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SPIN WAVES IN FLUIDS

by

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Lockheed Electronics Company, Inc.

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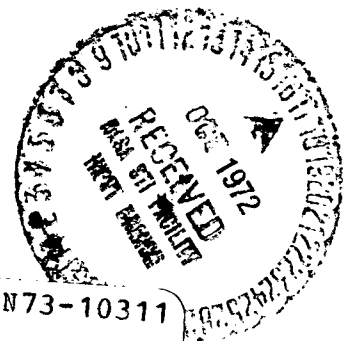Houston Aerospace Systems Division
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SUMMARY

This is a working report prepared in order to document early
results of research on the stability of laminar boundary
layers. The report shows that constitutive equations for
a structured continua may be derived by the simple technique
of reinterpreting velocity in the conventional stress to
rate-of-strain relationship so as to account for effects of
particle rotation.

The report also demonstrates that accounting for particle
structure even at a molecular level makes the fluid visco-
elastic with the ability to propagate vector waves. Finally,
it is shown that particle structure modifies the basic
stability equation for the system, which in turn would alter
values for critical Reynolds number.

It therefore has been demonstrated that conventional fluids
such as air actually are visco-elastic, and that the Navier-
stokes equations do not furnish a proper mathematical model
for the investigation of hydrodynamic stability.

ACKNOWLEDGEMENT

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LIST OF SYMBOLS

Symbol	Definition
a	Wave number for relaxation of molecular spin; Eqn. IV-55
$\hat{A}$	Pseudovector defined by Eqn. II-25
$\hat{A}$	Some undetermined vector in Eqn. III-7
A_1, A_2, A_3	Constants defined in Eqn. IV-3
B_1, B_2, B_3	Constants defined in Eqn. IV-3
c	Phase velocity defined in Eqn. IV-26
$\overset{\wedge\wedge}{C}$	Couple stress tensor; Eqn. II-23
$\overset{\wedge\wedge}{D}$	Viscosity tensor; Eqn. II-14, 15
D_{ijmn}	Components of viscosity tensor; Eqn. II-14, 15
$\hat{e}_i, \hat{e}_j, \hat{e}_k$	Base vectors
e	Compressed notation for $e^{i(\alpha x - \xi t)}$; Eqn. IV-29
$\hat{f}$	Body force; Eqn. II-1
f	Frequency of oscillation; Eqn. A-13
$\mathscr{F}$	Collection of terms; Eqns. III-5 vs. III-3
$\mathscr{G}$	Collection of terms; Eqns. III-6 vs. III-4
$\overset{\wedge\wedge}{I}$	Inertia tensor; average moment of inertia of the molecules; Eqn. II-3; also see Appendix A
I	Moment of inertia for isotropic fluid; Eqn. II-4
k	Average radius of gyration of the molecules; Eqn. A-8
$\hat{\ell}$	Intrinsic angular momentum of molecules; Eqn. II-2, 3

v

Symbol	Definition
L	Mean free path of molecules; Eqns. A-3, 11
n	Molecular number density; Eqn. A-10
p	Tensile force identified with average hydrostatic pressure; Eqn. II-21, IV-1, and IV-9
p'	Perturbation from mean hydrostatic pressure; Eqn. IV-9
P	Mean value of p ; Eqn. IV-9
$\hat{r}$	Position coordinate; $\hat{r} = \hat{i}x + \hat{j}y + \hat{k}z$
$\hat{R}$	Average radius of a molecule; Eqn. II-5
$\hat{R}$	Pseudovector defined as $R = 2\hat{\underset{\sim}{\omega}}$; Eqn. III-19
$\overset{\wedge\wedge}{S}$	Total stress tensor; Eqn. II-1, 21
t	Time
$\overset{\wedge\wedge}{T}$	Viscous stress tensor; Eqn. II-14, 20
T_{ij}	Components of viscous stress tensor; Eqn. II-14, 20
$\hat{u}$	Absolute translational velocity of center of mass of molecule; $\hat{u} = \hat{i}u + \hat{j}v + \hat{k}w$
u,v	Components of $\hat{u}$; Eqns. IV-6, 7
u',v'	Perturbations in u and v ; Eqn. IV-9
$\overline{v}$	Average molecular velocity; Eqn. A-10
$\hat{U}$	Total velocity at edge of a molecule; Eqn. II-5
U,V	Mean values of u and v ; Eqn. IV-9
V_{mn}	Index notation for symmetric part of velocity gradient dyadic; Eqn. II-13
x,y,z	Position coordinates; Eqns. IV-6, 7, 8

Greek Symbols	Definition
α	Dimensional wave number; ($\alpha = 2\pi/\text{wavelength}$) Eqn. IV-26
α	Independent constant, $\alpha \equiv \lambda$; Eqns. II-15, 16
β	Independent viscosity coefficient associated with lack of synchronization between vorticity and spin field $\hat{\omega}_o$; called vortex or spin viscosity; Eqn. II-16, 24
γ	Independent constant, $\gamma \equiv \mu - \beta$; Eqns. II-15, 16
$\hat{\zeta}$	Vorticity, defined by $\hat{\zeta} = \nabla \times \hat{u}$; Eqn. II-8
η	Complex frequency at molecular level; Eqn. IV-55
θ	Independent constant, $\theta \equiv \mu + \beta$; Eqns. II-15, 16
κ	Bulk coefficient of viscosity, $\kappa = 3\lambda + 2\mu$; Eqn. II-22
λ	Second coefficient, or dilitation coefficient of viscosity
λ	Wavelength at prime level; Eqn. IV-26
Λ	Wavelength at molecular level; Eqn. IV-55
μ	First coefficient or shear coefficient of viscosity
ν_1, ν_2	Independent viscosity coefficients associated with couple stresses; Eqn. II-23
ξ	Dimensional frequency; Eqn. IV-26
ρ	Density
σ	Kinetic cross section for binary collision; Eqn. A-10
τ	Relaxation time; Eqns. A-2, 3
$\phi(y)$	Perturbation function; Eqn. IV-25, 27, 29
ϕ	Compressed notation for perturbation function; Eqn. IV-29

Greek Symbols	Definition
ϕ'	Primes denote partial differentiation of $\phi(y)$ with respect to y ; Eqn. IV-27, 30
$\psi(x,y,t)$	Potential function, Eqn. IV-25, 27, 28
$\hat{\omega}_o$	Spin field taken as average angular velocity of the molecules; Eqn. II-3, 5
ω	z-component of spin field in 2D flow with spin mechanically polarized; Eqn. IV-8
ω'	Perturbation in vorticity associated with perturbations u' and v' ; Eqn. IV-13
$\hat{\underset{\sim}{\omega}}$	Vector difference between spin field and vorticity; a relaxation parameter; Eqn. II-13, III-8
$\underset{\sim}{\omega}$	z-component of $\hat{\underset{\sim}{\omega}}$ in 2D flow with spin mechanically polarized; Eqn. IV-8
$\underset{\sim}{\omega}'$	Perturbations in the molecular spin relaxation $\underset{\sim}{\omega}$; Eqn. IV-13
ω_e	Vorticity, when it is emphasized as equilibrium, or relaxed spin state; Eqn. A-2
$\underset{\sim}{\omega}_{mn}$	Index notation for antisymmetric part of velocity gradient dyadic; Eqn. II-13
Ω	Vorticity of the mean flow; Eqn. IV-13
$\underset{\sim}{\Omega}$	Mean value of relaxation parameter $\underset{\sim}{\omega}$; Eqn. IV-13

Subscripts

∞	Free stream
r	Real part
i	Imaginary part
e	Equilibrium or "relaxed" state

Mathematical Symbols	Definition
$\wedge$	Cap, used to denote vector or pseudovector
$\wedge\wedge$	Double cap, used to denote tensor
$\cdot$	Dot product
$\times$	Cross product
$\overset{\times}{\bullet}$	Nesting convention notation; $(\hat{a}\hat{b})\overset{\times}{\bullet}(\hat{c}\hat{d}) = (\hat{a}\times\hat{b})(\hat{c}\cdot\hat{d})$
$=$	Equals
$\Rightarrow$	Implies
$\bar{\pi}$	Defined as
$\equiv$	Identical to
$(\ )^{(s)}$	Symmetric part; Eqn. II-10
$(\ )^{(a)}$	Antisymmetric or skew-symmetric; Eqn. II-10
$\langle\ \rangle$	Pseudovector
∇	Nabla operator; $\nabla = \hat{i}\,\dfrac{\partial}{\partial x} + \hat{j}\,\dfrac{\partial}{\partial y} + \hat{k}\,\dfrac{\partial}{\partial z}$
∇^2	Laplacian operator; $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$
$\nabla\hat{u}$	Velocity gradient dyadic; grad $\hat{u}$; Eqn. II-10
$\hat{u}\nabla$	Conjugate of $\nabla\hat{u}$; Eqn. II-10
δ_{ij}	Kronecker delta
ε_{kij}	Levi-Civita tensor
U	Unit dyad; $\mathsf{U} = \hat{e}_i\hat{e}_j\delta_{ij}$
dual	Defined by $\ $ dual $\hat{\Omega} = -\mathsf{U}\times\hat{\Omega}$
	dual $\overset{\wedge\wedge}{E} = \dfrac{1}{2}\,\mathsf{U}\times\overset{\wedge\wedge}{E}$

I. INTRODUCTION

The location of boundary-layer transition along a surface is one of the factors necessary for the prediction of aerodynamic performance in highspeed flight. This is apparent when one considers that the condition of the boundary layer (i.e., whether laminar or turbulent) determines heat transfer, skin-friction drag, base-pressure drag, and effectiveness of control surfaces that are partially immersed in the body boundary layer. Furthermore, effects of the boundary layer on various types of "pressure" drags and control-surface effectiveness become more important as the flight speed increases, due to the fact that boundary-layer thickness increases with Mach number (at constant Reynolds number).

Around 1950, there was little quantitative information available regarding the effects of various parameters on boundary-layer transition, especially in supersonic flow. Research on subsonic flow had shown that the transition Reynolds number is influenced by surface temperature, surface roughness, pressure gradient, surface curvature, and free-stream turbulence. Investigations in transonic flow indicated that shock-wave/boundary-layer interaction also influences transition. However, only the qualitative effects of these variables were known; little quantitative information was available about the effect of these variables, either singly or interrelated. Quantitative information about boundary layer stability resulted from the theoretical work of Tollmien (1936) and Schlicting (1935). For years later, no experimental evidence was found to confirm the laminar oscillations predicted by Tollmien-Schlicting theory.

Finally, in 1947, the existence of Tollmien-Schlicting oscillations was confirmed by the experiments of Schubauer and Skramstad. However, this theory and its extensions only yield the Reynolds number above which disturbances may become sufficiently amplified to cause turbulence. The theories are useful, however, for indicating relative significance of various parameters on transition.

