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COMPARISON OF EXACT AND APPROXIMATE EVALUATIONS OF THE SINGLE-SCATTERING INTEGRAL IN NUCLEON-DEUTERON ELASTIC SCATTERING

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SUMMARY

The exact nucleon-deuteron elastic single-scattering integral has been calculated numerically so that errors in sticking-factor approximations can be evaluated. Evaluations of these errors are important since approximation of the multiple-scattering series appears to be at present the most practicable approach to nucleon-deuteron scattering at intermediate energy and the single-scattering integral is a principal contribution. Errors in sticking-factor approximations to the single-scattering integral made by using realistic interactions are typically 10 to 20 percent except at backward angles where errors are even larger. A similar analysis made by using S-wave separable potentials concluded that errors for these same approximations were negligible except near backward angles where they were found to be about 10 percent. The present results impose questions as to the usefulness of such unrealistic model calculations at these energies.

INTRODUCTION

The purpose of analyzing nucleon-deuteron elastic scattering is to study the internal structure of this bound nuclear state. There are three competing methods for anlysis of the nucleon-deuteron system. The usefulness of Faddeev and disperion calculations of nucleon-deuteron scattering is presently limited by simplifying assumptions which have made these calculations tractable. Faddeev calculations are made by the use of separable interactions (ref. 1) and usually allow incorporation of only low-energy shape-independent nucleon-nucleon data (ref. 2). Recent dispersion calculations (ref. 3) have neglected spin, and the spin correlations required to specify fully the scattering matrix are not determined. Currently, the most practicable approach to nucleon-deuteron scattering at intermediate energy is approximation via the multiple-scattering series. The multiple-scattering series relates the nucleon-deuteron scattering amplitude to sums of integrals over nuclear-wave functions, nucleon propagators, and in their fullest complexity, the off-shell two-body amplitudes (refs. 4 to 10), at least to the extent that they are known.

Extraction of deuteron wave functions from elastic nucleon-deuteron scattering data must be preceded by a careful analysis of theoretical uncertainties of the formalism used. The errors associated with the use of the multiple-scattering series are the uncertainty in obtaining off-shell values of the two-body amplitudes, those associated with approximating the integration of successive terms in the series, and those associated with the remainder since any practical calculation can explicitly include only a finite number of terms. Additional uncertainty owing to lack of knowledge of the deuteron wave function is not considered since a knowledge of the wave function is the ultimate goal. This paper is a first step in the evaluation of these errors.

The single-scattering integral, first nontrivial term of the multiple-scattering series, is a principal contribution of the elastic T-matrix and is nearly the sole contribution near forward scattering. This integral over wave functions and a two-body amplitude is often approximated by assuming little or no dependency of the two-body amplitude on the internal motion of the target nucleus (refs. 5, 7, and 8). Although model calculations made by using separable potentials indicate that some of these low-order approximations are adequate (ref. 9), an analysis of absolute error with realistic two-body amplitudes has not been made until now.

It is customary in multiple-scattering calculations to take the appropriate off-shell two-body amplitudes to be the on-shell amplitudes evaluated at the same momentum transfer and an energy which depends on the nucleon-nucleus laboratory energy, the momentum transfer, and internal target motion (refs. 5 to 10). This point of view is strengthened by the results for the so-called "linear approximation" of Kowalski and Feldman (ref. 5) for nucleon-deuteron scattering. However, in order to evaluate the single-scattering integrand throughout the integration domain, off-shell continuation must be considered.

In the present paper, the off-shell continuation of the two-body amplitudes is discussed. By using this continuation, the errors of various low-order approximations of the single-scattering integral are evaluated. The principal sources of error in low-order approximations are indicated, and alternate numerical techniques are suggested for eventual analysis of experimental elastic nucleon-deuteron scattering data.

SYMBOLS

a_{L_n}, b_{L_n} structure constants of deuteron wave function $n = 1, 2, 3$

D deuteron four momentum, amu

E_{lab} proton laboratory energy, MeV

$i = \sqrt{-1}$

j_β nucleon current matrix element, β = S, T, V, A, P, amu

L deuteron orbital angular momentum quantum numbers, dimensionless

l nucleon-nucleon orbital angular momentum quantum numbers, dimensionless

m nucleon mass, 1.0 amu

m_π pion mass, 0.14847 amu

P nucleon four momentum, amu

P_1 struck deuteron constituent four momentum, amu

P_2 spectator deuteron constituent four momentum, amu

$\vec{Q}$ momentum transfer in center of mass, amu

s,t,u Mandelstam variables for two-nucleon system, amu^2

$\hat{s}$ Lorentz invariant, amu^2

$S_{L,L}(Q)$ deuteron sticking factor, dimensionless

T_s single-scattering transition matrix, amu^{-1}

t_β Lorentz invariant two-body amplitudes in current × current basis, amu^{-3}

Y_L 2×2 representation of spin spherical harmonics with orbital angular momentum L, dimensionless

z off-shell two-body cosine of center-of-mass scattering angle, dimensionless

$\alpha = b_{L E_n} + b_{L F_m}$

δ nucleon mass difference, off-shell parameter, amu^2

θ_{cm} two-body center-of-mass scattering angle, rad

$\vec{\kappa}$ deuteron internal momentum, amu

$\vec{\pi}$ deuteron polarization vector, dimensionless

$\vec{\sigma}$ Pauli spin vector, dimensionless

ϕ_L deuteron radial wave function, $amu^{3/2}$

Subscripts:

