Reports of the Department of Geodetic Science

Report No. 177

IMPROVEMENT OF A GEODETIC TRIANGULATION THROUGH CONTROL POINTS ESTABLISHED BY MEANS OF SATELLITES OR PRECISION TRAVERSING

by

Narenda K. Saxena

Prepared for

National Aeronautics and Space Administration
Washington, D.C.
Contract No. NGR 36-008-093
OSURF Project No. 2514

The Ohio State University
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PREFACE

This project is under the supervision of Ivan I. Mueller, Professor of the Department of Geodetic Science at The Ohio State University, and it is under the technical direction of Jerome D. Rosenberg, Project Manager, Geodetic Satellites Program, NASA Headquarters, Washington, D. C. The contract is administered by the Office of University Affairs, NASA, Washington, D. C. 20546.

ABSTRACT

The subject of this investigation is to answer the question: Whether any significant increment to accuracy could be transferred from a super-control continental net (continental satellite net or super-transcontinental traverse) to the fundamental geodetic net (first-order triangulation). This objective was accomplished by evaluating the positional accuracy improvement for a triangulation station, which is near the middle of the investigated geodetic triangulation net, by using various station constraints over its geodetic position.

This investigation on a 1858 kilometer long triangulation chain shows that the super-control net can provide a useful constraint to the investigated geodetic triangulation net, and thus can improve it only when the accuracy of super-control net is at least 1 part in 500,000.

The preliminary accuracy of super-transcontinental traverse is already better than this limiting accuracy of 1 part in 500,000; however, the preliminary accuracy of continental satellite net is lower than this limiting accuracy of 1 part in 500,000. As such, continental satellite nets do not seem to provide any useful constraint, at least to this particular investigated triangulation chain.

ACKNOWLEDGEMENT

The author wishes to express his deep gratitude to Dr. Ivan I. Mueller, Project Supervisor, for his patience, constructive criticism, and encouragement during the entire period of this investigation.

Discussions with Doctors Peter Meissl, Helmut Moritz, Karl Rinner, Urho A. Uotila and William M. Wagner were of great help in reconstructing and confirming my conceptual approach.

The computer time for this investigation was provided partially free by the Instruction and Research Computer Center (IRCC) of The Ohio State University. This study was supported by NASA Research Grant No. NGR 36-008-093.

Working with the group at 231 Lord Hall on The Ohio State University campus has been a fruitful experience, especially because we shared our excitements over finding answers to our problems, exchanged ideas and enjoyed an academic atmosphere.

The difficult task of typing the formulae and this report was done with exceptional skill and expertise by Miss Barbara Beer and Mrs. Evelyn Rist.

This report was submitted to the Technical University of Graz, Austria in partial fulfillment of the requirements for the degree "Doctor der Technischen Wissenschaften."

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1. INTRODUCTION

Geodetic triangulation has been accepted as an accurate method of determining "precise" coordinates for the triangulation stations of relatively short chains. This well-accepted idea was also given in an article "How accurate is First-Order Triangulation?" [Simmons, 1950, pp. 53-56] with the following introductory words:

> The question is often asked, "How accurate is the position of a triangulation station," or "To what accuracy are the distances between triangulation stations known?" These questions are difficult to answer, principally because first-order triangulation gives the optimum accuracy in the measurement of great distances and there is at present no super yardstick to which it can be compared.

Two modern technological advancements, namely, satellites and electronic distance measuring (EDM) instruments, have questioned the first-order triangulation accuracy, especially if triangulation is extended to distances longer than 1000 km or more. In such extended triangulation systems systematic errors like lateral refraction, propagation of observational errors, residual polar motion effects on latitude, longitude and azimuth, etc. [Mueller, 1969, pp. 61, 86-87; Pellinen, 1970, pp. 34-35; Wolf, 1950, pp. 117], which cannot be eliminated, accumulate. Lately the question has been raised whether any significant increment to accuracy is "cascaded" from a 1:1 million 1000 km net through a 100 km net to local control over 10 km distances.

The satellite triangulation and super-transcontinental traverse, being of the highest achievable accuracy of today, i.e., super-control net of "zeroth" order, constitute a modern geodetic super structure, within which the classical geodetic triangulation is supposed to provide a geodetic control densification.

According to the classical geodetic concept a lower order system should be tied to a higher order system [Jordan/Eggert/Kneissl, 1958, Vol. IV.1, p. 112].

Statistically, this means that the variance-covariance of the higher order system, as a lower limit for accuracy, be at least compatible with the internal precision of the lower order system. For all practical reasons the accuracy of the higher order systems should be substantially better (by a factor of two to three) than the subordinated system, thus supplying a rigorous constraint in the reduction of the lower order system [Schmid, 1969, p. 4].

The objective of this investigation is to answer the question: Whether any significant increment to accuracy could be transferred from a super-control net to the basic geodetic net (first-order triangulation). This objective was accomplished by evaluating the positional accuracy improvement for a triangulation station, which is near the middle of the investigated geodetic triangulation net, by using various station constraints over its geodetic position.

2. DATA AND ACCURACY ESTIMATES

2.0 Data

For the purpose of the present investigation, the triangulation of the western-half of the United States has been considered, as this is more accurate than that of the eastern-half of the United States [Simmons, 1950, p. 54]. The investigation is done on the chain from Moses Lake, Washington to Chandler, Minnesota (Figure 2.0-1), as these two stations are also common on both the continental satellite net (CSN) and the super-transcontinental traverse (STT). The data used was supplied by the Triangulation Branch of Geodesy Division, and the Geodetic Research and Development Laboratory, both of the National Oceanic and Atmospheric Administration, Washington.

2.01 Geodetic Triangulation Net.

The observed data used is for the period 1897-1965. The approximate coordinates used are from 1965 adjustment. The details of Moses Lake - Chandler triangulation chain are as follows:

Number of stations		191
Number of bases	Taped	27
	Geodimeter	2
Laplace stations		13
Observed directions		919
Distance between two stations	Minimum	273 m
	Maximum	190 km
Total length of the chain		1858 km.

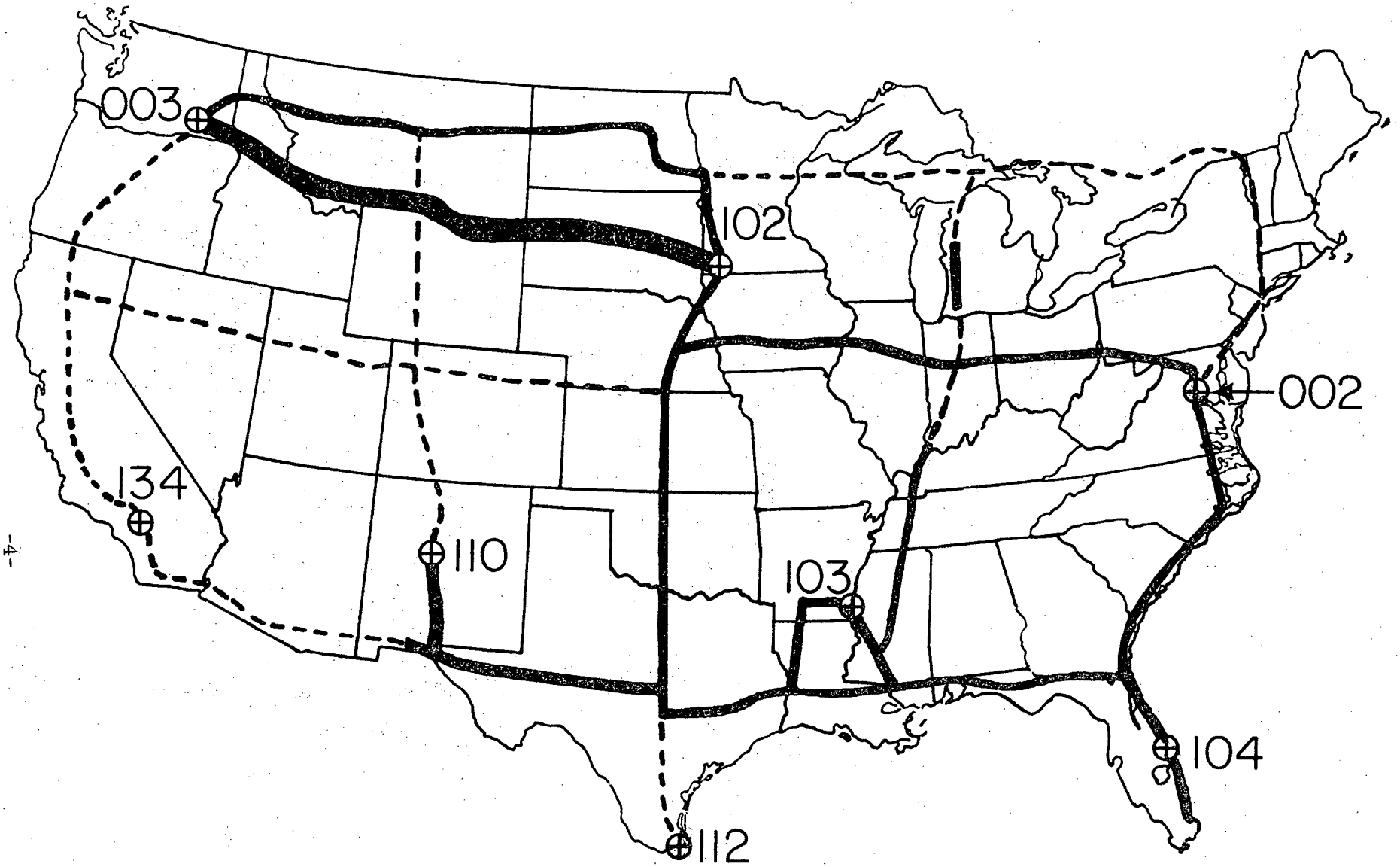

Figure 2.0-1 General Location of Geodetic Triangulation Chain from Moses Lake (003) to Chandler (102).

The supplied data was in U. S. sign conventions, i.e., longitude positive westwards and azimuth reckoned clockwise from south [Bouchard and Moffitt, 1964, pp. 94, 315; Mitchell, 1948], which was converted internally within the program to the conventional sign conventions, i.e., longitude positive eastwards, azimuth reckoned clockwise from north [Grossmann, 1964, pp. 5-6; Mueller, 1969, pp. 15-19]. It is assumed that the necessary reductions have been applied to the observed data and the weight function P is "a priori" known to be a sufficient good accuracy. The configurations and specifications for triangulation net are dealt with in [Bomford, 1965; Bouchard and Moffitt, 1964; Gossett, 1950]. A typical configuration of U.S. Coast and Geodetic Survey triangulation chain is shown in Figure 2.0-2.

2.02 Super-Control Nets.

Super-transcontinental traverse (STT) runs across the western-half and the eastern-half of the U.S.A. (Figure 2.0-3). Its specifications, configuration, reduction of data and instrumentation are dealt with by Meade [1967; 1969a; 1969b].

Continental satellite net (CSN) is, in general, planned in such a way so that the stations are around 1200 km apart and that these stations are evenly distributed over the entire area. CSN-stations are either the stations of first-order triangulation net or these are connected to them. Its specification and configuration are dealt with in [Deker, 1967; Mueller, 1964; Pellinen, 1970; Schmid, 1970; Shedlikh, 1970; Veis, 1960]. The continental satellite net of North American Continent comprises of twenty stations which can be anchored in the three world net stations; Thule, Greenland, Moses Lake, Washington, and Beltsville, Maryland. Furthermore, planned is a tie to a fourth world net station - Shemya (Figure 2.0-4) [Schmid, 1970].

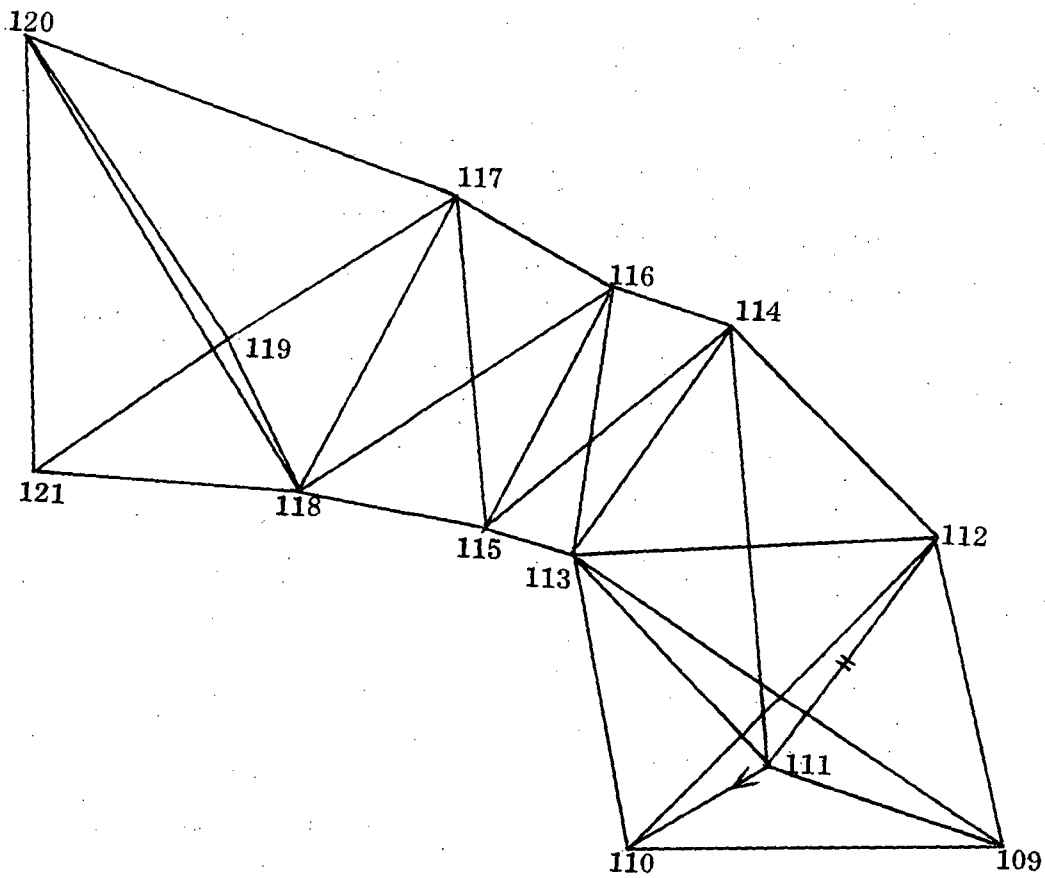

Figure 2.0-2 U. S. Coast and Geodetic Survey Triangulation Configuration.

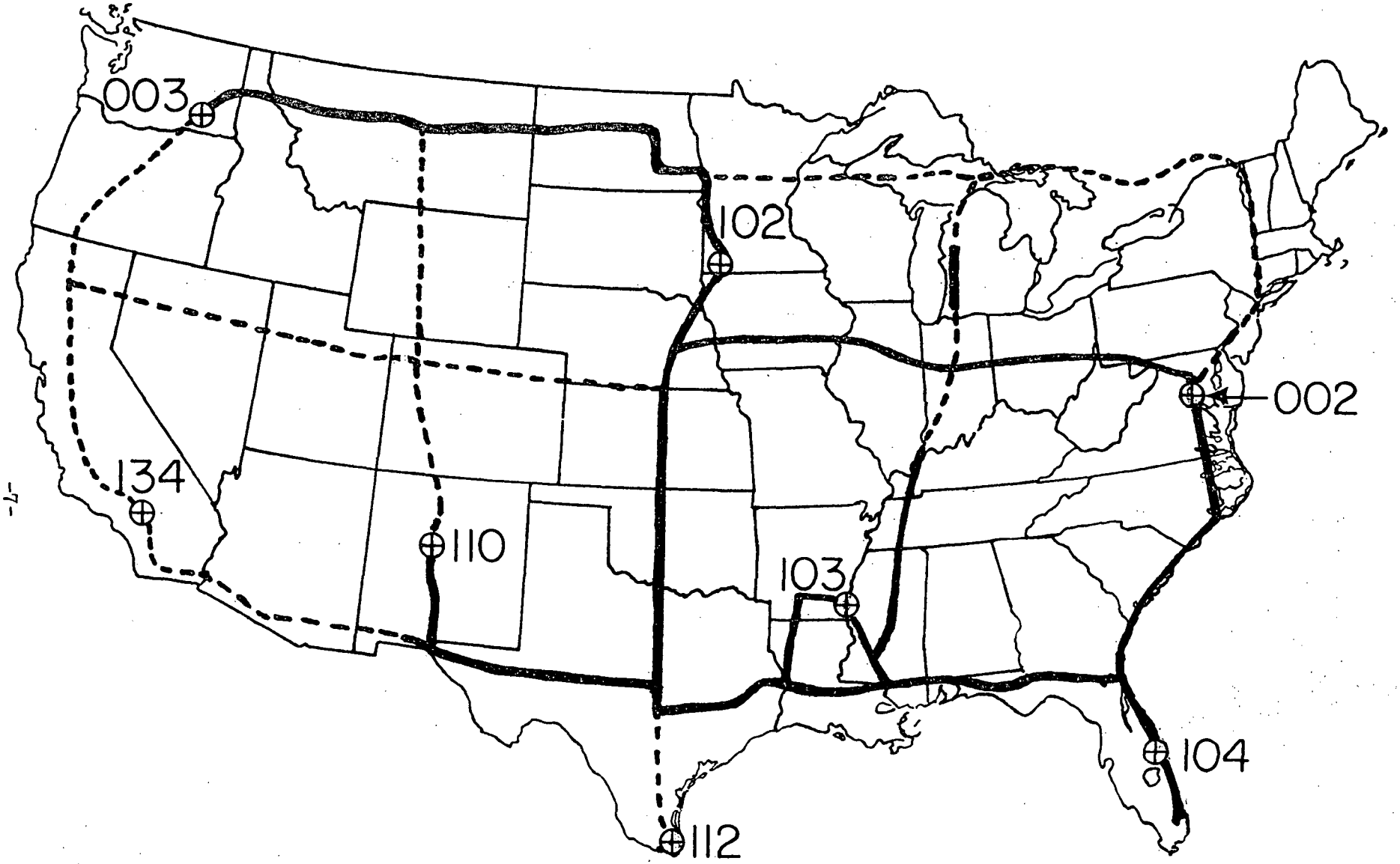

Figure 2.0-3 SUPER-TRANSCONTINENTAL TRAVERSE

—— Completed July 10, 1970

⊕ Satellite Triangulation Stations

--- Proposed

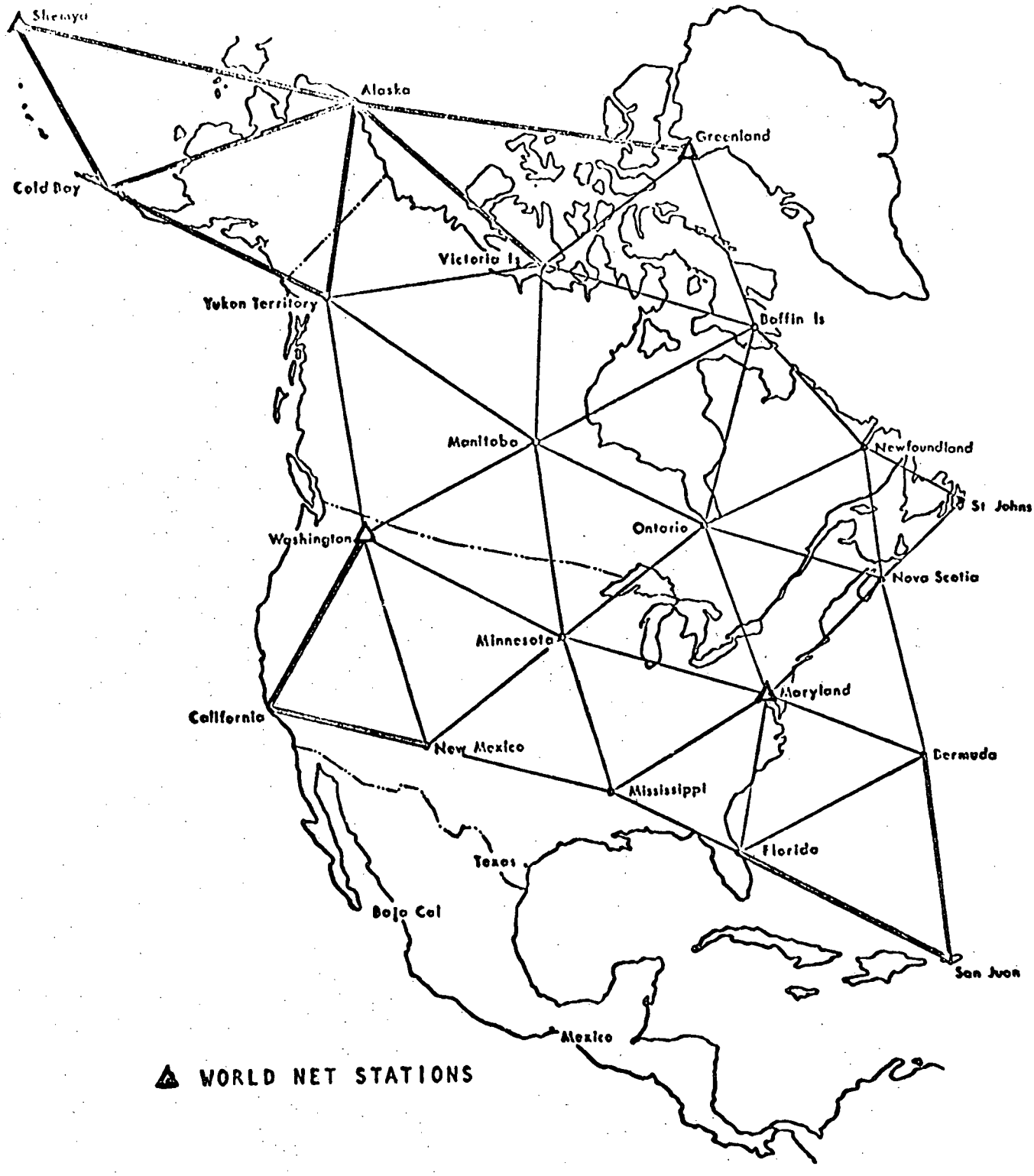

Figure 2.0-4 Continental Satellite Net of North America

2.1 Accuracy Estimates.

2.10 Data.

The following representative standard errors for observed data of Moses Lake - Chandler triangulation chain has been suggested [Meade, 1970]:

Directions	0.″4	
Azimuth	0.″8	
Base	Taped	1 part in 500,000
	Geodimeter	1 ppm for distance > 15 km
		1.5 cm for distances up to 15 km

2.11 Networks.

The mean of all section closures, which is the accuracy measure for the investigated geodetic triangulation net, is given as 1 part in 317,000 [Adams, 1930]. The standard position errors of the end stations of super-transcontinental traverse, which represent its accuracy measure, using actual data sets as given by different investigators differ too much from each other. The proportional error, which is the standard position error divided by the distance of the station from traverse-origin, is used for this investigation. The proportional errors of super-transcontinental traverse are given as follows: 1:740,000 over 318 kilometer long traverse and 1:1,100,000 over 1270 kilometer long traverse [Foreman, 1970]; 1:670,000 over 270 kilometer long traverse [Gergen, 1970] and 1:3,000,000 over 1858 kilometer long traverse [ESSA., 1969]. The preliminary accuracy (i.e., proportional error) of continental satellite net, as obtained from the supplied data, corresponds to 1:385,000 for Chandler station. Because of this wide range in preliminary accuracy measures of these two super-control nets, investigations using the following accuracies (station constraints), are made: 1:300,000; 1:400,000; 1:500,000; 1:600,000; 1:700,000; 1:1M; 1:1.5M; 1:3M. The use of these accuracy measures, which are within the limits of preliminary accuracies of the two super-control nets, will determine a limit on the accuracy requirement of the super-control net, which would be necessary to improve the geodetic triangulation net.

3. COMPUTATIONS AND RESULTS

During the earlier period of this investigation considerable thought was given to the selection and use of such formulas and methods which would not only provide high accuracies, but also minimize or eliminate loss of accuracy in computations. This resulted in using Helmert - Rainsford - Sodano's Iterative Solution for Inverse Problem (Appendix I), which are equally applicable for short and long lines, and Conjugate Gradient Method (Section 3.2) for the adjustment of the triangulation nets, where the original observation equation coefficient Matrix (A-Matrix) is used, thus avoiding direct formation of normal equations where certain properties of the original A-Matrix are lost. To minimize the round-off errors, computations are done in double-precision with double-precision storage [Müller-Merbach, 1970; Section 3.28].

3.0 Selection of Adjustment Method

From the two basic adjustment methods, i.e., Method of Observation Equations and Method of Condition Equations, the former has been preferred for the present investigation due to reasons of simplicity and clarity. The reasoning of this preference has been dealt with in many publications [Ashkenazi, 1967, p. 167; Gotthardt, 1968, p. 180; Grossmann, 1961, p. 174; Helmert, I. Teil, 1880, p. 556; Jordan/Eggert/Kneissl, Bd. IV. 1, 1958, p. 537; Wolf, 1968, p. 323; Wolfrum, 1969, p. 1]. Due to the large size of the triangulation net under investigation and the availability of digital computers, iterative methods were considered because (1) they are easier to program, (2) they require less storage space as the coefficient matrix of a triangulation net is very sparse, (3) they use directly the original set of equations throughout the process and hence rounding-off errors do not accumulate from one

iterative cycle to another. One important factor for deciding to use an iterative method is to know in advance whether the rate of convergence is rapid for the system. However, if the rate of convergence is found to be slow, which in general is the case in large systems, it can be accelerated by a certain process [Ashkenazi, 1969, p. 34; Fox, 1965, p. 194; Hilger, et al. 1967, p. 11; Ralston, 1965, p. 437]. While searching for a suitable adjustment method this investigator came across the Conjugate Gradient Method (Cg - Method) [Ralston, 1965; Schwarz, 1968 and 1970; Wolf, 1968; Zurmühl, 1958], which has the following distinct advantages over other iterative methods, such as Gauss-Seidl-, Jacobi-, Relaxation- and Gradient methods:

1. Original A-Matrix is used, thus avoiding the formation of normal equations, where certain useful characteristics of A-Matrix, such as very small coefficients may be lost.
2. Original A-Matrix, which has very few non-zero elements, is easily stored in comparatively much less computer space using an Index-Matrix.
3. No "mesh-point numbering technique" [Ashkenazi, 1967] to keep the band-width of the system a minimum is necessary. Thus stations can be added or taken out from the existing triangulation system without caring for their numbering.
4. It is a finite iterative process. Theoretically, the solution vector is obtained in a maximum of n-steps, n being the number of unknowns. Therefore, eigenvalues need not be calculated for determining the convergence. However, experimentation (Section 3.4) shows that the solution vector is not obtained in n-steps, as the orthogonality between the residue-vectors is not maintained exactly (Section 3.27). Consequently, the residue-vector $r^{(n)}$ after n-iterations is not zero (Section 3.29). This deviation from zero depends upon the condition of the system; the poorer the condition, the larger will be the deviation.
5. Even in case of a poorly conditioned system solution vector is obtainable after more iterations.

6. In large nets the effect of round-off errors depends upon the elimination sequence in normal matrix [Korhonen, 1954]. In Cg-Method the elimination sequence plays no role since adjustment is simultaneous, i.e., entire data as a whole is used.
7. Each approximation $x^{(j)}$ to the solution vector is closer to the true solution x than the preceeding one.
8. The ability to start anew at any point in the computation using the last $x^{(j)}$ as initial value so as to minimize the effects of round-off errors.

3.1 Mathematical Model (Method of Observation Equations).

Let L_i be the m independent observed quantities, v_i the residuals to the observed quantities (obtained from the adjustment) and x, y, z, ... the n unknown parameters to be determined. Each observation gives an observation equation, whose general from is

$$L_i + v_i = f_i(x, y, z, \ldots), \qquad (3.1\text{-}1)$$

where i = 1, 2, 3..., m and f represents a linear or non-linear function. The method of least squares however demands that (1) f should be linear, i.e., a linear relationship between the observations and the unknowns and (2) the number of observations (m) should be greater than those of the unknowns (n) i.e., $m > n$. In case of a non-linear function f this is linearized by using Taylor series about such good approximate values of the unknowns x_o, y_o, z_o, ... such that the second and higher order terms can be neglected. In this case, equation (3.1-1) can be written as

$$L_i + v_i = f_i(x_o + dx, y_o + dy, z_o + dz, \ldots)$$

or

$$L_i + v_i = f_i(x_o, y_o, z_o, \ldots) + a_i dx + b_i dy + c_i dz + \ldots \qquad (3.1\text{-}2)$$

i.e.

$$v_i = a_i dx + b_i dy + c_i dz + \ldots + \ell_i \qquad (3.1\text{-}3)$$

where

$$x = x_o + dx, \quad y = y_o + dy, \quad z = z_o + dz, \ldots$$

$$a_i = \frac{\partial f_i}{\partial x}, \quad b_i = \frac{\partial f_i}{\partial y}, \quad c_i = \frac{\partial f_i}{\partial z}, \ldots \qquad (3.1\text{-}4)$$

$$l_i = f_i(x_o, y_o, z_o, \ldots) - L_i.$$

Observation equation (3.1-3) can be written in the matrix form as

$$v = Ax + l. \qquad (3.1\text{-}5)$$

That we have preferred to use weighted constraints to the station Chandler can be seen at the end of this section. These "a priori" weighted constraints on the station position generate observation equations of the form

$$v_x = Fx \qquad (3.1\text{-}6)$$

where F is a rectangular matrix, whose elements are either zero or one. Thus the complete observation equation system can be written as

$$\left.\begin{aligned} v &= Ax + l \\ v_x &= Fx \end{aligned}\right\}, \qquad (3.1\text{-}7)$$

i.e.,

$$V = Bx + L, \qquad (3.1\text{-}7a)$$

where

$$V = \begin{bmatrix} v \\ v_x \end{bmatrix}; \quad B = \begin{bmatrix} A \\ F \end{bmatrix}; \quad L = \begin{bmatrix} l \\ 0 \end{bmatrix}. \qquad (3.1\text{-}8)$$

Observation equations for direction, Laplace azimuth, and distance are given in Appendix II [Grossmann, 1961, pp. 170, 177; Wolf, 1968, pp. 323-324]. Due to angular and linear (distance) observations the observed data in a triangulation net is of a **heterogeneous or dissimilar** nature.[1] This hetero-

[1] The term "heterogeneous or dissimilar" observations is used when the methods of their measurement are diverse; thus not only angles and distances, but also distances and heights are heterogeneous observations [Wolf, 1968, p. 56]; [Schmid and J. Schmid, 1965 p. 10] uses the term "hybrid systems" for "heterogeneous systems".

geneous data have not only more than one dimension but also different "a priori" standard errors. To make this data homogeneous, i.e., dimensionless and of unit weight, it is divided by the corresponding "a priori" standard error σ. For reasons of simplicity the mathematical model used is assumed to be uncorrelated. The resulting homogenized observation equation system can be written as

$$\left.\begin{aligned} \tilde{v} &= \tilde{A}x + \tilde{l} \\ \tilde{v}_x &= \tilde{F}x \end{aligned}\right\} \qquad (3.1\text{-}9)$$

where

$$\left.\begin{aligned} \tilde{v} = v/\sigma_l\,; \quad \tilde{A} = A/\sigma_l\,; \quad \tilde{l} = l/\sigma_l \\ v_x = v_x/\sigma_j\,; \quad F_j = F/\sigma_j \end{aligned}\right\} \qquad (3.1\text{-}10)$$

σ_l = standard error of L_l; σ_j = standard error of x_j.

Equations (3.1-9) and (3.1-10) can be written in the matrix form as

$$\tilde{V} = \tilde{B}x + \tilde{L} \qquad (3.1\text{-}9a)$$

where

$$\tilde{V} = \begin{bmatrix} \tilde{v} \\ \tilde{v}_x \end{bmatrix}; \quad B = \begin{bmatrix} \tilde{A} \\ \tilde{F} \end{bmatrix}; \quad \tilde{L} = \begin{bmatrix} \tilde{l} \\ 0 \end{bmatrix} \qquad (3.1\text{-}10a)$$

respectively.

Equation (3.1-9) is used directly for adjustment by conjugate gradients method in Sections 3.23 and 3.24 where these are used without $\sim$ sign, although the A-matrix and l-vector used there are homogenized. A complete algorithm for obtaining solution vector and N^{-1} are given in Sections 3.24 and 3.26 respectively, which give v^TPv and Q_{xx} or Q_{yy} for a particular column. Using these quantities the "a posteriori" variance of unit weight ($\hat{m}_o^2$), standard errors (m_x, m_y) of unknowns, standard positional error (m_p) and the elements θ, A, B of the error ellipse are computed [Gotthardt, 1968, pp. 121-125; Grossmann, 1961, pp. 163-168; Wolf, 1968, pp. 286-292]. Variance of unit weight $\hat{m}_o^2$ is given by

$$\hat{m}_o^2 = \frac{v^TPv}{m-n+c}. \qquad (3.1\text{-}11)$$

Standard errors (m_x, m_y) of unknowns are given by:

$$m_x = \hat{m}_o \sqrt{Q_{xx}} \; ; \quad m_y = \hat{m}_o \sqrt{Q_{yy}} \, , \tag{3.1-12}$$

and the standard position error

$$m_p = \sqrt{m_x^2 + m_y^2} = \hat{m}_o \sqrt{Q_{xx} + Q_{yy}} \, . \tag{3.1-13}$$

The elements of the error ellipse are given by

$$\theta = \frac{1}{2} \arctan \frac{2Q_{xy}}{Q_{xx} - Q_{yy}} \tag{3.1-14a}$$

$$A = \hat{m}_o \sqrt{Q_{max}} = \text{semi-major axis of error ellipse} \tag{3.1-14b}$$

$$B = \hat{m}_o \sqrt{Q_{min}} = \text{semi-minor axis of error ellipse} \tag{3.1-14c}$$

where

$$Q_{max,\,min} = \frac{Q_{xx} + Q_{yy}}{2} \pm \frac{1}{2} \sqrt{(Q_{xx} - Q_{yy})^2 + 4Q_{xy}^2} \tag{3.1-14d}$$

The standard errors of unknowns give the mean uncertainity of a station only in the direction of the coordinate axes, while the error ellipse gives this in any arbitrary direction. The standard positional error m_p as well as the error ellipse possess an important characteristic that these are independent of the coordinate system rotation.

The geodetic triangulation net is adjusted as an independent or free net, as it is not connected with other nets. For its unambiguoug determination, besides the observed data which includes directions, bases and astronomical observations, i. e., longitude and azimuth,one fixed station is required to serve as origin [Gotthardt, 1968, p. 167; Grossmann, 1961, p. 175; Jordan/Eggert/Kneissl, Bd. IV. 1, 1958, pp. 534-542]. Moses Lake station is kept as origin with its coordinates obtained from satellite triangulation results; these coordinates have been assumed to be the best known coordinates.

Combining the free triangulation net with super-control net of zero order, i. e., continental satellite net and/or super-transcontinental traverse means constraining the scale and/or orientation of the triangulation net. The effect of this combination is comparable with "tennis racket and string effect," where

the rigid outer racket frame (super-control) constrains the loose strings (triangulation net). If the strings are already constrained, there would be no "visible" effect of the additional constrain from the rigid outer frame. This is also the purpose of this investigation, i. e., to evaluate whether the existing geodetic triangulation is sufficiently "constrained" or needs to be constrained by additional super-control net. For the present investigation triangulation station Chandler, which is common to the three networks, provides constraint.

Geodetic triangulation net can be combined with the super-control net in either of the two ways:

(1) By using the actual data, i.e., by using the actual coordinates with their standard errors of Chandler as obtained from CSN and STT with the geodetic triangulation; or,

(2) By adding a weight constraint to Chandler with its coordinates from the geodetic triangulation.

For this investigation, the first way could not be used, as the super-control net coordinates of Chandler station are not compatible with those obtained from geodetic triangulation. As such, the second way has been preferred by using the actual preliminary accuracy estimates for Chandler, which are 1 part in 385,000 and 1 part in 3 million, as obtained from CSN ans STT, respectively. Further investigations are made by using hypothetical standard positional error accuracy estimates of Chandler station, which are 1:400,000; 1:500,000; 1:600,000; 1:700,000; 1:1 M; 1:1,5 M. These accuracy estimates are within the actual preliminary accuracy estimates of super-control nets. Thus, using these various accuracies of super-control net, a feeling for the accuracy limit of super-control net, which would be necessary to improve the investigated geodetic triangulation, can be obtained.

3.2 Conjugate Gradient Method (Cg-Method).

3.20 Introduction.

Although Cg-Method was developed by E. Stiefel and M. R. Hestenes independently from each other in 1952 [Stiefel, 1952, p. 23], this has been used only twice[1]) for geodetic computations [Hilger and Remmer, 1967; Wolf, 1968, p. 185]. Its basic algorithm given in most publications [Hilger and Remmer, 1967; Schwarz, 1968 and 1970; Stiefel, 1952] is good for well-conditioned systems. In case of ill-conditioned systems (Section 3.27) use of basic algorithm means too many iterations. However, use of certain formulas [Hestenes and Stiefel, 1952, p. 433] shows the suitability of Cg-Method even for very ill-conditioned systems.

An attempt is made here to derive the Cg-Method and put its algorithm together because this is too scattered in mathematical literature. The Method of Conjugate Gradients (Cg-Method) is a non-stationary relaxation method, which theoretically solves a system of simultaneous linear algebraic equations

$$Nx + u = 0 \qquad (3.2\text{-}1)$$

in n-iterative steps, where N is a symmetric and positive definite $n \times n$ coefficient matrix, x is a $n \times 1$ vector of unknowns and u is a $n \times 1$ vector of constants. In geodetic work as the columns of $m \times n$ A-matrix (observation equation coefficient matrix) are independent (A-matrix is of full-rank), its quadratic matrix $A^TA(=N)$ is symmetric and positive definite. Then the system (3.2-1) - known in geodesy as the Normal Equations - has a unique solution.

For the derivation of this method the matrix N of equation (3.2-1) will be considered symmetric and positive definite, and then the derived algorithm will be modified for an arbitrary N-matrix and for directly using the observation equations without explicit formation of normal equations.

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[1]) so far known to this investigator.

3.21 Basic Relaxation Method.

Certain terms of the basic relaxation methods, which are used later, will be discussed first:

To obtain the solution of (3.2-1) by relaxation methods a trial vector $x^{(o)}$ instead of x in (3.2-1) is used, which gives a residue vector $r^{(o)}$

$$r^{(o)} = Nx^{(o)} + u. \qquad (3.2\text{-}2)$$

The aim of relaxation methods is to change the trial vector $x^{(o)}$ until the residue vector $r^{(o)}$ disappears.