Because of the many factors that influence boundary-layer transition, the experimental study of the individual effect of any one of these variables on transition is difficult. For example, the effects of free-stream turbulence in wind-tunnel studies tend to mask out the effects of other variables. This is especially true of supersonic flow, since little is known about turbulence in supersonic flow or its effects; furthermore, the turbulence characteristics of most of the supersonic tunnels have not even been completely determined to this day.

The theory of stability of laminar flows was first formulated mathematically by Rayleigh (1887). Successful solutions for the laminar boundary layer on a flat plate were obtained by Tollmien and Schlicting, as already noted. The results of this theory indicate that instability in the laminar boundary-layer originates in small disturbances that are either damped or amplified depending on their frequency and the Reynolds number. This selective amplification is predicted only if the Reynolds number is greater than a certain value, called the "minimum critical Reynolds number". This Reynolds number is always found to be somewhat smaller than the transition Reynolds number, partially due to the fact that regular disturbances in the laminar boundary layer must have

time to be amplified to a magnitude sufficient to somehow produce the irregular oscillations characteristic of turbulence.

A somewhat different theoretical approach was suggested by Taylor (1936). The theory of Taylor, originally thought to be in opposition to that of Tollmien and Schlicting, involves the supposition that the free stream outside the boundary layer always carries at least a slight turbulence, and this turbulence imposes local adverse pressure gradients on the laminar boundary layer. These local adverse gradients are thought to cause local boundary-layer separation (insipient separation), with subsequent transition.

The apparent difference between these two theories was resolved by the experimental studies of Schubauer and Skramstad. In connection with a study of wind-tunnel turbulence, they observed the regular laminar boundary-layer oscillations predicted by Tollmien and Schlicting in a wind tunnel having an exceptionally low turbulence level. The predicted ranges of damping and amplification were closely confirmed. However, it was found that the oscillations and their damping or amplification to eventual turbulence could be detected only when the magnitude of the free-stream turbulent fluctuations is less than about 0.001 of the mean velocity. For larger magnitudes of the free-stream turbulence, the boundary-layer oscillations are difficult to identify because of the near coincidence of their appearance and the point of transition to turbulence.

With the aid of the Schubauer-Skramstad experiments, the
Tollmien-Schlicting theory and the Taylor theory can be
reconciled, and a unified picture of the mechanism leading
to transition in the boundary-layer can be obtained. In
the case of low free-stream turbulence $\left(\frac{u'}{U_\infty} < 0.001\right)$,
any small disturbances that are initially present may be
due to either internal or external disturbances such as
surface roughness, vibration, noise, or free-stream tur-
bulence. The amplified oscillations themselves cannot be
classified as turbulence because of their regularity; it
is only when these oscillations or waves become large enough
to roll up into random eddies that turbulence is initiated.
The eddies, formed close to the solid surface, break away
and dissipate in an irregular turbulent motion. The process
is well illustrated in the interferometer photographs of
Eckert (1951), and is somewhat analogous to the growth and
breaking of waves on an air-water interface. In the case
of a high free-stream turbulence level $\left(\frac{u'}{U_\infty} > 0.001\right)$, the
free stream fluctuations are large enough to produce local
adverse pressure gradients that cause local separation with
subsequent transition in the manner described by Taylor's
theory.

In recent years, numerous revisions and extensions to the
Tollmien-Schlicting theory have appeared. Lin (1946) has
made some revisions and mathematical clarifications.
Schlicting and Ulrich (1942) and Hahneman, Freeman, and
Finston (1948) extended the theory to flows with pressure
gradient and found that a positive pressure gradient
destabilized while a negative pressure gradient stabilized
the boundary layer.

Although the theories of the Tollmien-Schlicting type permit
prediction of the effect of various factors on laminar
boundary-layer stability and prediction of the Reynolds
number below which disturbances are not amplified, they give
no indication of the <u>transition</u> Reynolds number. As already
noted, presumably the original small distrubances must have
time to be amplified to the point where they become unstable.
While the theory permits calculation of the initial amplifica-
tion rate, it is strictly applicable only when the disturbances
are infinitesimally small. That this amplification takes
considerable time and distance is exemplified by a comparison
of the minimum critical Reynolds number with the transition
Reynolds number. For incompressible flow along a flat plate,
the minimum critical Reynolds number had been calculated
as 60,000; this value was confirmed by Schubauer and
Skramstad, who also found the Reynolds number range of the
<u>transition</u> region to be 2,800,000 to 3,900,000. For this
case, then, there is about a sixty-fold difference between
the Reynolds numbers at which amplification begins and at
which turbulence begins.

Liepmann (1945) has made an interesting attempt to compute
the transition Reynolds number using the maximum initial
rates of amplification given by the Tollmien-Schlicting
theory. He estimates that the transition to turbulence
will begin when the apparent shear stress of the amplified
disturbances becomes equal to existing laminar shear stress.
The resulting relationship gives the transition Reynolds
number as a function of the magnitude of the initial disturb-
ance having the frequency of maximum amplification. However,
this latter quantity is dependent on the magnitude and

frequency of the free-stream turbulence, of the surface roughness, and/or of noise and vibration; the quantitative dependence of this factor on these variables is not known. Therefore, Liepmann's calculation can't be compared directly with experiment; however, for reasonable values of the initial disturbances, the predicted ratio of transition Reynolds number to minimum critical Reynolds number is of the same order as that found experimentally.

Emmons (1951) has evolved a theory of transition based on the idea that every point on a body is turbulent part of the time, and the fraction of the time that a given spot is turbulent is derived from probability considerations. The results of this theory, while not enabling prediction of the absolute value of transition range of Reynolds numbers, give the variation of skin friction or heat-transfer rate in the transition region and the effect of flow geometry (flat plate, cone, swept wing) on the transition.

Basically, all work in recent years on boundary layer stability has followed the Tollmien-Schlicting pattern, based on better and better analysis of the Navier-Stokes equations as descriptive of the system. More recently, Kistler (1969, 1971) has reviewed existing literature and assembled clues to the effect that the Navier-Stokes equations themselves may not provide an adequate specification of the stability of the real fluid system being modeled. Based on a review of literature with a wider scope than conventional boundary-layer stability theory, a coupled-dynamics model (Kistler, 1971) based on the work of Dahler (1959) and associates has been suggested as more representative of the flow of diatomic gases. This report documents an initial investigation of the stability of that model.

The stage is set by noting it has been recognized for many years that internal degrees of freedom of molecules play important roles in determining microscopic thermodynamical properties of matter. However, the classical theories of fluid mechanics (i.e. Navier-Stokes equations) do not include explicitly the kinematical effects of these same internal degrees of freedom, apparently because it is not widely recognized that in some situations these kinematical effects may play an important role in stability of the flow and in nonequilibrium processes involving momentum transport.

This report presents first a simple derivation and interpretation of the constitutive equations for fluid species with a finite minimum scale and the ability to transfer intrinsic angular momentum between molecules. Diatomic or rough-sphere monatomic fluids would fall in this category.

In work on the theory of structured continua, Dahler (1959), Dahler and Scriven (1963), Condiff and Dahler (1964), and others derived these same equations by a more elegant but involved procedure, and showed that internal angular momentum is associated with configurational and kinetic structure of the continuum "particle". Condiff and Dahler assign a continuous spin field to the rotation or spin of molecular subunits. The interaction of this internal spin with fluid flow is described by antisymmetric stress, while couple stress accounts for viscous transport of internal angular momentum. Then, with constitutive equations appropriate to a linear, isotropic fluid they obtain generalized Navier-Stokes equations for the velocity and spin fields.

The macroscopic spin field is defined by the local average
rotational velocity of the molecules, and is a kinematical
representation of internal angular momentum of spinning
molecules. This spin field is dynamically coupled to the
fluid velocity by means of the collisional interactions of
the translating and rotating molecules. From a kinetic
theory point of view, this coupling is a result of noncentral
intermolecular forces. Condiff and Dahler note that Born
was evidently the first to suggest that antisymmetric stress
is the result if this internal spin is not "synchronized"
with the vorticity.

This document utilizes this background which establishes the
coupled-dynamics math model, and goes on to show that since
including the effects of molecular structure has coupled the
translational and rotational equations of motion, the system
now can actually support the propagation of vector waves
(transverse waves) within the interior of the fluid. These
waves are waves of angular momentum (or angular velocity),
and will be called "spin waves".

In classical theory of hydrodynamic stability of viscous
fluids such vector waves cannot exist. Specifically their
existence has been precluded by the neglect of molecular
substructure.

Finally, in this report it is shown that under assumptions
comparable to those leading to the classical Orr-Sommerfeld
equation, the analogous equation accounting for substructure
contains an exponential term which could lead to instability
of translational perturbations.

II. Constitutive Equations

Conservation equations for a structured continua have been
developed and presented in a number of journal articles and
texts. In the notation used by Condiff and Dahler (1964),
the equations for conservation of linear momentum and intrinsic
angular momentum of a steady incompressible flow are

$$\rho\left[\frac{\partial \hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u}\right] = \rho\hat{f} + \nabla\cdot\overset{\wedge\wedge}{S} \qquad \text{II-1}$$

$$\rho\left[\frac{\partial \hat{\ell}}{\partial t} + \hat{u}\cdot\nabla\hat{\ell}\right] = \rho\hat{G} + \nabla\cdot\overset{\wedge\wedge}{C} + \hat{A} \ . \qquad \text{II-2}$$

Constitutive equations for the stress and couple stress
tensors were developed by Dahler (1959), and Dahler and
Scriven (1963). It is found here that those same constitu-
tive equations can be obtained by a much easier procedure
if the relative velocity between two particles in the fluid
is properly interpreted.

From the theory of structured continua, the intrinsic angular
momentum $\hat{\ell}$ is taken as

$$\hat{\ell} = \overset{\wedge\wedge}{I}\cdot\hat{\omega}_o \ . \qquad \text{II-3}$$

where the tensor field $\overset{\wedge\wedge}{I} = \overset{\wedge\wedge}{I}(\hat{r},t)$ simulates the average
moment of inertia of the molecules. However, if the molecules
are rigid then $\overset{\wedge\wedge}{I}$ can be regarded as a constant dyadic, and
if the fluid is isotropic* then

$$\overset{\wedge\wedge}{I} = I\mathsf{U} \ , \qquad \text{II-4}$$

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*This is a critical assumption common to all models todate.
It allows relating an average angular velocity to voritcity
as opposed to an angular momentum relationship.

Here $\mathbf{U}$ is the unit dyadic and I is a constant scalar. See Appendix A for second or alternate interpretation of $\hat{\hat{I}}$.

If consideration of the constitutive relations for stress is restricted to the linear Stokesian regime, the stress-strain relationship is found by noting that the velocity field which determines the velocity gradient is not strictly defined by the translational velocity field $\hat{u}$, but instead it is represented by a local velocity $\hat{U}$ which may be slightly greater or less than $\hat{u}$ due to rotation of the molecule.