E entering state

F final state

MPE multipion exchange

OPE one-pion exchange

Three vectors are denoted by $\rightarrow$ above the quantity. Four vectors carry no special annotation. Dot products among four vectors $b = (b^o,\vec{b})$ and $c = (c^o,\vec{c})$ are as

$$b \cdot c = b^o c^o - \vec{b} \cdot \vec{c}$$

Dagger (†) denotes Hermitian conjugation. Circumflex (^) denotes unit vector.

SINGLE-SCATTERING INTEGRAL

The integral which represents the scattering of the incident nucleon from a constituent nucleon in the deuteron is the 2×2 matrix

$$T_S(P_F D_F; P_E D_E) = \sum_{L_E L_F} \int \tau(s,t,u,\delta)_{L_E L_F} \phi_{L_F}(\vec{\kappa}_F) \phi_{L_E}(\vec{\kappa}_E) \, d^3\kappa \qquad (1)$$

where subscripts E and F refer to the entering and final states, respectively. The sums over L include the two angular momentum states of the deuteron wave functions $\phi_L(\vec{\kappa})$ where $\vec{\kappa}_{E,F} = \vec{\kappa} \pm \frac{1}{4}\vec{Q}$ with $\vec{Q}$ the three momentum transfer in the center-of-mass frame. The deuteron spin spherical harmonics are included in the symbol $\tau(s,t,u,\delta)$ along with the two-body scattering amplitudes for simplicity of notation (spin

indices and deuteron polarization vectors are suppressed)

$$\tau(s,t,u,\delta)_{L_E L_F} = \sum_{\beta} t_\beta(s,t,u,\delta)\, j_\beta\left(P_F,P_E\right) \left\langle j_\beta\left(P1_F,P1_E\right) Y_{L_E}\left(\vec{\pi}_E,\vec{\kappa}_E\right) Y_{L_F}^{\dagger}\left(\vec{\pi}_F,\vec{\kappa}_F\right)\right\rangle \tag{2}$$

where $\langle\ \rangle$ denotes a trace and $j_\beta(P_1,P_2)$ are the five ordinary β-decay-type currents (given in refs. 11 and 12) which constitute a complete basis for the two-nucleon amplitudes and

$$Y_0\left(\vec{\pi},\vec{\kappa}\right) = \vec{\pi}\cdot\vec{\sigma}/\sqrt{2} \tag{3a}$$

$$Y_2\left(\vec{\pi},\vec{\kappa}\right) = \frac{3}{2}\vec{\pi}\cdot\hat{\kappa}\,\hat{\kappa}\cdot\vec{\sigma} - \frac{1}{2}\vec{\pi}\cdot\vec{\sigma} \tag{3b}$$

are the representation of the $L = 0, 2$ spin spherical harmonics used. The vector $\vec{\pi}$ is the deuteron polarization which depends on the deuteron spin projection. The arguments s, t, and u are the Mandelstam variables, and $t \equiv -\vec{Q}^2$ is a constant in the integral. The fourth argument of τ is linearly independent of s, t, and u and is taken to be the mass difference between the struck constituent in the entering and final states. These variables and the momenta of the two deuteron constituents are defined as

$$P1_E = \frac{1}{2}D_E + \vec{\kappa}_E$$

$$P2_E = D_E - P1_E$$

$$P1_F = \frac{1}{2}D_F + \vec{\kappa}_F$$

$$P2_F = D_F - P1_F$$

$$s = \left(P_E + P1_E\right)^2$$

$$t = \left(P_F - P_E\right)^2$$

$$u = \left(P1_F - P_E\right)^2$$

$$\delta = P1_F^{\,2} - P1_E^{\,2}$$

The spectator in the scattering is the target constituent not struck by the incident projectile and is taken always on-the-mass shell (ref. 10).

The value of equation (1) can be approximated or calculated numerically; however, off-shell values of the two-body amplitudes must be evaluated. In the sequel, the calculation of off-shell values is discussed. This off-shell continuation is then used in several low-order approximations of equation (1). The poor behavior of these low-order approximations is discussed in the context of the one-pion exchange interaction. Additional comparison with a numerical evaluation indicates that the low-order approximations are inadequate.

OFF-SHELL TWO-BODY AMPLITUDES

The two-body Lorentz invariant functions $t_\beta(s,t,u,\delta)$ of equation (2) depend on the four variables s, t, u, and δ which are linearly independent with two nucleons off shell. The behavior of the two-body amplitudes is generally not known except on the two-dimensional subspace

$$\delta = 0$$

$$s + t + u = 4m^2$$

corresponding to on-shell processes which are measured experimentally. A unique extension from this experimentally known subspace to the full space including off-shell processes cannot be made without a detailed knowledge of the strong force. A proposed extension is not completely arbitrary but must be consistent with what is known of the analytic structure (nature and location of singularities) of the two-body amplitudes.