This residue vector $r^{(o)}$ is the gradient of the quadratic function $F(x^{(o)})$ given by

$$F(x^{(o)}) = \frac{1}{2} x^{(o)T} N x^{(o)} + u^T x^{(o)}. \qquad (3.2\text{-}3)$$

Differentiating partially the quadratic function $F(x^{(o)})$ w.r.t. the trial vector $x^{(o)}$ we obtain

$$\frac{\partial F(x^{(o)})}{\partial x^{(o)}} = Nx^{(o)} + u, \qquad (3.2\text{-}4)$$

(3.2-2) and (3.2-4) give

$$r^{(o)} = \text{grad } F = Nx^{(o)} + u. \qquad (3.2\text{-}5)$$

Equation (3.2-5) indicates that the solution of (3.2-1) is synonymous with the problem of finding a minimum of the quadratic function $F(x^{(o)})$ given by (3.2-3) [Ralston, 1965, p. 439; Schwarz, 1968, p. 45]. Let h be an arbitrarily selected non-zero relaxation direction vector, which corrects the trial vector $x^{(o)}$ in the direction of h so as to achieve a minimum of the quadratic function $F(x^{(o)})$, giving a new trial vector x', which is a linear function of the last trial vector $x^{(o)}$, given by

$$x' = x^{(o)} + \Delta x^{(o)} = x^{(o)} + \lambda h \qquad (3.2\text{-}6)$$

where λ is the relaxation distance factor or the correction factor for the unknown vector, which is so determined that the quadratic function $F(x')$ be a minimum; $F(x')$ is considered a quadratic function of the only variable

λ with constant (fixed) $x^{(0)}$ and h. Thus

$$F(x') = F(x^{(0)} + \lambda h)$$

Using (3.2-3)

$$F(x') = \frac{1}{2}((x^{(0)} + \lambda h)^T N(x^{(0)} + \lambda h)) + u^T(x^{(0)} + \lambda h)$$

$$= \frac{1}{2}(x^{(0)T} N x^{(0)} + \lambda(x^{(0)T} N h) + \frac{1}{2}\lambda^2 h^T N h + u^T x^{(0)} + \lambda u^T h$$

Using (3.2-3) and (3.2-5)

$$F(x') = F(x^{(0)} + \lambda r^{(0)T} h + \frac{1}{2}\lambda^2 h^T N h. \qquad (3.2\text{-}7)$$

To find a minimum of $F(x')$ in which the only variable is λ, equation (3.2-7) gives

$$\frac{dF(x')}{d\lambda} = r^{(0)T} h + \lambda h^T N h = 0 \qquad (3.2\text{-}8)$$

i.e.,

$$\lambda_{min} = -\frac{r^{(0)T} h}{h^T N h}. \qquad (3.2\text{-}8a)$$

Equation (3.2-8a) means that the relaxation direction vector h cannot be selected orthogonal to the residue vector $r^{(0)}$, for in that case $\lambda_{min} = 0$ at the trial vector $x^{(0)}$. The new residue vector r' is given by

$$r' = Nx' + u \qquad (3.2\text{-}9)$$

It could easily be proved that the new residue vector r' must be orthogonal to the last relaxation direction vector h; in other words after each relaxation step the new residue vector $r^{(j)}$ must be orthogonal to the last relaxation direction vector $h^{(j-1)}$.

Proof:

From (3.2-8)

$$\frac{dF(x')}{d\lambda} = 0 = \frac{dF(x')}{dx'} \cdot \frac{dx'}{d\lambda}$$

using (3.2-4), (3.2-5) and (3.2-6)

$$\text{R.H.S.} = r'^{T} h = 0 \qquad (3.2\text{-}9a)$$

Alternative Proof:

Using (3.2-5), (3.2-6) and (3.2-9)

$$r' = Nx' + u = N(x^{(0)} + \lambda h) + u = r^{(0)} + \lambda Nh. \qquad (3.2\text{-}9b)$$

Hence

$$r'^{T} h = (r^{(0)} + \lambda Nh)^{T} h = r^{(0)T} h + \lambda (Nh)^{T} h$$

$$= r^{(0)T} h - \frac{r^{(0)T} h}{h^{T} Nh} \cdot (Nh)^{T} h = 0 \qquad \text{for } \lambda = \lambda_{min}.$$

i.e.,

$$r'^{T} h = 0.$$

As proved above the new residue vector r' must always be orthogonal to the last relaxation direction vector h; this can be considered as an Orthogonality Condition. Equations (3.2-9a) and (3.2-9b) give the following relaxation distance factor λ_{min}:

$$r'^{T} h = (r^{(0)} + \lambda Nh)^{T} h = r^{(0)T} h + \lambda h^{T} Nh = 0$$

$$\lambda_{min} = - \frac{r^{(0)T} h}{h^{T} Nh}$$

which is the same as given in equation (3.2-8a).

Going in the relaxation direction h with the correction factor λ_{min} a minimum for the quadratic function $F(x')$ is obtained; this can be proved by the second derivative of $F(x')$ obtained from (3.2-8):

$$\frac{d^2 F(x')}{d \lambda^2} = h^{T} Nh > 0. \qquad (3.2\text{-}10)$$

The second derivative in equation (3.2-10) is always positive for every non-zero relaxation direction vector h, as N is positive definite [Thomas, 1960, p. 130].

Let ΔF be the decrease in the quadratic function $F(x^{(o)})$ while going from $x^{(o)}$ to x' in the relaxation direction h using $\lambda = \lambda_{min}$. Considering (3.2-7) and (3.2-8a)

$$\Delta F = F(x^{(o)}) - F(x') = -[\lambda\, r^{(o)\tau} h + \frac{1}{2}\lambda^2 h^\tau N h] = +\frac{1}{2}\,\frac{(r^{(o)\tau} h)^2}{h^\tau N h} > 0\,, \quad (3.2\text{-}11)$$

for

$$r^{(o)\tau} h \neq 0.$$

(3.2-11) gives the largest decrease of $F(x^{(o)})$ in the direction h. Thus by each relaxation step the current quadratic function $F(x')$ decreases, which proves the convergence characteristic of the relaxation method. Geometrically the relaxation method can be interpreted as follows:

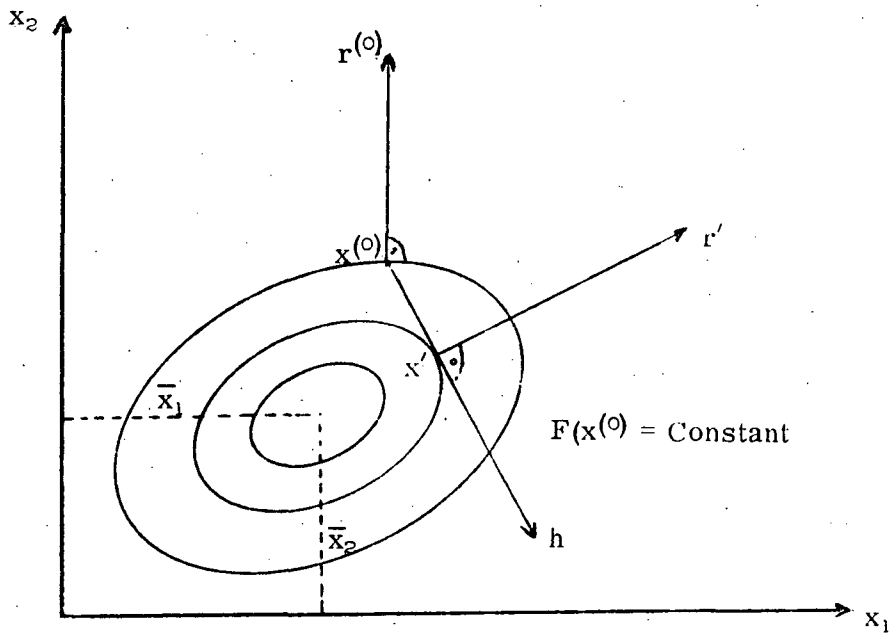

Figure 3.2-1

The quadratic function $F(x^{(o)})$ in the case n = 2 can be represented by its level lines $F(x^{(o)})$ = constant, which are concentric ellipses in rectangular (x_1, x_2) - coordinate system (Figure 3.2-1); the common center of these con-

centric ellipses coincides with the minimum point of $F(x^{(o)})$, whose coordinates are $(\bar{x}_1, \bar{x}_2)$, which represents the solution of the system. Let $x^{(o)}$ be the initial point; according to (3.2-5) the corresponding residue vector $r^{(o)}$ will be orthogonal to the level line passing through $x^{(o)}$. A relaxation step means first to choose the relaxation direction vector h and then to proceed in the direction of h until x', where the quadratic function along the relaxation direction is minimum. This point x' is naturally on a level line where it is tangential to the relaxation direction vector. At this point x', the new residue vector r' is orthogonal to the level line i. e. to the relaxation direction h (3.2-9a).

Based upon this basic relaxation method several relaxation methods have been developed, which differ in the selection of the relaxation direction vector h and the relaxation distance factor λ for each step.

3.22 Method of Conjugate Gradients (Cg-Method).

Based upon the basic relaxation method, gradient methods were developed where the relaxation direction vector h is not chosen arbitrarily but is a function of the current or previous residue vectors.

To find a minimum of the quadratic function $F(x^{(o)})$ (Figure 3.2-1) the quickest way obviously is to proceed orthogonally from $x^{(o)}$ in the direction h opposed to the grad F, i. e., opposite to current residue vector $r^{(o)}$. This can be written in a mathematical form as:

$$h^{(j)} = -r^{(j-1)}, \qquad (3.2\text{-}12)$$

Equation (3.2-12) constitutes the principle of the Method of Steepest Descent. Although the decrease in the quadratic function in each relaxation step is locally maximum, the convergence in general is not good [Schwarz, 1968, p. 68].

To improve the convergence, a method called the Method of Conjugate Gradients was developed by Stiefel and by Hestenes independent from each other, using not only the current residue vector but also the previous iteration results so that the system theoretically[1] has a solution in n-relaxa-

[1] In practice due to round-off errors this theoretical convergence is not achieved (Section 3.40).

tion steps, i.e. it converges in n-finite steps[2], n being the number of unknowns.

Cg-Method is thus a modification of the Method of Steepest Descent, where the relaxation direction vector $h^{(j)}$ is determined alone by the existing (current) residue vector $r^{(j-1)}$ according to equation (3.2-12), i.e. $h^{(j)} = -r^{(j-1)}$. In Cg-Method equation (3.2-12) is valid only for the first relaxation step but the further relaxation directions are calculated by

$$h^{(j)} = -r^{(j-1)} + \epsilon_{j-1} \cdot h^{(j-1)} \quad \text{for } j \geq 2 \qquad (3.2\text{-}13)$$

$$\left(h^{(j)} = -r^{(j-1)} \quad \text{for } j = 1 \right)$$

where ϵ_{j-1} is a Correction Factor for Relaxation Direction vector, which is proportional to the last relaxation direction vector $h^{(j-1)}$. The factor ϵ_{j-1} is so determined that the relaxation direction vectors $h^{(j)}$ and $h^{(j-1)}$ are conjugate[3] i.e.

$$h^{(j)\tau} N h^{(j-1)} = 0 \qquad (3.2\text{-}14)$$

(3.2-13) and (3.2-14) give

$$\epsilon_{j-1} = \frac{r^{(j-1)\tau} N h^{(j-1)}}{h^{(j-1)\tau} N h^{(j-1)}} . \qquad (3.2\text{-}15)$$

The relaxation distance factor λ_j is determined from (3.2-8a) as:

$$\lambda_j = -\frac{r^{(j-1)\tau} h^{(j)}}{h^{(j)\tau} N h^{(j)}} , \qquad (3.2\text{-}16)$$

and the solution vector $x^{(j)}$ is determined from (3.2-6):

$$x^{(j)} = x^{(j-1)} + \lambda_j h^{(j)} . \qquad (3.2\text{-}17)$$

The residue vector $r^{(j)}$ after j-th step is given by combining (3.2-2), (3.2-16) and (3.2-17):

$$r^{(j)} = Nx^{(j)} + u = N(x^{(j-1)} + \lambda_j h^{(j)}) + u = (Nx^{(j-1)} + u) + \lambda_j Nh^{(j)} = r^{(j-1)} + \lambda_j Nh^{(j)} \qquad (3.2\text{-}18)$$

---

[2] This is why it is also called the "n-Step Method".

[3] Due to this relationship this method is called the Method of Conjugate Gradients.

The Cg-Method is thus defined by equations (3.2-5), (3.2-12), (3.2-13), (3.2-15), (3.2-16), (3.2-17) and (3.2-18) after choosing a trial vector $x^{(0)}$; generally $x^{(0)} = 0$.

Proceeding in the same way as was used to obtain equation (3.2-9a), it can be easily proved that

$$r^{(j)\tau} h^{(j)} = 0. \qquad (3.2\text{-}19)$$

Similarly it can be proved that

$$r^{(j)\tau} h^{(j-1)} = 0. \qquad (3.2\text{-}20)$$

Using (3.2-13), (3.2-14), (3.2-19) and (3.2-20) we obtain

$$r^{(j)\tau} r^{(j-1)} = r^{(j)\tau}(-h^{(j)} + \epsilon_{j-1} h^{(j-1)}) = -r^{(j)\tau} h^{(j)} + \epsilon_{j-1} r^{(j)\tau} h^{(j-1)} = 0 \qquad (3.2\text{-}21)$$

Equations (3.2-14) and (3.2-21) can be generalized in the following form by the method of induction:

$$h^{(i)\tau} N h^{(j)} = 0 \qquad \text{for } i \neq j \qquad (3.2\text{-}14a)$$

$$r^{(i)\tau} r^{(j)} = 0 \qquad \text{for } i \neq j. \qquad (3.2\text{-}21a)$$

From equations (3.2-14a) and (3.2-21a), one can conclude that in the Cg-Method, the residue vectors $r^{(j)}$ constitute a mutually orthogonal system and the relaxation direction vector $h^{(j)}$ a mutually conjugate system. This gives an important characteristic of Cg-Method. As the residue vectors $r^{(j)}$ constitute a mutually orthogonal system in an n-dimensional vector space, this orthogonal system can contain a maximum of n non-zero vectors. Thus, at the latest, the (n+1)th residue vector $r^{(n)}$ must disappear, i.e., $r^{(n)}=0$. This proves that the Cg-Method provides the solution in a maximum of n-steps.

Numerical computations, however, show a deviation from the theoretical solution in n-steps, which may be due to (1) round-off errors and (2) an ill-

condition N-matrix; as these effect the mutual orthogonality of the residue vectors. Thus, the residue vector $r^{(n)}$ is not exactly zero. This deviation does not disturb the system. In this case where more than n-iterations are necessary to obtain the solution, the quadratic function decreases after every iteration.

Equations (3.2-15) and (3.2-16) can be written in an easier form using (3.2-13), (3.2-18), (3.2-19) and (3.2-21):

$$\text{Numerator of (3.2-16)} = -r^{(j-1)\tau} h^{(j)} = -r^{(j-1)\tau}(-r^{(j-1)} + \epsilon_{j-1} h^{(j-1)}$$

$$= +r^{(j-1)\tau} r^{(j-1)} - \epsilon_{j-1} r^{(j-1)\tau} h^{(j-1)} = r^{(j-1)\tau} r^{(j-1)}.$$

Hence (3.2-16) can be written as

$$\lambda_j = \frac{r^{(j-1)\tau} r^{(j-1)}}{h^{(j)\tau} N h^{(j)}}. \qquad (3.2\text{-}16a)$$

From (3.2-18)

$$N h^{(j-1)} = \frac{1}{\lambda_{j-1}} (r^{(j-1)} - r^{(j-2)}) \qquad (3.2\text{-}18a)$$

$$\text{Numerator of (3.2-15)} = r^{(j-1)\tau} N h^{(j-1)} = \frac{r^{(j-1)\tau}}{\lambda_{j-1}} (r^{(j-1)} - r^{(j-2)})$$

$$= \frac{1}{\lambda_{j-1}} (r^{(j-1)\tau} r^{(j-1)} - r^{(j-1)\tau} r^{(j-2)})$$

$$= \frac{r^{(j-1)\tau} r^{(j-1)}}{\lambda_{j-1}}.$$

Combining (3.2-15) and (3.2-16a) we obtain

$$\epsilon_{j-1} = \frac{r^{(j-1)\tau} r^{(j-1)}}{r^{(j-2)\tau} r^{(j-2)}} \qquad (j \geq 2). \qquad (3.2\text{-}15a)$$

Although the equations (3.2-15) and (3.2-16) are mathematically the same as equations (3.2-15a) and (3.2-16a) respectively, both sets of equations are equally good for well-conditioned systems. However, in case of ill-conditioned

systems equations (3.2-15) and (3.2-16) show better results [Hestenes and Stiefel, 1952, p. 433]. Consequently, these were the equations used in this investigation. Due to the simplicity of equations (3.2-15a) and (3.2-16a), they are the equations usually given in most of the publications.

3.23 Modification of Cg-Method.

1. If the A-matrix[1] is an arbitrary matrix and not assumed to be a positive definite symmetric matrix, Cg-Method can be used after multiplying the system by A^T, i.e.

$$(A^T A)x + A^T l = 0, \qquad (3.2\text{-}22)$$

for $A^T A$ is symmetric and positive definite, if A is non-singular matrix. The system of equations represented by (3.2-22) is equivalent to that given by (3.2-1).

2. To use the Cg-Method for observation equations directly without forming normal equations (N-Matrix), i.e., without forming a symmetric positive definite system the following procedure is used.

Let the observation equations be written as

$$Ax + l = v \qquad (3.2\text{-}23)$$

where A is a $m \times n$ matrix of observation equation coefficients, l is a $m \times 1$ vector of absolute terms and v is a $m \times 1$ vector of residuals. In case of observations with different weights, system (3.2-23) is homogenized by multiplying it with corresponding p_i, where p_i is the weight of the observations, thus the system becomes unitless and with unit weight. Normal equations to (3.2-23) can be written as

$$A^T Ax + A^T l = 0 \qquad (3.2\text{-}24)$$

which are equivalent to (3.2-22).

Similar to equation (3.2-9) the residue vector $r^{(j)}$ and the residual vector $v^{(j)}$ after j-th iterations are

[1] For reasons of consistency N and u of (3.2-1) has been replaced by A and l, respectively.

$$r^{(j)} = A^T A x^{(j)} + A^T l \qquad (3.2\text{-}25)$$

and

$$A x^{(j)} + l = v^{(j)} \qquad (3.2\text{-}26)$$

Combining (3.2-25) and (3.2-26) we obtain

$$r^{(j)} = A^T v^{(j)}. \qquad (3.2\text{-}27)$$

In the algorithm of Cg-Method N can be replaced by $A^T A$ in equations (3.2-15), (3.2-16) and (3.2-18) and thus the expression

$$h^{(j)} N h^{(j)} = h^{(j)} A^T A h^{(j)} = (A h^{(j)})^T (A h^{(j)}). \qquad (3.2\text{-}28)$$

Similar to equation (3.2-18) an expression for the residual vector $v^{(j)}$ after j-th iteration can be found:

$$v^{(j)} = Ax^{(j)} + l = A(x^{(j-1)} + \lambda_j h^{(j)}) + l = (Ax^{(j-1)} + l) + \lambda_j Ah^{(j)} = v^{(j-1)} + \lambda_j Ah^{(j)}. \qquad (3.2\text{-}29)$$

Thus, the residual vector $v^{(j)}$ after j-th iteration can be computed in two ways according to equations (3.2-26) and (3.2-29), thus providing a computational check. However, to increase the stability of the relaxation process $v^{(j)}$ should be computed anew in each step according to its definition given by equation (3.2-26) and not recursively by equation (3.2-29). Thus, the residual vector $r^{(j)}$ is also to be computed anew according to its definition by equation (3.2-27) and not recursively by equation (3.2-18) [Läuchli, 1959, p. 259].

