK=ZKP(IP)
       ZKK=-ZK
       IF(ZK.GT.-1.) GO TO 19
       ZI1(IP)=(0.,0.)
       ZI2(IP)=(0.,0.)
       GO TO 90
  19   CALL FINDI(ZK,Z1,Z2,ACAP,BCAP,CCAP,ZKA,NALP,NBETA,ZI1(IP),ZJP,NJS)
       CALL FINDI( ZKK,Z1,Z2,ACAP,BCAP,CCAP,ZKA,NALP,NBETA,ZI2(IP),ZJP,
      1NJS)
  90   CONTINUE
       DO 80 IT=1,NTHS
       ZK=ZKQ(IT)
       ZKK=-ZK
       IF(ZK .GT.-1.) GO TO 21
       TI1(IT)=(0.,0.)
       TI2(IT)=(0.,0.)
       GO TO 80
  21   CALL FINDI(ZK,0.,THNOT,DCAP,ECAP,FCAP,ZKAP,NALPP,NBETAP,TI1(IT),
      1THP,NJPS)
       CALL FINDI(ZKK,0.,THNOT,DCAP,ECAP,FCAP,ZKAP,NALPP,NBETAP,TI2(IT),
      1THP,NJPS)
  80   CONTINUE
       SMB=0.
       DO 60 IP=1,NZS
       DO 60 IT=1,NTHS
       SMB=SMB+ASM(IP,IT)*ZI1(IP)*TI1(IT)+ASMST(IP,IT)*ZI2(IP)*TI2(IT)
      1+BSMST(IP,IT)*ZI1(IP)*TI2(IT)+BSM(IP,IT)*ZI2(IP)*TI1(IT)
C      WRITE(6,35) IP,IT,SMB,ASM(IP,IT),ZI1(IP),TI1(IT),ASMST(IP,IT),
C     1ZI2(IP),TI2(IT),BSMST(IP,IT),ZI1(IP),TI2(IT),BSM(IP,IT)
  60   CONTINUE
       SRE(NM)=SMB*DCYL
 100   CONTINUE
       WRITE(6,37)
       DO 200 IR=1,NOFZ
       ZZ=Z(IR)*C0
       DO 210 JR=1,NTHA
       TH=TTH(JR)
       DO 207 NM=1,NMODES
       PST=CON(NM)
       NCOF=NCOFF(NM)
       DO 206 LL=1,NCOF
       L=(MCO(LL,NM)-1)/NKZ(NM)+1
       M=(L-1)/NTH(NM)+1
       L1=MOD(L-1,NTH(NM))
       L2=MOD(MCO(LL,NM)-1,NKZ(NM))
       P1=L1
       P2=L2
       L3=L2+1
       L4=L1+1
C      WRITE(6,31) NM,L1,L2,L3,L4,L,M,LL
       GO TO (2,3,4,5),M
```

```
      2 PSI=PSI+COS(P1*TH)*COF(LL,NM)*COS(P2*ZZ)
        GO TO 206
      3 PSI=PSI+COS(P1*TH)*COF(LL,NM)*SIN(P2*ZZ)
        GO TO 206
      4 PSI=PSI+SIN(P1*TH)*COF(LL,NM)*COS(P2*ZZ)
        GO TO 206
      5 PSI=PSI+SIN(P1*TH)*COF(LL,NM)*SIN(P2*ZZ)
    206 CONTINUE
C       WRITE(6,35) IP,JP,ZZ,TH,PSI
    207 PHI(NM)=PSI
        DO 220 LB=1,NFREQ
        SM=(0.,0.)
        PHAT=HCAP
        IF(NSS.LT.1) GO TO 224
        DO 221 IS=1,NSS
    221 PHAT=PHAT+ZKKS(IS)*CEXP(-OM(LB)*CM*TS(IS))
    224 IF(NETA.LT.1) GO TO 225
        DO 222 I=1,NETA
        CB1=1./(OM(LB)-ZNUCAP(I))
        CB2=CB1
        SM=ZNCAP(1,I)*CB1
        IF(NSIG.LT.2) GO TO 222
        DO 223 J=2,NSIG
        CB2=CB2*CB1
    223 SM=SM+ZNCAP(J,I)*CB2
    222 CONTINUE
        PHAT=PHAT+SM
    225 SM=(0.,0.)
        DO 208 NM=1,NMODES
        SM=SM+PHI(NM)*GHAT(LB,NM)*SPF(NM)
C       WRITE(6,35) NM,LB,SM,GHAT(LB,NM)   ,SPF(NM)
    208 CONTINUE
        RESPS=SM*PHAT
        X=REAL(RESPS)
        Y=AIMAG(RESPS)
        IF(ABS(X).GT.1.E-16) GO TO 243
        PHASE=1.5707963
        IF(Y.LT.0.) PHASE=PHASE+PI
        GO TO 242
    243 PHASE=ATAN(Y/X)
        IF(X*Y) 240,241,241
    241 IF(X.LT.0.)PHASE=PHASE+PI
        GO TO 242
    240 PHASE=PHASE+PI
        IF(X.GT.0.) PHASE=PHASE+PI
    242 WMAG=SQRT(X**2+Y**2)
        TI3=THCON*TH
        OMM=OM(LB)/PI2
        PHASE=PHASE*THCON
        WRITE(6,36) OMM,TI3,Z(IP),RESPS,WMAG,PHASE
    220 CONTINUE
    210 CONTINUE
    200 CONTINUE
        GO TO 1
```

```
   30 FORMAT(5E15.8)
   31 FORMAT(16I5)
   32 FORMAT(8E10.3)
   34 FORMAT(1X,10E12.5)
   35 FORMAT(2I5,8E12.5/(1X,10E12.5))
   36 FORMAT(13H0FREQUENCY = E13.6,11H,  THETA = E13.6,7H,  Z = E13.6,
     1 14H,  RESPONSE = 2E13.6/73X,12HMAGNITUDE = E13.6,11H,  PHASE =
     2E13.6)
   37 FORMAT(24H0***END OF INPUT DATA***//)
   38 FORMAT(17H1***INPUT DATA***)
      END
 @FOR,IS FINDIT
      SUBROUTINE FINDI(ZK  ,Z1,Z2,ACAP,BCAP,CCAP,ZKA,NALP,NBETA,FUNT,ZZP,
     1NJS)
      COMPLEX     ZKA,ZZ,COEZ,COKP,COE,TEM1,TEM2,COEX1,COEX2,TT1,TT2,
     1Z22,Z11,SM,SM2,SM3,ACAP,BCAP,CCAP
     1,ZKPT,FUNT
      DIMENSION  ZKA(1),COEZ(20),COKP(10,10),COE(10,10),FACT(11),
     1 Z11(20),Z22(20),CCAP(10,10),BCAP(1),ZZP(1)
      DATA FACT/1.,1.,2.,6.,24.,120.,720.,5040.,40320.,362880.,
     13628800./
      CM=6.2831853
      NBETA1=NBETA-1
      DO 10 I=1,NALP
      ZZ=ZK+ZKA(I)
      COKP(I,1)=(1.,0.)
      COEZ(I)=CEXP(ZZ)
      CN=1.
      DO 10 J=1,NBETA
      COKP(I,J+1)=ZZ*COKP(I,J)
      COE(I,J)=CN/COKP(I,J+1)*CCAP(I,J)
C     WRITE(6,30) I,J,ZK,ZZ,ZKA(I),COKP(I,J+1),COE(I,J),CN,CCAP(I,J),
C    1COEZ(I)
      CN=-CN
   10 CONTINUE
      TEM2=(0.,-1.)*Z2
      TEM1=(0.,-1.)*Z1
      TT1=1.
      TT2=1.
      DO 14 J=1,NBETA
      Z22(J)=TT2
      Z11(J)=TT1
      TT2=TT2*TEM2
      TT1=TT1*TEM1
C     WRITE(6,30) J,J,Z22(J),Z11(J),TT2,TT1
   14 CONTINUE
      SM2=0.
      DO 16 I=1,NALP
      SM=0.
      IF(NBETA .LE.0) GO TO 19
      COEX2=CEXP(-TEM2*(ZK+ZKA(I)))
      COEX1=CEXP(-TEM1*(ZK+ZKA(I)))
      DO 15 J=1,NBETA
      SM=SM+(Z22(J)*COEX2-Z11(J)*COEX1) *COKP(I,J)/FACT(J)
```

```
      SM2=SM2+COE(I,J)*SM
C     WRITE(6,30) I,J,SM,Z22(J),COEX2,Z11(J),COEX1,COKP(I,J)
   15 CONTINUE
   16 CONTINUE
   13 SM3=0.
      ZKPI=(0.,1.)*ZK
      IF(NJS.LT.1) GO TO 20
      DO 18 J=1,NJS
      SM3=SM3+BCAP(J)*CEXP(ZKPI*ZZP(J))
C     WRITE(6,30) J,J,SM3,BCAP(J),ZKPI,ZZP(J)
   18 CONTINUE
   20 IF(ZK)200,200,120
  200 FUNI=SM2+SM3+ACAP*(Z2-Z1)/CM
      RETURN
  120 FUNI=ACAP*(COEX2-COEX1)*EXP(ZK)/(ZKPI*CM)+SM3+SM2
C     WRITE(6,32) ACAP,COEX2,COEX1,ZK,ZKPI,CM,SM2,SM3
      RETURN
   30 FORMAT(2I5,(1X,9E12.6))
   32 FORMAT(1X,10E12.6)
      END
@MAP,IS ,ABS
 IN DEXCYL
@XQT ABS
   19    1    1   12    1    1    1    1    0    0    0    0    0
132.          0.          132.        55.
3.14159265  50.         0.          20.         40.         60.         80.         100.
120.        140.        160.        180.        200.        220.        240.        260.
280.        300.        320.        340.        360.
 0.          0.          0.          0.          1.          0.          -.1         .025
0.          0.
 0.          0.          0.          0.          1.          0.          0.          .05
0.          0.
 .015        0.          0.          0.          0.          0.          0.          0.
.02         .02         .02         .02         .02         .02         .02         .02
.02         .02         .02         .02
149.8409    149.8413    154.4314    154.4314    170.1672    187.4919    187.4922    258.45
258.4553    259.1931    280.3504    280.3506
149.9409
.5878779    .6263521    .5528155    .5654379    1.099990    .7388390    .6730367    .76422
.7751001    1.479049    1.021387    1.006094
    4    4    4   20   32   62   64
    .64891715-05  -.75502460+00   .25676082-01  -.61556394+00   .20933483-01
    4    4    4   20   32   62   64
    .11708251-03   .63559545+00  -.21614622-01  -.77966249+00   .25513896-01
    6    2    2    9   22   46
    .18283015-04  -.19143746-01   .98810329+00   .16254598+00
    6    2    2   22   33   46
    .12709915-03   .15226757+00   .19080995-01  -.98501851+00
    6    2    2   11   24
   -.54830542-06  -.20555947-01   .99730116+00
    4    4    4   26   29   58   60
    .15967466-05   .51843181+00  -.95938868-02   .83018614+00  -.12878921-01
    4    4    5   22   26   29   58   60
   -.10586593-03  -.49546157-03  -.79255639+00   .12295176-01   .59039415+00
```

```
   -.91580419-02
  5    3    3     14    33    69
   .15926782-09    .98410990-02   -.10017767+01    .53969285+00
  6    3    3     33    50    69
   -.14593651-08    .54169539+00   -.89123033-02    .10092856+01
  7    3    2     17    39
   -.19943995-09   -.10770447-01    .11265565+01
  4    4   11      7   18   22    24   30   37    39   54   56   58   62
   -.22611810-06    .71489229-04   -.14072712-03   -.48495815+00   -.20647036-02
   -.76995499-04    .22655226-01   -.22655292-01    .86073080+00    .11231609-01
   -.20228178-03   -.90219221-04
  4    4   11      5    7   18    22   24   26    30   39   54   56   58
   .22150605-06    .22537380-01   -.22537106-01    .13347271-03    .85426307+00
   .11224375-01   -.17934819-03   -.11332811-03   -.70947396-04    .48112981+00
  .20484      -02   -.11030     -03
```

3.3 Response to Random Excitation (RANCYL)

3.3.1 Program Description

The RANCYL computer program computes the dynamic response of nonuniform beams and cylindrical shells for a class of random pressure fields. The structural mode shapes in the form of combinations of trigonometric functions with arbitrary constant coefficients are assumed. These coefficients are obtained from the computer code "FITMSC."

3.3.2 Input

Section 3.3.9 presents the necessary input to be read into the code.

3.3.3 Governing Equations

The governing equations are given in Section 2.4.2.

3.3.4 Method of Solution

The computational procedure is described in Section 2.0.

3.3.5 Restart

The code contains a restart feature. The first time the code is run, the quantity "ZJUNK" is punched on cards. ZJUNK is the double sum involving rerms like $H_l^* I(-k_p, k_q, k_r, k_s)$. This part of the calculation represents the largest portion of the computation and hence is advantageous not to recalculate it if additional runs are to be performed.

3.3.6 Termination

The code will be terminated when the input data has been exhausted.

3.3.7 Computer Conversion

The program was written in FORTRAN V. It has been run and checked as thoroughly as is practical on the UNIVAC 1108 (EXEC 8). The code was designed to be as machine independent as possible. To run the codes on a CDC or IBM computer, the control cards at the beginning of each subroutine must be eliminated and/or replaced with appropriate control cards.

3.3.8 Equipment Requirements

The code uses no tape units other than 5 and 6 for input and output, respectively. The amount of core necessary to run is a strong function of the number of mesh points, modes, frequencies, etc. (See Section 3.3.14 for minimum storage requirements.) The present form of the code uses approximately 26,000 cells. There will be cards punched if ISHOT is less than or equal to 1. (See Section 3.3.9).

3.3.9 Program Input Requirements

NOTE: Certain real variable arrays were equivalenced to their complex counterparts to avoid the problem of format variation from machine to machine for complex variables.

3.3.10 Card Formats

NOTE: All integer variables must be right adjusted. Floating point fields must contain a decimal which may be arbitrarily located in the field width.

CARD GROUP 1

Columns	Description
1-5	NTHA, the response is computed for the points Z, θ, Z′, θ′. Z and Z′ use the same set of numbers. θ, θ′ use the same set of angles. NTHA is the number of angles in this set.
6-10	NOFZ, number of different values of Z (also Z′) for which the response is calculated. (Integer)
11-15	NFREQ, number of frequencies for which the response is calculated. (Integer)
16-20	NMODES, number of mode shapes for which fit data is available. (Integer)
21-25	NBETA, number of β terms in $Q_1(Z_0)$. (Integer)
26-30	NETA, number of η terms in $Q_2(\theta_0)$. (Integer)
31-35	NALP, number of α terms in $Q_1(Z_0)$. (Integer)
36-40	NKZET, number of ζ terms in $Q_2(\theta_0)$. (Integer)
41-45	NGAM, number of γ terms in $Q_1(Z_0)$. (Integer)
46-50	NEPS, number of ε terms in $Q_1(Z_0)$. (Integer)
51-55	NLAM, obsolete

Columns	Description
56-60	NSIG, obsolete
61-65	NSS, number of K_s's in $\hat{Q}_3(\omega)$. (Integer)
66-70	NETA1, number of η_1's in $\hat{Q}_3(\omega)$. (Integer)
71-75	NSIG1, number of ν_{σ_1} terms in $\hat{Q}_3(\omega)$. (Integer)
76-80	NETA2, number of η_2's in $\hat{Q}_3(\omega)$. (Integer)
Next Card, 1-5	NSIG2, number of ν_{σ_2} terms in $\hat{Q}_3(\omega)$. (Integer)
6-10	ISHOT, constant: set equal to 1 for an initial run and set equal to 2 for a continuation run. If ISHOT=2 additional data (ZJUNK, See Card Group 13) will be required. (Integer)
11-15	NCROS, flag: If NCROS = 1 no cross terms will be evaluated in W. If NCROS = 0 all terms will be included in the summation for W.

CARD GROUP 2

Columns	Description
1-10	ZNOT, length of the cylinder in inches. (Floating point)
11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	Z, the Z component of the points for which the cross spectral density of the response will be calculated. All combinations of Z, Z′, θ, θ′ will be used, where Z′ takes on the same set as Z. (Floating point, NOFZ number in all)

CARD GROUP 3

Columns	Description
1-10	RCYL, radius of the cylinder in inches. (Floating point)
11-20, 21-30, 31-40, 41-50, 51-60, 61-70 71-80	TTH, θ, angular values for which the cross spectral density of the response are calculated. (Floating point, NTHA numbers in all.)

NOTE: In the following three groups it is assumed the number of terms for each variable is one. If there are more terms the columns will be shifted appropriately.

CARD GROUP 4

Columns	Description
1-10, 11-20	AA, A_0, constant pressure field term in $Q_1(Z)$, real part followed by imaginary part. (Floating point)
21-30, 31-40	AA1, real and imaginary parts of A_1 used in $Q_1(Z)$. (Floating point)
41-50, 51-60	BB, $B_{\alpha,\beta}$, real and imaginary components of $B_{\alpha,\beta}$. There will be NALP pairs of numbers for $\beta=1$, followed by NALP pairs of numbers for $\beta=2$, etc. If NALP=0 there must be two fields containing zero. (Floating point, 2 x NALP x NBETA numbers in all.)

Columns	Description
61-70, 71-80	CC, $C_{\gamma,\epsilon}$, real and imaginary components of $C_{\gamma,\epsilon}$. There will be NGAM pairs of numbers for $\epsilon=1$, followed by NGAM pairs of numbers for $\epsilon=2$, etc. If NGAM=0 there must be two fields containing zero. (Floating point, 2 x NGAM x NEPS numbers in all.)
New Card	
1-10, 11-20	ZKALP, K_α, real and imaginary components of K_α. There will be NALP pairs of numbers. (Floating point)
21-30, 31-40	ZKGM, K_γ, real and imaginary parts of K_γ. There will be NGAM pairs of numbers. (Floating point)

CARD GROUP 5

Columns	Description
1-10, 11-20	DD, D_0, constant pressure field term in $Q_2(\theta)$, real part followed by the imaginary part. (Floating point)
21-30, 31-40	DD1, real and imaginary parts of D_1 used in $Q_2(\theta)$. (Floating point)
41-50, 51-60	EE, $E_{\zeta,\eta}$, real and imaginary components of $E_{\zeta,\eta}$. There will be NKZET pairs of numbers for $\eta=1$, followed by NKZET pairs of numbers for $\eta=2$, etc. If NKZET=0 there must be two fields containing zero. (Floating point, 2 x NKZET x NETA numbers in all.)

Columns	Description
61-70	ZKZETA, K_ζ, real and imaginary parts of K_ζ. There will be NKZET pairs of numbers. (Floating point)

CARD GROUP 6

Columns	Description
1-10	HCAP, H_0, constant pressure term in $\hat{Q}_3(\omega)$. (Floating point)
11-20, 21-30	ZKS, K_s, real and imaginary parts of K_s. (Float-point -- there will be NSS successive pairs of K_s. If NSS = 0 two zero fields must be present.)
31-40	TS, t_s, appearing in $\hat{Q}_3(\omega)$. There must be at least one field (which may be zero) and up to NSS numbers. (Floating point)
41-50, 51-60	ZNN, N_{η_1, σ_1}, real and imaginary components of N_{η_1, σ_1}. There will be NSIG1 pairs of numbers for $\eta_1 = 1$ followed by NSIG1 pairs of numbers for $\eta_1 = 2$, etc. until the list of η_1 is complete. If NSIG1 = 0 two fields containing zero must be present. (Floating point)
61-70, 71-80	ZNU, ν_{η_1}, real and imaginary components of ν_{η_1}. There must be at least two fields and up to NETA1 pairs of numbers in all. (Floating point)

Columns	Description
New Card	
1-10, 11-20	ZMM, M_{η_2, σ_2}, real and imaginary components of M_{η_2, σ_2}. There will be NSIG2 pairs of numbers for $\eta_2=1$ followed by NSIG2 pairs of numbers for $\eta_2=2$, etc. until the list of η_2 is complete. If NSIG2=0 two fields containing zero must be present. (Floating point)
21-30, 31-40	ZMU, ν_{η_2}, real and imaginary parts of ν_{η_2}. There must be at least two fields and up to NETA2 pairs of numbers in all. (Floating point)

CARD GROUP 7

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	AMN1, α_{nm}, damping constants. (Floating point- NMODES numbers in all)

CARD GROUP 8

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	WMN, ω_{nm}, modal frequency in cycles per second. (Floating point - eight numbers per card, NMODES in all.)

CARD GROUP 9

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	OM, ω frequencies for which the responses are to be calculated. (Floating point - eight numbers per card NFREQ in all.)

CARD GROUP 10

Columns	Description
1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80	ZMAS, $M_{n\,m}$, modal mass. (Floating point -- eight numbers per card NMODES numbers in all.)

CARD GROUP 11

Columns	Description
1-5	NTH, number of θ terms in the fitting of the modal data. (Integer)
6-10	NKZ, number of Z terms in the fitting of the modal data. (Integer)
11-15	NCOE, number of non-zero terms in the fit, excluding the constant term.
16-20, 21-25, 26-30, 31-35, 36-40, 41-45, 46-50, 51-55, 56-60, 61-65, 66-70, 71-75, 76-80	MCO, index tagging the fit coefficients with the appropriate trigonometric term. (Integer - NCOE numbers in all.)

CARD GROUP 12

Columns	Description
1-15	CON, constant term in the fits to the mode shapes. (Floating point)
15-30, 31-45, 46-60, 61-75	COE, fit coefficients to the mode shapes. (Floating point -- NCOE numbers in all.)

Card groups 11 and 12 are repeated once for each mode number. These two card groups are the direct output of the FITMSC computer code.

CARD GROUP 13

Columns	Description
1-15, 16-30	ZJUNK, SA, summations involving terms like $H_1^* I (K_p, K_q, K_r, K_s)$. There will be NMODES groups of cards with each group consisting of NMODES numbers, 5 per card. (Floating point)

NOTE: Card Group 13 will be present only for continuation runs when ISHOT ≥ 2. See Card Group 1.

NOTE: To run the random beam problem set

1) $R = RCYL = 1/2\pi$

2) $D_0 = D(1) + i \cdot D(2) = 1 + 0i$

3) NKZET = 0

4) $E_{\zeta,\eta} = EE(1) + i\,DD(2) = 0$

4) $D_0 = DD(1) + i\,DD(2) = 0$

6) NTH = 1 for all modes

3.3.11 Sample Input

The following is a listing of the input data used to obtain the sample output listed in Section 3.3.13. Model 1 fit data from FITMSC was used as input. Sample 1 is an initial run and sample 2 is a restart run.

SAMPLE 1 INITIAL RUN

```
   1    3   71   12    1    0    2    1    2    1    1    1    0    0    0    0
   0    1    1
172.          66.          88.          110.
 50.          0.
0.           0.           0.           0.           0.           -.5          0.           -.5
0.           .5           0.           .5           .1           .025         -.1          .025
.1           -.025        -.1          -.025
0.           0.           0.           0.           1.           0.           0.           2.865
.016         .0           0.           0.           0.           .0           .0           .0
0.           0.           0.           0.
.01          .01          .01          .01          .01          .01          .01          .01
.01          .01          .01          .01
149.8409     149.8413     154.4314     154.4314     170.1672     187.4919     187.4922     258.4653
258.4653     258.1931     280.3604     280.3606
100.         110.         120.         130.         135.         140.         142.         144.
146.         148.         149.         150.         151.         152.         153.         154.
155.         156.         158.         160.         162.         164.         166.         168.
169.         170.         171.         172.         173.         175.         176.         178.
180.         182.         184.         186.         187.         188.         189.         190.
192.         194.         196.         200.         210.         220.         230.         240.
245.         250.         254.         256.         258.         260.         265.         270.
272.         274.         276.         278.         280.         282.         284.         286.
290.         300.         310.         320.         330.         340.         350.
.5878779     .6263621     .5698155     .5664379     1.089990     .7388390     .6730367     .7642241
.7751001     1.479049     1.021387     1.006094
   4    4    4   30   32   62   64
 .64891715-05   -.75502460+00    .25676082-01   -.61556394+00    .20933483-01
   4    4    4   30   32   62   64
 .11708251-03    .63559545+00   -.21614622-01   -.77966249+00    .26513896-01
   5    2    3    9   22   46
 .18283015-04   -.19143746-01    .99810329+00    .16254598+00
   5    2    3   22   33   46
 .12709915-03    .16226757+00    .19080995-01   -.98501851+00
   6    2    2   11   24
-.54930642-06   -.20595947-01    .99770116+00
   4    4    4   26   28   58   59
 .16967466-05    .61843181+00   -.95938968-02    .83019614+00   -.12878921-01
   4    4    5   22   26   28   58   59
-.10586593-03   -.49546157-03   -.79255699+00    .12295176-01    .59039415+00
-.91599419-02
   6    3    3   14   33   69
 .15926782-09    .88414990-02   -.10017767+01    .53869285+00
   6    3    3   37   50   69
-.14583651-09    .54158579+00   -.99123033-02    .10092856+01
   7    3    2   17   39
-.19943995-09   -.10770447-01    .11255565+01
   4    4   11    7   18   22   24   30   37   39   54   56   58   62
-.22611810-06    .71489279-04   -.14072712-03   -.43495815+00   -.20647036-02
-.76995498-04    .22655226-01   -.22655292-01    .86077080+00    .11291609-01
-.20228179-03   -.90219221-04
   4    4   11    5    7   19   22   24   26   30   39   54   56   58
 .22150605-06    .22537380-01   -.22537196-01    .13347271-03    .85426307+00
```

```
  .11224375-01  -.17934919-03  -.11332811-03  -.70047396-04   .48112981+00
.20484       -02  -.11030    -03
```

SAMPLE 2 RESTART RUN

```
  1      3    71    12     1     0     2     1     2     1     1     1     0     0     0     0
  0      2     1
  0      2     0
132.        66.         88.         110.
 50.        0.
0.          0.          0.          0.          0.          -.5         0.          -.5
0.          .5          0.          .5          .1          .025        -.1         .025
.1          -.025       -.1         -.025
0.          0.          0.          0.          1.          0.          0.          2.865
.015        .0          0.          0.          0.          .0          .0          .0
0.          0.          0.          0.
 .01        .01         .01         .01         .01         .01         .01         .01
 .01        .01         .01         .01
149.8409    149.8413    154.4314    154.4314    170.1672    187.4919    187.4922    258.4653
258.4653    259.1931    280.3604    280.3606
 100.        110.        120.        130.        135.        140.        142.        144.
146.        148.        149.        150.        151.        152.        153.        154.
 155.        155.        158.        160.        162.        164.        166.        168.
 169.        170.        171.        172.        173.        175.        176.        178.
  180.        182.        184.        186.        187.        188.        189.        190.
  192.        194.        196.        200.        210.        220.        230.        240.
  245.        250.        254.        256.        258.        260.        265.        270.
  272.        274.        276.        278.        280.        282.        284.        286.
  290.        300.        310.        320.        330.        340.        350.
.5878779    .6263621    .5698155    .5654379    1.089890    .7398390    .6730367    .7642241
.7751001    1.479049    1.021387    1.006094
    4     4     4    30    32    62    64
   .64891715-05   -.75502460+00    .25676082-01   -.61556394+00    .20933483-01
    4     4     4    30    32    62    64
   .11708251-03    .63559545+00   -.21614622-01   -.77966249+00    .26513896-01
    6     2     3     9    22    45
   .18283015-04   -.19143746-01    .99810329+00    .16254598+00
    6     2     3    22    33    46
   .12709915-03    .16226757+00    .13080996-01   -.98501851+00
    6     2     2    11    24
  -.54830642-06   -.20595947-01    .99730116+00
    4     4     4    26    28    58    60
   .16967466-05    .61943181+00   -.95938869-02    .93018614+00   -.12878921-01
    4     4     5    22    26    28    58    60
  -.10586593-03   -.49546157-03   -.79255699+00    .12295176-01    .59039415+00
  -.91589419-02
    6     3     3    14    33    69
   .15926782-09    .88414990-02   -.10017767+01    .53869285+00
    6     3     3    33    50    69
  -.14583651-08    .54168539+00   -.89123033-02    .10092856+01
    7     3     2    17    39
  -.19943995-09   -.12770447-01    .11265665+01
    4     4    11     7    18    22    24    30    37    39    54    56    58    62
  -.22611810-05    .71489229-04   -.14072712-03   -.49495815+00   -.20647036-02
  -.75995498-04    .22555226-01   -.22655292-01    .96073080+00    .11291609-01
  -.20228178-03   -.90219221-04
    4     4    11     5     7    18    22    24    26    30    39    54    56    58
```

.22150605-05	.22537380-01	-.22537106-01	.13347271-03	.85426307+00
.11224375-01	-.17934819-03	-.11332811-03	-.70947396-04	.48112981+00
.20484 -02	-.11030 -03			
.61917104+03	.00000000	.26507292-01	.19073486-05	.24443792-05
-.35432442-14	.64142372-05	.35527137-14	.14263103-05	-.34694470-17
.28785843-05	-.69388939-17	-.57879925-05	.00000000	-.69900038-07
.00000000	-.34638900-08	.00000000	.45207499-07	.00000000
.56106725-01	-.86736174-18	.42572224-01	-.11630163-09	
.26507324-01	-.74505806-08	.61963097+03	.00000000	.10804297-04
-.35457748-14	.86471960-04	-.34694470-17	-.14421685-05	-.34694470-17
-.53918587-06	.00000000	-.64838254-04	.69388939-17	.19081074-07
.00000000	-.49460589-07	.00000000	-.36953352-07	.00000000
.10069292-01	.11368684-12	-.34553275-01	-.11618794-09	
.55519899-05	.00000000	.91800387-05	.00000000	.41866453+03
.29802322-07	-.59059068+00	.00000000	-.18881809-05	.88817842-15
-.39846752-05	.00000000	-.29959612-05	.00000000	-.11280961+02
.00000000	.86053009+01	.00000000	-.54937447-07	.00000000
-.23508185-04	-.13995749-13	.14686722-04	.00000000	
.48326910-05	.14210855-13	.83259181-04	-.71054274-14	.85095362+00
.00000000	.41953954+03	.29802322-07	-.60590499-06	-.71054274-14
-.55207267-05	-.14203916-13	-.77697399-04	.28421709-13	-.86236939+01
.00000000	-.11293942+02	.00000000	-.98204112-08	.00000000
-.18389996-03	.00000000	.18294711-03	-.27547409-13	
.21759330-05	-.69388939-17	-.20208235-05	.00000000	-.18195811-05
.00000000	-.79583867-06	-.35527137-14	.14426653+03	.00000000
-.49706685-05	-.71019579-14	.65237352-05	.00000000	.60091107-07
.00000000	-.25154474-07	.00000000	.52918745+01	.00000000
.37171139-05	.14432899-13	-.67997038-05	-.86736174-18	
.70518327-05	-.14183099-13	-.99799539-06	.27755576-16	-.23245121-05
.14210855-13	-.21757120-05	.00000000	-.30602802-05	.00000000
.78261200+03	.00000000	-.41428340-02	.18626451-08	.12379359-05
.00000000	.32952348-07	.00000000	-.97750167-07	.00000000
-.10426590+00	.69388939-17	-.12667939+00	-.13877788-16	
-.79318651-05	-.14238610-13	-.58513522-04	.00000000	-.63305283-05
-.71019579-14	-.76027782-04	.71088968-14	.45707555-05	.71088968-14
-.41428517-02	-.18626451-08	.78300469+03	.00000000	-.79778218-07
.00000000	.11761475-06	.00000000	.13880293-06	.00000000
.12989256+00	-.11367296-13	-.36632602+00	.46554760-09	
-.11509148-06	.00000000	.11795502-06	.00000000	-.11280970+02
.00000000	-.86236887+01	.00000000	.57679541-07	.00000000
.23033482-06	.00000000	-.12969964-06	.00000000	.72532459+03
.74505806-08	.13826454+01	.00000000	.29155202-05	.35527137-14
-.23813419-06	.00000000	.22217379-06	.00000000	
.93400517-07	.00000000	-.36038372-08	.00000000	.86053038+01
.00000000	-.11293952+02	.00000000	-.31586376-07	.00000000
.82825837-07	.00000000	.27419653-06	.00000000	-.21484949+00
.00000000	.72527235+03	-.74505806-08	-.16270818-05	.00000000
-.87648042-07	.00000000	-.24228745-06	.00000000	
.68763102-07	.00000000	-.55837823-07	.00000000	-.62969645-07
.00000000	-.16788833-07	-.44408921-15	.52918778+01	.00000000
-.16125866-06	.00000000	.20756735-06	.00000000	.30395119-05
.00000000	-.12971187-05	.00000000	.26709797+03	.00000000
.10382720-06	.00000000	-.18359025-06	.00000000	
.55106532-01	.19428903-15	.10097261-01	.00000000	-.21858523-04

.13998810-17	-.16220110-03	.00000000	.25124913-05	.14321877-13
-.10426783+00	.00000000	.12986550+00	.59894668-09	-.99853210-07
.00000000	-.28601637-07	.00000000	.65398529-07	.00000000
.84060091+03	.23841858-06	.35732599+01	.40046871-07	
.42578674-01	-.69388939-17	-.34579493-01	.11541538-09	.18039935-04
.00000000	.15757730-03	.71019579-14	-.45213664-05	.00000000
-.12667398+00	-.10408341-15	-.37071783+00	.93132257-09	.78377914-07
.00000000	-.10983193-06	.00000000	-.11558868-06	.00000000
-.35695557+01	-.29802322-07	.83971035+03	-.11129305-06	

3.3.12 Description of Output

The input is printed immediately after being read. No labels are printed. The order is the same as listed for the input cards.

<u>VARIABLE</u>

Frequency	ω, frequency of the calculation.	
Z	Z	Coordinates of the two points for which the cross spectral density of the response is computed.
Theta	θ	
Z1	Z'	
THETA1	θ'	
Magnitude	$\lvert W(Z, \theta, Z', \theta')\rvert$, magnitude of the cross spectral density of the response at two points on the cylinder	
PHASE	φ, Phase associated with $W(Z, \theta, Z', \theta')$	

3.3.13 Sample Output

The following sample case was run with the sample input as data.

