

P. J. Cefola**
 Senior Member of Technical Staff
 Computer Sciences Corporation
 Silver Spring, Maryland

Abstract

The matrizant of the two-body problem is developed in terms of elements that are free from singularities for zero eccentricities and zero- and ninety-degree inclinations. Retrograde equinoctial elements eliminate the singularity for inclinations near 180 degrees, with only minor changes in the expressions for the matrizant. The "single-averaged" variation-of-parameters equations for these elements are developed for third-body, oblateness, and drag effects. Higher order terms are included in the expansions for the third-body and oblateness potential. A computer program that uses these equations to predict orbital evolution is described. Numerical results are given for a near-circular orbit.

Introduction

This paper discusses the formulation and application of the variation-of-parameters equations for the equinoctial orbit elements. The equinoctial elements were the subject of a previous investigation that showed that the partial derivatives of position and velocity with respect to the elements, the Lagrange brackets, the Poisson brackets, and the partial derivatives of the elements with respect to position and velocity are all free from singularities for zero eccentricities and zero- and ninety-degree inclinations.⁽¹⁾ While we previously emphasized the application of the equinoctial orbit elements to general perturbation expansions of the variations in the rectangular coordinates, the present work concentrates on special perturbation techniques for the variation-of-parameters equations. Consideration is further restricted to the "single-averaged" variation-of-parameters equations in which the disturbing function is averaged over the orbital period of the satellite to eliminate the short period terms.⁽²⁾ The single-averaged classical orbit element variation-of-parameters equation is widely used in mission analysis^(3,4,5,6) and in selenodesy,^(7,8) where long-term orbit prediction is required at an economical cost.

The classical orbit element variation-of-parameters equation has apparent singularities for small eccentricities and small inclinations, but this has not limited its application. Only the singularities in the variational equations for Ω , ω , and M_0 are significant because the singularities in the equations for e and i are removed by the appearance of powers of e or $\sin i$ in the partial derivatives of the potential function.⁽⁹⁾

If consideration is restricted to the single-averaged variation-of-parameters equations, the question of singularities requires further attention because the averaging procedure eliminates certain singularities. For example, consider the perturbing potential of the satellite due to a third body. This potential, which is used to model the influence of the earth on a lunar orbiter or the influence of the sun on a planetary orbiter, can be expressed by

$$F^3 = \frac{\mu_3}{R_3} \sum_{n=2}^{\infty} F_n^3 \quad (1)$$

where

$$F_n^3 = \left(\frac{r}{R_3} \right)^n P_n(\cos \psi)$$

and

- μ_3 = gravitational constant of third body
- R_3 = distance from central body to third body
- r = distance from central body to the satellite
- P_n = nth order Legendre polynomial
- ψ = angle between the r and R_3 vectors

For artificial satellite problems, only F_2^3 , F_3^3 , and F_4^3 are of interest. To eliminate the short-period effects, the quantities F_2^3 , F_3^3 , and F_4^3 are averaged over the orbital period of the satellite to give $\bar{F}_2^3$, $\bar{F}_3^3$, and $\bar{F}_4^3$. Inspection of the $\bar{F}_2^3$, $\bar{F}_3^3$, and $\bar{F}_4^3$ terms shows that the partial derivatives of $\bar{F}_2^3$ and $\bar{F}_4^3$ with respect to the classical elements do not cause any singularities when the eccentricity is small. However, the $\bar{F}_3^3$ term does cause a singularity in the expressions for $\dot{\omega}$ and $\dot{M}_0$ for small eccentricities. For small inclinations, there is a singularity in the expressions for $\dot{\omega}$ and $\dot{\Omega}$. This singularity is absent if it is assumed that the third body moves in the equatorial plane of the central body.

For the oblateness perturbations in J_2 , J_3 , and J_4 , inspection of the single-averaged equations⁽⁴⁾ indicates that the terms in J_2 and J_4 do not cause any singularities for either small eccentricities or small inclinations.

* Work supported under Contract No. NAS5-11790 with NASA Goddard Space Flight Center, Greenbelt, Maryland.

** Member AIAA.

The term in J_3 does cause singularities for small eccentricities (in the expression for $\dot{\omega}$) and for small inclinations (in the expressions for $\dot{\Omega}$ and $\dot{\omega}$). The term in J_3 does not cause any singularities for inclinations of 90 degrees in the expressions for the time derivatives of the classical elements. However, the set of elements that is introduced in the classical literature to resolve the small eccentricity, small inclination problem⁽¹⁰⁾ is singular for orbits with inclinations of 90 degrees. If a more complete expression for the potential is used, there will be additional singularities.

Based on the preceding paragraphs, we can conclude that there will be singularities in the single-averaged classical orbit element variation-of-parameters equations. These singular cases include perturbations of practical importance in mission analysis and selenodesy. The singularities will cause rapid oscillations in some of the orbit elements when the orbit is near a singular case.⁽⁵⁾ These rapid oscillations are very undesirable in special perturbations procedures where the error at the end of one step contributes to the error at the end of the next step. The single-averaged equinoctial orbit element variation-of-parameters equations derived in this paper will not have these singularities. However, this is not the only advantage of the equinoctial orbit elements. The numerical integration of the single-averaged variation-of-parameters equations for the equinoctial elements will not require the evaluation of trigonometric functions at every step. This should prove advantageous from a computational point of view for all orbits, regardless of their eccentricity or inclination. Note that the simplified structure of the equinoctial variation-of-parameters equations has been used previously⁽⁹⁾ to aid in the study of the secular perturbations in planetary theory.

The next section of the paper provides the two-body results (including the components of the matrizant) required for the equinoctial variation-of-parameters equations. The following sections provide the variation-of-parameters equations for third-body perturbations, oblateness perturbations, and drag perturbations. While the variation-of-parameters equations, as given, are nonsingular for zero eccentricity and for inclinations equal to or greater than zero degrees and less than 180 degrees, the 180-degree inclination remains a problem. For inclinations near 180 degrees, an alternative set of elements (called the "retrograde equinoctial orbit elements") is required; this set of elements is described in the paper.

Finally, applications to current problems in mission analysis are considered. The Interplanetary Monitoring Platform-H (IMP-H) satellite mission conditions (perigee height of 35 Earth radii and apogee height of 39 Earth radii) will be used to demonstrate third-body effects. The Atmosphere Explorer (AE) satellite mission conditions (initial apogee height of 4000 km and perigee heights ranging from 120 to 150 km) will be used to demonstrate the combined effects of atmospheric drag and oblateness.

Other possible applications include long-term artificial satellite geodesy where near-circular or low inclination (or near-polar) orbits are sometimes included in the data base.

Two-Body Preliminaries

To model the effects of both conservative and non-conservative forces, the variation-of-parameters equations are written.

$$\frac{d a_{\alpha}}{dt} = - \sum_{\beta=1}^6 (a_{\alpha}, a_{\beta}) \frac{\partial R}{\partial a_{\beta}} + \frac{\partial a_{\alpha}}{\partial \vec{x}} \cdot \vec{Q} \quad (2)$$

where

a_{α} = orbit element ($\alpha = 1, \dots, 6$)

(a_{α}, a_{β}) = Poisson bracket

R = perturbing potential due to the conservative forces

$\vec{x}$ = velocity

$\vec{Q}$ = perturbing acceleration for the nonconservative forces

For the present application, the perturbing potential R will include the third-body and oblateness effects, and the perturbing acceleration $\vec{Q}$ will include the drag effects. Consideration of Equation (2) shows that the following two-body formulas are required:

- A transformation from classical elements to equinoctial elements
- A transformation from position and velocity to equinoctial elements
- A transformation from the equinoctial elements to position and velocity
- Poisson brackets for the equinoctial elements
- Partial derivatives of the equinoctial elements with respect to velocity

Formulas a and b are the basic definitions of the equinoctial elements. Formula c is required for $\partial a_{\alpha} / \partial \vec{x}$ and is also used in the numerical averaging procedure that is employed for the atmospheric drag perturbations. Because both direct and retrograde equinoctial elements are considered, two sets of formulas will be required. However, the functional dependence of the retrograde expressions on the retrograde elements is identical (with the exception of a few plus and minus signs) to the functional dependence of the direct expressions on the direct elements. Thus, the use of the direct and retrograde elements defined in this paper to compute orbits with inclinations $0 \leq i \leq 180$ has a definite advantage over alternate approaches⁽¹¹⁾ in which nonsymmetrical sets of orbit elements are used to span the space $0 \leq i \leq 180$ degrees.

