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INCOMPLETE STATE FEEDBACK FOR SYSTEMS WITH PARAMETER UNCERTAINTY AND RANDOM DISTURBANCES

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Department of Electrical Engineering

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I INTRODUCTION

1.1 INTRODUCTION AND HISTORICAL BACKGROUND

Satisfactory control of multi-input, multi-output nonlinear system with process uncertainty is a very basic and challenging problem in systems science. The problem is made more complex by unavailability of states of the system necessary to implement the control and by noise associated with the observations. Several basic questions still to be answered in this context are:

a) How large a control range is necessary to compensate for parameter uncertainty and unknown process nonlinearities?

b) What is the minimum number of measurements necessary to generate the required control signal?

c) What is an efficient way of controlling this class of systems and what is the relative effectiveness of various design techniques?

The present thesis will deal with these problems to some extent and will propose various design procedures via optimal theory to generate the control. Since the system involves parameter uncertainty, considerable attention has been directed to implement the control via identification, usually known as adaptive control in the literature[15]. The fundamental concept was introduced possibly by Whitaker _et. al._[62,63] in 1958, followed quickly by Osburn _et. al._[47]. The basic idea is illustrated in Figure 1.1. The plant structure is assumed to be known but its parameters are unknown. The model generates the desired per-

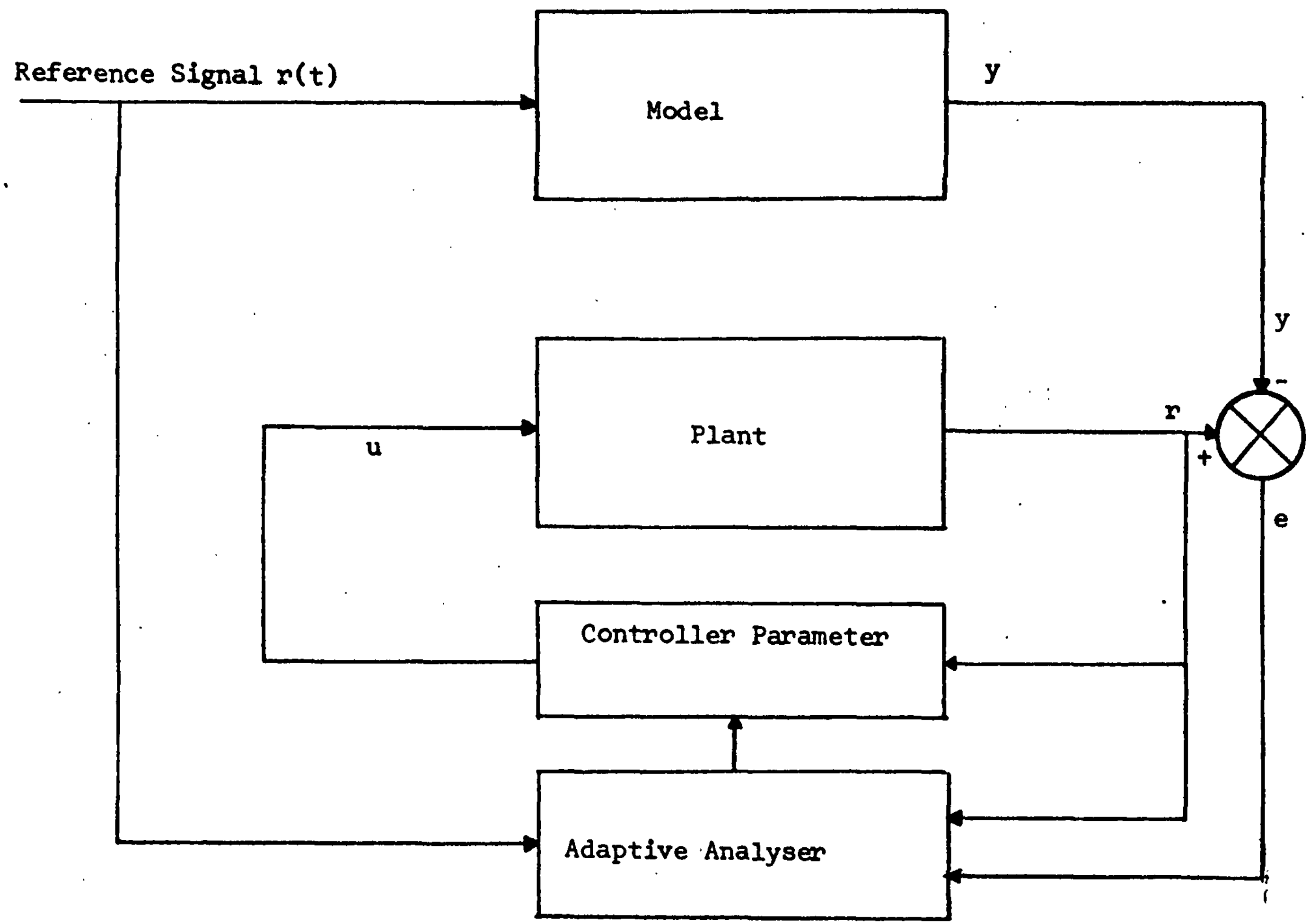

Figure 1.1 - A Typical Adaptive Control System

formance. The tracking error e is made small by adjusting controller parameters in an appropriate way. The adaptive analyser senses plant and model states and reference signal to provide proper variation of the controller parameters. The control structure design is based on the minimization of the integral square error between the plant and model outputs. This technique of adjusting controller parameters is referred to in the literature as the 'M.I.T.' rule. Similar techniques have been proposed by Krasovskii and may be found in the writings of Meerov[40]. These adaptive schemes may possess severe instrumentation and stability problems.

In another direction, linear regulator design has been shown to be an elegant design tool[4]. The control function $\underline{u}$, with $|u_i| \leq 1$, $i = 1,2,\ldots n$ is selected to minimize the performance index

$$\phi(u) = \int_0^\infty \underline{x}^T Q \underline{x}$$

subject to

$$\dot{x} = Ax + Bu.$$

Q is positive definite and the unforced system, $\dot{x} = Ax$, has been assumed to be stable. Bass[4] has also suggested a number of additional techniques for nonlinear regulator problems of the form:

$$\dot{x} = Ax + bf(\sigma)$$

where σ is a linear feedback of all the state variables. This technique says nothing about stability of the given system since the control function $f(\sigma)$ decreases to zero as the system approaches equilibrium. Another interesting technique has been suggested by Lee[32] and Geiss[19] for systems of the form:

$$\dot{x} = f(x) + u$$

where the unforced system is stable, but not asymptotically. The resulting control u is designed to insure asymptotic stability, as well as to maintain a maximum control amplitude or design for minimum time to reach the origin x=o. An alternative approach to satisfactory solution of this problem combines Lyapunov theory[26] with the model-reference concept. A control signal is generated via a design scheme, using a dynamic model to guarantee that the plant follows the model. It is not necessary to identify the plant to implement the control strategy, although this technique may be used as parameter identification scheme as suggested by Rang[53]. Rang assumes the process described by

$$\dot{x} = Ax + Bu$$

where A and B are constant but unknown matrices. The model is taken to be asymptotically stable and of the form

$$\dot{y} = A_o y + B_o r.$$

An error function defined as e = x-y satisfies the following

$$\dot{e} = A_o e + (A-A_o)x + Bu - B_o r$$

A Liapunov function,

$$V = e^T pe + ||S_1{}^T(D - A_o)S_2\ || + \ ||S_3{}^T(F - B_o)S_4\ ||,$$

is then chosen, where S_1, S_2, S_3, S_4 are constant column vectors and D,F will be determined to approximate A and B by constraining the time derivative $\dot{V}$ of V to be negative definite and setting indefinite terms to zero.

Grayson[21,22] synthesizes an algebraic (memory less) relay type controller by applying Liapunov's second method. Since Grayson's work, a number of generalizations and refinements of the technique have been made[5,42-46]. The efforts mainly have been centered on the elimination of higher order derivatives from the control signal. Motivation of this stems from the fact that it usually simplifies the hardware necessary to realize the controller and also reduces the noise level associated with differentiation involved in controller's implementation. Monopoli[42-45] and Lindorff[37] have shown that, in some cases, the following possibilities exist:

1) Some or all of the plant state variables may be replaced by the corresponding model state variables,
2) the need for some higher order derivatives may be eliminated entirely,
3) a reduction in the gain associated with the higher order derivatives will reduce the adverse effects of measurement noise.

Lindorff has also successfully extended the Liapunov synthesis technique to multivariable system when there are no input derivatives. A reduction in instrumentation noise level and the problem of noise rejection has also been treated by Lindorff[36]. This is especially directed to the systems which are not in phase variable form and where the parameter uncertainties exist. But it seems that the technique is restricted due to the requirement for special relationship. Nonlinearities in the system are not permitted. Confronted with the problem of controlling a plant which is imperfectly identified, Taylor[59,60] has obtained a realistic error bound for reduced order model-reference controller. Nikiforuk et. al.[46] extended the model reference control synthesis technique to plant with unknown nonlinearities and unknown parameters. The controller has been synthesized via a reduced order model. The resulting controller is highly nonlinear. The technique is applicable to single-input, single-output minimum phase* type systems. The limitation of the above techniques based on Liapunov's second method is not only the complexity of the controller structure, but also the lack of any insight as how to determine the control amplitude.

Winsor and Roy[64] combined optimal control and trajectory sensitivity to develop a design of desensitized model following control system. The control is generated by minimizing a quadratic performance index involving the error (between the plant and model output) and the control. Complete information regarding the plant, availability of all the state variables necessary to control the system, and model transfer relation being of same order as that of the plant, are assumed.

A somewhat different approach has been suggested by Donalson and Leondes[15]. They have selected a variable controller (controller has variable parameters) which causes the form of the plant's transfer function to match that of the model. The control law is synthesized by minimizing a quadratic performance index involving the error and its derivatives by following the path of steepest descent for f(e). This method has produced a controller with memory, i.e., the controller contains integrators. A discussion on overall stability of the system has been suggested in [15]. Shackcloth and Butchart[14] have also selected a variable controller by choosing a Liapunov function of the form:

$$V = e^T pe + Z^T MZ$$

where P and M are positive definite and symmetric and Z is the misalignment vector. The time derivative $\dot{V}$ of V, evaluated along the trajectory is constrained to be negative definite. Parks[48] has also suggested a similar technique and has shown clearly that the adaptive

*Minimum phase type system implies the system with lef--half plane zeros.

technique based on M.I.T. rule might result to an unstable controller. He designes the controller so as to insure, under certain conditions, asymptotic stability. Many extensions and generalizations of the above technique have been made and may be found in the literature[16,17,27,31,52,57,67,]

1.2 PROBLEM STATEMENT AND ORGANIZATION OF WORK

To counter various limitations of the techniques mentioned earlier, the present thesis deals initially with the problem of controlling a dynamic single input - single output system having parameter uncertainty using a minimax technique. Specifically, it treats the design of a controller using a nominal model to insure a satisfactory performance of the system in spite of ignorance of system parameters. The problem is posed with the additional constraint that the controller be linear and that it require only partial state feedback.

The system is described by n^{th} order differential equation

$$x_1^{(n)} + a_{n-1} x_1^{(n-1)} + \ldots + a_o x_1 = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_1 \dot{u} + b_o u + f(x_1, \ldots x_1^{(n-m-1)}, t)$$

where x_1 and u are the output and input to the system, superscript (n) denotes the number of derivatives, and $f(x_1, \dot{x}_1, \ldots x_1^{n-m-1}, t)$ is a nonlinear function.

Next we define a lower order stable model

$$y^{(n-m)} + \alpha_{n-m-1} y^{(n-m-1)} + \ldots + \alpha_o y = \beta_o r$$

where y and r are the model output and reference input respectively. A schematic diagram of this model reference system is shown in Figure 2.2.

Error $\underline{e}$ is defined as

$$\underline{e} = (x_1, \dot{x}_1 \ldots x_1^{n-m-1})^T - (y, \dot{y}, \ldots y^{n-m-1})$$

If the system is minimum phase type and $f(\cdot)$ is a bounded continuous nonlinear function, it is shown that the error $\underline{e}$ can be bounded with an arbitrarily small bound, despite imperfect knowledge of a_i, b_k, $i=0,1,\ldots,n, k=0,1,\ldots m$. Reduction of the error bound, however, requires in general greater control amplitudes. Furthermore this is achieved by a linear feedback obtained by minimizing with respect to control and maximizing with respect to a signal ξ relating to the uncertainty, a quadratic performance criterion of the form

$$J = \frac{1}{2} \int_0^\infty [e^T Q\underline{e} + u^T Ru - \xi^T L\xi \,] \, dt \quad .$$

The resulting control is linear and the number of states required to generate the control is equal to system order less the number of zeros. This discussion is the subject of Chapter 2.

One of the shortcomings of the minimax approach is that the controller requires the output of the plant and its derivatives up to (n-m-1). When some of the output derivatives are not available, a reduced order dynamic compensator is designed using a minimax technique. The input to the dynamic system is the available states or output of the plant and its output is the required control signal. If the output observation is noisy, differentiation of the output signal to generate the control is no longer possible.

In this case, estimation of states becomes complex due to lack of information regarding the system matrix. One way of approaching this problem is to obtain an optimal mean-square error estimate of the states under specified parameter uncertainty[39]. This class of adaptive estimation problem constitutes a class of nonlinear estimation problems and the resulting estimator gains require the solution of a set of simultaneous partial differential equations. In most cases, a closed form solution does not exist and hence the estimator is difficult to realize. Recently, a simplified closed form solution of this type of adaptive filter has been suggested[30]. This thesis reports an ad hoc estimation scheme to generate an estimate of the necessary states from the noisy observations using a deterministic model. The estimator is linear. Chapter 3 contains design of reduced order dynamic compensator and estimator for systems with parameter uncertainty. Another drawback of the present minimax procedure is the apparent difficulty in extending the concept to multivariable cases. The difficulty lies in the fact that a suitable canonical form for multivariable case is not currently available. The general situation becomes much more complex, due not only to multivariable nature of the problem, but also the presence parameter uncertainty, disturbance in the system and noisy measurement. One way of tackling the problem is to assume a form for the controller structure. Optimization techniques may then be employed to develop algorithms which yield optimum values of the parameters of the controller. This thesis will assume a linear structure as illustrated in Figure 1.2.

Athans, Levine, [33-35] Johnson[25] and Kosut[28] have discussed a simplified version of the above problem. A dynamic compensator of specified order is used along with the output feedback to provide the control. Matrices G,H,P,N are chosen so as to minimize an integral quadratic criterion.

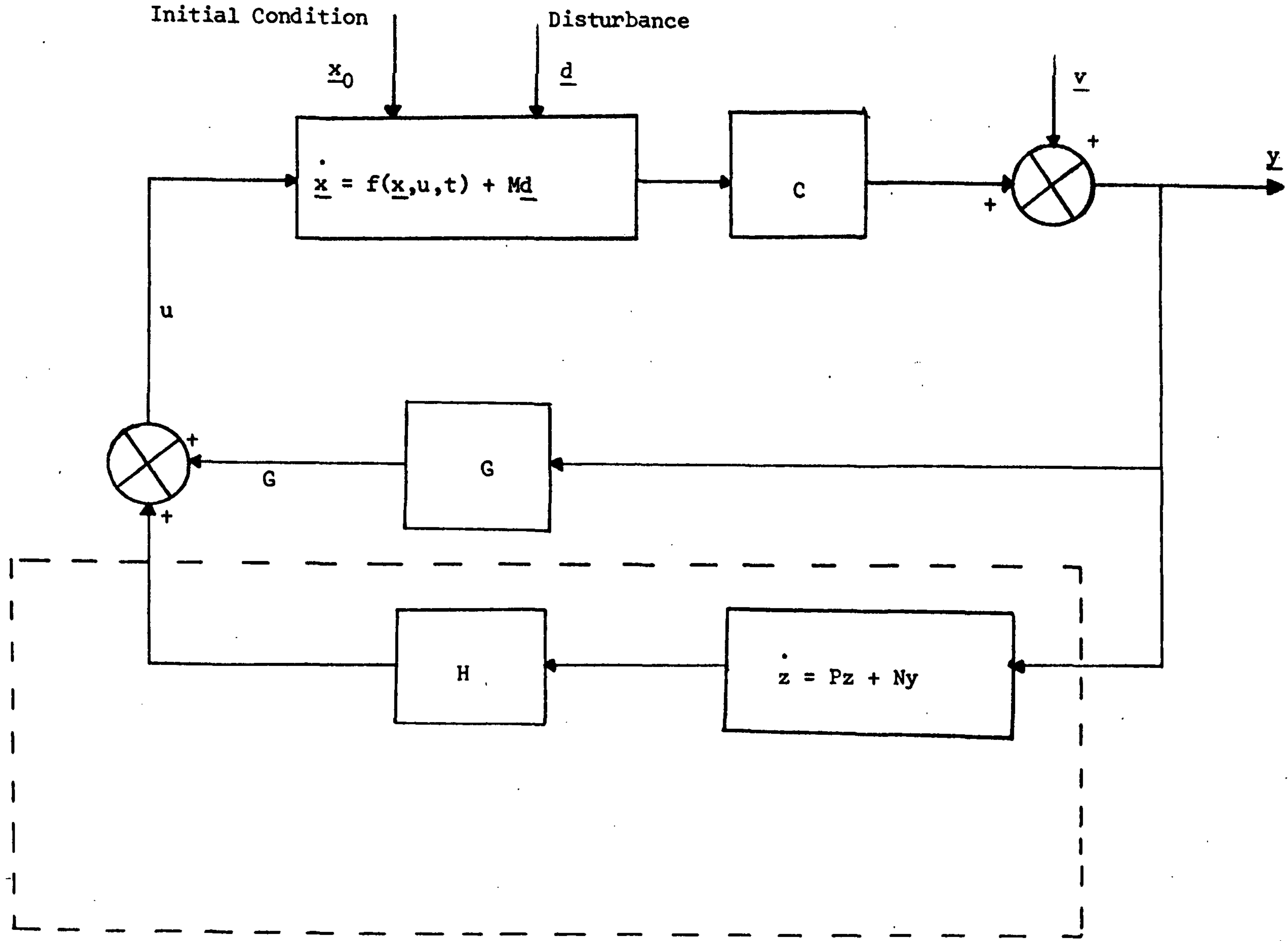
Initial Condition
$\underline{x}_0$
Disturbance
$\underline{d}$
$\dot{\underline{x}} = f(\underline{x},u,t) + M\underline{d}$
C
$\underline{v}$
+
+
$\underline{y}$
u
G
G
+
+
H
$\dot{z} = Pz + Ny$

Figure 1.2

The result is a set of simultaneous nonlinear algebraic matrix equations which must be solved recursively for G,H,P,N. Basic limitations of their approaches are:

1) A linearized system is assumed.

2) Disturbances, $\underline{d}$, and measurement noise $\underline{n}$ are not considered.

3) Parameter uncertainty is not considered.

4) Gain matrices depend on the initial state, $\underline{x}_o$ of the plant. Generally this is handled in [33,28]by assuming that the initial state is random variable with a known covariance matrix.

5) It is not generally known a prioiri whether a dynamic compensator can stabilize the system unless the dimension of $\underline{y}$ and $\underline{z}$ together equals that of $\underline{x}$.

Maclane[38] has considered a stochastic version of the problem to handle the disturbance. The tracking problem can be treated within this formulation by assuming that the desired inputs may be generated by the initial condition response of a linear system. This, however, has not been done. Goldstein[23] has designed a minimal order observer to yield an estimate of $\underline{x}$ which, in turn, is used to obtain the control. No disturbance or measurement noise is allowed. Ferguson and Rekasius[18], Pearson[50], Pearson and Ding[51], Brasch and Pearson[12] also use a dynamic compensator of suitable order to provide an optimal control. The gain matrices are independent of initial condition.

The present thesis deals with the problem as schematically shown in Figure 1.2. Specifically, the problems of parameter uncertainty and the presence of disturbance and measurement noise have been considered. The objective is to specify various gains G,H,P and N using optimization technique. The initial step is to design a controller for a linear dynamic multi input - multi output system having parameter uncertainty using a minimax procedure. The basic study here is to examine various minimax criteria so that the system behaves acceptably well over a wide range of parameter variation using only output feedback. Thus the dee sign procedure involves the specification of G assuming H = 0. Minimax controller design for this class of problems using complete state feedback has been suggested by many authors [29,41,56,58,61]. The present approach treats this problem by minimizing with respect to a feedback gain matrix and maximizing with respect to uncertainty, a quadratic performance index involving the system state, the control and a signal related to the uncertainty. The optimal feedback gain satisfies a set of nonlinear algebraic matrix equations. Several other minimax approaches are then considered to relax the conservativeness of the previous formulation. It has been demonstrated that the minimax design criteria, under wide parameter variation, yield better performance than a purely nominal design. This is the subject of Chapter 4.

Next the thesis treats the problem of designing a generalized controller for systems excited by white noise disturbance. The measurements are assumed to be contaminated with white noise of known variance. It is well known that if the system is linear, is excited by white gaussian noise and the measurement noise is gaussian, the estimator and the controller can be designed independently. This is due to the well-known separation theorem. The estimator is the well-known Kalman filter whose dimension is equal to that of the system. Sometimes the dimensionality of Kalman filter restricts its use in practice because of added computational difficulty. With these points in perspective, the thesis deals with the design of a generalized controller (combined estimator and controller) of specified dimension. It is assumed that the system is perturbed by white noise and the output observations are contaminated with white noise signal of known variance. No parameter uncertainty is assumed. The dynamic linear controller operating on the available noise corrupted outputs of the system generates the required control input to the system. The design involves determination of optimal values of G,H,P and N. Chapter 5 deals with this problem and presents a simplified analysis of estimator and controller combined. Various special cases are also discussed.

Chapter 6 summarizes the achievement of the present work, its shortcomings and possible extensions to more general problems.

II MINIMAX MODEL REFERENCE CONTROL IN SUBSTATE SPACE

2.1 INTRODUCTION

The design of a controller for a dynamic single input - single output system having parameter uncertainty will be undertaken in this chapter using a minimax technique. Specifically, it will be shown that a controller can be designed using a reduced order model to insure a satisfactory performance of the system in spite of ignorance of system parameters.[8] The problem is approached by minimizing and maximizing with respect to the control and an "uncertainty signal" respectively, a quadratic performance index involving the tracking error, the control, and the "uncertainty signal". The resulting controller is linear. The number of states required to generate the control is equal to the system order less the number of zeros. Bounded input - bounded output stability is guaranteed, provided the system transfer function in minimum phase type. The results also apply for systems with rather general nonlinearities that do not involve the control. It is also shown that the tracking error admits an upper bound and that the bound can be made arbitrarily small with adequate control levels.

2.2 MOTIVATION OF THE PROBLEM

Consider the single input - single output system described as in (2.1) - (2.3).

$$\frac{X_1(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)} \quad m<n \qquad (2.1)$$

where $X_1(s)$ and $U(s)$ represent the Laplace transforms of output $x_1(t)$ and input $u(t)$ respectively. The equation (2.1) may be expressed in the state variable form[3]

$$\underline{\dot{x}} = A\underline{x} + \underline{h}u = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ h_{n-m} \\ \vdots \\ h_n \end{bmatrix} u \qquad (2.2)$$

$$\text{where } h_k = \begin{cases} 0 & k<n-m \\ b_{n-k} - \sum\limits_{i=n-m}^{k-1} a_{n-k+i}h_i & k \geq n-m \; . \end{cases}$$

Since $x_{i+1} = \dot{x}_i$, $i=1,2,\ldots,n-m-1$, it can be seen from (2.1) that

$$\frac{X(s)}{U(s)} = (sI-A)^{-1}\underline{h} = \frac{1}{D(s)} \begin{bmatrix} N(s) \\ sN(s) \\ \vdots \\ s^{n-m-1}N(s) \\ N_{n-m}(s) \\ \vdots \\ N_n(s) \end{bmatrix} \qquad (2.3)$$

where $N_{n-k}(s)$, $k=0,1,\ldots,m$, are polynomials in s. Consider now the use of a linear feedback law

$$u = -k_0 [k_1x_1 + k_2x_2 + \cdots + k_{n-m}x_{n-m}] = -k_0\underline{k}^T\underline{x} \qquad (2.4)$$

where $\underline{k}^T = [k_1,k_2,\ldots,k_{n-m}, 0, \ldots 0]$ is a constant vector.

The eigenvalues of the closed loop system are then solutions of

$$0 = |sI - A + k_0\underline{hk}^T| = |sI-A|\,|I + k_0 (sI-A)^{-1}\underline{hk}^T|$$

$$= |sI-A|\,(1 + k_0\underline{k}^T (sI-A)^{-1}\underline{h}) \qquad (2.5)$$

where $|A|$ is the determinate of matrix A.

The last equality of (2.5) is obtained using the identity

$$|I + \underline{C}\,\underline{D}^T| = 1 + \underline{D}^T\underline{C} \; ; \qquad (2.6)$$

$\underline{C}$, $\underline{D}$ being vectors of compatible dimensions. Furthermore, combining (2.3) and the definition of $\underline{k}$ with (2.5) yields

$$|sI-A + k_0\underline{hk}^T| = D(s)\left[1 + k_0\frac{N(s)(k_1 + k_2 s + \dots + k_{n-m}s^{n-m-1})}{D(s)}\right]$$

$$= D(s)\left[1 + k_0\,\frac{N(s)\,k(s)}{D(s)}\right] = 0 \qquad (2.7)$$

(2.7) can also be obtained using Figure 2.1 which illustrates the system with feedback. The characteristic equation of the closed loop system is

$$1 - \text{Loop Gain} = 1 + k_0\underline{k}^T (sI-A)^{-1}\underline{h} = 0,$$

which agrees with (2.5) and (2.7). We know as $k_0 \to \infty$, that zeros of (2.7) approach the n-1 finite zeros of $N(s)\cdot k(s)$ and one zero at $-\infty$. Hence if $N(s)$, $k(s)$ are Hurwitz polynomials with b_m, $k_{n-m} > 0$, the system is stable for k_0 sufficiently large - regardless of the zeros of $D(s) = |sI-A|$. Furthermore, response to any bounded input $R(s)$ will be bounded. The problem is to choose the nonzero elements of $\underline{k}$ so that in addition to stability, the system exhibits behavior which is in some sense good. Furthermore we should like a design procedure which can yield this good behavior despite uncertainties in the system parameters and with minimal control effort.