```
***INPUT DATA***
    1     3    71    12     1     0     2     1     2     1     1     1     0     0     0
    0     2     0
   .13200+03   .66000+02   .88000+02   .11000+03
   .50000+02   .00000
   .00000      .00000      .00000      .00000      .00000     -.50000+00
   .00000      .50000+00   .10000+00   .25000-01  -.10000+00   .25000-01
   .00000      .00000      .00000      .00000      .10000+01   .00000
   .16000-01   .00000      .00000      .00000      .00000      .00000
   .00000      .00000
   .10000-01   .10000-01   .10000-01   .10000-01   .10000-01   .10000-01
   .10000-01   .10000-01
   .14984+03   .14984+03   .15443+03   .15443+03   .17017+03   .18749+03
   .28036+03   .28036+03
   .10000+03   .11000+03   .12000+03   .13000+03   .13500+03   .14000+03
   .14900+03   .15000+03   .15100+03   .15200+03   .15300+03   .15400+03
   .16200+03   .16400+03   .16600+03   .16800+03   .16900+03   .17000+03
   .17600+03   .17800+03   .18000+03   .18200+03   .18400+03   .18600+03
   .19200+03   .19400+03   .19600+03   .20000+03   .21000+03   .22000+03
   .25400+03   .25600+03   .25800+03   .26000+03   .26500+03   .27000+03
   .28000+03   .28200+03   .28400+03   .28600+03   .29000+03   .30000+03
   .35000+03
   .58788+00   .62636+00   .56982+00   .56544+00   .10899+01   .73884+00
   .10214+01   .10061+01
    4     4     4    30    32    62    64
  .64891715-05  -.75502460+00   .25676082-01  -.61556394+00   .20933483-01
    4     4     4    30    32    62    64
  .11708251-03   .63559545+00  -.21614622-01  -.77966249+00   .26513896-01
    6     2     3     9    22    46
  .18283015-04  -.19143746-01   .98810329+00   .16254598+00
    6     2     3    22    33    46
  .12709915-03   .16226757+00   .19080996-01  -.98501851+00
    6     2     2    11    24
 -.54830642-06  -.20595947-01   .98730116+00
    4     4     4    26    28    58    60
  .16967466-05   .61843181+00  -.95938868-02   .83018614+00  -.12878921-01
    4     4     5    22    26    28    58    60
 -.10586593-03  -.49546157-03  -.79255699+00   .12295176-01   .59039415+00
 -.91589418-02
    6     3     3    14    33    69
  .15926782-09   .88414990-02  -.10017767+01   .53869285+00
    6     3     3    33    50    69
 -.14583651-08   .54168539+00  -.89123033-02   .10092856+01
    7     3     2    17    39
 -.19943995-09  -.10770447-01   .11265665+01
    4     4    11     7    18    22    24    30    37    39    54    56    58    62
 -.22611810-06   .71489229-04  -.14072712-03  -.48495815+00  -.20647036-02
 -.76995498-04   .22655226-01  -.22655292-01   .86073080+00   .11291609-01
 -.20228178-03  -.90219221-04
    4     4    11     5     7    18    22    24    26    30    39    54    56    58
  .22150605-06   .22537380-01  -.22537106-01   .13347271-03   .85426307+00
  .11224375-01  -.17934819-03  -.11332811-03  -.70947395-04   .48112981+00
  .20484000-02  -.11030000-03
```

DYNAMIC RESPON

FREQUENCY	Z	THETA	Z1	THETA1	MAGNITUDE	P
.10000+03	.66000+02	.00000	.66000+02	.00000	.54099-06	.18
.10000+03	.66000+02	.00000	.88000+02	.00000	.45882-06	.18
.10000+03	.66000+02	.00000	.11000+03	.00000	.25281-06	.18
.10000+03	.88000+02	.00000	.66000+02	.00000	.45882-06	.18
.10000+03	.88000+02	.00000	.88000+02	.00000	.39959-06	.18
.10000+03	.88000+02	.00000	.11000+03	.00000	.22488-06	.18
.10000+03	.11000+03	.00000	.66000+02	.00000	.25281-06	.18
.10000+03	.11000+03	.00000	.88000+02	.00000	.22488-06	.18
.10000+03	.11000+03	.00000	.11000+03	.00000	.12864-06	.18
.11000+03	.66000+02	.00000	.66000+02	.00000	.75208-06	.18
.11000+03	.66000+02	.00000	.88000+02	.00000	.63756-06	.18
.11000+03	.66000+02	.00000	.11000+03	.00000	.35097-06	.18
.11000+03	.88000+02	.00000	.66000+02	.00000	.63756-06	.18
.11000+03	.88000+02	.00000	.88000+02	.00000	.55178-06	.18
.11000+03	.88000+02	.00000	.11000+03	.00000	.30884-06	.18
.11000+03	.11000+03	.00000	.66000+02	.00000	.35097-06	.18
.11000+03	.11000+03	.00000	.88000+02	.00000	.30884-06	.18
.11000+03	.11000+03	.00000	.11000+03	.00000	.17516-06	.18
.12000+03	.66000+02	.00000	.66000+02	.00000	.11772-05	.18
.12000+03	.66000+02	.00000	.88000+02	.00000	.99725-06	.19
.12000+03	.66000+02	.00000	.11000+03	.00000	.54822-06	.18
.12000+03	.88000+02	.00000	.66000+02	.00000	.99725-06	.18
.12000+03	.88000+02	.00000	.88000+02	.00000	.85719-06	.18
.12000+03	.88000+02	.00000	.11000+03	.00000	.47684-06	.18
.12000+03	.11000+03	.00000	.66000+02	.00000	.54822-06	.18
.12000+03	.11000+03	.00000	.88000+02	.00000	.47684-06	.18
.12000+03	.11000+03	.00000	.11000+03	.00000	.26783-06	.18
.13000+03	.66000+02	.00000	.66000+02	.00000	.22834-05	.19
.13000+03	.66000+02	.00000	.88000+02	.00000	.19322-05	.18
.13000+03	.66000+02	.00000	.11000+03	.00000	.10598-05	.18
.13000+03	.88000+02	.00000	.66000+02	.00000	.19322-05	.18
.13000+03	.88000+02	.00000	.88000+02	.00000	.16488-05	.18
.13000+03	.88000+02	.00000	.11000+03	.00000	.91073-06	.18
.13000+03	.11000+03	.00000	.66000+02	.00000	.10598-05	.18
.13000+03	.11000+03	.00000	.88000+02	.00000	.91073-06	.18
.13000+03	.11000+03	.00000	.11000+03	.00000	.50604-06	.18
.13500+03	.66000+02	.00000	.66000+02	.00000	.37086-05	.18
.13500+03	.66000+02	.00000	.88000+02	.00000	.31353-05	.18
.13500+03	.66000+02	.00000	.11000+03	.00000	.17166-05	.18
.13500+03	.88000+02	.00000	.66000+02	.00000	.31353-05	.18
.13500+03	.88000+02	.00000	.88000+02	.00000	.26653-05	.18
.13500+03	.88000+02	.00000	.11000+03	.00000	.14662-05	.19
.13500+03	.11000+03	.00000	.66000+02	.00000	.17166-05	.18
.13500+03	.11000+03	.00000	.88000+02	.00000	.14662-05	.19
.13500+03	.11000+03	.00000	.11000+03	.00000	.80989-06	.18
.14000+03	.66000+02	.00000	.66000+02	.00000	.74385-05	.19
.14000+03	.66000+02	.00000	.88000+02	.00000	.62792-05	.18
.14000+03	.66000+02	.00000	.11000+03	.00000	.34284-05	.18
.14000+03	.88000+02	.00000	.66000+02	.00000	.62792-05	.18
.14000+03	.88000+02	.00000	.88000+02	.00000	.53166-05	.18
.14000+03	.88000+02	.00000	.11000+03	.00000	.29105-05	.18
.14000+03	.11000+03	.00000	.66000+02	.00000	.34284-05	.19

3.3.14 Code Expansion

The following is a list of minimum storage requirements for running the code.

VARIABLES IN MAIN CODE

AA	(2)
AA1	(2)
AMN1	(2 · NMODES)
AMN2	(2 · NMODES)
ASM ASMST	$\left(\sum_{i=1}^{NMODES} NTH(i),\ NKZ_{max}\right)$
BB	(2 · NALP, NBETA)
BCAP	(NALP, NBETA)
BSM BSMST	$\left(\sum_{i=1}^{NMODES} NTH(i),\ NKZ_{max}\right)$
CC	(2 · NGAM, NEPS)
CCAP	(NGAM, NEPS)
COE	($NCOEE_{max}$, NMODES)
CON	(NMODES)
DD	(2)
DD1	(2)
ECAP	(NKZET, NETA)
EE	(2 · NKZET, NETA)
FF	(2 · NLAM, NSIG)
GHAT	(NFREQ, NMODES)
GHATS	(NFREQ, NMODES)
MCO	($NCOEE_{max}$, NMODES)
MCOEE	(NMODES)
NKZ	(NMODES)
NTH	(NMODES)

OM	(NFREQ)
OMEG1	(NMODES)
OMEG2	(NMODES)
P	$(4 \cdot NKZ_{max} \cdot NKZ_{max})$
PHI	(NOFZ • NTHA • NMODES)
TS	(NSS)
TTH	(NTHA)
WMN	(NMODES)
ZJUNK	(NMODES, NMODES)
ZKA	(NALP)
ZKALP	(2 • NALP)
ZKGAM	(NGAM)
ZKGM	(2 • NGAM)
ZKKS	(NSS)
ZKP	(NKZ_{max})
ZKPP	(NKZ_{max})
ZKQ	(NTH_{max})
ZKQQ	(NTH_{max})
ZKS	(2 • NSS)
ZKZET	(NKZET)
ZKZETA	(2 • NKZET)
* ZLC	(NB)
* ZLCS	(NB)
* ZLI	(NB, NB)
* ZLL	(NB)
* ZLLS	(NB, NB)
ZMASQ	(NMODES)
ZMCAP	(NSIG2, NETA2)
ZMM	(2 • NSIG2, NETA2)
ZMU	(2 • NETA2)
ZMUCAP	(NETA2)
ZNCAP	(NSIG1, NETA1)
ZNU	(2 • NETA1)
ZNUCAP	(NETA1)

(for the starred arrays ZLC through ZLLS) where NB = 1 + maximum of {NBETA, NEPS, NETA, NSIG}

VARIABLES IN FINDI

*	BCAP	(NALP, NBETA) or (NKZET, NETA) whichever is larger
*	CCAP	(NGAM, NEPS) or (NLAM, NSIG) whichever is larger .
*	FACT	(NBETA) or (NETA) whichever is larger.
*	SM2	(NBETA) or (NETA) whichever is larger.
*	SM4	(NBETA) or (NETA) whichever is larger.
*	ZKA	(1)
*	ZKGAM	(1)
*	ZLC	(1)
*	ZLG	(1)
*	ZLH	(1)
*	ZLI	(NB, 1)
*	ZLL	(1)

where NB is the largest of NBETA, NETA, NEPS, NSIG + 1

NOTE (*) variables may appear in Main and INIT.

NOTE: The following variables must be dimensioned the same to avoid errors in subroutine FINDI, (BCAP and ECAP). These arrays must be based on the largest of (NALP, NKZET), (NGAM, NLAM), (NBETA, NETA) and (NEPS, NSIG). The present form allows up to ten values for the above variables. The dimensions of ZLI should be based on the larger of (NBETA, NEPS, NETA, NSIG) + 1.

3.3.15 Code Listing

(following)

```
      COMPLEX ACAP,BCAP,CCAP,DCAP,ECAP,RESPS,ZKA,
     1 ZNCAP, ZNUCAP, OMEG1, OMEG2, PHAT, GHAT, CN, CM
     2,ASM,BSM,ASMST,BSMST,SM,CB1,CB2,GHATS
     3,ZKKS,ZJUNK,S2P, P, ZMCAP, ZMUCAP, A1CAP,ZKGAM, D1CAP, ZKZET,
     4 SM1
      COMMON /A1/NBETA,NEPS,NETA,NSIG
      COMMON /A2/ZLI(20,20)
      COMMON ZLL(20),ZLC(20),ZLLS(20),ZLCS(20)
     1,ZLLJ(20),ZLCJ(20),ZLLSJ(20),ZLCSJ(20)
      DIMENSION AA(2),BCAP(10,10),BB(20,10),CCAP(10,10),CC(20,10),DD(2),
     1 ECAP(10,10),EE(20,10),ZKA(20),
     2 ZKALP(40),TS(40), ZNCAP(10,10), ZMCAP(10,10), ZMM(20,10),
     3 ZNN(20,10), ZNUCAP(20), ZNU(40), OMEG1(50), AMN1(100), OMEG2(50),
     4 AMN2(100), ZKKS(20), ZKS(40), Z(100), TTH(100), WMN(50), OM(75)
     5,ZMAS(50),ASM(70,10),BSM(70,10),MCO(15,20),NTH(50),NKZ(50),CON(50)
     6,COE(15,30),NCOEE(30),ZKP(100),ZKQ(100),BSMST(70,10)
     7,ASMST(70,10),
     8 PHI(100),GHAT(75,25),P(800),S2P(800),ZMUCAP(20),ZMU(40),
     9 ZJUNK(20,20), GHATS(75,25), ZKPP(50), ZKQQ(50), AA1(2),
     * ZKGM(40), ZKGAM(20), DD1(2), ZKZETA(40), ZKZET(20)
      EQUIVALENCE (ACAP,AA), (BCAP,BB), (CCAP,CC), (DCAP,DD), (ECAP,EE),
     1 (ZKKS,ZKS),(ZKA,ZKALP),(ZKGAM,ZKGM),
     2 (ZNCAP,ZNN), (ZNUCAP,ZNU), (OMEG1,AMN1), (OMEG2,AMN2)
     3,(A1CAP,AA1),(D1CAP,DD1), (ZKZET,ZKZETA)
     4, (ZMCAP,ZMM), (ZMUCAP,ZMU)
      THNT=6.2831853
      THCON=57.29578
      CM=(0.,1.)
      PI2=6.28318530
      PI=3.14159265
    1 READ (5,31) NTHA,NOFZ,NFREQ,NMODES,NBETA,NETA,NALP,NKZET,NGAM,
     1 NEPS, NLAM, NSIG, NSS, NETA1, NSIG1, NETA2, NSIG2, ISHOT
     1,NCROS
      WRITE(6,38)
      WRITE(6,31) NTHA,NOFZ,NFREQ,NMODES,NBETA,NETA,NALP,NKZET,NGAM,
     1 NEPS, NLAM, NSIG, NSS, NETA1, NSIG1, NETA2, NSIG2, ISHOT
     1,NCROS
      LA=NALP*2
      LB=NBETA
      LD=2*NGAM
      LE=NEPS
      IF(LA.LT.2) LA=2
      IF(LB.LT.1) LB=1
      IF(LD.LT.2) LD=2
      IF(LE.LT.1) LE=1
      READ (5,32) ZNOT,(Z(I),I=1,NOFZ)
      WRITE(6,34) ZNOT,(Z(I),I=1,NOFZ)
      READ (5,32) RCYL,(TTH(I),I=1,NTHA)
      WRITE(6,34) RCYL,(TTH(I),I=1,NTHA)
      READ (5,32) AA,AA1,((BB(L,J),L=1,LA),J=1,LB),((CC(L1,L2),L1=1,LD),
     1L2=1,LE),(ZKALP(K),K=1,LA),(ZKGM(KK),KK=1,LD)
      WRITE(6,34) AA,AA1,((BB(L,J),L=1,LA),J=1,LB),((CC(L1,L2),L1=1,LD),
     1 L2=1,LE),(ZKALP(K),K=1,LA),(ZKGM(KK),KK=1,LD)
      LA=2*NKZET
```

```
      LB=NETA+1
      IF(LA.LT.2) LA=2
      IF(LB.LT.1) LB=1
      READ (5,32) DD,DD1,((EE(L,J),L=1,LA),J=1,LB),
     1(ZKZETA(K),K=1,LA)
      WRITE(6,34) DD,DD1,((EE(L,J),L=1,LA),J=1,LB),
     1(ZKZETA(K),K=1,LA)
      LJ=NSS
      LA=NETA1
      LB=NSIG1*2
      IF(LJ.LT.1) LJ=1
      LK=2*LJ
      IF(LA.LT.1) LA=1
      LC=2*LA
      IF(LB.LT.2) LB=2
      LE=NETA2
      LF=NSIG2*2
      IF(LE.LT.1) LE=1
      LG=2*LE
      IF(LF.LT.2)LF=2
      READ (5,32) HCAP,(ZKS(I),I=1,LK),(TS(J),J=1,LJ),((ZNN(L,K),L=1,LB)
     1,K=1,LA),(ZNU(M),M=1,LC),((ZMM(L1,L2),L1=1,LF),L2=1,LE),(ZMU(L3),L
     23=1,LG)
      WRITE(6,34) HCAP,(ZKS(I),I=1,LK),(TS(J),J=1,LJ),((ZNN(L,K),L=1,LB)
     1,K=1,LA),(ZNU(M),M=1,LC),((ZMM(L1,L2),L1=1,LF),L2=1,LE),(ZMU(L3),L
     23=1,LG)
      LJ=2*NMODES
      READ (5,32) (AMN1(I),I=2,LJ,2)
      WRITE(6,34) (AMN1(I),I=2,LJ,2)
      READ (5,32) (WMN(I),I=1,NMODES)
      WRITE(6,34) (WMN(I),I=1,NMODES)
      READ (5,32) (OM(I),I=1,NFREQ)
      WRITE(6,34) (OM(I),I=1,NFREQ)
      DO 14 I=1,NFREQ
   14 OM(I)=PI2*OM(I)
      J=0
      DO 15 I=1,NMODES
      WMN(I)=WMN(I)*PI2
      J=J+2
   15 AMN1(J)=AMN1(J)*WMN(I)
      CD=PI/ZNOT
      DO 42 I=1,NTHA
   42 TTH(I)=TTH(I)*.0174532925
      READ (5,32) (ZMAS(I),I=1,NMODES)
      WRITE(6,34) (ZMAS(I),I=1,NMODES)
      L=-1
      DO 20 I=1,NMODES
      L=L+2
      AMN1(L)=WMN(I)
      AMN2(L)=-WMN(I)
   20 AMN2(L+1)=AMN1(L+1)
      RR2=RCYL**2
      DO 43 I=1,NFREQ
      DO 43 NM=1,NMODES
      GHATS(I,NM)=1./((OM(I)+OMEG1(NM))*(OM(I)+OMEG2(NM))*ZMAS(NM))
```

```
   43 GHAT(I,NM)=-1./((OM(I)-OMEG1(NM))*(OM(I)-OMEG2(NM))*ZMAS(NM))
      CALL INIT(ZNOT,ZLL,ZLC,ZLLJ,ZLCJ)
      CALL INIT(PI,ZLLS,ZLCS,ZLLSJ,ZLCSJ)
      MAXK=-1.
      MAXQ=-1.
      LC3=0
      DO 102 NM=1,NMODES
      READ (5,31) NTH(NM),NKZ(NM),NCOE,(MCO(L,NM),L=1,NCOE)
      WRITE(6,31) NTH(NM),NKZ(NM),NCOE,(MCO(L,NM),L=1,NCOE)
      READ (5,30) CON(NM),(COE(L,NM),L=1,NCOE)
      WRITE(6,30) CON(NM),(COE(L,NM),L=1,NCOE)
      CO=.25/SQRT(ZMAS(NM))
      NCOEE(NM)=NCOE
      IF(ISHOT.GT.1) GO TO 102
      NZS=NKZ(NM)
      NZS1 = NZS - 1
      NTHS=NTH(NM)
      NTHS1 = NTHS - 1
      DO 110 K=1,NZS
      I=K+LC3
      DO 110 J=1,NTHS
      ASMST(I,J)=(0.,0.)
      BSMST(I,J)=(0.,0.)
      BSM  (I,J)=(0.,0.)
  110 ASM  (I,J)=(0.,0.)
      CA=CON(NM)*CO
      ASM(LC3+1,1)=CA
      BSM(LC3+1,1)=CA
      DO 6 LL=1,NCOE
      L=(MCO(LL,NM)-1)/NKZ(NM)+1
      M=(L-1)/NTH(NM)+1
      L1=MOD(L-1,NTH(NM))
      L2=MOD(MCO(LL,NM)-1,NKZ(NM))
      D1=L1
      D2=L2
      L3=L2+1+LC3
      L4=L1+1
      CB=COE(LL,NM)
C     WRITE(6,31) NM,L1,L2,L3,L4,L,M,LL
      GO TO (8,9,10,11),M
    8 ASM(L3,L4)=ASM(L3,L4)+CB
      BSM(L3,L4)=BSM(L3,L4)+CB
      ASMST(L3,L4)=ASMST(L3,L4)+CB
      BSMST(L3,L4)=BSMST(L3,L4)+CB
      GO TO 7
    9 CN=CM*CB
      ASM(L3,L4)=ASM(L3,L4)-CN
      BSM(L3,L4)=BSM(L3,L4)+CN
      ASMST(L3,L4)=ASMST(L3,L4)+CN
      BSMST(L3,L4)=BSMST(L3,L4)-CN
      GO TO 7
   10 CN=CM*CB
      ASM(L3,L4)=ASM(L3,L4)-CN
      BSM(L3,L4)=BSM(L3,L4)-CN
      ASMST(L3,L4)=ASMST(L3,L4)+CN
```

```
      BSMST(L3,L4)=BSMST(L3,L4)+CN
      GO TO 7
   11 ASM(L3,L4)=ASM(L3,L4)-CB
      BSM(L3,L4)=BSM(L3,L4)+CB
      ASMST(L3,L4)=ASMST(L3,L4)-CB
      BSMST(L3,L4)=BSMST(L3,L4)+CB
    7 CONTINUE
    6 CONTINUE
      DO 18 IP=1,NZS1
      L3=IP+1+LC3
      DO 18 IT=1,NTHS1
      L4=IT+1
      ASM(L3,L4)=ASM(L3,L4)*CO
      BSM(L3,L4)=BSM(L3,L4)*CO
      ASMST(L3,L4)=ASMST(L3,L4)*CO
   18 BSMST(L3,L4)=BSMST(L3,L4)*CO
      ASMST(LC3+1,1)=ASM(LC3+1,1)
      BSMST(LC3+1,1)=BSM(LC3+1,1)
      LC3=LC3+NZS
      IF(NZS.GT.MAXK) MAXK=NZS
      IF(NTHS.GT.MAXQ)MAXQ=NTHS
  102 CONTINUE
      IF(ISHOT.LE.1) GO TO 199
      DO 198 NM=1,NMODES
  198 READ (5,30) (ZJUNK(NM,NMM),NMM=1,NMODES)
      GO TO 321
  199 ZB=0.
      DO 265 I=1,MAXK
      ZKP(I)=ZB
      ZKPP(I)=ZB
  265 ZB=ZB+CD
      DO 201 I=1,MAXQ
      ZKQ(I)=FLOAT(I-1)
  201 ZKQQ(I)=ZKQ(I)
      IL=0
      DO 202 I=1,MAXK
      DO 204 M=1,2
      ZP=ZKP(I)
      DO 203 L=1,2
      DO 203 J=1,MAXK
      ZR=ZKPP(J)
      IL=IL+1
      CALL FINDI(ZP,ZR,ACAP,A1CAP,BCAP,CCAP,ZKA,NALP,NBETA,ZKGAM,NGAM,
     1NEPS,ZNOT,P(IL),ZLL,ZLC,ZLLJ,ZLCJ)
  203 ZKPP(J)=-ZKPP(J)
  204 ZKP(I)=-ZKP(I)
  202 CONTINUE
      IL=0
      DO 722 I=1,MAXQ
      DO 724 M=1,2
      ZP=ZKQ(I)
      DO 723 L=1,2
      DO 723 J=1,MAXQ
      ZR=ZKQQ(J)
      IL=IL+1
```

```
      CALL FINDT(ZP,ZR,DCAP,D1CAP,ECAP,ZKZET,NKZET,NETA,PI,S2P(IL),
     1ZLLS,ZLCS,ZLLSJ,ZLCSJ)
  723 ZKQQ(J)=-ZKQQ(J)
  724 ZKQ(I)=-ZKQ(I)
  722 CONTINUE
      IN=0
      NZ42=4*MAXK
      NT42=4*MAXQ
      DO 300 NM=1,NMODES
      IN2=0
      NP=NKZ(NM)
      NQ=NTH(NM)
      IN3=0
      IN1=0
      DO 301 NMM=1,NMODES
      NR=NKZ(NMM)
      NR2=MAXK+MAXK
      NR3=NR2+MAXK
      NQS=NTH(NMM)
      NQS2=MAXQ+MAXQ
      NQS3=NQS2+MAXQ
      SM=(0.,0.)
      IM=IN
      IN2=0
      DO 305 IP=1,NP
      IP1=IN2
      IM1=IN1
      IM=IM+1
      DO 304 IR=1,NR
      IP1=IP1+1
      IP11=IP1+MAXK
      IP12=IP1+NR2
      IP13=IP1+NR3
      IM1=IM1+1
      IN3=0
      DO 303 IQ=1,NQ
      IS1=IN3
      DO 302 IS=1,NQS
      IS1=IS1+1
      IS11=IS1+MAXQ
      IS12=IS1+NQS2
      IS13=IS1+NQS3
      SM=SM+ASM(IM,IQ)*(ASM(IM1,IS)*P(IP1)*S2P(IS1)+ASMST(IM1,IS)*P(IP11
     1 )*S2P(IS11) +BSM(IM1,IS)*P(IP11)*S2P(IS1)+BSMST(IM1,IS)*
     2P(IP1)*S2P(IS11))+BSM(IM,IQ)*(BSM(IM1,IS)*P(IP13)*S2P(IS1)+
     3 BSMST(IM1,IS)*P(IP12)*S2P(IS11)+ASM(IM1,IS)*P(IP12)*S2P(IS1)
     4+ASMST(IM1,IS)*P(IP13)*S2P(IS11))+ASMST(IM,IQ)*(ASMST(IM1,IS)*
     5P(IP13)*S2P(IS13)+ASM(IM1,IS)*P(IP12)*S2P(IS12)+BSMST(IM1,IS)*P(
     5IP12)*S2P(IS13)+BSM(IM1,IS)*P(IP13)*S2P(IS12))+BSMST(IM,IQ)*(BSMST
     6(IM1,IS)*P(IP1)*S2P(IS13)+BSM(IM1,IS)*P(IP11)*S2P(IS12)+ASMST(
     7IM1,IS)*P(IP11)*S2P(IS13)+ASM(IM1,IS)*P(IP1)*S2P(IS12))
  302 CONTINUE
      IN3=IN3+NT42
  303 CONTINUE
  304 CONTINUE
```

```
      IN2=IN2+NZ42
  305 CONTINUE
      ZJUNK(NM,NMM)=SM
      IN1=IN1+NR
  301 CONTINUE
      IN=IN+NP
  300 CONTINUE
      DO 322 I=1,NMODES
      PUNCH 30,(ZJUNK(I,J),J=1,NMODES)
  322 WRITE(6,34) (ZJUNK(I,J),J=1,NMODES)
  321 WRITE(6,37)
      NN=0
      DO 200 IB=1,NOFZ
      ZZ=Z(IB)*CD
      DO 210 JB=1,NTHA
      TH=TTH(JB)
      DO 507 NM=1,NMODES
      NN=NN+1
      PSI=CON(NM)
      NCOE=NCOEE(NM)
      DO 506 LL=1,NCOE
      L=(MCO(LL,NM)-1)/NKZ(NM)+1
      M=(L-1)/NTH(NM)+1
      L1=MOD(L-1,NTH(NM))
      L2=MOD(MCO(LL,NM)-1,NKZ(NM))
      D1=L1
      D2=L2
C     WRITE(6,31) NM,L1,L2,L3,L4,L,M,LL
      GO TO (2,3,4,5),M
    2 PSI=PSI+COS(D1*TH)*COE(LL,NM)*COS(D2*ZZ)
      GO TO 506
    3 PSI=PSI+COS(D1*TH)*COE(LL,NM)*SIN(D2*ZZ)
      GO TO 506
    4 PSI=PSI+SIN(D1*TH)*COE(LL,NM)*COS(D2*ZZ)
      GO TO 506
    5 PSI=PSI+SIN(D1*TH)*COE(LL,NM)*SIN(D2*ZZ)
  506 CONTINUE
C     WRITE(6,35) IB,JB,ZZ,TH,PSI
  507 PHI(NN)=PSI
  210 CONTINUE
  200 CONTINUE
      WRITE(6,36)
      DO 220 LB=1,NFREQ
      SM=(0.,0.)
      PHAT=HCAP
      IF(NSS.LT.1) GO TO 224
      DO 221 IS=1,NSS
  221 PHAT=PHAT+ZKKS(IS)*CEXP(-OM(LB)*CM*TS(IS))
  224 IF(NETA1.LT.1) GO TO 825
      DO 222 I=1,NETA1
      CB1=1./(OM(LB)-ZNUCAP(I))
      CB2=CB1
      SM=ZNCAP(1,I)*CB1
      IF(NSIG1.LT.2) GO TO 222
      DO 223 J=2,NSIG1
```