If the molecules are assumed rigid, then the tangential velocity of an atom due to rotation of the molecule is $\hat{R} \times \hat{\omega}_o$, where $\hat{R}$ is the average radius of a molecule in the flow. The atom then moves with an absolute velocity $\hat{U}$ which is

$$\hat{U} = \hat{u} + \hat{R} \times \hat{\omega}_o , \qquad\qquad \text{II-5}$$

and

$$\nabla\hat{U} = \nabla\hat{u} + \nabla(\hat{R} \times \hat{\omega}_o) = \nabla\hat{u} + [\nabla\hat{R} \times \hat{\omega}_o] - [\nabla\hat{\omega}_o \times \hat{R}] .$$

$$\text{II-6}$$

However,

$$\nabla\hat{R} \equiv \mathbf{U} . \qquad\qquad \text{II-7}$$

Thus,

$$\nabla\hat{U} = \nabla\hat{u} + (\mathbf{U} \times \hat{\omega}_o) + [\hat{R} \times \nabla\hat{\omega}_o] . \qquad\qquad \text{II-8}$$

Even though the spin field $\hat{\omega}_o$ may vary from particle to particle as does the translational velocity $\hat{u}$, the spin vector $\hat{\omega}_o$ is constant for any one particle if the particle is rigid. Hence $\nabla\hat{\omega}_o \equiv 0$ for an individual particle and does not influence the absolute velocity at the edge of the particle. Therefore, the absolute velocity gradient is just

$$\nabla\hat{U} = \nabla\hat{u} + (U \times \hat{\omega}_o) \ . \qquad\qquad II\text{-}9$$

The last term clearly is skew-symmetric, and may be combined with the skew part of the gradient of translational velocity. Thus

$$\nabla\hat{U} = (\nabla\hat{u})^{(s)} + (\nabla\hat{u})^{(a)} + (U \times \hat{\omega}_o)$$

$$= \frac{1}{2}(\nabla\hat{u} + \hat{u}\nabla) + \frac{1}{2}(\nabla\hat{u} - \hat{u}\nabla) + (U \times \hat{\omega}_o) \qquad II\text{-}10$$

$$= \frac{1}{2}(\nabla\hat{u} + \hat{u}\nabla) - \frac{1}{2}U \times \left\langle \frac{1}{2}(\nabla\hat{u} - \hat{u}\nabla)\right\rangle + (U \times \hat{\omega}_o) \ ,$$

where $\left\langle \frac{1}{2}(\nabla\hat{u} - \hat{u}\nabla)\right\rangle$ is the pseudovector of the skew part of translational velocity dyadic.

Now the pseudovectors of $\nabla\hat{u}$ and $\hat{u}\nabla$ are

$$\left\langle\nabla\hat{u}\right\rangle = \nabla\times\hat{u}$$

and

$$\left\langle\hat{u}\nabla\right\rangle = -\nabla\times\hat{u} \ , \qquad\qquad II\text{-}11$$

hence

$$\left\langle \frac{1}{2} (\nabla \hat{u} - \hat{u}\nabla) \right\rangle = \nabla \times \hat{u} . \qquad \text{II-12}$$

Thus,

$$\nabla \hat{U} = \frac{1}{2} (\nabla \hat{u} + \hat{u}\nabla) - \frac{1}{2} \mathsf{U} \times (\nabla \times \hat{u}) + (\mathsf{U} \times \hat{\omega}_o)$$

$$= \frac{1}{2} (\nabla \hat{u} + \hat{u}\nabla) - \frac{1}{2} \mathsf{U} \times [\nabla \times \hat{u} - 2\hat{\omega}_o] \qquad \text{II-13}$$

$$= \frac{1}{2} (\nabla \hat{u} + \hat{u}\nabla) - \frac{1}{2} \text{ dual } \hat{\underset{\sim}{\omega}} \Rightarrow V_{mn} + \underset{\sim}{\omega}_{mn}$$

where $\underset{\sim}{\hat{\omega}}$ is a relative angular velocity due to the intrinsic spin field $\hat{\omega}_o$ not being "synchronized" with the vorticity $\nabla \times \hat{u}$. It will be seen later that $\underset{\sim}{\omega}$ is like a relaxation parameter.

In the simple case where it is assumed that the relationship between stress and rate of strain is linear and isotropic, the constitutive equation is written as

$$T_{ij} = D_{ijmn} [V_{mn} + \underset{\sim}{\omega}_{mn}] , \qquad \text{II-14}$$

Here V_{mn} are the components of the symmetric velocity gradient dyadic, $\underset{\sim}{\omega}_{mn}$ are the components of the skew part, and

$$D_{ijmn} = \alpha \delta_{ij} \delta_{mn} + \theta \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm} , \qquad \text{II-15}$$

where δ_{pq} is the Kronecker delta.

The D_{ijmn} are components of the viscosity tensor, and this tensor is asymmetric, so it too may be written in terms of symmetric and skew parts. Let $\alpha = \lambda$; $\theta = \mu + \beta$; and $\gamma = \mu - \beta$. See discussion in Appendix B.

Then,

$$T_{ij} = \left[\lambda \delta_{ij} \delta_{mn} + \mu(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \right.$$

$$\left. + \beta(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \right] (V_{mn} + \underset{\sim}{\omega}_{mn}) . \qquad \text{II-16}$$

Performing the indicated multiplication gives

$$T_{ij} = \lambda \delta_{ij} \delta_{mn} V_{mn} + \mu(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) V_{mn}$$

$$+ \beta(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) V_{mn} + \lambda \delta_{ij} \delta_{mn} \underset{\sim}{\omega}_{mn}$$

$$+ \mu(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \underset{\sim}{\omega}_{mn} + \beta(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \underset{\sim}{\omega}_{mn}$$

$$\text{II-17}$$

Since the product of a symmetric and a skew-symmetric tensor is zero, the viscous stress tensor reduces to

$$T_{ij} = \lambda \delta_{ij} \delta_{mn} V_{mn} + \mu(\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) V_{mn}$$

$$+ \lambda \delta_{ij} \delta_{mn} \underset{\sim}{\omega}_{mn} + \beta(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \underset{\sim}{\omega}_{mn}$$

$$= \lambda \delta_{ij} V_{mm} + 2\mu V_{ij} + \lambda \delta_{ij} \delta_{mn} \underset{\sim}{\omega}_{mn} + \beta \varepsilon_{kij} \varepsilon_{kmn} \underset{\sim}{\omega}_{mn}$$

$$\text{II-18}$$

But δ_{mn} is symmetrical, so

$$\delta_{mn}\underset{\sim}{\omega}_{mn} \equiv 0 \ . \qquad\qquad \text{II-19}$$

Hence, the viscous stress tensor is just

$$\overset{\wedge\wedge}{T} = \lambda \mathbf{U} \nabla \cdot \hat{u} + \mu [\nabla \hat{u} + \hat{u} \nabla] + \beta \text{ dual } \underset{\sim}{\hat{\omega}} \ , \qquad \text{II-20}$$

and the full stress tensor is

$$\overset{\wedge\wedge}{S} = -p\mathbf{U} + \lambda \mathbf{U} \nabla \cdot \hat{u} + \mu [\nabla \hat{u} + \hat{u} \nabla] + \beta \text{ dual}(\nabla \times \hat{u} - 2\hat{\omega}_o) \ .$$

$$\text{II-21}$$

The coefficient μ is the usual coefficient of shear viscosity, and λ is second coefficient of viscosity. The third coefficient β is one half one what Condiff and Dahler have recently called a coefficient of vortex viscosity.

Contracting Eqn. II-18, identifying p with the average hydrostatic pressure*, and identifying $\kappa = 3\lambda + 2\mu$ as the usual coefficient of bulk viscosity** allows the stress tensor to be written as

$$\overset{\wedge\wedge}{S} = \mathbf{U}(-p + \frac{1}{3} \kappa \nabla \cdot \hat{u}) + \mu \left[(\nabla \hat{u} + \hat{u} \nabla) - \frac{2}{3} \mathbf{U} \nabla \cdot \hat{u} \right]$$

$$+ \beta \text{ dual}[\nabla \times \hat{u} - 2\hat{\omega}_o] \qquad\qquad \text{II-22}$$

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*Note that this is not really valid for problems involving finite disturbances, high frequencies, and relaxation effects. The analysis should be generalized further.

**See Appendix B

This is the same result derived by Dahler by a more elegant but involved procedure employing kinetic theory, whereas here it is obtained by a fairly simple procedure linked closely with intuition and without the need to resort to kinetic theory.

Condiff and Dahler take the couple stress tensor to be symmetric and to depend only upon the symmetrized spin gradient tensor. Hence, by a similar procedure to that above, the couple stress tensor is

$$\overset{\wedge\wedge}{C} = \nu_1 U \nabla \cdot \hat{\omega}_o + \nu_2 [\nabla \hat{\omega}_o + \hat{\omega}_o \nabla - \frac{2}{3} U \nabla \cdot \hat{\omega}_o] \qquad \text{II-23}$$

It is clear from Eqn. II-22 that the skew part of the stress tensor is

$$\overset{\wedge\wedge}{S}{}^{(a)} = \beta \, \text{dual}(\nabla \times \hat{u} - 2\hat{\omega}_o) \, , \qquad \text{II-24}$$

and the pseudovector of this tensor is

$$\hat{A} = 2\beta (\nabla \times \hat{u} - 2\hat{\omega}_o) \, , \qquad \text{II-25}$$

which appears in Equation II-2.

III. Spin Waves

Equations II-22 and II-23 are the constitutive equations for
a structured continua, and represent the relationship between
stress and rate-of-strain when the rigid-body rotational
degree of freedom is accounted for. When these expressions
are substituted into the kinematical equations for balance
of linear momentum and balance of intrinsic angular momentum,
i.e., Equations II-1 and II-2, then the resulting equations
for the fluid system are*

$$\rho\left[\frac{\partial\hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u}\right] = -\nabla p + \left(\kappa + \frac{1}{3}\mu - \beta\right)\nabla\nabla\cdot\hat{u}$$

$$+ (\mu + \beta)\nabla^2\hat{u} + 2\beta\nabla\times\hat{\omega}_0 \qquad\qquad \text{III-1}$$

$$\rho I\left[\frac{\partial\hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0\right] = \left(\nu_2 + \frac{1}{3}\nu_1\right)\nabla\nabla\cdot\hat{\omega}_0 + (\nu_2 + \nu_1)\nabla^2\hat{\omega}_0$$

$$+ 2\beta(\nabla\times\hat{u} - 2\hat{\omega}_0) \qquad\qquad \text{III-2}$$

This is a set of nonlinear partial differential equations
representing the dynamics of the system, and demonstrates
the coupling between the translational field $\hat{u}$ and the
molecular spin field $\hat{\omega}_0$. The vortex viscosity serves as a
coupling coefficient because if $\beta = 0$ then the spin field
does not influence the translational motion.