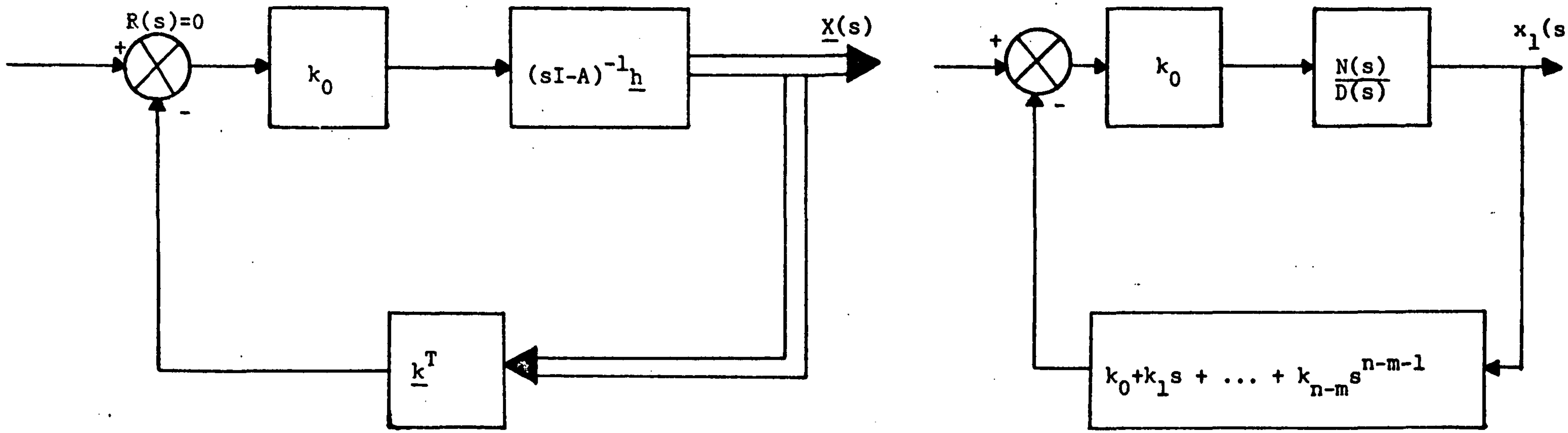

Figure 2.1 - A System with Incomplete State Feedback

2.3 SYSTEM DESCRIPTION, DEVELOPMENT AND PROBLEM FORMULATION

Define a stable model

$$\dot{\underline{y}} = A_0\underline{y} + \underline{\beta}_0 r = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-m-1} \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta_0 \end{bmatrix} r \qquad (2.8)$$

and a system

$$x_1^{(n)} + a_{n-1}x_1^{(n-1)} + \dots + a_0 x_1 = b_m u^m + b_{m-1}u^{m-1} + \dots + b_1\dot{u} + b_0 u + f(x_1,\dot{x}_1,\dots,x_1^{(n-m-1)};t) \qquad (2.9)$$

where $\underline{y}$ and r are the model output and reference input respectively, and $f(\cdot)$ is a nonlinear function. A block diagram representation of the model reference system is shown in Figure 2.2.

If

(1) $b_m \geq \beta > 0$ and $b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0$ is Hurwitz, and if

(2) $f(x_1,\dot{x}_1,\dots, x_1^{(n-m-1)};t)$ is a bounded continuous nonlinear function, the objective now is to show that the error

$$\underline{e} = (x_1,\dot{x}_1,\dots,x_1^{(n-m-1)})^T - \underline{y} \triangleq \overset{*}{\underline{x}} - \underline{y} \qquad (2.10)$$

can be bounded with an arbitrarily small bound, despite imperfect knowledge of a_i, b_k, $i = 0,1,\dots,n, k = 0,1,\dots m$. This will be achieved by a linear feedback law which requires only partial state feedback and is in a sense optimal. Furthermore u will be similar in form to (2.4) with k(s) Hurwitz.

Remark 2.1.

a) n,m, are integers which may be unknown but the difference (n-m)

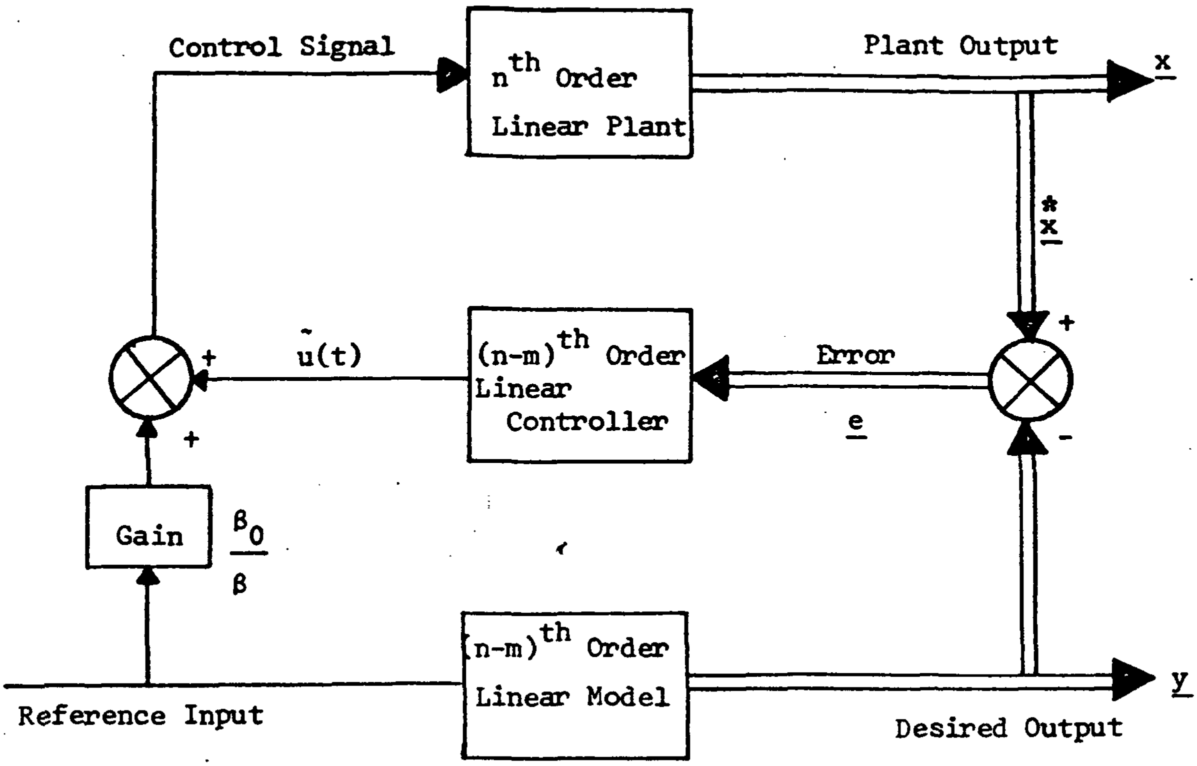

Figure 2.2 - Model Reference System

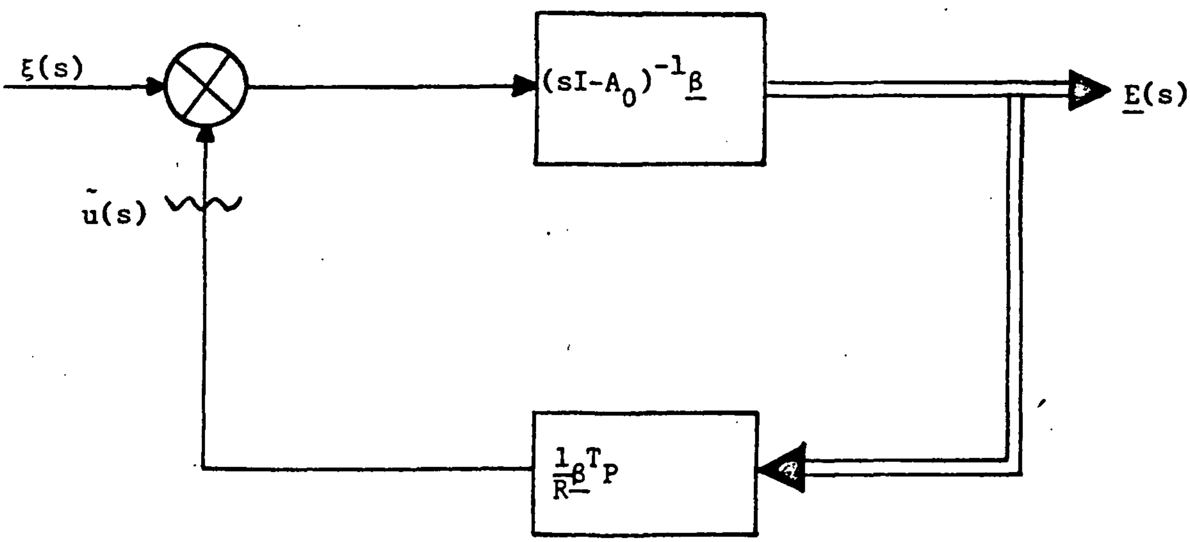

Figure 2.3

is assumed to be known. Also a lower bound of the highest order derivative coefficient of u is available.

b) Note that (2.9) is identical to the linear system (2.1) except for addition of the nonlinear term.

c) When a regulator is being designed, it is admissable to let $r=\underline{y}=0$.

We proceed by rewriting (2.9) to obtain

$$x_1^{(n)} + \alpha_{n-m-1} x_1^{(n-1)} + \ldots + \alpha_0 x_1^{(m)} - \beta u^m = \sum_{j=0}^{m-1} b_j u^{(j)} + (b_m - \beta) u^m + \sum_{j=m}^{n-1} \alpha_{j-m} x_1^{(j)} - \sum_{j=0}^{n-1} a_j x_1^{(j)} + f(x_1, \ldots, x_{n-m} t) \qquad (2.11)$$

Integrating each side m times, gives

$$x_1^{(n-m)} + \alpha_{n-m-1} x_1^{(n-m-1)} + \ldots + \alpha_0 x_1 = \beta u + \xi(t) \qquad (2.12)$$

where $\xi(t)$ is the m fold integral of the right side of (2.11) together with initial condition terms. In state variable form, (2.12) may be written

$$\dot{\underline{x}}^* = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_0 & -\alpha_1 & \alpha & \ldots & -\alpha_{n-m-1} \end{bmatrix} \underline{x}^* + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \beta \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \xi(t) \end{bmatrix}$$

$$= A_0 \underline{x}^* + \underline{\beta} u + \underline{\xi}(t), \qquad (2.13)$$

where $\overset{*}{\underline{x}}$ corresponds to the first n-m elements of $\underline{x}$ as in (2.10)

i.e. $$\overset{*}{\underline{x}} = [x_1, \dot{x}_1, \ldots, x_1^{(n-m-1)}]^T = [x_1, x_2, \ldots, x_{n-m}]^T \,. \qquad (2.14)$$

Consequently, subtracting (2.8) from (2.13) yields

$$\dot{\underline{e}} = (\dot{\overset{*}{\underline{x}}} - \dot{\underline{y}}) = A_0 e + \underline{\beta}\,(\tilde{u} + \tilde{\xi}),\ \tilde{u} = (u - \frac{\beta_0}{\beta} r), \tilde{\xi} = \frac{1}{\beta}\xi. \qquad (2.15)$$

$\tilde{\xi}(t)$, of course, generates differences between the model and $\overset{*}{\underline{x}}$.
If $\tilde{\xi} = 0 = \tilde{u}$, $\underline{e}(t)$ approaches zero asymptotically since A_0 has negative eigenvalues. Furthermore, even if $\tilde{u} = 0$, $\tilde{\xi} \neq 0$, $\underline{e}$ will be bounded for $\tilde{\xi}(t)$ bounded. The problem is to realize a $\tilde{u}$ which not only will retain stability but allow a bound on $\underline{e}$ to be made arbitrarily small.

2.4 CONTROLLER DESIGN

If the current and future values of $\xi(t)$ were known for all time, the 'ideal' optimal control would be obtained by minimizing

$$J = \frac{1}{2} \int_0^\infty (\underline{e}^T Q \underline{e} + Ru^2)dt \qquad (2.16a)$$

subject to (2.15), Unfortunately such a priori knowledge about $\xi(t)$ is not available. Although it may be argued that $\tilde{\xi}(t)$ can be generated from (2.15) once $\tilde{u}$ is known, one still needs to know $\xi(t)$ ahead of time to solve the optimization problem. If a bound on $\xi(t)$ were known, one way of designing a minimax controller would be to maximize w.r.t. $\xi(t)$ and then minimize w.r.t. $\tilde{u}$, the performance criterion (2.16a) subject to (2.15). But it is clear that an a priori bound on $\xi(t)$ is difficult to ascertain. An indirect way of penalizing $\xi(t)$ can be achieved by modifying the criterion (2.16a) to include $\tilde{\xi}$. The form used will be

$$J = \frac{1}{2} \int_0^\infty (\underline{e}^T Q \underline{e} + \tilde{u}^2 R - \tilde{\xi}^2 L)dt \tag{2.16b}$$

with R, L > 0 and Q positive definite. $\tilde{u}, \tilde{\xi}$ will be chosen to minimize and maximize J respectively. Although $\tilde{\xi}$ is not arbitrary, the design assumes that $\tilde{\xi}$ acts in the worst possible fashion and thus maximizes J. The term $-\tilde{\xi}^2 L$ is introduced to limit $\tilde{\xi}$. It is readily shown that the optimum $\tilde{u}$, $\tilde{\xi}$ are given by[20]

$$\overset{*}{\tilde{u}} = -\frac{1}{R} \underline{\beta}^T P \underline{e}, \quad \overset{*}{\tilde{\xi}} = \frac{1}{L} \underline{\beta}^T P \underline{e} \tag{2.17}$$

where P is the symmetric matrix satisfying

$$PA_0 + A_0^T P + Q - P\underline{\beta\beta}^T P \left(\frac{1}{R} - \frac{1}{L}\right) = 0 \quad . \tag{2.18}$$

Furthermore, (i) $A_0 - \frac{1}{R} \beta\beta^T P$ has negative eigenvalues

(ii) P is the unique positive definite steady state solution of $-\dot{P} = PA_0 + A_0^T P + Q - P\beta\beta^T P \left(\frac{1}{R} - \frac{1}{L}\right)$, $R \leq L$.

(iii) $J[\underline{e}_0; \overset{*}{\tilde{u}}, \overset{*}{\tilde{\xi}}] = \min_{\tilde{u}} \max_{\tilde{\xi}} J[\underline{e}_0; \tilde{u}, \tilde{\xi}] = \max_{\tilde{\xi}} \min_{\tilde{u}}$

$$J[\underline{e}_0; \tilde{u}, \tilde{\xi}] = \frac{1}{2} \underline{e}_0^T P \, \underline{e}_0$$

(2.17) will, in one sense, yield a conservative design since $\tilde{\xi}$ is assumed to act in the most perverse manner. In another sense, however, $\tilde{\xi}$ need not abide by the rules of the game and may be using a smaller L than assumed. This may require that R in turn be decreased.

Remark 2.2

It should be noted that a unique positive definite solution P of (2.18) exists[66] if $R \leq L$. In case, $R < L$ the implication is that we are trying to balance the effect of the "uncertainty signal" with a larger amplitude of control signal than what is needed with R = L. Consequently for much of the rest of the chapter, we shall restrict ourselves to the case R = L. In case R<L, it is still possible to show that proposition 2.1 (mentioned below) is true. In case R>L, nonuniqueness of Solution P of (2.18) poses some problem in the subsequent analysis of the error bound and is a subject of further investigation.

For convenience, define $\underline{k}$ as the last row or column of P. Then

$$P\underline{\beta} = [\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_{n-m}][00\ldots\beta]^T = \beta\underline{p}_{n-m} \triangleq \beta\underline{k}.$$

From (2.17), $\overset{*}{u}$ is given by

$$\overset{*}{u} = -\frac{1}{R}\beta\underline{p}^T_{n-m}\underline{e} = -\frac{\beta}{R}\underline{k}^T\underline{e} \tag{219a}$$

Accordingly

$$\overset{*}{U}(s) = -\frac{\beta}{R}\underline{k}^T[1,s\ldots,s^{n-m-1}]^T E(s) = -\frac{\beta}{R}k(s)E(s) \tag{2.19b}$$

where

$$k(s) = \underline{p}^T_{n-m}[1,s,s^2,\ldots,s^{n-m-1}]^T .$$

Now it will be helpful, at this stage, to establish an important property of this control signal $\tilde{u}$ that is outlined in the following proposition:

<u>Proposition 2.1</u>

With A_0 stable and as defined in (2.13), k(s) is a stable polynomial for R=L.

<u>Proof:</u>

Consider the system defined by (2.15) and represented in Figure (2.3), The open loop transfer function, z(s), between ξ(s) and u(s), is given by

$$z(s) = [\frac{\underline{\beta}^T}{R}P\,(sI-A_0)^{-1}\underline{\beta}] = \frac{\beta}{R}\underline{\beta}^T P/D_1(s)\,[1,s\ldots s^{n-m-1}] = \frac{\beta^2}{R}\,\frac{k(s)}{D_1(s)} \tag{2.20}$$

where $D_1(s) \triangleq |sI-A_0|$ is a stable polynomial and $k(s)$ is as defined in (2.19b). Now using (2.18) we obtain

$$\begin{aligned} 2\mathrm{Re}[z(s)] = z(\overset{*}{s}) + z(s) &= \frac{1}{R}\,[\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,P\underline{\beta} + \underline{\beta}^T P\,(sI-A_0)^{-1}\beta] \\ &= \frac{1}{R}\,\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,[P(s+\overset{*}{s}) + P\underline{\beta}^T\,(R^{-1}-L^{-1})\,\underline{\beta}P + Q](sI-A_0)^{-1}\underline{\beta} \\ &= 2\{\mathrm{Res}\}\,[\underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}\,P(sI-A_0)^{-1}\underline{\beta}] + \underline{\beta}^T(\overset{*}{s}I-A_0^T)^{-1}Q(sI-A_0)^{-1}\underline{\beta} \end{aligned}$$

for $R=L$.

Here $\overset{*}{s}$ denotes the complex conjugate of s. Since P, Q are positive definite, $\mathrm{Re}[z(s)]$ is nonnegative for $\mathrm{Re}(s)>0$. Therefore the transfer function $z(s)$ is positive real[1] which implies from (2.20) that $k(s)$ is Hurwitz.

2.5 DERIVATION OF A BOUND ON THE ERROR

In order to determine a bound on $\underline{e}$ when $\tilde{u}$ satisfies (2.17) but $\tilde{\xi}(t)$ is arbitrary, let

$$V(e) = \frac{1}{2}\underline{e}^T P\underline{e} \tag{2.21}$$

where P is the positive definite matrix satisfying (2.18). $\underline{e}$ must now satisfy the differential equation (2.15) with $\tilde{u}$ as in (2.19a); that is

$$\dot{\underline{e}} = [A_0 - \frac{\beta}{R}\underline{\beta}^T P]\,\underline{e} + \underline{\beta}\;\tilde{\xi}(t). \tag{2.22}$$

The time derivative of (2.21) is

$$\dot{V}(e) = \frac{1}{2}e^T[A_0 - \frac{1}{R}\underline{\beta\beta}^T P)^T P + P(A_0 - \frac{1}{R}\underline{\beta}\,\underline{\beta}^T P)2]\,\underline{e} + \tilde{\xi}(t)\underline{\beta}^T P\underline{e}$$

$$= -\frac{1}{2}\underline{e}^T Q\underline{e} - \frac{1}{2}\beta^2(\underline{k}^T\underline{e})^2(\frac{1}{L} + \frac{1}{R}) + \tilde{\xi}(t)\beta(\underline{k}^T\underline{e}) \qquad (2.23)$$

where (2.18), (2.19a). and (2.22) have been used to refine the result. Clearly if $|\tilde{\xi}(t)|$ is bounded, (2.23) will be negative for $||\underline{e}||$ sufficiently large and admit an upper bound $||e||$. It is necessary to examine $\tilde{\xi}(t)$, therefore, by considering the behavior of the full system described by (2.9), (2.10), (2.21). It will then be possible to complete examination of (2.23).

2.6 STABILITY OF THE OVERALL SYSTEM

Turning now to the total system, (2.9) may be written in state form basically as in (2.2, (2.14),

$$\dot{\underline{x}} = A\underline{x} + \underline{h}u + \underline{f}(\overset{*}{\underline{x}}, t) \qquad (2.25)$$

with $\underline{f}(\overset{*}{\underline{x}},t) = [0,0,\ldots,0,f(\overset{*}{\underline{x}},t)]^T$.

Now define

$$\underline{x} = [\frac{\overset{*}{\underline{x}}}{\underline{v}}],\ A = \begin{bmatrix} A_* & A_{*v} \\ A_{v*} & A_v \end{bmatrix}$$

$$\underline{Z} = [\frac{e}{\underline{v}}],\ \underline{v} = (v_1, v_2, \ldots v_m)^T \qquad (2.26)$$

and combine (2.25), (2.8) and (2.10), to eliminate $\overset{*}{\underline{x}}$. The result is

$$\dot{\underline{Z}} = A\underline{Z} + \underline{h}u + \begin{bmatrix} A_* & -A_0 \\ A_{v_*} & \end{bmatrix} \underline{y} - \begin{bmatrix} \underline{\beta}_0 \\ \underline{0} \end{bmatrix} r + \underline{f}(\underline{e} + y, t)$$

$$= A\underline{Z} + \underline{h}u + \underline{\nu} \qquad (2.27)$$

where

$$\underline{\nu} = [0,0,\ldots,0, \sum_{i=0}^{n-m} \alpha_{i-1}y_i - \beta_0 r, 0, 0, .., f(\underline{y} + \underline{e}, t) - \sum_{i=1}^{n-m} a_{i-1}y_i]^T.$$

$$(n-m)^{+th} \text{ entry} \tag{2.28}$$

$\underline{\nu}$ is bounded if the model input r, is bounded as assumed. If the "optimal" u given by (2.19a) is used in (2.27), the closed loop system satisfies

$$\dot{\underline{Z}} = [A - \frac{\beta}{R}\underline{h}\,\underline{\hat{k}}^T]\,\underline{Z} + \underline{h}\,\frac{\beta_0}{\beta} r + \underline{\nu}$$

$$= [A - \frac{\beta}{R}\underline{h}\,\underline{\hat{k}}^T]\,\underline{Z} + \underline{\nu}' \tag{2.29}$$

with

$$\underline{\hat{k}}^T = [k_1, k_2, \ldots k_{n-m}, 0,0,..,0]$$

$$\tag{2.30}$$

$$\underline{\nu}' = \underline{\nu} + \underline{h}\,\frac{\beta_0}{\beta} r \overset{\Delta}{=} [0,0,\ldots 0, \nu'_{n-m}\ldots\nu'_n] \;.$$

The characteristic equation of the closed loop system is obtained as in (2.5) - (2.7), i.e.

$$0 = |sI-A + \frac{\beta}{R}\underline{h}\,\underline{\hat{k}}^T| = |sI-A| \quad |I + (sI-A)^{-1}\frac{\beta}{R}\underline{h}\,\underline{\hat{k}}^T|$$

$$= |sI-A|\ (1 + \frac{\beta}{R}\hat{k}^T(sI-A)^{-1}\underline{h})$$

$$= D(s)\ [1 + \frac{\beta}{R}\frac{N(s)}{D(s)}(k_1 + k_2 s + \ldots + k_{n-m}s^{(n-m-1)}]$$

$$= D(s) + \frac{\beta}{R}N(s)\,k(s) \;. \tag{2.31}$$

Now as $R \to 0$, (n-1) roots of (2.31) approach zeros of N(s)k(s), and the last root goes to infinity along the negative real axis. Since N(s), k(s) are "stable polynomials", then (2.30) is stable for R adequately small.

Consequently as the penalty R on control is reduced, permitting larger control amplitudes, the system (2.1) - (2.2) or (2.7) is stable for the feedback law (2.19) provided N(s) is a stable polynomial with $b_m > 0$.

Now since (2.29) yields a bounded $\underline{Z}$ with a bounded input $\underline{\nu}'$, all the elements of $\underline{Z}$ are bounded. In fact, bounds on $\underline{\nu}'$ do not depend on $\underline{Z}$ but are determined mainly by $\underline{y}$. From (2.29) - (2.31) it can be seen that

$$\mathcal{L}(e_i) = E_i(s) = \frac{1}{D(s) + \frac{\beta}{R} N(s)k(s)} \sum_{j=n-m}^{n} f_j^i(s)\, \nu_j'(s) \quad i=1,2,\ldots,(n-m-1)$$

$$\mathcal{L}(v_i) = V_i(s) = \frac{1}{D(s) + \frac{\beta}{R} N(s)k(s)} \sum_{j=n-m}^{n} [g_j^i(s) + \frac{1}{R} \ell_j^i(s)]\nu_j'(s),$$

$$i=1,2,\ldots,m.$$

where $f_j^i(s)$, $g_j^i(s)$ and $\ell_j^i(s)$ are polynomials in s, independent of R, and of order $\leq$ (n-1). $\mathcal{L}(\cdot)$ is the laplace transform of $(\cdot)$.

As $R \to 0$, $|E_i(s)| \to 0$, $i=1,2,\ldots,(n-m-1)$

and

$$|V_i(s)| \to \frac{\sum_{j=n-m}^{n} \ell_j^i(s)\, \nu_j'(s)}{N(s)k(s)} \quad i=1,2,\ldots,m.$$ Thus an ultimate bound on V_1 exists and is independent of R as $R \to 0$. Furthermore the error bound, i.e., the bound on $||\underline{e}||$ goes to zero as $R \to 0$. We explore this further.

2.7 FURTHER RESULTS ON THE BOUND OF THE ERROR

Let us now examine $\xi(t)$. From (2.22)

$$\xi(t) = \beta\tilde{\xi}(t) = \dot{e}_{n-m} + \sum_{i=1}^{n-m} a_{i-1} e_i + \frac{\beta^2}{R} \underline{k}^T \underline{e}$$

$$= \dot{e}_{n-m} + \sum_{i=1}^{n-m} a_{i-1}e_i + \frac{\beta 2}{R} k_i)e_i \quad .$$

From (2.29), on the other hand

$$\dot{z}_{n-m} = \dot{e}_{n-m} = v_1 - \frac{\beta}{R} b_m \underline{k}^T \underline{e} + \sum_{i=0}^{n-m} \alpha_{i-1} y_i - \beta_0 r + b_m \frac{\beta_0}{\beta} r .$$

Thus

$$\xi(t) = v_1 + \sum_{i=1}^{n-m} [a_{i-1}e_i + \alpha_{i-1}y_i] + \beta_0 \left(\frac{b_m}{\beta} - 1\right) r + \frac{\beta}{R} (\beta - b_m) \underline{k}^T \underline{e}$$

$$= \xi_1 + \frac{\beta}{R} (\beta - b_m) \underline{k}^T \underline{e} \tag{2.32}$$

where

$$\xi_1 = v_1 + \sum_{i=1}^{n-m} [a_{i-1}e_i + \alpha_{i-1}y_i] + \beta_0 \left(\frac{b_m}{\beta} - 1\right) r$$

has an upper bound which is independent of R and exists as $R \to 0$. Returning to (2.23) with (2.32) replacing $\xi(t)$,

$$\dot{V}(e) = -\frac{1}{2} \underline{e}^T Q \underline{e} - (\underline{k}^T \underline{e})^2 \left(\frac{b_m \beta}{R} - \frac{\beta^2}{2R} + \frac{\beta^2}{2L} \right) + \xi_1 \underline{k}^T \underline{e}$$

$$\leq -\frac{1}{2} \underline{e}^T Q \underline{e} - \frac{1}{2} (\underline{k}^T \underline{e})^2 \beta^2 \left(\frac{1}{R} + \frac{1}{L} \right) + \xi_1 \underline{k}^T \underline{e}$$

$$\leq -\frac{1}{2} \underline{e}^T Q \underline{e} - \frac{1}{2} (\underline{k}^T \underline{e})^2 \beta^2 \left(\frac{1}{R} + \frac{1}{L} \right) + |\xi_1|_{max} | \underline{k}^T \underline{e} | \tag{2.33}$$

$$\leq -\frac{1}{2} \underline{e}^T Q \underline{e} + \frac{1}{2} \frac{|\xi_1|^2_{max}}{\left(\frac{1}{L} + \frac{1}{R}\right)\beta^2} = W(\underline{e})$$

The last inequality is obtained by maximizing the last two terms in the equality with respect to $|\underline{k}^T \underline{e}|$.