The singularities of the two-body amplitudes which lie closest to the physical region are those corresponding to the exchange of a meson with the smallest mass (a pion). The one-pion exchange (OPE) contributions are simple poles shown in figure 1 along with the multipion exchange (MPE) contributions which are branch-cut singularities at points farther removed from the physical region. The variables chosen for the figure are defined by

$$t = \left(4m^2 - \hat{s}\right)(1 - z)/2$$

$$u = \left(4m^2 - \hat{s}\right)(1 + z)/2$$

where z is the off-shell cosine of the center-of-mass scattering angle and

$$\hat{s} \equiv 4m^2 - t - u$$

so that $\hat{s} \longrightarrow s$ for on-shell processes. In accordance with figure 1

$$t(s,t,u,\delta) = t_{OPE}(t,u) + t_{MPE}(s,t,u,\delta) \tag{4}$$

which is true on or off shell. Any off-shell continuation must be consistent with equation (4).

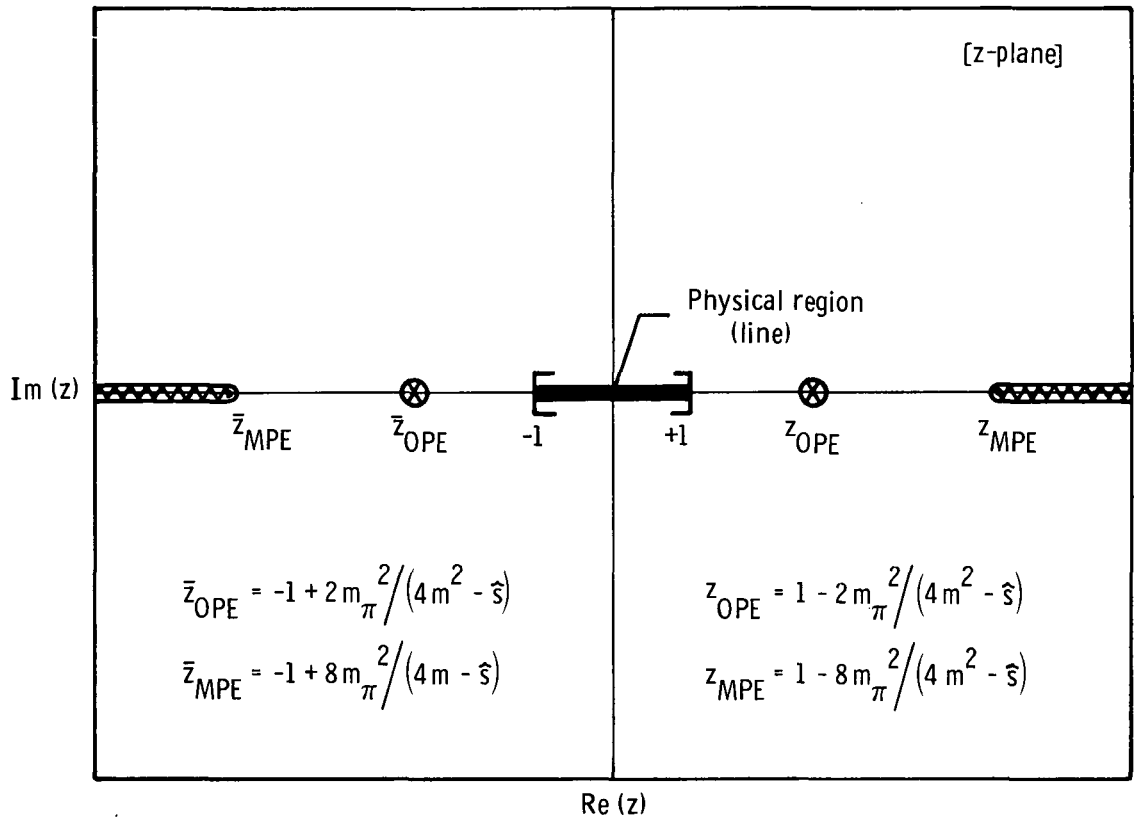


Figure 1.- Singularities of two-body amplitudes as seen in z-plane. $z = \cos \theta_{cm}$.

Now the off-shell continuation used in the present calculation is discussed. The off-shell values of the OPE contribution are exactly known. The singularities of the MPE part are far removed from the physical region as seen in figure 1. Since the MPE amplitude is analytic in the neighborhood of the physical region, the off-shell continuation can be accomplished by a Taylor series expansion

$$t_{MPE}(s,t,u,\delta) = t_{MPE}(\hat{s},t,u,\delta_0) + \frac{\partial}{\partial s} t_{MPE}(\hat{s},t,u,\delta_0)(s - \hat{s}) + \frac{\partial}{\partial \delta} t_{MPE}(\hat{s},t,u,\delta_0)(\delta - \delta_0) + \ldots \tag{5}$$