That the norm of $v^{(j)}$ decreases monotonously while solving the observation equations (3.2-23) by Cg-Method can be proved as follows from (3.2-29):

$$v^{(j-1)} = v^{(j)} - \lambda_j Ah^{(j)}$$

$$v^{(j-1)T} v^{(j-1)} = v^{(j)T} v^{(j)} - 2\lambda_j v^{(j)} (Ah^{(j)}) + \lambda_j^2 (Ah^{(j)})^T (Ah^{(j)}) \qquad (3.2\text{-}30)$$

From (3.2-27) and (3.2-19)

$$v^{(j)T} (Ah^{(j)}) = (A^T v^{(j)})^T h^{(j)} = r^{(j)T} h^{(j)} = 0$$

(3.2-30) can be now written as

$$v^{(j)T} v^{(j)} = v^{(j-1)T} v^{(j-1)} - \lambda_j^2 (Ah^{(j)})^T (Ah^{(j)}) \qquad (3.2\text{-}31)$$

Until $x^{(j-1)}$ is not the solution x of (3.2-23), $r^{(j-1)} \neq 0$, hence $\epsilon_{j-1} \neq 0$ and $h^{(j)} \neq 0$; thus $Ah^{(j)} \neq 0$ and $\lambda_j > 0$, hence

$$\lambda_j^2 (Ah^{(j)})^T (Ah^{(j)}) > 0. \qquad (3.2\text{-}32)$$

Equation (3.2-31) together with (3.2-32) proves that the norm of residual vector $v^{(j)}$ decreases monotonously.

3.24 Algorithm of Cg-Method Using A-Matrix.

Now a complete algorithm of Cg-Method for obtaining the solution vector (x) using directly the homogenized observation equations can be summarized in the following systematic way:

Given: Homogenized observation equations $Ax + l = v$.

Select: Initial Trial Vector $x^{(0)} = 0$

Compute:

(1) $v^{(0)} = Ax^{(0)} + l$

Relaxation steps $j = 1, 2, \ldots n$

(2) $r^{(j-1)} = A^T v^{(j)}$

(3) $\epsilon_{j-1} = \dfrac{(Ar^{(j-1)})^T (Ah^{(j-1)})}{(Ah^{(j-1)})^T (Ah^{(j-1)})}$ (for $j \geq 2$)

(4) $h^{(j)} = \begin{cases} -r^{(0)} & (\text{for } j = 1) \\ -r^{(j-1)} + \epsilon_{j-1} h^{(j-1)} & (\text{for } j \geq 2) \end{cases}$

(5) $\lambda_j = -\dfrac{r^{(j-1)T} h^{(j)}}{(Ah^{(j)})^T (Ah^{(j)})}$

(6) $x^{(j)} = x^{(j-1)} + \lambda_j h^{(j)}$

(7) $v^{(j)} = Ax^{(j)} + l$

(7a) $v^{(j)}_{check} = v^{(j-1)} + \lambda_j Ah^{(j)}$

Tests:

(8) Orthogonality Test:

$$r^{(j)T} h^{(j)} = 0$$

$$r^{(j-1)T} r^{(j)} = 0$$

$$\cos\theta_1 = \frac{r^{(j-1)T} r^{(j)}}{\|r^{(j-1)}\| \cdot \|r^{(j)}\|} = 0$$

$$\cos\theta_2 = \frac{r^{(j)T} h^{(j)}}{\|r^{(j)}\| \cdot \|h^{(j)}\|} = 0$$

(9) $v^{(j)} = v^{(j)}_{check}$

3.25 Termination of Iterations.

Based upon the theory of Cg-Method and the geodetic requirements, iterations should be terminated as soon as any of the following conditions are fulfilled:

(a) if the improvement in the solution vector between two consecutive iterations is negligibily small, i.e., $|x^{(j)} - x^{(j-1)}| = 1.0 \cdot 10^{-4}$ seconds (i.e. $1.0 \cdot 10^{-4}$ second in φ or $\lambda \hat{=} 3.0$ mm),

(b) if $r^{(j)T} r^{(j)} = 0$,

(c) if $(Ah^{(j)})^T (Ah^{(j)}) = 0$;

(d) if the given number of iterations is reached;

(e) if the round-off error (RFE) during iterations exceeds a certain accuracy limit, which is given by the vector difference $|r_{true} - r_{comp}|^2$, where $r_{true} = A^T Ax_j + A^T l = A^T v^{(j)}$ and $r_{comp} = A^T v^{(j)}_{check}$.

$$RFE = |A^T (v^{(j)} - v^{(j)}_{check})|^2$$

The iterations should be terminated if $r^{(j)T} r^{(j)} \leq 3 \cdot RFE$.

3.26 N^{-1} - Inverse of Normal Matrix.

To analyse a geodetic network variance-covariance matrix (S) is needed. This requires the computation of N^{-1}, the inverse of the coefficient matrix of the normal equations,

$$S = \hat{m}_o^2 \cdot Q \qquad \text{where } Q = N^{-1} \qquad (3.2\text{-}33)$$

By Cg-Method N^{-1} is computed column by column and for each column of N^{-1} the computational steps needed are the same as those required to obtain the solution vector of (3.2-1). For huge systems the computational time for obtaining one column of N^{-1} or to obtain the solution vector of (3.2-1) is large and hence in such cases only a few necessary column vectors of N^{-1} need to be computed.

The algorithm for computing N^{-1} is obtained in the following way:
The inversion of N basically amounts to solving the system

$$N \cdot N^{-1} = E \qquad (3.2\text{-}34)$$

or

$$N \cdot Q = E \qquad (3.2\text{-}34a)$$

where E is a unit matrix.
Let q_k be the k-th column vector of Q (= N^{-1}) and e_k the k-th column vector of E, then q_k is the solution to the system

$$N \cdot q_k - e_k = 0. \qquad (3.2\text{-}35)$$

q_k, which is now the solution vector of (3.2-35), can be obtained by equations (3.2-5), (3.2-12), (3.2-13), (3.2-15), (3.2-16), (3.2-17) and (3.2-18) after choosing a trial vector $q_k^{(o)}$; generally $q_k^{(o)} = 0$.

Modifying this algorithm for using the A-matrix directly without explicitly forming the normal equations (N-matrix) N is replaced by $A^T A$ in the above mentioned equations. From (3.2-35):

$$A^T A q_k - e_k = 0.$$

Let $q_k^{(o)}$ be the initial trial vector, then

$$r^{(0)} = A^T A q_k^{(0)} - e_k. \qquad (3.2\text{-}36)$$

The complete algorithm of Cg-Method for k-th column vector of N^{-1}- inverse of normal matrix - using directly the homogenized observation equation coefficient matrix A can be given as follows:

Given: Homogenized observation equation coefficient matrix A.

Select: Initial trial vector $q_k^{(0)} = 0$

Compute:

(1) $r^{(0)} = -e_k$

Relaxation Steps $j = 1, 2, \ldots n$

(2) $$\epsilon_{j-1} = \frac{(Ar^{(j-1)})^T (Ah^{(j-1)})}{(Ah^{(j-1)})^T (Ah^{(j-1)})} \qquad (\text{for } j \geq 2)$$

(3) $$h^{(j)} = \begin{cases} -r^{(0)} & (\text{for } j = 1) \\ -r^{(j-1)} + \epsilon_{j-1} h^{(j-1)} & (\text{for } j \geq 2) \end{cases}$$

(4) $$\lambda_j = -\frac{r^{(j-1)T} h^{(j)}}{(Ah^{(j)})^T (Ah^{(j)})}$$

(5) $$q_k^{(j)} = q_k^{(j-1)} + \lambda_j h^{(j)}$$

(6) $$r^{(j)} = r^{(j-1)} + \lambda_j A^T (Ah^{(j)})$$

Test:

(7) Orthogonality Test:

Same as (8) of Section 3.24.

Termination of Iterations: Same as in Section 3.25.

3.27 Ill-Conditioning.

If small errors in the coefficients of equation (3.2-1), i.e., equation (3.2-24) or in the solving process have little or no effect on the solution the system is called well-conditioned; if the effect is large it is called an ill-conditioned system. Ill-conditioned systems have a very poor rate of convergence. A system can be evaluated if some information about its condition (condition number) is known. Condition numbers can be computed by using equations given in [Fox, 1965, p. 142; Schwarz, 1968, pp. 22-23; Zurmühl, 1964,

pp. 212-214]. For the present investigation condition number (K) is the ratio of the largest eigenvalue (λ_{max}) of N to its smallest eigenvalue (λ_{min}), i.e.,

$$K = \lambda_{max} / \lambda_{min} \tag{3.2-37}$$

This condition number K should be near unity for a very well-conditioned system from the point of view of solving linear equations [Fox, 1965, p. 199].

As mentioned in Section 3.1 eigenvalues need not to be calculated for determining the convergence as Cg-Method is a finite iterative method. This characteristic (i.e., no computation of eigenvalues) is of special importance in the case of modified Cg-Method as the original A-matrix is directly used, where due to the lack of a square matrix (N-matrix) computation of eigenvalues is not possible. In spite of this a condition measure can be derived by using the Cg-Method algorithm in the following way:

In [Hestenes and Stiefel, 1952, pp. 416-420] it is shown that

$$\frac{1}{\lambda_{max}} < \lambda_j < \frac{1}{\lambda_{min}} \tag{3.2-38}$$

$$\frac{r^{(j-1)T} r^{(j)}}{\| r^{(j-1)} \|^2} = \frac{\lambda_j}{\lambda_{j-1}} \cdot \frac{r^{(j-2)T} r^{(j-1)}}{\| r^{(j-2)} \|^2} \tag{3.2-39}$$

$$\frac{h^{(j)T} N h^{(j+1)}}{h^{(j)T} N h^{(j)}} = \frac{\lambda_j}{\lambda_{j-1}} \cdot \frac{h^{(j-1)T} N h^{(j)}}{h^{(j-1)T} N h^{(j-1)}} \tag{3.2-40}$$

The last two equations show that if, at the beginning of any iteration, $r^{(j-2)T} r^{(j-1)} \neq 0$ and $h^{(j-1)T} N h^{(j)} \neq 0$ then the computed values of $r^{(j-1)T} r^{(j)}$ and $h^{(j)T} N h^{(j+1)}$ will also deviate from zero. This deviation will depend upon the magnitude of ALPHA = $\lambda_j / \lambda_{j-1}$; the larger this ratio is the greater will be the disturbance of the orthogonality relations, and more rapidly the rounding-off errors accumulate. According to equation (3.2-38) the condition number K ($= \lambda_{max} / \lambda_{min}$) is an upper bound of the critical ratio $\lambda_j / \lambda_{j-1}$, which

determines the stability of the process. When K is near one, that is, when N is near a multiple of the identity, the Cg-Method is relatively stable [Beckman, 1960, p. 70].

3.28 Round-Off Errors.

Round-off errors could be mainly due to two reasons:

(1) For adjustment, the approximate values of the station coordinates are to be used in decimal degrees, where these decimal values are rounded-off to a certain significant digit;

(2) Due to the limited number of digits per storage location in computers, round-off error occurs during arithmetic operations.

A detailed study on round-off errors is given in [Müller-Merbach, 1970], wherein the investigations show that only double precision storage combined with double precision computations will keep the round-off errors to a minimum. To minimize the computer-error while adding small numbers to large numbers, the summation of scalar products $r^{(j)T}r^{(j)}$, $v^T P v$ and $(Ah^{(j)})^T(Ah^{(j)})$, which are used for iterations, is done in blocks. Investigations on a system of 66 equations and 39 unknowns show that $r^{(j)T}r^{(j)}$ and $(Ah^{(j)})^T(Ah^{(j)})$ are very sensitive to the block size while $v^T Pv$ is quite insensitive to it (Table 3.2-1). It has been found that the best convergence, that is, solution vector for $r^{(j)T}r^{(j)}=0$ after least iterations, is obtained when the ratio of blocksize (KMM) for $(Ah^{(j)})^T(Ah^{(j)})$ to that (KM) for $r^{(j)T}r^{(j)}$ is a function of the ratio of the number of equations to that of unknowns. This has been programmed in Subroutine PATSUM. The only parameter to be determined in this subroutine for individual problem is the basic blocksize (KM) for $r^{(j)T}r^{(j)}$.

A characteristic measure for the round-off error (RFE), which is required for the termination of the iterations, is the norm [Ginsburg, 1963, pp. 197-198]

$$RFE = |r_{true} - r_{comp}|^2 \qquad (3.2\text{-}41)$$

where r_{true} is the true residual vector by definition according to equation (3.2-9) or (3.2-27) and r_{comp} the recursively computed residual vector according to equation (3.2-18). Hence

$$RFE = |A^T(v^{(j)}-v^{(j)}_{check})|^2 \qquad (3.2\text{-}41a)$$

which provides a condition for termination of iterations, given by

$$r^{(j)T}r^{(j)} < 3.RFE. \qquad (3.2\text{-}42)$$

Table 3.2-1

Effect of Block Size on $r^{(j)T}r^{(j)}$, $(Ah^{(j)})^T(Ah^{(j)})$ and V^TPV at 80th Iteration

Experiment	Block Size		$r^{(j)T}r^{(j)}$	$(Ah^{(j)})^T(Ah^{(j)})$	V^TPV
	KM for RTR	KMM for $(Ah^{(j)})^T(Ah^{(j)})$ or V^TPV			
1	10	10	0.4127 D03	0.2020 D10	0.2529 D01
2	10	15	0.1594 D02	0.7879 D08	0.2529 D01
3	10	20	0.1087 D03	0.1286 D10	0.2529 D01
4	10	$KM_x\frac{NE}{NU}$	0.1247 D03	0.6080 D09	0.2529 D01

NE = Number of Equations

NU = Number of Unknowns

3.29 Residue Vector (r).

As mentioned earlier each approximation $x^{(j)}$ to the solution vector is

closer to the true solution vector x than the preceeding one, the error vector $(x - x^{(j)})$ also decreases at every step. However, the residue vector r may not decrease; normally the "residue square" $|r|^2$ oscillates and may even increase. This oscillating behaviour of the residue square $|r|^2$ could be due to round-off errors. Investigations have shown that the solution vector $x^{(j)}$ is no longer improved substantially as soon as the norm $|r|^2$ comes down to the range of the round-off errors. This occurs when the iterations are continued beyond the number of unknowns, i.e., $j > n$, when the residue vector $r^{(j)}$ will sooner or later begin to decrease sharply such that ultimately $x^{(j)}$ will be as good an approximation to x as it may be expected from the condition of the system [Beckman, 1960, p. 69; Ginsburg, 1963, pp. 192.199; Hestenes and Stiefel, 1952, p. 411]. The following systems were investigated:

System I: 39 unknowns, 66 equations (Figure 3.2-2)
System II: 84 unknowns, 138 equations (Figure 3.2-3)
System III: 573 unknowns, 965 equations (Figure 3.2-4)

In Figures 3.2-2, 3.2-3 and 3.2-4 asterisk marks (*) show that the current solution vector has no substantial improvement over the last solution vector, which happens after such n-iterations where the round-off errors stop increasing. Thus, any of these solutions are as good an approximation to the final solution vector as could be expected from the condition of the system.

3.3 Programming

All programs were written and tested by the investigator, except the SUBROUTINE DUMMY, by F. Fajemirokun.

3.30 A-Matrix, Index Matrix and Storage Requirement

Structure of A-Matrix and F-Matrix.