Providing $\underline{e}$ is sufficiently large, the first component of W(e) will dominate. Next $\frac{1}{2} \underline{e}^T \underline{e}$ is maximized subject to $\dot{W}=0$, in order to establish an upper bound on $\underline{e}^T \underline{e}$. For this purpose, it is required to maximize the Hamiltonian

$$H = \frac{1}{2}\underline{e}^T\underline{e} + \eta[-\frac{1}{2}\underline{e}^TQ\underline{e} + c], \quad c \stackrel{\Delta}{=} \frac{1}{2}\frac{|\xi_1|^2_{max}}{(\frac{1}{L}+\frac{1}{R})\beta^2} \tag{2.34}$$

where η is a constant multiplier. Equating $\frac{\partial H}{\partial \underline{e}}$ to zero yields

$$Q\underline{e} = \frac{1}{\eta}\underline{e} = \lambda_Q\underline{e} \tag{2.35}$$

Thus (2.35) shows that $\underline{e}$ and $\frac{1}{\eta}$ are an eignevector and eigenvalue of Q respectively. Since

$$\underline{e}^TQ\underline{e} = 2c = \lambda_Q\underline{e}^T\underline{e}$$

is required,

$$(\underline{e}^T\underline{e})_{max} = 2c/(\lambda_Q)\ min$$

where $(\lambda_Q)_{min}$ is the minimum eignevalue of Q.

Thus $\dot{V} < 0$ for all $\underline{e}^T\underline{e} \geq \dfrac{|\xi_1|^2_{max}}{(\frac{1}{L}+\frac{1}{R})\beta^2\,(\lambda_Q)_{min}}$,

i.e., $\dot{V} < 0$ for all $||\underline{e}|| \geq R_a = \dfrac{|\xi_1|_{max}}{\beta}\sqrt{\dfrac{1}{(\lambda_Q)_{min}\,(\frac{1}{L}+\frac{1}{R})}}$.

(2.36)

This, of course, is conservative. Since we have permitted $\underline{e}$ to both maximize certain terms in (2.33) and to maximize $||\underline{e}||$ subject to $W(\underline{e}) \geq 0$.

The bound on $||\underline{e}||$ can now be found in standard fashion,

(1) Determine V_o = maximum V(e) subject to $||\underline{e}|| \leq R_a$

(2) Determine R_b = max $||e||$ subject to $V(e) \leq V_o$

Thus, to find V_o, we proceed as follows:

Minimize the Hamiltonian H given by

$$H = \underline{e}^T P \underline{e} + \nu_1 [R_a^2 - \underline{e}^T \underline{e}] \tag{2.37}$$

or $P\underline{e} = \nu_1 \underline{e}$

indicating that $\underline{e}$ and ν_1 are eigenvector and eigenvalue of P respectively.

Thus $\underline{e}^T P \underline{e} = \nu_1 \underline{e}^T \underline{e} = \nu_1 R_a^2$

$$\text{and} \quad V_o \overset{\Delta}{=} V_{max} = (\lambda_p)_{max} R_a^2 \tag{2.38}$$

where $(\lambda_p)_{max}$ is the maximum eigenvalue of P.

Now to find R_b, we maximize $R_b^2 = \underline{e}^T \underline{e}$ subject to

$\underline{e}^T P \underline{e} = (\lambda_p)_{max} R_a^2$. Thus

$$H = \underline{e}^T \underline{e} + \nu_2 [(\lambda_p)_{max} R_a^2 - \underline{e}^T P \underline{e}] \tag{2.39}$$

and $\frac{\partial H}{\partial \underline{e}} = 0 = \underline{e} - \nu_2 P \underline{e}$

or $P\underline{e} = \frac{1}{\nu_2} \underline{e}$.

Hence $\underline{e}$ and $\frac{1}{\nu_2}$ are eigenvector and eigenvalue at P.

Furthermore $\underline{e}^T P \underline{e} = (\lambda_p)_{max} R_a^2 = \lambda_p \underline{e}^T \underline{e}$

i.e. $\underline{e}^T \underline{e} = (\lambda_p)_{max} R_a^2 / \lambda_p$

Thus the required R_b is given by

$$R_b^2 = \frac{(\lambda_p)_{max}}{(\lambda_p)_{min}} R_a^2 \tag{2.40}$$

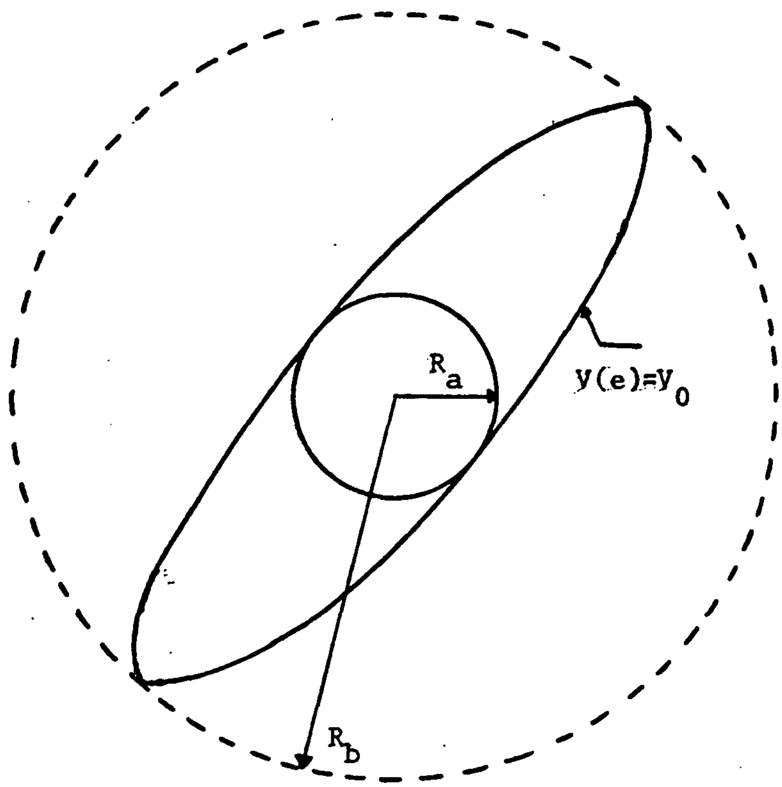

Figure 2.4

and

$$R_b = R_a \sqrt{(\lambda_p)_{max} / (\lambda_p)_{min}}$$

$$= \frac{|\xi_1|_{max}}{\beta} \sqrt{\frac{1}{(\lambda_Q) \min(\frac{1}{L} + \frac{1}{R})}} \tag{2.41}$$

R_b represents an ultimate bound on $||\underline{e}||$ since $\underline{e}$ asymptotically approaches the region defined by $v(\underline{e}) \leq v_o$ but $\dot{v}(e)$ may be indefinite therein. Thus $||\underline{e}||$ is bounded by R_b as shown in Figure (2.4). This bound can be made arbitrarily small by relaxing the penalty R on the control u since $|\xi_1|_{max}$ admits an upper bound as can be seen from the previous section and (2.32).

2.8 EXAMPLES

To illustrate the preceding analysis, consider the following open loop unstable system described by

$$\dddot{x} + \ddot{x} - \dot{x} + x - \sin x + x^2 = \ddot{u} + 2\dot{u} + u \tag{2.42}$$

with a first order model described by

$$\dot{y} + 2y = r \tag{2.43}$$

where r is a step input. It is to be noted that the minimum order of the model is specified by the difference between the number of poles and zeros of the plant. This, in this example, is one. With a little manipulation and integrating (2.42) twice, the error equation can be expressed as

$$\dot{e} + 2e = \xi + \tilde{u} \tag{2.44}$$

$$\tilde{u} = u - r$$

It is desired to find an optimal control $\overset{*}{\tilde{u}}$ by maximizing w.r.t. ξ and minimizing w.r.t $\tilde{u}$ the performance criterion

$$J = \frac{1}{2} \int_0^\infty (4e^2 + R\tilde{u}^2 - L\xi^2)\, dt, \; R=L \tag{2.45}$$

subject to (2.44).

The resulting control is given by

$$\overset{*}{\tilde{u}} = -R^{-1}e \tag{2.46}$$

and is applied to the original system (2.41).

The outputs of the plant and model are shown in Figure (2.5) for different values of R. It can be seen that the error decreases monotonically with the decrease of penalty R. Also plotted are the remaining states $x_2 = \dot{x}$, $x_3 = \ddot{x}$ for different values of R . These are bounded as can be seen in Figure (2.7) and (2.8). The control signal characteristics are shown in Figure (2.9).

2.9 CONCLUSIONS

A linear time invariant controller has been designed for a single input - single output system with parameter uncertainty. The number of states required to generate the control signal is equal to the system order less the number of zeros. The feedback gains are obtained by minimizing a quadratic performance index involving the tracking error, the control signal and the "uncertainty signal". This, however, yields a conservative design since the "uncertainty signal" is assumed to act in the most unfavorable manner. Bounded input - bounded output stability is guaranteed, provided the transfer function is of minimum-phase type. If the uncertainties are bounded,

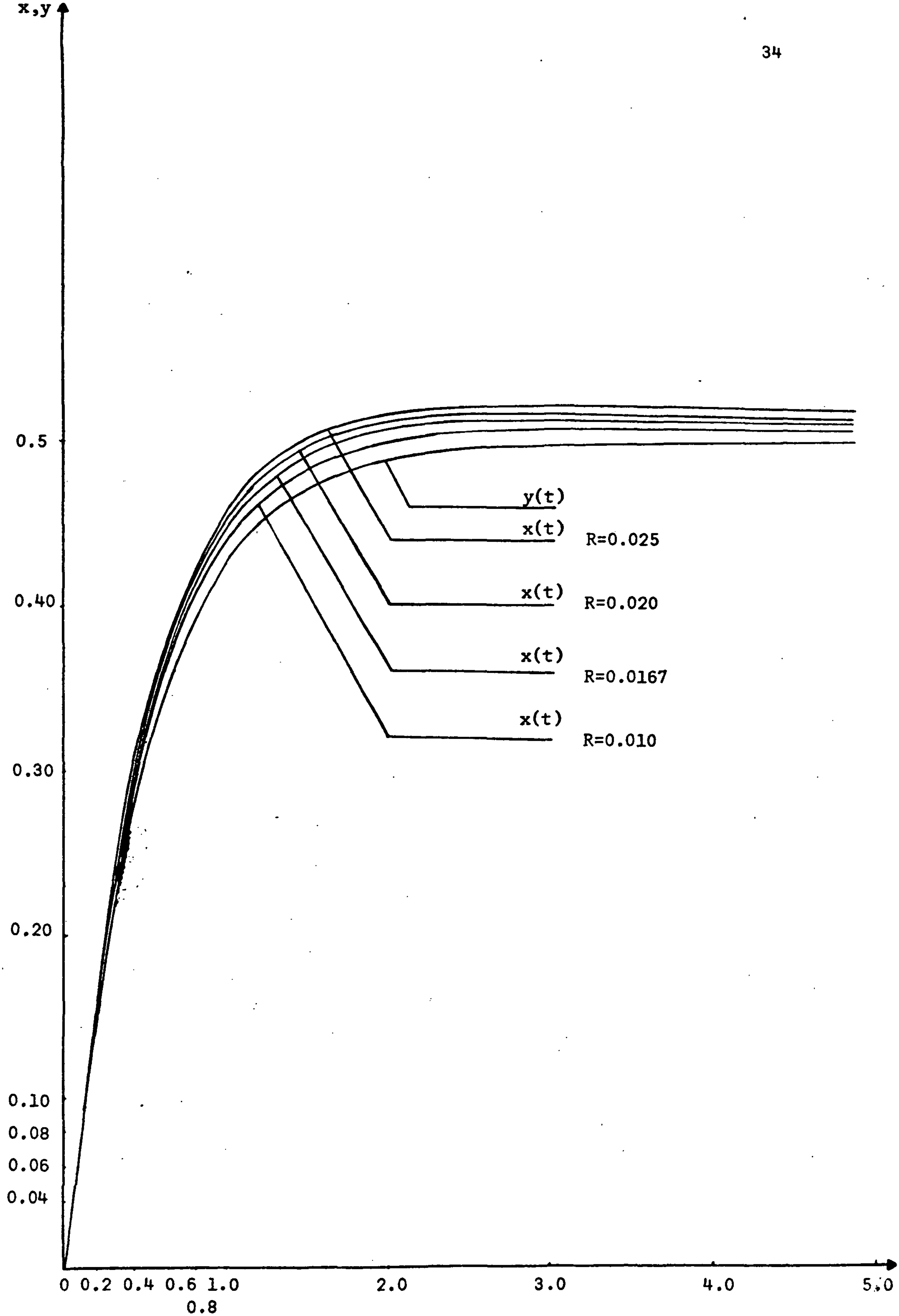

Figure 2.5 System Response for Different Values of R

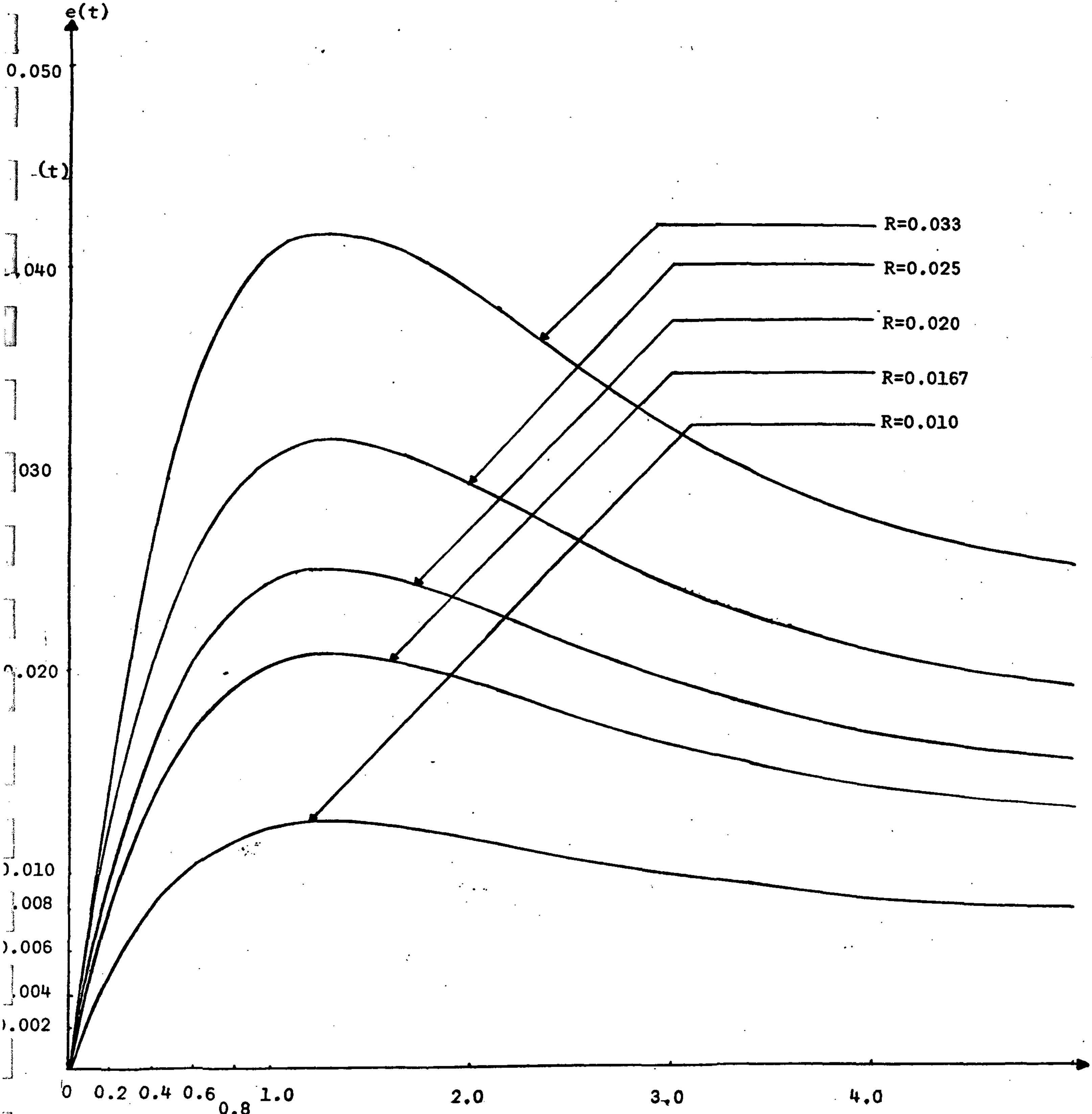

Figure 2.6 - Error Between System Output and Model Output for Different Values of R

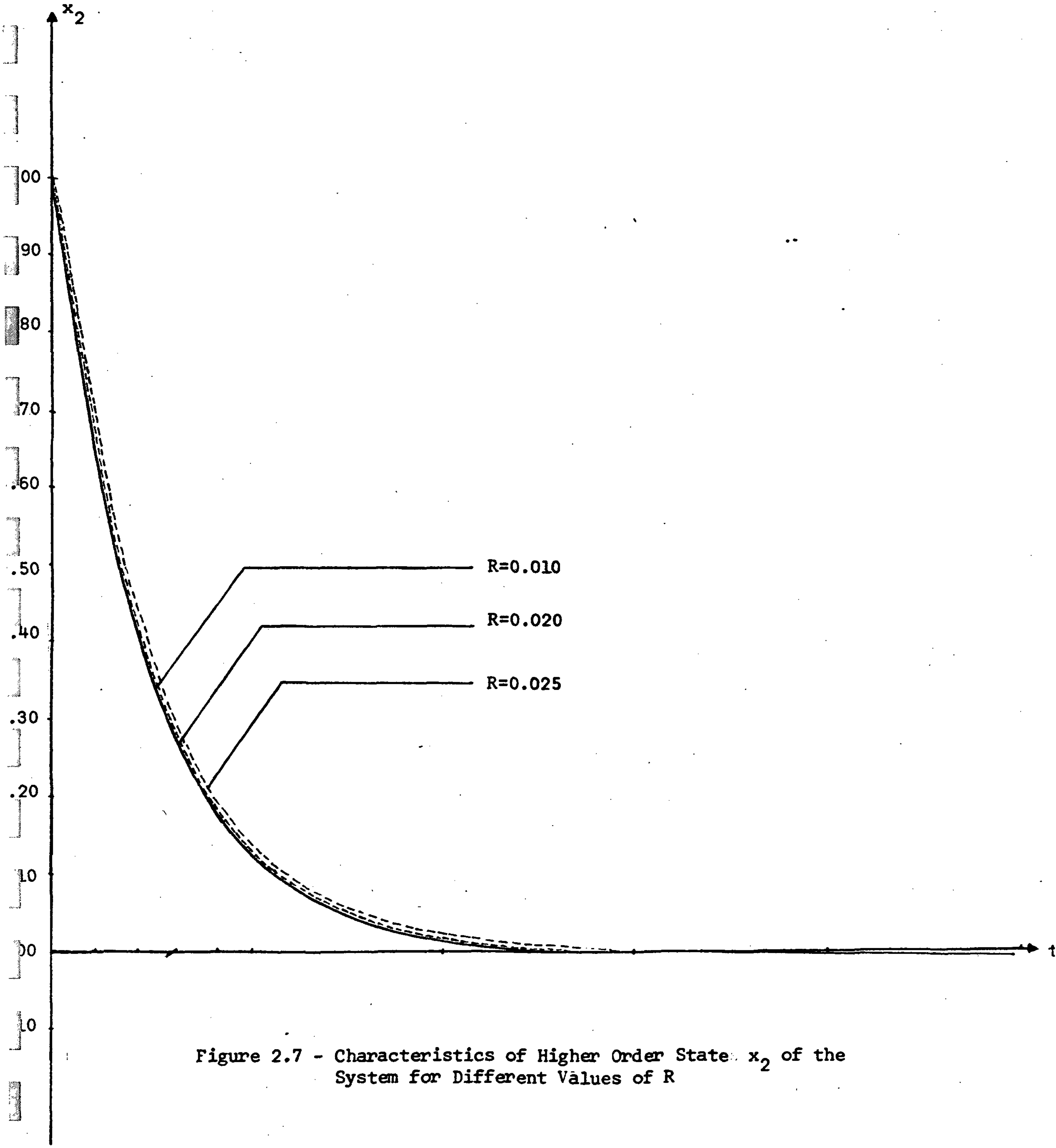

Figure 2.7 - Characteristics of Higher Order State x_2 of the System for Different Values of R

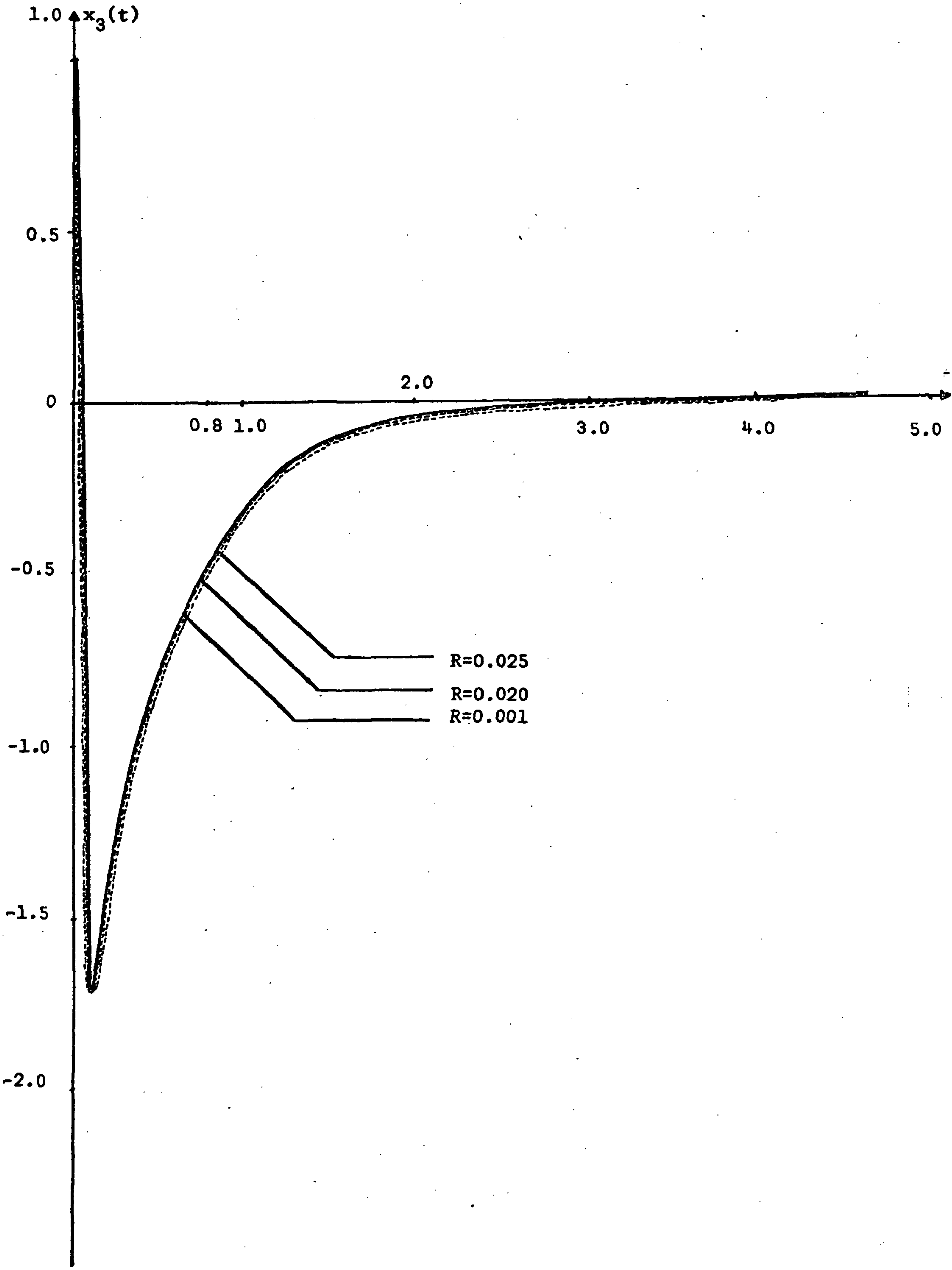

Figure 2.8 - Characteristics of Higher Order State x_3 of the System for Different Values of R

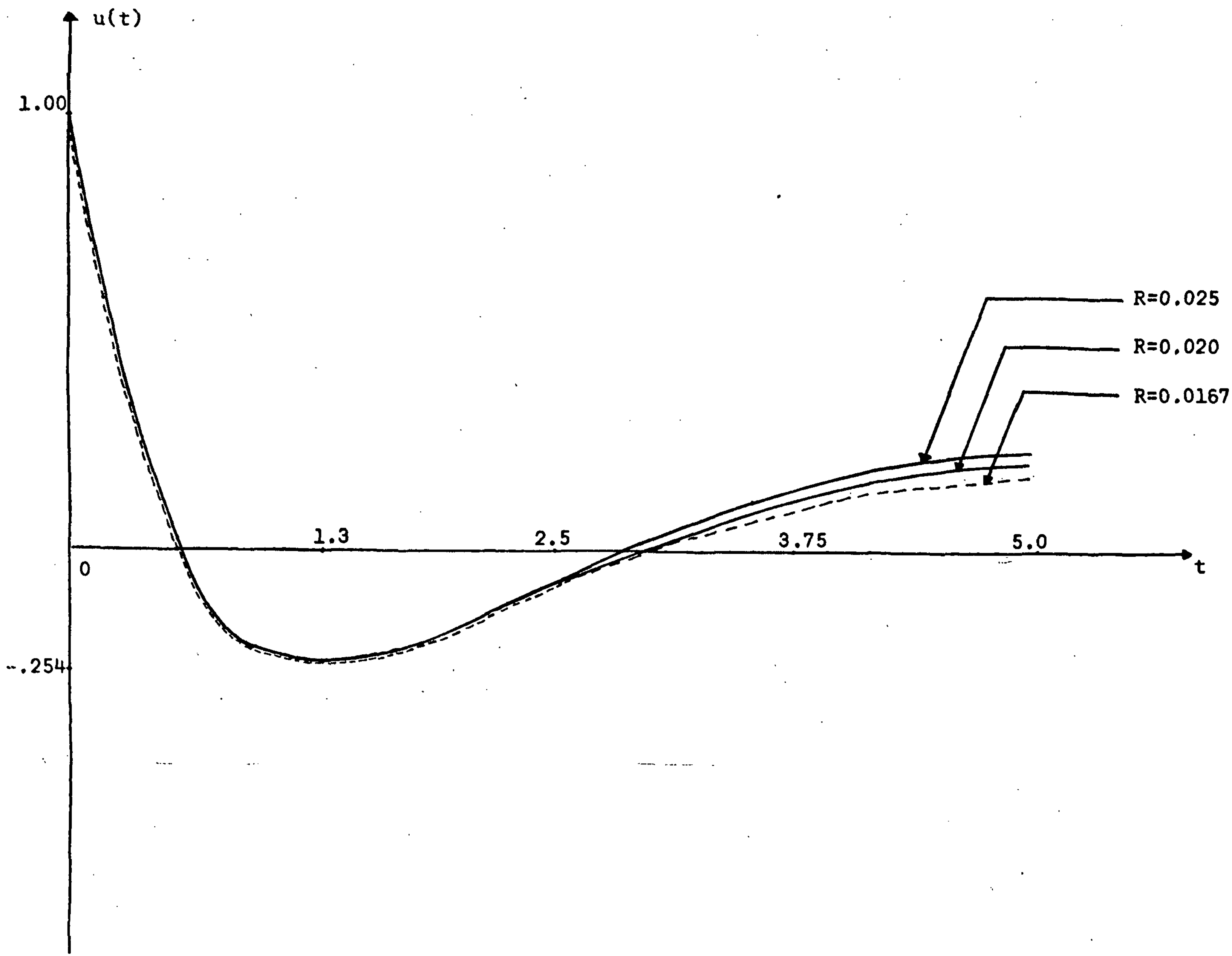

Figure 2.9 - Control Signal Characteristics for Different Values of R

it has been shown that the system can always be stabilized if sufficient control amplitude is available. These results also apply for systems with rather general nonlinearities that do not involve the control. It has also been shown that the tracking error admits an upper bound and that the bound can be made arbitrarily small. Since very little a priori knowledge about the system is assumed, the feedback controller may be forced to supply excessive feedback gain in order to insure stability. This is, it seems, a logical compromise under the present situation. It is also clear that the controller uses partial state feedback from the states of a certain companion form. In practice, it is possible that some of these state variables may not be available. This requires reconstruction of the states from the available states or output. State reconstruction via well known Luenberger observer is difficult, because of the uncertainty in the system parameters. The problem may be compounded by measurement noise. Differentiation of the output to provide unavailable states together with a high gain design is likely to yield unsatisfactory performance in the presence of measurement noise. These problems will be tackled in the next chapter.