```
      CB2=CB2*CB1
  223 SM=SM+ZNCAP(J,I)*CB2
  222 CONTINUE
  825 IF(NETA2.LT.1) GO TO 826
      DO 822 I=1,NETA2
      CB1=1./(OM(LB)-ZMUCAP(I))
      CB2=CB1
      SM=SM+ZMCAP(1,I)*CB1
      IF(NSIG2.LT.2) GO TO 822
      DO 823 J=2,NSIG2
      CB2=CB2*CB1
  823 SM=SM+ZMCAP(J,I)*CB2
  822 CONTINUE
  826 PHAT=(PHAT+SM)*RR2
  225 NMDX=NMODES*NOFZ*NTHA
      IP=0
      DO 913 IB=1,NMDX,NMODES
      IP=IP+1
      IP2=(IP-1)/NTHA+1
      IP4=MOD(IP-1,NTHA)+1
      IP1=0
      DO 914 JB=1,NMDX,NMODES
      IP1=IP1+1
      IP3=(IP1-1)/NTHA+1
      IP5=MOD(IP1-1,NTHA)+1
      SM=(0.,0.)
      IL=IB
      IF(NCROS.EQ.1) GO TO 918
      DO 915 LZ=1,NMODES
      JL=JB
      SM1=(0.,0.)
      DO 916 LY=1,NMODES
  134 FORMAT(4I5,8E12.6)
      SM1=SM1+PHI(JL)*GHATS(LB,LY)*ZJUNK(LZ,LY)
C     WRITE(6,134) LB,LY,LZ,JL,PHI(JL),SM1,GHATS(LB,LY),ZJUNK(LZ,LY)
      JL=JL+1
  916 CONTINUE
      SM=SM+SM1*PHI(IL)*GHAT(LB,LZ)
C     WRITE(6,134) LB,LB,IL,LZ,SM,SM1,PHI(IL),GHAT(LB,LZ)
      IL=IL+1
  915 CONTINUE
      GO TO 919
  918 DO 920 LZ=1,NMODES
      JL=JB+LZ-1
      SM=SM+PHI(JL)*GHATS(LB,LZ)*ZJUNK(LZ,LZ)*PHI(IL)*GHAT(LB,LZ)
      IL=IL+1
  920 CONTINUE
  919 RESPS=SM*PHAT
C     WRITE(6,34)  SM,PHAT,RESPS
      X=REAL(RESPS)
      Y=AIMAG(RESPS)
      IF(ABS(X).GT.1.E-16) GO TO 243
      PHASE=1.5707963
      IF(Y.LT.0.) PHASE=PHASE+PI
      GO TO 242
```

```
243 PHASE=ATAN(Y/X)
    IF(X*Y) 240,241,241
241 IF(X.LT.0.)PHASE=PHASE+PI
    GO TO 242
240 PHASE=PHASE+PI
    IF(X.GT.0.) PHASE=PHASE+PI
242 WMAG=SQRT(X**2+Y**2)
    PHASE=PHASE*THCON
    OMM=OM(LB)/PI2
    TT1=TTH(IP4)*THCON
    TT2=TTH(IP5)*THCON
    WRITE(6,34) OMM,Z(IP2),TT1,Z(IP3),TT2,WMAG,PHASE
914 CONTINUE
913 CONTINUE
220 CONTINUE
    GO TO 1
 30 FORMAT(5E15.8)
 31 FORMAT(16I5)
 32 FORMAT(8E10.3)
 34 FORMAT(1X,10E12.5)
 35 FORMAT(2I5,9E12.5/(1X,10E12.5))
 36 FORMAT(1H1,64X,16HDYNAMIC RESPONSE /4X,9HFREQUENCY,6X,1HZ,8X,
   15HTHETA,9X,2HZ1,8X,6HTHETA1,6X,9HMAGNITUDE,5X,5HPHASE)
 37 FORMAT(24H0***END OF INPUT DATA***//)
 38 FORMAT(17H1***INPUT DATA***)
    END
```

```
@FOR,IS INIT
      SUBROUTINE INIT(ZL,ZLL,ZLC,ZLLJ,ZLCJ)
      COMMON/A1/NBETA,NEPS,NETA,NSIG
      COMMON /A2/ZLI(20,20)
      DIMENSION ZLL(1),ZLC(1),ZLLJ(1),ZLCJ(1)
      NB=MAX0(NBETA,NEPS,NETA,NSIG)+1
      ZL1=1.
      ZL1J=1.
      ZN=-1.
      DO 10 I=1,NB
      ZLLJ(I)=ZL1J
      ZLL(I)=ZL1
      ZLCJ(I)=ZL1J*ZN
      ZLC(I)=ZL1*ZN
      ZLI(I,1)=ZN
      FL=ZN
      IB=I
      DO 11 L=2,NB
      FL=FL-1.
      ZLI(IB+1,L)=ZLI(IB,L-1)*FL
   11 IB=IB+1
      ZL1=ZL1*ZL
      ZL1J=-ZL1J*ZL
```

```
      ZN=ZN-1.
   10 CONTINUE
      RETURN
      END
@FOR,IS FINDIT
      SUBROUTINE FINDI(ZKP,ZKR,ACAP,A1CAP,BCAP,CCAP,ZKA,NALP,NBETA,
     1ZKGAM,NGAM,NEPS,ZL,FUNI,ZLL,ZLC,ZLLJ,ZLCJ)
      COMPLEX ACAP,A1CAP,BCAP,CCAP,ZKA,FUNI,ZZ1,ZZ2,Z3,ZLH,CC,Z4,
     1 SMAL,CON1,CON2,CB,ZLG,CON3,SM,SM1,SM2,SM3,ZZ3,
     2SMGM,     SN,CON6,CON7,ZTL,CON4,CD,ZKGAM,SM4
     3,CON5
      COMMON /A2/ZLT(20,20)
      DIMENSION ZLG(20),ZLH(20),SM4(20),FACT(11),BCAP(10,10),
     1CCAP(10,10),ZKA(20),SM2(20),ZKGAM(20)
     1,ZLL(1),ZLC(1),ZLLJ(1),ZLCJ(1)
      DATA FACT/1.,1.,2.,6.,24.,120.,720.,5040.,40320.,362880.,3628800./
      CD=(1.,0.)
      CC=(0.,1.)
      ZZ3=(ZKP+ZKR)*(0.,1.)
      CON2=CEXP(ZZ3*ZL)
      IF(ZKP+ZKR)10,6,10
    6 CON3=ZL
      GO TO 11
   10 CON3=(CON2-CD)/ZZ3
   11 SMAL=(0.,0.)
      DO 100 IAL=1,NALP
      ZZ1=(ZKR-ZKA(IAL))*(0.,-1.)
      ZZ2=(ZKP+ZKA(IAL))*(0., 1.)
      CON1=CEXP(ZZ2*ZL)
      Z3=1./ZZ1
      Z4=1./ZZ2
      ZZ1=Z3
      ZZ2=Z4
      DO 12 I=1,NBETA
      ZLG(I)=Z3
      ZLH(I)=Z4
      Z3=Z3*ZZ1
      Z4=Z4*ZZ2
   12 CONTINUE
      CB=CC
      SM1=(0.,0.)
      DO 106 II=1,NBETA
  106 SM2(II)=(0.,0.)
      SM3=(0.,0.)
      CN=-1.
      ZI=0.
      DO 102 I=1,NBETA
      IF(I.LT.2) GO TO 104
      ZI=ZI-1.
      SM1=SM1*ZI*ZLH(1)+ZLG(I-1)*ZLH(2)
      ZIL=ZI*ZLG(1)
      I1=I-2
      J=I-1
      IF(I1.LE.1) GO TO 107
```

```
      DO 105 II=1,I1
      SM2(J)=SM2(J-1)*ZIL
  105 J=J-1
  107 SM2(1)=ZLT(I-1,1)*ZLG(2)
      IF(I.LT.3) GO TO 104
      SM3=SM3*(ZI+1.)*ZLH(1)+ZLC(I-2)*ZLH(2)
  104 SM4(I)=SM3
      CN=-CN
      CNN=CN
      I1=I-1
      SN=(0.,0.)
      IF(I1.LT.1) GO TO 109
      L1=I+1
      L2=I
      DO 108 J=1,I1
      LI=L1-I
      CNN=-CNN
      L2=L2-I
      SN=SN+SM2(J)*(CON1*(ZLL(L2)*ZLH(1)+SM4(L1))-CNN*ZLH(L2)*FACT(L2))
  108 CONTINUE
  109 SM=ZLG(1)*(CON1*(ZLL(I)*ZLH(1)+SM1)-CN*ZLH(I)*FACT(I))+SN
     1-CN*ZLG(I)*CON3*FACT(I)
      SMAL=SMAL+BCAP(IAL,I)*SM*CB/FACT(I)
      CB=CB*CC
  102 CONTINUE
  100 CONTINUE
      SMGM=(0.,0.)
      DO 200 IGM=1,NGAM
      ZZ2=(ZKP+ZKGAM(IGM))*CC
      ZZ1=(ZKR-ZKGAM(IGM))*(0.,-1.)
      CON1=CEXP(ZZ2*ZL)
      CON4=CEXP(ZZ1*ZL)
      CON5=CEXP(ZL*CC*(ZKR-ZKGAM(IGM)))
      Z3=1./ZZ1
      Z4=1./ZZ2
      ZZ1=Z3
      ZZ2=Z4
      DO 212 I=1,NEPS
      ZLG(I)=Z3
      ZLH(I)=Z4
      Z3=Z3*ZZ1
      Z4=Z4*ZZ2
  212 CONTINUE
      CB=CC
      SM1=(0.,0.)
      DO 206 II=1,NEPS
  206 SM2(II)=(0.,0.)
      SM3=(0.,0.)
      ZI=0.
      CN=-1.
      DO 202 I=1,NEPS
      IF(I.LT.2) GO TO 204
      ZI=ZI-1.
      SMI=SM1*ZI*ZLH(1)+ZLCJ(I-1)*ZLH(2)
```

```
      ZIL=ZI*ZLG(1)
      I1=I-2
      J=I-1
      IF(I1.LE.1) GO TO 207
      DO 205 II=1,I1
      SM2(J)=SM2(J-1)*ZIL
  205 J=J-1
  207 SM2(1)=ZLI(I-1,1)*ZLG(2)
      IF(I.LT.3) GO TO 204
      SM3=SM3*(ZI+1.)*ZLH(1)+ZLCJ(I-2)*ZLH(2)
  204 SM4(I)=SM3
      CN=-CN
      CNN=CN
      I1=I-1
      SN=(0.,0.)
      IF(I1.LT.1) GO TO 209
      L1=I+1
      L2=I
      DO 208 J=1,I1
      L1=L1-1
      CNN=-CNN
      L2=L2-1
      SN=SN+SM2(J)*(CON5*(ZLLJ(L2)*ZLH(1)+SM4(L1))-CNN*ZLH(L2)*FACT(L2)
     1*CON2)
  208 CONTINUE
  209 SM=ZLG(1)*(CON5*(ZLLJ(I)*ZLH(1)+SM1)-CN*ZLH(I)*FACT(I))+SN+CN*ZLG(
     1I)*CON3*FACT(I)
      SMGM=SMGM+CCAP(IGM,I)*SM*CB/FACT(I)
      CB=CB*CC
  202 CONTINUE
  200 CONTINUE
      CON6=CEXP(ZKP*ZL*CC)
      CON7=CEXP(ZKR*ZL*CC)
      IF(ZKP)241,240,241
  240 CON6=CC*ZL
      GO TO 242
  241 CON6=(CON6-CD)/ZKP
  242 IF(ZKR)244,243,244
  243 CON6=CC*ZL*CON6
      GO TO 245
  244 CON6=(CON7-CD)*CON6/ZKR
  245 FUNI=A1CAP*CON3-ACAP*CON6+SMAL-SMGM
      RETURN
      END
@FOR,IS FINDT
      SUBROUTINE FINDT(ZKP,ZKR,ACAP,A1CAP,BCAP,ZKA,NALP,NBETA,ZL,FUNI,
     1ZLL,ZLC,ZLLJ,ZLCJ)
      COMPLEX ACAP,A1CAP,BCAP,ZKA,FUNI,ZZ1,ZZ2,Z3,ZLH,CC,Z4,
     1 SMAL,CON1,CON2,ZLG,CON3,SM,SM1,SM2,SM3,ZZ3,
     2 SM6,SN,CON6,CON7,ZIL,CD,SM4
     1,YL,SM5,CON8,SM8,SM9,SM7,YZ1,YZ2,Y3,Y4,YON1,YLG,YLH,YIL,
     2 SM10,YON3,YN,YBON,YBON1,YM,CON11
      COMMON /A2/ZLI(20,20)
      DIMENSION ZLG(20),ZLH(20),SM4(20),FACT(11),BCAP(10,10),
```

```
      1 ZKA(20),SM2(20)
      1,ZLL(1),ZLC(1),ZLLJ(1),ZLCJ(1)
      1,SM7(20),YLG(20),YLH(20),SM10(20)
      DATA FACT/1.,1.,2.,6.,24.,120.,720.,5040.,40320.,362880.,3628800./
      CD=(1.,0.)
      CC=(0.,1.)
      YBON=CEXP(2.*ZL*ZKR*(0.,1.))
      YBON1=CD/YBON
      ZZ3=(ZKP+ZKR)*(0.,1.)
      CON2=CEXP(ZZ3*ZL)
      NBET1=NBETA+1
      IF(ZKP+ZKR)10,6,10
    6 CON3=ZL
      CON11=ZL*2.
      GO TO 11
   10 CON3=(CON2-CD)/ZZ3
      CON11=(CEXP(ZZ3*ZL*2.)-CD)/ZZ3
   11 SMAL=(0.,0.)
      DO 100 IAL=1,NALP
      ZZ1=(ZKP-ZKA(IAL))*(0.,-1.)
      ZZ2=(ZKP+ZKA(IAL))*(0., 1.)
      CON1=CEXP(ZZ2*ZL)
      CON8=CEXP(ZZ1*ZL)
      Z3=1./ZZ1
      Z4=1./ZZ2
      ZZ1=Z3
      ZZ2=Z4
      YZ1=(ZKP-ZKA(IAL))*(0.,-1.)
      YZ2=(ZKR+ZKA(IAL))*(0.,1.)
      YON1=CEXP(YZ2*ZL)
      Y3=1./YZ1
      Y4=1./YZ2
      YON8=CEXP(YZ1*ZL)
      YZ1=Y3
      YZ2=Y4
      DO 12 I=1,NBET1
      ZLG(I)=Z3
      ZLH(I)=Z4
      Z3=Z3*ZZ1
      Z4=Z4*ZZ2
      YLG(I)=Y3
      YLH(I)=Y4
      Y3=Y3*YZ1
      Y4=Y4*YZ2
   12 CONTINUE
      SM1=(0.,0.)
      SM5=(0.,0.)
      SM8=(0.,0.)
      SM9=(0.,0.)
      DO 106 II=1,NBET1
      SM7(II)=(0.,0.)
  106 SM2(II)=(0.,0.)
      SM3=(0.,0.)
      SM6=(0.,0.)
```

```
      CN=-1.
      ZI=0.
      DO 102 IJ=1,NBET1
      I=IJ-1
      IF(I.GE.1) GO TO 103
      SM=ZLG(1)*((CON1-CD)*ZLH(1)*(CD-YBON)+CON3*(CON8*(CON2+YBON)-(CD+
     1CON2)))
      YM=CON3* (YON1*(CD+YBON1*CON2)-CD)*YLH(1)+CON2*(CON2*YLG(1)*
     2(YON8-CD)*(CD-YBON1)-CON3)*YLH(1)
      GO TO 115
  103 ZI=ZI-1.
      SM1=SM1*ZI*ZLH(1)+ZLC(I)*ZLH(2)
      SM5=SM5*ZI*ZLG(1)+ZLC(I)*ZLG(2)
      SM9=SM9*ZI*YLH(1)+ZLC(I)*YLH(2)
      SM8=SM8*ZI*YLG(1)+ZLC(I)*YLG(2)
      ZIL=ZI*ZLG(1)
      YIL=ZI*YLH(1)
      I1=I-1
      J=I
      IF(I1.LE.1) GO TO 107
      DO 105 II=1,I1
      SM2(J)=SM2(J-1)*ZIL
      SM7(J)=SM7(J-1)*YIL
  105 J=J-1
  107 SM2(1)=ZLI(I,1)*ZLG(2)
      SM7(1)=ZLI(I,1)*YLH(2)
      IF(I.LT.2) GO TO 104
      SM3=SM3*(ZI+1.)*ZLH(1)+ZLC(I-1)*ZLH(2)
      SM6=SM6*(ZI+1.)*YLG(1)+ZLC(I-1)*YLG(2)
  104 SM4(I)=SM3
      SM10(I)=SM6
      CN=-CN
      CNN=CN
      I1=I
      SN=(0.,0.)
      YN=(0.,0.)
      L1=I1+1
      L2=I+1
      IF(I1.LT.1) GO TO 109
      DO 108 J=1,I1
      L1=L1-1
      CNN=-CNN
      L2=L2-1
      SN=SN+SM2(J)*(CON1*(ZLL(L2)*ZLH(1)+SM4(L1))+CNN*ZLH(L2)*FACT(L2))
      YN=YN+SM7(J)*(YON8*(ZLL(L2)*YLG(1)+SM10(L1))+CNN*YLG(L2)*FACT(L2))
  108 CONTINUE
  109 SM=(ZLG(1)*(CON1*(ZLL(I+1)*ZLH(1)+SM1)+CN*ZLH(I+1)*FACT(I+1))+SN)
     1*(CD-YBON)+CON3*CON8*(ZLL(I+1)*ZLG(1)+SM5)*(CON2+YBON)+CN*ZLG(I+1)
     2*CON3*(CD+CON2)*FACT(I+1)
      YM=CON3*(YON1*(ZLL(I+1)*YLH(1)+SM9)*(CD+YBON1*CON2)+CN*YLH(I+1)*
     1FACT(I+1))+CON2*(YN*CON2*(CD-YBON1)+CN*YLH(I+1)*CON3*FACT(I+1)
     2+CON2*YLH(1)*(YON8*(ZLL(I+1)*YLG(1)+SM8)+CN*YLG(I+1)*FACT(I+1))
     3*(CD-YBON1))
  115 SMAL=SMAL+BCAP(IAL,IJ)*(SM+YM)
```