where $\hat{s}$ and δ_0 are taken to be the on-shell-like values

$$\delta_0 = 0 \tag{6a}$$

$$\hat{s} + t + u = 4m^2 \tag{6b}$$

Since the MPE contribution is a slowly varying function in the neighborhood of the physical region (e.g., in modern phase-shift analysis of refs. 13 and 14, the OPE amplitude is used for all partial waves with $l \gtrsim 5$ with the MPE contributing to the phase shifts with $l \lesssim 5$) it is reasonable to assume that

$$t_{MPE}(s,t,u,\delta) \approx t_{MPE}(\hat{s},t,u,\delta_0) \equiv t_{exp}(\hat{s},t) - t_{OPE}(t,u) \tag{7}$$

where subscript exp denotes an experimentally measured quantity. This continuation is accomplished by subtracting the correct off-shell OPE part from the $t_{exp}(s,t)$ amplitude and replacing the energy variable s by the new quantity $\hat{s}$.

The off-shell continuation given by equations (4) and (7) is now used to study approximations of equation (1). Note that this choice of continuation preserves the location of the branch cuts of the MPE part as seen in figure 1 as well as preserves the proper symmetry of the amplitudes under interchange of identical nucleons (i.e., left-right symmetry in fig. 1). From the relation of δ to the integration variable of equation (1)

$$\delta = P_{1_F}^2 - P_{1_E}^2 = -2\vec{Q} \cdot \vec{\kappa}$$

It follows that

$$\int \phi_L(\vec{\kappa}_F)\, \delta\phi_L(\vec{\kappa}_E)\, d^3\kappa = 0$$

and generally

$$\int \phi_{L'}(\vec{\kappa}_F)\, \delta\phi_L(\vec{\kappa}_E)\, d^3\kappa \approx -\frac{1}{2}\vec{Q}^2 f(Q)$$

Hence, when the entering and final nuclear states are the same, the linear term in δ of expansion in equation (5) is expected to be small for all values of $\vec{Q}$. Even when the entering and final states are different, this off-shell effect is vanishingly small for forward scattering or $\vec{Q}^2 \approx 0$.

EVALUATION OF SINGLE-SCATTERING INTEGRAL

The single-scattering integral (eq. (1)) can be evaluated numerically by using the off-shell continuation (eq. (7)). As noted previously, the variable t is fixed by the momentum transfer while the remaining variables s, u, and δ are functions of the integration variable $\vec{\kappa}$. To aid in evaluation, the deuteron wave function is approximated by a sum of Gaussians

$$\phi_L(\vec{\kappa}) = \sum_n a_{L_n} \exp\left(-b_{L_n}\vec{\kappa}^2\right) \tag{8}$$

The coefficients given in table I are estimates for the Hamada-Johnston wave function (ref. 15) and are reasonable fits to distances of about 0.28 fm. First some low-order approximations to the integral (eq. (1)) are examined.

TABLE I.- FITTED STRUCTURE PARAMETERS FOR GAUSSIAN APPROXIMATION OF HAMADA-JOHNSTON WAVE FUNCTION

$$\left[\text{Normalization } \int_0^\infty \left[\phi_0^2(q) + \phi_2^2(q)\right] q^2\, dq = 1; \quad \phi_L(q) = \sum_n a_{L_n} \exp\left(-b_{L_n}q^2\right)\right]$$

n	a_{0_n}, amu$^{-3/2}$	b_{0_n}, amu^{-2}	a_{2_n}, amu$^{-3/2}$	b_{2_n}, amu^{-2}
1	92.41770	204.15580	-1.683110	11.27690
2	12.74490	25.77230	-.971535	2.85898
3	-.35298	1.29835	2.659846	465.61200

The principal assumption normally used in developing approximations to the integral (eq. (1)) is that the product of two-body amplitudes and spin spherical harmonics (eq. (2)) are slowly varying functions of the integration variable $\vec{\kappa}$ compared to the radial wave functions. Hence, a Taylor series expansion of equation (2) is expected to be rapidly converging when placed into the integral (eq. (1)), and various low-order approximations are obtained by choice of expansion point and the number of terms retained. The development of the more successful approximations is given in the order of increasing accuracy, as well as a discussion of their shortcomings. The Chew impulse approximation (ref. 7) is discussed in reference 5.

Kottler-Kowalski Prescription

The Kottler-Kowalski prescription as described in reference 8 has the advantage of requiring a single evaluation of the two-body amplitudes at $\vec{\kappa} = 0$, thereby leaving an integral over wave functions (sticking factors or form factors) to be performed

$$T_S = \sum \int \tau(s,t,u,\delta)_{L_E L_F} \phi_{L_E}\left(\vec{\kappa}_E\right) \phi_{L_F}\left(\vec{\kappa}_F\right) d^3\kappa$$

$$\approx \sum \tau\left(s_0,t,u_0,\delta_0\right)_{L_E L_F} S\left(\vec{Q}\right)_{L_E L_F} \tag{9}$$

where

$$S\left(\vec{Q}\right)_{L_E L_F} = \int \phi_{L_F}\left(\vec{\kappa} - \frac{1}{4}\vec{Q}\right) \phi_{L_E}\left(\vec{\kappa} + \frac{1}{4}\vec{Q}\right) d^3\kappa$$

and s_0, u_0, and δ_0 are values at $\vec{\kappa} = 0$. These form factors differ from those normally used in electron-deuteron elastic scattering calculations. The value δ_0 is equal to its on-shell value so that off-shell effects appear through the variables s_0 and u_0 only. Had the nonrelativistic Galilean invariants been chosen to describe the two-body amplitudes, then $\vec{\kappa} = 0$ is sufficient to place the two-body reaction on-the-energy shell (ref. 5).