The homogenized A-Matrix, in general, contains m rows and n columns

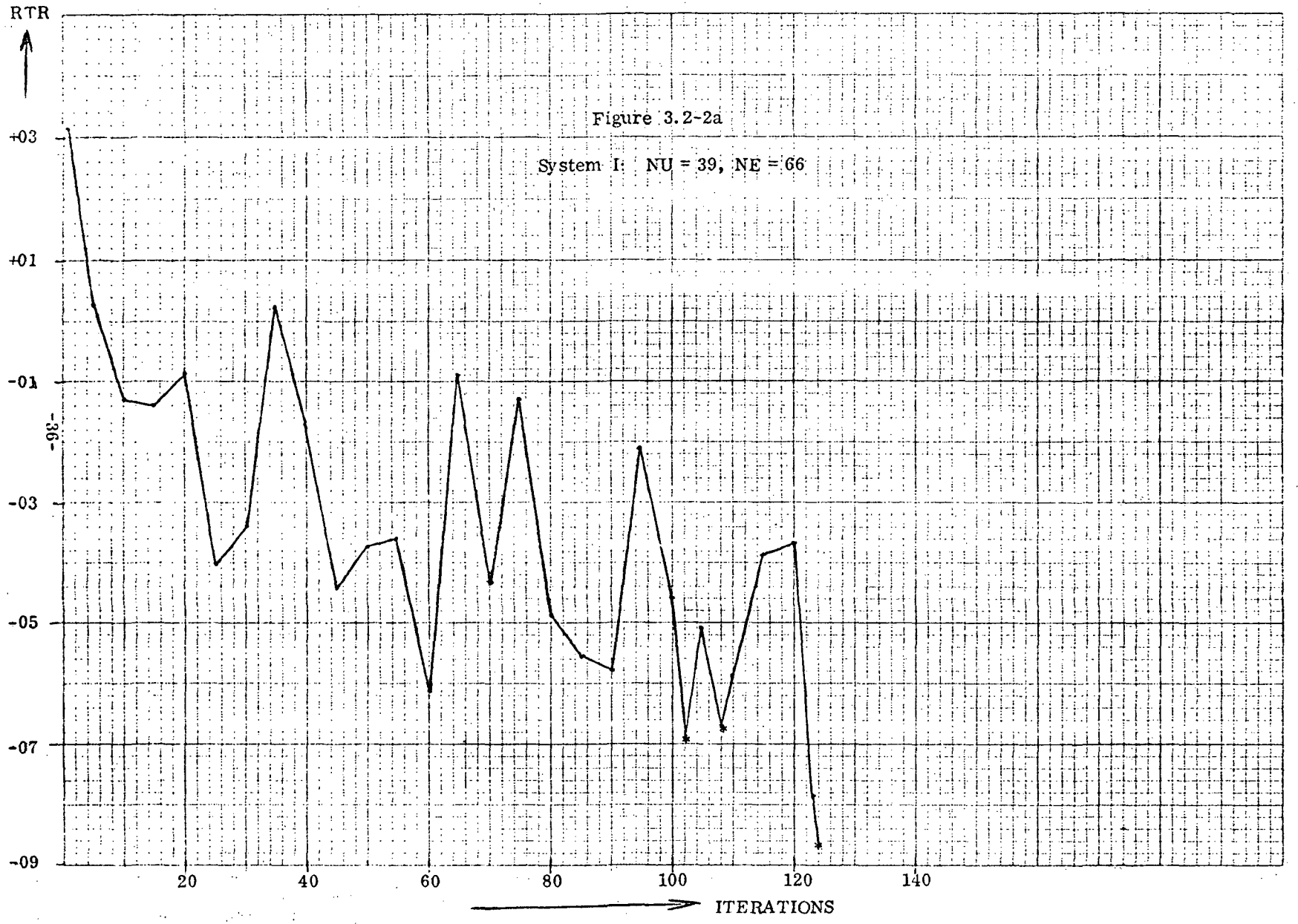

Figure 3.2-2a

System I: NU = 39, NE = 66

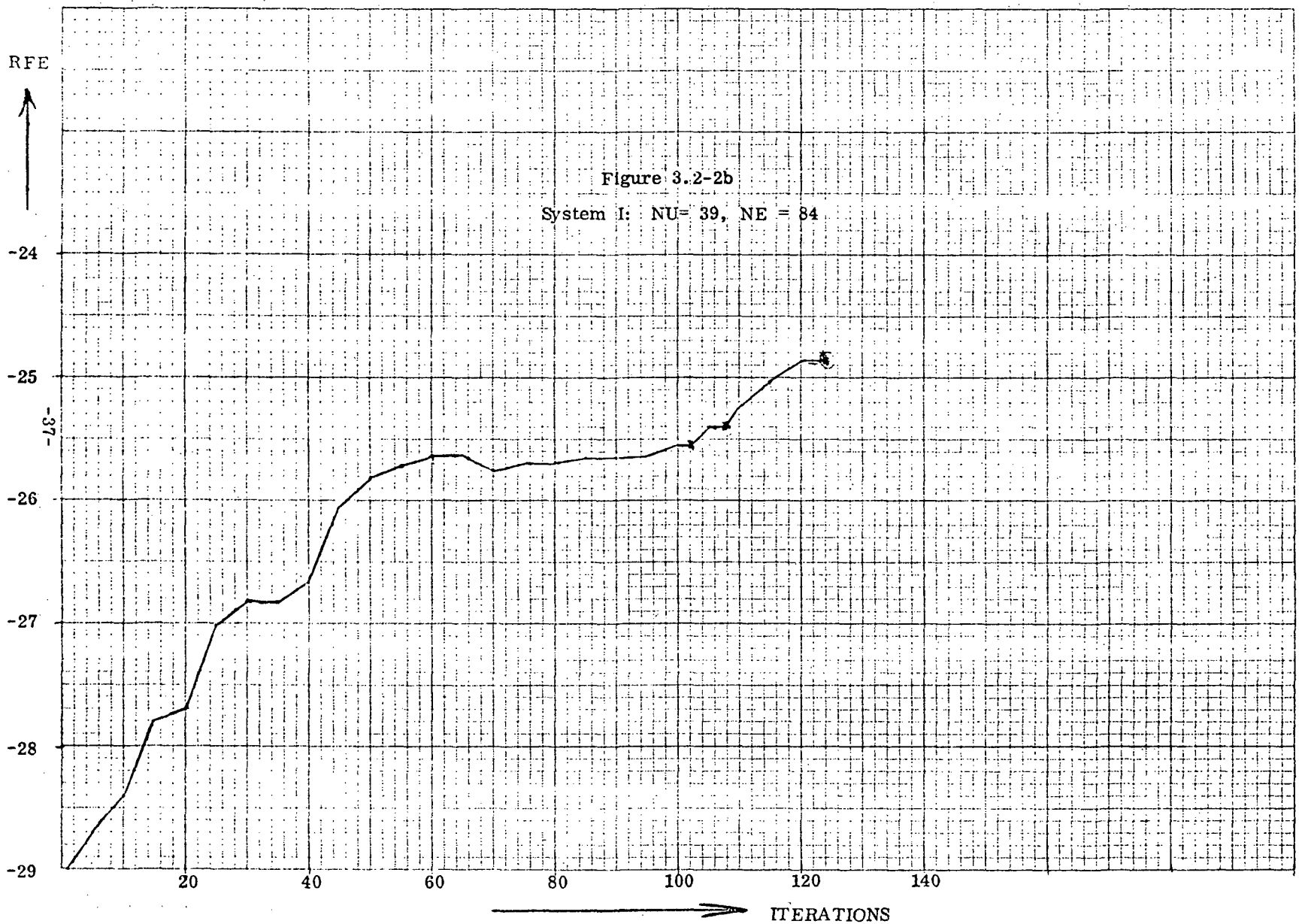

Figure 3.2-2b

System I: NU= 39, NE = 84

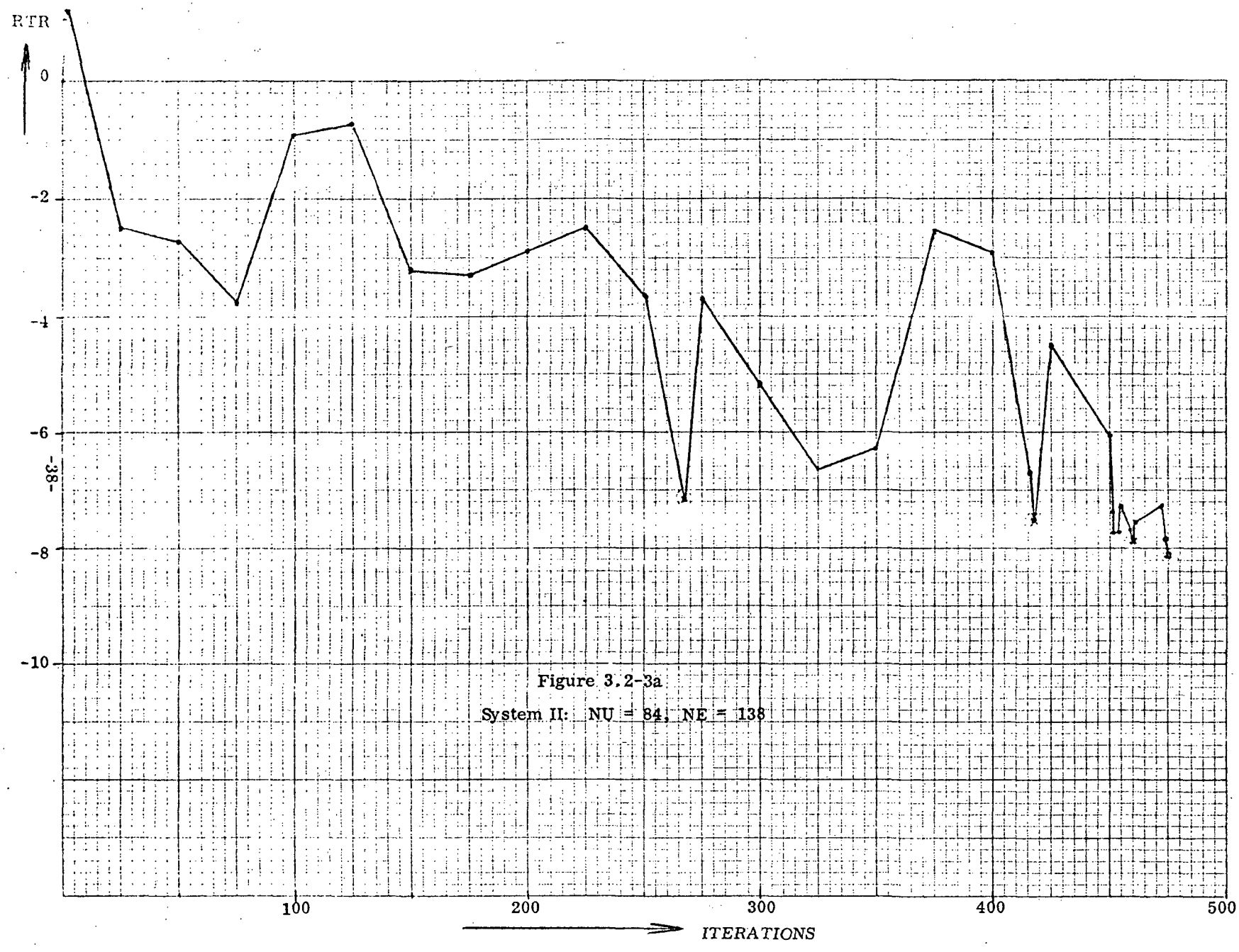

Figure 3.2-3a

System II: NU = 84, NE = 138

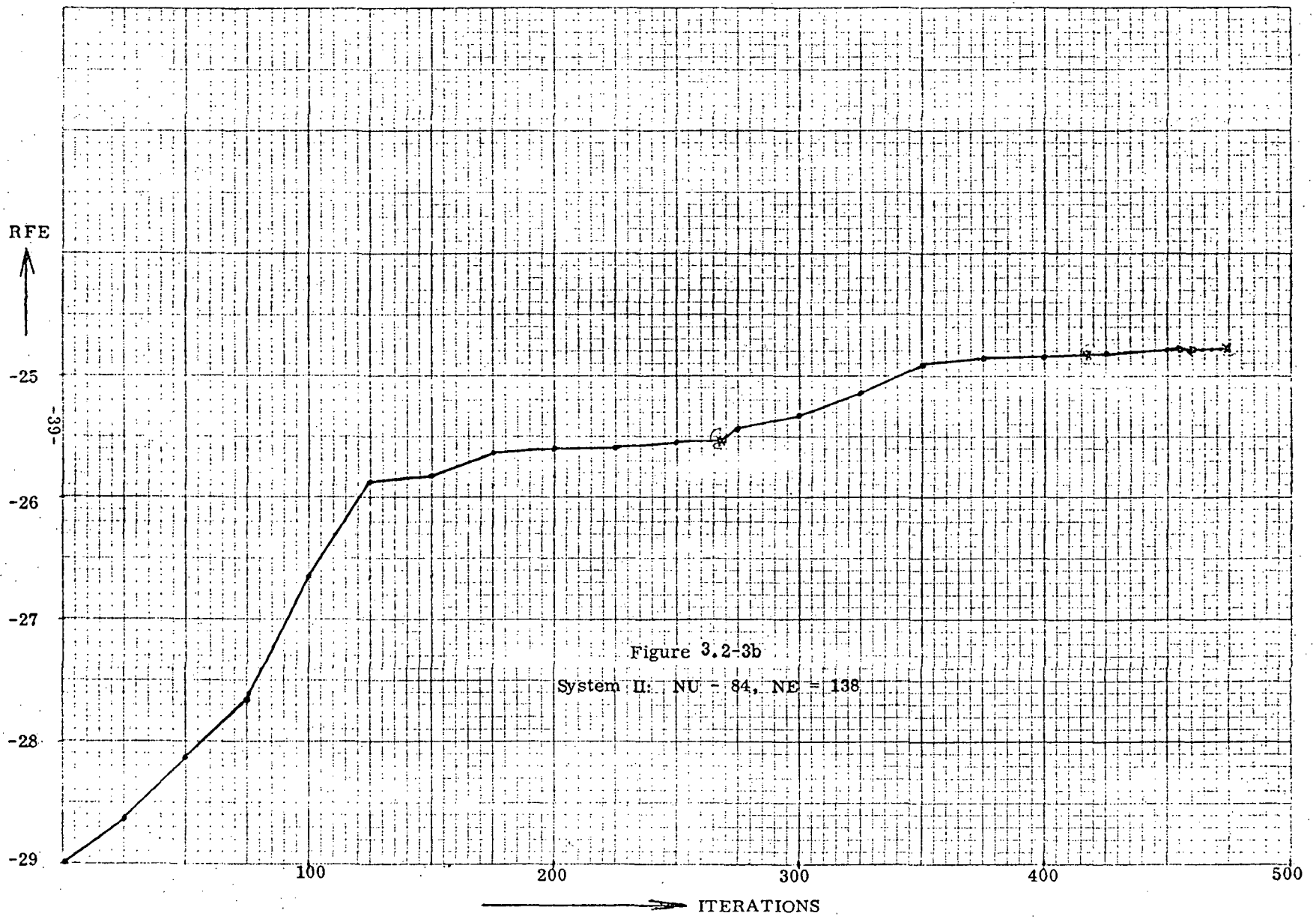

Figure 3.2-3b

System II: NU = 84, NE = 138

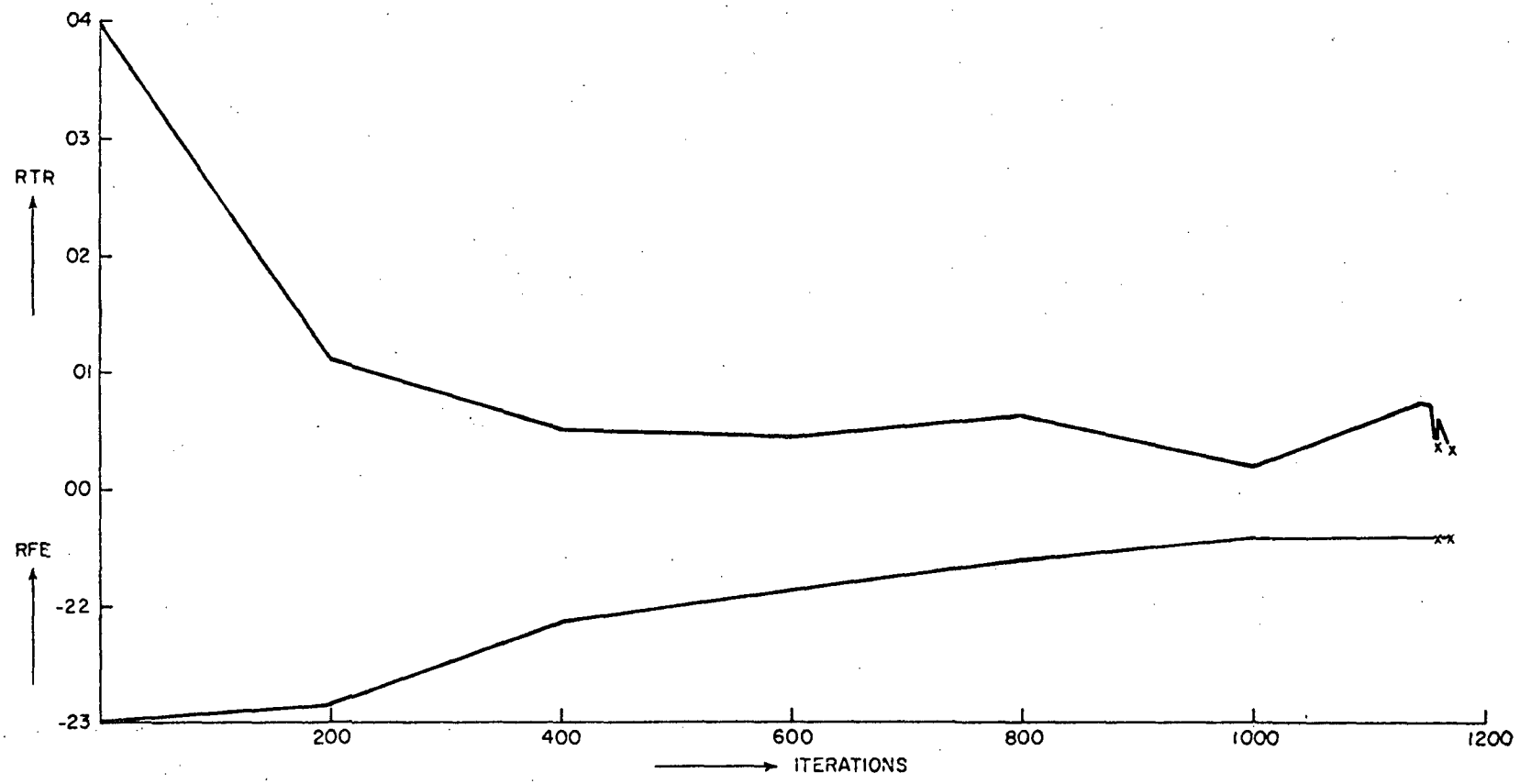

Figure 3.2-4

System III: NU = 573, NE = 965

where m = number of observations and n = number of unknowns, which is equal to 3 times number of stations, as for each station there are three unknowns, namely orientation correction (dz) and the two coordinate corrections ($d\varphi$ and $d\lambda$). Each row of A-Matrix contains a maximum of five non-zero elements, which are so arranged that the first term is the coefficient of orientation correction, the other four terms are the coefficients of station coordinates corrections. Each row of F-Matrix contains a maximum of one non-zero element. Thus for a large geodetic triangulation net there will be much more zero elements in each row than the non-zero elements, i.e., A-Matrix is a sparse matrix. The observation equations for the same station are put together one after another.

Index Matrix and Storage Requirement.

The original A-matrix is a sparse matrix and contains a maximum of five non-zero elements in its each row. Thus in a large triangulation system of 1400 observations and 820 unknowns the non-zero elements are only 0.6% of the original A-matrix. To save the computer storage space for storing the original A-matrix, Index Matrix (I1) has been used so that the "reduced" A-matrix has only five columns. The elements of the Index Matrix are positive integers. A very simple algebraic expression was deduced to obtain the elements of Index Matrix from the A-matrix; the lowest element of I1-matrix is equal to the lowest station number and its largest element is equal to 3 times the largest station number. A program gives I1-matrix from A-matrix. Thus Index Matrix, also called Reference or Guide Matrix, reduces an original (m x n) A-matrix to a "reduced" (m x 5) A-matrix. Both the "reduced" A-matrix and the I1-matrix together need much less storage than the original A-matrix. For example for an original (1400 x 820) A-matrix 114800 storage locations and for N-matrix 336610 storage locations stored in compressed form are needed, whereas the "reduced" A-matrix together with I1-matrix needs 14000 storage locations, which is 1.2% of the original A-matrix and 4.2% of the N-matrix. This would mean that the same computer, which cannot handle an original A-matrix or N-matrix, can easily handle

the same system using Index Matrix, as well as can also handle much larger systems. In future discussion, the term A-matrix will be used for "reduced" A-matrix.

3.31 Minimizing Round-Off Error.

The ideas of Section 3.28 were used to minimize the round-off errors. Double precision storage was used together with double precision computation; for integers single precision was used. Summation of scalar products $r^{(j)T}r^{(j)}$, $(Ah^{(j)})^T(Ah^{(j)})$ and v^TPv was done by using the subroutine PATSUM, which minimizes the effect of adding small numbers to large numbers.

3.32 Solution Vector Program and N^{-1}-Program.

The algorithm of Sections 3.24 and 3.25 was programmed as a SUBROUTINE SOLN. The algorithm of Section 3.26 was programmed as a SUBROUTINE QSOLN. These programs were initially tested on five data sets. The results obtained by these two subroutines were found to be the same (within computational accuracy) as those obtained by using Gauss-Algorithm. The five data sets used were: (3, 2), (10, 4), (16, 12), (36, 24) and (66, 39) - the first number = number of observation equations and the second number = number of unknowns.

The solution vector or N^{-1}-column will be printed after every N-iterations (N=Interval for writing computed output) besides after the fulfillment of the usual conditions.

3.33 Universality of Programs.

Both subroutines, namely SUBROUTINE SOLN and SUBROUTINE QSOLN, can be used for any feasible size of data, which can be accomodated on the available computer, after changing KM, which is the PATSUM Basic Block Size for RTR.

The main program used together with these subroutines has dimension statements and a data card for Number of Unknowns (NU), Number of Equations (NE) and Number of Columns of Index Matrix (NI), which can be changed if there is need for it.

The program is universal in the sense that it can be used for varying

data without much change and that "mesh-point numbering technique" is not required. Therefore, stations can be added or taken out from the triangulation system without worrying about the band-width and size of blocks. These programs have been tested on systems from as small as 2 unknowns, 3 equations up to as large as 804 unknowns, 1397 equations.

3.4 Number of Iterations and Computer Time

3.40 Number of Iterations.

As mentioned earlier in Sections 3.1 and 3.22, the Cg-Method theoretically gives the solution vector at n-iterative steps (n = number of unknowns) [Läuschli, 1959, pp. 257; Schwarz, 1968, pp. 74 and 1970 pp. 133; Wolf, 1968, pp. 184]. Investigations, however, show that the solution vector is not achieved in n-iterations due to round-off errors, ill-conditioning of the system, disturbances of the orthogonality and of the conjugacy relations. In [Beckman, 1960, pp. 69; Fox, 1965, pp. 210; Hestenes and Stiefel, 1952, pp. 411] it is mentioned that frequently (n + 1)th solution vector is significantly better than the nth one. According to [Ginsburg, 1963, pp. 192] up to 3n iterations may be needed in case of bad conditioned system; while [Hilger and Remmer, 1967, pp. 13-14] mentions that n to $\frac{3}{2}$n iterations are needed in case of large systems for 4 decimal accurate solution vector and that number of required iterations strongly depends upon the condition of the matrix A^TA.

This investigation, using the actual data set (Section 2), shows that the number of iterations required to obtain the solution vector by Cg-Method using directly the A-matrix without explicitly forming the N-matrix depends upon two factors: (1) condition of the system, and (2) accuracy of the solution vector required. The dependence of the number of iterations on the condition of the system has already been discussed in Section 3.27. Using the geodetic triangulation data (573 unknowns, 963 equations), the program went up to 5778 iterations without giving any 7 decimal accurate solution vector, while 4 decimal accurate solution vector was obtained after 1161

iterations, i.e., 2.1 times number of unknowns (Table 3.40-1).

Each column vector q_k of N^{-1} is generally computed in less than 1.2 n-iterations (Table 3.40-1).