```
  102 CONTINUE
  100 CONTINUE
      YL=2.*ZL*CC
      CON6=CEXP(ZKP*YL)
      CON7=CEXP(ZKR*YL)
      IF(ZKP)241,240,241
  240 CON6=CC*ZL*2.
      GO TO 242
  241 CON6=(CON6-CD)/ZKP
  242 IF(ZKR)244,243,244
  243 CON6=CC*ZL*2.*CON6
      GO TO 245
  244 CON6=(CON7-CD)*CON6/ZKR
  245 FUNT=A1CAP*CON11-ACAP*CON6+SMAL
      RETURN
      END
@MAP,IS ,ABS
  IN RANCYL
@XQT ABS
```

4.0 SAMPLE PROBLEM - COMPARISON WITH NASTRAN CODE

In this section, we apply the results of this study to a specific engineering problem in order to test the usefulness, accuracy and efficiency of the developed approach as compared to strictly numerical methods. The NASTRAN code is selected as representative of such methods. The problem consists of a ring-stiffened cylindrical shell excited by a random pressure field whose correlation properties resemble those of pressure fluctuations beneath a turbulent boundary layer, Figure 2.

Two discrete models of the shell were constructed:* Model 1, having 360 degrees of freedom, is shown in Figures 3, 4; Model 2 is a more detailed one (720 degrees of freedom) and is shown in Figure 5. The NASTRAN eigenvalue routine was used to obtain the corresponding normal frequencies, modal masses and values of the associated mode shapes for the first few modes. Table 1 lists the computed natural frequencies for both shell models, while Figures 6 and 7 show typical NASTRAN plots of the deformed shell in its natural modes. The solution to the eigenvalue problem is common to both the present approach and the NASTRAN calculation. At this point, the present method departs from numerical procedures. The NASTRAN-generated mode-shape data was first curve-fitted using the FITMSC code (Section 3.1) which produced values for the coefficients in Equation (19) (see Figure 1). Typical FITMSC plots are shown in Figures 8 and 9. The points in these figures are the NASTRAN values. Each curve represents a cross-section of the shell. For clarity in presentation, the curves have been displaced relative to each other. The undeformed ends of the shell are shown by the inner and outer circles in each plot.

---

*The modeling was performed by Mr. G. Jones, Goddard Space Flight Center, who is also responsible for the NASTRAN calculation of the eigenvalue and response problems.

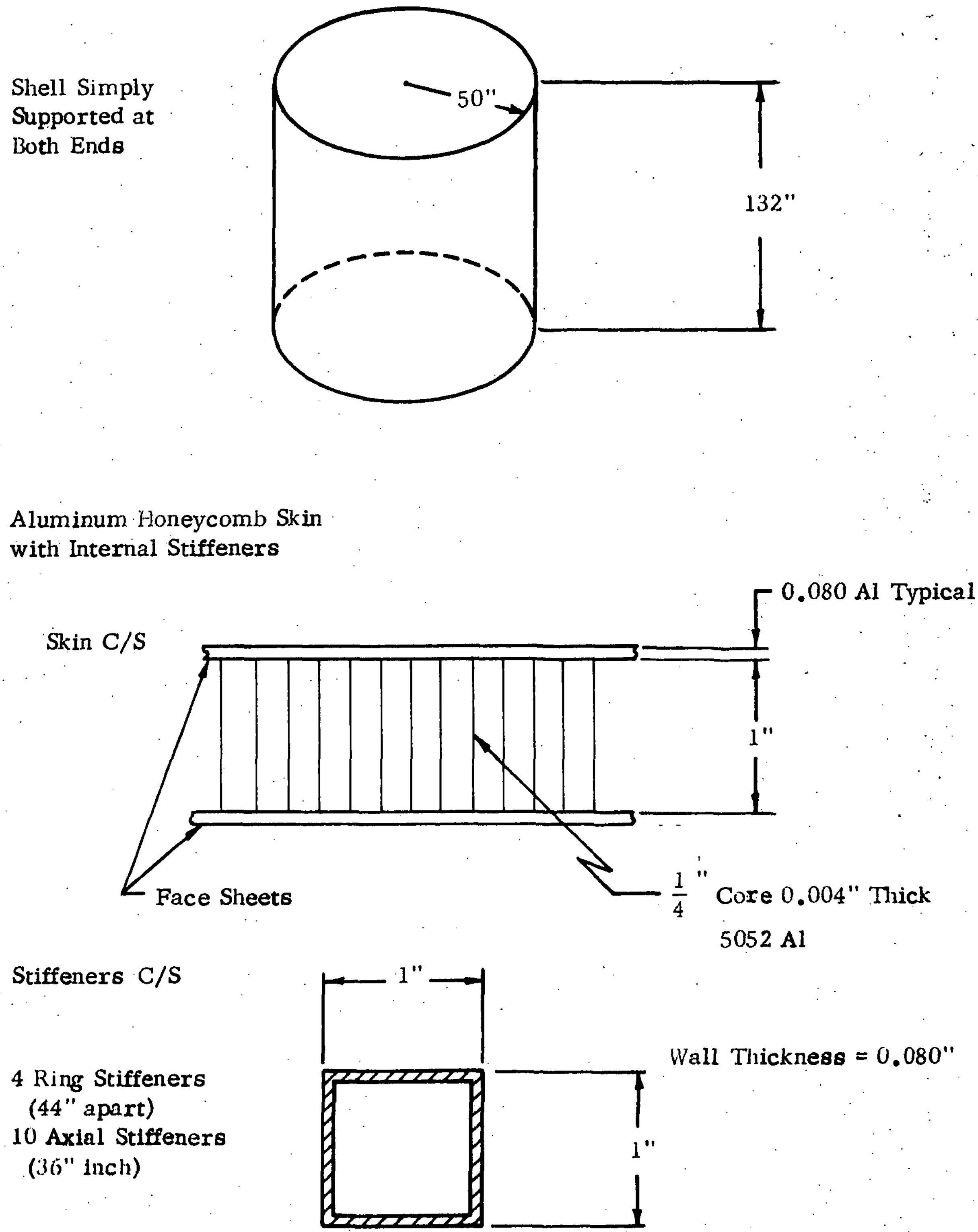

Figure 2. Construction of Shell Structure Used in Sample Problem.

m	n	Model 1	Model 2
4	1	154.4, 154.4 Hz	130.7, 130.7 Hz
3	1	149.8, 149.8 Hz	131.4, 131.4 Hz
5	1	170.2 - Hz	160.7, 167.4 Hz
2	1	187.5, 187.5 Hz	175.8, 175.8 Hz
4	2	258.4, 258.4 Hz	214.5, 214.5 Hz
6	1	- -	217.3, 217.3 Hz
5	2	259.2 - Hz	219.0, 228.5 Hz
3	2	- -	243.7, 243.7 Hz
6	2	- -	259.9, 259.9 Hz
1	1	280.4, 280.4 Hz	276.8, 276.8 Hz

TABLE I. Modal Frequencies of shell model 1 and 2 (m = number of circumferential waves, n = number of axial half waves).

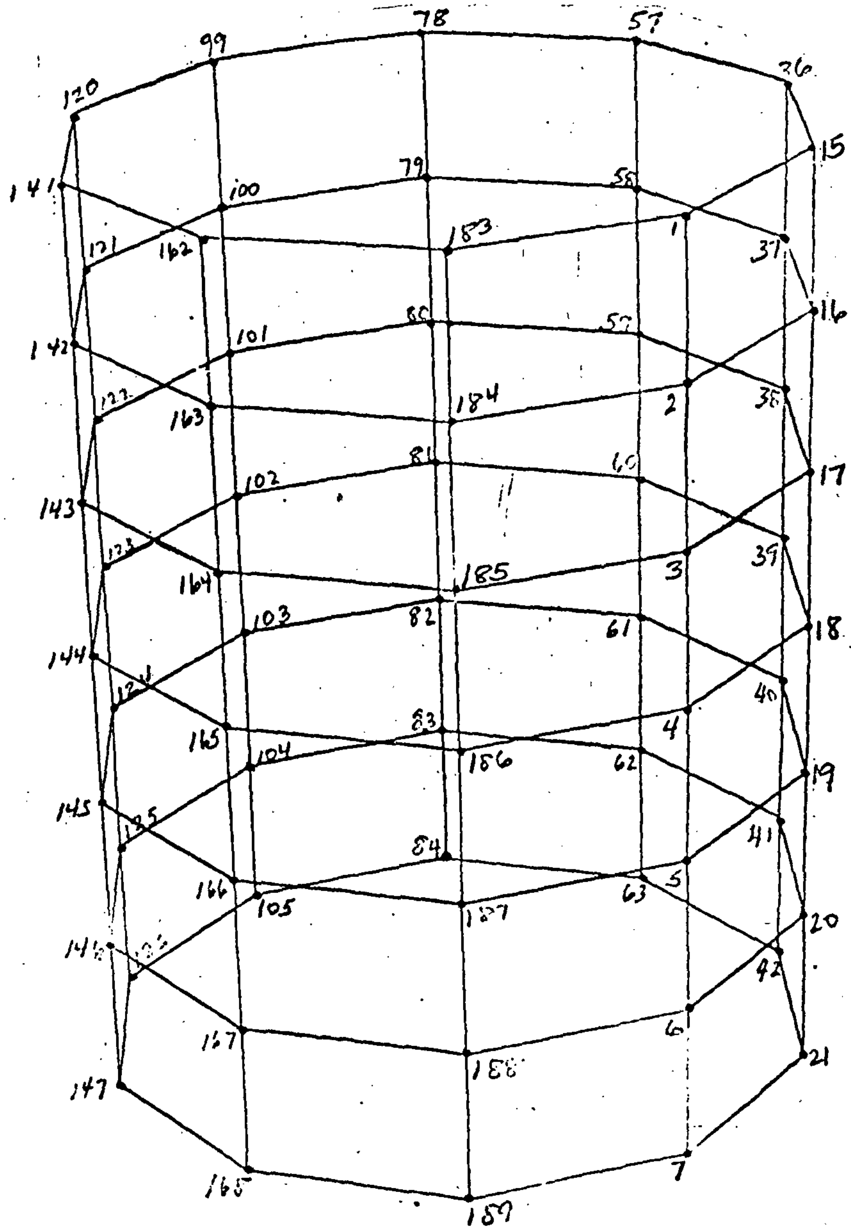

Figure 3. Shell Model 1 - Grid Point Location

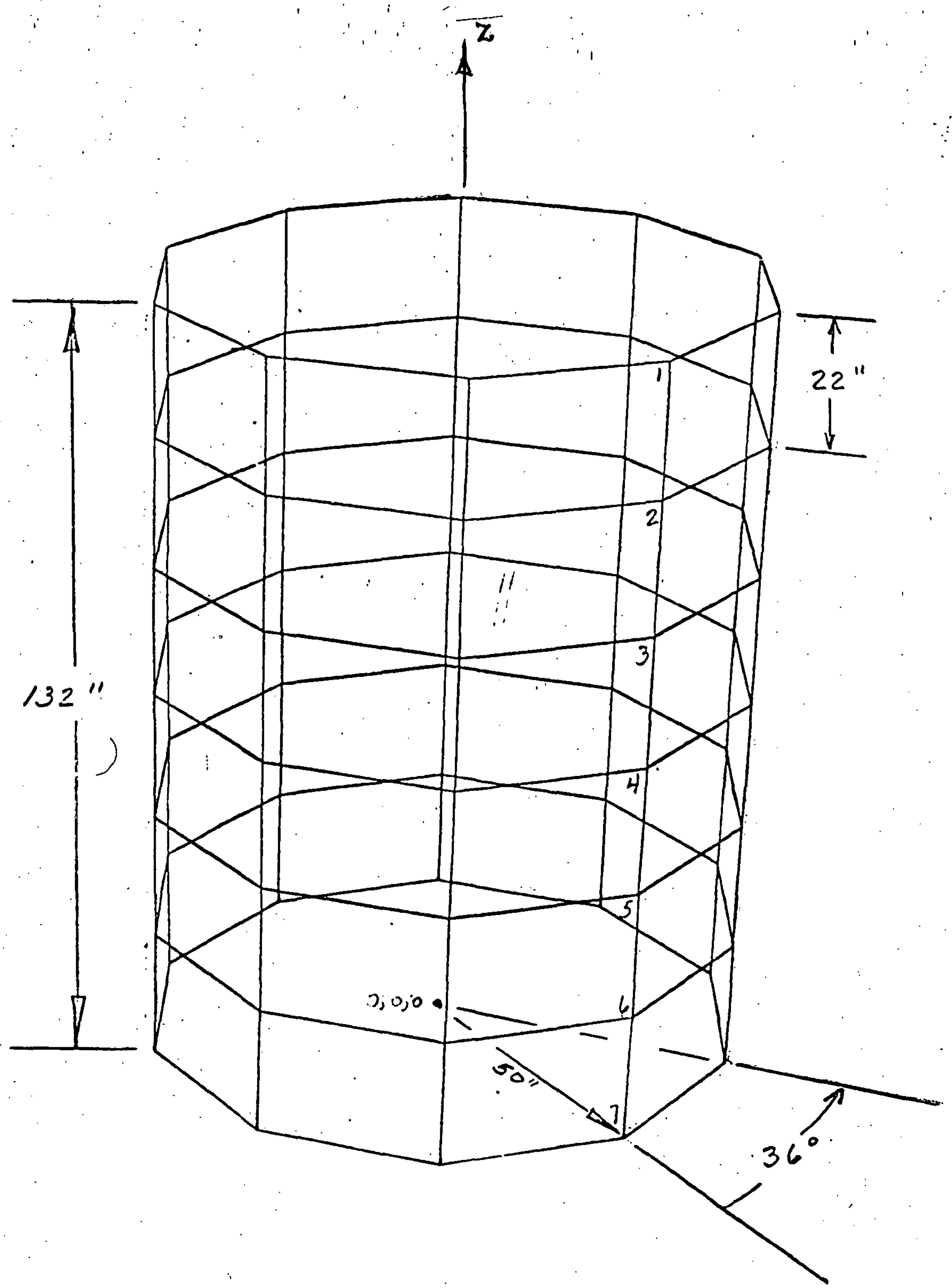

Figure 4. Shell Model 1 - Grid Point Layout and Dimensions

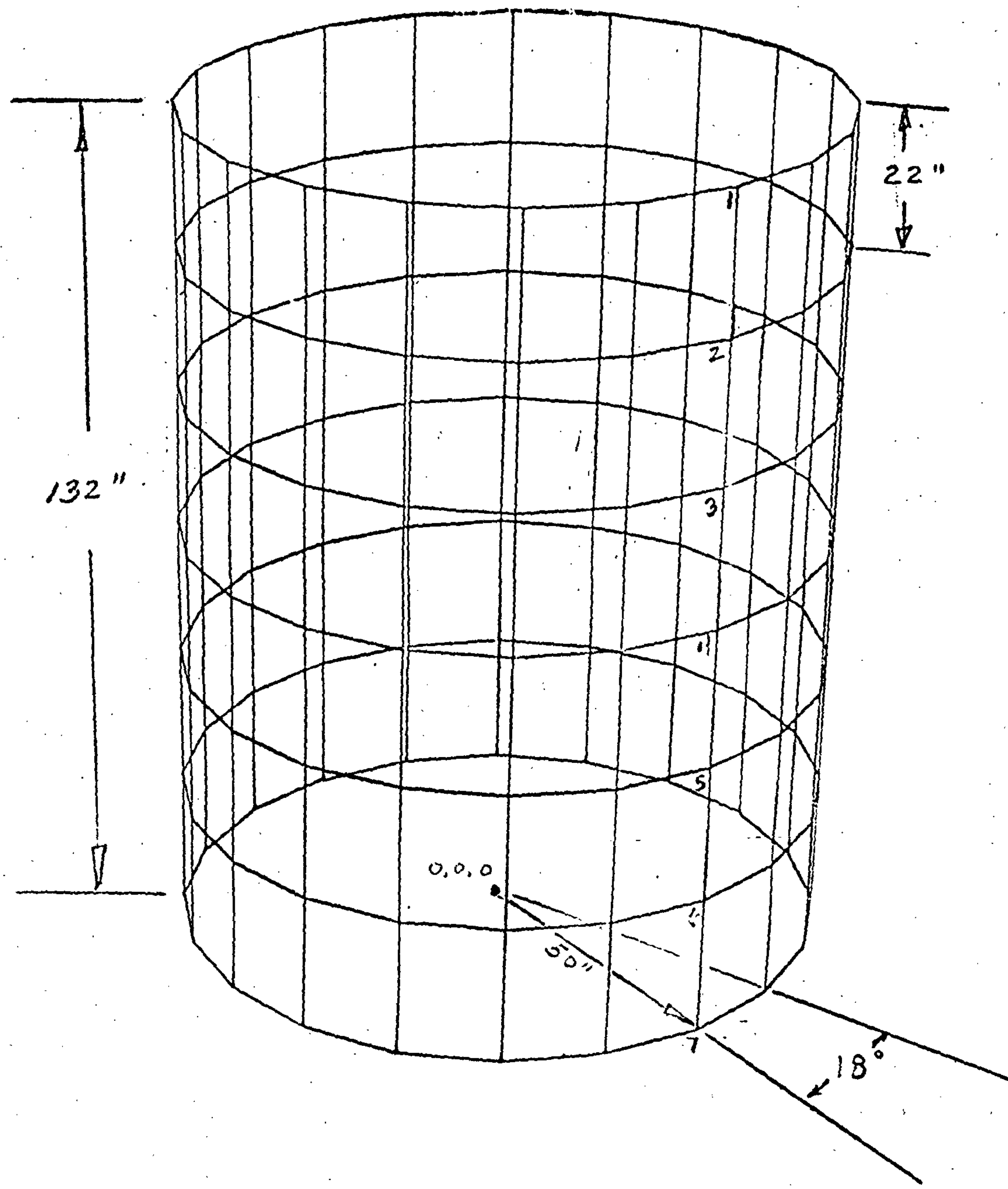

Figure 5. Shell Model 2 - Grid Point Layout and Dimensions

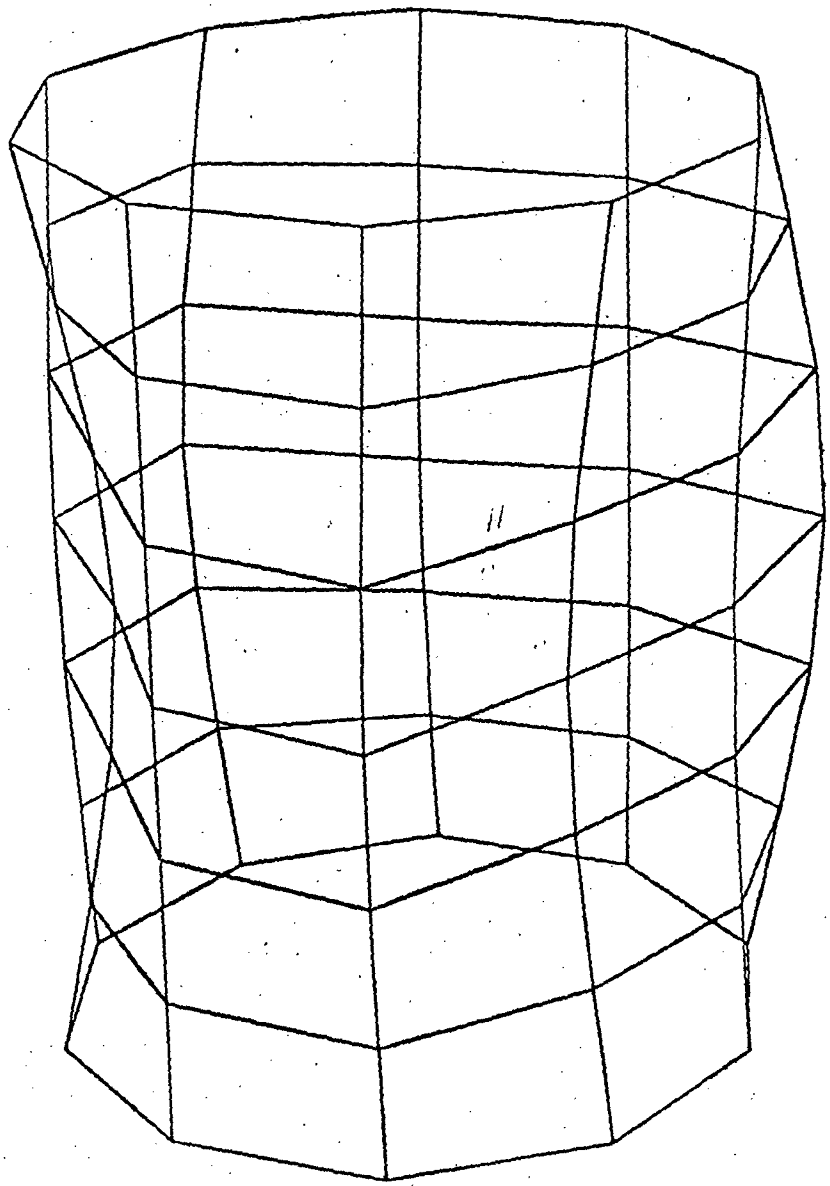

Figure 6. Shell Model 1 in Natural Vibration
(f = 149.8 Hz)

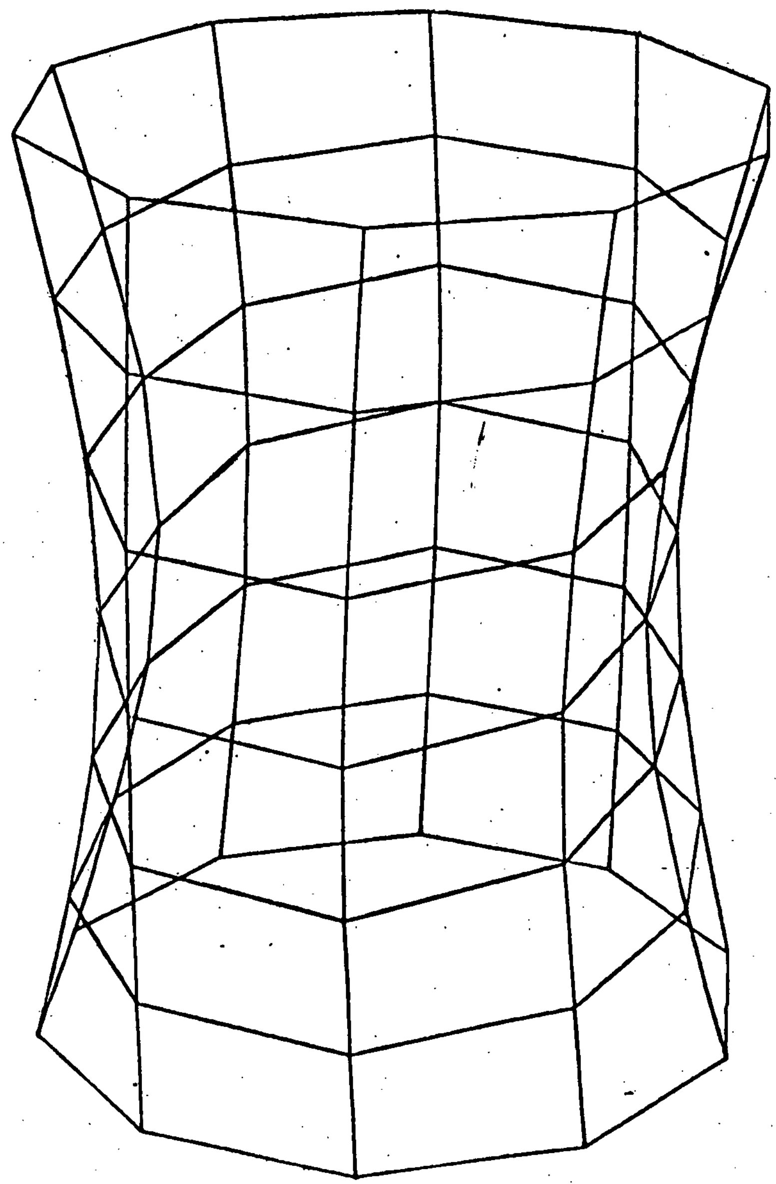

Figure 6. - continued (f = 187.5 Hz)

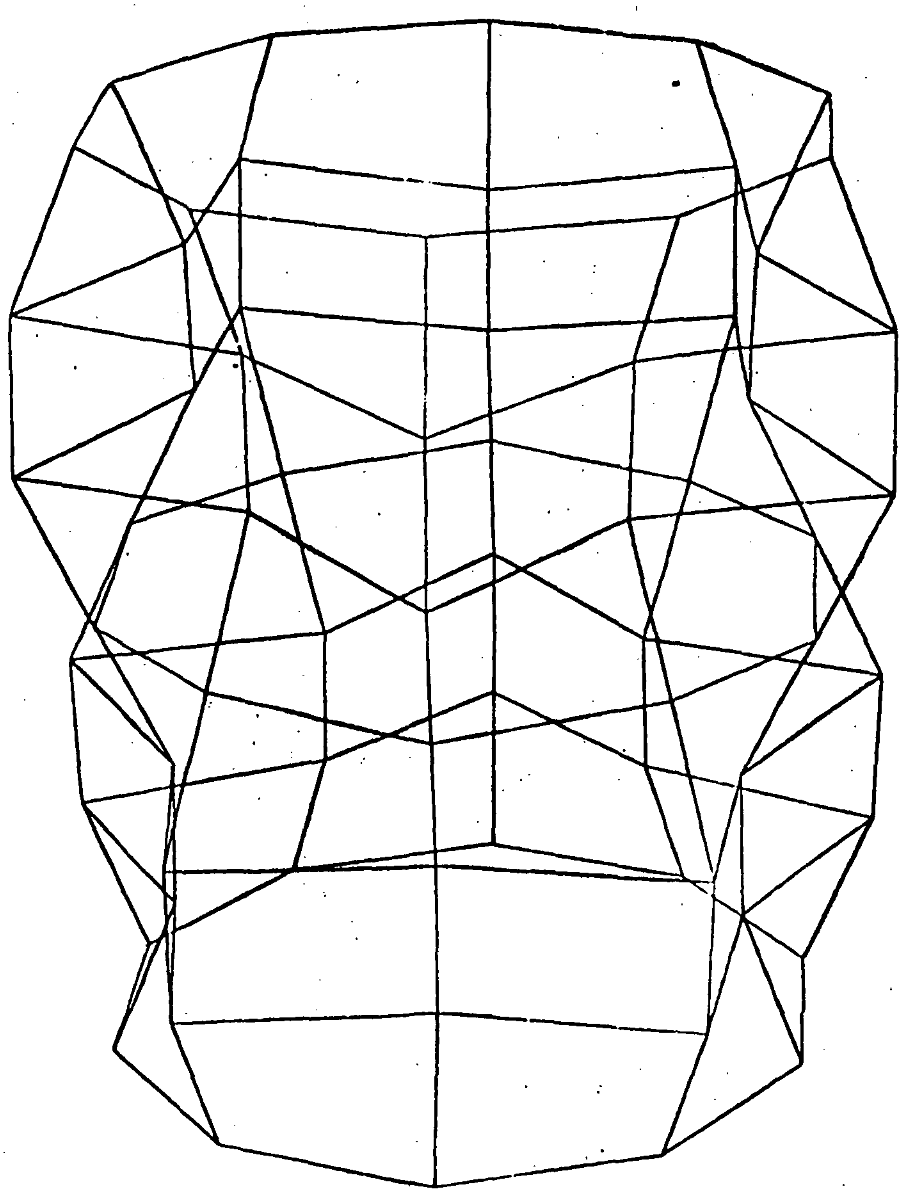

Figure 6. - continued (f = 259.2 Hz)

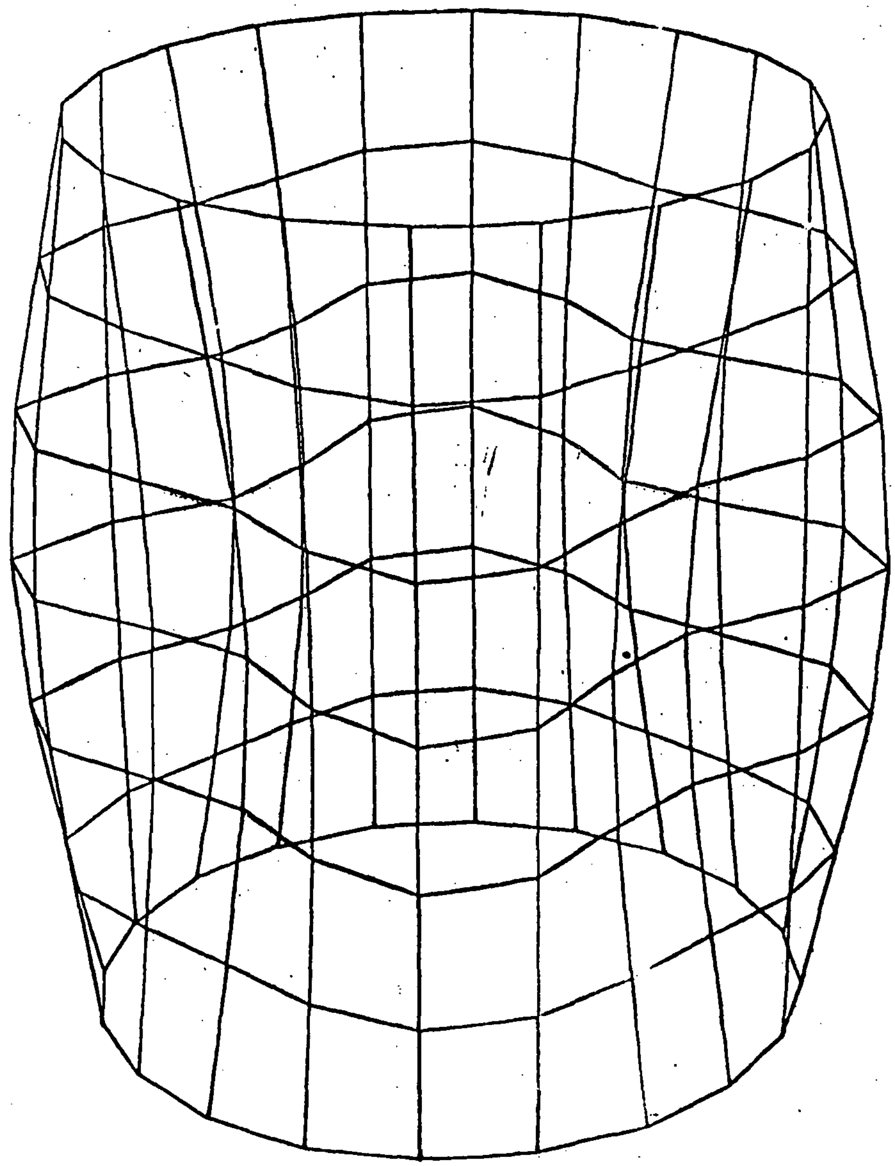

Figure 7. Shell Model 2 in Natural Vibration (f = 130.7 Hz)

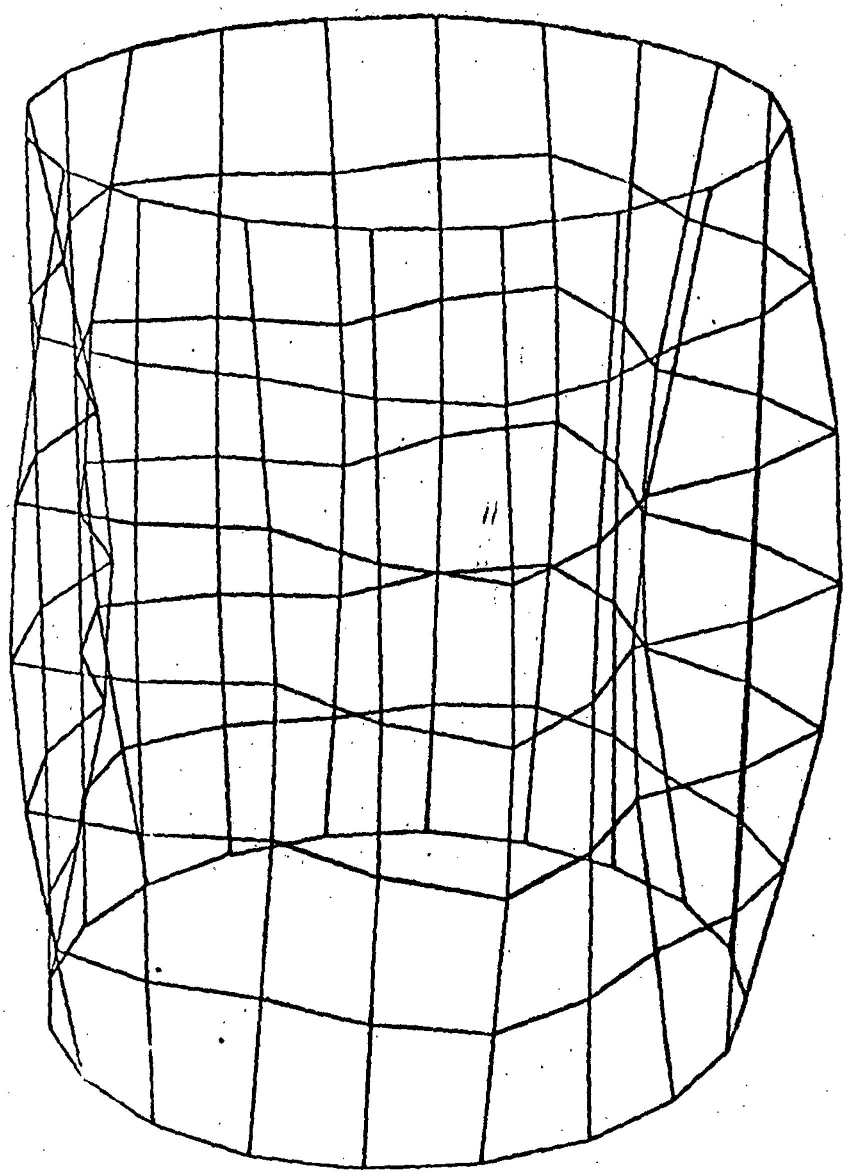

Figure 7. - continued (f = 167.4 Hz)

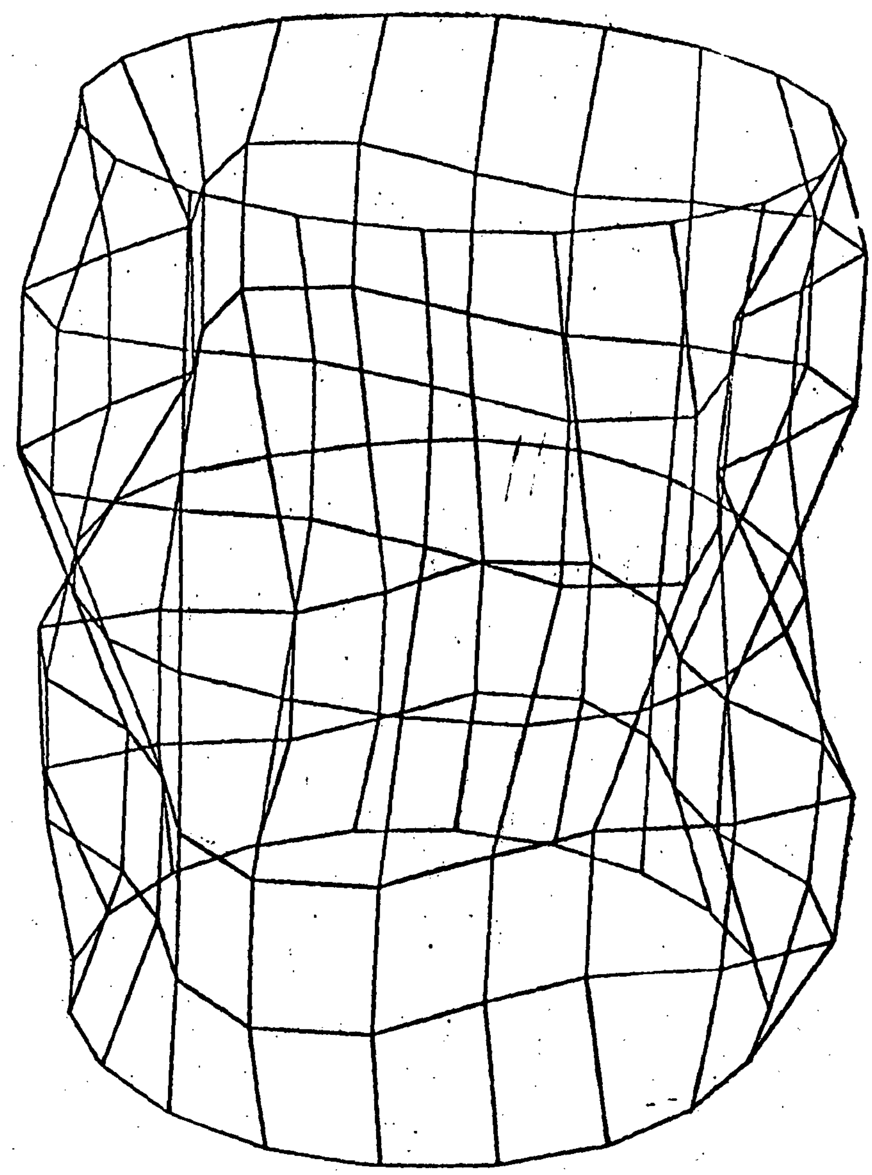

Figure 7. - continued (f = 214.5 Hz)

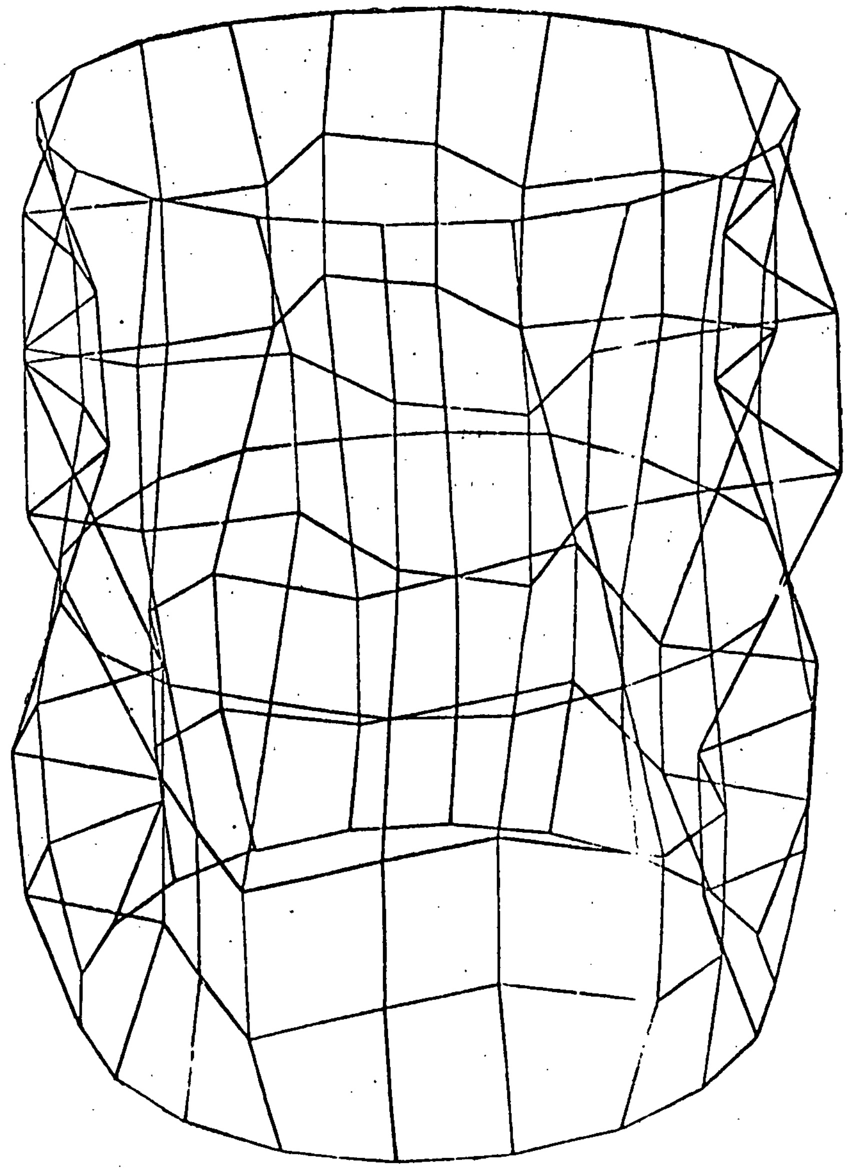

Figure 7. - continued (f = 259.9 Hz)

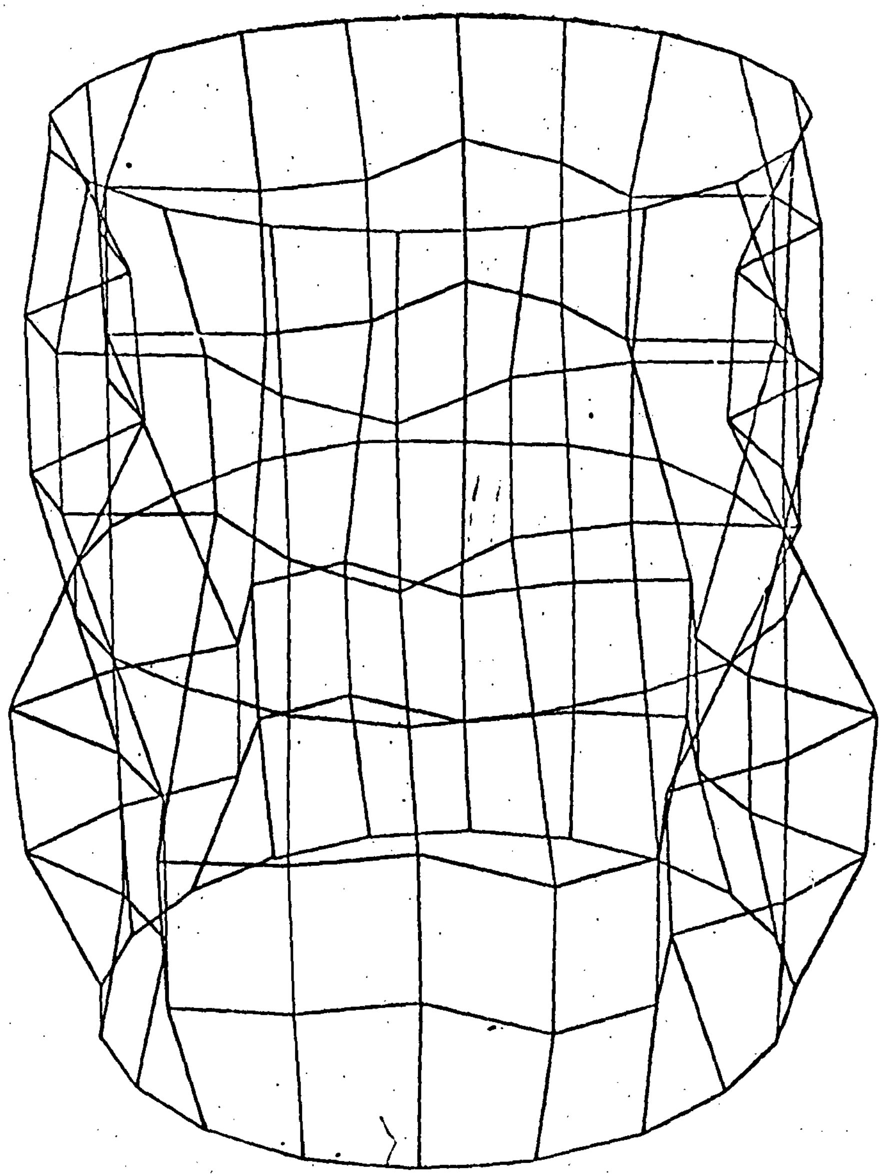

Figure 7. - continued (f = 259.9 Hz)

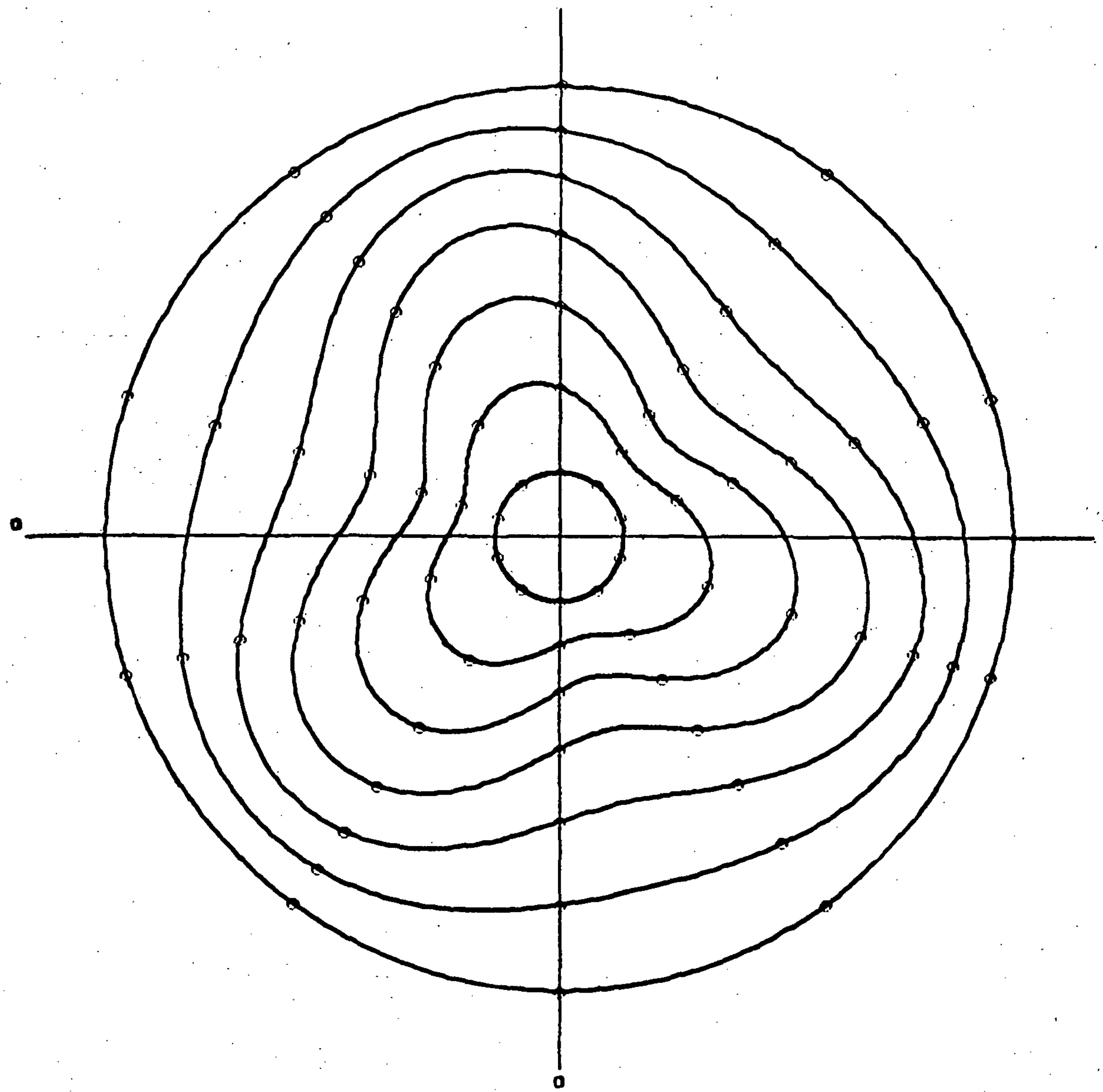

Figure 8. Trigonometric Interpolation of Mode Shape Data. Shell Model 1 (f = 149.8 Hz)

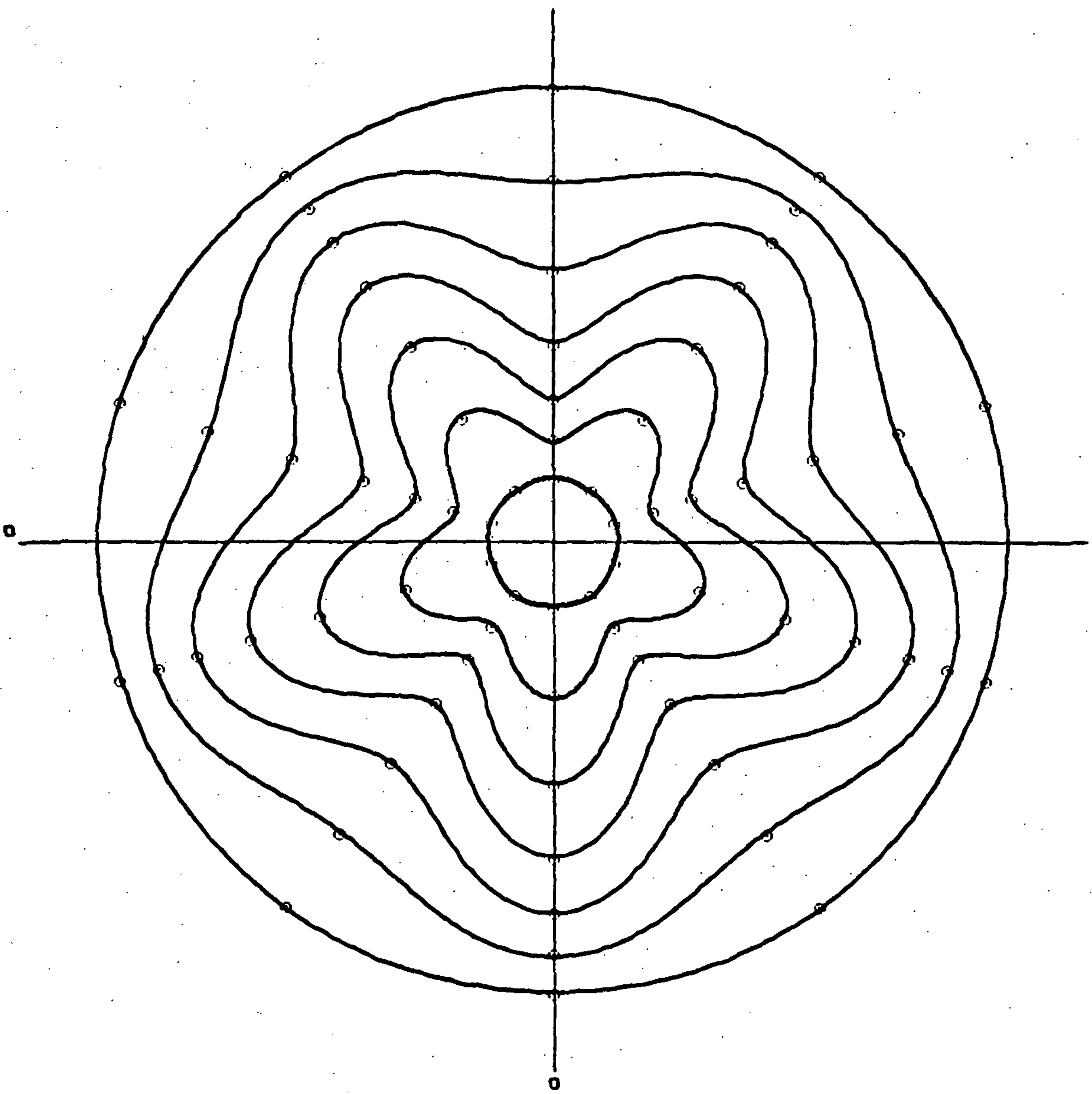

Figure 8. - continued (f = 170. 2 Hz)

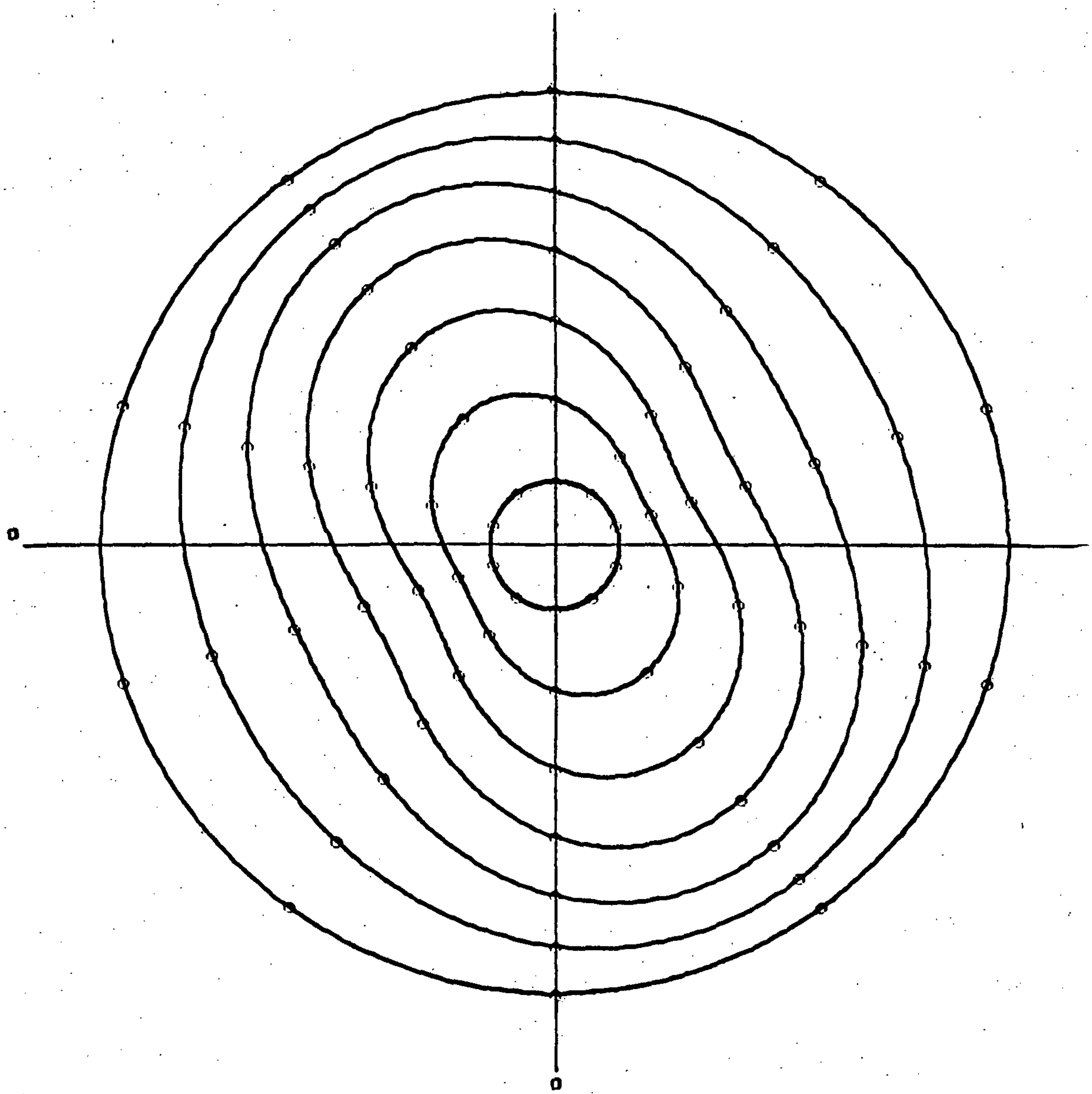

Figure 8. - continued (f = 187.5 Hz)

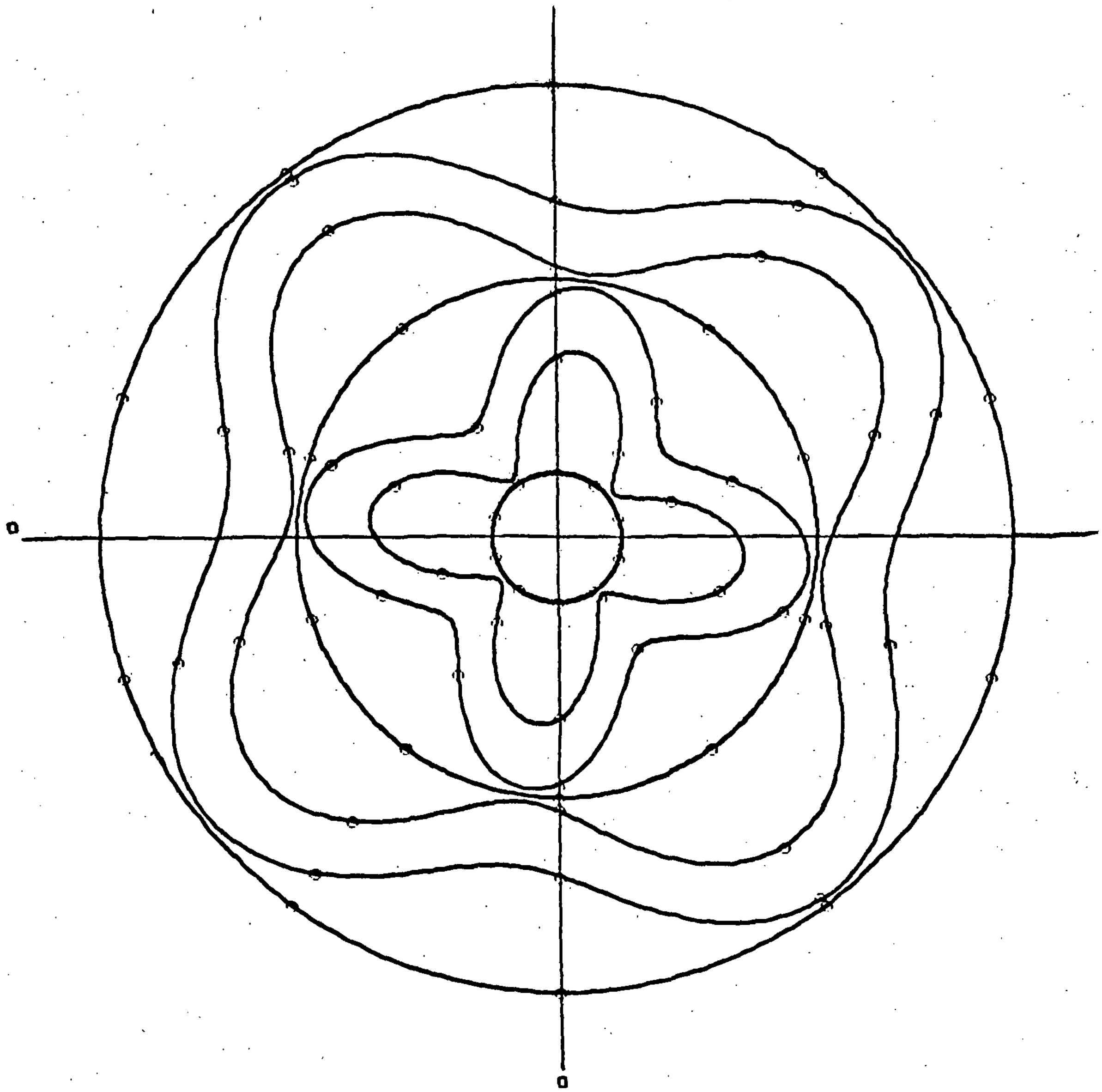

Figure 8. - continued (f = 258.4 Hz)

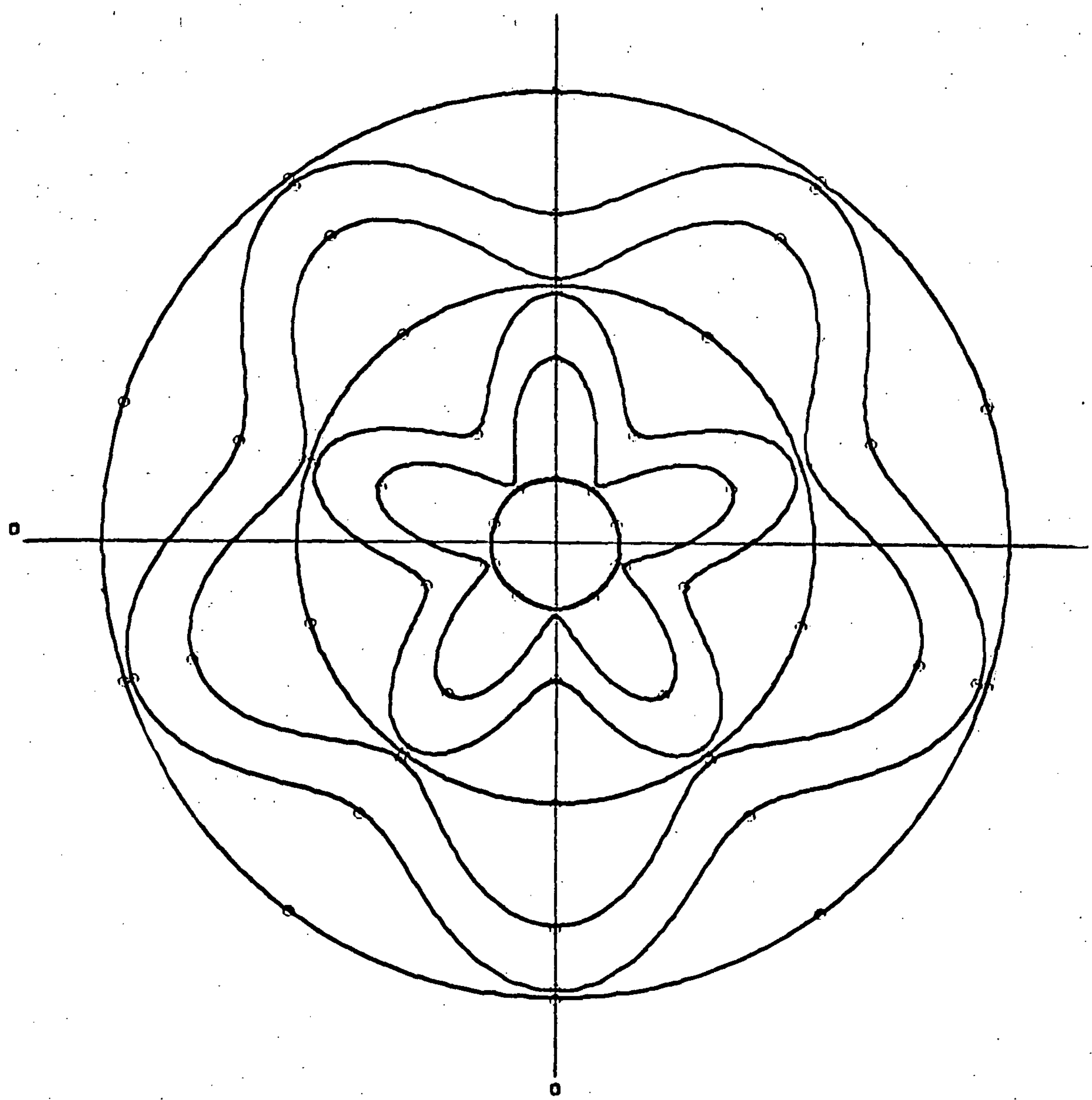

Figure 8. - continued (f = 259. 2 Hz)

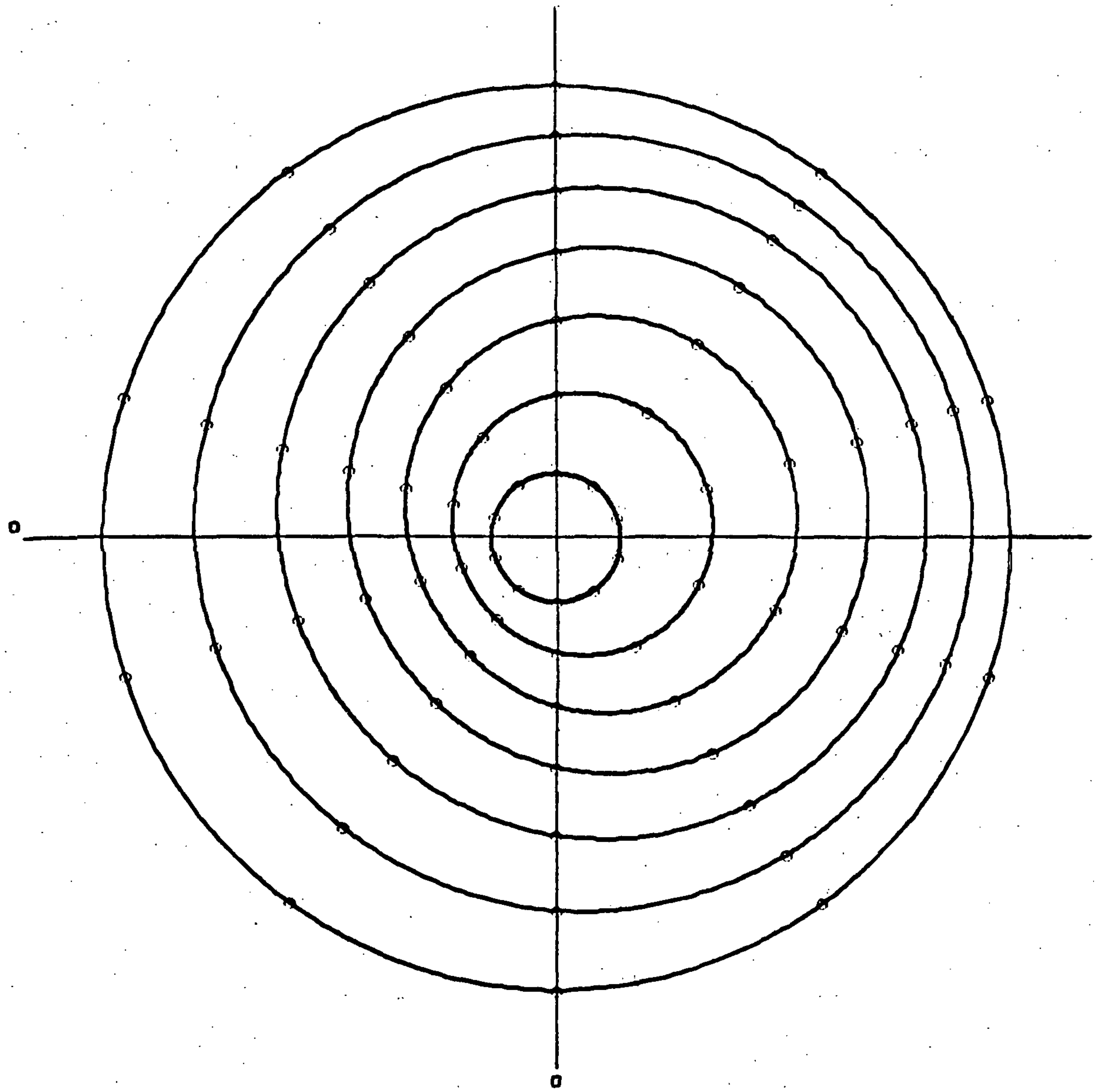

Figure 8. - continued (f = 280.4 Hz)

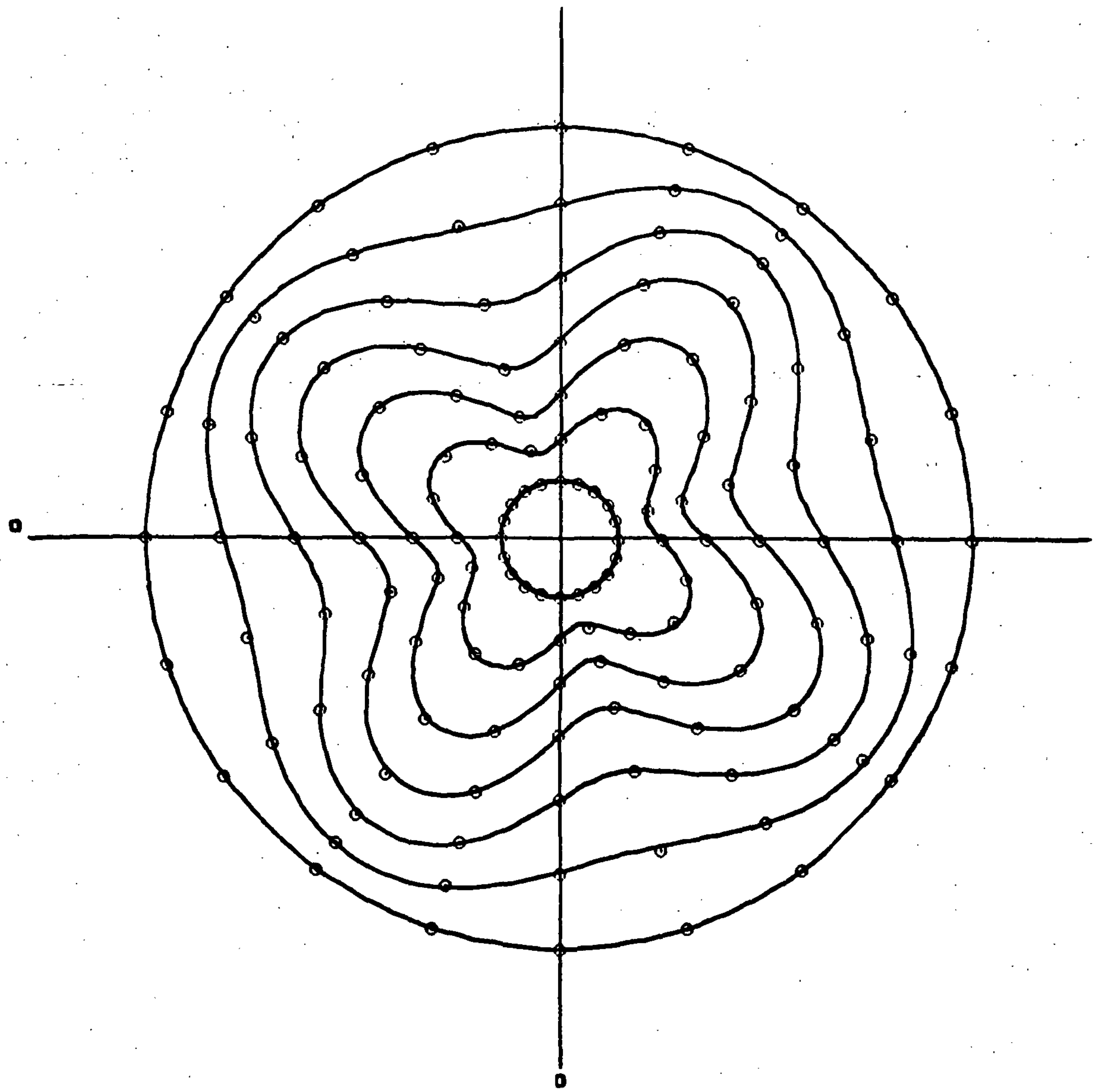

Figure 9. Trigonometric Interpolation of Mode Shape Data. Shell Model 2 (f = 130.7 Hz)

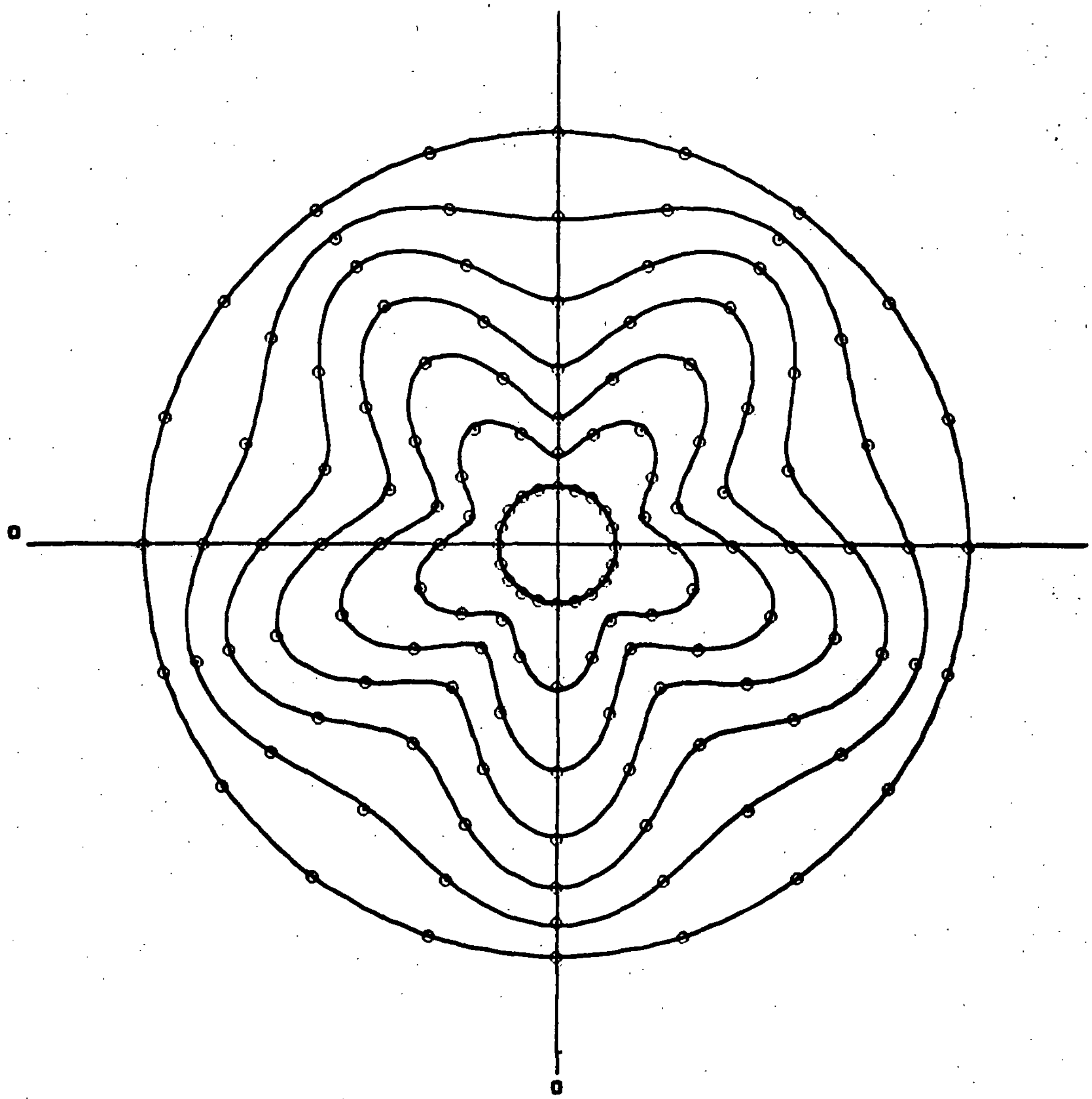

Figure 9. - continued (f = 160.7 Hz)

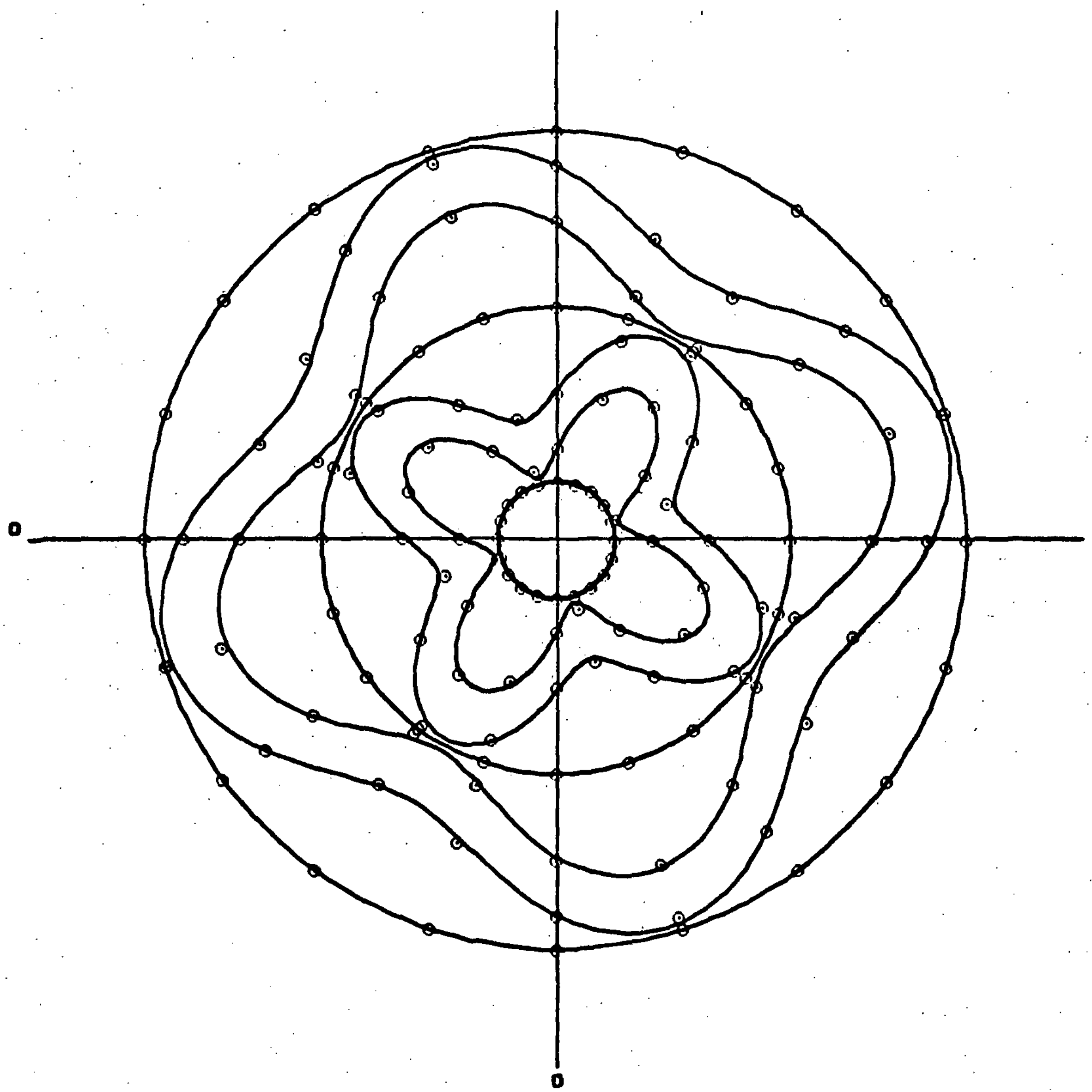

Figure 9. - continued (f = 214. 5 Hz)

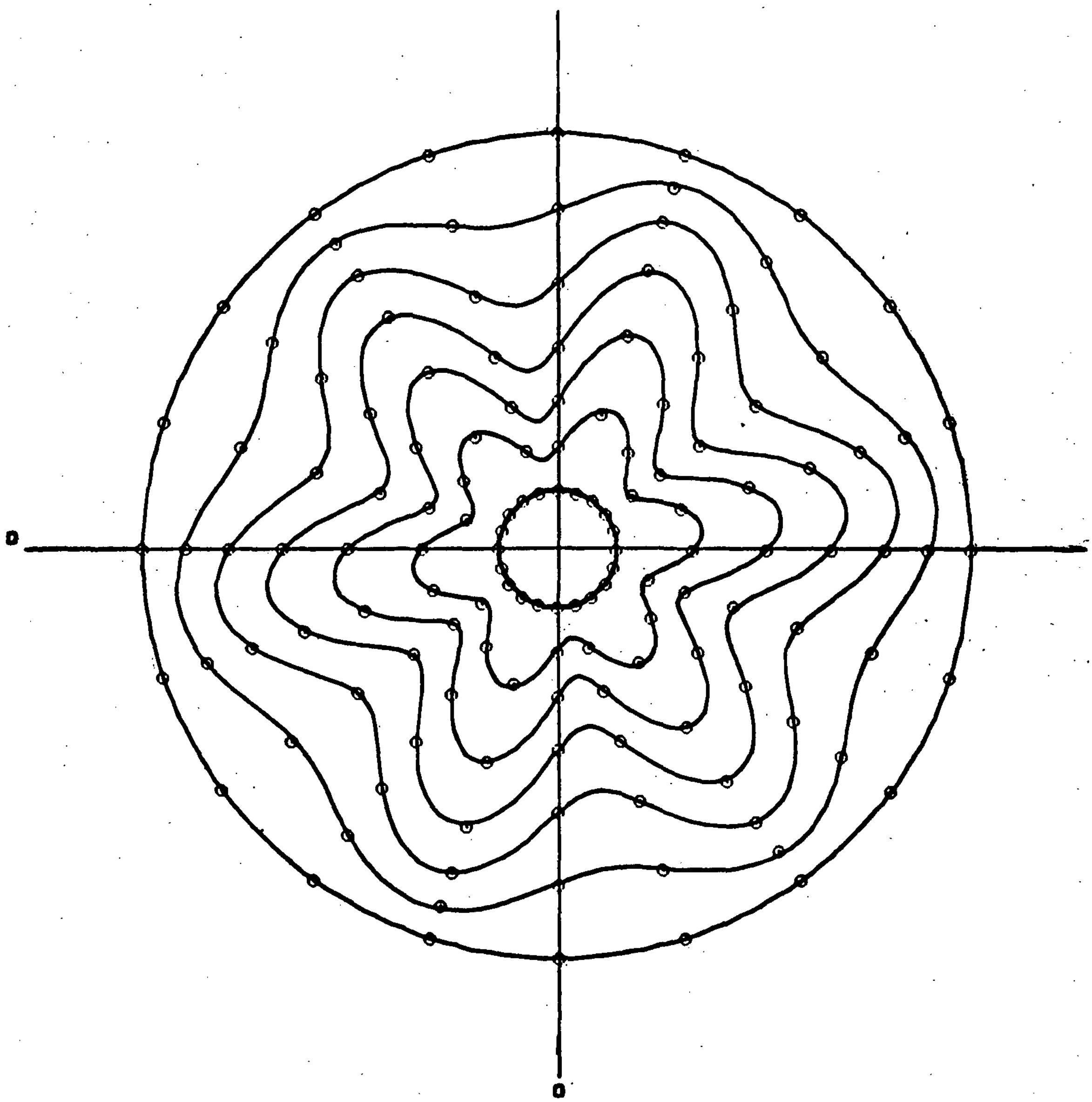

Figure 9. - continued (f = 217.3 Hz)

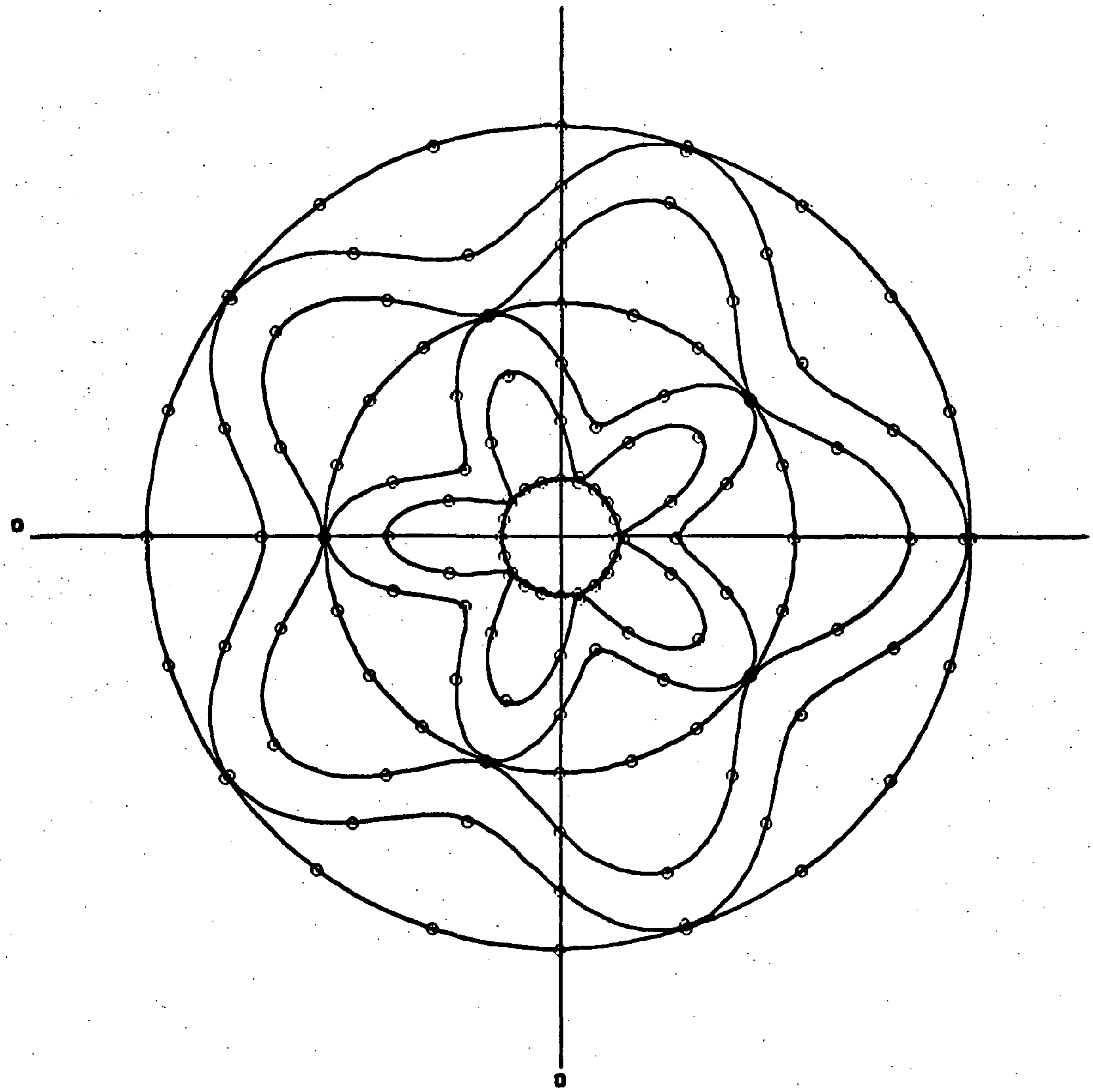

Figure 9. - continued (f = 228.5 Hz)

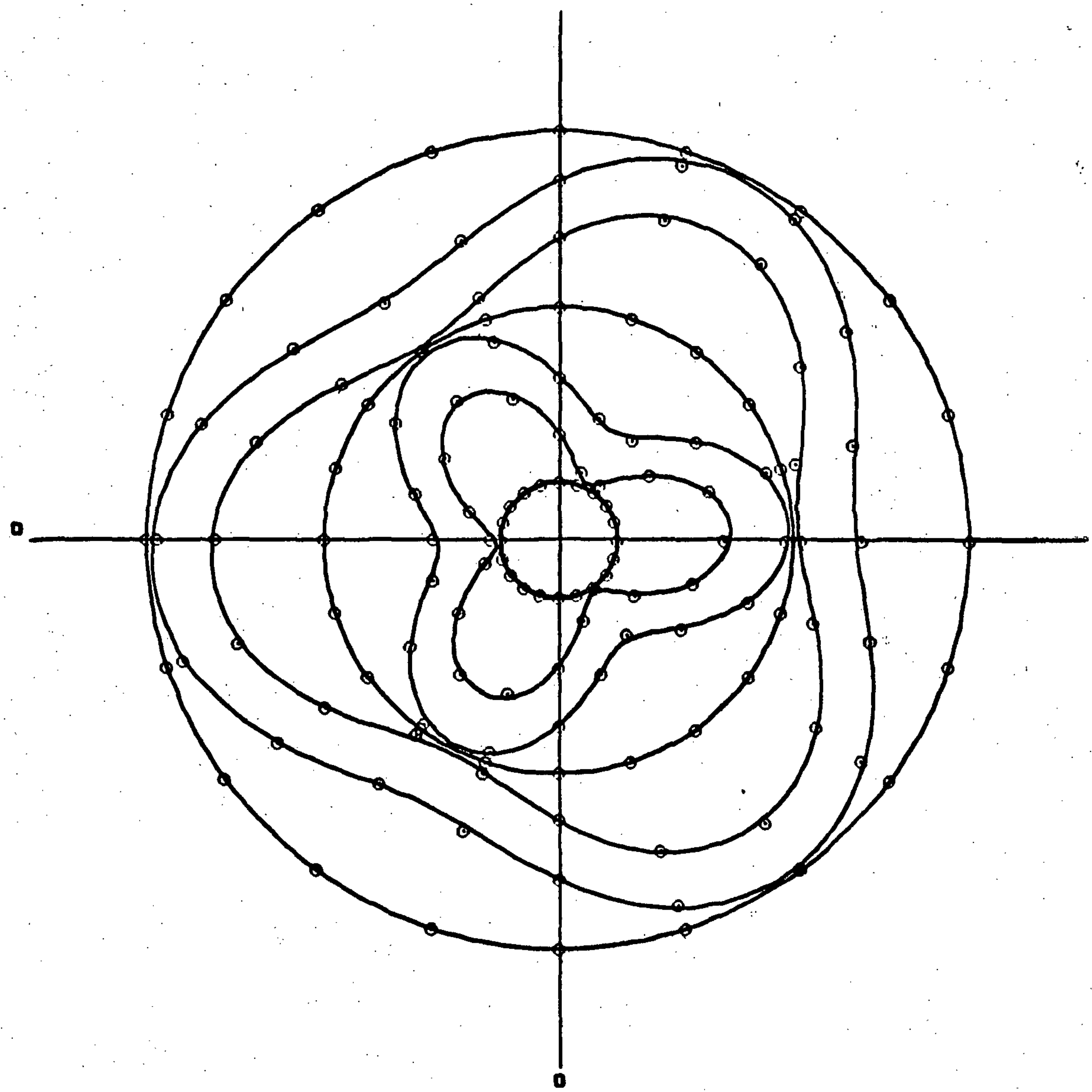

Figure 9. - continued (f = 243.7 Hz)

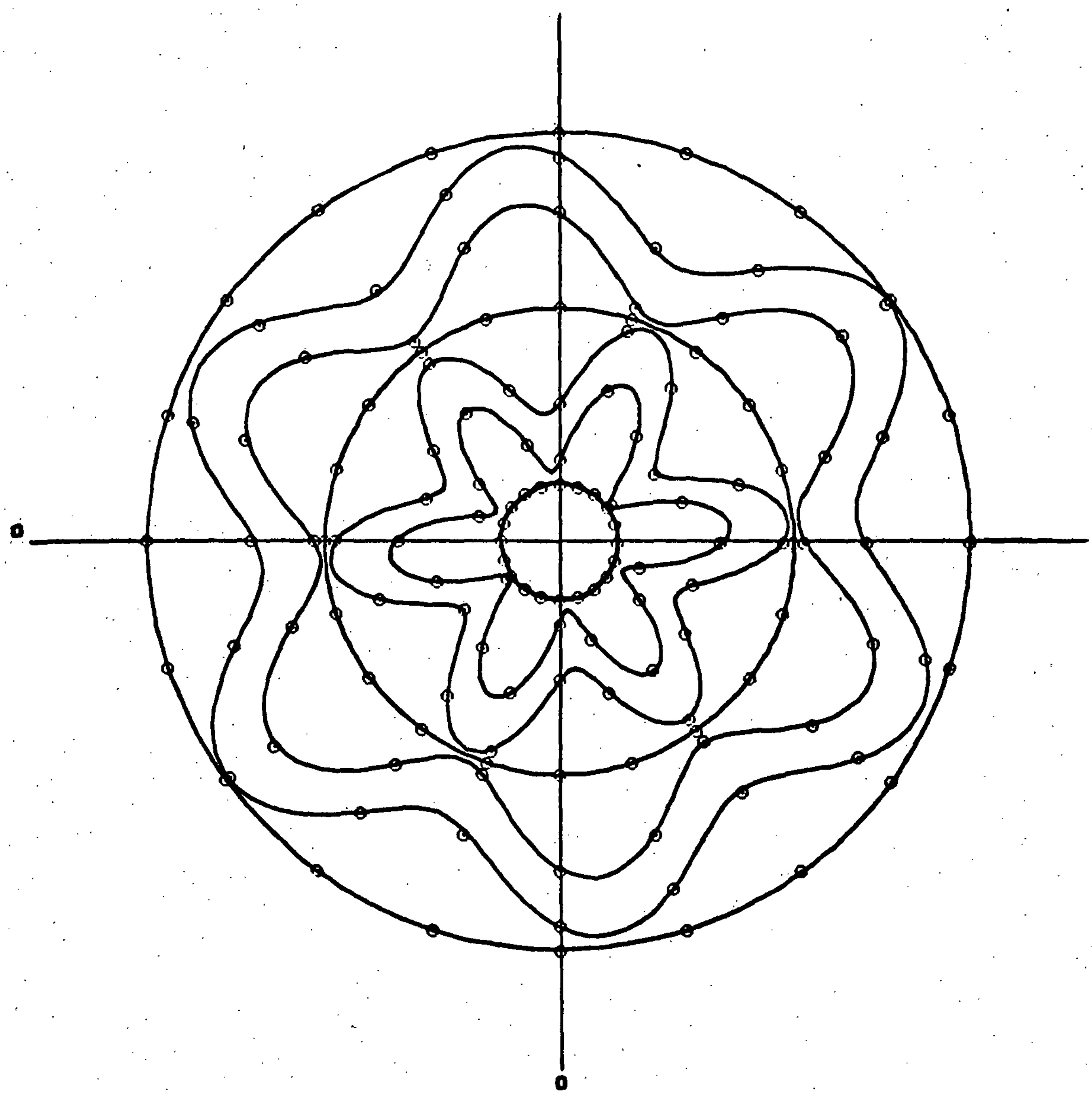

Figure 9. - continued (f = 259.9 Hz)

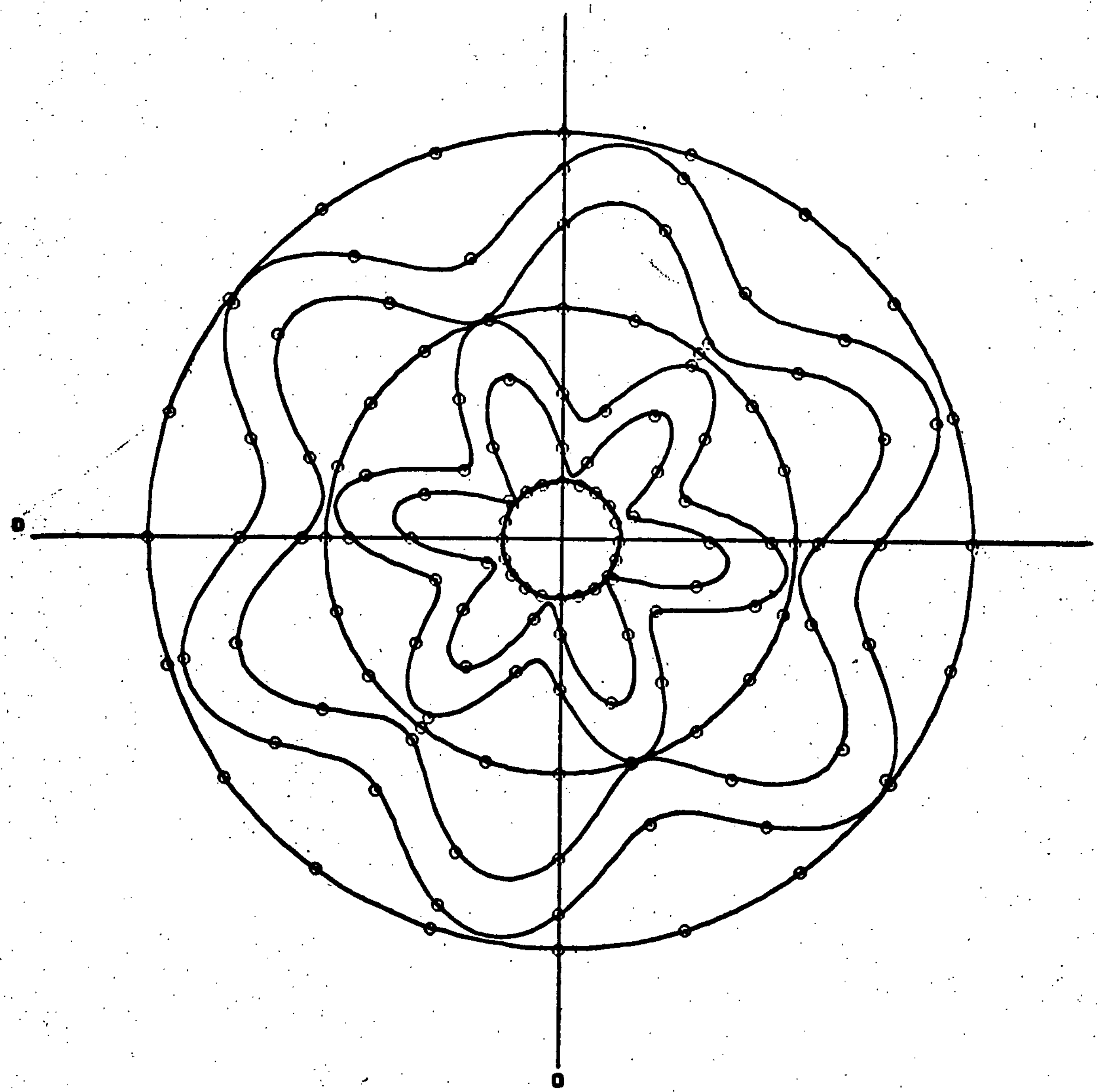

Figure 9. - continued (f = 259. 9 Hz)

The output of the FITMSC code, along with the other structural parameters (e.g. modal frequencies and masses, length, radius, damping*, et cetera) constitute part of the input to the RANCYL code (Section 3.3). The remaining parameters depend on the input function.

The following form for the pressure cross-spectral density was chosen:

$$\hat{Q}(Z,\Theta,\omega) = 0.016 \exp[-0.05\Theta - 0.025|Z|] \cos 0.1Z; \ (\text{psi})^2/\text{Hz} \quad 0<\Theta<\pi \tag{59}$$

With reference to Equation (25),

$$\begin{aligned} Q_1(Z) &= e^{-0.025|Z|} \cos 0.1\,Z \\ &= \frac{1}{2}\,[e^{i(0.1 + 0.025i)Z} + e^{i(-0.1 + 0.025i)Z}]\,U(Z) \\ &+ \frac{1}{2}\,[e^{i(0.1 - 0.025i)Z} + e^{i(-0.1 - 0.025i)Z}]\,U(-Z) \end{aligned} \tag{60}$$

$$Q_2(\Theta) = e^{-0.05\Theta}; \quad 0<\Theta<\pi \tag{61}$$

$$\hat{Q}_3(\omega) = 0.016 \tag{62}$$

Expanding Equations (34) and (36),