A supporting argument for this procedure is that

$$\phi_L\left(\vec{\kappa} + \frac{1}{4}\vec{Q}\right) \phi_{L'}\left(\vec{\kappa} - \frac{1}{4}\vec{Q}\right)$$

has an extreme at $\vec{\kappa} = 0$, and $L = L'$ and is most pronounced at low energies. Usually other extremum in this product of radial wave functions give an even larger contribution at higher energies. For $L = L'$, this prescription yields the same result as the linear approximation.

Linear Approximation

The linear approximation as developed in reference 5 makes use of the fact that the two-body amplitudes are analytic over the domain of integration and can be expanded in a Taylor series

$$\tau(s,t,u,\delta) = \tau_A + \left[\vec{\nabla}_\kappa \tau(s,t,u,\delta)\right]_A \cdot \left(\vec{\kappa} - \vec{\kappa}_A\right) + \ldots \tag{10}$$

where subscript A denotes evaluation at $\vec{\kappa}_A$ chosen such that

$$\vec{\kappa}_A S_{L_E L_F}(\vec{Q}) = \int \vec{\kappa}\, \phi_{L_F}\left(\vec{\kappa} - \frac{1}{4}\vec{Q}\right) \phi_{L_E}\left(\vec{\kappa} + \frac{1}{4}\vec{Q}\right) d^3\kappa \tag{11}$$

which is an average value $\vec{\kappa}_A \equiv \langle \vec{\kappa} \rangle$. Then

$$T_S = \sum \tau_{A L_E L_F} S(Q)_{L_E L_F} + 0\left[\left(\langle \vec{\kappa}^2 \rangle - \vec{\kappa}_A^{\,2}\right)\right] \tag{12}$$

since the terms linear in $\vec{\kappa}$ of the expansion vanish by virtue of the choice of $\vec{\kappa}_A$. When the remainder is neglected, an approximation which is correct to the linear term is obtained.

The error of the linear approximation is estimated by comparing the quadratic term of the remainder with the amplitude of equation (12). Estimates using the OPE amplitude and neglecting spin show the remainder to dominate the amplitude near backward scattering. This large error indicates the poor convergence of the series of equation (10). Note that

$$\left|\tau_{OPE}(s_0,t_0,u_0,\delta_0)\right| \geq \left|\tau_{OPE}(s_A,t_A,u_A,\delta_A)\right|$$

so that the more accurate linear approximation generally yields results smaller than the Kottler-Kowalski prescription.

Asymptotic Method

The asymptotic method was devised to eliminate some of the defects of the previous approximations. Since the radial wave functions exhibit many extreme values, the approximation should contain sums of two-body amplitudes evaluated at the extremes times an appropriate function of $\vec{Q}$. Such an approximation is easily accomplished by using the Gaussian approximated wave functions which reduce equation (1) to the form

$$T_S = \sum_{\substack{L_E L_F \\ n \;\; m}} a_{L_{E_n}} a_{L_{F_m}} \exp\left[-\alpha\left(1 - \beta^2\right)\vec{Q}^2/16\right] \int \tau(s,t,u,\delta)_{L_E L_F} \exp\left[-\alpha\left(\vec{\kappa} - \frac{1}{4}\beta\vec{Q}\right)^2\right] d^3\kappa \tag{13}$$

where

$$\alpha = b_{L_{E_n}} + b_{L_{F_m}} \tag{14}$$

$$\beta = \left(b_{L_{F_m}} - b_{L_{E_n}}\right)\Big/\alpha \tag{15}$$

Aside from the two-body amplitudes, the integrand exhibit maxima at

$$\vec{\kappa} = \frac{1}{4}\beta\vec{Q} \tag{16}$$

Thus, by using the Chebyshev-Hermite and Chebyshev-Laguerre integration and retaining the first term which is Laplace's method (ref. 16), the next higher approximation to the linear approximation is to factor out of the integrand of equation (13) the function τ evaluated at the corresponding maximum. The result is

$$T_S \approx \sum_{\substack{L_E L_F \\ n\ m}} a_{L_{E_n}} a_{L_{F_m}} \exp\left[-\alpha\left(1 - \beta^2\right)\vec{Q}^2/16\right] \tau\left(s_\beta, t, u_\beta, \delta_\beta\right)(\pi/\alpha) \tag{17}$$

By neglecting for the moment the spin spherical harmonics in equation (13), the accuracy of equation (17) can be examined by finding under what conditions the next term in Laplace's expansion is small. By using the OPE amplitude, equation (17) is found to be a good approximation if either $\alpha^{-1} << m_\pi^2$ or $\left[3\vec{P}^2/2 + (\beta\vec{Q}/4)^2\right] >> m_\pi^2$ where $\vec{P} = \left(\vec{P}_E + \vec{P}_F\right)/2$. Unfortunately, the terms of equation (17) for which this is true do not dominate the sum. In fact, the terms with $\alpha^{-1} > m_\pi^2$ dominate for large values of Q.