Transformation from Classical Elements to Equinoctial Elements

The direct equinoctial elements are given by

$$\begin{aligned} a &= a \\ h &= e \sin(\omega + \Omega) \\ k &= e \cos(\omega + \Omega) \\ \lambda_o &= M_o + \omega + \Omega \\ p &= \tan(1/2) \sin \Omega \\ q &= \tan(1/2) \cos \Omega \end{aligned} \quad (3)$$

where a , e , i , M_o , ω , and Ω are the classical orbit elements. The elements h and k are the components (in the orbital frame) of the eccentricity vector that points toward the perigee and has the magnitude e . Elements p and q are required in the rotation matrix between the inertial frame and the orbital frame. The element λ_o is the mean longitude in the classical literature. The retrograde equinoctial elements are given by

$$\begin{aligned} a &= a \\ h_r &= e \sin(\omega - \Omega) \\ k_r &= e \cos(\omega - \Omega) \\ \lambda_{or} &= M_o + \omega - \Omega \\ p_r &= \cot(1/2) \sin \Omega \\ q_r &= \cot(1/2) \cos \Omega \end{aligned} \quad (4)$$

The quantities h_r and k_r are the components of the eccentricity vector relative to the retrograde orbital frame. The elements p_r and q_r are required in the rotation matrix between the inertial frame and the retrograde orbital frame.

The orbital coordinate frames can be defined in terms of the classical orbital elements; this is done in Figure 1 for the direct case and in Figure 2 for the retrograde case. In Figures 1 and 2, unit vector $\hat{w}$ is the normal to the orbit plane. For the direct case the $\hat{f}$ and $\hat{g}$ unit vectors are in the orbital plane. The direction of the $\hat{f}$ unit vector depends on the classical orbit elements Ω and i . The unit vector $\hat{g}$ completes the right-handed triad of $\hat{f}$, $\hat{g}$, and $\hat{w}$. For the retrograde case, the $\hat{f}^*$, $\hat{g}^*$, and $\hat{w}$ unit vectors comprise the right-handed triad. Mathematically, the unit vectors can be expressed in terms of the equinoctial elements by

$$\hat{f} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2pI \end{pmatrix} \quad (5)$$

$$\hat{g} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2pqI \\ (1 + p^2 - q^2)I \\ 2q \end{pmatrix} \quad (6)$$

$$\hat{w} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2p \\ -2q \\ (1 - p^2 - q^2)I \end{pmatrix} \quad (7)$$

Equations (5), (6), and (7) require some comment. If $I = +1$, the p and q elements defined in Equation (3) must be used in Equations (5), (6), and (7). The $\hat{f}$, $\hat{g}$, and $\hat{w}$ unit vectors computed with $I = +1$ have the meaning indicated in Figure 1. If $I = -1$, the retrograde p and q defined in Equation (4) must be used in Equations (5), (6), and (7). The $\hat{f}$, $\hat{g}$, and $\hat{w}$ unit vectors computed with $I = -1$ have the meaning indicated in Figure 2. This notation will reduce the repetition of almost identical formulas in the remainder of the paper.

Transformation from Position and Velocity to Equinoctial Elements

This paragraph gives the formulas required to compute the equinoctial elements from the position and velocity. The semi-major axis is

$$a = \left(\frac{1}{|\vec{x}|} - \frac{|\dot{\vec{x}}|^2}{\mu} \right)^{-1} \quad (8)$$

The eccentricity vector is given by

$$\vec{e} = -\frac{\vec{x}}{|\vec{x}|} - \frac{(\vec{x} \times \dot{\vec{x}}) \times \dot{\vec{x}}}{\mu} \quad (9)$$

The unit vector normal to the orbital plane is given by

$$\hat{w} = \frac{\vec{x} \times \dot{\vec{x}}}{|\vec{x} \times \dot{\vec{x}}|} \quad (10)$$

Unit vector $\hat{w}$ has the exact same meaning in Equation (10) as in Equation (7). These relationships lead to

$$p = \frac{\hat{w} \cdot \vec{x}}{1 + \hat{w} \cdot \vec{e}} \quad (11)$$

$$q = \frac{-\hat{w} \cdot \dot{\vec{x}}}{1 + \hat{w} \cdot \vec{e}} \quad (12)$$

In Equations (11) and (12), if $I = +1$, the p and q (defined in Equation (3)) result. If $I = -1$, the retrograde p and q (defined in Equation (4)) result. The unit vectors $\hat{f}$ and $\hat{g}$ (or $\hat{f}^*$ and $\hat{g}^*$) may now be computed using Equations (5) and (6). The equinoctial orbital elements h and k are computed using the formulas

$$h = \vec{e} \cdot \hat{g} \quad (13)$$

$$k = \vec{e} \cdot \hat{f} \quad (14)$$

If I was set equal to -1 in Equations (5) and (6), $\hat{f}^*$ and $\hat{g}^*$ were computed and Equations (13) and (14) give h_r and k_r :

$$h_r = \vec{e} \cdot \hat{g}^* \quad (15)$$

$$k_r = \vec{e} \cdot \hat{f}^* \quad (16)$$

The only remaining element to be computed is the mean longitude, λ . We first compute the position coordinates X_1 and Y_1 relative to the orbital frame $\hat{f}$, $\hat{g}$, $\hat{w}$ by

$$X_1 = \vec{r} \cdot \hat{f} \quad (17)$$

$$Y_1 = \vec{r} \cdot \hat{g} \quad (18)$$

Then we compute

$$\cos F = k + \frac{(1 - k^2 \beta) X_1 - hk\beta Y_1}{a \sqrt{1 - h^2 - k^2}} \quad (19)$$

$$\sin F = h + \frac{(1 - h^2 \beta) Y_1 - hk\beta X_1}{a \sqrt{1 - h^2 - k^2}} \quad (20)$$

where the auxiliary variable β is

$$\beta = \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \quad (21)$$

The mean longitude λ is given by

$$\lambda = F - k \sin F + h \cos F \quad (22)$$

For the retrograde case, the quantities $\hat{f}^*$, $\hat{g}^*$, h_r , and k_r should replace $\hat{f}$, $\hat{g}$, h , and k in Equations (17) through (22). The result of Equation (22) is the retrograde mean longitude defined in Equation (4). The derivation of Equations (19), (20), and (22) is explained in the next paragraph.