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*See Kistler (1971) for references on the derivation of
these equations.

First note that for the uncoupled case ($\beta = 0$) the equations are

$$\rho\left[\frac{\partial\hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u}\right] = -\nabla p + \left(\kappa + \frac{1}{3}\mu\right)\nabla\nabla\cdot\hat{u} + \mu\nabla^2\hat{u} \qquad \text{III-3}$$

$$\rho I\left[\frac{\partial\hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0\right] = \left(\nu_2 + \frac{1}{3}\nu_1\right)\nabla\nabla\cdot\hat{\omega}_0 + (\nu_2 + \nu_1)\nabla^2\hat{\omega}_0 \ .$$

$$\text{III-4}$$

These are of the form

$$\nabla^2\hat{u} - \left(\frac{\rho}{\mu}\right)\frac{\partial\hat{u}}{\partial t} = \mathscr{F} \qquad \text{III-5}$$

$$\nabla^2\hat{\omega}_0 - \frac{\rho I}{(\nu_2 + \nu_1)}\frac{\partial\hat{\omega}_0}{\partial t} = \mathscr{G} \ , \qquad \text{III-6}$$

which are parabolic and representative of diffusion phenomena. Arguments applied to Equation III-4 produce the classical result of conservation of vorticity, and time rate of change of vorticity via diffusion is obtained from the curl of Equation III-3. By Helmholtz's theorem,

$$\hat{u} = \text{curl } \hat{A} + \text{grad } B \ , \qquad \text{III-7}$$

where $\hat{A}$ is some vector and B is a scalar, it can be shown from Equation III-3 that there exists a scalar wave equation for the system, representing wave propagation of potential B . However, the equation resulting from the vector potential $\hat{A}$ is again a diffusion equation, so that the system will not

support propagation of transverse (i.e. vector) waves.
The scalar wave equation from this procedure is just the
equation for a longitudinal wave, and describes propagation
of sound in the fluid. See Morse and Feshbach (1953).

At first glance the presence of coupling terms $2\beta\nabla\times\hat{\omega}_0$
and $2\beta(\nabla\times\hat{u}-2\hat{\omega}_0)$ does not appear to change this situation.
Since

$$2\hat{\omega}_0 = \nabla\times\hat{u} - 2\underset{\sim}{\hat{\omega}}$$

$$= \hat{\zeta} - 2\underset{\sim}{\hat{\omega}} \qquad\qquad III-8$$

and

$$2\beta\nabla\times\hat{\omega}_0 = \beta\nabla\times\nabla\times\hat{u} - 2\beta\nabla\times\underset{\sim}{\hat{\omega}} \qquad\qquad III-9$$

where

$$\nabla\times\nabla\times\hat{u} = \nabla\nabla\cdot\hat{u} - \nabla^2\hat{u} \quad, \qquad\qquad III-10$$

it appears that Equation III-1 remains parabolic and the
nature of the diffusion process is only modified slightly by
the term $2\beta\nabla\times\underset{\sim}{\hat{\omega}}$.

Since $\underset{\sim}{\hat{\omega}}$ is taken to be a molecular rotational relaxation
parameter and β is the associated spin viscosity coefficient
as discussed in Section II, the magnitude of this coupling
term is expected to be very small (see Appendix A). One
would then conclude that the dynamics of the system are
adequately revealed by the uncoupled case expressed by
Equation III-3. This equation of course is the Navier-Stokes
equation, and is classically taken as the equation of motion

for the system. From it one obtains the Orr-Sommerfeld equation which is used to evaluate the stability of the system.

At this point, it seems that the skew-symmetric part of the stress tensor has not added anything significant to the problem. However, it will be shown now that this is not the case. Clearly, the skew-symmetric term in Equation III-1 did couple this equation to Equation III-2. Thus, if $\beta \neq 0$, then another mode of energy transfer has been established. If the translation and rotation are oscillatory there is a potential resonance between the two modes, and even a very small coupling term can exchange significant energy over a sufficient number of oscillations.

This is particularly true when the shear coefficient of viscosity is small, and then the system is near neutral stability, i.e. near transition. Then the dynamics of the system could be tremendously influenced by coupling an additional degree of freedom. Experience with lightly damped systems having large inertia terms indicates that the dynamics of these systems can be profoundly influenced by normally trivial terms if the system is near neutral stability. This suggests there may be a difficulty in numerical computation of the stability of boundary layers near transition due strictly to the techniques used to suppress numerical instabilities since these may actually mask physical instability characteristics which are being studied.

Taking $\frac{\partial}{\partial t}$ of Equation III-2 gives an equation describing spin response of the system to a jerk which provides sudden

change in torque on a fluid element (recall that the element now has dimensions and inertia):

$$\frac{\partial}{\partial t}\left\{\rho I\left(\frac{\partial \hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0\right)\right\} = \left(\nu_2 + \frac{1}{3}\nu_1\right)\nabla\nabla\cdot\frac{\partial \hat{\omega}_0}{\partial t} + (\nu_2 + \nu_1)\nabla^2 \frac{\partial \hat{\omega}_0}{\partial t}$$

$$+ 2\beta \frac{\partial}{\partial t}(\nabla\times\hat{u}) - 4\beta \frac{\partial \hat{\omega}_0}{\partial t} \quad . \qquad \text{III-11}$$

This appears still to yield no further information, but merely represent a diffusion equation for the vector quantity $\left(\frac{\partial \hat{\omega}_0}{\partial t}\right)$ by virtue of the first term on the left side and the second term on the right side. But consider now the coupling term (third term on RHS) and see what it does to the equation. For simplicity, take the fluid to be steady and incompressible. Then using

$$\nabla^2\hat{u} = \nabla\nabla\cdot\hat{u} - \nabla\times\nabla\times\hat{u} \quad , \qquad \text{III-12}$$

Equation III-1 becomes

$$\rho\left[\frac{\partial \hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u}\right] = -\nabla p + \left(\kappa + \frac{4}{3}\mu\right)\nabla\nabla\cdot\hat{u} - (\beta + \mu)\nabla\times\hat{\zeta} + 2\beta\nabla\times\hat{\omega}_0 \quad ,$$

$$\text{III-13}$$

Taking the curl of Equation III-13 removes the pressure term and gives

$$\rho \frac{\partial}{\partial t}(\nabla\times\hat{u}) + \rho\nabla\times(\hat{u}\cdot\nabla\hat{u}) = -(\beta + \mu)\nabla\times\nabla\times\hat{\zeta} + 2\beta\nabla\times\nabla\times\hat{\omega}_0 \quad .$$

$$\text{III-14}$$

Solving for $\frac{\partial}{\partial t}(\nabla \times \hat{u})$ from Equation III-14 and substituting into Equation III-11 yields

$$\rho I \left\{ \frac{\partial}{\partial t}\left(\frac{\partial \hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0 \right) \right\} = \left(\nu_2 + \frac{1}{3}\nu_1 \right)\nabla\nabla\cdot\frac{\partial \hat{\omega}_0}{\partial t} + (\nu_2 + \nu_1)\nabla^2 \frac{\partial \hat{\omega}_0}{\partial t}$$

$$- 4\beta \frac{\partial \hat{\omega}_0}{\partial t} + 2\beta \left\{ -\nabla\times(\hat{u}\cdot\nabla\hat{u}) \right.$$

$$\left. -\frac{1}{\rho}(\beta + \mu)\nabla\times\nabla\times\hat{\zeta} + \frac{2\beta}{\rho}\nabla\times\nabla\times\hat{\omega}_0 \right\}$$

$$\text{III-15}$$

This equation is rearranged to give

$$\nabla\times\nabla\times\hat{\omega}_0 - \left(\frac{\rho^2 I}{4\beta^2} \right)\frac{\partial^2 \hat{\omega}_0}{\partial t^2} - \left(\frac{\rho}{\beta} \right)\frac{\partial \hat{\omega}_0}{\partial t} - \left(\frac{\rho^2 I}{4\beta^2} \right)\frac{\partial}{\partial t}[\hat{u}\cdot\nabla\hat{\omega}_0]$$

$$+ \left(\frac{\rho}{4\beta^2} \right)\left(\nu_2 + \frac{1}{3}\nu_1 \right)\nabla\nabla\cdot\frac{\partial \hat{\omega}_0}{\partial t} + \left(\frac{\rho}{4\beta^2} \right)(\nu_2 + \nu_1)\nabla^2 \frac{\partial \hat{\omega}_0}{\partial t}$$

$$- \left(\frac{\rho}{2\beta} \right)\nabla\times(\hat{u}\cdot\nabla\hat{u}) - \left(\frac{\beta + \mu}{2\beta} \right)\nabla\times\nabla\times\hat{\zeta} = 0 \quad , \qquad \text{III-16}$$

which describes the rotational motion of the system. Since

$$\nabla\times\nabla\times\hat{\omega}_0 = \nabla\nabla\cdot\hat{\omega}_0 - \nabla^2\hat{\omega}_0 \quad , \qquad \text{III-17}$$

it is clear that this equation has combined features of parabolic <u>and hyperbolic</u>, and the hyperbolic character is

representative of wave propagation! The parameter being propagated is $\hat{\omega}_o$, which is a vector quantity! Thus, this system can actually support the propagation of vector or transverse waves, in contrast to all modern classical theory about the mechanics of viscous fluids.

Note that a suitable combination of the third and eighth terms in Equation III-16 represents <u>diffusion propagation</u> of the vorticity $\hat{\zeta}$ because of Equation III-8, as would be expected to exist in accordance with classical theory. However, note also that a suitable combination of the second and eighth terms represent <u>wave propagation</u> of vorticity. Finally, note that a suitable combination of the second and sixth terms represents a diffusion of the vector quantity $\left(\frac{\partial \hat{\omega}_0}{\partial t}\right)$.