As for the linear approximation, the two-body amplitudes could have been expanded into a Taylor series in equation (13) about each relative extreme, and higher order terms could be retained. Similar to results for the linear approximation, the second-order terms are found to dominate the terms retained in approximation of equation (17) near backward scattering.

Exact Evaluation

The form of the single-scattering integral given in equation (13) suggests evaluation by use of standard numerical procedures. Computation time is greatly reduced by a proper choice of variables and integration scheme. Variables are chosen to be those of a cylindrical system with $\hat{z}$ along $\vec{Q}$. This choice reduces equation (13) to a form for direct application of Gauss integration by using Hermite, Laguerre, and Legendre polynomials for components of $\vec{\kappa}$ along $\vec{Q}$, perpendicular $\vec{Q}$, and the angular variable, respectively, as described in reference 16. Some optimization will allow computation of the integral in about 10 minutes on a Control Data 6000 series computer.

COMPARISON OF RESULTS

The modulus squared of the single-scattering integral as approximated by the Kottler-Kowalski and asymptotic methods are shown in figure 2 in comparison with results

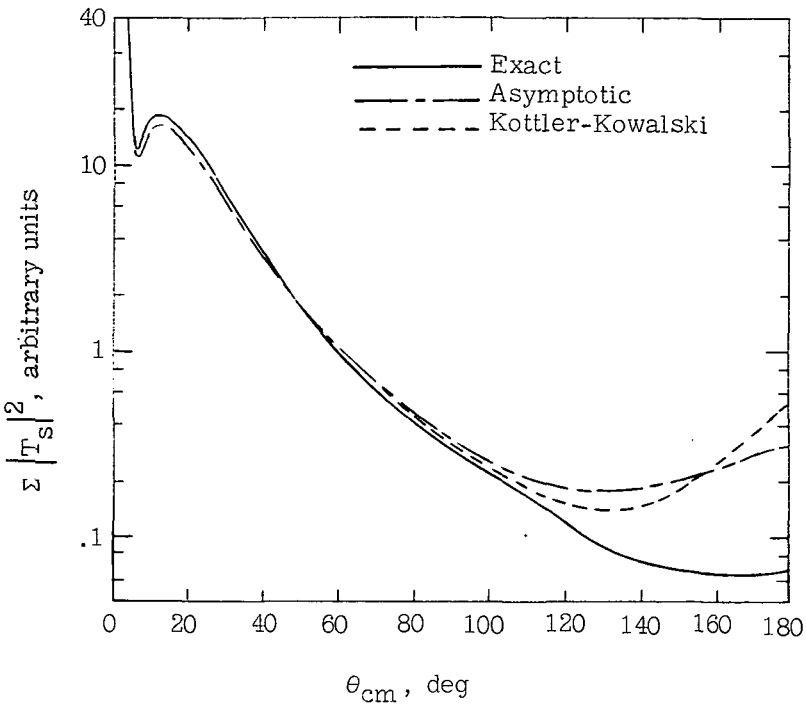


Figure 2.- Modulus squared of single-scattering integral as function of center-of-mass scattering angle for 146-MeV proton laboratory energy.

of an exact evaluation of equation (13). The two-body amplitudes were generated from the Livermore phase shifts (ref. 13) with the off-shell continuation given by equation (7). These results show that, even at forward angles, errors of several percent enter these low-order approximations, while errors in the backward hemisphere are on the order of one magnitude. As noted previously, the source of error comes from the assumption that the product of the two-body amplitudes and spin spherical harmonics is a slowly varying function of $\vec{\kappa}$ compared to the radial wave functions.

The errors caused by the rapid variation of spin spherical harmonics alone are shown in figure 3. These results were obtained by setting the two-body invariants to a constant in equation (13) and approximating the resulting integral over spin functions. The results in figure 3 clearly indicate that any quantitative calculation must accurately integrate the rapidly varying D-state harmonic function. The spin-dependent nucleon currents of equation (2) are slowly varying functions of $\vec{\kappa}$, and to assume them to be constant results in less than a 4-percent error in the modulus squared of equation (13). The results in figures 2 and 3 clearly indicate that the errors of low-order approximations cannot be attributed entirely to poor approximation of the integral over spin spherical harmonics. That the total error is not due to poor approximation to the integral over spherical harmonics is especially true for backward scattering where low-order approximations of the single-scattering integral are in error by an order of magnitude.