Transformation from Equinoctial Elements to Position and Velocity

The key to this formulation is the use of the longitudes λ , F , and L , defined by

$$\begin{aligned} \lambda &= M + \omega + \Omega \\ F &= E + \omega + \Omega \\ L &= v + \omega + \Omega \end{aligned} \quad (23)$$

where M , E , and v are the classical anomalies. Elementary manipulations show that Kepler's equation can be written in terms of the eccentric longitude F

$$\lambda = F + h \cos F - k \sin F \quad (24)$$

Once Kepler's equation has been solved, the position and velocity vectors can be expressed as

$$\vec{r} = X_1 \hat{f} + Y_1 \hat{g} \quad (25)$$

and

$$\dot{\vec{r}} = \dot{X}_1 \hat{f} + \dot{Y}_1 \hat{g} \quad (26)$$

In these equations the unit vectors $\hat{f}$ and $\hat{g}$ are computed using Equations (5) and (6). The coordinates X_1 , Y_1 , $\dot{X}_1$, and $\dot{Y}_1$ relative to the equinoctial frame are given by

$$X_1 = a \left[(1 - h^2 \beta) \cos F + hk\beta \sin F - k \right] \quad (27)$$

$$Y_1 = a \left[(1 - k^2 \beta) \sin F + hk\beta \cos F - h \right] \quad (28)$$

$$\dot{X}_1 = \frac{na^2}{r} \left[hk\beta \cos F - (1 - h^2 \beta) \sin F \right] \quad (29)$$

$$\dot{Y}_1 = \frac{na^2}{r} \left[(1 - k^2 \beta) \cos F - hk\beta \sin F \right] \quad (30)$$

where the auxiliary equation

$$\frac{r}{a} = 1 - k \cos F - h \sin F \quad (31)$$

is necessary for the velocity coordinates. The coordinates can also be expressed in terms of the true longitude by

$$X_1 = r \cos L \quad (32)$$

$$Y_1 = r \sin L \quad (33)$$

$$\dot{X}_1 = \frac{-na}{\sqrt{1 - h^2 - k^2}} (h + \sin L) \quad (34)$$

$$\dot{Y}_1 = \frac{na}{\sqrt{1 - h^2 - k^2}} (k + \cos L) \quad (35)$$

$$r = a(1 - h^2 - k^2)/(1 + k \cos L + h \sin L) \quad (36)$$

Equations (32) through (36) will be used in the numerical averaging procedure for the atmospheric drag. Also, the right-hand side of Equation (32) can be set equal to the right-hand side of Equation (27). This relation and one involving Equations (33) and (28) can be solved simultaneously to give Equations (19) and (20).

For the retrograde case, the longitudes are defined by

$$\begin{aligned} \lambda^* &= M + \omega - \Omega \\ F^* &= E + \omega - \Omega \\ L^* &= v + \omega - \Omega \end{aligned} \quad (37)$$

and the quantities λ^* , F^* , L^* , $\hat{f}^*$, $\hat{g}^*$, h_r , and k_r replace the direct variables in Equations (24) through (36).

Poisson Brackets

In the present application, the Poisson brackets must be given in terms of the equinoctial elements. The results are obtained by direct substitution into the previous results by Broucke and Cefola⁽¹⁾ and are listed in Table 1.

Partial Derivatives of the Equinoctial Elements with Respect to Velocity

The partial derivatives $\partial a / \partial \dot{\vec{r}}$, $\partial p / \partial \dot{\vec{r}}$, and $\partial q / \partial \dot{\vec{r}}$ are obtained directly as functions of the equinoctial elements by using the results of Broucke and Cefola.⁽¹⁾ However, the expressions for $\partial h / \partial \dot{\vec{r}}$, $\partial k / \partial \dot{\vec{r}}$, and $\partial \lambda_0 / \partial \dot{\vec{r}}$ in terms of the classical orbit elements are not so easily translated

into the equinoctial elements. To compute these quantities, we have to use the relationship

$$\frac{\partial a_\alpha}{\partial \bar{x}} = - \sum_{\beta=1}^6 (a_\alpha, a_\beta) \frac{\partial \bar{x}}{\partial a_\beta} \quad (38)$$

which requires the Poisson brackets from Table 1 and the partial derivatives of the position vector. For $\partial \bar{x}/\partial h$ and $\partial \bar{x}/\partial k$, we need the partial derivatives of X_1 and Y_1 , which are

$$\begin{aligned} \frac{\partial X_1}{\partial h} &= a \left[-(\lambda - F) \left(\beta + \frac{h^2 \beta^3}{1 - \beta} \right) - \frac{a}{r} \cos F (h\beta - \sin F) \right] \\ \frac{\partial X_1}{\partial k} &= -a \left[(\lambda - F) \frac{hk\beta^3}{1 - \beta} + 1 + \frac{a}{r} \sin F (\sin F - h\beta) \right] \\ \frac{\partial Y_1}{\partial h} &= a \left[(\lambda - F) \frac{kh\beta^3}{1 - \beta} - 1 + \frac{a}{r} \cos F (k\beta - \cos F) \right] \\ \frac{\partial Y_1}{\partial k} &= a \left[(\lambda - F) \left(\beta + \frac{k^2 \beta^3}{1 - \beta} \right) + \frac{a}{r} \sin F (\cos F - k\beta) \right] \end{aligned} \quad (39)$$

With these results, the position partials can be specified, as shown in Table 2. Substitution of the results of Tables 1 and 2 into Equation (38) gives the desired results, which are listed in Table 3.

Third-Body Potential

This section gives the single-averaged perturbing potential due to a third body in terms of the equinoctial orbit elements. The first two terms in the expansion (Equation (1)) are given. The partial derivatives of the potential and the resulting variation-of-parameters equations are also given.

Substitution of the expressions for the Legendre polynomials in Equation (1) gives

$$F_2^3 = \left(\frac{r}{R_3} \right)^2 \left(\frac{3 \cos^2 \psi - 1}{2} \right) \quad (40)$$

and

$$F_3^3 = \left(\frac{r}{R_3} \right)^3 \left(\frac{5 \cos^3 \psi - 3 \cos \psi}{2} \right) \quad (41)$$

In Kaufman⁽⁴⁾ the procedure is to express $\cos \psi$ as a function of the true anomaly of the satellite and then average the potential over one orbit. In the present development, we proceed in a similar manner except we use the $\hat{f}, \hat{g}, \hat{w}$ coordinate frame instead of the $\hat{P}, \hat{Q}, \hat{R}$ frame and the true longitude instead of the true anomaly. Thus

$$\cos \psi = \alpha_1 \cos L + \beta_1 \sin L \quad (42)$$

where L is the true longitude defined in Equation (23) and

$$\begin{aligned} \alpha_1 &= \hat{f} \cdot \hat{R}_3 \\ \beta_1 &= \hat{g} \cdot \hat{R}_3 \\ \gamma_1 &= \hat{w} \cdot \hat{R}_3 \end{aligned} \quad (43)$$

where $\hat{R}_3$ is the unit vector from the central body to the perturbing body. For the retrograde elements, we use the $\hat{f}^*, \hat{g}^*, \hat{w}$ coordinate frame (Figure 2) with

$$\cos \psi = \alpha_r \cos L^* + \beta_r \sin L^* \quad (44)$$

where L^* is the retrograde true longitude defined in Equation (37) and

$$\begin{aligned} \alpha_r &= \hat{f}^* \cdot \hat{R}_3 \\ \beta_r &= \hat{g}^* \cdot \hat{R}_3 \\ \gamma_r &= \hat{w} \cdot \hat{R}_3 \end{aligned} \quad (45)$$

Substitution of Equation (42) into Equations (40) and (41) gives

$$\begin{aligned} F_2^3 &= \frac{1}{2} \left(\frac{r}{R_3} \right)^2 \left[\left(\frac{3}{2} \alpha_1^2 + \frac{3}{2} \beta_1^2 - 1 \right) \right. \\ &\quad \left. + 3 \alpha_1 \beta_1 \sin 2L + \frac{3}{2} (\alpha_1^2 - \beta_1^2) \cos 2L \right] \end{aligned} \quad (46)$$

and

$$\begin{aligned} F_3^3 &= \frac{1}{2} \left(\frac{r}{R_3} \right)^3 \left[\frac{5 \alpha_1^3}{4} \cos 3L - \frac{5 \beta_1^3}{4} \sin 3L \right. \\ &\quad \left. + \frac{15}{2} \alpha_1^2 \beta_1 \cos L \sin 2L + \frac{15}{2} \alpha_1 \beta_1^2 \sin L \sin 2L \right. \\ &\quad \left. + \left(\frac{15}{4} \alpha_1^3 - 3 \alpha_1 \right) \cos L + \left(\frac{15}{4} \beta_1^3 - 3 \beta_1 \right) \sin L \right] \end{aligned} \quad (47)$$