Table 3.40-1

Experiment Number*	Number of		Solution Vector			Covariance Vector for Column 8		
	Unknowns	Equations	Iterations	Time** m	sec	Iterations	Time** m	sec
1	573	963	1161	9	37.13	640	3	45.96
2	573	965	1177	9	23.27	657	3	31.91
3	573	965	1175	5	45.97+	659	2	12.59+
4	573	965	1176	9	22.32	682	3	45.64
5	573	965	1164	5	53.44+	674	2	1.77+
6	573	965	1162	5	41.16+	675	2	0.00+
7	573	965	1166	9	09.46	631	3	20.03
8	573	965	1159	9	24.29	648	3	19.29
9	573	965	1169	9	29.41	608	3	11.51

*Refer to Table 3.5-1

**Time is the Execution time on H-Compiler, Option = 2 (IBM 360/75) except those marked with a plus (+) sign, which is the Execution time on H-Compiler, Option = 0 (IBM 370/165).

3.41 Computer Time.

As for this investigation, IBM 360/75 and 370/165 are used. Factors influencing the computer time are valid only for these types of computers.

For large systems, H-Compiler with Option = 2 (IBM 360/75) is found to be approximately 10 times faster in execution than the G-Compiler,

although compilation takes a longer time in H-Compiler than in G-Compiler. For (819 x 1395) system, G-Compiler took 9.58 minutes for 98 iterations, while the H-Compiler took 7.00 minutes for 791 iterations. The time given in Table 3.40-1 is thus for H-Compiler. It is noticeable that time required for one column covariance vector is much less than the time required for solution vector.

3.5 Results.

The results of the investigation are given in Tables 3.5-1 and 3.5-2, wherein the improvement of the particular geodetic triangulation by super-control net is visible only when its accuracy is at least 1 part in 500,000. The positional improvements of Wyola (95), which is in the middle of the triangulation chain, using various station constraints for Chandler (3) are shown in Figure 3.5-1. These positional improvements are relative to free net adjustment results. As the preliminary accuracy of continental satellite net is lower than 1 part in 500,000, this cannot be useful as a "constraint" to the geodetic triangulation net. On the other hand, the high accuracy of super-transcontinental traverse, which is one part in 3 million, makes it most suitable as a "constraint" to the geodetic triangulation net.

Worth mentioning is that the longitude terms, which are Q_{yy} and m_y^2 in Table 3.5-1, remain practically uneffected during the entire investigation. This could be explained by the fact that station Wyola is very close to Laplace stations, which control the azimuth error accumulation, thus effecting the longitude error [Bomford, 1965, pp. 90, 519]. Hence, due to closeness of Laplace stations, the longitude terms remain practically uneffected.

Table 3.5-1

Experiment Number	Accuracy 1 in	$\hat{m}_o$	WYOLA (95)				Remarks
			Q_{xx}	Q_{yy}	m_x^2	m_y^2	
1		2.42	6.0	0.5	35.2	2.9	Free Net
2	300,000	2.41	6.7	0.5	38.9	2.9	
3	400,000	2.41	5.9	0.5	34.3	2.9	
4	500,000	2.41	4.1	0.5	23.8	2.9	
5	600,000	2.41	4.1	0.5	23.8	2.9	
6	700,000	2.41	4.1	0.5	23.8	2.9	
7	1,000,000	2.41	3.7	0.5	21.5	2.9	
8	1,500,000	2.41	3.2	0.5	18.6	2.9	
9	3,000,000	2.41	2.1	0.5	12.2	2.9	

Q_{xx}, Q_{yy} and m_x^2, m_y^2 are given in 10^{-4} seconds^2.

Table 3.5-2

Experiment Number	Accuracy 1 in	WYOLA (95)				
					Positional Improvement Relative to Experiment 1	
		m_x	m_y	m_p	Meters	%
1	Free Net	1.83	0.37	1.9		
2	300,000	1.93	0.37	2.0	-0.1	- 5
3	400,000	1.81	0.37	1.8	0.1	5
4	500,000	1.51	0.37	1.5	0.4	21
5	600,000	1.51	0.37	1.5	0.4	21
6	700,000	1.51	0.37	1.5	0.4	21
7	1,000,000	1.43	0.37	1.5	0.4	21
8	1,500,000	1.33	0.37	1.4	0.5	26
9	3,000,000	1.08	0.37	1.1	0.8	42

Standard Errors of Unknowns (m_x, m_y) and Standard Positional Error (m_p) are given in meters.

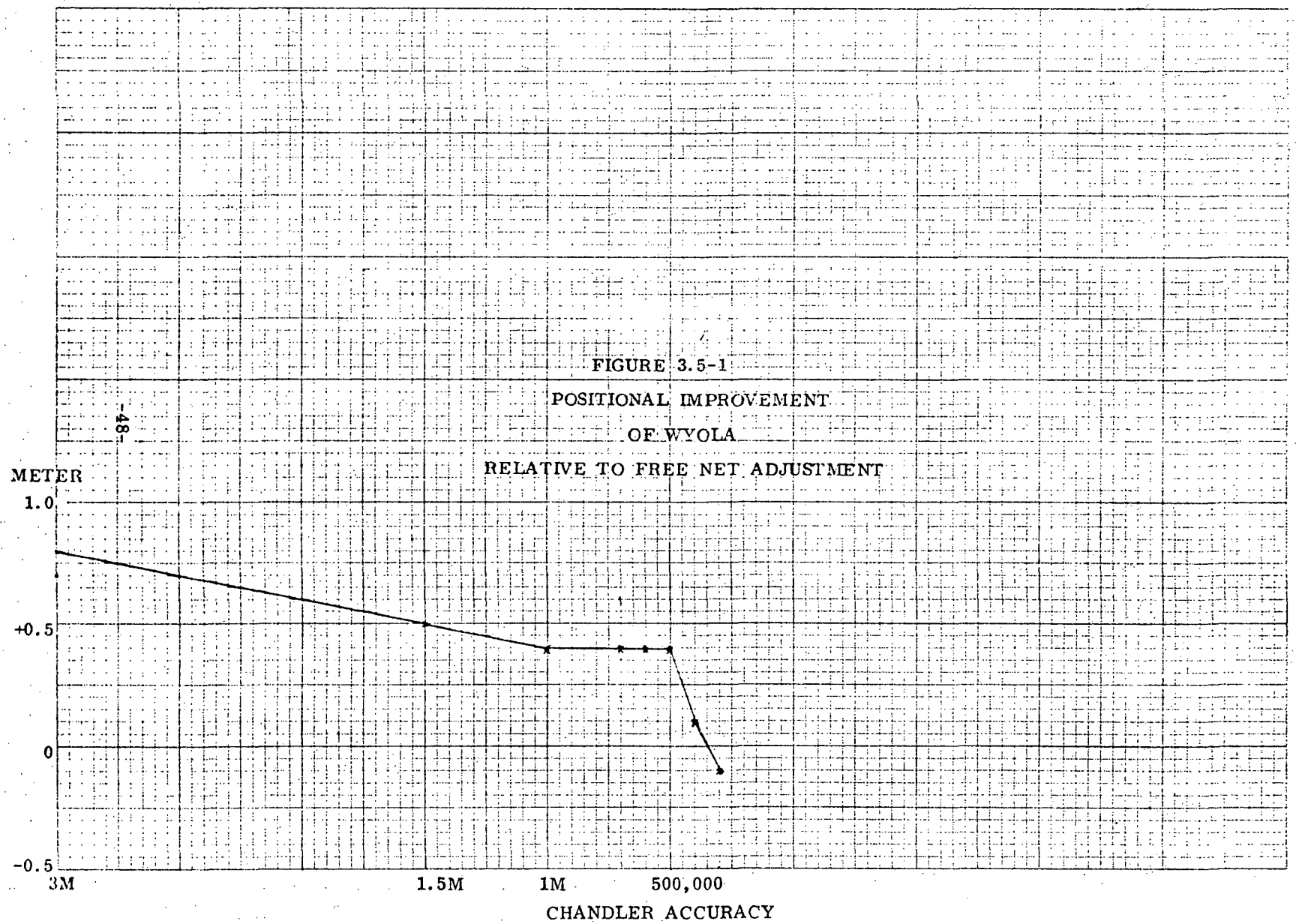

FIGURE 3.5-1

POSITIONAL IMPROVEMENT

OF WYOLA

RELATIVE TO FREE NET ADJUSTMENT

4. SUMMARY AND CONCLUSIONS

The super-control net, i.e., continental satellite net or super-trans-continental traverse, can provide a useful constraint to the investigated geodetic triangulation net, and thus can improve it only when the accuracy of super-control net is at least 1 part in 500,000; in this case, this corresponds to ±3.7m standard position error for the station Chandler.

The preliminary accuracy of super-transcontinental traverse is already better than this limiting accuracy of 1 part in 500,000. The preliminary accuracy of continental satellite net is, however, lower than the limiting accuracy of 1:500,000; the preliminary standard position error for Chandler as obtained from continental satellite net corresponds to ±4.8m, i.e., 1:385,000. The future will show whether the limiting accuracy could be achieved by continental satellite net, especially because numerous spatial triangulations of CSN have produced accuracies within the range of 1 part of 400,000 and 1 part in 700,000 [Schmid, 1965, p. 22].

Schmid [1970, pp. 23-24] indicates that continental satellite net will fall short of an optimum solution with respect to both its coverage and its accuracy. The three-dimensional positions of CSN-stations will probably be determined to no better than ±4 meters in all components, which does not seem to be good enough at least for this particular investigation.

It might be useful to have a "block constrain" instead of "chain constrain", that is, to use four well separated satellite stations, namely 003, 102, 112 and 134 (Figure 2.0-3) over a very large area, thus constraining the triangulation of the western-half of the United States instead of one triangulation chain ("chain constrain") between stations 003 and 102.

Super-transcontinental traverse can provide a better constraint, if more then two of its stations are common to the stations of geodetic triangulation net.

Also, a "block constrain", as explained above, might be more useful instead of a "chain constrain".

The development tendencies of instrumentation indicate that the future super-control nets will use VLBI (Very Long Baseline Interferometry) and Laser ranging systems.

APPENDIX I

Helmert-Rainsford-Sodano Inverse Problem Solution

As the distance between stations varies from 273m to 190km Sodano's concept which gives formulas applicable to very short as well as very long lines to solve the Inverse Problem has been used. The formula renders accuracies on the order of 0.″00001 in azimuths and a millimeter in distance for any length of lines [Helmert, 1880, Vol. 1; Jordan/Eggert/Kneissl, 1959, IV.2; Rainsford, 1955; Rapp, 1969; Sodano, 1958, 1963]. The iteration process stops when the value of $(\lambda - L)$ (i.e., longitude on the reduced sphere - longitide on the ellipsoid) does not differ by its preceeding value by 0.″00001.

GIVEN:

B_1, L_1 = Geodetic latitude and longitude of a point P_1

B_2, L_2 = Geodetic latitude and longitude of another point P_2 (longitude positive eastward, latitude positive northward).

REQUIRED:

Geodetic distance S_{12}, direct azimuth A_{12} and reverse azimuth A_{21} (Azimuth reckoned clockwise from north).

NOTATIONS USED:

$L = L_2 - L_1$ = Difference of longitude on the Ellipsoid

λ = Difference of longitude on the reduced sphere, for which a progressively better value is calculated after each iteration (for first approximation $\lambda = L$)

α = Azimuth of the specific geodesic at the equator.

CALCULATIONS:

1. Reduced latitude β_1 for each point:

$$\beta_1 = \arctan [(1 - f) \tan B_1].$$

2. Spherical arc σ between two points

$$\sin\sigma = \left[(\sin\lambda\cos\beta_2)^2 + (\sin\beta_2\cos\beta_1 - \sin\beta_1\cos\beta_2\cos\lambda)^2\right]^{\frac{1}{2}}$$

For first approximation let $\lambda = L = L_2 - L_1$

3. $$\sin\alpha = \frac{\sin\lambda_1\cos\beta_1\cos\beta_2}{\sin\sigma}$$

4. $$\cos 2\sigma_m = \cos\sigma - \frac{2\sin\beta_1\sin\beta_2}{\cos^2\alpha}$$

From this relation further relations for $\cos 4\sigma_m$, $\cos 6\sigma_m$ and $\cos 8\sigma_m$ can be deduced:

$$\cos 4\sigma_m = 2\cos^2 2\sigma_m - 1;\quad \cos 6\sigma_m = 4\cos^3 2\sigma_m - 3\cos 2\sigma_m;$$

$$\cos 8\sigma_m = 2\cos^2 4\sigma_m - 1.$$

5. $$(\lambda - L) = f\sin\alpha(A_0\sigma + A_2\sin\sigma\cos 2\sigma_m + A_4\sin 2\sigma\cos 4\sigma_m + A_6\sin 3\sigma\cos 6\sigma_m + \cdots)$$

where

$$A_0 = 1 - \frac{1}{4}f(1+f+f^2)\cos^2\alpha + \frac{3}{16}f^2(1+\frac{9}{4}f)\cos^4\alpha - \frac{25}{128}f^3\cos^6\alpha$$

$$A_2 = \frac{1}{4}f(1+f+f^2)\cos^2\alpha - \frac{1}{4}f^2(1+\frac{9}{4}f)\cos^4\alpha + \frac{75}{256}f^3\cos^6\alpha$$

$$A_4 = \frac{1}{32}f^2(1+\frac{9}{4}f)\cos^4\alpha - \frac{15}{256}f^3\cos^6\alpha$$

$$A_6 = \frac{5}{768}f^3\cos^6\alpha.$$

6. Iterate till $(\lambda - L)$ does not differ with its last computed value by 1.10^{-12} radians (as double-precision computation is done).

7. $$\lambda = L + (\lambda - L)$$

8. $$\tan A_{12} = \frac{\sin\lambda\cos\beta_2}{\sin\beta_2\cos\beta_1 - \cos\lambda\sin\beta_1\cos\beta_2}$$

$$\tan A_{21} = \frac{\sin\lambda\cos\beta_1}{\sin\beta_2\cos\beta_1\cos\lambda - \sin\beta_1\cos\beta_2}$$

The following table is used in the assignment of the quadrants for azimuths:

Sign of L $L = L_2 - L_1$	Sign of tan A_{12}	Quadrant of A_{12}	Sign of tan A_{21}	Quadrant of A_{21}
+	+	I	+	III
+	-	II	-	IV
-	+	III	+	I
-	-	IV	-	II

9. $$S_{12} = b(B_0\sigma + B_2 \sin\sigma \cos 2\sigma_m + B_4 \sin 2\sigma \cos 4\sigma_m + B_6 \sin 3\sigma \cos 6\sigma_m + B_8 \sin 4\sigma \cos 8\sigma_m + \cdots)$$

where

$$B_0 = 1 + \frac{1}{4}u^2 - \frac{3}{64}u^4 + \frac{5}{256}u^6 - \frac{175}{16384}u^8$$

$$B_2 = -\frac{1}{4}u^2 + \frac{1}{16}u^4 - \frac{15}{512}u^6 + \frac{25}{2048}u^8$$

$$B_4 = -\frac{1}{128}u^4 + \frac{3}{512}u^6 - \frac{35}{8192}u^8$$

$$B_6 = -\frac{1}{1536}u^6 + \frac{5}{6144}u^8$$

$$B_8 = -\frac{5}{65536}u^8$$

where

$$u^2 = K^2 = e'^2 \cos^2\alpha$$

b = semi-minor axis of the ellipsoid.

This completes the Inverse Problem solution.

APPENDIX II

Observation Equations (for Ellipsoidal Geodetic Coordinate System).

NOTATIONS USED:

B_j, L_j = adjusted coordinates of station j (where j = i or k)

B_j^o, L_j^o = approximate coordinates of station j

M_j, N_j = meridional - , normal radius of curvature at station j

Z_j = adjusted orientation unknown at station j

L_{ik} = observed direction from station i to station k

α_{ik}^*, λ_i^* = observed astronomical azimuth and longitude at station i

s_{ik} = measured distance of the geodesic ik

A_{ik}^o = computed ellipsoidal azimuth of the geodesic ik

s_{ik}^o = computed distance of the geodesic ik from approximate coordinates of stations i and k

Z_j^o = computed value of orientation unknown Z_j by "mean-orientation-method"

dB_j, dL_j = corrections to approximate coordinates of station j

dZ_j = correction to computed orientation unknown Z_j^o at station j

ν_j = number of observed directions at station j

ℓ_{ik} = absolute term of the observation equation

v_{ik} = residual to the observed quantity.

Direction Observation Equation [Grossmann, 1961, p.177; Wolf, 1968, pp:275, 323]

$$v_{ik(Dir)} = -dZ_i + \frac{M_i}{s^o_{ik}} \cdot \sin A^o_{ik} \cdot dB_i - \left(\frac{N_i}{s^o_{ik}} \cdot \cos A^o_{ik} \cdot \cos B^o_i - \sin B^o_i\right) \cdot dL_i$$

$$+ \frac{M_k}{s^o_{ik}} \cdot \sin A^o_{ki} \cdot dB_k - \frac{N_k}{s^o_{ik}} \cdot \cos A^o_{ki} \cdot \cos B^o_k \cdot dL_k + \ell_{ik} \qquad \text{(II. 1)}$$

where

$$\left.\begin{aligned} \ell_{ik} &= A^o_{ik} - (L_{ik} + Z^o_i) \\ Z^o_i &= \frac{1}{\nu_i} [A^o_{ik} - L_{ik}] \end{aligned}\right\} \qquad \text{(II. 2)}$$

Laplace Azimuth Observation Equation [Grossmann, 1961, p.177; Wolf, 1968, p.323].

$$v_{ik(Lap)} = \frac{M_i}{s^o_{ik}} \cdot \sin A^o_{ik} \cdot dB_i - \frac{N_i}{s^o_{ik}} \cdot \cos A^o_{ik} \cdot \cos B^o_i \cdot dL_i + \frac{M_k}{s^o_{ik}} \cdot \sin A^o_{ki} \cdot dB_k$$

$$- \frac{N_k}{s^o_{ik}} \cdot \cos A^o_{ki} \cdot \cos B^o_k \cdot dL_k + \ell_{ik} \qquad \text{(II. 3)}$$

where

$$\ell_{ik} = (A^o_{ik} - \alpha^*_{ik}) - (L^o_i - \lambda^*_i) \sin B^o_i \qquad \text{(II. 4)}$$

Distance Observation Equation [Grossmann, 1961, p.170; Wolf, 1968, p.324].

$$v_{ik(Dis)} = -\frac{M_i}{\rho} \cdot \cos A^o_{ik} \cdot dB_i - \frac{N_i}{\rho} \cdot \sin A^o_{ik} \cdot \cos B^o_i \cdot dL_i - \frac{M_k}{\rho} \cdot \cos A^o_{ki} \cdot dB_k$$

$$- \frac{N_k}{\rho} \cdot \sin A^o_{ki} \cdot \cos B^o_k \cdot dL_k + \ell_{ik} \qquad \text{(II. 5)}$$

where

$$\ell_{ik} = s^o_{ik} - s_{ik} \qquad \text{(II. 6)}$$

The corrections dB_j, dL_j and dZ_j are obtained by the method of least squares; thus the adjusted station coordinates (B_j, L_j) and orientation unknown Z_j are given by

$$\left.\begin{aligned} Z_j &= Z_j^o + dZ_j \\ B_j &= B_j^o + dB_j \\ L_j &= L_j^o + dL_j \end{aligned}\right\} \qquad \text{(II.7)}$$

DIMENSIONS:

All angular quantities are in seconds of arc and linear quantities in meters.