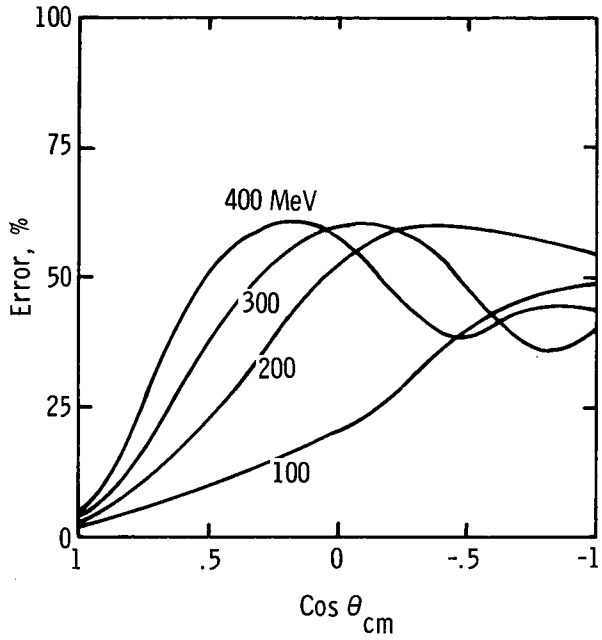


Figure 3.- Error in modulus squared of integral over spin functions in asymptotic method for indicated proton laboratory energies.

To examine the errors contributed by rapid variation of the invariant amplitudes, the spin dependence in equation (2) was totally neglected and the two-body amplitudes were approximated by a sum of poles in t and u as follows. The lowest pole is the OPE with the more distant poles as approximation to branch cuts of the MPE part. Thus

$$\tau(s,t,u,\delta) = \frac{1}{8}\left(\frac{15}{t - m_\pi^2} + \frac{60i}{t - 4m_\pi^2} - \frac{120}{t - 16m_\pi^2}\right) \pm (t \rightarrow u)$$

where the $\pm$ applies for total isospin $I = 0, 1$ so that T_S is properly symmetrized under interchange of incident and constituent nucleons. Each of the low-order approximations is easily evaluated for this form of function and an exact evaluation for the OPE terms gives

$$T^{OPE}_{L_E L_F} = \frac{15}{8}\left\{\frac{1}{t - m_\pi^2} S(\vec{Q})_{L_E L_F} + \sum_{n,m} a_{L_{E_n}} a_{L_{F_m}} \exp\left[-\alpha(1 - \beta^2)\vec{Q}^2/16\right] \int_{-\infty}^{\infty} e^{-\alpha z^2} E_1\left[f(z)\right] dz\right\}$$

where

$$f(z) = \alpha(z - a)^2 + \alpha m_\pi^2$$

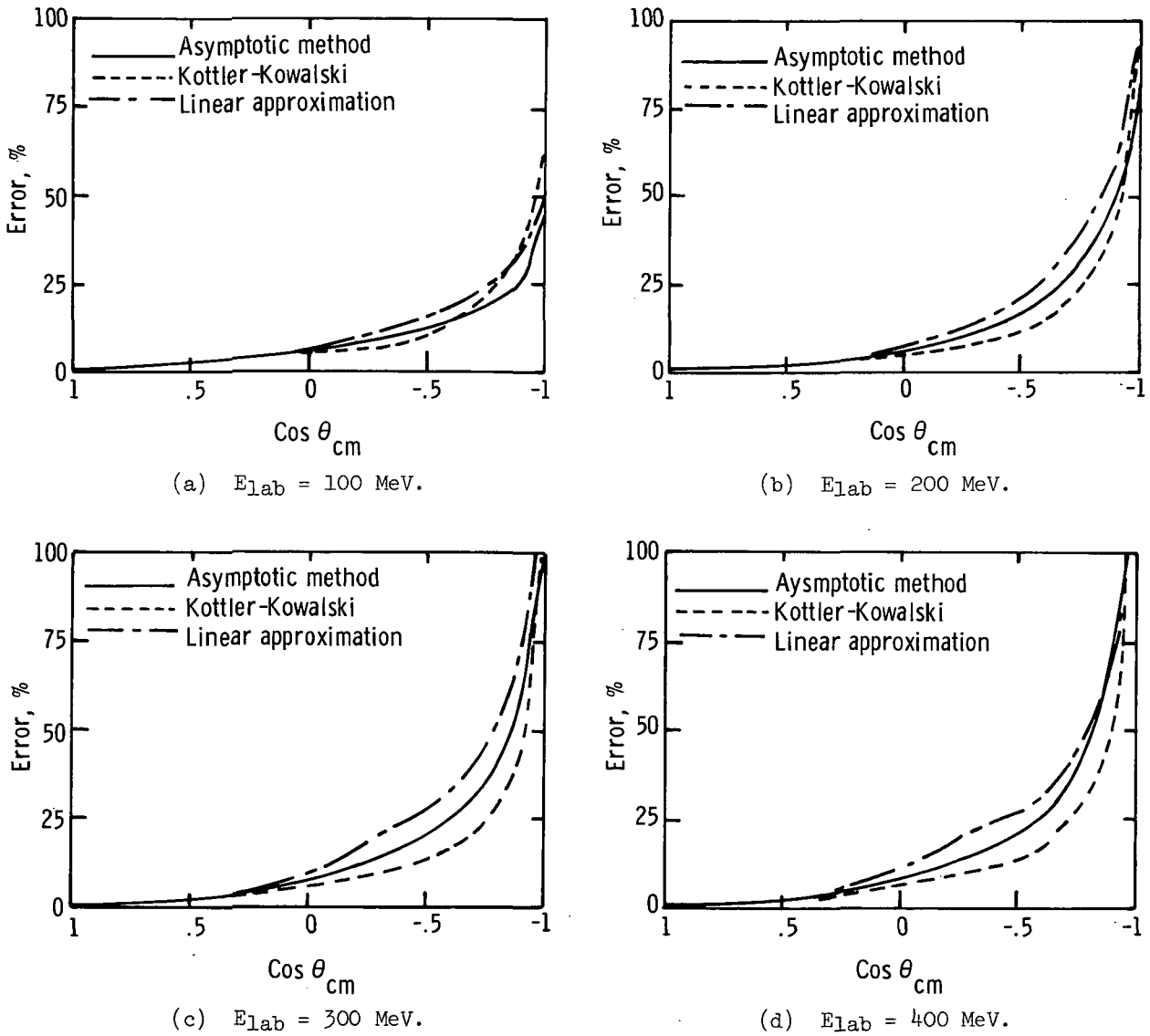
$$a = \left[\left(\frac{3}{2}\vec{P}\right)^2 + \left(\frac{1}{4}\beta\vec{Q}\right)^2\right]^{1/2}$$