Consider now the simplified case where the system is still coupled ($\beta \neq 0$), but let the couple stresses vanish ($\nu_1 = \nu_2 = 0$). Equation III-16 then reduces to

$$\nabla \times \nabla \times \hat{\omega}_0 - \left(\frac{\rho^2 I}{4\beta^2}\right) \frac{\partial^2 \hat{\omega}_0}{\partial t^2} - \left(\frac{\rho}{\beta}\right) \frac{\partial \hat{\omega}_0}{\partial t} - \left(\frac{\rho^2 I}{4\beta^2}\right) \frac{\partial}{\partial t} [\hat{u} \cdot \nabla \hat{\omega}_0]$$

$$- \frac{1}{2\beta} (\beta + \mu) \nabla \times \nabla \times \hat{\zeta} - \frac{\rho}{2\beta} \nabla \times [\hat{u} \cdot \nabla \hat{u}] = 0 \quad . \qquad \text{III-18}$$

Replace the spin vector $\hat{\omega}_0$ by

$$2\hat{\omega}_0 = \nabla \times \hat{u} - 2\hat{\omega} = \nabla \times \hat{u} - \hat{R} \quad , \qquad \text{III-19}$$

to give

$$\left\{ \nabla \times \nabla \times \hat{\zeta} + \left(\frac{\rho^2 I}{4\beta\mu}\right) \frac{\partial^2 \hat{\zeta}}{\partial t^2} + \left(\frac{\rho}{\mu}\right) \frac{\partial \hat{\zeta}}{\partial t} + \left(\frac{\rho^2 I}{4\beta\mu}\right) \frac{\partial}{\partial t} [\hat{u} \cdot \nabla \hat{\zeta}] + \left(\frac{\rho}{\mu}\right) \nabla \times [\hat{u} \cdot \nabla \hat{u}] \right\}$$

$$+ \left(\frac{\beta}{\mu}\right) \left\{ \nabla \times \nabla \times \hat{R} - \left(\frac{\rho^2 I}{4\beta^2}\right) \frac{\partial^2 \hat{R}}{\partial t^2} - \left(\frac{\rho}{\beta}\right) \frac{\partial \hat{R}}{\partial t} - \left(\frac{\rho^2 I}{4\beta^2}\right) \frac{\partial}{\partial t} [u \cdot \nabla \hat{R}] \right\} = 0$$

$$\text{III-20}$$

This still is a vector wave equation in $\hat{R}$, and in $\hat{\zeta}$.
Note that if the fluid is "relaxed" and $\hat{R} \equiv 0$, then
Equation III-20 reduces to

$$\nabla \times \nabla \times \hat{\zeta} + \left(\frac{\rho^2 I}{4\beta\mu}\right) \frac{\partial^2 \hat{\zeta}}{\partial t^2} + \left(\frac{\rho}{\mu}\right) \frac{\partial \hat{\zeta}}{\partial t} + \left(\frac{\rho^2 I}{4\beta\mu}\right) \frac{\partial}{\partial t} [\hat{u} \cdot \nabla \hat{\zeta}]$$

$$+ \left(\frac{\rho}{\mu}\right) \nabla \times [\hat{u} \cdot \nabla \hat{u}] = 0 , \qquad \text{III-21}$$

which appears to say that even when the rotational mode is
relaxed the system still can support a vector wave, but such
is not the case. Under the condition where $\hat{R} \equiv 0$, and
with the assumption that $\nu_1 = \nu_2 = 0$, it is clear from
Equation III-2 that the classical case of conservation of
vorticity has been retrieved from this more general problem.
That is,

$$\frac{d\hat{\zeta}}{dt} = \frac{\partial \hat{\zeta}}{\partial t} + \hat{u} \cdot \nabla \hat{\zeta} = 0 , \qquad \text{III-22}$$

so Equation III-21 reduces to

$$\nabla \times \nabla \times \hat{\zeta} + \left(\frac{\rho}{\mu}\right) \frac{\partial \hat{\zeta}}{\partial t} + \left(\frac{\rho}{\mu}\right) \nabla \times [\hat{u} \cdot \nabla \hat{u}] = 0 , \qquad \text{III-23}$$

which is not a vector wave equation, but merely a vector
analog of the vorticity diffusion equation.

Classical sound waves are scalar and are basically waves of compression and rarefaction, but the only component of the particle displacement that is propagated as a wave motion is that lying in the direction of propagation. For this reason they are called longitudinal.

In contrast, the transverse waves have a vectorial character in that it is necessary to specify the orientation of the displacement in the yz-plane (i.e. a plane normal to the direction of propagation). It is clear now that a fluid described by Equations III-1 and III-2 can support both longitudinal and transverse waves, and that they may be coupled. This then is at least one mechanism whereby strictly 2D disturbances can and do become 3D. Except for the obvious additional possibility that no experimenter can ever produce a completely 2D disturbance, the mechanism discovered here is the first answer ever provided for 2D to 3D transfer in the linear regime of the problem.

Longitudinal waves are added algebraically, as scalar quantities. Transverse waves are added vectorially, with components in two mutually perpendicular transverse directions added separately. Now this vector property gives rise to the various phenomena of polarization, and so it is reasonable to expect that if sufficiently accurate experiments were performed on the boundary layer region they would indicate that the molecules become mechanically polarized due to flow along a boundary.

Before moving on to the question of stability of the system, it should be noted that the classical absence of transverse

waves in a fluid is essentially attributed to its inability
to support shear stress. In saying that transverse waves
do not exist in a fluid, this of course is speaking of the
interior of a mass of fluid and beyond reach of any surface
effect, because it clearly is a matter of common observation
that transverse waves can exist on the surface of a liquid.

There is a clear-cut distinction between the longitudinal
and transverse waves in the system. There is no possibility
of regarding the longitudinal wave as merely the first com-
ponent of some more general vector wave whose second and
third components constitute the transverse wave for $\hat{\omega}_0$.
The two waves are separate solutions for the dynamics of the
system, and the two waves travel at different velocities of
propagation. Furthermore, if the solution starts out longi-
tudinal it will remain longitudinal and if it starts out
transverse it will remain transverse if the coefficients in
the equation have everywhere the same value. However, if
the coefficients can vary in a region of space then wave
reflections will occur and the longitudinal and transverse
waves may become intermingled.

In electromagnetic theory, transverse waves are associated
with the propagation of angular momentum. Transverse waves
also exist in theory of elasticity, and again are associated
with angular momentum. However, angular motion in a solid
is constrained since a given element is not completely free
to rotate. However, it may exhibit a rotational oscillation
(Brillioun, 1964) with spring stiffness proportional to
angular displacement. Counter rotations of adjacent elements
(Brillioun; and Kistler, 1969) represent a shear phenomena,
and so transverse waves in a solid are also called shear waves.

It is suspected that this is the same type of phenomena exhibited by the system under discussion here, and that in essence what has been discovered here is that without anything like specification of frequency-dependent viscosity coefficients, etc. it has been established here that the inclusion of molecular characteristics which allow transmittal of angular momentum have in effect shown that such a real fluid has the properties known generally as visco-elastic!

Visco-elasticity normally is <u>assumed</u> to exist, via postulates about the viscosity coefficients in the constitutive equations. Here, in the development of Sections II and III, no postulates about visco-elastic or frequency dependent viscosity coefficients have been resorted to. All that has been postulated is that the fluid does have a structure, and can support angular momentum exchange, which of course is well-established even for air. Now, it further is well-established that visco-elasticity changes the dynamics of the system, and particularly the critical Reynolds number obtained.*

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*See a discussion of this in Kistler (1971).

IV. Stability Analysis

The governing equations for the system were given as

$$\rho\left[\frac{\partial\hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u}\right] = -\nabla p + \left(\kappa + \frac{1}{3}\mu - \beta\right)\nabla\nabla\cdot\hat{u} + (\mu + \beta)\nabla^2\hat{u} + 2\beta\nabla\times\hat{\omega}_0$$

$$\text{IV-1}$$

$$\rho I\left[\frac{\partial\hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0\right] = \left(\nu_2 + \frac{1}{3}\nu_1\right)\nabla\nabla\cdot\hat{\omega}_0 + (\nu_2 + \nu_1)\nabla^2\hat{\omega}_0$$

$$+ 2\beta(\nabla\times\hat{u} - 2\hat{\omega}_0) \qquad \text{IV-2}$$

To show the stability effects of the coupling, it is suffi-
cient to consider a simplified case. Take the flow to be
incompressible so that $\nabla\cdot\hat{u} = 0$, and let

$$A_1 = \frac{1}{\rho}\left(\kappa + \frac{1}{3}\mu - \beta\right) \quad ; \quad B_1 = \frac{1}{\rho I}\left(\nu_2 + \frac{1}{3}\nu_1\right)$$

$$A_2 = \frac{1}{\rho}(\mu + \beta) \qquad\quad ; \quad B_2 = \frac{1}{\rho I}(\nu_2 + \nu_1) \qquad \text{IV-3}$$

$$A_3 = \frac{2\beta}{\rho} \qquad\qquad\quad ; \quad B_3 = \frac{2\beta}{\rho I} = \frac{A_3}{I} \quad .$$

Then for $\nu_1 = \nu_2 = 0$, the governing equations are

$$\frac{\partial\hat{u}}{\partial t} + \hat{u}\cdot\nabla\hat{u} = -\frac{\nabla p}{\rho} + A_2\nabla^2\hat{u} + A_3\nabla\times\hat{\omega}_0 \qquad \text{IV-4}$$

and

$$\frac{\partial \hat{\omega}_0}{\partial t} + \hat{u}\cdot\nabla\hat{\omega}_0 = B_3(\nabla\times\hat{u} - 2\hat{\omega}_0) = 2B_3\tilde{\omega} \quad . \qquad \text{IV-5}$$

Assume 2D parallel flow with the vorticity and spin vectors aligned due to mechanical polarization. The governing equations in component form then are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + A_3\left[\frac{\partial \omega}{\partial y}\right] \qquad \text{IV-6}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - A_3\left[\frac{\partial \omega}{\partial x}\right] \qquad \text{IV-7}$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 2B_3\tilde{\omega} \, , \qquad \text{IV-8}$$

where ω and $\tilde{\omega}$ are the z-components respectively of $\hat{\omega}_0$ and $\hat{\tilde{\omega}}$.

Now make the assumption that velocities and pressure may be represented by the sum of a mean term plus a small perturbation:

$$u = U + u' \quad ; \quad v = V + v' \quad ; \quad p = P + p' \qquad \text{IV-9}$$

The spin field ω also is composed of a mean field plus perturbations of the same scale as u', v', and p' , as well as a relaxation perturbation $\tilde{\omega}$ which could be of a greatly smaller scale. This would suggest that there should be mini-scale perturbations u'', v'', and p'' comparable to $\tilde{\omega}$. The work of Molo-Christensen (1970, 1971) shows how these mini-scale perturbations influence dynamics of the large-scale motion. For the purpose of this document the complexity

of the double-prime scale is not needed, so the analysis retains only the single-prime scale plus the relaxation parameter $\underset{\sim}{\omega}$ which is at a molecular scale or possibly at the scale of molecular clusters.