III DESIGN OF REDUCED ORDER COMPENSATOR AND ESTIMATOR FOR SYSTEM WITH PARAMETER UNCERTAINTY

3.1 INTRODUCTION

In the previous chapter, it has been shown that a linear time-invariant controller can be designed to stabilize a single input - single output nonlinear system provided that the system output and its lower order derivatives up to (n-m-1) are available. Here n is the system order and m is the number of zeros. In many practical situations, these state variables may not be available. In addition, output measurements may also be noisy. The inherent difficulty associated with differentiation and possibly high gains limits the minimax procedure described in previous chapter.

This chapter will deal initially with the generation of the required control signal as the response of a dynamic system to the available states or output for deterministic case. The dimension of dynamic compensator is arbitrary. Because of the previous theoretical development in Chapter 2, the highest order dynamic compensator will be (n-m-1). The problem is to design a dynamic compensator which is in some sense best to generate the required control signal. It is shown that the constant parameters that specify the dynamic compensator can be obtained by solving a set of simultaneous nonlinear algebraic equations.

The second problem in this chapter deals with the estimation of states of a dynamical system, given noise-corrupted observation, when there is parameter uncertainty in the dynamical system. The deterministic case will be treated first.

3.2 DETERMINISTIC CASE: DESIGN OF DYNAMIC COMPENSATOR

One way of estimating the states or generating an optimal control would be to assume that the observation is contaminated with noise and then use a usual Kalman filter. In addition to the dimensionality problem of Kalman filter, it is difficult to specify various covariance matrices of plant disturbances and measurement noise in an essentially deterministic situation. Thus an alternative approach of generating the optimal control will be suggested below. It is to be noted that the gains of the dynamic compensator should be independent of initial state of plant and compensator, otherwise the compensator gains will have to be changed with the change of plant state due to a disturbance.

3.2.1 PROBLEM FORMULATION

The model reference system described in previous chapter is shown to be described by

$$\dot{e} = A_0 e + \underline{B}\tilde{u} + \underline{\beta}\xi \tag{3.1}$$

$$\underline{\zeta} = C\underline{e} \tag{3.2}$$

where $\underline{\zeta}$ is r-dimensional output vector .

The dynamic compensator of specified order, s, is described by

$$\dot{z}(t) = Fz(t) + G\zeta(t) = Fz(t) + GC\underline{e}(t) \tag{3.3}$$

$$u(t) = hz(t) + n\zeta(t) = hz(t) + nC\underline{e}(t) \tag{3.4}$$

The input to the dynamic compensator is the available output whereas its output is the required control signal. It is easy to see that

C is r x n-m matrix

F is s x s matrix

G is s x r matrix

h is 1 x s vector

n is 1 x r vector

The present formulation of the problem is similar to the one reported by Johnson and Athans[25].

To design via optimization, the following cost function is chosen:

$$J(F,G,h,n;\xi) = \frac{1}{2} \int_{t_o}^{t_f} [\underline{e}^T Q \underline{e} + \zeta^T (g^T R_2 g + n^T R_1 n)\zeta + z^T (F^T R_2 F + h^T R_1 h) z - L\xi^2]\, dt \qquad (3.5)$$

The problem is to minimize the above criterion with respect to F, G, h, n and maximize w.r.t. ξ , subject to (3.3) and

$$\begin{aligned} \dot{e} &= A_0 e + \beta \tilde{u} + \beta\xi \\ &= (A_0 + \beta n c)\underline{e} + \beta h z + \underline{\beta}\xi \end{aligned} \qquad (3.6)$$

The inclusion of the second and third terms in the performance criterion avoids placing the poles of the compensator at $-\infty$ and thus allow the high frequency plant noise not to pass through the system as mentioned in [25].

Now defining

$$P \triangleq \begin{bmatrix} n & h \\ G & F \end{bmatrix}, \quad \hat{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\beta} \triangleq \begin{bmatrix} \underline{\beta} & 0 \\ 0 & I \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{T} \triangleq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \quad \underline{w} \triangleq \begin{bmatrix} \underline{e} \\ \underline{z} \end{bmatrix}, \quad \overset{*}{\underline{\beta}} = \begin{bmatrix} \underline{\beta} \\ 0 \end{bmatrix},$$

it can be easily seen that (3.3) and (3.6) can be represented as

$$\dot{\underline{w}} = \begin{bmatrix} \dot{\underline{e}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + \beta nC & \beta h \\ GC & F \end{bmatrix} \begin{bmatrix} \underline{e} \\ \underline{z} \end{bmatrix} + \begin{bmatrix} \underline{\beta} \\ 0 \end{bmatrix} \xi$$

$$\text{i.e. } \dot{\underline{w}} = (\hat{A} + \hat{\beta}PC)\underline{w} + \overset{*}{\underline{\beta}}\, \xi \tag{3.7}$$

and (3.5) can be written as

$$J(P,\xi, \underline{w}(to)) = \frac{1}{2} \int_{t_o}^{t_f} [\underline{w}^T\tilde{Q}\underline{w} - L\xi^2]\, dt \tag{3.8}$$

$$\text{where } \tilde{Q} = \hat{Q} + \frac{1}{2} (\hat{C}^T P^T \hat{R} P \hat{C} + T\hat{C}^T \hat{R} P \hat{C} T) \quad . \tag{3.9}$$

Thus the problem is to minimize and maximize (3.8) with respect to P and ξ subject to (3.7). The resulting optimal P will depend on initial condition $\underline{w}(to)$. In order to remove this restriction w(to) can be treated as a random vector in which case J may be replaced by

$$J(P,\xi) = E[J(P,\xi, \underline{w}(to)].$$

Thus the requirement for optimal solution yields[2]

$$\frac{\partial \hat{J}}{\partial P} = \frac{\partial}{\partial P} E[J(P,\xi, \underline{w}(to)] = E[\frac{\partial J}{\partial P} (P,\xi, \underline{w}(to)] = 0 \tag{3.10}$$

$$\frac{\partial \hat{J}}{\partial \underline{\xi}} = 0 \tag{3.11}$$

The interchange of order of expectation (i.e., integration) and differentiation is crucial here and is valid under rather general conditions[10].

3.2.2 ANALYSIS

Applying the technique presented in Chapter 4, it can be easily seen that the optimal gain matrix P is given by

$$P = \hat{R}^{-1}\hat{b}^T K L \hat{C}^T [\frac{1}{2}\hat{C}(L+\tilde{T}L\tilde{T})\hat{C}^T]^{-1} \tag{3.12}$$

where

$$M = E[\int_{t_o}^{t_f} (\underline{w}\,\underline{w}^T)dt] = \int_{t_o}^{t_f} \phi(t)\, E[\underline{w}(to)\underline{w}^T(to)]\phi^T(t)dt$$

$$\text{i.e., } -\dot{M} = (\hat{A}+\hat{\beta}P\hat{C})M+M(\hat{A}+\hat{\beta}P\hat{C})^T+E[\underline{w}(to)\underline{w}^T(to)] \tag{3.13}$$

and K satisfies

$$-\dot{K} = (\hat{A}+\hat{\beta}P\hat{C})^T K+K(\hat{A}+\hat{\beta}P\hat{C})+\tilde{Q}-K\underline{\hat{\beta}}L^{-1}\underline{\hat{\beta}}^T K \tag{3.14}$$

It $t_f \to \infty$, $t_o \to 0$, M and K are the steady state solutions of (3.13) and (3.14) respectively.

If

$$\underline{E}\begin{bmatrix}\underline{e}(o)\\ z(o)\end{bmatrix} \stackrel{\Delta}{=} E[\underline{w}(to)] = \underline{w}_o$$

and $E[(\underline{w}(to) - \underline{w}_o)(\underline{w}(to) - \underline{w}_o] \stackrel{\Delta}{=} W_o$

Then $E[w(to)w^T(to)] = W_o + \underline{w}_o\underline{w}_o^T$ (3.15)

The optimal cost can be seen to be

$$\hat{J} = \frac{1}{2}\,\mathrm{Tr}\,\{KE[w(to)w^T(to)]\} = \frac{1}{2}\,\mathrm{Tr}\,\{K(w_o + \underline{w}_o\underline{w}_o^T)\} \tag{3.16}$$

Thus the optimal signal requires only the available states $C\underline{e}$ and the states of dynamic compensator. This control signal is now applied to the original system (2.9). Overall stability of the system is not apparent and is the subject of future investigation. Computational schemes to solve the simultaneous nonlinear equations have been discussed in Chapter 4.

3.3 STOCHASTIC CASE: DESIGN OF ESTIMATOR

3.3.1 PROBLEM FORMULATION

It is assumed that a continuous record of a realization of the (n-m)x1 vector observable process $\{z_i(t)\}$ is available where

$$\underline{z}(t) = H\overset{*}{\underline{x}}(t) - Hy(t) + n(t) = H\underline{e}(t) + \underline{n}(t) \qquad (3.17)$$

and $\underline{n}(t)$ is white noise.

Note that this means that the model is used in the estimation process to generate $\underline{z}(t)$. Moreover, the process $\{e_i\}$ assumed for mathematical treatment to be a random one, is modeled as the output of a dynamic system, excited by a pseudo-random signal $\xi(t)$, uncorrelated with $n(t)$; that is,

$$\dot{\underline{e}} = A_0 e + \underline{\xi}(t) + \underline{g}(t)$$
$$g(t) = \underline{\beta}\tilde{u} \qquad (3.18)$$

where $\underline{\xi}(t)$ and $\underline{n}(t)$ are assumed to be zero mean, random processes having covariance matrices Q and R respectively. In case, $\xi(t)$ has a bias, the system equation (3.18) can be augmented to take into account this factor.

The problem is to determine the initial state $\underline{e}(to)$ and $\xi(t)$ for all t_f $[t_o,t_f]$ which minimize

$$\frac{1}{2}\int_{t_o}^{t_f}[(\underline{z}-H\underline{e})^T R_1^{-1}(\underline{z}-H\underline{e}) + \underline{\xi}^T Q_1^{-1}\underline{\xi}] + \frac{1}{2}[\underline{e}(to)-\overset{*}{\underline{e}}(to)]^T P^{-1}[\underline{e}(to)-\overset{*}{\underline{e}}(to)] \qquad (3.19)$$

subject to (3.18).

$\overset{*}{\underline{e}}(to)$ is the expected value of $\underline{e}(to)$. It is well known that, if $\xi(t)$ and $n(t)$ are sample functions of white, zero-mean, uncorrelated random processes, this procedure will give a maximum a posteriori estimate. If, in addition, $\xi(t)$ and $n(t)$ are gaussian, this will give a least mean-square error estimate. Even if $\underline{\xi}(t)$ is not random, the above performance index carries meaning in the sense that an integral square error in the estimates is being minimized. Since

$\underline{z}(t)$ and $H\underline{e}(t)$ are the actual observation and predicted observation respectively, $[\underline{z}(t)-He(t)]$ is the error in the estimate and R_1^{-1} provides relative weighting (related to the covariance of the noise) on this component.

3.3.2 ANALYSIS

The estimate $\underline{e}(t_f)$ of $\underline{e}(t)$ at $t=t_f$ is the solution of equation (2.18) using $\underline{e}(to)$ and $\xi(t)$ as estimated. To obtain $\underline{e}(to)$, $\underline{\xi}(t)$, the usual Hamiltonian

$$H = \frac{1}{2}(\underline{z}-H\underline{e})^T R_1^{-1}(\underline{z}-H\underline{e}) + \frac{1}{2}\underline{\xi}^T Q_1^{-1}\underline{\xi} + \underline{\lambda}^T[A_0\underline{e} + \xi(t) + \underline{g}(t)] \tag{3.20}$$

is introduced with the costate equations and necessary conditions

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{e}} = H^T R_1^{-1}(\underline{z}-H\underline{e}) - A_o^T\underline{\lambda} \tag{3.21}$$

$$\frac{\partial H}{\partial \underline{\xi}} = \underline{\lambda} + Q_1^{-1}\underline{\xi}(t) = 0 \tag{3.22}$$

$$\dot{\underline{e}} = A_0\underline{e} + \underline{\xi}(t) + \underline{g}(t) \tag{3.23}$$

and the boundary conditions

$$\underline{\lambda}(to) = -P^{-1}[\underline{e}(to) - \overset{*}{\underline{e}}(to)] \tag{3.24}$$

$$\underline{\lambda}(t_f) = 0 \quad . \tag{3.25}$$

Using (3.22), (3.23) reduces to

$$\dot{\underline{e}} = A_0\underline{e} = Q_1\lambda + g(t) \quad . \tag{3.26}$$

Now we claim a feedback solution of the form

$$\lambda(t) = K(t)\,[e(t) - v(t)] \quad . \tag{3.27}$$

Using (3.21), (3.26) and (3.27), it is clear that K and v satisfy the following equations:

$$-\dot{K} = KA_0 + A_0^T K - KQ_1 K + H^T R_1^{-1} H \tag{3.28}$$

$$\dot{\underline{v}} = A_0 \underline{v} - K^{-1} H^T R_1^{-1} [\underline{z} - H\underline{v}] + \underline{g}(t) \; . \tag{3.29}$$

Again as in (3.25),

$$\lambda(t_f) = K(t_f)\,[e(t_f) - \underline{v}(t_f)] = 0 \; . \tag{3.30}$$

Thus

$\underline{e}(t_f) = \underline{v}(t_f)$ is the estimate $\hat{\underline{e}}(t_f)$ of $\underline{e}(t)$ at t_f given $\underline{z}(t)$ on $[t_o, t_f]$.

Also from (3.24) and (3.27)

$$K(to) = -P^{-1} \; , \; \underline{v}(to) = \overset{*}{\underline{e}}(to) \; . \tag{3.31}$$

Thus

$$\dot{\hat{\underline{e}}}(t_f/t_f) = \dot{v}\,(t_f) = A_0 \hat{e}\,(t_f/t_f) - K^{-1} H^T R_1^{-1} [z(t_f) - H\hat{e}(t_f/t_f)] + g(t) \tag{3.32}$$

where K is given by (3.28).

Now defining

$$M = -K^{-1} \, ,$$

it follows that $\dot{M} = K^{-1} \dot{K} K^{-1}$.

Thus the estimator is given by

$$\dot{\hat{\underline{e}}}\,(t_f/t_f) = A_0 \hat{\underline{e}}\,(t_f/t_f) + MH^T R_1^{-1} [\underline{z}(t_f) - H\hat{\underline{e}}(t_f/t_f)] + \underline{g}(t)$$

$$\hat{\underline{e}}\,(to/to) = \overset{*}{\underline{e}}(to) \tag{3.33}$$

where the positive definite matrix M satisfies

$$\dot{M} = A_0 M + MA_0^T + Q - MH^T R_1^{-1} HM \tag{3.34}$$

$$M(to) = P(to) \tag{3.35}$$

3.4 EXAMPLE

Case 1 : Deterministic Case

Following example will illustrate the design of a first order dynamic compensator for a third order nonlinear system described by

$$x^{(3)} + x^{(2)} + 2\dot{x} - x - \sin\dot{x} = \dot{u} + u \qquad (3.36)$$

with a second order model

$$y^{(2)} + 2\dot{y} + y = r \quad . \qquad (3.37)$$

A second order model is chosen since the difference between plant order and the number of zeros is two. (3.36) and (3.37) can be combined to yield the error equation

$$e^{(2)} + 2\dot{e} + e = \tilde{u} + \xi \qquad (3.38)$$

$$\tilde{u} = u - r \ .$$

It is clear from the previous chapter that the optimal controller requires measurements of e and $\dot{e}$. It will be assumed that only e is available. Next a dynamic compensator

$$\dot{z} = fz + ge \qquad (3.39)$$

is defined. The output of dynamic compensator is the required control signal given by

$$\tilde{u} = hz + ne \quad . \qquad (3.40)$$

With $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_1 = .01$, $R_2 = .015$ and $L = .01$

(3.12) - (3.14) are solved using the algorithms reported in next chapter to obtain

$$f=-11.68 \quad g= 8.2 \quad h=-51.2 \quad n=-47.68.$$

The control signal is then applied to the original system (3.36). The outputs of model and the plant are shown in Figure 3.1. The control signal characteristic is shown in Figure 3.2. Thus the example establishes the fact that a reduced order dynamic compensator can be effectively used to generate the control when some of the necessary states of certain companion form are not available.

Case 2: Stochastic Case

In this example, a first order estimator will be designed for the above model-reference system to estimate e and $\dot{e}$ from the observation

$$\begin{aligned} z(t) &= e(t) + n(t) \\ &= h\underline{e}(t) + n(t) \qquad (3.41) \\ h &= (1\ 0), \end{aligned}$$

where $n(t)$ is a gaussian noise with standard deviation .01.

Using $Q_1 = \begin{vmatrix} 10 & 0 \\ 0 & 55.36 \end{vmatrix}$ $R_1 = 1$, the estimator is given by

$$\dot{\hat{\underline{e}}}(t_f) = \begin{vmatrix} -24.56 & 1 \\ 1.32 & -2 \end{vmatrix} \hat{\underline{e}}(t_f) + \begin{vmatrix} 24.56 \\ -2.36 \end{vmatrix} z(t) + \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tilde{u} \qquad (3.42)$$

The controller is designed using the procedure of the previous chapter with $Q = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, R=L=0.01 and is given by

$$\tilde{u} = -R^{-1}(\ .5,\ .5)\hat{\underline{e}}\ . \qquad (3.43)$$

The control signal is then applied to the system (3.36). The outputs of the plant and model are shown in Figure (3.3). Noise corrupted observation $z(t)$ and its estimate $\hat{e}(t)$ are shown in Figure 3.4. The control signal is plotted in Figure 3.5. Thus a reduced order estimator can be designed to implement the controller for a class of model reference system.