$$\begin{aligned} Q_1(Z) = A_o + A_1\,\delta(Z) + U(Z) \sum_{\alpha=1} e^{i\kappa_\alpha Z} \left\{ iB_{\alpha,1} + iB_{\alpha,2}(iZ) + \ldots\ldots \right\} \\ + U(-Z) \sum_{\gamma=1} e^{i\kappa_\gamma Z} \left\{ -iC_{\gamma,1} - iC_{\gamma,2}(iZ) \ldots\ldots \right\} \end{aligned} \tag{63}$$

*For this example, the ratio of damping to critical damping was chosen to be 0.01. In terms of this ratio, the modal damping a_n is given by $a_n = 0.01\,\omega_n$.

$$Q_2(\Theta) = D_o + D_1\, \delta(\Theta) + \sum_{\eta=1} e^{i\kappa_\eta \Theta} \left\{ E_{\eta,0} + E_{\eta,1}\Theta + \ldots \right\} \qquad (64)$$

Comparing Equations (60) and (63),

$A_o = A_1 = 0$

$\alpha = 1, 2;\ \beta = 1$ $\qquad$ $\gamma = 1, 2;\ \epsilon = 1$

$B_{1,1} = B_{2,1} = \frac{1}{2i}$ $\qquad$ $C_{1,1} = C_{2,1} = -\frac{1}{2i}$

$\kappa_{\alpha=1} = 0.1 + 0.025i$ $\qquad$ $\kappa_{\gamma=1} = 0.1 - 0.025i$

$\kappa_{\alpha=2} = -0.1 + 0.025i$ $\qquad$ $\kappa_{\gamma=2} = -0.1 - 0.025i$

$B_{\alpha,\beta} = \kappa_\alpha = 0$, all $\alpha > 2$, $\beta > 1$ $\qquad$ $C_{\gamma,\epsilon} = \kappa_\gamma = 0$, all $\gamma > 2$, $\epsilon > 1$

Similarly, comparison of Equation (61) with (64) yields

$D_o = D_1 = 0$

$\eta = 1,\ \sigma = 0$

$E_{1,0} = 1;\ E_{\eta,\sigma} = 0$, all $\eta > 1$, $\sigma > 0$

$\kappa_{\eta=1} = 0.05i;\ \kappa_\eta = 0$, all $\eta > 1$

From Equations (62) and (33), we further deduce

$H_o = N_{\eta,\sigma} = M_{\xi,\rho} = 0$, all η, σ, ξ, ρ

$H_1 = 0.016$

The above values for the pressure field parameters constituted the remaining portion of the input to the RANCYL code, which was run to obtain values for the response cross-spectral and power spectral densities at selected grid points of shell models 1 and 2. For this purpose, grid points 2, 3 and 4 were selected (see Figures 3 - 5). The power spectral density at the same grid points was also computed using the NASTRAN computer program (rigid format 11). The pressure was input in the form of discrete values at the grid points. Rigid format 11, modal random response, makes use of modal data to derive a dynamic transfer function for a set of input frequencies and then using the input random loads to calculate the structural response at those frequencies.

The results of these computations are shown in Figures 10 - 19. Power spectral density plots at grid points 2, 3, and 4 of shell model 1 are shown in Figures 10 - 12 along with the NASTRAN values. The power spectral density at the same grid points of shell model 2 is plotted in Figures 13 - 15. Due to difficulties encountered in the NASTRAN analysis of shell model 2, a limited number of values were obtained, as shown in these figures. Typical cross-spectral density results obtained using the RANCYL code are shown in Figures 16 - 19. A NASTRAN calculation of response cross-spectral density was not performed.

Inspection of the results show very good agreement, in general. The mode-shape fits, as exemplified in Figures 8 and 9, are indeed excellent. It should be noted that, at most, eight terms of Eq. (19) were necessary to produce any one mode shape. The power spectral density results based on shell model 1 show reasonable agreement in general trend

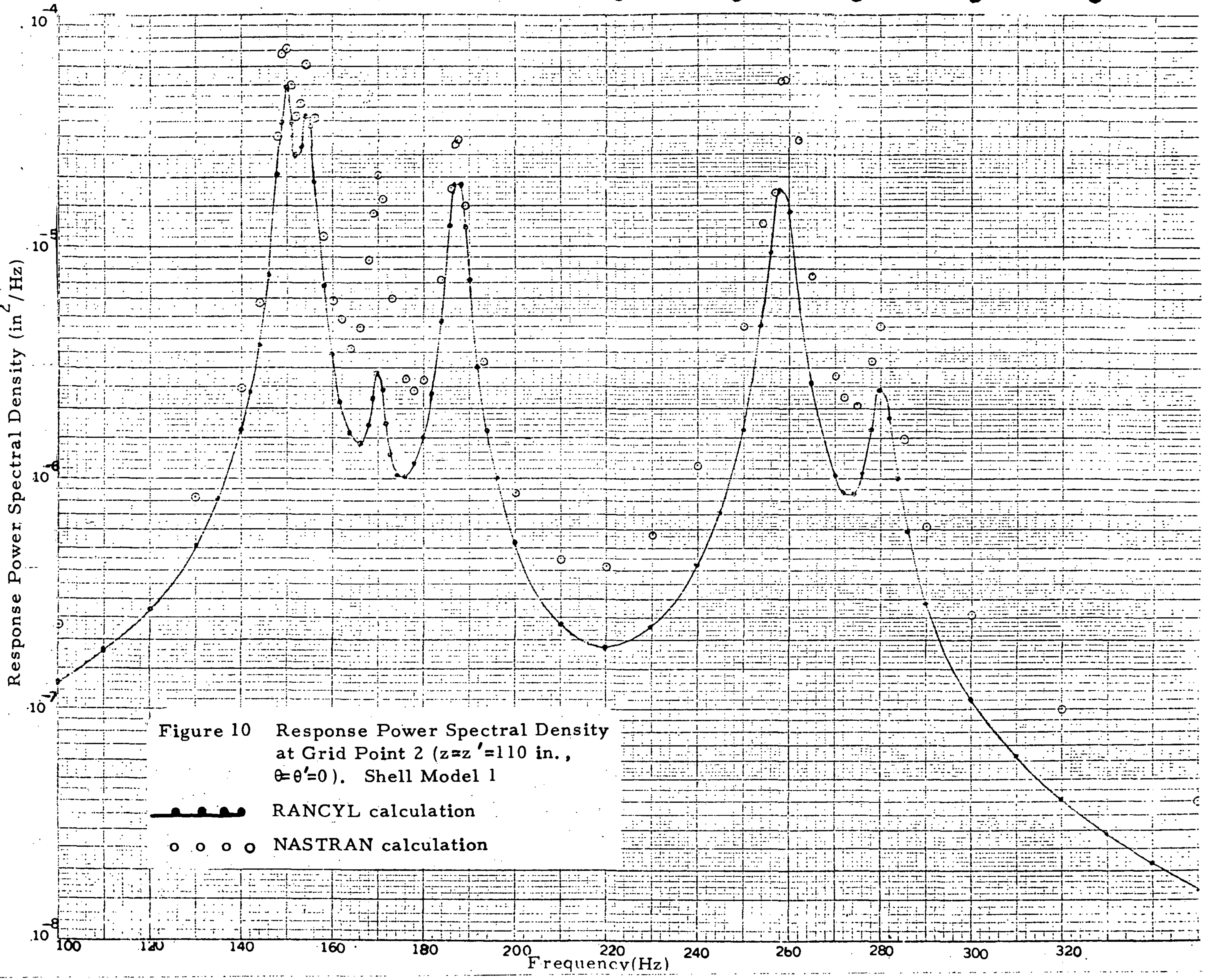

Figure 10 Response Power Spectral Density at Grid Point 2 ($z=z'=110$ in., $\theta=\theta'=0$). Shell Model 1

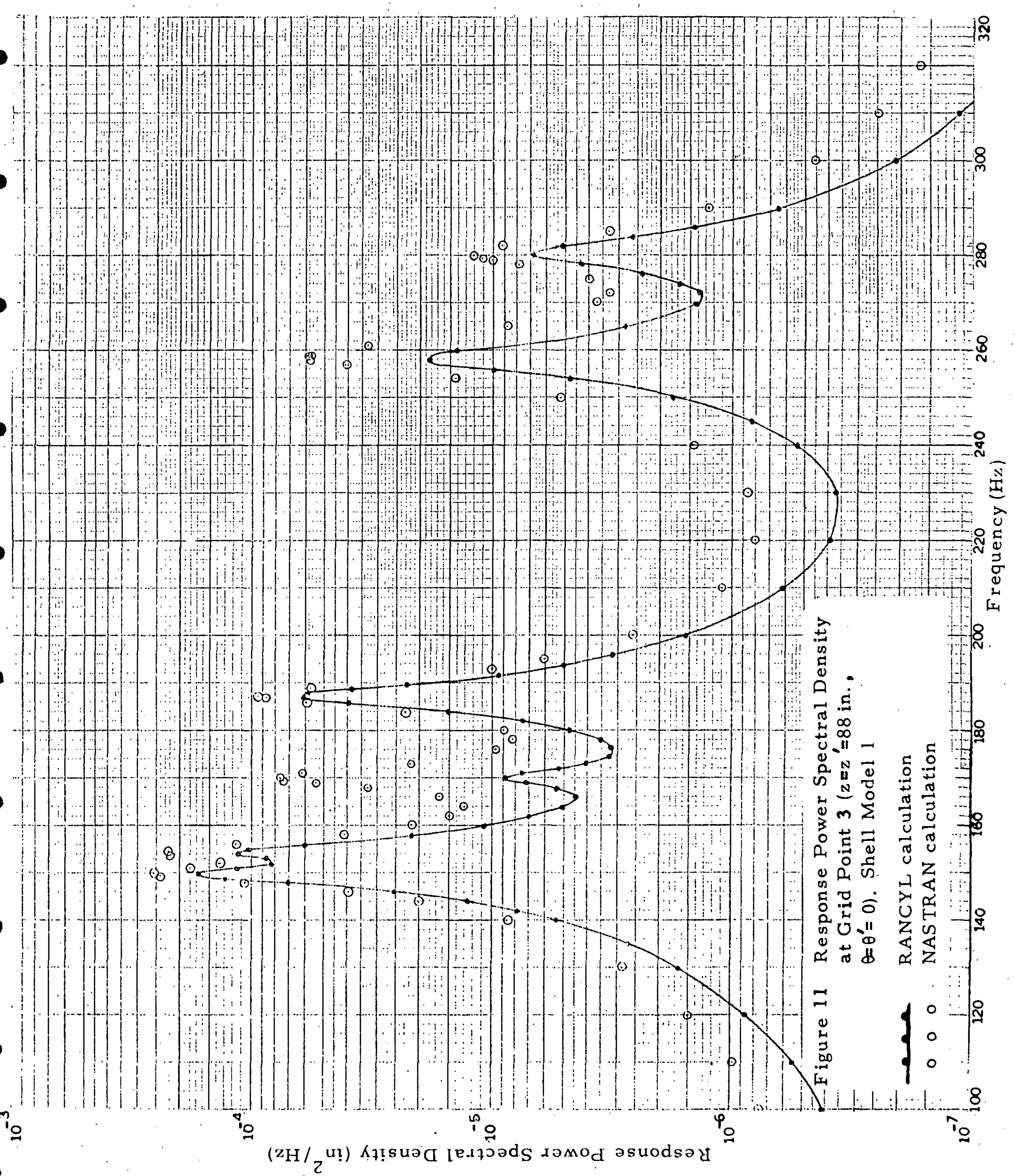

Figure 11 Response Power Spectral Density at Grid Point 3 ($z=z'=88$ in., $\theta=\theta'=0$). Shell Model 1

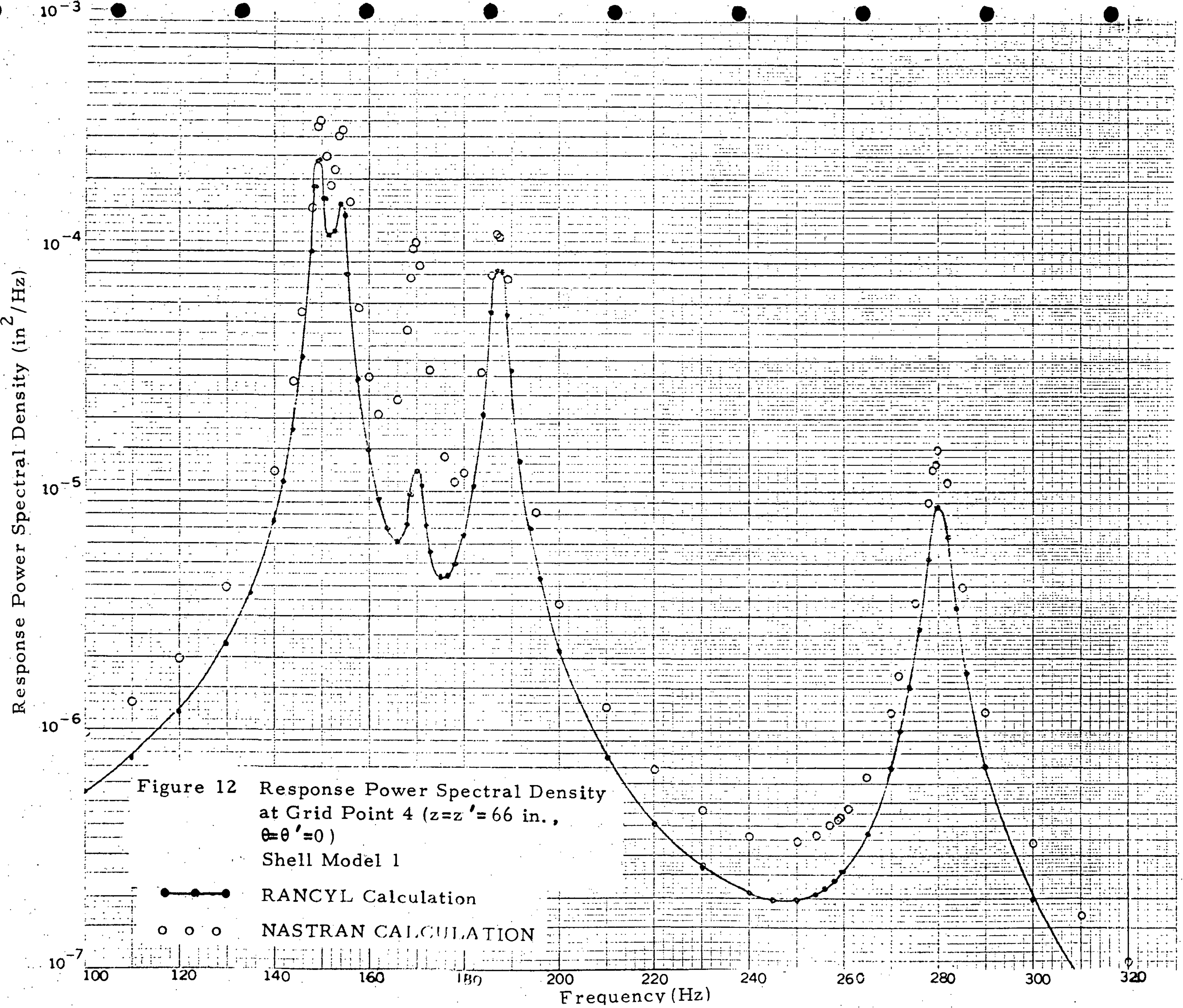

Figure 12 Response Power Spectral Density at Grid Point 4 ($z = z' = 66$ in., $\theta = \theta' = 0$)
Shell Model 1

●—●—● RANCYL Calculation

o o o NASTRAN CALCULATION

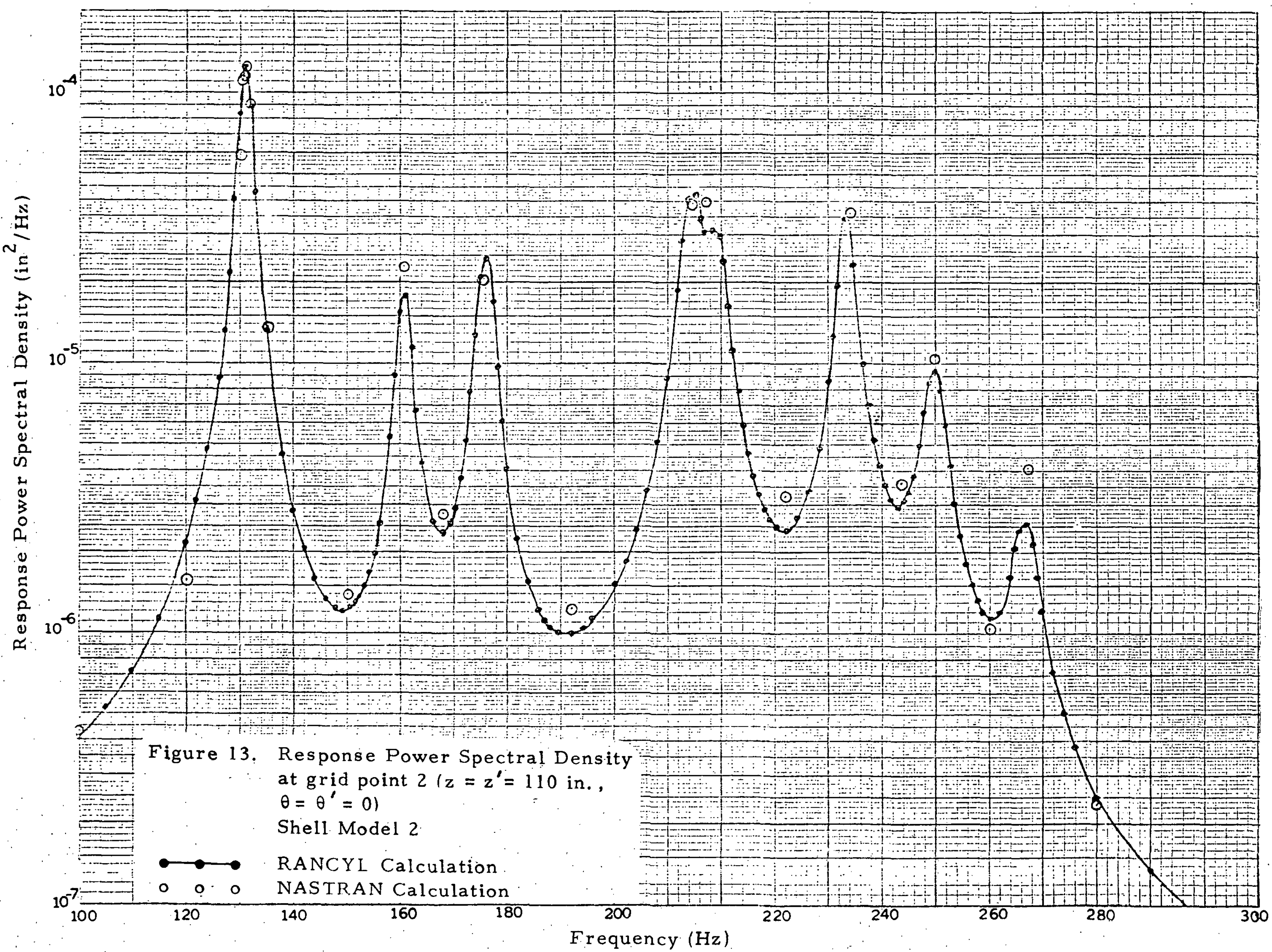

Figure 13. Response Power Spectral Density at grid point 2 ($z = z' = 110$ in., $\theta = \theta' = 0$) Shell Model 2

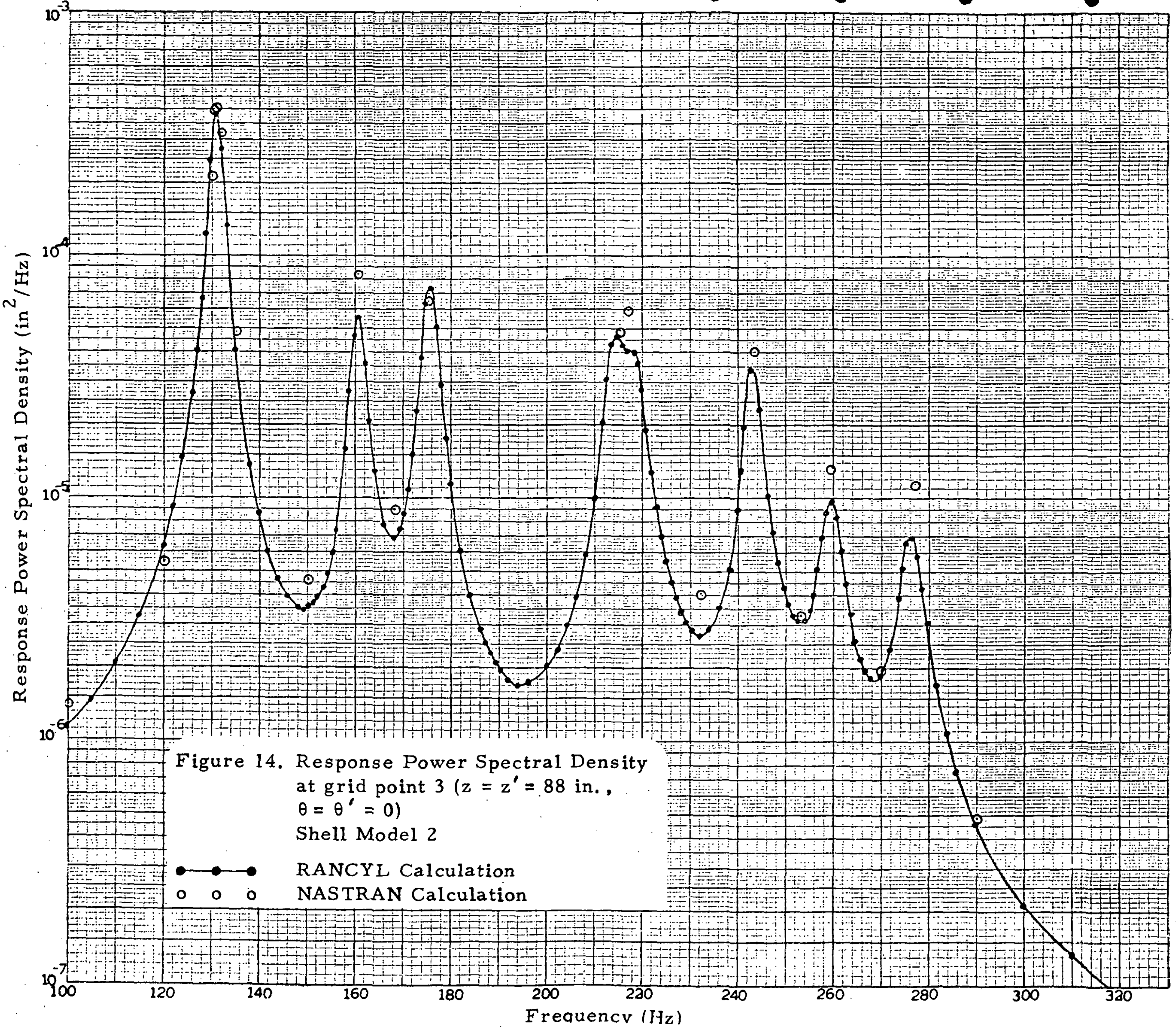

Figure 14. Response Power Spectral Density at grid point 3 ($z = z' = 88$ in., $\theta = \theta' = 0$) Shell Model 2

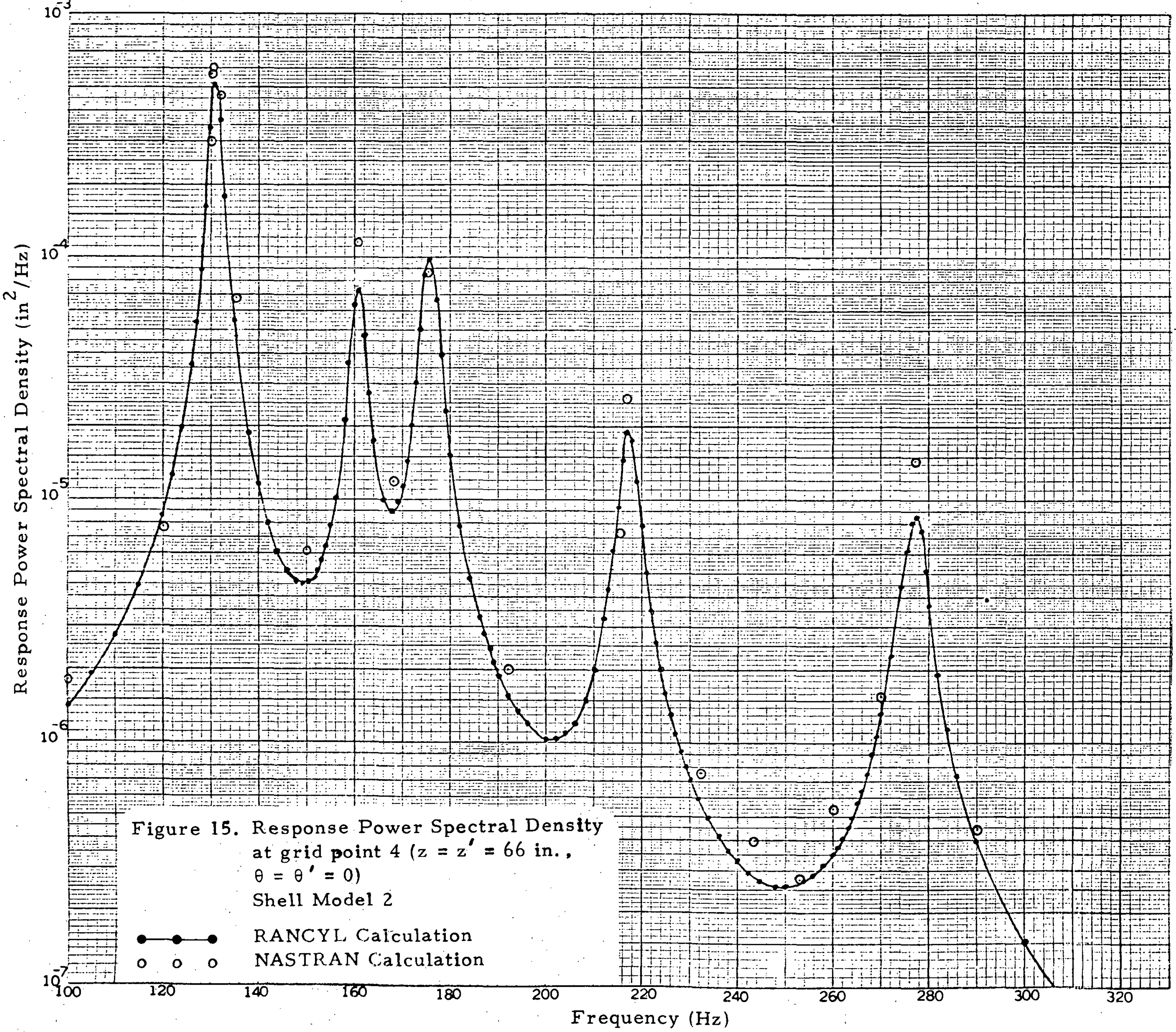

Figure 15. Response Power Spectral Density at grid point 4 ($z = z' = 66$ in., $\theta = \theta' = 0$) Shell Model 2

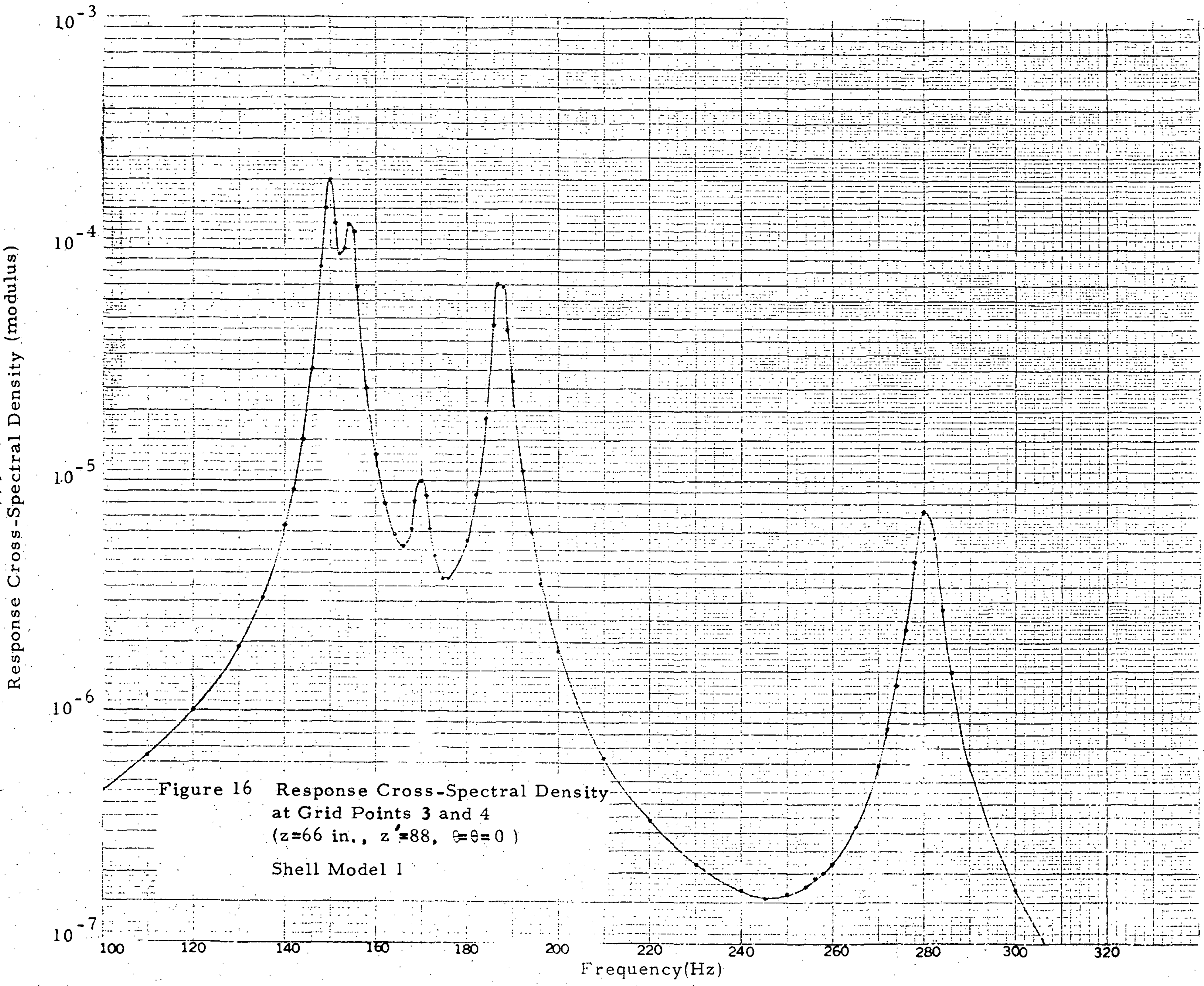

Figure 16 Response Cross-Spectral Density at Grid Points 3 and 4 ($z=66$ in., $z'=88$, $\theta=\theta=0$)

Shell Model 1

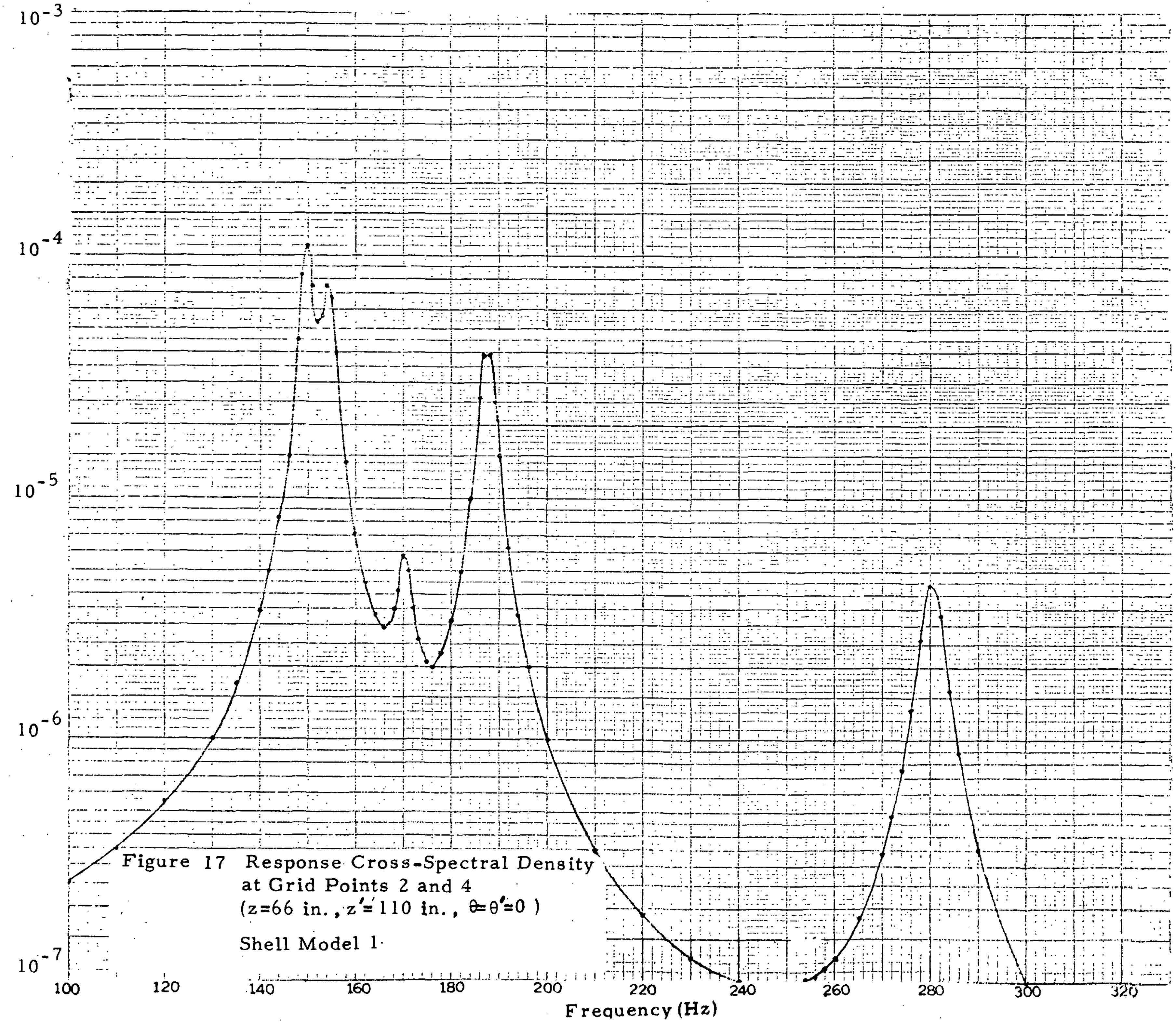

Figure 17 Response Cross-Spectral Density at Grid Points 2 and 4 ($z=66$ in., $z'=110$ in., $\theta=\theta'=0$)

Shell Model 1

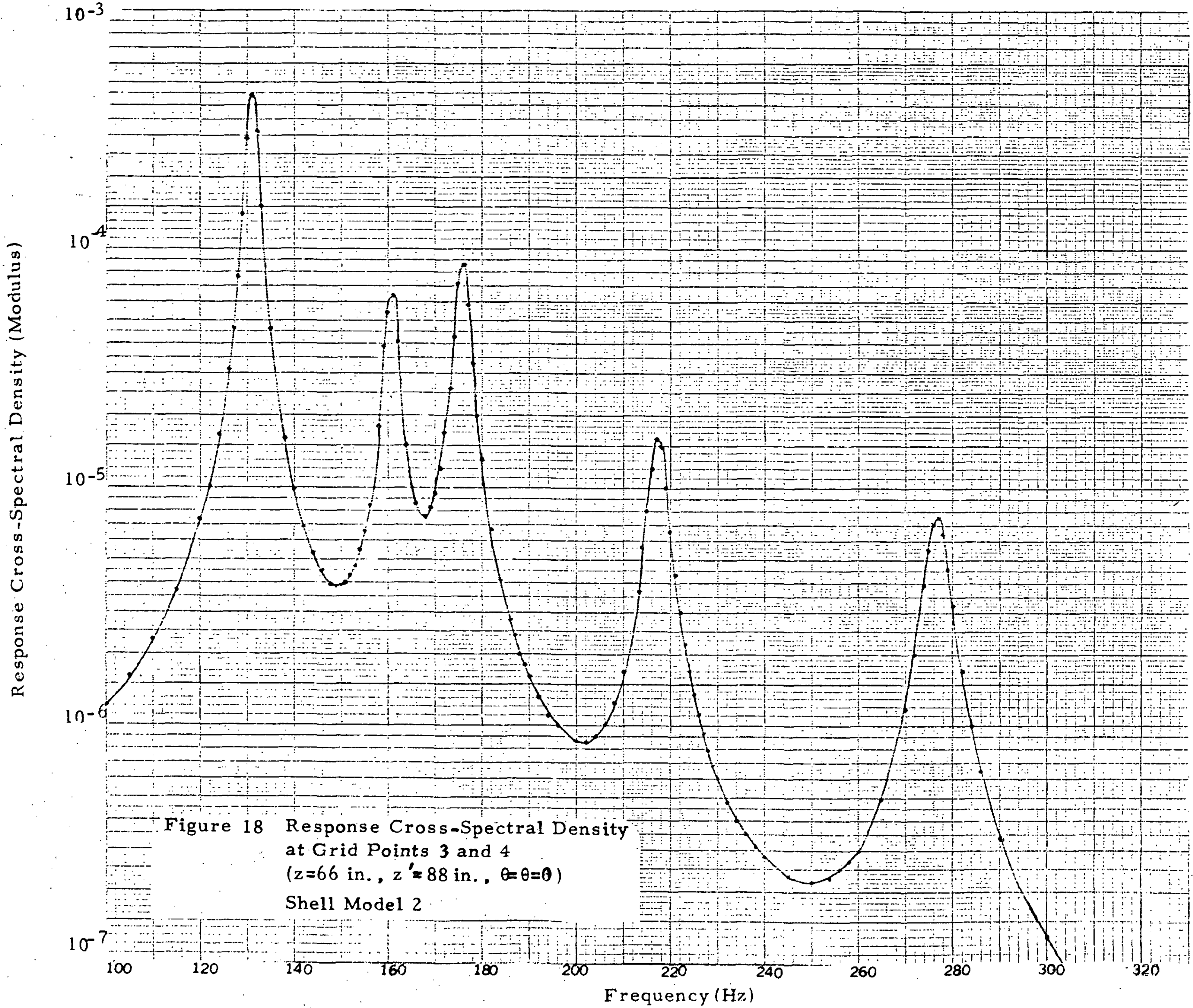

Figure 18 Response Cross-Spectral Density at Grid Points 3 and 4 ($z=66$ in., $z'=88$ in., $\theta=\theta=\theta$)

Shell Model 2

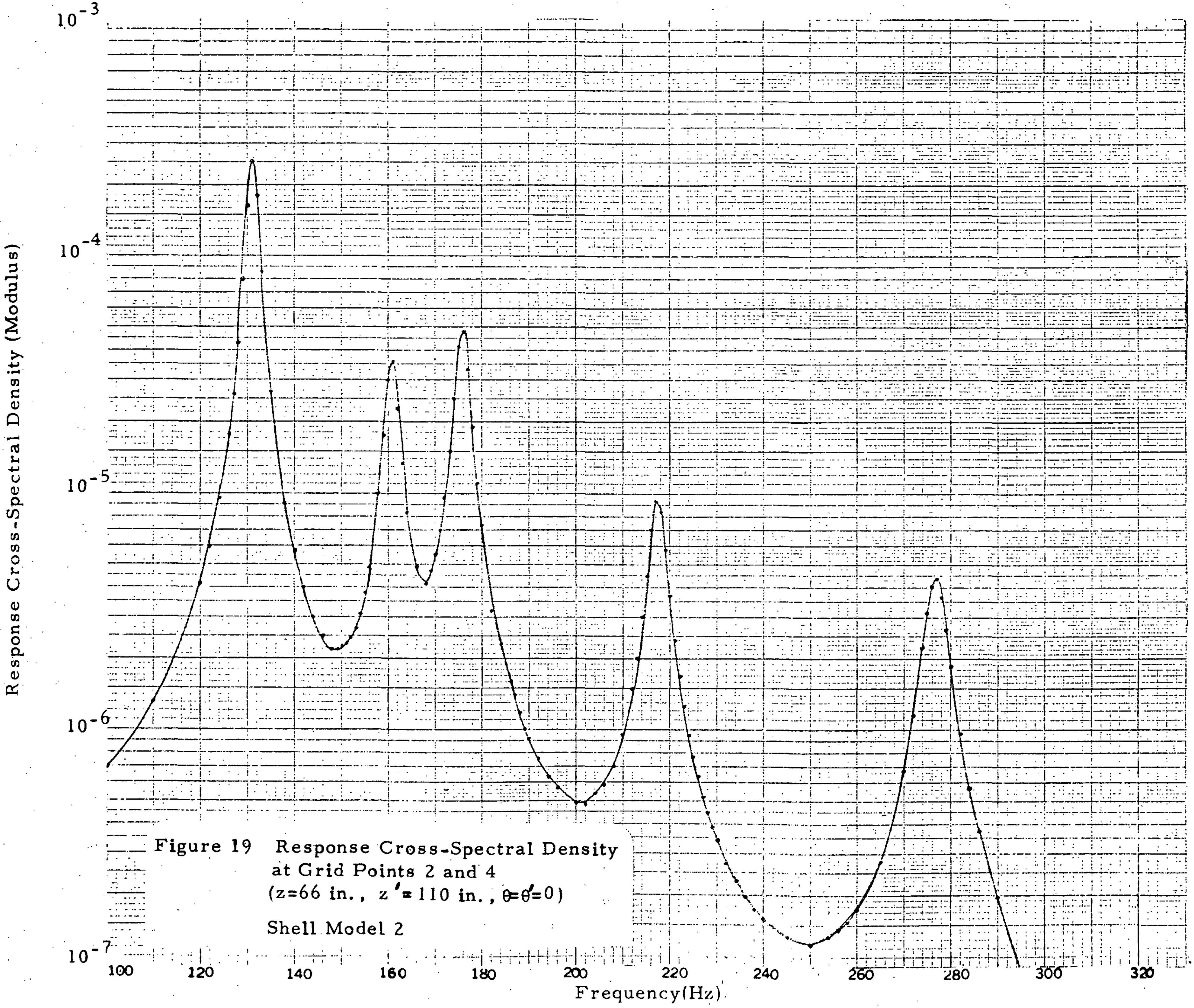

Figure 19 Response Cross-Spectral Density at Grid Points 2 and 4 ($z=66$ in., $z'=110$ in., $\theta=\theta'=0$)

Shell Model 2

and shape of the curves, though magnitude differences exist. These differences are reduced considerably when a more detailed model is used (Shell Model 2) resulting in the very good agreement shown in Figures 13 - 15. That is, the NASTRAN and RANCYL results converge to a particular value when more detailed discrete models are used, as expected. Differences in the high frequency range still persist, however, as one would also expect in going from a discrete to a continuous representation of the structure. The importance of modeling becomes clear by comparing the RANCYL calculation based on Models 1 and 2. Typical of the existing differences is Figure 20 where the power spectral density at grid point 3 of Shell Models 1 and 2 is plotted.