APPENDIX III

Programs for Solution Vector and N^{-1} by Cg-Method

Due to comment cards, the programs are self-explanatory.

```
C      MAIN PROGRAM OF CG-METHOD FOR COVARIANCE MATRIX
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION A(990,5),II(990,5),X(600),E(600),        L(990),M(990),
      1LK(990),MK(990),IL(990),KL(990),R(600),H(600),AH(990),RT(1,600),
      2RLST(600),XLST(600),AR(990)
C      VARIABLES : DIMENSION STATEMENT IN THE MAIN PROGRAM
       VARIABLE  : DATA CARD FOR  NU  NE  NI
C      NU = NUMBER OF UNKNOWNS
C      NE = NUMBER OF OBSERVATION EQUATIONS
C      NI = NUMBER OF COLUMNS IN INDEX MATRIX (II)
       READ(5,5000) NU,NE,NI
 5000 FORMAT(3I5)
       CALL          OSOLN(A,II,X,E,  L,M,LK,MK,IL,KL,R,H,AH,RT,RLST,XLST,
      1AR,NU,NE,NI)
       STOP
       END
C
       SUBROUTINE OSOLN(A,II,X,E,  L,M,LK,MK,IL,KL,R,H,AH,RT,RLST,XLST,
      1AR,NU,NE,NI)
C
C      COVARIANCE MATRIX BY CG-METHOD
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION A(NE,NI),II(NE,NI),X(NU),E(NU),        L(NE),M(NE),
      1LK(NE),MK(NE),IL(NE),KL(NE),R(NU),H(NU),AH(NE),RT(1,NU),RLST(NU)
      2,XLST(NU),AR(NE)
C      VARIABLES IN SUBROUTINE :  N , KM
       DIFF = 1.0 D-07
       NE3 = 3*NE
       NE6 = 6*NE
       EMIN = 1.0 D-09
C      N = INTERVAL FOR WRITE-OUTPUT   (STATEMENT NR.1115)
       N = 400
C      KD = CODE FOR ADJUSTED STATION COORD. PUNCHED OUTPUT
       KD = 95
C
C      KM = PATSUM BASIC BLOCKSIZE (FOR RTR)
       KM = 50
       FKTR = DFLOAT(NE)/DFLOAT(NU)
       KMM =IDINT(FKTR*DFLOAT(KM))
C      INPUT : A-MATRIX AND II-MATRIX
       REWIND 1
       DO 11 J = 1,NE
       READ(1)          L(J),M(J),(A(J,K),K=1,5),IL(J),LK(J),
      1MK(J),(II(J,K),K=1,5),KL(J)
    11 CONTINUE
       DO 10  J = 1,NE
       IF(L(J).NE.LK(J).OR.M(J).NE.MK(J).OR.IL(J).NE.KL(J)) GO TO 2500
    10 CONTINUE
       WRITE(6,6000)  (L(J),M(J),(A(J,K),K=1,5),IL(J),LK(J),
      1MK(J),(II(J,K),K=1,5),KL(J),J=1,NE)
       WRITE(6,6001)  NU,NE,NI
       CALL MSCALE(1.0 D-02,A,NE,NI)
     1 READ(5,5200)  NEND
       IF(NEND.EQ.2999)  GO TO 9000
       NK = NEND
       WRITE(6,612)  NK
       DO  12   J = 1,NU
    12 E(J) = 0.0 D0
       E(NK) = 1.0 D0
       DO  16  J = 1,NU
       R(J) = 0.0 D0
```

```
      RLST(J) = 0.0 D0
   16 X(J) = 0.0 D0
      CALL MSCALE(1.0 D-04,E,NU,1)
      DO 15  J = 1,NU
   15 R(J) = -E(J)
      FLMLST = 0.0 D0
      KOUNT = 1
      KKOUNT = 1
      RTR = 0.0 D0
      RTRLST = 0.0 D0
      DO 46  K = 1,NU
   46 XLST(K) = 0.0 D0
      CALL PATSUM(R,NU,KM,RTR)
      WRITE(6,602)  RTR,KOUNT
      IF(RTRLST.EQ.0.0) GO TO 81
   45 ELAST = ARTAH/AHTAH
   81 RTRLST = RTR
      IF(KOUNT-1) 85,85,88
   85 DO 86  I=1,NU
   86 H(I) = -R(I)
      GO TO 90
   88 DO 89  I = 1,NU
   89 H(I) = -R(I) + ELAST*H(I)
   90 CONTINUE
      JJ = 1
   92 AH(JJ) = 0.0 D0
      DO 95  KK = 1,NI
      MS=I1(JJ,KK)
      IF(MS.EQ.0) GO TO 95
      AH(JJ) = AH(JJ)+A(JJ,KK)*H(MS)
   95 CONTINUE
      JJ = JJ+1
      IF(JJ.GT.NE) GO TO 97
      GO TO 92
   97 CONTINUE
      CALL PATSUM(AH,NE,KMM,AHTAH)
      IF(AHTAH.LE.1.0 D-12)  GO TO 2300
      FLAMJ = 0.0 D0
      HTR = 0.0 D0
      DO 31  J = 1,NU
   31 HTR = HTR + H(J)*R(J)
      ALPHA = 0.0 D0
      FLAMJ = -(HTR/AHTAH)
      IF(FLMLST.EQ.0.0 D0) GO TO 96
      ALPHA = FLAMJ/FLMLST
      IF(KOUNT.EQ.2)  GO TO 93
      GO TO 96
   93 WRITE(6,617)  KOUNT,ALPHA
   96 FLMLST = FLAMJ
      DO 100  I = 1,NU
  100 X(I) = X(I)+FLAMJ*H(I)
      IF(KOUNT.GE.NU) GO TO 106
      GO TO 105
  106 DO 104  I = 1,NU,3
      IF(DABS(XLST(I+1)-X(I+1)).LE.1.0D-07.AND.DABS(XLST(I+2)-X(I+2))
     1.LE.1.0 D-07)  GO TO 104
      GO TO 105
  104 CONTINUE
      WRITE(6,606)  KKOUNT
      WRITE(6,613)  RTR,KOUNT
      WRITE(7,613)  RTR,KOUNT
      WRITE(6,617)  KOUNT,ALPHA
```

```
      WRITE(6,6006) NK
      WRITE(7,7006) NK,KOUNT
      JS = 1
      DO 302  I = 1,NU,3
      WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  302 CONTINUE
      IF(KKOUNT.EQ.5) GO TO 18
      GO TO 19
   18 WRITE(7,613)  RTR,KOUNT
      WRITE(6,6006) NK
      WRITE(7,7006) NK,KOUNT
      JS = 1
      DO 303  I = 1,NU,3
      WRITE(7,7600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  303 CONTINUE
   19 KKOUNT = KKOUNT + 1
      IF(KKOUNT.GE.6)  GO TO 17
      GO TO 105
   17 WRITE(6,606) KKOUNT
      GO TO 1
  105 CONTINUE
      DO 107  I = 1,NU
  107 RLST(I) = R(I)
      DO 110  I = 1,NU
  110 R(I) = 0.0 D0
      J = 1
  102 DO 101  K = 1,NI
      MS = I1(J,K)
  101 R(MS) = R(MS) + FLAMJ*A(J,K)*AH(J)
      J = J + 1
      IF(J-NE)  102,102,103
  103 CONTINUE
      DO 108  I = 1,NU
  108 R(I) = R(I) + RLST(I)
      CALL PATSUM(R,NU,KM,RTR)
   48 CONTINUE
      WRITE(6,602)  RTR,KOUNT
      RTH = 0.0 D0
      DO 91  J = 1,NU
   91 RTH = RTH +  R(J)*H(J)
   84 IF(DABS(RTH).GT.1.0 D-09)  GO TO 201
      GO TO 82
  201 WRITE(6,603) RTH,KOUNT
   82 CONTINUE
      RLSTR = 0.0 D0
      DO 47  I = 1,NU
   47 RLSTR = RLSTR + RLST(I)*R(I)
      IF(DABS(RLSTR).GT.1.0 D-09)  GO TO 200
      GO TO 49
  200 WRITE(6,604)  RLSTR,KOUNT
   49 DO 50  I = 1,NU
   50 XLST(I) = X(I)
      JJ = 1
   35 AR(JJ) = 0.0 D0
      DO 33  KK = 1,NI
      MS=I1(JJ,KK)
      IF(MS.EQ.0) GO TO 33
      AR(JJ) = AR(JJ) + A(JJ,KK)*R(MS)
   33 CONTINUE
      JJ = JJ+1
```

```
      IF(JJ.GT.NE) GO TO 34
      GO TO 35
   34 CONTINUE
      ARTAH = 0.0 D0
      DO 36  I = 1,NE
   36 ARTAH = ARTAH + AR(I)*AH(I)
      IF(KOUNT.EQ.1)  GO TO 115
  113 IF(RTR.LE.EMIN)  GO TO 2100
      IF(KOUNT-(KOUNT/N)*N)  114,115,114
  115 WRITE(6,6900) KOUNT
      WRITE(6,602)  RTR,KOUNT
      WRITE(6,6041)  RLSTR
      WRITE(6,6031)RTH
      WRITE(6,6400)  KOUNT,AHTAH
      WRITE(6,617)  KOUNT,ALPHA
      WRITE(6,6006) NK
      WRITE(7,7006) NK,KOUNT
      JS = 1
      DO 301  I = 1,NU,3
      WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  301 CONTINUE
  114 CONTINUE
      KOUNT = KOUNT + 1
      IF(KOUNT.EQ.NE6)  GO TO 2000
      IF(KOUNT.LT.NE6)  GO TO 45
 2100 WRITE(6,611)  KOUNT,RTR
 2300 WRITE(6,614)  KOUNT,AHTAH
 2000 WRITE(6,610) KOUNT,RTR
      WRITE(6,6041)  RLSTR
      WRITE(6,6031)RTH
      WRITE(6,6400)  KOUNT,AHTAH
      WRITE(6,617)  KOUNT,ALPHA
      WRITE(6,6006) NK
      WRITE(7,7006) NK,KOUNT
      JS = 1
      DO 300  I = 1,NU,3
      WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
      WRITE(7,7600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  300 CONTINUE
      GO TO 1
 2500 WRITE(6,6100)  L(J),LK(J),M(J),MK(J),IL(J),KL(J),J
  602 FORMAT(1H ,'RTR=',D25.16,10X,'KOUNT= ',I5)
  603 FORMAT(1H ,'BAD CONJUGACY      RTH= ',D25.16,'      KOUNT= ',I5)
  604 FORMAT(1H ,'RLAST NOT CONJUGATE TO R',10X,'RLSTR= ',D25.16,10X,
     1'KOUNT= ',I5)
  606 FORMAT(1H1,'**<<**  SOLUTION VECTOR DOES NOT IMPROVE  **>>**',
     1I5,5X,'TIMES')
  610 FORMAT(1H1,'ITERATIONS COMPLETED',20X,'KOUNT= ',I5,5X,'RTR= ',
     1D25.16)
  611 FORMAT(1H1,'RTR VANISHES',20X,'KOUNT= ',I5,D25.16)
  612 FORMAT(1H1,'COLUMN VECTOR= ',I5)
  613 FORMAT(1X,'SOLUTION VECTOR DOES NOT IMPROVE',5X,'RTR= ',D17.8,
     12X,'KOUNT= ',I5)
  614 FORMAT(1H1,'TERMINATE FOR AHTAH=0'//' KOUNT=',I5,10X,'AHTAH= ',
     1D25.16)
  617 FORMAT(1H ,'KOUNT= ',I5,10X,'ALPHA= ',D25.16,'<<<<UPPER BOUND
     1CONDITION NUMBER <<<<<<')
 5200 FORMAT(I5)
 6000 FORMAT(1X,2I4,F7.2,4E12.5,I3,7I4,I3)
 6001 FORMAT(1H0,'   NU    NE    NI'//3I5)
```

```
6006 FORMAT(1H0,2X,'COVARIANCE VECTOR FOR COLUMN= ',I5//)
6031 FORMAT(1H ,'RTH=',D25.16)
6041 FORMAT(1H ,'RLSTR= ',D25.16)
6100 FORMAT(1H ,'SOMETHING WRONG'//7I5)
6400 FORMAT(1H ,'KOUNT= ',I5,10X,'AHTAH= ',D25.16)
6600 FORMAT(1H ,I5,3E23.16,4X,I2)
6900 FORMAT(1H1,'  ITERATION  ',I5)
7006 FORMAT(10X,'COVARIANCE VECTOR FOR COLUMN= ',I5,5X,'KOUNT= ',I5)
7600 FORMAT(I5,3E23.16,4X,I2)
9000 RETURN
     END
```

```
C      MAIN PROGRAM OF CG-METHOD FOR SOLUTION VECTOR
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION A(990,5),I1(990,5),X(600),FL(990),V(990),AX(990),L(990),
      1M(990),LK(990),MK(990),LN(990),ML(990),MN(990),IL(990),KL(990),
      2LM(990),R(600),H(600),AH(990),VCHECK(990),RT(1,600),RLST(600),
      3XLST(600),R1(600),CHV(990),AR(990)
       VARIABLE :  DIMENSIONS IN THE MAIN PROGRAM
C      VARIABLE :  DATA CARD FOR NU  NE  NI
C      NU = NUMBER OF UNKNOWNS
C      NE = NUMBER OF OBSERVATION EQUATIONS
C      NI = NUMBER OF COLUMNS IN INDEX MATRIX (I1)
       READ(5,5000) NU,NE,NI
 5000  FORMAT(3I5)
       CALL        SOLN(A,I1,X,FL,V,AX,L,M,LK,MK,LN,ML,MN,IL,KL,LM,R,H,
      1 AH,VCHECK,RT,RLST,XLST,R1,CHV,AR,NU,NE,NI)
       STOP
       END
C
       SUBROUTINE SOLN(A,I1,X,FL,V,AX,L,M,LK,MK,LN,ML,MN,IL,KL,LM,R,H,
      1 AH,VCHECK,RT,RLST,XLST,R1,CHV,AR,NU,NE,NI)
C      SOLUTION OF OBSERVATION EQUATIONS BY CG-METHOD
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION A(NE,NI),I1(NE,NI),X(NU),FL(NE),V(NE),AX(NE),L(NE),
      1M(NE),LK(NE),MK(NE),LN(NE),ML(NE),MN(NE),IL(NE),KL(NE),LM(NE),
      2R(NU),H(NU),AH(NE),VCHECK(NE),RT(1,NU),RLST(NU),XLST(NU),R1(NU)
      3,CHV(NE),AR(NE)
       REWIND 1
C      VARIABLES IN SUBROUTINE ARE : N , IM , IR , KM .
       DIFF = 1.0 D-07
       NU2 = 2*NU
       NE2 = 2*NE
       NE3 = 3*NE
       NE4 = 4*NE
       NE6 = 6*NE
       FMIN = 1.0 D-09
C      N = INTERVAL FOR WRITE-OUTPUT   (STATEMENT NR.115)
       N = 400
C      IM = INTERVAL FOR CALCULATING REE  (STATEMENT NR. 72)
       IM = 200
C      IR = NUMBER OF CONSTRAINS
       IR = 13
       IKOUNT = 1
       KKOUNT = 1
C      KO = CODE FOR ADJUSTED STATION COORD. PUNCHED OUTPUT
       KO = 9
C      IREE = CODE FOR REE-PUNCHED OUTPUT
       IREE = 80
C      IRTR = CODE FOR RTR-PUNCHED OUTPUT
       IRTR = 81
C
C      KM = PATSUM BASIC BLOCKSIZE (FOR RTR)
       KM = 50
       FKTR = DFLOAT(NE)/DFLOAT(NU)
       KMM =IDINT(FKTR*DFLOAT(KM))
C      INPUT : A-MATRIX, I1-MATRIX AND L-VECTOR
       DO 2  J = 1,NE
       READ(1)          L(J),M(J),(A(J,K),K=1,5),IL(J),LK(J),
      1MK(J),(I1(J,K),K=1,5),KL(J),ML(J),MN(J),FL(J),LM(J)
    2  CONTINUE
       DO 10  J = 1,NE
       IF(L(J).NE.LK(J).OR.M(J).NE.MK(J).OR.IL(J).NE.KL(J)) GO TO 2500
```

```
      IF(                    L(J).NE.ML(J).OR.M(J).NE.MN(J)) GO TO 2600
   10 CONTINUE
      WRITE(6,6000)  (L(J),M(J),(A(J,K),K=1,5),IL(J),LK(J),
     1IPK(J),(II(J,K),K=1,5),KL(J),ML(J),MN(J),FL(J),FM(J),J=1,NE)
      WRITE(6,6001)  NU,NE,NI
      CALL MSCALE(1.0 D-02,A,NE,NI)
      CALL MSCALE(1.0 D-02,FL,NE,1)
      DO11 J = 1,NU
   11 X(J) = 0.0 D0
C     AFTER THIS CARD READ X-VECTOR IF MORE ITERATIONS ARE NEEDED WITH GIVEN X
   12 CONTINUE
      DO  16  J = 1,NU
   16 R(J) = 0.0 D0
      DO 15  J = 1,NE
      VCHECK(J) = 0.0 D0
   15 V(J) = 0.0 D0
      J = 1
   20 AX(J) = 0.0 D0
      DO 30  K = 1,NI
      MS = II(J,K)
      IF(MS.EQ.0) GO TO 30
      AX(J) = AX(J) + A(J,K)*X(MS)
   30 CONTINUE
      J = J+1
      IF(J.GT.NE) GO TO 40
      GO TO 20
   40 CONTINUE
      CALL DGMADD(AX,FL,V,NE,1)
      VPV = 0.0 D0
      CALL PATSUM(V,NE,KMM,VPV)
      CALL MSCALE(1.0 D 04,VPV,1,1)
      WRITE(6,6070) VPV
      ELAST = 0.0 D0
      ELMLST = 0.0 D0
      REE = 0.0 D0
      KOUNT = 1
      RTR = 0.0 D0
   45 RTRLST = RTR
      DO 46  K = 1,NU
      XLST(K) = X(K)
   46 RLST(K) = R(K)
      DO 50  K = 1,NU
   50 R(K) = 0.0 D0
      J = 1
   65 DO 60  K = 1,NI
      MS = II(J,K)
   60 R(MS) = R(MS) + A(J,K)*V(J)
      J = J+1
      IF(J-NE) 65,65,70
   70 CONTINUE
      IF(IKOUNT-(IKOUNT/IM)*IM)  71,72,71
   72 DO 73  K = 1,NU
   73 R1(K) = 0.0 D0
      J = 1
   74 DO 75  K= 1,NI
      MS = II(J,K)
   75 R1(MS) = R1(MS) + A(J,K)*VCHECK(J)
      J = J + 1
      IF(J-NE) 74,74,76
   76 CONTINUE
      REE = 0.0 D0
      DO 77  I = 1,NU
```

```
   77 REE = REE + (R(I)-R1(I))**2
      WRITE(6,601) IKOUNT,REE,KOUNT
      REE = 3*REE
      IF(RTR.LE.REE)  GO TO 14
      GO TO 71
   14 WRITE(6,607)  IKOUNT,REE,RTR
      GO TO 9000
   71 CALL PATSUM(R,NU,KM,RTR)
      WRITE(6,602)  RTR,KOUNT,IKOUNT
      IF(IKOUNT-(IKOUNT/N)*N)  43,48,43