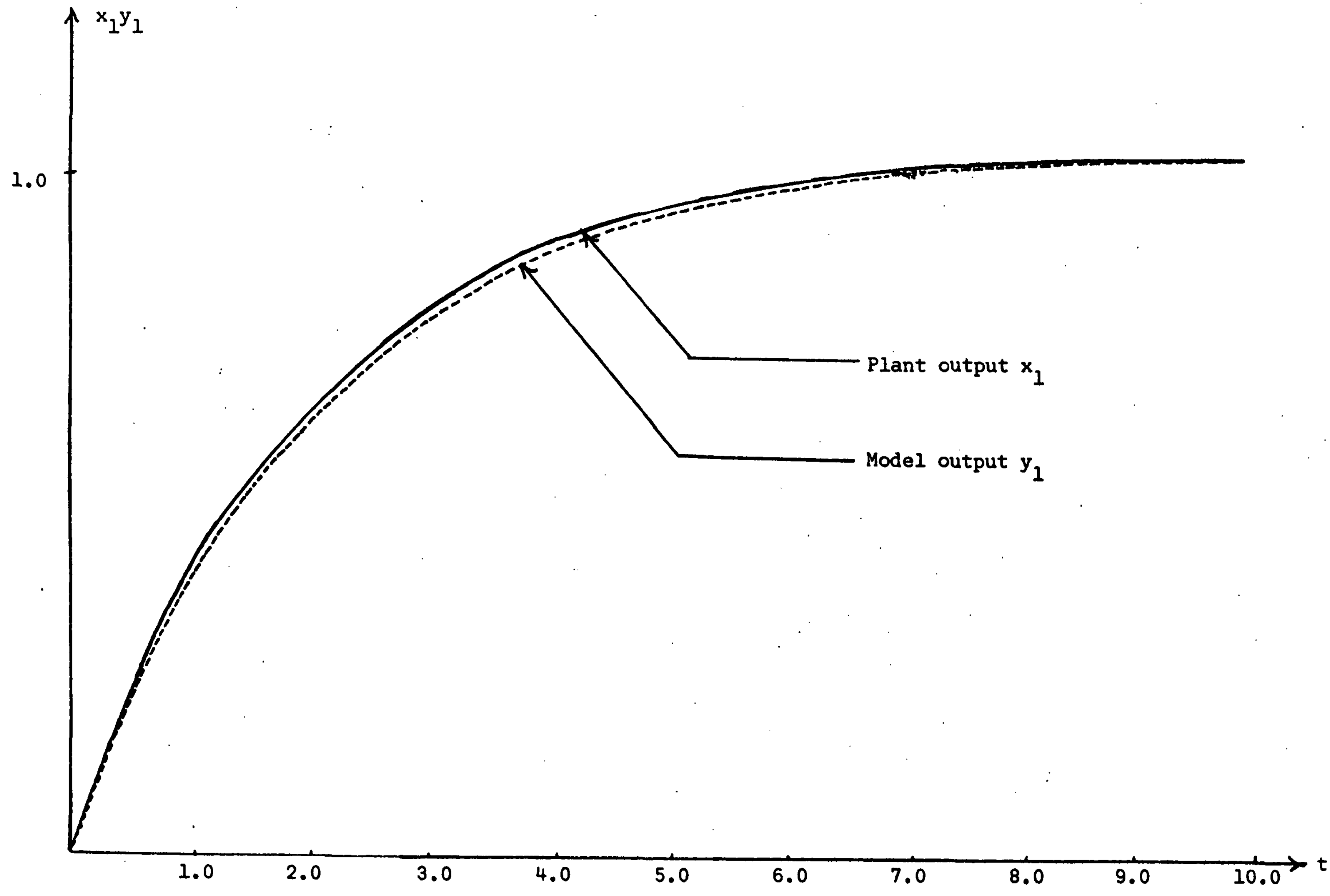

Figure 3.1 Output Characteristics of Plant and Model

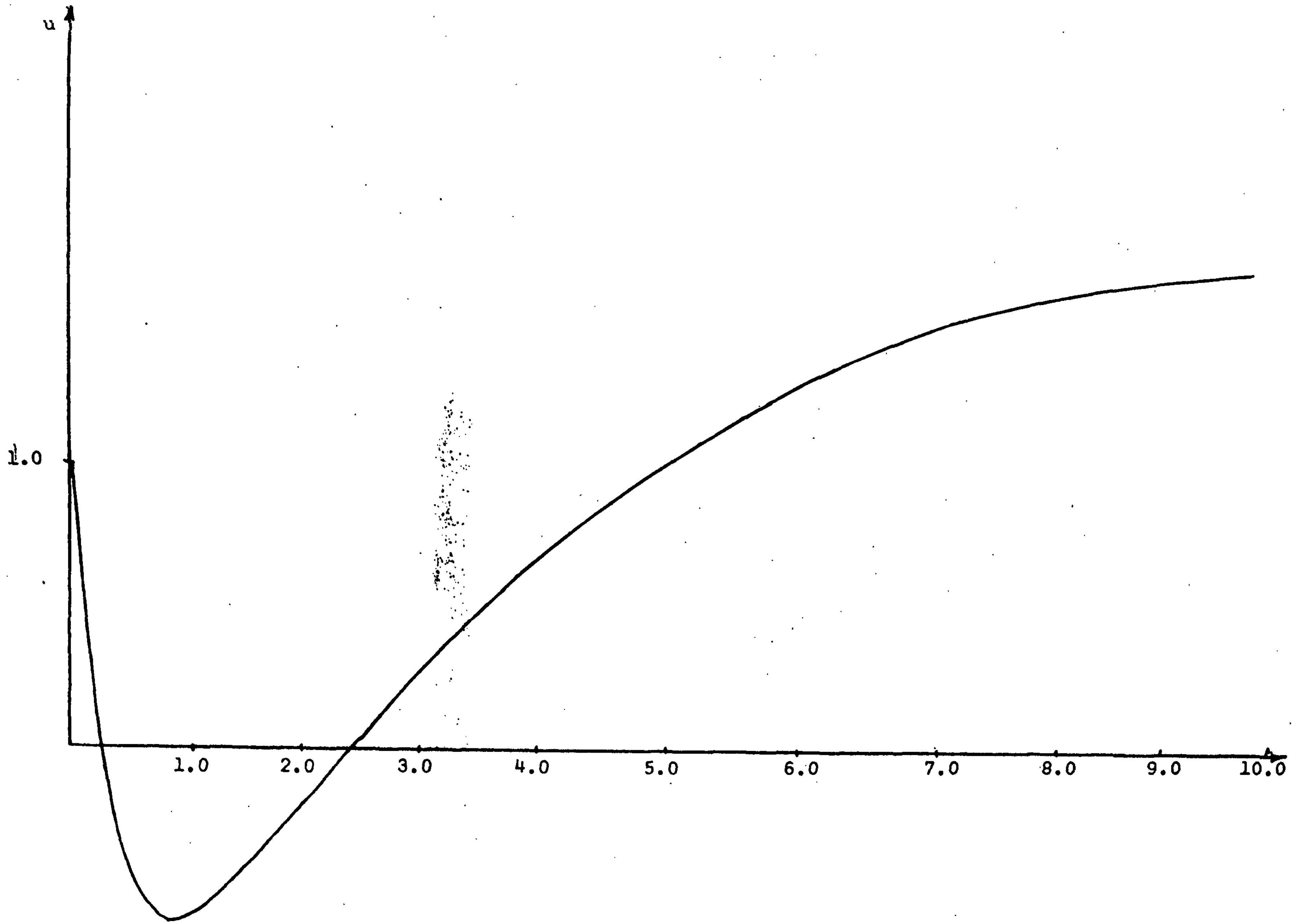

Figure 3.2 Control Signal Characteristics

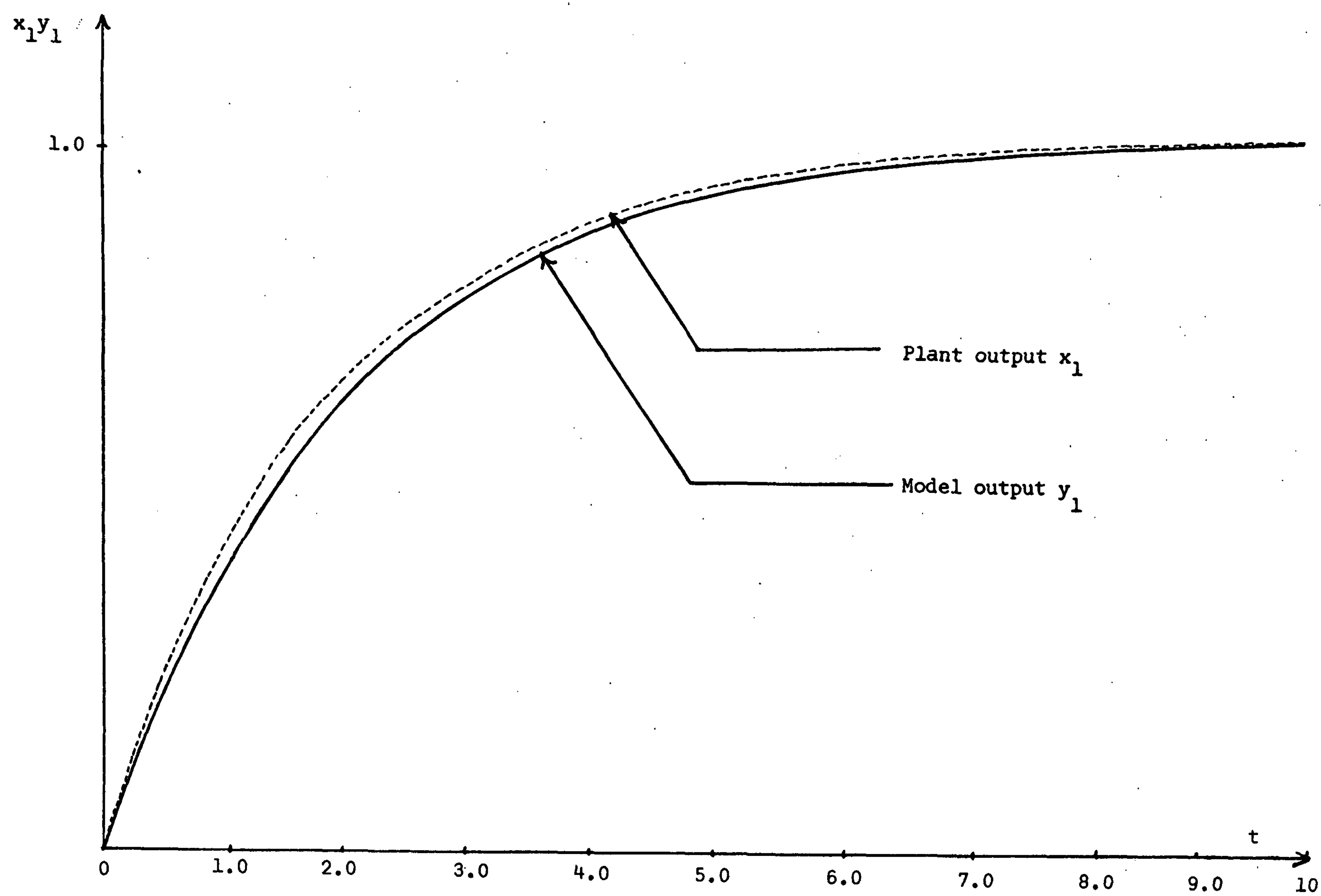

Figure 3.3 Output Characteristics (x_1y_1) of Model and Plant

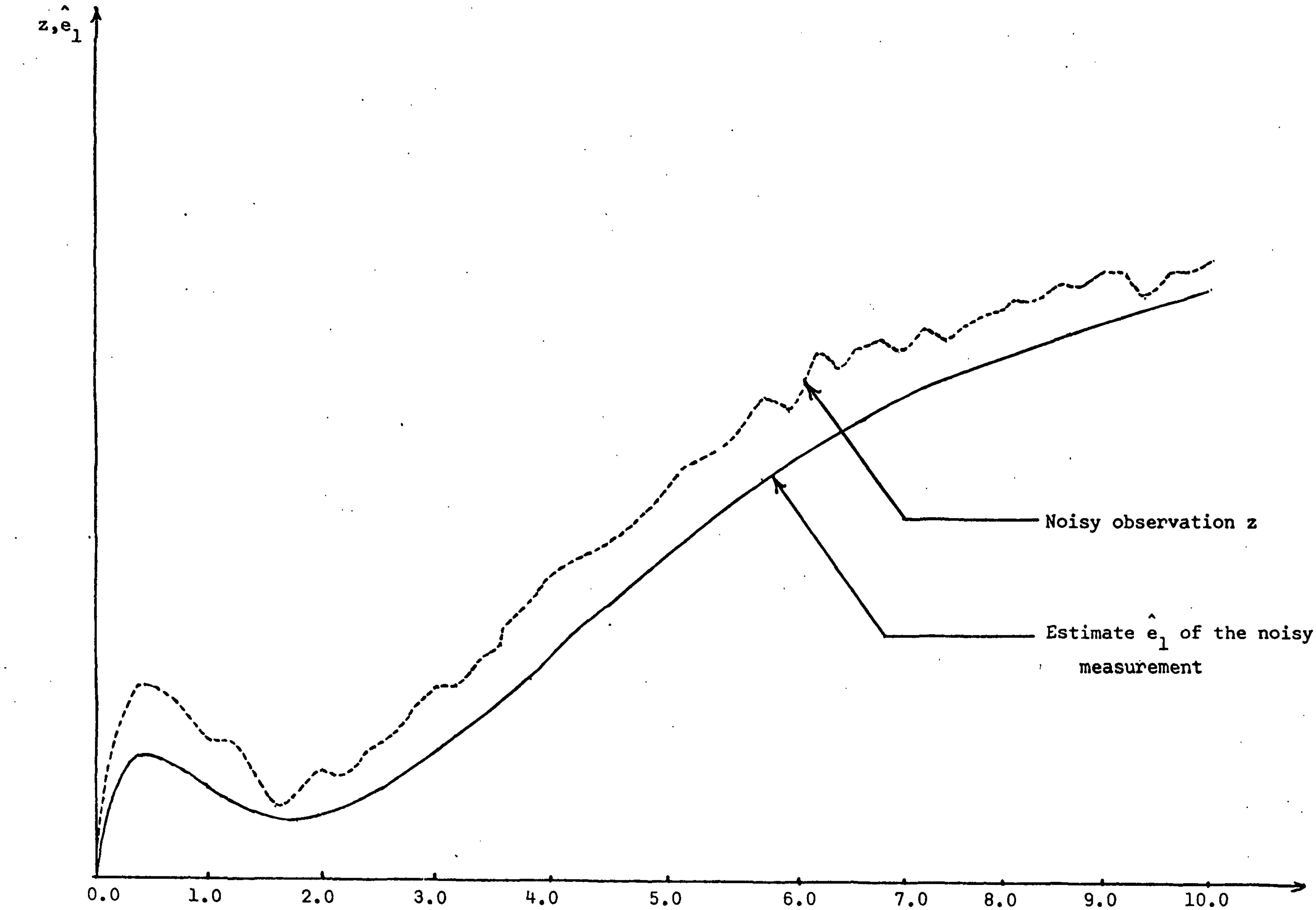

Figure 3.4 Moise corrupted observation z and its estimate $\hat{e}_1$

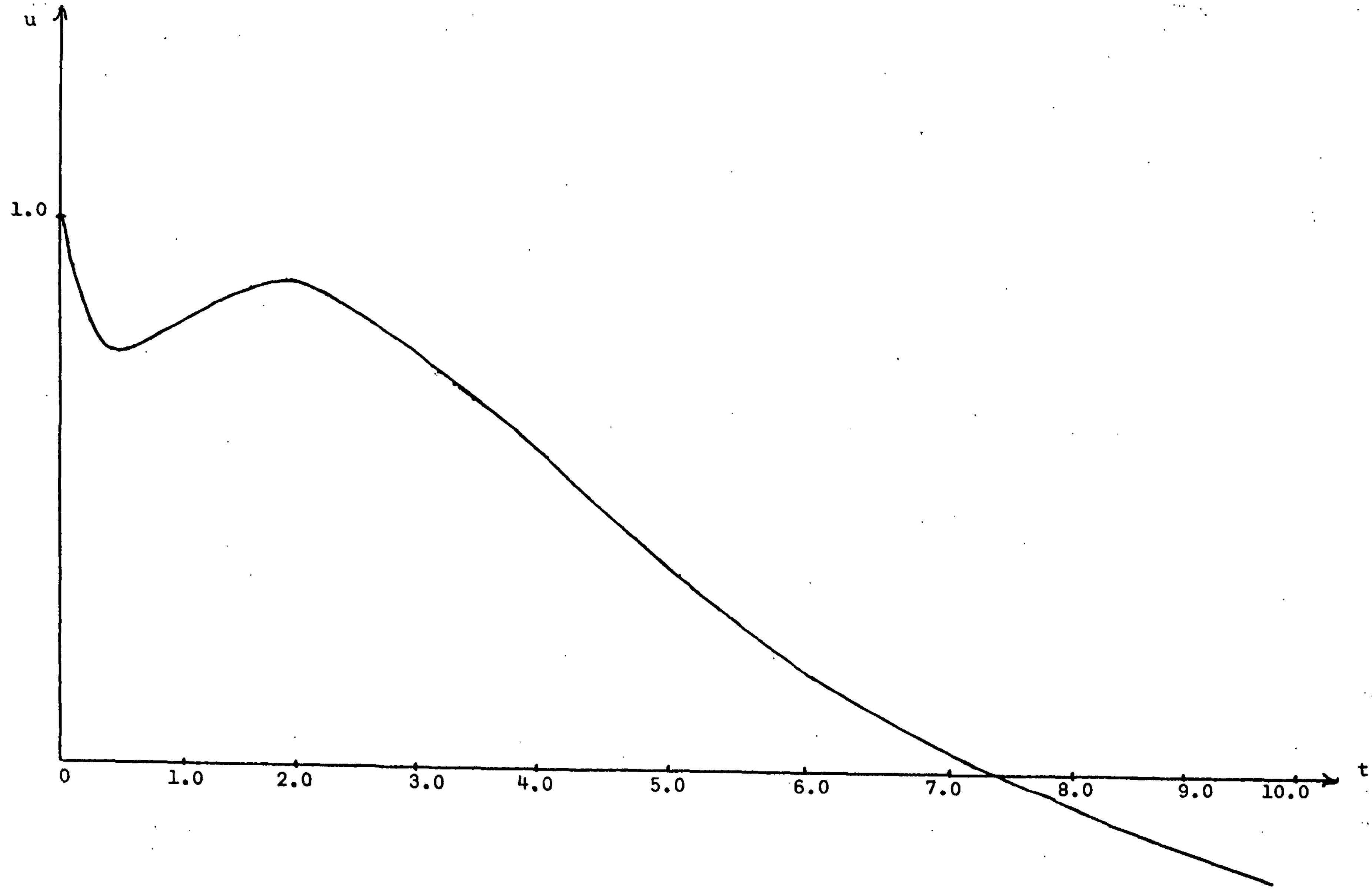

Figure 3.5 Control Characteristics

3.5 CONCLUSION

In this chapter, the problem of generating the optimal control as the output of a dynamic compensator is treated for system with parameter uncertainty. The input to the dynamic compensator is the available output of the system, whereas its output is the required control signal. The minimax technique of the previous chapter has been extended to the case where the state variables are not in phase variable form and also the necessary state variables are not available.

Finally an ad hoc scheme for estimating the necessary states of the system with process uncertainty has been developed using a reduced order deterministic model. Very little a priori knowledge of the parameters is assumed. If the signal related to the uncertainty and the measurement noise are uncorrelated, the resulting estimator is linear and is optimal in the sense that it minimizes a quadratic criterion involving estimation error and a signal related to uncertainty. The minimax technique presented in Chapter 2 and 3 has some limitations. First, it is difficult to extend this basic concept directly to general multivariable system. The difficulty is due to the fact that a suitable canonical form for multivariable case is not available. Second, the uncertainty signal $\underline{\xi}(t)$ is related to the system parameters in a complicated way. Thus an ultimate bound on parameter variation to insure system stability is difficult to ascertain. The problem of controlling multivariable system with parameter uncertainty will be reformulated and will be treated in the next chapter.

IV MINIMAX OUTPUT FEEDBACK CONTROLLER

4.1 INTRODUCTION

The design of a controller for a linear multivariable system having parameter uncertainty is explored in this chapter. Linear output feedback is employed with the feedback gains determined by minimizing one of several criteria. The problem is treated initially by minimizing with respect to the feedback gain matrix and maximizing with respect to uncertainty, a quadratic performance index involving the system state, the control and the "uncertainty signal"[9].

The optimal gain matrix satisfies a set of simultaneous nonlinear algebraic equations. The design procedure often leads to a pessimistic result, either because the uncertainty does not act as perversely as assumed, or because the control often makes an effort to reduce the cost where it is high, even with perfect knowledge of parameter. To meet this objection, other criterion and in particular, a minimax sensitivity criterion[11] are also examined. The optimal feedback gain matrix for the so-called "regret criterion" is shown to satisfy a set of nonlinear equations similar to those obtained for the standard criterion. It is demonstrated that various minimax design criteria yield better system performance under wide range of parameter variation that a purely nominal design.

4.2 SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider an n^{th} order linear system with state vector $\underline{x}(t)\epsilon R_m$ and output vector $y(t)\epsilon R_q$ defined by

$$\underline{\dot{x}} = A_o\underline{x} + B_o\underline{U} + (A-A_o)\underline{x} + (B-B_o)\underline{u} \tag{4.1}$$

$$y = C\underline{x} \tag{4.2}$$

with a controller

$$u = -Fy = -FC\underline{x} \; . \tag{4.3}$$

where A_0, B_0 are nominal matrices. Using (4.3), (4.1) can be represented as

$$\begin{aligned}\underline{\dot{x}} &= (A_0-B_0FC)\underline{x} + [(A-A_0)-(B-B_0)FC]\underline{x} \\ &= (A_0-B_0FC)\underline{x} + (W-W_0)\underline{x} = (A_0-B_0FC)\underline{x} + D\underline{\xi}\end{aligned} \tag{4.4}$$

where $\underline{\xi}$ represents the effect of uncertainty.
Since the uncertainty is assumed to be limited, $\underline{\xi}$ will likewise be constrained. In order to place any restriction on the form of $(W-W_0)$ let $W-W_0 = DGC_1$. D, C_1 are fixed and G contains variable terms. An example is

$$B = B_0,\ A-A_0 = \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ (\alpha_0-\hat{\alpha}_0) & \cdots & & (\alpha_{n-1}-\hat{\alpha}_{n-1}) \end{bmatrix}$$

$$C_1 = I,\ G = [\hat{\alpha}_0-\alpha_0,\ \hat{\alpha}_1-\alpha_1 \ \ldots \ \hat{\alpha}_{n-1}-\alpha_{n-1}],\ D= [0,0,\ldots 1]^T$$

Thus the uncertainty vector is specified as

$$\underline{\xi} = GC_1\underline{x} \tag{4.5}$$

where G is the gain matrix associated with the uncertainty vector and C_1 has rank n or less. Both C, C_1 are assumed to have maximum rank, i.e., rank equal to number of rows.

Substitution of (4.5) in (4.4) gives

$$\dot{\underline{x}} = (A_0 - B_0FC + DGC_1)\underline{x}. \tag{4.6}$$

In order to achieve a design through optimization, the feedback matrices F and G will initially be chosen to minimize and maximize, respectively, the performance criterion

$$J(F,G) = \frac{1}{2}\int_0^\infty [\underline{x}^T Q\underline{x} + \underline{u}^T R\underline{u} - \underline{\xi}^T L\underline{\xi}]\, dt \Big|_{\underline{u}=FC\underline{x},\ \underline{\xi}=GC\underline{x}}$$

$$= J_0(\underline{u},\underline{\xi})\Big|_{\underline{u}=FC\underline{x},\ \underline{\xi}=GC_1\underline{x}}$$

$$= \frac{1}{2}\int_0^\infty \underline{x}^T [Q + C^T F^T RFC - C_1^T G^T LGC_1]\ \underline{x}\ dt; \tag{4.7}$$

i.e.

(i) Find F and G such that

$$\min_F \max_G J(F,G) = \max_G \min_F J(F,G) \quad . \tag{4.8a}$$

We shall also consider the following minimax procedures to obtain optimal gain matrix F :

(ii) $\min J_0(u,\xi^*)\Big|_{\underline{u}=FC\underline{x}}$

where $\underline{\xi}^*$ is obtained from $\max\limits_{\underline{\xi}^*} [\min\limits_{\underline{u}^*} J_0(\underline{u}^*,\ 0) - \frac{1}{2}\int_0^\infty \underline{\xi}^T L\underline{\xi}\ dt]$

and $\underline{u}^*$ minimizes J_0 assuming G=0. (4.8b)

(iii) $\min\limits_F \max\limits_G [J(F,G) - J_1^*(G)]$ (4.8c)

where $J_1^*(G) = \min\limits_F [J(F,G)]$ given G.

(iv) $$\min_F \max_G \, [J(F,G) - J_2^*(G)] \tag{4.8d}$$

where $J_2^*(G) = \min_u \, [J_0(\underline{u}\ \underline{\xi})]_{\underline{\xi}=GC_1\underline{x}}$

Criterion (ii) is less pessimistic in the sense that $\underline{\xi}$ is given the first play and, in making its play, assumes that $\underline{u}(\underline{x})$ is obtained by an optimal full state design (with $\underline{\xi}=0$) for the nominal plant. Matrix F is then chosen to minimize the criterion based on the announced strategy of $\underline{\xi}$.

In criterion (iii), (iv) the best control with perfect parameter information, i.e., $\underline{\xi}$ known, is obtained with output feedback and full state feedback respectively. Matrices F, G then minimize and maximize respectively the difference between the actual cost and cost with perfect parameter information.

4.3 MINIMAX PERFORMANCE CONTROL WITH DIRECT CONFLICT OF INTEREST

In this case, the saddle point is defined by the following inequality

$$J(F^*,G) \leq J(F^*,G^*) \leq J(F,G^*) \ . \tag{4.9}$$

It is clear from (4.6) and (4.8) that J is determined by the initial state, $\underline{x}(t_0)$ as well as matrices F and G. That is,

$$J = J(F,G,\underline{x}\,(t_0)). \tag{4.10}$$

In order to make the optimum F and G independent of $\underline{x}(t_0)$, $\underline{x}(t_0)$ can be treated as a random vector in which case J may be replaced by

$$\hat{J}(F,G) = E[J(F,G,x(t_0)]. \tag{4.11}$$

E $(\cdot)$ denotes expectation with respect to $\underline{x}(t_0)$. The necessary condition that F and G should minimize and maximize $\hat{J}(F,G)$, respectively, requires

$$\frac{\partial \hat{J}}{\partial F} = \frac{\partial}{\partial F} E[J(F,G,\underline{x}(t_0))] = E[\frac{\partial J}{\partial F}(F,G,\underline{x}(t_0))] = 0 \quad (4.12a)$$

$$\frac{\partial \hat{J}}{\partial G} = \frac{\partial}{\partial G} E[J(F,G,\underline{x}(t_0))] = E[\frac{\partial J}{\partial G}(F,G,\underline{x}(t_0))] = 0 \quad (4.12b)$$

The interchange of order of expectation (i.e., integration) and differentiation is critical here and is valid under rather general conditions[10].

The partial derivatives of (4.12) will be evaluated by the application of following Lemma:

Lemma 4.1

If

$$J=J(\underline{x}(t_0))= W(\underline{x}(t_0)) + \int_{t_0}^{t_f} L(\underline{x},t)\, dt ,$$

where

$\dot{\underline{x}} = f(\underline{x},t)$ and $W(\underline{x}(t_0))$ is the penalty on the initial states $\underline{x}(t_0)$, then

$$\frac{\partial J}{\partial x_i(t_0)} = \lambda_i(t_0) + \frac{\partial W}{\partial x_i}(\underline{x}(t_0)) ,$$

where

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} = -\frac{\partial}{\partial \underline{x}}[L + \underline{\lambda}^T \underline{f}], \underline{\lambda}(t_f) = 0 .$$

This Lemma follows from the variational calculus where the first variation of J with respect to $\underline{x}(t_0)$ is $[\underline{\lambda}(t_0) + \partial W/\partial \underline{x}]^T \delta\underline{x}(t_0)$.[13] In order to apply the lemma, the elements of F and G are treated as additional "states" which satisfy

$$\dot{F} = 0, \dot{G} = 0. \quad (4.13)$$

Vector multiplier $\underline{\lambda}_x$ will be used for the regular state constraint and matrix multipliers $\Lambda_F(t)$ and $\Lambda_G(t)$ will be used for matrices

F and G respectively. It is to be noted that the Hamiltonian H will be independent of $\Lambda_F(t)$ and $\Lambda_G(t)$ due to (4.13). Thus the Hamiltonian H for (4.6), (4.7) is

$$\begin{aligned} H &= \underline{\lambda}_x^T[(A_0-B_0FC_0 + DGC_1)\underline{x}] + \tfrac{1}{2}\underline{x}^T\,[Q + C^TF^TRFC - C_1^TG^TLGC_1]\underline{x} \\ &= \mathrm{Tr}[(A_0-B_0FC_0 + DGC_1)\underline{x}\,\underline{\lambda}_x^T + \tfrac{1}{2}\,(Q + C^TF^TRFC - C_1^TG^TLGC_1)\underline{xx}^T]. \end{aligned} \tag{4.14}$$

Tr denotes the trace and

$$\dot{\underline{\lambda}}_x = -\frac{\partial H}{\partial \underline{x}} = -\,(A_0-B_0FC + DGC_1)^T\underline{\lambda}_x - (Q + C^TF^TRFC-C^TG^TLGC)\underline{x}, \underline{\lambda}_x(t_f)=0; \tag{4.15}$$

$$\dot{\Lambda}_F(t) = \frac{\partial H}{\partial F} = -\,RFC\underline{xx}^TC^T + B_0^T\underline{\lambda}_x\underline{x}^TC^T,\ \underline{\Lambda}_F(t_f) = 0\ ; \tag{4.16}$$

$$\dot{\Lambda}_G(t) = -\frac{\partial H}{\partial G} = LGC_1\underline{xx}^TC_1^T - D^T\underline{\lambda}_x\underline{x}^TC_1^T,\ \underline{\Lambda}_G(t_f) = 0. \tag{4.17}$$

According to the lemma, the necessary condition (4.12), and integrated forms of (4.16) and (4.17), we obtain

$$0 = E[\frac{\partial J}{\partial F}] = E[\Lambda_F(t_0)] = E\int_{t_0}^{t_f} [RFC\underline{xx}^TC^T - B_0^T\underline{\lambda}_x\underline{x}^TC^T]\,dt \tag{4.18}$$

$$0 = E[\frac{\partial J}{\partial G}] = E[\Lambda_G(t_0)] = E\int_{t_0}^{t_f} [LGC_1\underline{xx}^TC_1^T - D^T\underline{\lambda}_x\underline{x}^TC_1^T]\,dt\ . \tag{4.19}$$

Thus if R and L are constants, (4.18) and (4.19) yield

$$F = R^{-1}\int_{t_0}^{t_f} B_0^T\,E[\underline{\lambda}_x\underline{x}^T]C^T\,dt\ \left[\int_{t_0}^{t_f} C\,E[\underline{xx}^T]C^T\,dt\right]^{-1}\ ; \tag{4.20}$$

$$G = L^{-1}\int_{t_0}^{t_f} D^T\,E[\underline{\lambda}_x\underline{x}^T]\,C_1^T\,dt\ \left[\int_{t_0}^{t_f} C_1\,E[\underline{xx}^T]C_1^T\,dt\right]^{-1}. \tag{4.21}$$

(4.20) and (4.21) can now be simplified. If $\underline{\lambda}_x = K(t)\underline{x}$ is assumed, then(4.6) and (4.15) give

$$-\dot{K} = A_*^T K + KA_* + Q + C^T F^T RFC - C_1^T G^T LGC_1, \; K(t_f) = 0 \qquad (4.22)$$

$$\text{or } K(t) = \int_t^{t_f} \phi_*^T(\tau,t) \, [Q + C^T F^T RFC - C_1^T G^T LGC_1] \, \phi_*(\tau,t) \, d\tau \qquad (4.23)$$

where ϕ_* is the transition matrix corresponding to $A_* \stackrel{\Delta}{=} (A_0 - B_0 FC + DGC_1)$. Limiting attention to the time invariant case ($Q, A_0, B_0, C, D,$ constant) with $t_f = \infty$, $t_0 = 0$, equations (4.20) - (4.23) yield

$$F = R^{-1} B_0^T KMC^T [CMC^T]^{-1} \qquad (4.24)$$

$$G = L^{-1} D^T KMC_1^T \, [C_1 MC_1^T]^{-1} \qquad (4.25)$$

where

$$K = \int_t^{\infty} e^{A_*^T(\tau - t)} \, (Q + C_1^T F^T RFC - C_1^T G^T LGC_1) \, e^{A_*(\tau - t)} \, d\tau$$

$$= \int_0^{\infty} e^{A_*^T \sigma} \, (Q + C^T F^T RFC - C_1^T G^T LGC_1) \, e^{A_* \sigma} \, d\sigma \; , \qquad (4.26a)$$

or

$$K(A_0 - B_0 FC + DGC_1) + (A_0 - B_0 FC + DGC_1)^T K + Q + C^T F^T RFC -$$
$$C_1^T G^T LGC_1 = 0 \quad , \qquad (4.26b)$$

and

$$M \stackrel{\Delta}{=} \int_0^{\infty} E[\underline{x}\underline{x}^T] \, dt = \int_0^{\infty} e^{A_* t} \, E[\underline{x}(t_0)\underline{x}^T(t_0)] \, e^{A_*^T t} \, dt, \qquad (4.27a)$$

or

$$(A_0 - B_0 FC + DGC_1) \, M + M(A_0 - B_0 FC + DGC_1)^T + E[\underline{x}(t_0)\underline{x}^T(t_0)] = 0.. \qquad (4.27b)$$

$[CMC^T]^{-1}$ and $[C_1MC_1^T]^{-1}$ exist because C, C_1 have maximum rank and M is positive definite.

If $E[\underline{x}(t_0)] \triangleq \underline{x}_0$, $E[(\underline{x}(t_0)-\underline{x}_0)(\underline{x}(t_0)-\underline{x}_0)^T] \triangleq X_0$ (4.28)

then

$$E[\underline{x}(t_0)\underline{x}^T(t_0)] = X_0 + \underline{x}_0\underline{x}_0^T \tag{4.29}$$

is positive definite for $X_0 \neq 0$. Thus M is a positive definite solution of (4.27b) if $X_0 \neq 0$, M is positive semi-definite if $X_0 = 0$.

The optimal cost can be seen to satisfy

$$J = \frac{1}{2} E[\underline{x}^T(t_0) K\underline{x}(t_0)] = \frac{1}{2} \mathrm{Tr}\,[KE(\underline{x}(t_0)\underline{x}^T(t_0))] =$$

$$= \frac{1}{2} \mathrm{Tr}\,[K(X_0+\underline{x}_0\underline{x}_0^T)] = \frac{1}{2} \mathrm{TR}\,[K] \text{ when } E[\underline{x}(t_0)\underline{x}^T(t_0)]=I.$$

Remark 4.1

It can be easily seen that $\min_F \max_G \hat{J}(F,G) = \max_G \min_F \hat{J}(F,G)$

4.4 COMPUTATION OF F* AND G*

The feedback gain matrices F and G are specified by (4.24) and (4.25), where K and M are given by (4.26b) and (4.27b) respectively. These equations must be solved numerically and the following algorithm similar to that presented in [34] can be conveniently used for this purpose.