A comparison of computer time required by the two methods was also performed in order to assess the efficiency of the present approach as compared to strictly numerical techniques. With reference to Eq. (48), it is noted that the terms within the brackets (i.e., the terms summed over p, q, r, and s) are independent of frequency and position. For a set of input values, the evaluation of these terms is carried out, once and for all, regardless of the number of spatial points and frequencies at which the response is calculated. The machine time required for the evaluation of these terms was found to be 5.04 sec. and 70 sec. for Shell Model 1 and 2, respectively. In addition to this fixed time, 0.009 sec. per point (i.e., per set of values for z, θ, z', θ' and ω) was required in the case of Model 1 and 0.0107 sec. per point in the case of Model 2. Thus, the machine time, t_m, required by the RANCYL code is related linearly to the number of points at which the response is calculated through the formula

$$t_m = a + bn \quad \text{sec.} \qquad (65)$$

where n is the number of points and where $a = 5.04$, $b = 0.009$ in the case of Model 1 and $a = 70$, $b = 0.0107$ in the case of Model 2. The above

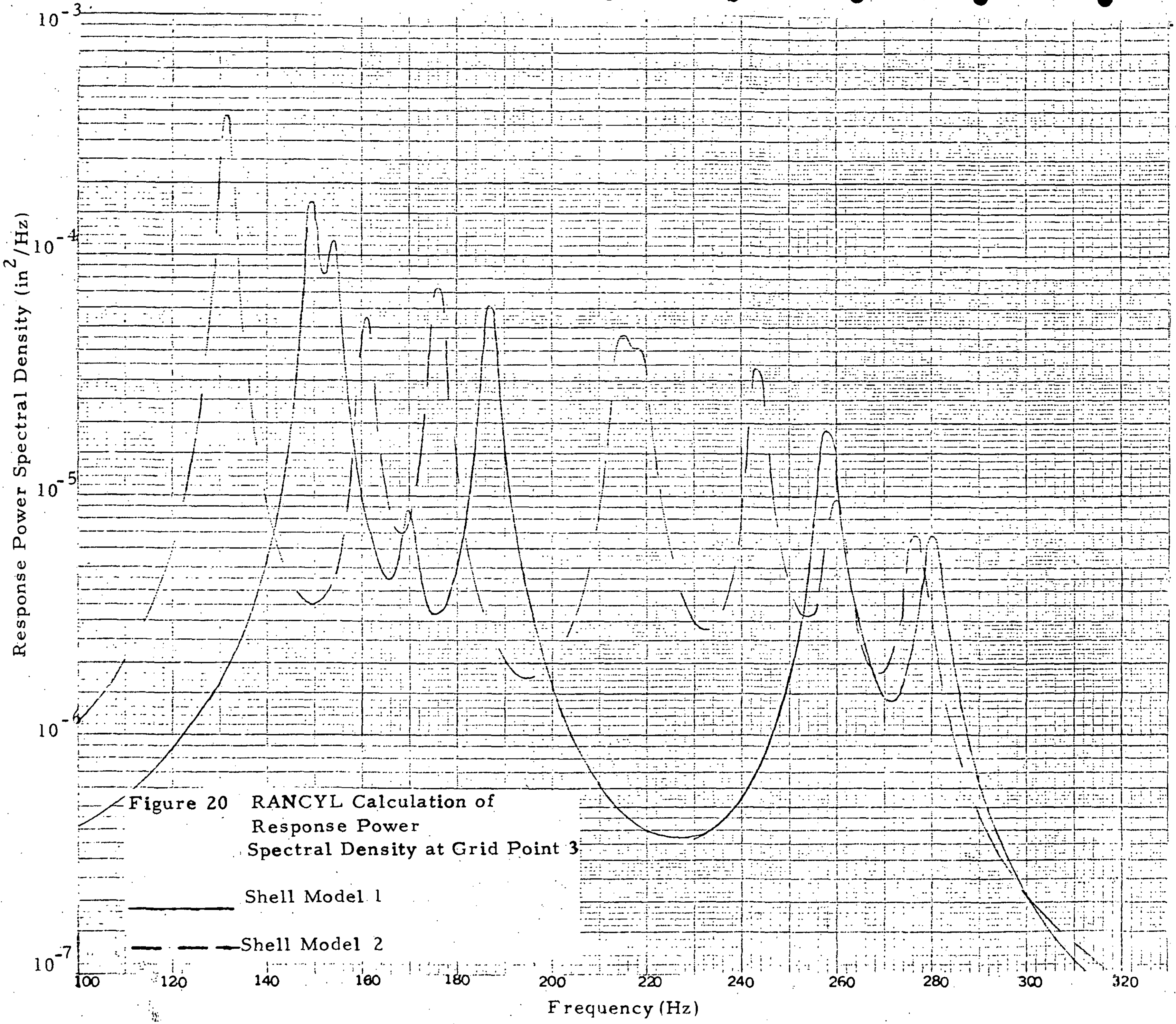

Figure 20 RANCYL Calculation of Response Power Spectral Density at Grid Point 3

time requirement is the same in both power spectral and cross-spectral density calculations. Typically, each curve in Figures 10-19 represents part of a single computer run involving approximately 75 points (i. e., 75 frequency values at the coordinates of the chosen grid point), resulting in a total machine time of 5.72 sec. and 70.8 sec. for Model 1 and 2, respectively. In this particular application of the NASTRAN code, up to nine frequencies per run were input at each grid point in the Model 1 power spectral density analysis and two frequencies per run were input in the case of Model 2. The corresponding machine times are shown in Table 2. The RANCYL time requirements for the same number of points, are also shown. These values are based on Eq. (65).

MODEL	NO. of POINTS	MACHINE TIME	
		NASTRAN (sec)	RANCYL (sec)
1	7	636	5.1
1	9	750	5.12
2	2	444	70.02

Table 2 Machine Time Requirements in the Response Evaluation of the Sample Problem

It should be noted that the NASTRAN code was run on IBM 360-95 computer, while RANCYL was run on a Univac 1108 computer. The latter machine is slower by at least a factor of three. On the other hand, some fraction of the time required by the FITMSC code for the trigonometric interpolation of the mode-shape data should be added on to the RANCYL entries in Table 2. In this particular sample problem, the curve-fitting routine required a total of 77 sec. for twelve modes of Model 1 and 115 sec. for eighteen modes of Model 2. The time requirement of this routine depends strictly on the number of modes and on the number of points per mode. It is an initial time investment, independent of the evaluation of Eq. (48) and the number of points at which the response is evaluated.

5.0 CONCLUDING REMARKS

1. A methodology for the calculation of structural response has been presented and applied to nonuniform beams and cylindrical shells. The method rests on the use of parametric representations for the impulse response and input functions. Use of these representations makes possible the derivation of analytic response solutions applicable to a host of structures of the above geometries and pressure fields. The arbitrary parameters in the response solutions can be fixed once results of the numerical solution to the eigenvalue problem and a particular input function are available.

2. Use of the developed computer codes, based on the analytical results, is simple and efficient. For instance, for an initial run of the RANCYL code, it is necessary to manually keypunch about a dozen cards only (the remaining cards are direct output of the FITMSC code). For a restart run, only a few cards need to be punched manually. In general, only those cards containing the grid point coordinates or frequency values are changed. In its present form, given two or more spatial points, the code computed the response power spectral and cross-spectral densities at these points simultaneously for up to seventy-five frequencies per run. The dimensions of the pertinent code variables may still be increased significantly without exceeding core storage to allow for more frequencies, mode shapes, grid points, et cetera.

3. The spatial-temporal distribution of pressure (or, the corresponding spectrum) must be given in one of two forms for the analyst to determine values for the relevant input parameters to the code: if an analytic function is given, values for these parameters can easily be specified by direct comparison of the given input function to the class of functions considered in this study. Such has been the case in the sample problem (Section 4.0). If pressure field data is given, then the analyst must either empirically

fit a function to the data, or employ some interpolation technique. In the first instance, the procedure for identifying input values for the pressure field parameters is the same as before, namely, by simple comparison. In the latter instance, readily available interpolation routines involve either polynomials or trigonometric functions, both of which belong to the class of input functions considered. Such a routine can easily be coupled to the response codes so that its output (i. e., values for the coefficients) can be directly input to the RANCYL or DEXCYL codes.

4. The FITMSC code has been most successful in its application to the sample problem, both from the standpoint of the goodness of the fits to the given mode-shape data and the minimal computer time necessary. This, however, may not always be the case. Difficulties may arise in the application of the code to shells of unusual construction, especially for high frequency modes. It is anticipated, however, that in the case of most common engineering applications, this will not be the case.