The substitution of Equation (44) into Equation (40) and (41) produces the exact same result except α_r, β_r , and L^* replace α_1, β_1 , and L . Next, F_2^3 and F_3^3 are averaged over the period of the orbit according to

$$\bar{F}_n^3 = \frac{1}{2\pi} \int_0^{2\pi} F_n^3 d\lambda, \quad n = 2, 3 \quad (48)$$

which results in

$$\begin{aligned} \bar{F}_2^3 &= \frac{1}{2} \left(\frac{a}{R_3} \right)^2 \left[\left(\frac{3}{2} \alpha_1^2 + \frac{3}{2} \beta_1^2 - 1 \right) \left[1 + \frac{3}{2} (h^2 + k^2) \right] \right. \\ &\quad \left. + 15 \alpha_1 \beta_1 h k + \frac{15}{4} (\alpha_1^2 - \beta_1^2) (k^2 - h^2) \right] \end{aligned} \quad (49)$$

and

$$\begin{aligned} \bar{F}_3^3 &= \frac{5}{4} \left(\frac{a}{R_3} \right)^3 \left[\frac{5}{8} \beta_1 h [3 \alpha_1^2 (5h^2 - 9k^2 - 2) \right. \\ &\quad \left. + \beta_1^2 (15k^2 + h^2 - 6)] + \frac{5}{8} \alpha_1 k [3 \beta_1^2 (5k^2 - 9h^2 - 2) \right. \\ &\quad \left. + \alpha_1^2 (15h^2 + k^2 - 6)] - \frac{3}{4} (k \alpha_1 + h \beta_1) (3h^2 + 3k^2 - 4) \right] \end{aligned} \quad (50)$$

where these results have been computed using the definite integrals given in Table 4. For the retrograde case, the quantities α_r , β_r , h_r , k_r , L^* , and λ^* replace α_1 , β_1 , h , k , L , and λ in Equations (48), (49), and (50) and in Table 4. Note that α_1 and β_1 depend only on the equinoctial elements p and q , so that the quantities $\bar{F}_2^3$ and $\bar{F}_3^3$ are functions of a , h , k , p , and q . For the variation of parameters equations, the partial derivatives of $\bar{F}_2^3$ and $\bar{F}_3^3$ are required; the next portion of this section is devoted to their computation.

Partial derivatives of the $\bar{F}_n^3$ with respect to a , h , and k require no special comment. To differentiate the $\bar{F}_n^3$ with respect to p and q , it is necessary to evaluate $\partial\alpha/\partial p$, $\partial\beta/\partial p$, $\partial\alpha/\partial q$, and $\partial\beta/\partial q$. From Equations (43) and (45):

$$\begin{aligned}\frac{\partial\alpha}{\partial p} &= \frac{\partial\hat{f}}{\partial p} \cdot \hat{R}_3 & \frac{\partial\alpha}{\partial q} &= \frac{\partial\hat{f}}{\partial q} \cdot \hat{R}_3 \\ \frac{\partial\beta}{\partial p} &= \frac{\partial\hat{g}}{\partial p} \cdot \hat{R}_3 & \frac{\partial\beta}{\partial q} &= \frac{\partial\hat{g}}{\partial q} \cdot \hat{R}_3\end{aligned}\quad (51)$$

The derivatives of $\hat{f}$ and $\hat{g}$ are evaluated most simply by using the results in Table 3:

$$\begin{aligned}\frac{\partial\hat{f}}{\partial p} &= \frac{-2}{1+p^2+q^2} [q\hat{g}I + \hat{w}] \\ \frac{\partial\hat{f}}{\partial q} &= \frac{2}{1+p^2+q^2} p\hat{g}I \\ \frac{\partial\hat{g}}{\partial p} &= \frac{2}{1+p^2+q^2} q\hat{f}I \\ \frac{\partial\hat{g}}{\partial q} &= \frac{-2}{1+p^2+q^2} [p\hat{f} - \hat{w}]I\end{aligned}\quad (52)$$

Substitution of Equation (52) into (51) with the definitions from (43) and (45) results in

$$\begin{aligned}\frac{\partial\alpha}{\partial p} &= \frac{-2}{1+p^2+q^2} (q\beta I + \gamma) \\ \frac{\partial\alpha}{\partial q} &= \frac{2p\beta I}{1+p^2+q^2} \\ \frac{\partial\beta}{\partial p} &= \frac{2q\alpha I}{1+p^2+q^2} \\ \frac{\partial\beta}{\partial q} &= \frac{-2(p\alpha - \gamma)I}{1+p^2+q^2}\end{aligned}\quad (53)$$

Equation (53) could be used to compute the partial derivatives $\partial\bar{F}_n^3/\partial p$ and $\partial\bar{F}_n^3/\partial q$. However, the variation-of-parameters equations can be simplified by substituting Equation (53) directly. We also note that $\bar{F}_2^3$ and $\bar{F}_3^3$ do not depend on λ . First, we assume a disturbing function

R that is a function of a , h , k , α , and β and write

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial p} + \frac{\partial R}{\partial \beta} \frac{\partial \beta}{\partial p} \quad (54)$$

and

$$\frac{\partial R}{\partial q} = \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial q} + \frac{\partial R}{\partial \beta} \frac{\partial \beta}{\partial q} \quad (55)$$

Then

$$\begin{aligned}p \frac{\partial R}{\partial p} + q \frac{\partial R}{\partial q} &= \frac{\partial R}{\partial \alpha} \left(p \frac{\partial \alpha}{\partial p} + q \frac{\partial \alpha}{\partial q} \right) \\ &\quad + \frac{\partial R}{\partial \beta} \left(p \frac{\partial \beta}{\partial p} + q \frac{\partial \beta}{\partial q} \right)\end{aligned}\quad (56)$$

Substitution of Equation (53) into (56) gives the identity

$$p \frac{\partial R}{\partial p} + q \frac{\partial R}{\partial q} = \frac{-2\gamma}{1+p^2+q^2} \left(p \frac{\partial R}{\partial \alpha} - I q \frac{\partial R}{\partial \beta} \right) \quad (57)$$

Substitution of Table 1 and Equation (57) into Equation (2) (with $\bar{Q}$ in Equation (2) set equal to zero) results in the third-body variation-of-parameters equations given in Table 5. In the expressions for dp/dt and dq/dt we have used the identity

$$k \frac{\partial R}{\partial h} - h \frac{\partial R}{\partial k} - \beta_1 \frac{\partial R}{\partial \alpha_1} + \alpha_1 \frac{\partial R}{\partial \beta_1} = 0 \quad (58)$$

Equation (58) is a consequence of the fact that motion of the third body in the osculating orbit plane of the satellite does not perturb the satellite orbit plane. Specifically, it can be shown that Equation (58) holds for the quantities $\bar{F}_2^3$ and $\bar{F}_3^3$ given in Equations (49) and (50). The partials of $\bar{F}_2^3$ and $\bar{F}_3^3$ are given in Tables 6 and 7.

For the retrograde case, $I = -1$ and h_r , k_r , p_r , q_r , $\hat{f}^*$, $\hat{g}^*$, α_r , β_r , and γ_r replace h , k , p , q , $\hat{f}$, $\hat{g}$, α , β , and γ in Equations (51) through (58) and in Tables 4 through 7.