F_{n+1}, G_{n+1} and M_{n+1} are computed by simultaneous solution of the following equations:

$$F_{n+1} = R^{-1}B_0^TK_{n+1}C^T(CM_{n+1}C^T)^{-1} \tag{4.30}$$

$$G_{n+1} = L^{-1}D^TK_{n+1}M_{n+1}C_1^T\ (C_1M_{n+1}C_1^T)^{-1} \tag{4.31}$$

$$(A_0-B_0F_{n+1}C + DG_{n+1}C_1)\ M_{n+1} + M_{n+1}(A_0-B_0F_{n+1}C + DG_{n+1}C_1)^T + I = 0 \tag{4.32}$$

where K_{n+1} is given by the following equation:

$$(A_0-B_0F_nC + DG_nC_1)^T\ K_{n+1} + K_{n+1}\ (A_0-B_0F_nC+DG_nC_1) + Q + C^TF_n^TRF_nC - C_1^TG_n^TLG_nC_1 = 0. \tag{4.33}$$

Observe that (4.33) is approximate, while (4.30) - (4.32) are exact. The iteration starts with an initial guess of F_0 and G_0 such that $(A_0-B_0F_0C + DG_0C_1)$ is stable and also $(Q + C^TF_0^TFR_0C - C_1^TG_0^TLG_0C)$ is positive definite. Then K_1 is the positive definite solution of (4.33). With this value of K_1, (4.30) - (4.32) can be solved simultaneously to get F_1, G_1, M_1 which, in turn, give new estimate, K_2, and the iteration proceeds. Alternately (4.24) - (4.27) can be solved simultaneously.

Lemma 4.2

If $(B_0R^{-1}B_0^T - DL^{-1}D^T) \geq 0$ and $C = C_1$, the above algorithm will converge in the sense that $Tr[K_n-K_{n+1}] \geq 0$ for all n.

Proof:

The proof closely follows [34].

M_n can be expressed as

$$M_n = \int_0^\infty \phi_{*_n}\ \phi_{*_n}^T\ dt \overset{\Delta}{=} \psi_n\psi_n^T \tag{4.34}$$

If $C = C_1$, then

$$(B_0F_n - DG_n)\, C = (B_0R^{-1}B_0^T - DL^{-1}D^T)\, K_nM_nC^T(CM_nC^T)^{-1}C \tag{4.35}$$

and

$$C^T(F_n^TRF_n - G_n^TLG_n)C = C^T(CM_nC^T)^{-1}CM_nK_n(B_0R^{-1}B_0^T - DL^{-1}D^T)\, K_nM_n\, C^T(CM_nC^T)^{-1}C \tag{4.36}$$

Substitution of (4.35) and (4.36) into (4.32) and (4.33) yield equations identical to those of [34] for which $Tr[K_n - K_{n+1}] \geq 0$ is proven except that $B_0R^{-1}B_0^T - DL^{-1}D^T$ replaces $B_0R^{-1}B_0^T$. Thus Lemma 4.1 holds. Proof of convergence under less restrictive assumptions is the subject of further investigation.

4.5 MINIMAX PERFORMANCE CONTROL WITH INDIRECT CONFLICT OF INTEREST

In the previous formulation, the feedback matrix F has been chosen in a most favorable way after the uncertainty vector was allowed to take its "worst" value. This will lead to a very conservative design approach. On the other hand, it may be assumed that nature is not perverse enough to alter its strategy with that of the control. Under this situation of indirect conflict of interest, the previous formulation may be modified as follows.

The game is, as usual, defined by

$$\dot{\underline{x}} = A_0\underline{x} + B_0\underline{u} + D\underline{\xi}\ . \tag{4.37}$$

To start with, let us assume $\underline{\xi}=0$. The optimal control $\underline{u}_0^*$ is obtained by minimizing

$$J = \frac{1}{2}\int_0^\infty (\underline{x}^TQ\underline{x} + \underline{u}^TR\underline{u})\, dt \tag{4.38}$$

subject to (4.37).

Thus the resulting control is given by

$$\underline{u}^*_0 = - R^{-1}B_0^T P_0 \underline{x} \tag{4.39}$$

where P_0 is given by

$$A_0^T P_0 + P_0 A_0 + Q - P_0 B_0 R^{-1} B_0^T P_0 = 0 \tag{4.40}$$

Substitution of (4.39) in (4.37) yields

$$\dot{\underline{x}} = (A_0 - B_0 R^{-1} B_0^T P_0)\underline{x} + D\underline{\xi} \; . \tag{4.41}$$

To limit the uncertainty at this stage, the performance criterion (4.38) is modified as

$$J = \frac{1}{2} \int_0^\infty [\underline{x}^T (Q + P_0 B_0 R^{-1} B_0^T P_0)\underline{x} - \underline{\xi}^T L \underline{\xi}] \, dt \tag{4.42}$$

The 'worst' value of $\underline{\xi}$ is obtained by maximizing (4.42) with respect to $\underline{\xi}$, subject to (4.41), and is given by

$$\underline{\xi}^*_0 = L^{-1} D^T P_1 \underline{x} \tag{4.43}$$

where P_1 is the solution of

$$(A_0 - B_0 R^{-1} B_0^T P_0)^T P_1 + P_1 (A_0 - B_0 R^{-1} B_0^T P_0) + Q + P_0 B_0 R^{-1} B_0^T P_0$$
$$- P_1 D L^{-1} D^T P_1 = 0 \tag{4.44}$$

Using the estimate of $\underline{\xi}$ as in (4.43), the original system is reduced to

$$\dot{\underline{x}} = (A_0 + D L^{-1} D^T P_1)\underline{x} + B_0 \underline{u} \tag{4.45}$$

with the controller

$$\underline{u} = -Fy = -FC\underline{x} \quad . \tag{4.46}$$

Now F can be chosen to minimize $E[J] = \frac{1}{2} E \int_0^\infty \underline{x}^T [Q + C^T F^T R F C]\underline{x} \, dt$

subject to (4.45). The optimal F is given by

$$F = -R^{-1}B_0^T PMC^T(CMC^T)^{-1} \tag{4.47}$$

where P and M are given by

$$(A_0 + DL^{-1}D^TP_1 - B_0FC)^T P + P (A_0 + DL^{-1}D^TP_1-B_0FC) + Q + C^TF^TRFC=0 \tag{4.48}$$

$$(A_0 +DL^{-1}D^TP_1-B_0FC)M + M(A_0 + DL^{-1}D^TP_1 - B_0FC)^T+I=0 \tag{4.49}$$

Remark 4.2

(a) To be more general, u_0^* in (4.31) and $\underline{\xi}^*$ in (4.43) may be constrained to the form

$$\underline{u}_0^* = F_0^*C\underline{x} \text{ , } \underline{\xi}_0^* = G_0^*C_1\underline{x}$$

(b) It should be noted that this formulation assumes the existence of matrices P_0, P_1 and F that stabilize $(A_0+DL^{-1}D^TP_1-B_0FC)$ and $(A_0-B_0R^{-1}B_0{}^TP_0-DL^{-1}D^TP_1)$. Under this condition, (4.47) - (4.49) can be solved using basically the same algorithm as described in section 4.4.

4.6 MINIMAX SENSITIVITY (OR LOSS) CONTROL

If G as defined in (4.5) were known, the ideal optimal control would be obtained by minimizing

$$J = \frac{1}{2} \int_0^\infty \{\underline{x}^T(Q-C_1^TG^TLGC_1)\underline{x} + \underline{u}^TR\underline{u}\} \, dt \tag{4.50}$$

with respect to F_1 subject to

$$\dot{\underline{x}} = (A_0 + DGC_1)\underline{x} + B\underline{u} \; . \tag{4.51}$$

where

$$u = -F_1C\underline{x}$$

The resulting optimal F_1^* is given by

$$F_1^* = R^{-1}B_0^T\overset{*}{K}\overset{*}{M}C^T(C\overset{*}{M}C^T)^{-1} \tag{4.52}$$

where $\overset{*}{K}$ and $\overset{*}{M}$ satisfy

$$N_1(F_1^*,G,\overset{*}{K}) \triangleq (A_0 + DGC_1-B_0F_1^*C)^T\overset{*}{K} + \overset{*}{K}(A_0 + DGC_1-B_0F_1^*C)$$

$$+ Q + C^TF_1^{*T}RF_1^*C - C_1^TG^TLGC_1 = 0 \qquad (4.53)$$

$$N_2(F_1^*,G,\overset{*}{M}) \triangleq (A_0 + DGC_1 -B_0F_1^*C)\overset{*}{M} + \overset{*}{M}(A_0 +DGC_1-B_0F_1^*C)^T+I=0$$

$$(4.54)$$

The ideal optimal control using output feedback is then

$$u = -F_1^*C\overset{*}{x} \qquad (4.55)$$

where $\overset{*}{\underline{x}}$ satisfies

$$\dot{\overset{*}{\underline{x}}} = (A + DGC_1 - BF_1^*C)\overset{*}{\underline{x}} \qquad (4.56)$$

and the resulting cost $J_1^*(G) = \min_{F_1} J$ is given by

$$J_1^*(G) = \frac{1}{2}\int_0^\infty \underline{x}^{*T} [Q + C^TF_1^{*T}RF_1^*C - C_1^TG^TLGC_1]\overset{*}{\underline{x}}\, dt \quad . \qquad (4.57)$$

This is the best that can be achieved with constrained feedback (4.55) and perfect parameter information (G) .

Note that

$$E[\overset{*}{\underline{x}}(to)\overset{*}{x}^T(to)] = E[\underline{x}(to)\underline{x}^T(to)] = I \qquad (4.58)$$

has been assumed in (4.54) .

Now the following performance sensitivity or "regret loss" criterion is considered:

$$S(F,G) = f[J(F,G),J_1^*(G)] \qquad (4.59)$$

Definition 4.1

S(F,G) as defined in (4.59) is a performance sensitivity function if [11,54,56]

1) $f(\cdot)$ is continuous jointly in its two arguments

2) $f>0 \rightarrow J(F,G) > J_1^*(G)$

3) $f=0 \rightarrow J(F,G) = J_1^*(G)$

In this thesis, attention has been confined to the following sensitivity function

$$\hat{S}(F,G) = J(F,G) - J_1^*(G) \tag{4.60}$$

The immediate problem is to minimize and maximize $\hat{S}$ with respect to F and G respectively, subject to (4.6), (4.56), (4.53) and (4.54). (4.60) modified to include the equality constraints (4.53) and (4.54) is

$$\begin{aligned} S = \mathrm{Tr}\ [N_1(F_1^*,G,\overset{*}{K})P_1 + N_2(F_1^*,G,\overset{*}{M})P_2] + \\ \frac{1}{2}E \int_0^\infty \underline{x}^T[Q + C^TF^TRFC - C_1{}^TG^TLGC_1]\underline{x}\ dt \\ - \frac{1}{2}E \int_0^\infty \overset{*}{\underline{x}}{}^T[Q + C^TF_1^{*T}RF_1^*C - C_1^TG^TLGC_1]\overset{*}{\underline{x}}\ dt \end{aligned} \tag{4.61}$$

where P_1 and P_2 are matrix Lagrange multipliers.

Thus the problem reduces to minimizing and maximizing (4.61) with respect to F and G respectively, subject to (4.6), (4.56) and

$$\dot{F} = 0,\ \dot{G} = 0,\ \dot{F}_1^* = 0 \tag{4.62}$$

The Hamiltonian H for this case is given by

$$\begin{aligned} H = \frac{1}{2}\,\mathrm{Tr}\ [(Q + C^TF^TRFC - C_1^TG^TLGC_1)\underline{x}\,\underline{x}^T] - \frac{1}{2}\,\mathrm{Tr}\ [(Q + C^TF_1^{*T}RF_1^*C - \\ C^TG^TLGC_1)\overset{*}{\underline{x}}\,\overset{*}{\underline{x}}{}^T] + \mathrm{Tr}\ [(A_0-B_0FC + DGC_1)\underline{x}\,\underline{\lambda}_x^T] + \\ \mathrm{Tr}\ [(A_0-B_0F_1^*C + DGC_1)\overset{*}{\underline{x}}\,\underline{\lambda}_{\overset{*}{x}}^T] \end{aligned} \tag{4.63}$$

with the 'costate' equations

$$\dot{\underline{\lambda}}_x = -\frac{\partial H}{\partial \underline{x}} = -(Q + C^T F^T RFC - C_1^T G^T LGC_1)\overset{*}{\underline{x}} - (A_0 - B_0 FC + DGC_1)^T \underline{\lambda}_x,$$

$$\underline{\lambda}_x(\infty) = 0 . \qquad (4.64)$$

$$\dot{\underline{\lambda}}_{\overset{*}{x}} = -\frac{\partial H}{\partial \overset{*}{\underline{x}}} = (Q + C^T F_1^{*T} R F_1^* C - C_1^T G^T LGC_1)\overset{*}{\underline{x}} - (A_0 + DGC_1 - B_0 \overset{*}{F} C)^T \underline{\lambda}_{\overset{*}{x}} ,$$

$$\underline{\lambda}_{\overset{*}{x}}(\infty) = 0, \qquad (4.65)$$

$$\dot{\Lambda}_F = -\frac{\partial H}{\partial F} = -RFC\,\underline{x}\underline{x}^T C^T + B_0^T \underline{\lambda}_x \underline{x}^T C^T, \qquad \Lambda_F(\infty) = 0 \qquad (4.66)$$

$$\dot{\Lambda}_G = -\frac{\partial H}{\partial G} = LGC_1 \underline{x}\underline{x}^T C^T - LGC_1 \overset{**}{\underline{x}\underline{x}} C_1^T - D^T \underline{\lambda}_x \underline{x}^T C_1^T - D^T \underline{\lambda}_{\overset{*}{x}} \overset{*}{\underline{x}}^T C_1^T,$$

$$\Lambda_G(\infty) = 0 \qquad (4.67)$$

$$\dot{\Lambda}_{\overset{*}{F}_1} = -\frac{\partial H}{\partial \overset{*}{F}_1} = R\overset{*}{F}_1 C \overset{**}{\underline{x}\underline{x}}^T C^T + B^T \underline{\lambda}_{\overset{*}{x}} \overset{*}{\underline{x}}^T C^T \qquad \Lambda_{\overset{*}{F}_1}(\infty) = 0 \qquad (4.68)$$

Now according to Lemma 4.1 and the necessary conditions (4.12), it can be easily seen after integrating (4.66) - (4.68) from 0 to ∞ that

$$0 = E[\Lambda_F(0)] = E\int_0^\infty [RFC\underline{x}\underline{x}^T C^T - B_0^T \underline{\lambda}_x \underline{x}^T C^T]\, dt \qquad (4.69)$$

$$0 = E[\Lambda_G(0) + \frac{\partial}{\partial G} \mathrm{Tr}\,\{N_1(\overset{*}{F}_1, G, \overset{*}{K})\, P_1 + N_2(\overset{*}{F}_1, G, \overset{*}{M}) P_2\}\ ]$$

$$= D^T \overset{*}{K} (P_1 + P_1^T) C_1^T - LGC_1(P_1 + P_1^T) C^T + D^T (P_2 + P_2^T) \overset{*}{M} C_1^T$$

$$- LG \int_0^\infty E(C_1 \underline{x}\underline{x}^T C_1^T)\, dt + LGC_1 \int_0^\infty E(\overset{**}{\underline{x}\underline{x}}^T)\, dt\; C_1^T$$

$$= D^T \int_0^\infty \underline{\lambda}_x \underline{x}^T C_1^T \, dt + D^T \int_0^\infty \underline{\lambda}_{\overset{*}{x}} \overset{*}{\underline{x}}^T C_1^T \qquad (4.70)$$

$$0 = E[\Lambda_{\overset{*}{F}_1}(0) + \frac{\partial}{\partial \overset{*}{F}_1} \operatorname{Tr}\{N_1(\overset{*}{F}_1,G,\overset{*}{K}) P_1 + N_2(\overset{*}{F}_1,G,M)P_2\}]$$

$$= -B_0^T\overset{*}{K}(P_1 + P_1^T)C^T + R\overset{*}{F}_1 C(P_1 + P_1^T)C^T - B_0^T(P_2 + P_2^T)\overset{*}{M}C^T$$

$$- R\overset{*}{F}_1 \int_0^\infty E(C\overset{*}{\underline{x}}\,\overset{*}{\underline{x}}^T C^T)\, dt - B_0^T \int_0^\infty E(\underline{\lambda}_{\overset{*}{x}}\overset{*}{\underline{x}}^T)C^T\, dt \qquad (4.71)$$

As usual, the following feedback solution is assumed:

$$\underline{\lambda}_x = \tilde{K}\underline{x}\ , \quad \underline{\lambda}_{\overset{*}{x}} = \overset{*}{\tilde{K}}\overset{*}{\underline{x}} \qquad (4.72)$$

This can be verified by (4.64) and (4.65) to be valid provided $\tilde{K}$ and $\overset{*}{K}$ satisfy

$$(A_0-B_0FC+DGC_1)^T\tilde{K}+\tilde{K}(A_0-B_0FC+DGC_1)+(Q+C^TF^TRFC-C_1^TG^TLGC_1) = 0, \qquad (4.73)$$

$$(A_0-B_0\overset{*}{F}_1C+DGC_1)^T\overset{*}{\tilde{K}}+\overset{*}{\tilde{K}}(A_0-B_0\overset{*}{F}_1C+DGC_1)-(Q+C^T\overset{*}{F}{}_1^TR\overset{*}{F}_1C-C_1^TG^TLGC_1)=0 \cdot \qquad (4.74)$$

Furthermore

$$\tilde{M} \triangleq \int_0^\infty E[\underline{x}\underline{x}^T]\, dt, \quad \overset{*}{\tilde{M}} \triangleq \int_0^\infty E[\overset{*}{\underline{x}}\,\overset{*}{\underline{x}}^T]\, dt \qquad (4.75)$$

are given by

$$(A_0-B_0FC+DGC_1)\tilde{M}+M(A_0-BFC+DGC_1)^T + I = 0 \quad , \qquad (4.76)$$

$$(A_0-B_0\overset{*}{F}_1C+DGC_1)\overset{*}{\tilde{M}}+\overset{*}{\tilde{M}}(A_0-B_0\overset{*}{F}_1C+DGC_1)^T + I = 0 \quad . \qquad (4.77)$$

It can be easily seen from (4.53), (4.54), (4.75) and (4.77) that

$$\overset{*}{\tilde{K}} = -\overset{*}{K}, \quad \overset{*}{M} = \overset{*}{\tilde{M}} \quad . \qquad (4.78)$$

Using (4.77), (4.72), and (4.75), (4.71) reduces to

$$0 = -B_0^T\overset{*}{K}\,(P_1+P_1^T)C^T+R\overset{*}{F}_1C\,(P_1+P_1^T)C^T - B_0^T(P_2+P_2^T)\overset{*}{M}C^T-R\overset{*}{F}_1C\overset{*}{\tilde{M}}C^T-B^T\overset{*}{\tilde{K}}\overset{*}{M}C^T \tag{4.79}$$

Using (4.52), (4.79) reduces to

$$0 = -B_0^T\overset{*}{K}\,(P_1+P_1^T)C^T+R\overset{*}{F}_1C\,(P_1+P_1^T)C^T - B_0^T\,(P_2+P_2^T)\overset{*}{M}C^T, \tag{4.80}$$

Now two cases may arise.

Case 1. $(P_1 + P_1^T) \neq 0$

Solving for $\overset{*}{F}_1$, (4.80) yields

$$\overset{*}{F}_1 = R^{-1}B_0^T\,[(P_2+P_2^T)\overset{*}{M} + \overset{*}{K}(P_1+P_1^T)]C^T[C(P_1+P_1^T)C^T]^{-1}. \tag{4.81}$$

Comparing (4.81) with (4.52) yields

$$(P_1+P_1^T) = \overset{*}{M} \text{ and } (P_2+P_2^T) = 0 \tag{4.82}$$

Using (4.82), (4.72) and (4.75), (4.69) and (4.70) give basically the same result as obtained via minimax performance control.

Case 2. $(P_1 + P_1^T) = 0$ (4.83)

Using (4.83), (4.80) gives

$$(P_2 + P_2^T) = 0 \tag{4.84}$$

Substitution of (4.83), (4.84), (4.72) and (4.75) into (4.69) and (4.70) gives

$$P = R^{-1}B_0^T\,\tilde{K}\tilde{M}C^T\,(C\tilde{M}C^T)^{-1} \tag{4.85}$$

$$G = L^{-1}D^T(\tilde{K}\tilde{M}+\overset{*}{K}\,\overset{*}{\tilde{M}})C_1^T\,[C_1(\tilde{M}-\overset{*}{\tilde{M}})C_1^T]^{-1} \tag{4.86}$$

Thus it is clear that soltuion of F requires simultaneous solution (4.85) - (4.86) together with (4.73) - (4.74) and (4.76) - (4.77).

Remark 4.3

It can be easily verified that

(a) The optimal cost $S = \frac{1}{2}\,\mathrm{Tr}\,(\tilde{K} + \overset{*}{\tilde{K}})$, $E[\underline{x}(to)\underline{x}^T(to)] =$

$$E[\overset{*}{\underline{x}}(to)\overset{*}{x}^T(to)] = I \tag{4.87}$$

(b) $$\min_F \max_G \hat{S}(F,G) = \max_G \min_F \hat{S}(F,G) \tag{4.88}$$

It can be seen from (4.55) that $\overset{*}{u}$ is constrained. A rather optimistic situation will be to allow $\overset{*}{u}$ to have complete state feedback. The problem here is to minimize and maximize with respect to F and G respectively. The following sensitivity criterion

$$S = J(F,G) - \overset{*}{J}_2(G) \tag{4.89}$$

where J(F,G) is given by (4.7) and $\overset{*}{J}_2(G)$ is given by

$$\overset{*}{J}_1(G) = \min_u \frac{1}{2} \int_0^\infty [x^T(Q - C_1^T G^T L G C_1)\underline{x} - \underline{u}^T R \underline{u}]\, dt \tag{4.90}$$

This is a special case of the constrained feedback problem and the required result is obtained by setting C=I in (4.55). Thus the required feedback matrices F and G are given by

$$F = R^{-1} B_0^T \tilde{K} M C^T (CMC^T)^{-1} \tag{4.91}$$

$$G = L^{-1} D^T (\tilde{K}M + \overset{*}{\tilde{K}}\overset{*}{M}) C_1^T\, [C_1(M - \overset{*}{M})C_1^T]^{-1} \tag{4.92}$$

where $\tilde{K}$, $\overset{*}{\tilde{K}}$, M, $\overset{*}{M}$ are given by

$$(A_0 - B_0 FC + DGC_1)^T \tilde{K} + \tilde{K}(A_0 - B_0 FC + DGC_1) + Q + C^T F^T RFC - C_1^T G^T LGC_1 = 0 \tag{4.93}$$

$$(A_0 + B_0 R^{-1} B_0^T \overset{*}{\tilde{K}} + DGC_1)^T \overset{*}{\tilde{K}} + \overset{*}{\tilde{K}}(A_0 + B_0 R^{-1} B_0^T \overset{*}{\tilde{K}} + DGC_1) - (Q + \overset{*}{\tilde{K}} B_0 R^{-1} B_0^T \overset{*}{\tilde{K}} - C_1^T G^T LGC_1) = 0 \tag{4.94}$$

$$(A_0-B_0FC+DGC_1)M + M(A_0-B_0FC+DGC_1)^T + I = 0 , \tag{4.95}$$

$$(A_0+B_0R^{-1}B_0^T\overset{*}{\tilde{K}} + DGC_1)\overset{*}{M} + \overset{*}{M}(A_0+B_0R^{-1}B_0^T\overset{*}{\tilde{K}}+DGC_1)^T + I = 0 \tag{4.96}$$

4.7 COMPUTATION OF F, $\overset{*}{F}_1$, G

An algorithm similar to that mentioned earlier can be used to solve for the feedback matrices. As before, at iteration n, positive definite matrix $\tilde{K}_n$ and negative definite matrix $\overset{*}{\tilde{K}}_n$ are obtained from the following linearized equations:

$$(A-BF_nC+DG_nC)^T\tilde{K}_{n+1}+\tilde{K}_{n+1}(A-BF_nC+DG_nC)+Q+C^TF_n^TRF_nC-C_1^TG_n^TLG_nC = 0 \tag{4.97}$$

$$(A-B\overset{*}{F}_{1n}C+DG_nC_1)^T\overset{*}{\tilde{K}}_{n+1} + \overset{*}{\tilde{K}}_{n+1}(A-B\overset{*}{F}_{1n}C+DG_nC_1)-(Q+C^T\overset{*}{F}_{1n}R\overset{*}{F}_{1n}C-C_1^TG_n^TLG_nC_1)=0 \tag{4.98}$$

F_n $\overset{*}{F}_{1n}$, G_n, M_n, $\overset{*}{M}_n$ are then obtained by simultaneous solution of the following nonlinear equations:

$$\overset{*}{F}_{1n} = R^{-1}B_0^T\overset{*}{\tilde{K}}_n\overset{*}{M}_nC^T(C\overset{*}{M}_nC^T)^{-1} \tag{4.99}$$

$$(A_0+DG_nC_1-B_0\overset{*}{F}_{1n}C)^T\overset{*}{\tilde{K}}_n + \overset{*}{\tilde{K}}_n(A_0+DG_nC_1-B\overset{*}{F}_{1n}C)+Q+C^T\overset{*}{F}_{1n}{}^TR\overset{*}{F}_{1n}C$$
$$-C_1^TG^TLGC_1 = 0, \tag{4.100}$$

$$(A_0+DG_nC_1-B_0\overset{*}{F}_1C)\overset{*}{M}_n + \overset{*}{M}_n(A_0+DG_nC_1-B_0\overset{*}{F}_1C)^T+I = 0, \tag{4.101}$$

$$F_n = R^{-1}B_0^T\overset{*}{\tilde{K}}_n\overset{*}{M}_nC^T(CM_nC^T)^{-1} \tag{4.102}$$

$$G_n = L^{-1}D^T(\tilde{K}_nM_n + \overset{*}{\tilde{K}}_n\overset{*}{M}_n)C_1^T\,[C_1(M_n-\overset{*}{M}_n)C_1^T]^{-1} \tag{4.103}$$

The nonlinear equations (4.99) - (4.103) can be solved at each iteration n, by standard conjugate gradient technique. The algorithm starts as follows:

a) Assume initial guesses F_0, F_{10}, G_0 such that $(A_0-B_0F_0C+DG_0C)$ and $(A_0-B_0\overset{*}{F}_{10}C+DG_0C)$ are stable and also $(Q+C^TF_0^TRF_0C-C_1^TG_0^TLG_0C)$ and $(Q+C^T\overset{*}{F}{}_{10}^TRF_{10}C-C^TG_0^TLG_0C)$ are positive definite.

b) Solve (4.97) and (4.96) and store the values $\bar{K}_{n+1}$ and $\overset{*}{\tilde{K}}_{n+1}$

c) Using $\bar{K}_{n+1}$ and $\overset{*}{\tilde{K}}_{n+1}$, solve (4.99) - (4.103) by conjugate gradient technique to obtain F_{n+1}, $\overset{*}{F}_{1n+1}$, G_{n+1}, M_{n+1}, and $\overset{*}{M}_{n+1}$.

d) With these values, $\bar{K}_n$ and $\overset{*}{\tilde{K}}_n$ are updated and the iteration continues till the specified stopping criterion is met.

4.8 SOME STABILITY BOUNDS IN TERMS OF PARAMETER VARIATION

The perturbed system (4.1) can be represented as

$$\dot{\underline{x}} = A_0\underline{x} + B_0\underline{u} + (A-A_0)\underline{x} + (B-B_0)\underline{u} \tag{4.104}$$

$$= [(A_0-B_0FC) + \Delta A + \Delta BFC]\underline{x} \tag{4.105}$$

where F is given by (4.24). The following analysis is also true for F, given by (4.47), (4.85), and (4.31) .