   43 IF(IKOUNT-(IKOUNT/IM)*IM)  48,44,48
   44 WRITE(7,701)  IKOUNT,RTR,KOUNT,IRTR
      WRITE(7,701)  IKOUNT,REE,KOUNT,IREE
   48 RLSTR = 0.0 D0
      DO 47  I = 1,NU
   47 RLSTR = RLSTR + RLST(I)*R(I)
      IF(DABS(RLSTR).GT.1.0 D-09)  GO TO 200
      GO TO 49
  200 WRITE(6,604)  RLSTR,IKOUNT
   49 IF(KOUNT.EQ.1)  GO TO 83
      RTH = 0.0 D0
      DO 91  J = 1,NU
   91 RTH = RTH +  R(J)*H(J)
      GO TO 84
   83 RTH = 0.0 D0
   84 IF(DABS(RTH).GT.1.0 D-09)  GO TO 201
      GO TO 82
  201 WRITE(6,603) RTH,IKOUNT
   82 IF(RTRLST.EQ.0.0) GO TO 81
      JJ = 1
   35 AR(JJ) = 0.0 D0
      DO 33  KK = 1,NI
      MS=I1(JJ,KK)
      IF(MS.EQ.0) GO TO 33
      AR(JJ) = AR(JJ) + A(JJ,KK)*R(MS)
   33 CONTINUE
      JJ = JJ+1
      IF(JJ.GT.NE) GO TO 34
      GO TO 35
   34 CONTINUE
      ARTAH = 0.0 D0
      DO 36  I = 1,NE
   36 ARTAH = ARTAH + AR(I)*AH(I)
      ELAST = ARTAH/AHTAH
   81 IF(KOUNT-1) 85,85,88
   85 DO 86  I=1,NU
   86 H(I) = -R(I)
      GO TO 90
   88 DO 89  I = 1,NU
   89 H(I) = -R(I) + ELAST*H(I)
   90 CONTINUE
C     ORTOGONALITY  TEST BET. RLST AND R, AND R,AND H
      HTH = 0.0 D0
      DO 87  I = 1,NU
   87 HTH = HTH + H(I)*H(I)
      ENRMR1 = DSQRT(RTR)
      ENRMR2 = DSQRT(RTRLST)
      ENRMH = DSQRT(HTH)
      ORTH1 = (RLSTR)/(ENRMR1)*(ENRMR2)
      ORTH2 = (RTH)/(ENRMR1)*(ENRMH)
      IF(DABS(ORTH1).GT.1.0 D-05.OR.DABS(ORTH2).GT.1.0 D-05) GO TO 98
      GO TO 99
```

```
 98 WRITE(6,620)  ORTH1,ORTH2,IKOUNT
 99 JJ = 1
 92 AH(JJ) = 0.0 D0
    DO 95  KK = 1,NI
    MS=I1(JJ,KK)
    IF(MS.EQ.0) GO TO 95
    AH(JJ) = AH(JJ)+A(JJ,KK)*H(MS)
 95 CONTINUE
    JJ = JJ+1
    IF(JJ.GT.NE) GO TO 97
    GO TO 92
 97 CONTINUE
    CALL PATSUM(AH,NE,KMM,AHTAH)
    IF(AHTAH.LE.1.0 D-12)  GO TO 2300
    FLAMJ = 0.0 D0
    HTR = 0.0 D0
    DO 31  J = 1,NU
 31 HTR = HTR + H(J)*R(J)
    ALPHA = 0.0 D0
    FLAMJ = -(HTR/AHTAH)
    IF(FLMLST.EQ.0.0 D0) GO TO 96
    ALPHA = FLAMJ/FLMLST
 51 IF(KOUNT.EQ.2)  GO TO 93
    IF(IKOUNT-(IKOUNT/50)*50)  94,93,94
 93 WRITE(6,617)  IKOUNT,ALPHA
    WRITE(6,622)  KOUNT,FLAST,FLAMJ
 94 CONTINUE
    GO TO 96
 96 FLMLST = FLAMJ
    DO 100  I = 1,NU
100 X(I) = X(I)+FLAMJ*H(I)
    IF(IKOUNT.GE.NU2)  GO TO 107
    GO TO 103
107 DO 102  I = 1,NU,3
    IF(DABS(XLST(I+1)-X(I+1)).LE.1.0D-04.AND.DABS(XLST(I+2)-X(I+2))
   1.LE.1.0 D-04)  GO TO 102
    GO TO 103
102 CONTINUE
    WRITE(6,606)  KKOUNT
    WRITE(6,612) IKOUNT,RTR
    WRITE(7,701)  IKOUNT,RTR,KOUNT,IRTR
    WRITE(6,601) IKOUNT,REE,KOUNT
    WRITE(7,701)  IKOUNT,REE,KOUNT,IREE
    WRITE(7,712) IKOUNT,RTR
    WRITE(6,6041)  RLSTR
    WRITE(6,6021)RTH
    WRITE(6,622) IKOUNT,FLAST,FLAMJ
    WRITE(6,6400) IKOUNT,AHTAH
    WRITE(6,617) IKOUNT,ALPHA
    WRITE(6,6006)
    JS = 1
    DO 303  I = 1,NU,3
    WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
    WRITE(7,7600) JS,X(I),X(I+1),X(I+2),KD
    JS = JS + 1
303 CONTINUE
    DO 304  I = 1,NE
304 V(I) = AX(I) + FL(I)
    WRITE(6,6007)
    WRITE(6,6700) (V(I),I=1,NE)
    CALL PATSUM(V,NE,KMM,VPV)
    CALL MSCALE(1.0 D 04,VPV,1,1)
```

```
      WRITE(6,6070) VPV
      FM = DSQRT(VPV/(NE-NU+IR))
      WRITE(6,605)  FM
      KKOUNT = KKOUNT + 1
      IF(KKOUNT.GE.6)  GO TO 17
      GO TO 103
   17 WRITE(6,606) KKOUNT
      GO TO 9000
  103 CONTINUE
      IF(KOUNT.EQ.1)  GO TO 101
      GO TO 104
  101 DO 110  I = 1,NE
  110 VCHECK(I) = V(I) + FLAMJ*AH(I)
      GO TO 106
  104 DO 105  I= 1,NE
  105 VCHECK(I) = VCHECK(I) + FLAMJ*AH(I)
  106 VCHCKV = 0.0 D0
      CALL PATSUM(VCHECK,NE,KMM,VCHCKV)
      CALL MSCALE(1.0 D 04,VCHCKV,1,1)
      JJ = 1
  120 AX(JJ) = 0.0 D0
      DO 130  KK = 1,NI
      MS = I1(JJ,KK)
      IF(MS.EQ.0) GO TO 130
      AX(JJ) = AX(JJ) + A(JJ,KK)*X(MS)
  130 CONTINUE
      JJ = JJ + 1
      IF(JJ.GT.NE) GO TO 140
      GO TO 120
  140 CONTINUE
      VPVLST = VPV
      DO 150  I = 1,NE
  150 V(I) = AX(I) + FL(I)
      VPV = 0.0 D0
      CALL PATSUM(V,NE,KMM,VPV)
      CALL MSCALE(1.0 D 04,VPV,1,1)
      CHNVPV = DABS(VPV-VPVLST)
      IF(CHNVPV.LE.1.0 D-04)  GO TO 2200
      GO TO 119
 2200 WRITE(6,618)  KOUNT,CHNVPV
  119 CONTINUE
      DO 118  I = 1,NE
      CHV(I) = DABS(V(I)-VCHECK(I))
      IF(CHV(I).GE.1.0 D-02)  GO TO  116
      GO TO 117
  116 WRITE(6,623)  I,V(I),VCHECK(I)
  118 CONTINUE
  117 CONTINUE
      IF(DABS(VCHCKV-VPV).GT.1.0D-02)  GO TO 2800
      GO TO 113
 2800 WRITE(6,616)  KOUNT,VCHCKV,VPV
  113 CONTINUE
      IF(RTR.LE.FMIN)      GO TO 2100
      IF(RTR.LE.1.0 D-08)  GO TO 115
      IF(IKOUNT.EQ.1) GO TO 115
      IF(IKOUNT-(IKOUNT/N)*N)  114,115,114
  115 WRITE(6,6900) KOUNT,IKOUNT
      WRITE(6,602)  RTR,KOUNT,IKOUNT
      WRITE(7,701)  IKOUNT,RTR,KOUNT,IRTR
      WRITE(6,601) IKOUNT,REE,KOUNT
      WRITE(7,701)  IKOUNT,REE,KOUNT,IREE
      WRITE(6,6041)  RLSTR
```

```
      WRITE(6,6031)RTH
      WRITE(6,622) IKOUNT,ELAST,FLAMJ
      WRITE(6,6400) IKOUNT,AHTAH
      WRITE(6,617) IKOUNT,ALPHA
      WRITE(6,6006)
      JS = 1
      DO 301  I = 1,NU,3
      WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  301 CONTINUE
      WRITE(6,6007)
      WRITE(6,6700) (V(I),I=1,NE)
      WRITE(6,6070)  VPV
      WRITE(6,6080)  VCHCKV
      FM = DSQRT(VPV/(NE-NU+IR))
      WRITE(6,605)  FM
  114 CONTINUE
      IF(IKOUNT-(IKOUNT/IM)*IM)  79,80,79
   80 IF(RTR.LE.REE)  GO TO 78
      GO TO 79
   78 WRITE(6,613) IKOUNT,RTR,REE,KOUNT
      IKOUNT = IKOUNT + 1
      GO TO 12
   79 KOUNT = KOUNT + 1
      IKOUNT = IKOUNT + 1
      IF(IKOUNT.EQ.NE6)  GO TO 2000
      IF(KOUNT.LT.NE6)   GO TO 45
 2000 WRITE(6,610) IKOUNT
 2300 WRITE(6,614) IKOUNT,AHTAH
 2100 WRITE(6,611) IKOUNT,RTR
      WRITE(7,701)  IKOUNT,RTR,KOUNT,IRTR
      WRITE(6,601) IKOUNT,REE,KOUNT
      WRITE(7,701)  IKOUNT,REE,KOUNT,IREE
      WRITE(6,6041)  RLSTR
      WRITE(6,6031)RTH
      WRITE(6,622) IKOUNT,ELAST,FLAMJ
      WRITE(6,6400) IKOUNT,AHTAH
      WRITE(6,617) IKOUNT,ALPHA
      WRITE(6,6006)
      JS = 1
      DO 300  I = 1,NU,3
      WRITE(6,6600) JS,X(I),X(I+1),X(I+2),KD
      WRITE(7,7600) JS,X(I),X(I+1),X(I+2),KD
      JS = JS + 1
  300 CONTINUE
      WRITE(6,6007)
      CALL DSCALE(1.0 D 02,V,NE,1)
      WRITE(6,6700) (V(I),I=1,NE)
      WRITE(6,6070)  VPV
      WRITE(6,6080)  VCHCKV
      FM = DSQRT(VPV/(NE-NU+IR))
      WRITE(6,605)  FM
      GO TO 9000
 2500 WRITE(6,6100)  L(J),LK(J),M(J),MK(J),IL(J),KL(J),J
 2600 WRITE(6,6200)                L(J),ML(J),M(J),MM(J) ,J
  601 FORMAT(1H ,'IKOUNT= ',I5,5X,'REE= ',D25.16,5X,'KOUNT= ',I5)
  602 FORMAT(1H ,'RTR=',D25.16,10X,'KOUNT= ',I5,10X,'IKOUNT= ',I5)
  603 FORMAT(1H ,'BAD CONJUGACY      RTH= ',D25.16,'    IKOUNT= ',I5)
  604 FORMAT(1H ,'RLAST NOT CONJUGATE TO R',10X,'RLSTR= ',D25.16,10X,
     1'IKOUNT= ',I5)
  605 FORMAT(1H ,'STANDARD ERROR OF UNIT WEIGHT = ',D12.4)
  606 FORMAT(1H1,'**<<** SOLUTION VECTOR DOES NOT IMPROVE **>>**',
```

```
     1I5,5X,'TIMES')
  607 FORMAT(1H ,'TERMINATION OF ITERATIONS DUE TO RTR.LE.RFE'//
     1'IKOUNT= ',I5,5X,'REE= ',D25.16,5X,'RTR= ',D25.16)
  610 FORMAT(1H1,'ITERATIONS COMPLETED',20X,'IKOUNT= ',I5)
  611 FORMAT(1H1,'RTR VANISHES',20X,'IKOUNT= ',I5,5X,'RTR= ',D25.16)
  612 FORMAT(1H ,'SOLUTION VECTOR DOES NOT IMPROVE',10X,'IKOUNT= ',I5,
     15X,'RTR= ',D25.16)
  613 FORMAT(1H ,'IKOUNT=',I5,5X,'RTR=',D25.16,5X,'REE=',D25.16,5X,
     1'KOUNT= ',I5)
  614 FORMAT(1H1,'TERMINATE FOR AHTAH=0'//'IKOUNT=',I5,10X,'AHTAH= ',
     1D25.16)
  616 FORMAT(1H ,'VCHCKV.NE.VPV',10X,I5,2D25.16)
  617 FORMAT(1H ,'IKOUNT=',I5,10X,'ALPHA= ',D25.16,'<<<<UPPER BOUND
     1CONDITION NUMBER <<<<<<')
  618 FORMAT(1H ,'******   IKOUNT= ',I5,10X,'VPV INCREASE= ',D25.16)
  620 FORMAT(1H ,'ORTHOGONALITY TEST: ','ORTH1=RLSTR=',D16.9,5X,
     1'ORTH2=RTH= ',D16.9,5X,'IKOUNT= ',I5)
  622 FORMAT(1H ,' KOUNT= ',I5,5X,'ELAST= ',E20.10,5X,'ELAMJ= ',E20.10)
  623 FORMAT(1H ,'I= ',I5,          'V= ',E20.10,10X,'VCHECK= ',E20.10)
  701 FORMAT(I5,D25.10,I10,3E8X,I2)
  712 FORMAT(5X,'IKOUNT= ',I5,'    RTR= ',E20.10)
 6000 FORMAT(1X,2I4,F7.2,4F12.5,I3,7I4,I3,2I4,F11.4,I3)
 6001 FORMAT(1H0,'   NU   NE   NI'//3I5)
 6006 FORMAT(1H0,2X,'I',12X,'Z',22X,'PHI',20X,'LAMBDA')
 6007 FORMAT(1H0,'****V(I)****')
 6031 FORMAT(1H ,'RTH=',D25.16)
 6041 FORMAT(1H ,'RLSTR= ',D25.16)
 6070 FORMAT(1H ,'VPV=',4X,D25.16)
 6080 FORMAT(1H ,'VCHECKV=',D25.16)
 6100 FORMAT(1H ,'SOMETHING WRONG'//7I5)
 6200 FORMAT(1H ,'NOT MATCHING'//5I5)
 6400 FORMAT(1H ,'IKOUNT=',I5,10X,'AHTAH= ',D25.16)
 6600 FORMAT(1H ,I5,3F23.16,5X,I1)
 6700 FORMAT(6F20.10)
 6900 FORMAT(1H1,'  ITERATION  ',2I5)
 7600 FORMAT(I5,3F23.16,5X,I1)
 9000 RETURN
      END
```

```
C
      SUBROUTINE PATSUM(R,NU,KM,RTR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION R(NU)
      RTR = 0.0 D0
      NP = NU/KM
      NL = NU-(NP*KM)
      IF(NL.LE.0)  GO TO 25
      II = (KM*NP) + 1
      RTR1 = 0.0 D0
      DO 24   J = II,NU
   24 RTR1 = RTR1 + R(J)*R(J)
      RTR = RTR + RTR1
   25 CONTINUE
      IF(KM.GT.NU) GO TO 28
      DO 21    I = 1,NP
      II = I*KM
      JJ = II - (KM-1)
      RTR1 = 0.0 D0
      DO 22    J = JJ,II
   22 RTR1 = RTR1 + R(J)*R(J)
   21 RTR = RTR + RTR1
      RETURN
   28 WRITE(6,61)
   61 FORMAT(5X,'IN PATSUM,  KM  GT  NU ')
      RETURN
      END
```

```
C
      SUBROUTINE DUMMY%A,B,C,P,T,SIGN,SCALE,N1<
      DOUBLE PRECISION A%1<,B%1<,C%1<,P%1<,T%1<,SIGN,SCALE
      ENTRY MWRITE%A,IA,JA,NAME<
      DIMENSION NAME%3<
      WRITE%6,1<NAME%1<,IA,JA
    1 FORMAT %A4,27X,I5,5H ROWS%X,I5,6H COL@S<
      JJ#IA*JA
      DO 2 II#1,IA
    2 WRITE%6,3<II,%A%JJ<,JJ#II,JJ,IA<
    3 FORMAT%/4H ROWI5,//%1X,1P5D20.8<<
      RETURN
      ENTRY MPUNCH%A,IA,JA,NAM<
      JJ#%IA*JA-1</3&1
      WRITE%7,4<%A%II*3-2<,A%JI*3-1<,A%II*3<,NAM,II,II#1,JJ<
    4 FORMAT%3D24.16,A4,I4<
      RETURN
      ENTRY MMULT%A,B,N,M,L,C<
      ENTRY GMPRD%A,B,C,N,M,L<
      IR#0
      IK#-M
      DO 10 K#1,L
      JK#IK&M
      DO 10 J#1,N
      IR#IR&1
      JI#J-N
      IB#IK
      C%IR<#0.0D0
      DO 10 I#1,M
      JI#JI&N
      IB#IB&1
   10 C%IR<#C%IR<&A%JI<*B%IB<
      RETURN
      ENTRY GTPRD%A,B,C,N,M,L<
      N1#N
      ENTRY T1MULT%A,N,M,B,N1,L,C<
      IF%N-N1<60,8,60
    8 IR#0
      IK#-N
      DO 20 K#1,L
      IJ#0
      IK#IK&N
      DO 20 J#1,M
      IB#IK
      IR#IR&1
      C%IR<#0.0D0
      DO 20 I#1,N
      IJ#IJ&1
      IB#IB&1
   20 C%IR<#C%IR<&A%IJ<*B%IB<
      RETURN
      ENTRY T2MULT%A,N,M,B,L,M1,C<
      IF%M-M1<50,9,50
    9 IR#0
      DO 30 K#1,L
      DO 30 J#1,N
      IB#K-L
      IR#IR&1
      JI#J-N
      C%IR<#0.0D0
      DO 30 I#1,M
      JI#JI&N
```

```
      IB#IB&L
   30 C%IR<#C%IR<&A%JT<*B%IB<
      RETURN
   50 N#M
      N1#M1
   60 WRITE%6,70<N,N1
   70 FORMAT%30H1***NOT A VALID OPERATION--JA#I5,4H IB#I5,
     * 16H.  CALL IGNORED.<
      RETURN
      ENTRY MSCALE%SCALE,A,IA,JA<
      JJ#IA*JA
      DO 100 J#1,JJ
  100 A%J<#SCALE*A%J<
      RETURN
      ENTRY MADD%A,B,IA,JA,C<
      SIGN#1.0D0
      GO TO 200
      ENTRY MSUBT%A,B,IA,JA,C<
      SIGN#-1.0D0
  200 JJ#IA*JA
      DO 300 J#1,JJ
  300 C%J<#A%J<&SIGN*B%J<
      RETURN
      ENTRY MDUP%A,IA,JA,B<
      JJ#IA*JA
      DO 400 J#1,JJ
  400 B%J<#A%J<
      RETURN
      ENTRY MATMA%A,IA,JA,P,C,T<
      IR#0
      JK#-IA
      DO 600 K#1,JA
      JK#JK&IA
      IT#0
      DO 500 J#1,IA
      IM#J-IA
      IB#JK
      IT#IT&1
      T%IT<#0.0D0
      DO 500 I#1,IA
      IM#IM&IA
      IB#IB&1
  500 T%IT<#T%IT<&P%IM<*A%IB<
      IB#0
      DO 600 J#1,JA
      IR#IR&1
      C%IR<#0.0D0
      DO 600 I#1,IA
      IB#IB&1
  600 C%IR<#C%IR<&A%IB<*T%I<
      RETURN
      ENTRY MAMAT%A,IA,JA,P,C,T<
      IR#0
      DO 800 K#1,IA
      IK#K-IA
      IT#0
      DO 700 J#1,JA
      IB#IK
      IM#J-JA
      IT#IT&1
      T%IT<#0.0D0
      DO 700 I#1,JA
```

```
      IM=IM&JA
      IB=IB&IA
  700 T%IT<=T%IT<&P%IM<*A%IB<
      DO 800 J=1,IA
      IB=J-IA
      IR=IR&1
      C%IR<=0.0D0
      DO 800 I=1,JA
      IB=IB&IA
  800 C%IR<=C%IR<&A%IB<*T%I<
      RETURN
      END
```

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