Oblateness Potential

This section gives the single-averaged perturbing potential due to oblateness effects. The basic results are obtained by direct transformations of previous results. (12) The terms $\bar{F}_{20}$ and $\bar{F}_{30}$ in the oblateness potential can be obtained as functions of the equinoctial elements defined by Equation (3) by elementary manipulations. Partial derivatives of $\bar{F}_{20}$ and $\bar{F}_{30}$ with respect to the equinoctial elements are straightforward. The variation-of-parameters equations for oblateness (J_2 and J_3) are given in Tables 8 and 9. Additional results have been obtained for the oblateness perturbation. (13)

Atmospheric Drag

To complete modeling of the forces that control the long-term behavior of artificial satellite orbits, atmospheric drag must be considered. Since drag is a non-conservative force, a potential function does not exist and the Gaussian form of the variation-of-parameters equations is used:

$$\frac{da_\alpha}{dt} = \frac{\partial a_\alpha}{\partial \dot{\mathbf{x}}} \cdot \dot{\mathbf{Q}} \quad (59)$$

where the quantities a_α , $\dot{\mathbf{x}}$, and $\dot{\mathbf{Q}}$ were defined previously. Note that the partial derivatives $\partial a_\alpha / \partial \dot{\mathbf{x}}$ for the equinoctial elements are given in Table 3. For atmospheric drag, the perturbing acceleration is given by

$$\dot{\mathbf{Q}} = \frac{-C_D A \rho}{2m} |\dot{\mathbf{x}}| \dot{\mathbf{x}} \quad (60)$$

where

- C_D = aerodynamic drag coefficient
- A = cross-sectional area of the satellite
- ρ = density
- m = mass of satellite

Substitution of the results from Table 3 and Equation (60) into Equation (59) eventually results in

$$\frac{da}{dt} = - \left(\frac{C_D A}{m} \right) \frac{\rho |\dot{\mathbf{x}}|^3}{n^2 a} \quad (61)$$

$$\frac{dh}{dt} = - \left(\frac{C_D A}{m} \right) \rho |\dot{\mathbf{x}}| (h + \sin L) \quad (62)$$

$$\frac{dk}{dt} = - \left(\frac{C_D A}{m} \right) \rho |\dot{\mathbf{x}}| (k + \cos L) \quad (63)$$

$$\frac{d\lambda}{dt} = n + \frac{C_D A}{m} \rho |\dot{\mathbf{x}}| (k \sin L - h \cos L) \quad (64)$$

$$\left[\frac{\sqrt{1 - h^2 - k^2}}{1 + k \cos L + h \sin L} - \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \right]$$

$$\frac{dp}{dt} = 0 \quad (65)$$

$$\frac{dq}{dt} = 0 \quad (66)$$

where n is the mean motion. Note that dp/dt and dq/dt are zero because the drag acceleration model given in Equation (60) has no component in the direction of the $\hat{\mathbf{w}}$ unit vector. Because λ does not appear in the right-hand sides of the averaged variational equations, we do not consider it further. The elements a , h , and k do change

in a secular fashion, and these changes should be considered. The Gaussian-quadrature approach is used to average the right-hand sides of Equations (61), (62), and (63). First, we obtain expressions for da/dL , dh/dL , and dk/dL (where L is the true longitude). We then compute the changes in the elements over one orbit (for

example, $\Delta a = \int_{-\pi}^{+\pi} \frac{da}{dL} dL$). The resulting expressions

are divided by the period of the orbit to obtain the average rates:

$$\bar{a} = \frac{-C_D A a^2 (1 - h^2 - k^2)^{3/2}}{2\pi m \mu} \int_{-\pi}^{+\pi} \frac{\rho |\dot{\mathbf{x}}|^3}{(1 + k \cos L + h \sin L)^2} dL \quad (67)$$

$$\bar{h} = \frac{-C_D A (1 - h^2 - k^2)^{3/2}}{2\pi m} \int_{-\pi}^{+\pi} \frac{\rho |\dot{\mathbf{x}}| (h + \sin L)}{(1 + k \cos L + h \sin L)^2} dL \quad (68)$$

$$\bar{k} = \frac{-C_D A (1 - h^2 - k^2)^{3/2}}{2\pi m} \int_{-\pi}^{+\pi} \frac{\rho |\dot{\mathbf{x}}| (k + \cos L)}{(1 + k \cos L + h \sin L)^2} dL \quad (69)$$

In Equations (67) through (69), the density is to be considered a function of the true longitude. In practice, the density is a function of position computed from the orbit elements and the true longitude, using Equations (32), (33), and (36).

For the retrograde case, h_r , k_r , and L^* replace h , k , and L in Equations (67), (68), and (69). The position is still computed using Equations (32), (33), and (36), but the reader should recall the remark following Equation (37).

Numerical Studies

The Earth Satellite Mission Analysis Program (ESMAP) was developed to demonstrate the practical application of the analytical results given in this paper. This program integrates numerically the variation-of-parameters equations for the equinoctial elements. Perturbations included are third-body, oblateness, and drag.

The numerical studies currently in progress have two goals:

1. To verify numerically the correctness of the analytical results*
2. To determine limits in the applicability of this formulation to mission analysis problems

Several test cases have been specified to assist in reaching the above goals.

*A small experimental program has been made for the case of a satellite perturbed by the sun. The results have been compared with classical variation of parameter integrations, and an agreement of 10 digits obtained. (14)

Case 1

A low altitude near-circular orbit is being used to validate the oblateness modeling. The initial conditions are

$$\begin{aligned}a &= 7086.85 \text{ km} \\e &= .001 \\M &= 0^\circ \\ \Omega &= 0^\circ \\i &= 20^\circ, 40^\circ, 60^\circ, \text{ and } 80^\circ \\\omega &= 0^\circ\end{aligned}$$

This case has been studied in detail in the literature.⁽¹⁵⁾ The ESMAP results are given in Figures 3 and 4.

Case 2

An Interplanetary Monitoring Platform-G (IMP-G) orbit is being used to validate the third-body modeling. The initial conditions are:

$$\begin{aligned}a &= 94,820.03 \text{ km} \\e &= .8940664 \\M &= 0^\circ \\ \Omega &= 103.62458^\circ \\i &= 83.864809^\circ \\\omega &= -157.19876^\circ\end{aligned}$$

Both lunar and solar perturbations are important for this case. The Goddard Trajectory Determination Subsystem (GTDS) program is being used to generate reference trajectories.

Case 3

The IMP-H satellite mission conditions (perigee height of 35 Earth radii and apogee height of 39 Earth radii) will also be used to demonstrate third-body effects. Note that this near-circular mission will require the inclusion of the second-term $\bar{F}_3^3$ in the third-body potential.

Case 4

The Atmospheric Explorer (AE) satellite mission conditions (initial apogee height of 4000 km and perigee heights ranging from 120 to 150 km) will be used to demonstrate the combined effects of atmospheric drag and oblateness.

Conclusions

Uphoff⁽⁶⁾ notes two problem areas in the study of the long-term orbit evolution of an artificial satellite:

1. The lack of high-order terms in the expansion of the third-body disturbing function
2. The lack of high-order nonsphericity effects

Lorell and Liu⁽¹²⁾ provide these analytical results in

terms of the classical orbit elements (a , e , i , M_0 , ω , and Ω). However, some of the high-order terms contain singularities associated with small eccentricities and inclinations. The equinoctial orbit elements (Equation (3)) resolve these problems. The results for the third-body perturbing potential are developed in detail (Equations (49) and (50)). The second term in this expansion, which causes singularities in the variation-of-parameters equations for the classical orbit elements, does not cause singularities in the variational equations for the equinoctial elements. Detailed results are given for the oblateness potential (Tables 8 and 9) and for the perturbing acceleration due to atmospheric drag (Equations (67), (68) and (69)).

The retrograde equinoctial elements are introduced (Equation (4)) to resolve problems associated with inclinations near or equal to 180 degrees. It is shown that the variational equations describing the direct and retrograde equinoctial elements have the same form except for plus and minus signs. Thus duplication of functionally similar computer routines can be avoided.

Finally, the status of the numerical studies currently in progress is given. The primary emphasis is on mission analysis applications.

Acknowledgement

The author wishes to express gratitude to Professor R. Broucke, UCLA, who tested some of the results in an experimental program; to Dr. A. Deprit, Cincinnati Observatory, who suggested consideration of the true frame of reference and who commented on a preliminary draft; to G. Holloway and B. Lamers, Computer Sciences Corporation, who assisted in the analysis and completed the programming of ESMAP; and to R. Coady and R. Barbieri, Goddard Space Flight Center, who suggested the particular numerical examples.