Define the Liapunov function V(x) as

$$V(\underline{x}) = \frac{1}{2}\underline{x}^TK\underline{x} \tag{4.106}$$

where K, a positive definite matrix, satisfies (4.26). The time derivative $\dot{V}(\underline{x})$ of $V(\underline{x})$, evaluated along the trajectory (4.105), is given by

$$\dot{V}(\underline{x}) = -\frac{1}{2}\underline{x}^T[-(A_0-B_0FC)^TK-K(A_0-B_0FC)-2K\Delta A-2K\Delta BFC]\underline{x} \tag{4.107}$$

Using (4.26), (4.107) reduces to

$$\dot{V}(\underline{x}) = -\frac{1}{2}\underline{x}^T[(Q+C^TF^TRFC-C_1^TG^TLGC_1)-2K\Delta A + 2K(DGC_1K^{-1} + \Delta BFCK^{-1})K]\underline{x} \tag{4.108}$$

Let the norms of $\underline{x}$ and matrix A are defined as follows

$$||\underline{x}|| \triangleq (\underline{x}^T\underline{x})^{1/2}$$

$$||A|| \triangleq \sup_{||\underline{x}||=1} ||A\underline{x}|| \text{ so that } ||A|| = \lambda_{max}^{1/2}[A^TA]$$

where $\lambda_{max}(\cdot)$ is the maximum eigenvalue of a symmetric positive definite matrix $(\cdot)$. Restricting terms in the bracket in (4.108) to be at least p.s.d. to guarantee stability of perturbed system (4.105), the bounds on ΔA and ΔB can be found as

$$||\Delta A|| \leq \frac{\lambda_{min}(Q + C^TF^TRFC-C_1^TG^TLGC_1)}{2||K||} \tag{4.109}$$

$$||\Delta\ || \leq \frac{||DGC_1K^{-1}||}{||FCK^{-1}||} \tag{4.110}$$

It should be noted that $(Q + C^TF^TRFC-C_1^TG^TLGC_1)$ is at least positive semidefinite under the condition mentioned in Lemma 4.2

4.9 EXAMPLE

Following example will be considered to illustrate various theoretical formulation discussed earlier.

Let the system be described by

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u = A\underline{x} + \underline{b}_o u \tag{4.111}$$

$$y = [0\ 1]\ \underline{x} = c\underline{x} \tag{4.112}$$

with controller

$$u = -\underline{f}y = -f\,[0\ 1]\ \underline{x} \tag{4.113}$$

'a' in (4.11) is the uncertain parameter. Let the nominal system correspond to the one with a = 0. Thus (4.111) can be written as

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi$$

$$= A_0\underline{x} + \underline{b}_0 u + \underline{d}\ \xi \tag{4.114}$$

with ξ constrained to be

$$\xi = gx_2 = gy = gc\underline{x} \tag{4.115}$$

where g is the gain (i.e., an estimate of the uncertainty) to be determined.

Consider the following performance criterion

$$\hat{J} = \min_{f}\ \max_{g} \left[E\ 1/2 \int_0^{\infty} [\underline{x}^T Q \underline{x} + Ru^2 - L\xi^2]dt \right] \tag{4.116}$$

with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = 1, \quad E[\underline{x}(0)\underline{x}^T(0)] = I$$

f optimal for the nominal system (i.e., with no parameter uncertainty) is determined to be 0.816. f and g for different values of L are obtained through minimax procedures (i), (ii) and (iii) and using algorithms of sections IV and VII. Simultaneous nonlinear algebraic equations, e.g., equations (4.30) - (4.32) of minimax procedure (i) and (ii) have been solved at each iteration using a conjugate gradient technique. The computed values of f for different values of L are tabulated for various minimax procedures.

Table 4.1

L^{-1} \ f	Minimax Performance Control		Minimax Sensitivity Control
	Criterion (i)	Criterion (ii)	Criterion (iii)
0.1	.878	.914	.824
0.2	.947	.96	.844
0.3	1.03	1.1	1.08
0.5	1.265	1.277	1.354
0.7	1.69	1.71	1.815

To study the effect of uncertainty, J is computed for different values of 'a' using f as tabulated above and

$$J = \frac{1}{2} \operatorname{Tr} K = \frac{1}{2} E \int_0^\infty [\underline{x}^T Q \underline{x} + Rf^2 x^2] dt \qquad (4.117)$$

where K is the solution of

$$(A-b_0\ fc)^T K + K(A-b_0\ fc) + Q + c^T f^2 c = 0 \qquad (4.118)$$

and are plotted as shown in Figures (4.1) - (4.2). In Figure (4.1), cost J is plotted as a function of the uncertain parameter 'a', using the feedback gain as determined in minimax performance sensitivity criterion (N), for difference values of L. For comparison, we have also plotted the 'optimal' cost as a function of parameter 'a' if it were known. In Figure 2, different design criterion are compared as "a" varies from nominal. It can be seen that the minimax procedure effects the design of f in such a way that the system will operate acceptably over a wider range of parameters than a purely nominal design.

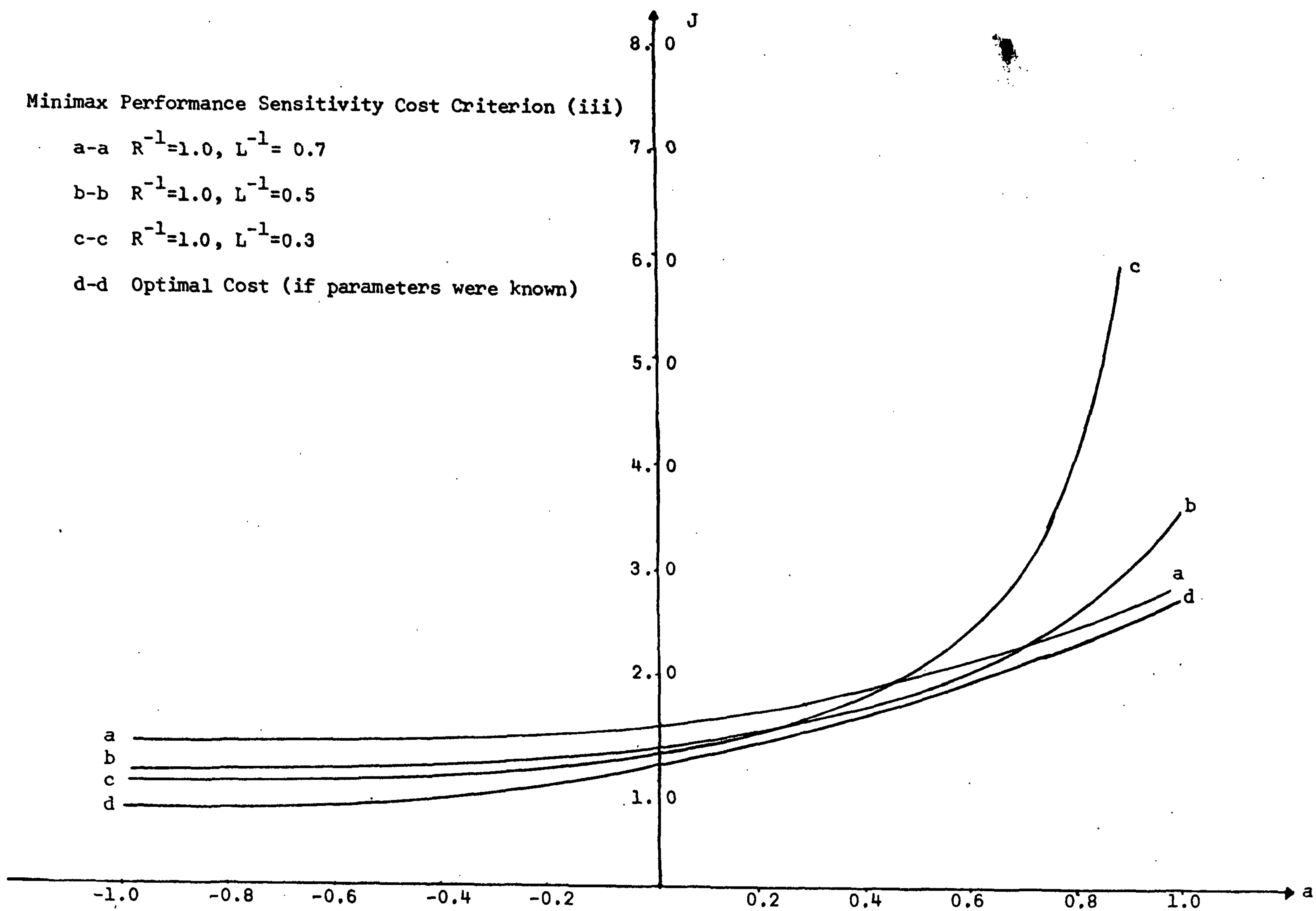
Minimax Performance Sensitivity Cost Criterion (iii)
a-a R^{-1}=1.0, L^{-1}= 0.7
b-b R^{-1}=1.0, L^{-1}=0.5
c-c R^{-1}=1.0, L^{-1}=0.3
d-d Optimal Cost (if parameters were known)
J
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
a
-1.0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1.0
a
b
c
d

Figure 4.1

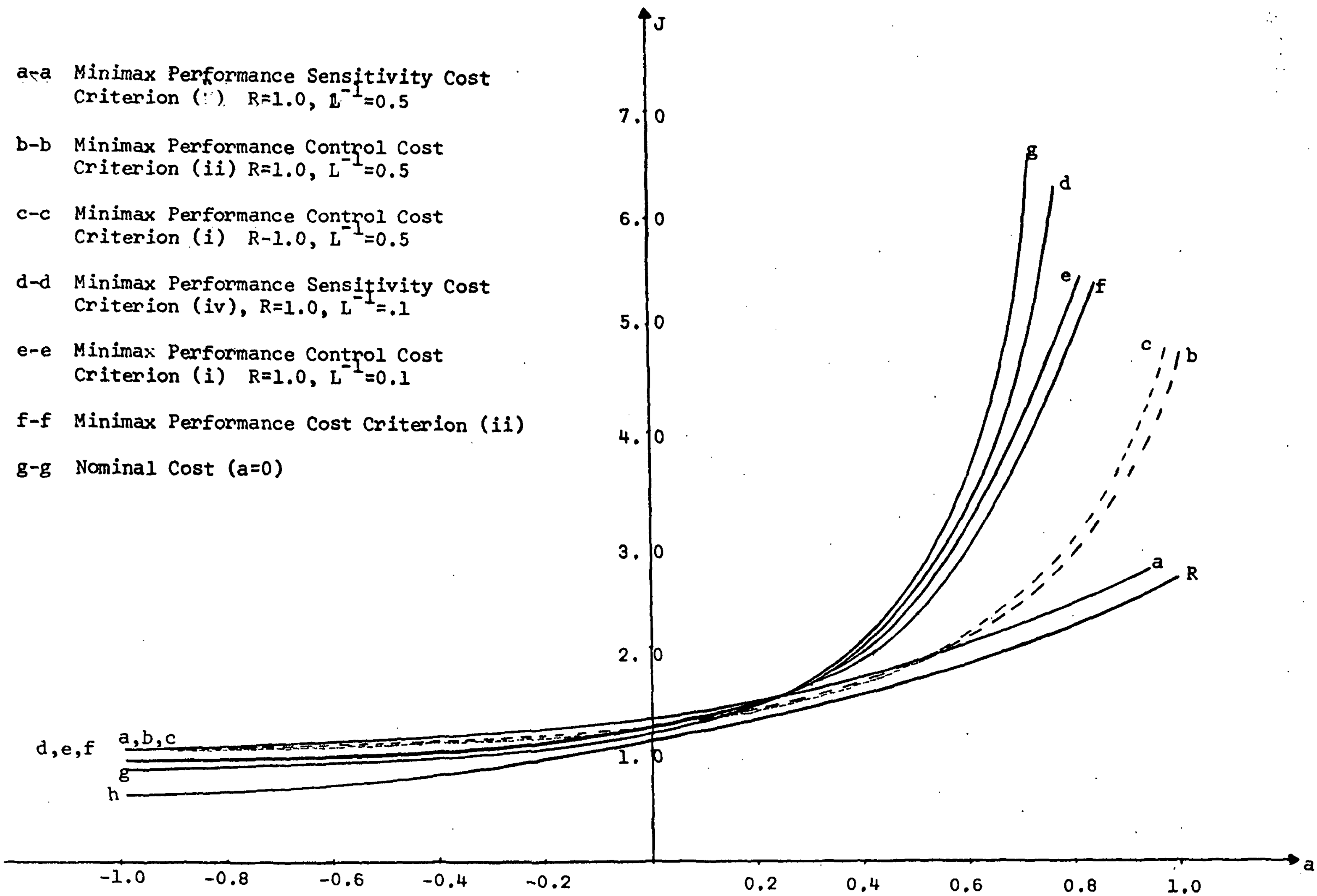
a-a Minimax Performance Sensitivity Cost Criterion (i) R=1.0, L^{-1}=0.5
b-b Minimax Performance Control Cost Criterion (ii) R=1.0, L^{-1}=0.5
c-c Minimax Performance Control Cost Criterion (i) R-1.0, L^{-1}=0.5
d-d Minimax Performance Sensitivity Cost Criterion (iv), R=1.0, L^{-1}=.1
e-e Minimax Performance Control Cost Criterion (i) R=1.0, L^{-1}=0.1
f-f Minimax Performance Cost Criterion (ii)
g-g Nominal Cost (a=0)
J
7.0
6.0
5.0
4.0
3.0
2.0
1.0
a
-1.0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
1.0
a,b,c
d,e,f
g
h
a
R
b
c
d
e
f
g

Figure 4,2

For any particular parameter set, however, the nominal design may be superior. It is also evident from Figure (4.1) - (4.2) that the penalty on the uncertainty should be relaxed to accommodate larger parameter variation. For limited parameter variation, different design approaches nearly identical performance whereas the minimax performance sensitivity control offers better design when the parameter variation is large.

4.10 CONCLUSION

The problem of controlling a system with parameter uncertainty has been treated using only available output feedback. Since the controller is designed with incomplete state feedback, the uncertainty is likewise constrained. To achieve a design via optimization, a quadratic cost function involving the system state, the control and the uncertainty vector, is defined and the optimal feedback matrices relating the control and the uncertainty are chosen to minimize and maximize, respectively, the performance criterion. The resulting controller is linear, the optimal feedback matrix being specified by a set of simultaneous nonlinear equations. The above procedure usually leads to a conservative design. To meet this objection, a sensitivity or loss criterion is defined. Minimaximization of the sensitivity function with respect to feedback matrices yields a linear controller. The optimal feedback matrices must satisfy a set of nonlinear simultaneous algebraic equations. Some algorithms to solve these algebraic minimax problems and their convergence properties are discussed. An example has been treated to illustrate the various formulations presented in this chapter.

It is assumed throughout this chapter that the nominal system is stabilized with output feedback. Even if the nominal system is stabilizable with output feedback, various minimax design procedures allow only certain parameters in system matrices to vary in order to maintain stability of the perturbed system. To relax these limitations, the required control can be generated as the response of a linear dynamic system whose input is the available outputs. Various gain matrices specifying the dynamic compensator can be determined in the similar as reported in previous chapter. An important limitation of the various design techniques presented in this chapter is the fact that the measurements are assumed to be noise-free and also the system is not subjected to any disturbances. In the next chapter, the stochastic version of the output feedback problem will be explained with and without an estimator.

V OPTIMAL INCOMPLETE STATE FEEDBACK CONTROLLERS FOR STOCHASTIC SYSTEM

5.1 INTRODUCTION

The problem of optimal output feedback for system with parameter uncertainty has been explored in the previous chapter where it was assumed that the measurements are noise-free and the disturbances in the system are negligible. This chapter will treat, among other things, the determination of the optimal output feedback when the system is excited by a white noise disturbance with and without measurement noise. No parameter uncertainty is assumed. Next a design procedure is developed for generating an optimal control as the output of a dynamic compensator. The input to the dynamic compensator is the available noisy output measurements of the system. It is well known[55] that if the system is linear, is excited by white gaussian noise and the measurement noise is also gaussian, the estimator and controller can be designed independently. This is due to so-called separation theorem. The estimator is the well-known Kalman filter whose dimension is equal to that of the system. The present chapter deals with the problem of designing a combined estimator and controller. The formulation used is more general than designing Kalman filter since the dimension of the estimator or dynamic compensator is arbitrary. Since the dimensionality of

Kalman filter is often a practical limitation, this problem of designing a reduced order estimator is not only challenging but may have substantial practical benefit. An important question to be answered, in this context, is whether the control is composed of both the estimator output and the noisy observations or only a linear feedback of the output of the estimator as in the design procedure via separation principle. The present design procedure involves i) a precise formulation of mathematical optimization problem, ii) determination of various gains specifying the dynamic compensator and feedback controller. The various gains should be independent of initial plant state so that compensator gains do not have to be tuned up every time the disturbance changes the plant state. Because of the above problem, a design procedure that is optimal only "on the average" will be presented.

The state and output equations are given by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} + D\underline{w}_1 \tag{5.1}$$

$$\underline{y} = C\underline{x} + \underline{v} \tag{5.2}$$

where the state $\underline{x}$ (an n-vector) is the signal process; the output y (an m-vector) is the observation process; $\underline{u}$ (an r-vector) is the control and the vectors $\underline{w}_1(t)$ and $\underline{v}(t)$ are zero mean white noise processes of respective dimensions r and m. The covariances of these processes are given by

$$E\,\{\underline{w}_1(t_1)\underline{w}_1^T(t_2)\} = Q_1\delta(t_1-t_2) \tag{5.3a}$$

$$E\,\{\underline{v}(t_1)\underline{v}^T(t_2)\} = R_1\delta(t_1-t_2) \tag{5.3b}$$

$$E\,\{\underline{v}(t_1)\underline{w}_1^T(t_2)\} = L_1\delta(t_1-t_2) \tag{5.3c}$$

where Q_1 and R_1 are positive definite matrices, and $E(\cdot)$ represents the expected value of $(\cdot)$. The following constraints on the control will be explored in this chapter:

Case 1: Optimal output feedback with no measurement noise

$$\underline{u} = -N\underline{y} \tag{5.4}$$

$$R_1 = 0 \tag{5.5}$$

Case 2: Optimal output feedback controller with dynamic compensator

$$\underline{u} = Hz + N\underline{y} \tag{5.6}$$

where $\underline{z}$ is the compensator state

$$\dot{\underline{z}} = F\underline{z} + G\underline{y} \tag{5.7}$$

A schematic diagram is shown in Figure 5.1

Case 3: Optimal output feedback with white measurement noise

$$\underline{u} = N\underline{y} \tag{5.8}$$

Case 4: Optimal output feedback with nonwhite measurement noise

The controller is given by (5.8) and the output equation is described as

$$\underline{y} = C\underline{x} + H\underline{z} \tag{5.9}$$

where the non-white noise $\underline{z}$ is generated as the response of a linear dynamic system to the white noise $\underline{v}$

$$\dot{\underline{z}} = F\underline{z} + G\underline{v} \quad . \tag{5.10}$$

F, G, H, C are all specified but N is not.

5.2 OPTIMAL OUTPUT FEEDBACK WITH NO MEASUREMENT NOISE

5.2.1 STATEMENT AND FORMULATION OF THE PROBLEM

The initial problem in this chapter is the control of the time-invariant system (5.1) with outputs

$$\underline{y}(t) = C\underline{x}(t) \tag{5.11}$$

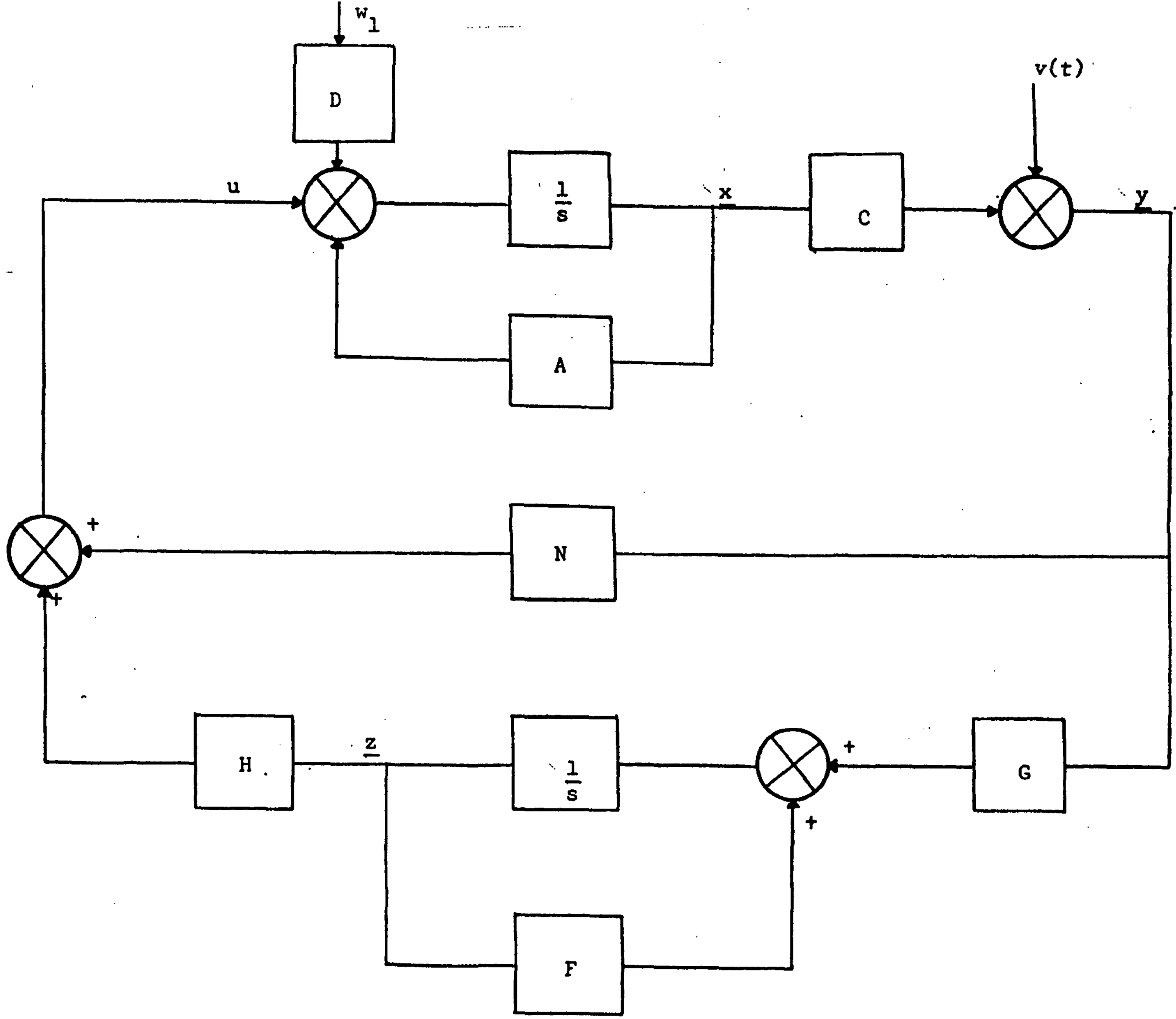

Figure 5.1 - Dynamic Compensator for Stochastic System

and with a controller

$$u(t) = -Ny(t) = -NC\underline{x}(t) \tag{5.12}$$

The closed-loop system is given by

$$\dot{\underline{x}} = [A-BNC]\underline{x} + D\underline{w}_1 \ . \tag{5.13}$$

The solution of (5.13) may be written

$$\underline{x}(t) = \phi(t-to)x(to) + \int_{to}^{t} \phi(t-\tau)Dw_1(\tau)d\tau \tag{5.14}$$

where the state transition matrix $\phi(t)$ satisfies

$$\dot{\phi} = (A-BNC)\phi \ . \tag{5.15}$$

The problem is to determine N by minimizing

$$J = \frac{1}{2} E \int_{to}^{t_f} [\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u}]dt = \frac{1}{2} E \int_{to}^{t_f} \underline{x}^T [Q+C^T N^T RNC]\underline{x} \ dt$$

with respect to N and subject to (5.13). (5.16)

It is clear from (5.16) that J is determined by the initial state $\underline{x}(to)$ as well as matrix N. In order to make the optimum N independent of $\underline{x}(to)$, the expectation operation will be carried out over the initial conditions also. The necessary condition that N should minimize (5.16), requires[2]

$$\frac{\partial J}{\partial N} = \frac{1}{2} \frac{\partial}{\partial N} E \int_{to}^{t_f} \underline{x}^T [Q+C^T N^T RNC]\underline{x} \ dt$$

$$= \frac{1}{2} E \int_{to}^{t_f} \frac{\partial}{\partial N} \underline{x}^T [Q+C^T N^T RNC]\underline{x} \ dt = 0 \tag{5.17}$$

The interchange of order of expectation (i.e., integration) and differentiation is assumed to be valid[10].

5.2.2 RESULT

The optimal feedback gain

$$N = R^{-1}B^T KPC^T [CPC^T]^{-1} \tag{5.18}$$

where K and P are the solution of

$$(A-BNC)^T K+K(A-BNC)+Q+C^T N^T RNC = 0 \tag{5.19}$$

$$P(A-BNC)^T + (A-BNC)^T P + DQ_1 D^T = 0 \tag{5.20}$$

5.2.3 DERIVATION OF THE RESULT

The partial derivative of (5.17) will be evaluated by the application of Lemma 4.1 and by treating the elements of N as additional "states" which satisfy

$$\dot{N} = 0 \tag{5.21}$$

Vector multiplier $\underline{\lambda}_x$ will be used for the regular state constraint (5.12) and matrix multiplier Λ_N will be used for constraint (5.21). Thus the Hamiltonian for (5.16) and (5.13) is

$$\begin{aligned} H &= \frac{1}{2}\, \underline{x}^T[Q+C^T N^T RNC]\underline{x} + \underline{\lambda}_x^T[(A-BNC)\underline{x} + D\underline{w}_1] \\ &= \mathrm{Tr}\,[\, \frac{1}{2}\,(Q+C^T N^T RNC)\underline{x}\,\underline{x}^T + \{(A-BNC)\underline{x} + D\underline{w}\}\,\underline{\lambda}_x^T] \end{aligned} \tag{5.22}$$

with the costate equations

$$\dot{\underline{\lambda}}_x = -\frac{\partial H}{\partial \underline{x}} = -\,(A-BNC)^T\underline{\lambda}_x - (Q+C^T N^T RNC)\underline{x}\,, \qquad \underline{\lambda}_x(t_f) = 0; \tag{5.23}$$

$$\dot{\Lambda}_N = -\frac{\partial H}{\partial N} = -\,RNC\underline{x}\,\underline{x}^T C^T + B^T\underline{\lambda}_x x^T C^T, \qquad \Lambda(t_f) = 0. \tag{5.24}$$

According to the Lemma 4.1, the necessary condition and integrated forms of (5.24), it can be seen that

$$0 = E\,[\,\frac{\partial J}{\partial N}\,] = E\int_{to}^{t_f}[RNC\underline{x}\,\underline{x}^T C^T - B^T\underline{\lambda}_x\underline{x}^T C^T]\,dt \tag{5.25}$$

Thus (5.25) yields

$$N = R^{-1} \int_{to}^{t_f} B^T E[\underline{\lambda}_x \underline{x}^T] C^T \, dt \left[\int_{to}^{t_f} CE[\underline{x}\ \underline{x}^T] C^T dt \right]^{-1} . \qquad (5.26)$$

Now assume a solution for $\underline{\lambda}_x$, of the form

$$\underline{\lambda}_x = K\underline{x} + \underline{n} \quad ; \qquad (5.27)$$

Then (5.13), (5.23) and (5.27) give

$$-\dot{K} = (A-BNC)^T K + K(A-BNC) + Q + C^T N^T RNC, \quad K(t_f) = 0 \ ; \qquad (5.28)$$

$$\dot{\eta} = -(A-BNC)^T \eta + KDw_1 \quad , \qquad \eta(t_f) = 0 \ . \qquad (5.29)$$

The solution of (5.29) is given by

$$\eta(t) = - \int_{t_o}^{t} \phi(t-\tau) KDw_1(\tau) d\tau \ . \qquad (5.30)$$

In order to complete (5.22) we now obtain the required averages.