Substituting Equations IV-9 into Equations IV-6, 7 gives

$$\frac{\partial (U + u')}{\partial t} + (U + u') \frac{\partial (U + u')}{\partial x} + (V + v') \frac{\partial (U + u')}{\partial y}$$

$$= -\frac{1}{\rho} \frac{\partial (P + p')}{\partial x} + A_2 \left[\frac{\partial^2}{\partial x^2} (U + u') + \frac{\partial^2}{\partial y^2} (U + u') \right] + A_3 \left(\frac{\partial \omega}{\partial y} \right)$$

$$\text{IV-10}$$

$$\frac{\partial (V + v')}{\partial t} + (U + u') \frac{\partial (V + v')}{\partial x} + (V + v') \frac{\partial (V + v')}{\partial y}$$

$$= -\frac{1}{\rho} \frac{(P + p')}{\partial y} + A_2 \left[\frac{\partial^2}{\partial x^2} (V + v') + \frac{\partial^2}{\partial y^2} (U + u') \right] - A_3 \left(\frac{\partial \omega}{\partial x} \right)$$

$$\text{IV-11}$$

$$\frac{\partial \omega}{\partial t} = 2B_3 \underset{\sim}{\omega} - (U + u') \left(\frac{\partial \omega}{\partial x} \right) - (V + v') \left(\frac{\partial \omega}{\partial y} \right) .$$

$$\text{IV-12}$$

It is clear (from Equation III-8) that the z-component of spin is

$$\omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \underset{\sim}{\omega}$$

$$= \frac{1}{2} \left[\frac{\partial}{\partial x} (V + v') - \frac{\partial}{\partial y} (U + u') \right] - \underset{\sim}{\omega}$$

$$= \frac{1}{2} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) - \underset{\sim}{\omega}$$

$$= (\Omega + \omega') - (\underset{\sim}{\Omega} + \underset{\sim}{\omega}')$$

$$= \omega_e - \underset{\sim}{\omega}$$

$$\text{IV-13}$$

The term of Ω is identified as the vorticity of the mean flow, ω' is identified as a perturbation in vorticity associated with the perturbations u' and v' , and $\underset{\sim}{\Omega}$ is identified as a mean relaxation parameter with an oscillatory modification given by $\underset{\sim}{\omega'}$. Even though the prime quantities are perturbations from the mean flow, they are of a completely different scale than $\underset{\sim}{\omega'}$, and the sum of $\underset{\sim}{\Omega}$ and $\underset{\sim}{\omega'}$ may be taken as an equilibrium value ω_e when considered relative to $\underset{\sim}{\omega}$. For an estimate of the scale of $\underset{\sim}{\omega'}$ see Appendix A.

Equation IV-12 then expands to

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} = 2B_3 \underset{\sim}{\omega} - (U + u')\left(\frac{\partial \Omega}{\partial x} + \frac{\partial \omega'}{\partial x} - \frac{\partial \underset{\sim}{\Omega}}{\partial x} - \frac{\partial \underset{\sim}{\omega}}{\partial x}\right)$$

$$- (V + v')\left(\frac{\partial \Omega}{\partial y} + \frac{\partial \omega'}{\partial y} - \frac{\partial \underset{\sim}{\Omega}}{\partial y} - \frac{\partial \underset{\sim}{\omega'}}{\partial y}\right) \qquad \text{IV-14}$$

Substituting Equation IV-13 into Equation IV-10 and separating mean from perturbation terms gives

$$\left\{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} + \left\{\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial U}{\partial x} + u'\frac{\partial u'}{\partial x}\right.$$

$$\left. + V\frac{\partial u'}{\partial y} + v'\frac{\partial U}{\partial y} + v'\frac{\partial u'}{\partial y}\right\} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$+ A_2\left\{\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right\} + A_3\left\{\frac{\partial \Omega}{\partial y} + \frac{\partial \omega'}{\partial y} - \frac{\partial \underset{\sim}{\Omega}}{\partial y} - \frac{\partial \underset{\sim}{\omega'}}{\partial y}\right\}$$

$$\text{IV-15}$$

Doing the same thing for Equation IV-11 gives

$$\left\{\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right\} + \left\{\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + u'\frac{\partial V}{\partial x} + u'\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y}\right.$$

$$\left. + v'\frac{\partial V}{\partial y} + v'\frac{\partial v'}{\partial y}\right\} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{1}{\rho}\frac{\partial p'}{\partial y} + A_2\left\{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 v'}{\partial x^2}\right.$$

$$\left. + \frac{\partial^2 u'}{\partial y^2}\right\} - A_3\left\{\frac{\partial \Omega}{\partial x} + \frac{\partial \omega'}{\partial x} - \frac{\partial \tilde{\Omega}}{\partial y} - \frac{\partial \tilde{\omega}'}{\partial y}\right\}$$

IV-16

Now, paralleling the usual assumption that the mean flow by itself also satisfies the Navier-Stokes equations, the same assumption will be invoked here for modified Navier-Stokes equations with mean vorticity and relaxation terms. Thus, subtracting out the mean flow terms allows Equations IV-15 and IV-16 to reduce to

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial U}{\partial x} + u'\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} + v'\frac{\partial U}{\partial y} + v'\frac{\partial u'}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + A_2\left\{\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right\} + A_3\left\{\frac{\partial \omega'}{\partial y} - \frac{\partial \tilde{\omega}'}{\partial y}\right\} \quad \text{IV-17}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + u'\frac{\partial V}{\partial x} + u'\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} + v'\frac{\partial V}{\partial y} + v'\frac{\partial v'}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial p'}{\partial y} + A_2\left\{\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right\} - A_3\left\{\frac{\partial \omega'}{\partial x} - \frac{\partial \tilde{\omega}'}{\partial x}\right\} \quad \text{IV-18}$$

where

$$\omega' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \quad \text{IV-19}$$

The boundary layer assumptions are now invoked, so

$$V = 0 \quad ; \quad \frac{\partial U}{\partial x} = 0 \quad ; \quad \frac{\partial P}{\partial y} = 0 \quad ; \qquad \text{IV-20}$$

and since the perturbations are assumed small their products will be assumed negligible. It is emphasized that these several assumptions are not necessarily concurred with, and they may significantly influence the problem. However, they are made here in order to compare features of the structured fluid with the classical model used for hydrodynamic stability analyses. With the boundary layer assumptions, and dropping terms such as

$$u' \frac{\partial u'}{\partial x} \quad ; \quad v' \frac{\partial u'}{\partial y} \quad ; \quad \text{etc.} \qquad \text{IV-21}$$

Equations IV-17 and IV-18 reduce to

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + A_2 \left[\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right]$$

$$+ A_3 \left[\frac{\partial}{\partial y} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) - \frac{\partial \tilde{\omega}'}{\partial y} \right] \qquad \text{IV-22}$$

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} + A_2 \left[\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right]$$

$$- A_3 \left[\frac{\partial}{\partial x} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) - \frac{\partial \tilde{\omega}'}{\partial x} \right] \qquad \text{IV-23}$$

Taking $\frac{\partial}{\partial y}$ of Equation IV-22 and subtracting from it $\frac{\partial}{\partial x}$ of Equation IV-23 eliminates the pressure terms:

$$\frac{\partial^2 u'}{\partial y \partial t} + U \frac{\partial^2 u'}{\partial y \partial x} + \frac{\partial U}{\partial y} \frac{\partial u'}{\partial x} + v' \frac{\partial^2 U}{\partial y^2} + \frac{\partial v'}{\partial y} \frac{\partial U}{\partial y}$$

$$- \frac{\partial^2 v'}{\partial x \partial t} - U \frac{\partial^2 v'}{\partial x^2} - \cancel{\frac{\partial U}{\partial x} \frac{\partial v'}{\partial x}} = \cancel{-\frac{1}{\rho} \frac{\partial^2 p'}{\partial y \partial x}}$$

$$+ A_2 \left[\frac{\partial^3 u'}{\partial y \partial x^2} + \frac{\partial^3 u'}{\partial y^3} \right] + A_3 \left[\frac{\partial^2}{\partial y^2} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) - \frac{\partial^2 \tilde{\omega}'}{\partial y^2} \right]$$

$$+ \cancel{\frac{1}{\rho} \frac{\partial^2 p'}{\partial x \partial y}} - A_2 \left[\frac{\partial^3 v'}{\partial x^3} + \frac{\partial^3 v'}{\partial x \partial y^2} \right] + A_3 \left[\frac{\partial^2}{\partial x^2} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) - \frac{\partial^2 \tilde{\omega}'}{\partial x^2} \right]$$

$$\text{IV-24}$$

It is now assumed that the disturbed motion is oscillatory* with the stream function given by

$$\psi(x,y,t) = \phi(y) e^{i(\alpha x - \xi t)} \qquad \text{IV-25}$$

where

$$\alpha = \frac{2\pi}{\lambda} = \text{dimensional wave number}$$

$$\xi = \xi_r + i\xi_i = \text{complex frequency} \qquad \text{IV-26}$$

$$c = \frac{\xi}{\alpha} = c_r + ic_i = \text{complex phase velocity}$$

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*See Kistler (1971)

and

$$u' = \frac{\partial \psi}{\partial y} = \phi'(y)e^{i(\alpha x - \xi t)} \qquad \text{IV-27}$$

$$v' = -\frac{\partial \psi}{\partial x} = -i\alpha\phi(y)e^{i(\alpha x - \xi t)} \qquad \text{IV-28}$$

Note that primes on u and v indicate perturbation quantities, whereas primes on ϕ indicate differentiation with respect to y .

Then the necessary derivatives are

$$\frac{\partial u'}{\partial x} = i\alpha\phi'(y)e^{i(\alpha x - \xi t)} = i\alpha\phi'e \qquad \text{IV-29}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial x}\right) = i\alpha\phi''e \qquad \text{IV-30}$$

$$\frac{\partial v'}{\partial x} = \alpha^2\phi e \qquad \text{IV-31}$$

$$\frac{\partial^2 v'}{\partial x^2} = i\alpha^3\phi e \qquad \text{IV-32}$$

$$\frac{\partial u'}{\partial y} = \phi''e \qquad \text{IV-33}$$

$$\frac{\partial^2 u'}{\partial y^2} = \phi'''e \qquad \text{IV-34}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 u'}{\partial y^2}\right) = \phi''''e \qquad \text{IV-35}$$

$$\frac{\partial v'}{\partial y} = -i\alpha\phi'e \qquad \text{IV-36}$$

$$\frac{\partial^2 v'}{\partial y^2} = -i\alpha\phi''e \qquad \text{IV-37}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial y^2}\right) = \alpha^2\phi''e \qquad \text{IV-38}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial x^2}\right) = -\alpha^4\phi e \qquad \text{IV-39}$$

$$\frac{\partial^2 u'}{\partial x^2} = -\alpha^2\phi'e \qquad \text{IV-40}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 u'}{\partial x^2}\right) = -\alpha^2\phi''e \qquad \text{IV-41}$$

$$\frac{\partial u'}{\partial t} = -i\xi\phi'e \qquad \text{IV-42}$$

$$\frac{\partial v'}{\partial t} = \alpha\xi\phi e \qquad \text{IV-43}$$

$$\frac{1}{i\alpha}\frac{\partial^2 u'}{\partial y\partial t} = -\frac{i\xi}{i\alpha}\phi''e = -c\phi''e \qquad \text{IV-44}$$

$$\frac{1}{i\alpha}\frac{\partial^2 v'}{\partial x\partial t} = \frac{i\alpha^2\xi}{i\alpha}\phi e = c\alpha^2\phi e \qquad \text{IV-45}$$