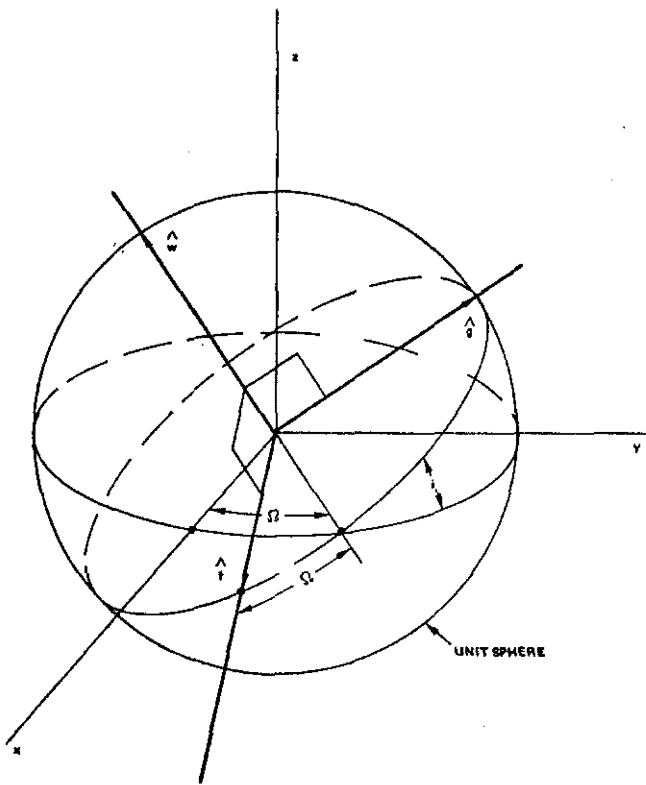


Figure 1. Equinoctial Coordinate Frame

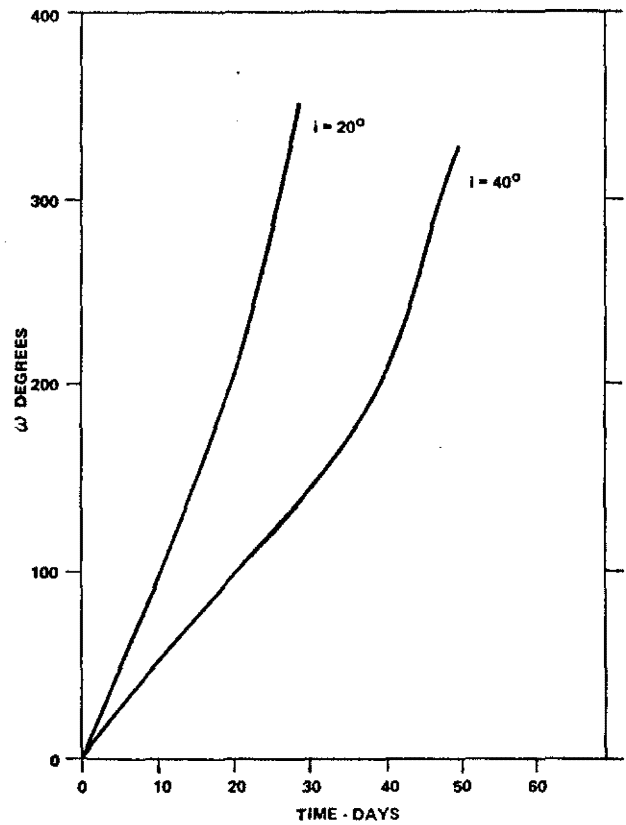


Figure 3. Argument of Perigee Versus Time for Two Orbits With Initial $e = .001$, $R_e/a = .9$, J_2 and J_3

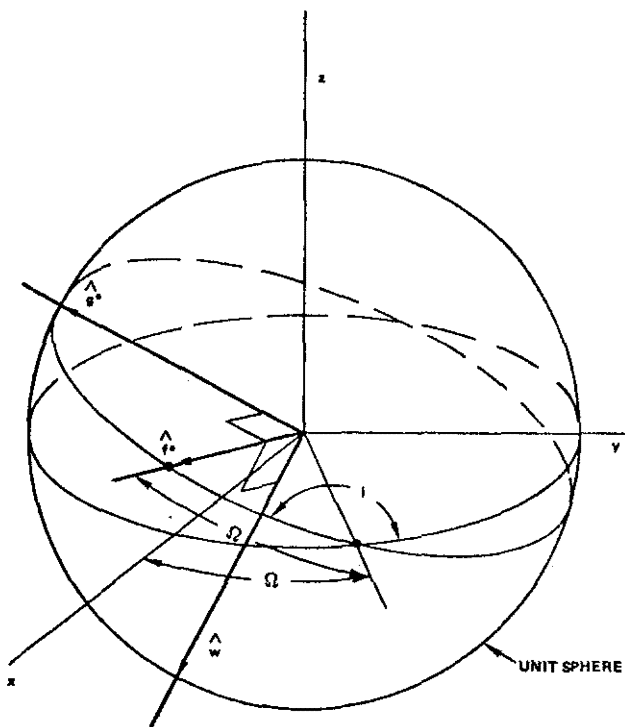


Figure 2. Retrograde Equinoctial Coordinate Frame

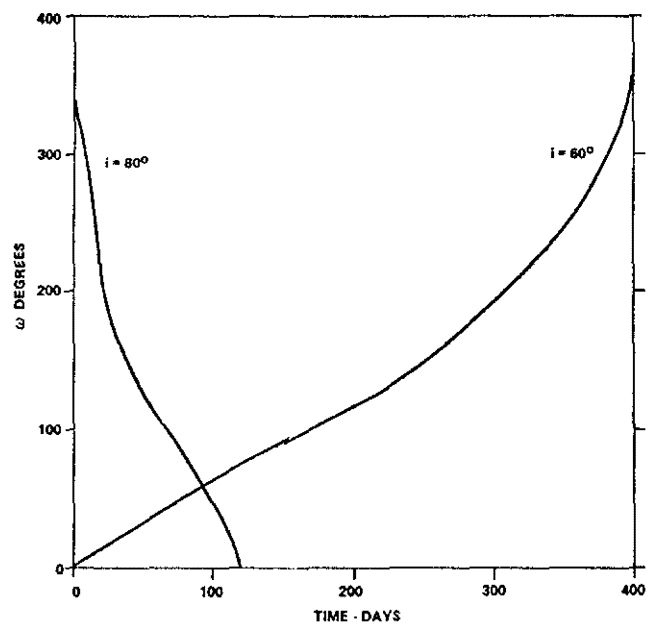


Figure 4. Argument of Perigee Versus Time for Two Orbits With Initial $e = .001$, $R_e/a = .9$, J_2 and J_3

Table 1. Poisson Brackets of Equinoctial Elements*, **

$$\begin{aligned}
(a, \lambda_0) &= -2a s_1 & (n, p) &= -kp s_5 \\
(\lambda_0, h) &= -h s_4 & (h, q) &= -kq s_5 \\
(\lambda_0, k) &= -k s_4 & (k, p) &= hp s_5 \\
(\lambda_0, p) &= -p s_5 & (k, q) &= hq s_5 \\
(\lambda_0, q) &= -q s_5 & (p, q) &= -\frac{1}{2} s_2 s_5 I \\
(h, k) &= -s_1 s_3
\end{aligned}$$

*Auxiliary variables:

$$\begin{aligned}
s_1 &= 1/na^2 \\
s_2 &= 1+p^2+q^2 \\
s_3 &= \sqrt{1-h^2-k^2} \\
s_4 &= s_1 s_3 / (1+s_3) \\
s_5 &= s_1 s_2 / (2s_3)
\end{aligned}$$

**Note: If $I = +1$, the elements have the meaning of Equation (3). If $I = -1$, the elements have the meaning of Equation (4).