5. The sample problem has demonstrated some of the advantages of the tools developed in this study. Although general conclusions valid for all applications should not be drawn on this basis, some features of the comparison with the NASTRAN computation are indicative of the usefulness of the developed codes. Perhaps the most striking features are the efficiency, and hence low cost of operation (Table 2), the ease with which the analyst can use RANCYL and the number of spatial-frequency points per run at which the response can be evaluated (e. g., in the case of shell model 1, a RANCYL run accommodated seventy-five frequency values. NASTRAN accommodated nine). Comparison of the results of the two calculations (Figures 10-15) points out the importance of modeling in response calculations and the usefulness of the developed

codes as guides in the application of the discrete element approach. The differences in the results stem from both modeling of the discrete structure and modeling of the driving pressure field. In the discrete element approach (e.g., NASTRAN), the pressure field is specified by discrete values only at the structural grid points; the present method requires a continuous representation of the pressure, i.e., a function. As the number of structural grid points considered increases, the model better approximates the continuous structure. Moreover, pressure field values at a larger number of points are specified, so that the discrete pressure field representation better approximates a continuous function. Indeed, the disagreement in the results, exemplified particularly in the vicinity of 165 Hz, is considerably reduced when a more detailed discrete model is used. In fact, the agreement in results based on model 2 is surprisingly good.

6. The results of this study indicate that the developed codes can provide a valuable supplement to numerical analyses of complex structures, especially in the early stages of design of a vehicle where ease and speed of computation is crucial. The application of the developed methodology to other structural shapes (coordinate systems) should further increase its usefulness and versatility. Such an extension follows the same procedure, (Figure 1). However, the mathematical manipulation can be greatly complicated depending upon the coordinate system used. This is particularly true if the structure and driving pressure field can not be represented most conveniently in the same coordinate system.

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1. No attempt is made here to give an exhaustive list of references on the subject of structural dynamics. For a comprehensive bibliography the reader is referred to:

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APPENDIX

Evaluation of $P_{pr}(L)$, $S_{qs}(2\pi)$

Consider Equation (51). Using Equation (34) and letting

$$f(z, z') = \varphi_p(z)\, \varphi_r(z') = \exp\left[i(k_p z + k_r z')\right]$$

$$P_{pr}(L) = A_o \int_o^L dz_o \int_o^L dz'_o \; f(z_o, z'_o) + A_1 \int_o^L dz_o \int_o^L dz'_o \; f(z_o, z'_o)\, \delta(z_o - z'_o)$$

$$+ \sum_{\alpha,\beta} \frac{B_{\alpha,\beta}}{(\beta-1)!}\, i^{\beta} \int_o^L dz_o \int_o^L dz'_o \; f(z_o, z'_o)\, e^{i\kappa_\alpha Z_o}\, Z_o^{\beta-1}\, U(Z_o)$$

$$- \sum_{\gamma,\epsilon} \frac{C_{\gamma,\epsilon}}{(\epsilon-1)!}\, i^{\epsilon} \int_o^L dz_o \int_o^L dz'_o \; f(z_o, z'_o)\, e^{i\kappa_\gamma Z_o}\, Z_o^{\epsilon-1}\, U(-Z_o) \qquad \text{(A1)}$$

where $Z = z - z'$. The first two terms are evaluated trivially giving rise to the first two terms in Equation (53). To evaluate the third integral in Equation (A1), let $z'_o = z_o - Z_o$ to obtain

$$\int_o^L dz_o \int_{z_o - L}^{z_o} dZ_o \; f(z_o, z_o - Z_o)\, e^{i\kappa_\alpha Z_o}\, Z_o^{\beta-1}\, U(Z_o)$$

Since the range of z_o is always less than L, the lower limit of the Z_o integration is always negative. In view of the step function $U(Z_o)$, it follows that the integral to be evaluated is

$$\int_0^L dz_o \int_0^{z_o} dZ_o\ f(z_o, z_o - Z_o)\ e^{i\kappa_\alpha Z_o}\ Z_o^{\beta-1}$$

$$= \int_0^L dz_o\ \varphi_p(z_o) \int_0^{z_o} dZ_o\ \varphi_r(z_o - Z_o)\ e^{i\kappa_\alpha Z_o}\ Z_o^{\beta-1}$$

$$= \int_0^L dz_o\ e^{i(k_p + k_r)z_o} \int_0^{z_o} dZ_o\ e^{-i(k_r - \kappa_\alpha)Z_o}\ Z_o^{\beta-1}$$

Repeated application of the formula

$$\int dx\ e^{ax}\ x^n = e^{ax} \left\{ \frac{x^n}{a} + \sum_{k=1}^{n} \frac{(-1)^k n(n-1)(n-2)\ldots(n-k+1)}{a^{k+1}}\ x^{n-k} \right\} \tag{A2}$$

gives rise to the term within the α, β-sum in Equation (53). Similarly, to evaluate the fourth integral in Equation (A1), let $z_o' = z_o - Z_o$ to obtain

$$\int_0^L dz_o \int_{z_o - L}^{z_o} dZ_o\ f(z_o, z_o - Z_o)\ e^{i\kappa_\gamma Z_o}\ Z_o^{\epsilon-1}\ U(-Z_o)$$

Since the integrand vanishes for $Z_o > 0$ and the upper limit of the Z_o-integration is always positive, the integral to be evaluated is

$$\int_0^L dz_o \int_{z_o - L}^{0} dZ_o\ f(z_o, z_o - Z_o)\ e^{i\kappa_\gamma Z_o}\ Z_o^{\epsilon-1}$$

$$= \int_0^L dz_o\ \varphi_p(z_o) \int_{z_o - L}^{0} dZ_o\ \varphi_r(z_o - Z_o)\ e^{i\kappa_\gamma Z_o}\ Z_o^{\epsilon-1}$$

$$= \int_0^L dz_o\ e^{i(k_p + k_r)z_o} \int_{z_o - L}^{0} dZ_o\ e^{-i(k_r - \kappa_\gamma)Z_o}\ Z_o^{\epsilon-1}$$

Repeated use of Equation (A2) yields the term within the γ, ϵ-sum in Equation (53).

In the evaluation of $S_{qs}(2\pi)$, we note that the first two terms of Equation (36) in (52) give rise to the first two terms of Equation (54) in a straightforward manner. The evaluation for the third term of Equation (36) is somewhat more complex because of the conditions on $Q_2(\theta - \theta')$ given in Equation (31). These conditions can be combined in a single equation by introducing appropriate step functions:

$$Q_2(\Theta) = Q_2(|\Theta|)\ U(\pi - |\Theta|) + Q_2(2\pi - |\Theta|)\ U(|\Theta| - \pi) \qquad \text{(A3)}$$

where $\Theta = \theta - \theta'$. Using Equation (A3) in Equation (52) and letting

$$f(\theta, \theta') = \varphi_q(\theta)\ \varphi_s(\theta') = \exp\left[i(k_q \theta + k_s \theta')\right], \qquad \text{(A4)}$$

we obtain

$$S_{qs}(2\pi) = \int_0^{2\pi} d\theta_o \int_0^{2\pi} d\theta'_o\ f(\theta_o, \theta'_o)\ Q_2(|\Theta_o|)\ U(\pi - |\Theta_o|)$$

$$+ \int_0^{2\pi} d\theta_o \int_0^{2\pi} d\theta'_o\ f(\theta_o, \theta'_o)\ Q_2(2\pi - |\Theta_o|\ U(|\Theta_o| - \pi) \qquad \text{(A5)}$$