Inserting the proper derivatives into Equation IV-24 gives

$$(-i\alpha c\phi''e) + U(i\alpha\phi''e) + \frac{\partial U}{\partial y} (i\alpha\phi'e)$$

$$+ (-i\alpha\phi e) \frac{\partial^2 U}{\partial y^2} + (-i\alpha\phi'e) \frac{\partial U}{\partial y} + (i\alpha^3 c\phi e)$$

$$- U(i\alpha^3\phi e) = A_2\left[(-\alpha^2\phi''e) + (\phi''''e) - (-\alpha^4\phi e)\right.$$

$$\left. - (\alpha^2\phi''e)\right] + A_3\left[(\alpha^2\phi''e) - (\phi''''e) + (-\alpha^4\phi e)\right.$$

$$\left. - (-\alpha^2\phi''e)\right] - A_3\left\{\frac{\partial^2\underset{\sim}{\omega}'}{\partial x^2} + \frac{\partial^2\underset{\sim}{\omega}'}{\partial y^2}\right\} \qquad \text{IV-46}$$

This reduces to

$$(U - c)(\phi'' - \alpha^2\phi)e - \phi e \frac{d^2 U}{dy^2} = \frac{(A_2 - A_3)}{i\alpha}\left[\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi\right]e$$

$$- \frac{A_3}{i\alpha}\left\{\frac{\partial^2\underset{\sim}{\omega}'}{\partial x^2} + \frac{\partial^2\underset{\sim}{\omega}'}{\partial y^2}\right\}.$$

$$\text{IV-47}$$

or

$$\boxed{\begin{array}{l}(U - c)(\phi'' - \alpha^2\phi) - \phi \frac{d^2 U}{dy^2} = \frac{(A_2 - A_3)}{i\alpha}\left[\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi\right] \\[2em] \qquad\qquad - \frac{A_3}{i\alpha}\left\{\frac{\partial^2\underset{\sim}{\omega}'}{\partial x^2} + \frac{\partial^2\underset{\sim}{\omega}'}{\partial y^2}\right\} e^{-i(\alpha x - \xi t)}\end{array}}$$

$$\text{IV-48}$$

To retrieve the classical Orr-Sommerfeld stability equation
it is only necessary to let $A_3 = 0$:

$$(U - c)(\phi'' - \alpha^2\phi) - \phi U'' = -\frac{i\mu}{\alpha\rho} [\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi] \quad .$$

IV-49

Dividing all velocities by the maximum velocity U_m of the
laminar flow (or for boundary layers, use the free-stream
velocity U_∞) and dividing all lengths by an appropriate
reference length such as boundary layer thickness, the
above equation is nondimensionalized to give

$$(U - c)(\phi'' - \alpha^2\phi) - \phi U'' = -\frac{i}{\alpha R} [\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi] \quad \text{IV-50}$$

which is the usual form of the Orr-Sommerfeld equation
(Schlicting, 1955) with R as the Reynolds number.

The two differences between Equation IV-49 and the more
general equation for a structured continua (Equation IV-48)
are the modified coefficient of the usual viscous part and
then the additional terms in $\underset{\sim}{\omega}'$. Now the coefficient A_3
may or may not be small depending on what it is assumed to
represent (Appendix A). Supposing A_3 is extremely small,
as it would be if spin $\hat{\omega}_o$ is taken as the spin of individual
molecules, then the modification to the usual viscous solu-
tion is negligible. However, the additional term

$$\frac{A_3}{i\alpha} \left\{ \frac{\partial^2 \underset{\sim}{\omega}'}{\partial x^2} + \frac{\partial^2 \underset{\sim}{\omega}'}{\partial y^2} \right\} e^{-i(\alpha x - \xi t)} \qquad \text{IV-51}$$

could have a strong influence on system dynamics even if A_3 is very small. This term may be written as

$$-\frac{iA_3}{\alpha}\left\{\frac{\partial^2 \tilde{\omega}'}{\partial x^2} + \frac{\partial^2 \tilde{\omega}'}{\partial y^2}\right\} e^{-i\alpha x} e^{i(\xi_r + i\xi_i)t}$$

$$= -\frac{iA_3}{\alpha}\left\{ \right\} e^{-i(\alpha x - \xi_r t)} e^{-\xi_i t}$$

$$= -\frac{iA_3}{\alpha}\left\{ \right\}\left[e^{-\xi_i t}\right]\left[\cos(\alpha x - \xi_r t) - i\sin(\alpha x - \xi_r t)\right] \quad \text{IV-52}$$

As with the classical theory, physical meaning is attached only to the real part, which is

$$RE = -\frac{A_3}{\alpha}\left\{\frac{\partial^2 \tilde{\omega}'}{\partial x^2} + \frac{\partial^2 \tilde{\omega}'}{\partial y^2}\right\} e^{-\xi_i t} \sin(\alpha x - \xi_r t) .$$

$$\text{IV-53}$$

From Equation IV-53 it appears that the physically significant part of the additional term is oscillatory and always damped. However, $\tilde{\omega}'$ also will be oscillatory, and of the form

$$\tilde{\omega}' = \gamma(y)e^{i(ax - \eta t)} , \qquad \text{IV-54}$$

where

$$a = \frac{2\pi}{\Lambda} = \text{wave number for relaxation of molecular spin}$$

$$\Lambda = \text{wavelength}$$

$$\eta = \eta_r + i\eta_i = \text{complex frequency on order of magnitude of molecular collision frequency}$$

IV-55

Thus

$$\frac{\partial \tilde{\omega}'}{\partial x} = ia\gamma(y)e^{i(ax-\eta t)} \qquad \text{IV-56}$$

$$\frac{\partial^2 \tilde{\omega}'}{\partial x^2} = -a^2\gamma(y)e^{i(ax-\eta t)} \qquad \text{IV-57}$$

$$\frac{\partial^2 \tilde{\omega}'}{\partial y^2} = \gamma'(y)e^{i(ax-\eta t)} \qquad \text{IV-58}$$

$$\frac{\partial^2 \tilde{\omega}'}{\partial y^2} = \gamma''(y)e^{i(ax-\eta t)} \qquad \text{IV-59}$$

and Equation IV-53 becomes

$$-\frac{A_3}{\alpha}\left\{-a^2\gamma + \gamma''\right\} e^{i(ax-\eta t)}e^{-\xi_i t} \sin(\alpha x - \xi_r t) \quad , \qquad \text{IV-60}$$

or

$$-\frac{A_3}{\alpha}\left\{\gamma'' - a^2\gamma\right\} e^{(\eta_i-\xi_i)t} e^{i(ax-\eta_r t)} \sin(\alpha x - \xi_r t) \quad . \quad \text{IV-61}$$

Equation IV-61 can be expanded and the real part is

$$RE = -\frac{A_3}{\alpha}\left\{\gamma'' - a^2\gamma\right\} e^{(\eta_i-\xi_i)t} \cos(ax - \eta_r t)\sin(\alpha x - \xi_r t) \quad .$$

$$\text{IV-62}$$

Now it is clear that certain multiples of the spin frequencies η_i may enforce the prime level perturbations and cause oscillations in the system to amplify. It should be noted that since η_i is associated with molecular collisions it should be expected to be much greater than ξ_i and therefore able to drive this term to significant influence even though A_3 and γ are small.

While it is hard to believe that molecular level oscillations could feed much energy into macroscopic perturbations at the prime level, the work of Mollo-Christensen has clearly shown that there is actually an infinity of scales in between, and each need act only on its neighbor to produce a cascade type process in reverse. All theory aside, Mollo-Christensen also notes physical proof of the strong interaction between disparate scales of motion in nature. As one example furnishing dramatic proof, he was able to completely change the dynamics of the gravity waves on the ocean surface by covering it with thin surface slick which eliminated the tiny capillary waves.

V. Conclusions

1. It is concluded from Section II that the constitutive equations for a structured continua can be obtained by proper interpretation of the velocity to which the stress tensor is proportional. This technique quickly yields the results of Dahler (1959), Dahler and Scriven (1963), and Condiff and Dahler (1964), and does not have to resort to kinetic theory.

2. It is concluded from Section III that the spin viscosity term which couples the system equations for balance of linear momentum and balance of intrinsic angular momentum is important since the coupling transforms the system from parabolic to parabolic plus hyperbolic, and allows the system to support vector wave propagation in the interior of the fluid in addition to the usual diffusion phenomena.

3. From the development of the vector wave equation in Section III, it is concluded further that the system can be mechanically polarized due to flow.

4. It also is concluded that classical sound waves can be coupled to the transverse waves. Thus, sound waves can excite transverse oscillations in the flow, and in turn the propagation of high frequency sound can be attenuated not only by classical dissipation but also by dissipating their energy into the transverse waves which they excite.

5. Since the constitutive and kinematic equations discussed
    in Sections II and III are representative of fluids such
    as air, it is concluded that air can support transverse
    waves.

6. Since longitudinal and transverse waves in a physical
    media are usually called respectively compressional and
    shear waves, and since the ability to support both com-
    pressional and shear waves is representative of what is
    usually called visco-elastic, it is concluded that this
    work reasonably establishes a visco-eleastic character
    for air.

7. From the stability analysis of Section IV it is concluded
    that the basic stability equation for the system has an
    additional term multiplied by a positive exponential term
    in time. Thus, a part of the viscous solution is
    inherently divergent and the system will be unstable
    unless it possesses sufficient damping in the other
    (i.e. the usual) part of the viscous solution.

8. Since the unstable part has a very slow growth rate
    (i.e. β = small), it appears that in digital computations
    the methods used to suppress numerical instability would
    normally completely overshadow this physical instability.
    This would tend to imply that the instability due to
    skew-symmetric stress is a negligibly weak effect.

    Obviously, if the flow is tripped or has large free-
    stream disturbances, then the effects described herein
    are not going to influence the transition location much.

However, in cases of natural transition for which the free stream disturbances are very small and the viscosity is also small, then it is suggested that skew stress effects can be very significant in adjusting the actual transition point.

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Appendix A: Order of Magnitude Considerations

From Equation III-2, the case for no stress couples is

$$\frac{d\hat{\omega}_o}{dt} = \frac{2\beta}{\rho I} (\nabla \times \hat{u} - 2\hat{\omega}_o) \quad , \qquad \text{A-1}$$

which can be written as

$$\frac{d\hat{\omega}_o}{dt} = \frac{1}{\tau} \left(\frac{1}{2} \nabla \times \hat{u} - \hat{\omega}_o \right) = \frac{1}{\tau} (\hat{\omega}_e - \hat{\omega}_o) \qquad \text{A-2}$$

Note that this is like a simple reaction rate equation where $\hat{\omega}_e = \frac{1}{2} \nabla \times \hat{u}$ is considered to be an equilibrium or relaxed level of spin, and $\tau = \frac{\rho I}{4\beta}$ is the relaxation time.