Table 2. Partial Derivatives of Position

$$\begin{aligned}
\frac{\partial \vec{x}}{\partial a} &= \frac{1}{a} \left(\vec{x} - \frac{3}{2} \frac{\dot{\vec{x}}}{x} t \right) \\
\frac{\partial \vec{x}}{\partial h} &= \frac{\partial X_1}{\partial h} \hat{f} + \frac{\partial Y_1}{\partial h} \hat{g} \\
\frac{\partial \vec{x}}{\partial k} &= \frac{\partial X_1}{\partial k} \hat{f} + \frac{\partial Y_1}{\partial k} \hat{g} \\
\frac{\partial \vec{x}}{\partial \lambda_0} &= \frac{\dot{\vec{x}}}{x} / n \\
\frac{\partial \vec{x}}{\partial p} &= \frac{2}{1+p^2+q^2} \left[q(Y_1 \hat{f} - X_1 \hat{g}) I - X_1 \hat{w} \right] \\
\frac{\partial \vec{x}}{\partial q} &= \frac{2}{1+p^2+q^2} \left[p(X_1 \hat{g} - Y_1 \hat{f}) + Y_1 \hat{w} \right] I
\end{aligned}$$

Table 3. Partial Derivatives of the Equinoctial Elements With Respect to Velocity

$$\begin{aligned}
\frac{\partial a}{\partial \dot{x}} &= \frac{2}{n^2 a} \frac{\dot{\vec{x}}}{x} \\
\frac{\partial h}{\partial \dot{x}} &= \frac{\sqrt{1-h^2-k^2}}{na^2} \left[\left(\frac{\partial X_1}{\partial k} - h\beta \frac{\dot{X}_1}{n} \right) \hat{f} + \left(\frac{\partial Y_1}{\partial k} - h\beta \frac{\dot{Y}_1}{n} \right) \hat{g} \right] + \frac{k(IqY_1 - pX_1)}{na^2 \sqrt{1-h^2-k^2}} \hat{w} \\
\frac{\partial k}{\partial \dot{x}} &= -\frac{\sqrt{1-h^2-k^2}}{na^2} \left[\left(\frac{\partial X_1}{\partial h} + k\beta \frac{\dot{X}_1}{n} \right) \hat{f} + \left(\frac{\partial Y_1}{\partial h} + k\beta \frac{\dot{Y}_1}{n} \right) \hat{g} \right] - \frac{h(IqY_1 - pX_1)}{na^2 \sqrt{1-h^2-k^2}} \hat{w} \\
\frac{\partial \lambda_0}{\partial \dot{x}} &= \frac{-2}{na^2} \left[\vec{x} - \frac{3}{2} \frac{\dot{\vec{x}}}{x} t \right] + \frac{\sqrt{1-h^2-k^2}}{na^2} \beta \left[\left(h \frac{\partial X_1}{\partial h} + k \frac{\partial X_1}{\partial k} \right) \hat{f} + \left(h \frac{\partial Y_1}{\partial h} + k \frac{\partial Y_1}{\partial k} \right) \hat{g} \right] \\
&\quad + \frac{1}{na^2 \sqrt{1-h^2-k^2}} (IqY_1 - pX_1) \hat{w} \\
\frac{\partial p}{\partial \dot{x}} &= \frac{1+p^2+q^2}{2na^2 \sqrt{1-h^2-k^2}} Y_1 \hat{w} \\
\frac{\partial q}{\partial \dot{x}} &= \frac{(1+p^2+q^2)I}{2na^2 \sqrt{1-h^2-k^2}} X_1 \hat{w}
\end{aligned}$$

Table 4. Integrals

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 d\lambda &= 1 + \frac{3}{2} (h^2 + k^2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 \sin 2L d\lambda &= 5hk \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 \cos 2L d\lambda &= \frac{5}{2} (k^2 - h^2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \cos 3L d\lambda &= \frac{35}{8} k (3h^2 - k^2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \sin 3L d\lambda &= \frac{35}{8} h (h^2 - 3k^2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \cos L \sin 2L d\lambda &= \frac{5}{8} h (5h^2 - 9k^2 - 2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \sin L \sin 2L d\lambda &= \frac{5}{8} k (5k^2 - 9h^2 - 2) \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \sin L d\lambda &= \frac{5}{2} h \left[\frac{3}{4} (h^2 + k^2) - 1 \right] \\
\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^3 \cos L d\lambda &= \frac{5}{2} k \left[\frac{3}{4} (h^2 + k^2) - 1 \right]
\end{aligned}$$

*Notes:

1. The quantities L and λ are defined in Equation (23).
2. The quantities h and k are defined in Equation (3).
3. These integrals have the exact same form if L^* , λ^* , h_r , and k_r replace L, λ , h, and k.

Table 6. Partial Derivatives of $\bar{F}_2^3$

$$\begin{aligned}
\frac{\partial \bar{F}_2^3}{\partial a} &= \frac{2}{a} \bar{F}_2^3 \\
\frac{\partial \bar{F}_2^3}{\partial h} &= -\frac{3}{2} \left(\frac{a}{R_3}\right)^2 \left[(\alpha_1^2 - 4\beta_1^2 + 1) h - 5\alpha_1 \beta_1 k \right] \\
\frac{\partial \bar{F}_2^3}{\partial k} &= -\frac{3}{2} \left(\frac{a}{R_3}\right)^2 \left[(\beta_1^2 - 4\alpha_1^2 + 1) k - 5\alpha_1 \beta_1 h \right] \\
\frac{\partial \bar{F}_2^3}{\partial \alpha_1} &= -\frac{3}{2} \left(\frac{a}{R_3}\right)^2 \left[(h^2 - 4k^2 - 1) \alpha_1 - 5\beta_1 hk \right] \\
\frac{\partial \bar{F}_2^3}{\partial \beta_1} &= -\frac{3}{2} \left(\frac{a}{R_3}\right)^2 \left[(k^2 - 4h^2 - 1) \beta_1 - 5\alpha_1 hk \right]
\end{aligned}$$

Table 5. Third Body Variation of Parameters Equations

$$\begin{aligned}
\frac{da}{dt} &= 0 \\
\frac{dh}{dt} &= \frac{1}{na^2 \sqrt{1-h^2-k^2}} \left[(1-h^2-k^2) \frac{\partial R}{\partial k} - k\gamma_1 \left(p \frac{\partial R}{\partial \alpha_1} - Iq \frac{\partial R}{\partial \beta_1} \right) \right] \\
\frac{dk}{dt} &= \frac{-1}{na^2 \sqrt{1-h^2-k^2}} \left[(1-h^2-k^2) \frac{\partial R}{\partial h} - h\gamma_1 \left(p \frac{\partial R}{\partial \alpha_1} - Iq \frac{\partial R}{\partial \beta_1} \right) \right] \\
\frac{d\lambda_0}{dt} &= \frac{-2}{na} \frac{\partial R}{\partial a} + \frac{\sqrt{1-h^2-k^2}}{na^2 (1+\sqrt{1-h^2-k^2})} \left(h \frac{\partial R}{\partial h} + k \frac{\partial R}{\partial k} \right) - \frac{\gamma_1}{na^2 \sqrt{1-h^2-k^2}} \left(p \frac{\partial R}{\partial \alpha_1} - Iq \frac{\partial R}{\partial \beta_1} \right) \\
\frac{dp}{dt} &= \frac{\gamma_1 (1+p^2+q^2)}{2na^2 \sqrt{1-h^2-k^2}} \frac{\partial R}{\partial \beta_1} \\
\frac{dq}{dt} &= \frac{\gamma_1 (1+p^2+q^2)}{2na^2 \sqrt{1-h^2-k^2}} \frac{\partial R}{\partial \alpha_1} I
\end{aligned}$$