Under the change of variable $\theta'_o = \theta_o - \Theta_o$, Equation (A5) becomes

$$S_{qs}(2\pi) = \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(|\Theta_o|) \, U(\pi - |\Theta_o|)$$

$$+ \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(2\pi - |\Theta_o|) \, U(|\Theta_o| - \pi)$$

$$= \int_0^{2\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(\Theta_o) \, U(\pi - \Theta_o)$$

$$+ \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(-\Theta_o) \, U(\pi + \Theta_o)$$

$$+ \int_0^{2\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(2\pi - \Theta_o) \, U(\Theta_o - \pi)$$

$$+ \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(2\pi + \Theta_o) \, U(-\Theta_o - \pi) \qquad \text{(A6)}$$

Let

$$S_1 = \int_0^{2\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(\Theta_o) \, U(\pi - \Theta_o) \qquad \text{(A7)}$$

$$S_2 = \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(-\Theta_o) \, U(\pi + \Theta_o) \qquad \text{(A8)}$$

$$S_3 = \int_0^{2\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \, f(\theta_o, \theta_o - \Theta_o) \, Q_2(2\pi - \Theta_o) \, U(\Theta_o - \pi) \qquad \text{(A9)}$$

$$S_4 = \int_0^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{o} d\Theta_o \; f(\theta_o, \theta_o - \Theta_o)\, Q_2(2\pi + \Theta_o)\, U(-\Theta_o - \pi) \qquad \text{(A10)}$$

To establish the regions of integration in Equations (A7) - (A10), one need only plot the limits of the θ_o, Θ_o integrations and the domain in which the integrands do not vanish, that is, where the step functions are unity. This is done in Figure 21. From this figure, it follows that

$$S_1 = \int_0^{\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \; F(\theta_o, \Theta_o) + \int_{\pi}^{2\pi} d\theta_o \int_0^{\pi} d\Theta_o \; F(\theta_o, \Theta_o) \qquad \text{(A11)}$$

$$S_2 = \int_0^{\pi} d\theta_o \int_{-\pi}^{o} d\Theta_o \; G(\theta_o, \Theta_o) + \int_{\pi}^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{0} d\Theta_o \; G(\theta_o, \Theta_o) \qquad \text{(A12)}$$

$$S_3 = \int_{\pi}^{2\pi} d\theta_o \int_{\pi}^{\theta_o} d\Theta_o \; H(\theta_o, \Theta_o) \qquad \text{(A13)}$$

$$S_4 = \int_0^{\pi} d\theta_o \int_{\theta_o - 2\pi}^{-\pi} d\Theta_o \; J(\theta_o, \Theta_o) \qquad \text{(A14)}$$

where

$$F(\theta_o, \Theta_o) = f(\theta_o, \theta_o - \Theta_o)\; Q_2(\Theta_o) \qquad \text{(A15)}$$

$$G(\theta_o, \Theta_o) = f(\theta_o, \theta_o - \Theta_o)\; Q_2(-\Theta_o) \qquad \text{(A16)}$$

$$H(\theta_o, \Theta_o) = f(\theta_o, \theta_o - \Theta_o)\; Q_2(2\pi - \Theta_o) \qquad \text{(A17)}$$

$$J(\theta_o, \Theta_o) = f(\theta_o, \theta_o - \Theta_o)\; Q_2(2\pi + \Theta_o) \qquad \text{(A18)}$$

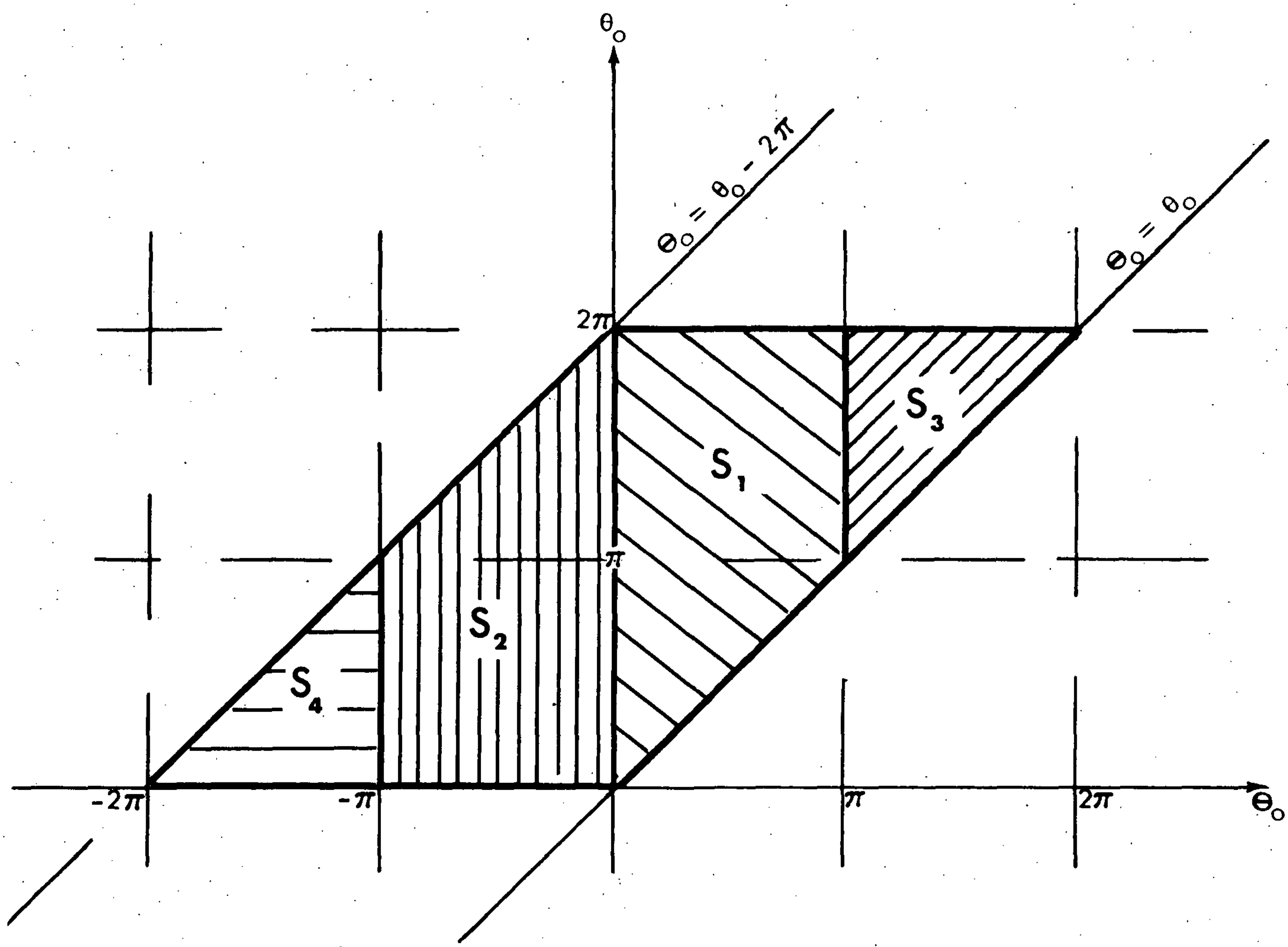

FIGURE 21. Regions of integration in the evaluation of S_1, S_2, S_3, S_4; Equations (A7) - (A10)

Consider Equation (A11). Using Equations (A15), (A4) and the third term of Equation (36),

$$S_1 = \int_0^{\pi} d\theta_o \int_0^{\theta_o} d\Theta_o \; \varphi_q(\theta_o) \; \varphi_s(\theta_o - \Theta_o) \; Q_2(\Theta_o)$$

$$+ \int_{\pi}^{2\pi} d\theta_o \int_0^{\pi} d\Theta_o \; \varphi_q(\theta_o) \; \varphi_s(\theta_o - \Theta_o) \; Q_2(\Theta_o)$$

$$= \sum_{\eta, \sigma} E_{\eta, \sigma} \Bigg[\int_0^{\pi} d\theta_o \; e^{i(k_q + k_s)\theta_o} \int_0^{\theta_o} d\Theta_o \; e^{-i(k_s - \kappa_\eta)\Theta_o} \; \Theta_o^{\sigma}$$

$$+ \int_{\pi}^{2\pi} d\theta_o \; e^{i(k_q + k_s)\theta_s} \int_0^{\pi} d\Theta_o \; e^{-i(k_s - \kappa_\eta)\Theta_o} \; \Theta_o^{\sigma} \Bigg]$$

Repeated application of Equation (A2) yields Equation (55). Similarly, with Equations (A16) and (A4) in (A12)

$$S_2 = \int_0^{\pi} d\theta_o \int_{-\pi}^{0} d\Theta_o \; \varphi_q(\theta_o) \; \varphi_s(\theta_o - \Theta_o) \; Q_2(-\Theta_o)$$

$$+ \int_{\pi}^{2\pi} d\theta_o \int_{\theta_o - 2\pi}^{0} d\Theta_o \; \varphi_q(\theta_o) \; \varphi_s(\theta_o - \Theta_o) \; Q_2(-\Theta_o)$$

Letting $y = -\Theta_o$ and using Equation (36), we have

$$S_2 = \int_0^{\pi} d\theta_o \; \varphi_q(\theta_o) \int_0^{\pi} dy \; \varphi_s(\theta + y) \; Q_2(y)$$

$$+ \int_{\pi}^{2\pi} d\theta_o \; \varphi_q(\theta_o) \int_0^{2\pi - \theta_o} dy \; \varphi_s(\theta_o + y) \; Q_2(y)$$

$$= \sum_{\eta,\sigma} E_{\eta,\sigma} \left[\int_0^{\pi} d\theta_o \, e^{i(k_q+k_s)\theta_o} \int_0^{\pi} dy \, e^{i(k_s+\kappa_\eta)y} \, y^{\sigma} \right.$$

$$\left. + \int_{\pi}^{2\pi} d\theta_o \, e^{i(k_q+k_s)\theta_o} \int_0^{2\pi-\theta_o} dy \, e^{i(k_s+\kappa_\eta)y} \, y^{\sigma} \right] ,$$

which can be evaluated by application of Equation (A2) to yield Equation (56). To evaluate S_3, let $y = 2\pi - \Theta_o$ in Equation (A13) to obtain

$$S_3 = \int_{\pi}^{2\pi} d\theta_o \int_{2\pi-\theta_o}^{\pi} dy \; H(\theta_o, 2\pi - y)$$

$$= \int_{\pi}^{2\pi} d\theta_o \int_{2\pi-\theta_o}^{\pi} dy \; f(\theta_o, \theta_o + y - 2\pi) \; Q_2(y)$$

$$= \int_{\pi}^{2\pi} d\theta_o \; \varphi_q(\theta_o) \int_{2\pi-\theta_o}^{\pi} dy \; \varphi_s(\theta + y - 2\pi) \; Q_2(y)$$

$$= \sum_{\eta,\sigma} E_{\eta,\sigma} \, e^{-2\pi i k_s} \int_{\pi}^{2\pi} d\theta_o \, e^{i(k_q+k_s)\theta_o} \int_{2\pi-\theta_o}^{\pi} dy \, e^{i(k_s+\kappa_\eta)y} \, y^{\sigma} \qquad \text{(A19)}$$

where use has been made of Equations (A17), (A4) and (36). Similarly, let $y = 2\pi + \Theta_o$ in Equation (A14) to obtain

$$S_4 = \int_0^{\pi} d\theta_o \int_{\theta_o}^{\pi} dy \; J(\theta_o, y - 2\pi) = \int_0^{\pi} d\theta_o \int_{\theta_o}^{\pi} dy \; f(\theta_o, \theta_o - y + 2\pi) \; Q_2(y)$$

$$= \int_0^{\pi} d\theta_o \; \varphi_q(\theta_o) \int_{\theta_o}^{\pi} dy \; \varphi_s(\theta_o - y + 2\pi) \; Q_2(y)$$

$$= \sum_{\eta,\sigma} E_{\eta,\sigma} \, e^{2\pi i k_s} \int_0^{\pi} d\theta_o \, e^{i(k_q+k_s)\theta_o} \int_{\theta_o}^{\pi} dy \, e^{-i(k_s-\kappa_\eta)y} \, y^{\sigma} \qquad \text{(A20)}$$

where Equations (A18), (A4) and (36) have been used. Equations (A19) and (A20) can now be evaluated using Equation (A2) to yield Equations (57) and (58), respectively.