Since the relaxation is by molecular collisions, it is reasonable to conclude that the order of magnitude of the relaxation time is given by

$$\tau = \frac{\rho I}{4\beta} = 0(\text{collision time}) = 0\left(\frac{\text{mean free path}}{\text{avg. molec. speed}} \right)$$

$$= 0\left(\frac{L}{\bar{v}} \right) \quad .$$

Now the ordinary coefficient of shear viscosity may be taken as

$$\mu = 0.499 \ \rho \bar{v} L \quad , \qquad \text{A-4}$$

and so

$$\bar{v} \;=\; \frac{\mu}{0.499\ \rho L} \;\cdot \qquad\qquad \text{A-5}$$

Thus

$$\tau \;=\; \frac{\rho I}{4\beta} \;=\; 0\left(\frac{L}{\bar{v}}\right) \;=\; 0\left(\frac{\rho L^2}{\mu}\right) \qquad\qquad \text{A-6}$$

which implies that

$$\beta \;=\; 0\left(\frac{I}{L^2}\cdot\mu\right) \;\cdot \qquad\qquad \text{A-7}$$

In Section II, it was noted that in the theory of structured continua the $\overset{\wedge\wedge}{I}$ is a constant dyadic representing average moment of inertia of the rigid molecules in the flow. Thus Equation A-7 be written in terms of the average radius of gyration of the rigid molecules, and this interpretation of $\overset{\wedge\wedge}{I}$ would yield

$$\beta \;=\; 0\left(\frac{k^2}{L^2}\cdot\mu\right) \;, \qquad\qquad \text{A-8}$$

which is a very small quantity since mean free path is so much greater than the dimensions of a molecule. The coefficient of spin viscosity, representing resistance to rotation of individual molecules, is therefore recognized to be quite small. Whether or not this small quantity and the effects it represents can be omitted is still open to question and must await extensive carefully analysis of the coupled model.

There is second possible interpretation of $\hat{I}$ that must be considered. In Equation II-3, the parameter $\hat{I}$ actually is a sort of "moment of inertia of the collision", and for Equation A-7 this was interpreted to be the average moment of inertia of the molecules. Dr. R. C. Ried has noted another interpretation is that $\hat{I}$ could be an inertia tensor associated with molecular correlations or a kinetic theory formulation based on a doublet distributed function. In this case the average instantaneous radius of gyration would be approximately the mean free path. Hence $k/L \approx 1$, and

$$\beta_2 = 0(\mu) , \qquad\qquad A-9$$

which means that the spin coupling would be very significant in the problem.

It is possible that both interpretations yield valid and significant terms, and that in Equation II-3 the intrinsic angular momentum must account for both the angular momentum change of a pair during collision plus the change of angular momentum of individual molecules about their own axis of rotation.

The relaxation time associated with the first interpretation of I can be estimated as follows. For air at NTP, assume

n = number density $\approx 3 \times 10^{19}$ molecules/cm^3.

σ = kinetic cross section $\approx 3 \times 10^{15}$ cm. $\qquad$ A-10

$\bar{v}$ = avg. molecular velocity $\approx 5 \times 10^4$ cm/sec.

Then

$$L = \frac{1}{n\sigma} = \text{mean free path}$$

$$\approx 10^{-5} \text{ cm} \qquad\qquad \text{A-11}$$

and

$$\tau = 0\left(\frac{L}{\bar{v}}\right) = 0\left(\frac{10^{-5}}{5\times10^4}\right) = 0(2\times10^{-9}) \text{ sec.} \qquad \text{A-12}$$

This of course is a very short relaxation time. However, it is noted that propagation of ultrasonic waves in a gas is known to produce stress or temperature changes with a time scale on the order of at least 10^{-7} sec. For example, at a frequency of 30 Mc, the period is

$$t = \frac{1}{f} = \frac{1}{3\times10^7 \text{ cy/sec}} = 3\times10^{-8} \text{ sec.} \qquad \text{A-13}$$

Thus, high frequency noise radiating from a wind-tunnel boundary layer conceivably could excite such high frequencies. It is noted further that the flow under consideration will generate such high frequency waves as its own boundary condition, because at the wall the collision frequency is on the same order of magnitude as in the flow, and each colli-sion at the wall must be considered as collision with a rough surface and the generator of spin for the colliding molecules. It is postulated then that this boundary condition amounts to the excitation of the system with random disturb-ances and at frequencies of all magnitudes up to and including the maximum collision frequency.

Section III showed that if the system can support the transfer
of intrinsic angular momentum (via either spin of individual
molecules, or spin of a colliding pair) then the rotation
can propagate as a wave. It is thus postulated further
that such spin originated at the boundary (or elsewhere)
propagates into the flow and "pre-excites" local molecules
so that they become more and more susceptible to clustering
and random cluster rotation which could then be the physical
mechanism of transition.

It should be noted that if the $\hat{I}$ is interpreted as a
moment of inertia of the binary collisions, then considerable
care must be used in developing the analysis since $\hat{I}$ should
be a function of time and other parameters.

Appendix B: Bulk Coefficient of Viscosity

In recent years there has developed a controversy regarding
the bulk viscosity of fluids. In some ways this has been
healthy, because previously there was a widespread practice
in classical fluid mechanics of accepting Stoke's relation
as established truth. The matter is not a trivial academic
argument since magnitude and functional dependence of any of
the viscosity coefficients (μ, λ, or κ) could have significant
influence on stability of the fluid system (Kistler; 1971).
The manner in which these coefficients are determined experi-
mentally (both as to accuracy and interpretation of the data)
needs to be carefully re-examined with more attention on
features important to a stability problem and not just with
concern for thermodynamic considerations and computation of
mean flows.

For Newtonian fluids it generally is assumed that the elements
of $\hat{D}$ in Equation II-14 are constants or functions of thermo-
dynamic state, but that they do not depend on stress, rate of
deformation, frequency, etc. However, in the field of ultra-
sonics where dilitation effects are important, it now is
customary to treat one or more of the viscosity coefficients
as complex and frequency dependent as a means of adjusting
hydrodynamic theory to fit data exhibiting relaxation
effects (Tisza; 1952).

Truesdell (1952, 1953) questions the physical meaning of com-
plex viscosity coefficients and of theory to fit certain
data. Furthermore, he has shown that hydrodynamic theory
cannot account for relaxation effects in general by a simple
adjustment of bulk viscosity, because when it is adjusted

to make the theory fit absorption data in the linear range
then the absorption is too great at higher frequencies.

According to Truesdell, relaxation theory is only an ad hoc
theory created to explain ultrasonic absorption and disper-
sion. He holds that it is isolated from mechanics in general,
and gives no idea how the fluid will behave for conditions
other than infinitesimal oscillations. Truesdell claims the
real implication of the theory is that fluids might have
some purely mechanical properties not included in the classical
hydrodynamical model given by the Navier-Stokes equation.
This supports the intuition of Kistler (1969, 1971), and is
in agreement with the equations of Dahler (1953), Grad (1952,
1953), and others.

Now if such properties do exist, then they should manifest
themselves in a variety of mechanical situations and not just
in ultrasonics. Truesdell attributes this point of view to
Gemant (1935), who suggested that fluids exhibiting relaxation
effects are in fact visco-elastic substances. Truesdell also
indicates that formulae similar to the proposal by Tisza do
result from the visco-elastic theories of Gemant and others.
Furthermore, he has shown that when such formula are applied
to plane infinitesimal waves, they yield an equation identical
in form with the wave equation of classical elasticity theory
except that the elastic moduli are replaced by certain func-
tions of $i\omega$.

For the hydrodynamic stability problem it is likely that
accuracy requirements on the coefficients are more stringent
than those suitable for mean flow calculations, and any
frequency dependence of the coefficients as already noted
could be tremendously important. It is reasoned that experi-
ments should be conducted up to very high ultrasonic fre-
quencies because relaxation effects and mechanical polarization
would be greatest under these conditions. It has already
been suggested (Appendix A) that flow along the boundary pro-
duces extremely high frequency waves (approaching the colli-
sion frequency with the wall), because each collision with
the wall produces a translational and rotational perturbation
to the system.

Rosenhead (1954) notes that the significance of λ does not
enter into the usual phenomena of hydrodynamics. This is
because it drops out of Equation III-1 when the flow is
incompressible. Furthermore, the acceptance of Stoke's
relation of course means that the bulk viscosity κ is
assumed to be zero, and it too does not enter the usual
hydrodynamics problem.

Regarding the coefficient λ , when dilitation is signifi-
cant (and even in cases where very small dilitation is
responsible for phenomena of interest), λ cannot be neg-
lected. This is well known to be the situation for absorption
of sound waves in liquids and gases, and based on the results
of Sections II and III of this document it is felt also to be
the case for stability of laminar flows nearing transition.

The coefficient λ is usually taken as a constant connected
with divergence of the velocity vector, but Andrade (1954)

points out that it is not a physical constant in the ordinary
sense because no experiment can be devised which will measure
it directly. In experiments where volume is compressed
without change in shape, the coefficient involved actually
is the bulk viscosity, $\kappa = (2\mu + 3\lambda)$.

Rosenhead comments further that both μ and λ are intro-
duced as quite <u>independent</u> phenomenological coefficients.
It is only the <u>assumption</u> of $\kappa = 0$ that leads to Stoke's
relation $\lambda = -\frac{2}{3}\mu$, which is equivalent to an assumption
that the molecules have no internal structure. This is <u>not</u>
generally valid, and actual fluids have values of λ/μ that
may be positive rather than negative, with the magnitude of
this ratio at times being as great as 200.

It does seem to be reasonably well established now that μ
and λ indeed are independent, but there still is considerable
uncertainty about what should be used for bulk viscosity.
The problem lies in the difficulty of devising experiments
that clearly separate and distinguish influences of the
various dissipative mechanisms. For example, Truesdell (1954)
indicates that despite crude agreement between values of bulk
viscosity calculated from absorption and from streaming,
neither absorption nor streaming measurements, (in the linear
range) can distinguish between the effects of various possible
dissipative mechanisms such as bulk viscosity and non-linear
shear viscosity.

According to Andrade, the streaming and sound absorption are interrelated rather than independent, and streaming at sonic frequencies seems to be a boundary-layer circulation due to sound absorption and explainable without resort to a second coefficient of viscosity.

Clearly the whole problem of hydrodynamic stability is opening to a new line of thinking, and much work needs to be done in both the theoritical and the experimental areas to evaluate effects previously believed to be unimportant. Without significant experimental activities there is going to remain an inadequacy in the phenomenological data going into the mathematical models, and there will not be any suitable experimental results with which the analytical results can be compared. Without a rapid improvement in experimental activities there soon will be a gap very analogous to the gap between the theoritical work of Orr, Sommerfeld, Tollmien and Schicting, and the much later experimental confirmation by Schubeauer and Skramstad.