Table 7. Partial Derivatives of $\bar{F}_3^3$

$$\begin{aligned}\partial \bar{F}_3^3 / \partial a &= \frac{3}{a} \bar{F}_3^3 \\ \partial \bar{F}_3^3 / \partial h &= \frac{5}{4} \left(\frac{a}{R_3} \right)^3 \left\{ \frac{15}{8} \beta_1 \left[\alpha_1^2 (15h^2 - 9k^2 - 2) + \beta_1^2 (h^2 + 5k^2 - 2) \right] \right. \\ &\quad \left. - \frac{3}{4} \beta_1 (12h^2 + 3k^2 - 4) + \frac{3}{4} hk \alpha_1 (25 \alpha_1^2 - 45 \beta_1^2 - 9) \right\} \\ \partial \bar{F}_3^3 / \partial k &= \frac{5}{4} \left(\frac{a}{R_3} \right)^3 \left\{ \frac{15}{8} \alpha_1 \left[\beta_1^2 (15k^2 - 9h^2 - 2) + \alpha_1^2 (k^2 + 5h^2 - 2) \right] \right. \\ &\quad \left. - \frac{3}{4} \alpha_1 (12k^2 + 3h^2 - 4) + \frac{3}{4} hk \beta_1 (25 \beta_1^2 - 45 \alpha_1^2 - 9) \right\} \\ \partial \bar{F}_3^3 / \partial \alpha_1 &= \frac{5}{4} \left(\frac{a}{R_3} \right)^3 \left\{ \frac{15}{4} h \alpha_1 \beta_1 (5h^2 - 9k^2 - 2) + \frac{15}{8} k \beta_1^2 (5k^2 - 9h^2 - 2) \right. \\ &\quad \left. + \frac{15}{8} k \alpha_1^2 (15h^2 + k^2 - 6) - \frac{3}{4} k (3h^2 + 3k^2 - 4) \right\} \\ \partial \bar{F}_3^3 / \partial \beta_1 &= \frac{5}{4} \left(\frac{a}{R_3} \right)^3 \left\{ \frac{15}{4} k \alpha_1 \beta_1 (5k^2 - 9h^2 - 2) + \frac{15}{8} h \alpha_1^2 (5h^2 - 9k^2 - 2) \right. \\ &\quad \left. + \frac{15}{8} h \beta_1^2 (15k^2 + h^2 - 6) - \frac{3}{4} h (3h^2 + 3k^2 - 4) \right\}\end{aligned}$$

Table 8. J_2 Variation of Parameters Equations

$$\begin{aligned}\frac{dh}{dt} &= \frac{3\mu R_e^2 J_2 k \left[1 - 6(p^2 + q^2) + 3(p^2 + q^2)^2 \right]}{2na^5 (1 - h^2 - k^2)^2 (1 + p^2 + q^2)^2} \\ \frac{dk}{dt} &= -\frac{3\mu R_e^2 J_2 h \left[1 - 6(p^2 + q^2) + 3(p^2 + q^2)^2 \right]}{2na^5 (1 - h^2 - k^2)^2 (1 + p^2 + q^2)^2} \\ \frac{dp}{dt} &= -\frac{3\mu R_e^2 J_2 q (1 - p^2 - q^2)}{2na^5 (1 - h^2 - k^2)^2 (1 + p^2 + q^2)^2} I \\ \frac{dq}{dt} &= \frac{3\mu R_e^2 J_2 p (1 - p^2 - q^2)}{2na^5 (1 - h^2 - k^2)^2 (1 + p^2 + q^2)^2} I\end{aligned}$$

Table 9. J_3 Variation of Parameters Equations

$$\begin{aligned}\frac{dh}{dt} &= -\frac{3\mu R_e^3 J_3 I}{2na^6 (1 - h^2 - k^2)^3 (1 + p^2 + q^2)^3} \left\{ 2 \left[1 - 3(p^2 + q^2) + (p^2 + q^2)^2 \right] \right. \\ &\quad \left[p - h(kqI + hp) - 4k(hq - kp) \right] - k(hq - kp) \left[1 - 13(p^2 + q^2) + (p^2 + q^2)^2 \right] \right. \\ &\quad \left. (1 - p^2 - q^2) \right\} \\ \frac{dk}{dt} &= -\frac{3\mu R_e^3 J_3 I}{2na^6 (1 - h^2 - k^2)^3 (1 + p^2 + q^2)^3} \left\{ 2 \left[1 - 3(p^2 + q^2) + (p^2 + q^2)^2 \right] \right. \\ &\quad \left[q - k(kq + hp) + 4h(hq - kp) \right] + h(hq - kp) \left[1 - 13(p^2 + q^2) + (p^2 + q^2)^2 \right] \right. \\ &\quad \left. (1 - p^2 - q^2) \right\} \\ \frac{dp}{dt} &= \frac{3\mu R_e^3 J_3 (1 - p^2 - q^2)}{4na^6 (1 - h^2 - k^2)^3} \left\{ h - 5 \frac{[p(kq + hp) + 3q(hq - kp)]}{(1 + p^2 + q^2)^2} \right\} \\ \frac{dq}{dt} &= \frac{3\mu R_e^3 J_3 (1 - p^2 - q^2)}{4na^6 (1 - h^2 - k^2)^3} \left\{ k - 5 \frac{[q(kq + hp) - 3p(hq - kp)]}{(1 + p^2 + q^2)^2} \right\}\end{aligned}$$

References

1. R. Broucke and P. Cefola, "On The Equinoctial Orbit Elements," Celestial Mechanics, vol. 5, no. 3, pp 303-310
2. Bogolibov and Mitropolsky, Non-Linear Oscillations (translated from the Russian). Delhi, India: Hindustan Publishing Corporation, 1961
3. J. L. Stricker and W. W. Miessner, "Satellite Lifetime Program," Proceedings of the 1966 Fall Joint Computer Conference, November 7-10, pp. 789-798
4. B. Kaufman, "Variation of Parameters and the Long-Term Behavior of Planetary Orbiters" (AIAA Paper 70-1055, presented at the AAS/AIAA Astrodynamics Conference, Santa Barbara, California, August 19-21, 1970
5. J. Murphy, R. Nankervis, and C. Doll, An Eccentricity Analysis for the Second Radio Astronomy Explorer (RAE-B), NASA Goddard Space Flight Center, Document X-552-70-408, October 1970
6. C. Uphoff, "The Long-Term Motion of Artificial Planetary Satellites," AAS 70-038, presented to the AAS 16th Annual Meeting, June 8-10, 1970. Also in Advances in the Astronautical Sciences, vol 28, pp. 393-435
7. J. Lorell, "Lunar Orbiter Gravity Analysis," The Moon, 1970, vol. 1, no. 2, pp. 190-231
8. A. Liu, and P. Laing, "Lunar Gravity Analysis from Long-Term Effects," Science, September 10, 1971, vol. 173, pp. 1017-1020
9. D. Brouwer and G. Clemence, Methods of Celestial Mechanics, New York: Academic Press, 1961
10. W. de Sitter, Astronomische Nachrichten, 1903, vol. 163, p. 106
11. A. G. Lubowe, "Application of Lagrange's Planetary Equations to Orbits with Low Eccentricities, or Low Inclinations, or Both," Journal Astronautical Science, Spring 1965, vol. 21, no. 1, pp. 7-17
12. J. Lorell and A. Liu, Method of Averages Expansions for Artificial Satellite Applications, Jet Propulsion Laboratory, Technical Report 32-1513, April 1971.
13. B. Lamers, Variational Equations due to the Oblateness of the Central Body for a Potential in Terms of the J_2 and J_3 Harmonic Coefficients, Computer Sciences Corporation, April 1972 (copy available from author)
14. R. Broucke and P. Cefola, Numerical Integration of Satellite Orbits with Equinoctial Elements, Jet Propulsion Laboratory, JPL Section 391 TM 248, November 5, 1971
15. G. E. Cook, "Perturbations of Near-Circular Orbits by the Earth's Gravitational Potential," Planetary Space Science, 1966, vol. 14, pp. 433-444