and, similarly, for the more distant poles. The relative error ϵ in the amplitudes is calculated by forming incoherent sums and differences of various S- and D-state contributions. Thus

$$\epsilon = \sum_{L_E L_F} \left| T_{L_E L_F} - T^{A}_{L_E L_F} \right| \Big/ \sum_{L_E L_F} \left| T_{L_E L_F} \right|$$

where superscript A denotes an approximate amplitude.

The amplitude errors, neglecting spin, for the three approximations at several energies are shown in figure 4. The errors tend to be small in the forward hemisphere



(a) E_{lab} = 100 MeV.

(b) E_{lab} = 200 MeV.

(c) E_{lab} = 300 MeV.

(d) E_{lab} = 400 MeV.

Figure 4.- Errors in integral over spinless amplitudes.

with a sharp rise at backward angles past 130° for energies of one to a few hundred MeV. At lower angles, the errors decrease slowly to a small value at 90° and remain nearly constant at lower angles and independent of method of approximation. Over most of the angular range, the estimated errors for these spinless calculations are less than 10 percent. Each approximation greatly overestimates the amplitude at these backward angles. Recall that the high-momentum components of the wave functions (small values of α) dominate in this backward region where these approximations were expected to fail. These general features are retained at higher energies as shown in figures 4(b) to 4(d). The backward angle errors increase as a function of energy, thereby indicating greater difficulty in approximating the integral as the deuteron is probed at smaller distances (small values of α).

The difficulty in calculating the single-scattering integral at backward angles arises from exchange symmetry terms owing to the identity of the incident nucleon and the constituent nucleons of the deuteron. When the incident particle and constituents are not identical, the left-hand singularities shown in figure 1 do not appear in the two-body amplitudes; thus, lower order approximations are expected to be more accurate.

The errors due to identity effects cause a backward peaking in the nucleon-deuteron single-scattering amplitude as seen in figure 2. This backward peaking has erroneously been identified in references 17 and 18 as the cause of the observed backward peaking in proton-deuteron elastic scattering at high energies.

Model calculations with separable potentials (ref. 9) have shown the Kottler-Kowalski prescription to be far more accurate than indicated herein. Those calculations, however, have only S-wave contributions in both the deuteron wave function and two-body amplitudes and are therefore essentially spinless. Such a simple separable model should not be expected to yield results similar to the realistic model used in the present study.

CONCLUDING REMARKS

The spin-dependent errors of low-order approximations to the single-scattering integral are large over most of the angular range and decrease to a few percent at forward scattering.

Even when spin is neglected, the simple approximations to the single-scattering integral, such as the Kottler-Kowalski prescription, linear approximation, or asymptotic method, are inadequate for scattering angles above 130° in the center of mass for energies of one to a few hundred MeV. At lower angles, the errors decrease slowly to a small value at 90° and remain nearly constant at lower angles and independent of method of approximation. Over most of the angular range, the estimated errors for these spinless calculations are less than 10 percent.

At backward angles, much of the difficulty in calculating the single-scattering integral arises from exchange symmetry terms owing to the identity of the incident nucleon and the constituent nucleons of the deuteron. When the incident particle and constituents are not identical, lower order approximations are expected to be more accurate.

In applying these results to multiple-scattering analysis, an adequate approximation of single scattering short of numerical integration appears nonexistent. Much computation speed can be gained with only a 4-percent loss of accuracy by assuming that the current matrix elements of the two-body amplitudes are constant. The off-shell amplitudes will probably not be well known for a long time so the most profitable work will be to examine the higher order multiple-scattering terms.

Others have suggested that the identity effects in single scattering account for the backward peaking in proton-deuteron scattering at high energies. The present error estimates make these conclusions based on low-order approximations suspect. The present results indicate that the backward peak cannot be produced by single scattering.

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National Aeronautics and Space Administration,
Hampton, Va., August 18, 1972.

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