First, it can be easily seen from (5.14) and (5.3) that

$$P \overset{\Delta}{=} E[\underline{x}\ \underline{x}^T] = \phi(t-to)P(to)\phi^T(t-to) + \int_{t_f}^{t} \phi(t-\tau)DQ_1 D^T \phi^T(t-\tau) d\tau \ , \qquad (5.31)$$

where $E[\underline{x}(to)\underline{x}^T(to)] = P(to) \quad .$ (5.32)

Solving (5.31) is equivalent to solving the differential equation

$$\dot{P} = (A-BNC)P + P(A-BNC)^T + DQ_1 D^T \qquad (5.33)$$

which may be verified by differentiation of P with respect to t.

Next,

$$E[\underline{\lambda}_x \underline{x}^T] = KE[\underline{x}\,\underline{x}^T] + E[\eta \underline{x}^T]$$

$$= KP + E\,[\{\int_{t_f}^{t} \phi(t-\tau)KDw_1(\tau)d\tau\}$$

$$\{\underline{x}^T(to)\phi^T(t-to) + \int_{to}^{t} w_1^T(\tau_1)D^T\phi^T(t-\tau)d\tau_1\}]$$

$$= KP \,, \tag{5.34}$$

since the above integrals do not overlap.

Thus (5.26) reduces to

$$N = R^{-1} \int_{to}^{t_f} B^T KPC^T dt \; [\int_{to}^{t_f} CPC^T dt]^{-1} . \tag{5.35}$$

As $t_f \to \infty$, $t_o = 0$, K and P are the steady state solutions of (5.28) and (5.33), respectively. Consequently (5.35) is indeterminate. Applying L' Hospital Rule as $t_f \to \infty$, (5.35) reduces to

$$N = R^{-1}B^T KPC^T [CPC^T]^{-1} \tag{5.36}$$

where K and P are the steady state solutions of (5.28) and (5.33) respectively.

5.2.4 COMMENTS

It should be noted that if C = I,

$$N = R^{-1}B^T K . \tag{5.37}$$

(5.37) implies that the optimal feedback for the deterministic case (no plant disturbance) is the same as for stochastic case (without measurement noise) if all the states are available for feedback. This is not true with incomplete feedback. (5.18) - (5.20) can be solved basically with the same algorithm as suggested in Chapter 4.

5.3 OPTIMAL DYNAMIC COMPENSATOR

5.3.1 STATEMENT AND FORMULATION OF THE PROBLEM

Consider now a control law

$$\underline{u} = H\underline{z} + N\underline{y} = H\underline{z} + NC\underline{x} + N\underline{v} \tag{5.38}$$

where $\underline{z}$ is the state of a time-invariant dynamic compensator of fixed order (s_1),

$$\dot{\underline{z}} = F\underline{z} + G\underline{y} = Fz + GC\underline{x} + G\underline{v} \quad . \tag{5.39}$$

The problem is to determine the time-invariant matrices $F(s_1 x s_1)$, $G(s_1 x m)$, $H(r x s_1)$ and $N(r x m)$ by minimizing the quadratic criterion

$$J = \frac{1}{2}E \int_{to}^{t_f} \{\underline{x}^T Q\underline{x} + \underline{u}^T R\underline{u}\}dt \tag{5.40}$$

where Q is positive semidefinite and R is positive definite. Note that this formulation is general enough to include the Kalman filter. In order to avoid the dependence of various gain matrices to be determined on the initial state (both the plant and the compensator), the expectation operation in (5.40) will also be carried out over the initial states by treating $\underline{x}(o)$ and $\underline{z}(o)$ to be random variable with

$$E\{\underline{x}(o)\underline{x}^T(o)\} = X_0 \tag{5.41}$$

$$E\{\underline{z}(o)\underline{z}^T(o)\} = Z_0 \tag{5.42}$$

Using (5.28), (5.1) and (5.39) can be written as

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{z}} \end{bmatrix} = \begin{bmatrix} A+BNC & BH \\ GC & F \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} + \begin{bmatrix} BN & D \\ G & 0 \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{w}_1 \end{bmatrix} \tag{5.43}$$

and (5.40) becomes

$$J = \frac{1}{2} E \int_{to}^{t_f} [\underline{x}^T, \underline{z}^T] \begin{bmatrix} Q+C^T N^T RNC & C^T N^T RH \\ H^T RNC & H^T RH \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix}$$

$$+ [\underline{v}^T, \underline{w}_1^T] \begin{bmatrix} N^T RNC & N^T RH \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} + [\underline{x}^T z^T] \begin{bmatrix} C^T N^T RN & 0 \\ H^T RN & 0 \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{w}_1 \end{bmatrix}$$

$$+ [v^T w_1^T] \begin{bmatrix} N^T RN & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{w}_1 \end{bmatrix} \tag{5.44}$$

Defining

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} N & H \\ G & F \end{bmatrix}, \quad \hat{I} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{s} = \begin{bmatrix} \underline{v} \\ \underline{w}_1 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \tag{5.45}$$

(5.43) and (5.44) reduce to

$$\dot{\underline{w}} = (\hat{A}+\hat{B}P_1\hat{C})\underline{w} + (\hat{D}+\hat{B}P_1\hat{I})\underline{s} \tag{5.46}$$

$$J = \frac{1}{2} E \int_{to}^{t_f} \{\underline{w}^T[\hat{Q}+\hat{C}^T P_1^T \hat{R} P_1 \hat{C}]\underline{w}+\underline{w}^T \hat{C}^T P_1^T \hat{R} P_1 \hat{I}\underline{s} + \underline{s}^T \hat{I} P_1^T \hat{R} P_1 \hat{C}\underline{w}+\underline{s}^T \hat{I} P_1^T \hat{R} P_1 \hat{I}\underline{s}\}dt$$

$$\stackrel{\Delta}{=} \frac{1}{2} E \int_{to}^{t_f} V(t)dt \tag{5.47}$$

Thus the problem is to minimize of (5.47) with respect to P_1 subject to (5.45). The necessary condition that P_1 should minimize J requires

$$\frac{\partial J}{\partial P_1} = \frac{1}{2} E \int_{to}^{t_f} \frac{\partial V}{\partial P_1} dt = 0 \quad . \tag{5.48}$$

5.3.2 RESULTS

The optimal gain matrix P_1 is the solution of

$$RP_1[\hat{C}P\hat{C}^T + \frac{\hat{C}}{2}(\hat{D}+\hat{B}P_1\hat{I})\tilde{R}_1\hat{I} + \frac{\hat{I}}{2}\tilde{R}_1(\hat{D}+\hat{B}P_1\hat{I})^T\hat{C}^T] + \hat{B}^TKP\hat{C}^T = 0 \tag{5.49}$$

with $N = 0$ (5.50)

where K and P are the solutions of

$$(\hat{A}+\hat{B}P_1\hat{C})^TK+K(\hat{A}+\hat{B}P_1\hat{C})+ \hat{Q}+\hat{C}^TP_1^T\hat{R}P_1\hat{C} = 0 \tag{5.51}$$

$$(\hat{A}+\hat{B}P_1\hat{C})P+P(\hat{A}+\hat{B}P_1\hat{C})^T+(\hat{D}+\hat{B}P\hat{I})\tilde{R}_1(\hat{D}+\hat{B}P\hat{I})^T = 0 \tag{5.52}$$

respectively.

5.3.3 DERIVATION OF THE RESULTS

Once again the elements of the matrix P_1 will be treated as additional "states" which satisfy

$$\dot{P}_1 = 0 \; . \tag{5.53}$$

Vector multiplier $\underline{\lambda}_w$ will be used for state constraint (5.46) and matrix multiplier Λ_{P_1} for (5.53). Thus the Hamiltonian H for (5.46) and (5.47) is now

$$\begin{aligned}\tilde{H} = & \frac{1}{2}\underline{w}^T[\hat{Q}+\hat{C}^TP_1^T\hat{R}P_1\hat{C}\underline{w} + \frac{1}{2}\underline{w}^T\hat{C}^TP_1^T\hat{R}P_1\hat{I}\underline{s}] \\ & + \frac{1}{2}\underline{s}^T IP_1^T\hat{R}P_1\hat{C}\underline{w} + \frac{1}{2}\underline{s}^T \hat{I}P_1^T\hat{R}P_1\hat{I}\underline{s} \\ & + \underline{\lambda}_w^T[(\hat{A}+\hat{B}P_1\hat{C})\underline{w} + (\hat{D}+\hat{B}P_1\hat{I})\underline{s}] \; ,\end{aligned} \tag{5.54}$$

with

$$\dot{\underline{\lambda}}_w = -\frac{\partial H}{\partial \underline{w}} = -(\hat{Q}+\hat{C}^T P_1^T \hat{R} P_1 \hat{C})\underline{w} - \hat{C}^T P_1^T \hat{R} P_1 \hat{I}\underline{s} - (\hat{A}+\hat{B}P_1\hat{C})^T \underline{\lambda}_w,$$

$$\underline{\lambda}_w(t_f) = 0 \tag{5.55}$$

$$\dot{\Lambda}_{P_1} = -\frac{\partial \tilde{H}}{\partial P_1} = -\hat{R}P_1\hat{C}\underline{w}\,\underline{w}^T\hat{C}^T - \hat{R}P_1\hat{I}\underline{s}\,\underline{w}^T\hat{C}^T - \hat{R}P_1\hat{C}\underline{w}\,\underline{s}^T\hat{I} -$$

$$- \hat{R}P_1\hat{I}\underline{s}\,s^T\hat{I} - \hat{B}^T\underline{\lambda}_w w^T\hat{C}^T - \hat{B}^T\underline{\lambda}_w\underline{s}^T\hat{I},$$

$$\Lambda_{P_1}(t_f) = 0 \tag{5.56}$$

According to the Lemma 4.1 and the necessary condition (5.48),

$$E[\partial J/\partial P_1] = E[\Lambda_{P_1}(to)] = 0$$

or

$$0 = \hat{R}P_1\hat{C}\int_{to}^{t_f} E[\underline{w}\,\underline{w}^T]\,dt\,\hat{C}^T + \hat{R}P_1\hat{I}\int_{to}^{t_f} E[S\,\underline{w}^T]dt\,\hat{C}^T$$

$$+ \hat{R}P_1\hat{C}\int_{to}^{t_f} E[\underline{w}\,\underline{s}^T]dt\,\hat{I} + \hat{R}P_1\hat{I}\int_{to}^{t_f} E[\underline{s}(t)\underline{s}^T(t)]dt\hat{I}$$

$$+ \hat{B}^T\int_{to}^{t_f} E[\underline{\lambda}_w(\underline{w}^T\hat{C}^T+\underline{s}^T\hat{I}]\,dt\,. \tag{5.57}$$

Since $E[\underline{s}(t)\underline{s}^T(t)]$ is infinite for white noise, (5.57) can only be true if

$$\hat{R}P_1\hat{I} = \begin{bmatrix} RN & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{5.58}$$

(5.58) implies that unless $R = 0$ then

$$N = 0 \tag{5.59}$$

The result (5.59) is rather interesting. It implies that the observation contaminated with white noise must be filtered irrespective of the dimension of the dynamic compensator. (5.59) is, of course, true for the Kalman filter where its dimension is the same as that of the plant.

(5.57) is the basic result which can now be simplified. For this purpose, a solution for $\underline{\lambda}_w$ of the form

$$\underline{\lambda}_w = K\underline{w} + \underline{n} \tag{5.60}$$

is assumed.

Substituting (5.60) into (5.55) yields

$$-\dot{K} = (\hat{A}+\hat{B}P_1\hat{C})^T K + K(\hat{A}+\hat{B}P_1\hat{C}) + (\hat{Q}+\hat{C}^T P_1^T \hat{R} P_1 \hat{C}), \quad K(t_f) = 0 \tag{5.61}$$

$$\dot{\underline{n}} = -(\hat{A}+\hat{B}P_1\hat{C})^T \underline{n} - [K(\hat{D}+\hat{B}P_1\hat{I}) + \hat{C}^T P_1^T \hat{R} P_1 \hat{I}]\underline{s}, \quad n(t_f)=0. \tag{5.62}$$

Proceeding as previously to evaluate the averages in (5.57), define

$$P = E[\underline{w}\,\underline{w}^T] . \tag{5.63}$$

The solutions of (5.61) and (5.62) are given by

$$\underline{w} = \phi(t-to)\underline{w}(to) + \int_{to}^{t} \phi(t-\tau)(\hat{D}+\hat{B}P_1\hat{I})s(\tau)d\tau, \tag{5.64}$$

$$\underline{n} = -\int_{t_f}^{t} \psi(t-\tau)[K(\hat{D}+\hat{B}P_1\hat{I}) + \hat{C}^T P_1^T \hat{R} P_1 \hat{I}]s(\tau)d\tau \tag{5.65}$$

where $\phi(t)$ and $\psi(t)$ are the state transition matrix satisfying

$$\dot{\phi}(t) = (\hat{A}+\hat{B}P_1\hat{C})\phi(t) \quad . \tag{5.66}$$

$$\dot{\psi}(t) = -(\hat{A}+\hat{B}P_1\hat{C})\psi(t) \; . \tag{5.67}$$

Using (5.64) and assuming $\underline{w}(to)$ is independent of $\underline{s}(t)$ for all time t, P is given by

$$P = \phi(t-to)P(to)\phi^T(t-to) + \int_{to}^{t} \phi^T(t-\tau)(\hat{D}+\hat{B}P_1\hat{I})\tilde{R}_1(\hat{D}+\hat{B}P_1\hat{I})^T\phi^T(t-\tau)d\tau \tag{5.68}$$

where $E[\underline{w}(to)w^T(to)] = P(to)$ (5.69)

and $\tilde{R}_1\delta(t_1-t_2) = E[\underline{s}(t_1)\underline{s}^T(t_2)] = \begin{bmatrix} R_1 & L_1 \\ L_1^T & Q_1 \end{bmatrix} \delta(t_1-t_2)$. (5.70)

(5.68) can be seen to satisfy

$$\dot{P} + (\hat{A}+\hat{B}P_1\hat{C})P+P(\hat{A}+\hat{B}P_1\hat{C})^T+(\hat{D}+\hat{B}P\hat{I})\tilde{R}_1(\hat{D}+\hat{B}P\hat{I})^T. \tag{5.71}$$

Similarly

$$\begin{aligned} E[\underline{s}\ \underline{w}^T] &= [E\{\underline{s}(t)\underline{w}^T(to)\}\ \phi^T(t-to)t \\ &\quad + \int_{to}^{t} E\{\underline{s}(t)s^T(\tau_1)\}(\hat{D}+\hat{B}P_1\hat{I})^T\phi^T(t-\tau_1)d\tau_1] \\ &= \int_{to}^{t} \tilde{R}_1\delta(t-\tau_1)(\hat{D}+\hat{B}P_1\hat{I})^T\phi^T(t-\tau_1)d\tau_1 \\ &= \frac{\tilde{R}_1}{2}\ (\hat{D}+\hat{B}P_1\hat{I})^T\ , \end{aligned} \tag{5.72}$$

$$E[\underline{w}(t)s^T(t)] = (\hat{D}+\hat{B}P_1\hat{I})\ \frac{\tilde{R}_1}{2}\ , \tag{5.73}$$

$$\begin{aligned} E[\underline{\eta}(t)\underline{w}^T(t)] = \{ &\int_{t}^{t_f} \psi(t-t_f)[K(\hat{D}+\hat{B}P_1\hat{I})+\hat{C}^TP_1^T\hat{R}P_1\hat{I}]s(\tau)d\tau\} \\ &\{\underline{w}(to)\phi^T(t-to) + \int_{to}^{t} s^T(\tau)(\hat{D}+\hat{B}P_1\hat{I})^T\phi^T(t-\tau)d\tau\}-0, \end{aligned} \tag{5.74}$$

since the integrals do not overlap.

$$E[\underline{\lambda}_w \underline{w}^T(t)] = KE[\underline{w}(t)\underline{w}^T(t)] + E[\underline{\eta}(t)\underline{w}^T(t)]$$

$$= KP \; . \tag{5.75}$$

$$E[\underline{\lambda}_w s^T] = KE[\underline{w}\;\underline{s}^T] + E[\underline{\eta}\;\underline{s}^T]$$

$$= \frac{K}{2}(\hat{D}+\hat{B}P\hat{I})\tilde{R}_1 - E\int_{t_f}^{t} \psi(t-\tau\{k(\hat{D}+\hat{B}P_1\hat{I})+\hat{C}^T P_1^T \hat{R} P_1 \hat{I}\}\underline{s}(\tau)s^T(t)d\tau$$

$$= -\frac{1}{2}\hat{C}^T P_1^T \hat{R} P_1 \hat{I} R_1 \tag{5.76}$$

with $t_f \to \infty$, $t_o = 0$, K and P are steady state solution of (5.61) and (5.71) respectively. Under this condition and using (5.59) and (5.72) - (5.76), (5.57) reduces to

$$\hat{R}P_1\,[\hat{C}P\hat{C}^T + \frac{\hat{C}}{2}(\hat{D}+\hat{B}P_1\hat{I})\tilde{R}_1\hat{I} + \frac{\hat{I}}{2}\tilde{R}_1(\hat{D}+\hat{B}P_1\hat{I})^T\hat{C}^T] + \hat{B}^T K P\hat{C}^T = 0 \tag{5.77}$$

Partitioning K and P as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}, \qquad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \tag{5.78}$$

(5.77) reduces to equations involving the original variables:

$$RH(P_{12}^T C^T + \frac{1}{2}GR_1) + B^T(K_{11}P_{11} + K_{12}P_{12}^T)C^T = 0 \tag{5.79}$$

$$(K_{12}^T P_{11} + K_{22}P_{12}^T)C^T = 0 \tag{5.80}$$

$$RHP_{22} + B^T(K_{11}P_{12} + K_{12}P_{22}) = 0 \tag{5.81}$$

$$K_{12}^T P_{12} + K_{22}P_{22} = 0 \tag{5.82}$$

Using (5.59) and (5.78), (5.61) and (5.71) reduce to

$$A^T K_{11} + C^T G^T K_{12}^T + K_{11} A + K_{12} GC + Q = 0 \tag{5.83}$$

$$H^T B^T K_{11} + F^T K_{12}^T + K_{12}^T A + K_{22} GC = 0 \tag{5.84}$$

$$H^T B^T K_{12} + F^T K_{22} + K_{12}^T BH + K_{22} F + H^T RH = 0 \tag{5.85}$$

$$AP_{11} + BHP_{12}^T + P_{11} A^T + P_{12} H^T B^T + DQ_1 D^T = 0 \tag{5.86}$$

$$GCP_{11} + FP_{12}^T + P_{12}^T A^T + P_{22} H^T B^T = 0 \tag{5.87}$$

$$GCP_{12} + FP_{22} + P_{12}^T C^T G^T + P_{22} F^T + GR_1 G^T = 0 \tag{5.88}$$

In general, evaluation of F, G, H requires simultaneous solution of (5.79) - (5.88).

5.3.4 RECURSIVE ALGORITHMS FOR COMPUTING FEEDBACK GAINS

Various gains of the dynamic compensator can be computed using basically the same algorithm as reported in previous chapters.

P_1^{n+1} and P^{n+1} are computed using

$$\hat{R}P_1^{n+1}[\hat{C}P^{n+1}\hat{C}^T + \frac{\hat{C}}{2}(\hat{D}+\hat{B}P_1^{n+1}\hat{I})\tilde{R}_1\hat{I} + \frac{\hat{I}}{2}\tilde{R}_1(\hat{D}+\hat{B}P_1^{n+1}\hat{I})^T\hat{C}^T] + \hat{B}^T K^{n+1} P^{n+1}\hat{C}^T = 0, \tag{5.89}$$

$$(\hat{A}+\hat{B}P_1^{n+1}\hat{C})P^{n+1} + P^{n+1}(\hat{A}+\hat{B}P_1^{n+1}\hat{C})^T + (\hat{D}+\hat{B}P^{n+1}\hat{I})\tilde{R}_1(\hat{D}+\hat{B}P^{n+1}\hat{I}), \tag{5.90}$$

where K_{n+1} is the solution of

$$K_{n+1}(\hat{A}+\hat{B}P_1^n\hat{C}) + (\hat{A}+\hat{B}P_1^n\hat{C})K_{n+1} + \hat{Q}+\hat{C}^T P_1^{n\,T}\hat{R}P_1^n\hat{C} = 0 \tag{5.91}$$

The iteration starts with an initial guess $P_1^{\,o}$ such that the augmented system, i.e., $(\hat{A}+\hat{B}P_1^n\hat{C})$ is stable. K_1 is the positive definite solution of (5.91). With this value of K^1, (5.89) and (5.90) are solved

5.4 OPTIMAL OUTPUT FEEDBACK WITH WHITE MEASUREMENT NOISE

5.4.1 PROBLEM STATEMENT

The basic problem is to determine matrix N which minimize

$$\frac{1}{2}E\int_{t_o}^{t_f}(\underline{x}^T Qx + u^T Ru)dt \tag{5.92}$$

subject to

$$\dot{\underline{x}} = A\underline{x} + Bu + D\underline{w} \tag{5.93}$$

$$\underline{y} = C\underline{x} + \underline{v}(t) \tag{5.94}$$

and with the controller

$$u = -Ny = -NC\underline{x} - N\underline{v} \tag{5.95}$$

5.4.2 RESULT

The optimal feedback gain

$$N = 0 \tag{5.96}$$

5.4.3 DERIVATION OF RESULT AND COMMENTS

This is a special case of the previous problem and the result follows by setting

$$F = 0,\ G = 0,\ H = 0 \tag{5.97}$$

Note that condition (5.50) will still have to be satisfied. This implies (5.96) although (5.96) seems surprising but is not difficult to reason out.

For any nonzero N, (5.92) is infinte since the quadratic term in the control involves the term $E[\underline{v}(t)\underline{v}^T(t)]$ which is infinte. Thus the performance index is infinite. (5.92) is finite if $N = 0$ for t_f is finite. Thus $N = 0$ is the optimal solution. This does not necessarily imply that a nonzero optimal solution does not exist. This only points out the mathematical optimization problem is ill-posed.

5.5 OPTIMAL OUTPUT FEEDBACK WITH NONWHITE MEASUREMENT NOISE

5.5.1 STATEMENT AND FORMULATION OF THE PROBLEM

If measurement noise is not white, which is of course reasonable, then $N \neq 0$. The controller without dynamics is given by

$$u = Ny \tag{5.98}$$

where the output y

$$y = C\underline{x} + H\underline{z} \tag{5.99}$$

is contaminated with nonwhite noise $\underline{z}$ described by

$$\dot{\underline{z}} = F\underline{z} + G\underline{v} \quad . \tag{5.100}$$

F, G, H, and C are all specified. The problem is to determine N.

Using (5.98) and (5.99), (5.1) becomes

$$\dot{\underline{x}} = (A+BNC)\underline{x} + BNH\underline{z} + D\underline{w}_1 \tag{5.101}$$

Using (5.98) and (5.99), the performance criterion becomes

$$J = \frac{1}{2}E\int_{t_0}^{t_f} \{\underline{x}^T[Q+C^TN^TRNC]\underline{x} + \underline{x}^TC^TRNH\underline{z} + \underline{z}^TH^TRNC\underline{x}$$

$$+ \underline{z}^TH^TN^TRNH\underline{z}\}dt.$$

$$= \frac{1}{2}E\int_{t_0}^{t_f} (\underline{x}^T,\underline{z}^T) \begin{bmatrix} Q+C^TN^TRNC & C^TN^TRNH \\ H^TN^TRNC & H^TN^TRNH \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} dt \; . \tag{5.102}$$

Now defining

$$\overset{*}{A} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}, \quad \overset{*}{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \overset{*}{C} = \begin{bmatrix} C & H \\ 0 & 0 \end{bmatrix}, \quad \overset{*}{\underline{s}} = \begin{bmatrix} \underline{w}_1 \\ \underline{v} \end{bmatrix}, \quad \underline{w} = \begin{vmatrix} \underline{x} \\ \underline{z} \end{vmatrix}$$

$$\overset{*}{N} = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad \overset{*}{D} = \begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix}, \quad \overset{*}{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overset{*}{R}_1 \delta(t_1 - t_2) = E[\underline{\overset{*}{s}}(t_1) \overset{*}{s}{}^T(t_2)] = \begin{bmatrix} Q_1 & L_1^T \\ L_1 & R_1 \end{bmatrix} \delta(t_1 - t_2) \; ; \tag{5.103}$$

(5.101) and (5.102) reduce to

$$\dot{\underline{\overset{*}{w}}} = (\overset{*}{A} + \overset{*}{B}\overset{*}{N}\overset{*}{C})\underline{w} + \overset{*}{D}\underline{\overset{*}{s}} \tag{5.104}$$

$$J = \frac{1}{2} E \int_{to}^{t_f} \underline{w}^T [\overset{*}{C}{}^T \overset{*}{N}{}^T \overset{*}{R}\overset{*}{N}\overset{*}{C} + \hat{Q}] \underline{w} \, dt. \tag{5.105}$$

Thus the problem reduces to the minimization of (5.105) with respect to $\overset{*}{N}$ subject to (5.104).

5.5.2 RESULT

The feedback matrix $\overset{*}{N}$ satisfies

$$\overset{*}{R}\overset{*}{N}\overset{*}{C}\overset{*}{P}\overset{*}{C}{}^T + \overset{*}{B}{}^T \overset{*}{K}\overset{*}{P}\overset{*}{C}{}^T = 0 \tag{5.106}$$

where $\overset{*}{K}$ and $\overset{*}{P}$ satisfy

$$\overset{*}{K}(\overset{*}{A} + \overset{*}{B}\overset{*}{N}\overset{*}{C}) + (\overset{*}{A} + \overset{*}{B}\overset{*}{N}\overset{*}{C})^T \overset{*}{K} + \overset{*}{Q} + \overset{*}{C}{}^T \overset{*}{N}{}^T \overset{*}{R}\overset{*}{N}\overset{*}{C} = 0 \tag{5.107}$$

$$\overset{*}{P}(\overset{*}{A} + \overset{*}{B}\overset{*}{N}\overset{*}{C})^T + (\overset{*}{A} + \overset{*}{B}\overset{*}{N}\overset{*}{C})\overset{*}{P} + \overset{*}{D}\overset{*}{R}_1 \overset{*}{D}{}^T = 0 \tag{5.108}$$

respectively.

5.5.3 DERIVATION AND SIMPLIFICATION OF THE RESULT

Proceeding exactly in the same manner as in Case 1, it can be easily seen that the feedback matrix $\overset{*}{N}$ satisfy (5.106) - (5.108). Partitioning $\overset{*}{K}$ and $\overset{*}{P}$ as in (5.78) and expanding (5.107) and (5.108), it can be seen that

$$P_{11}(A+BNC)^T + P_{12}H^TN^TB^T+(A+BNC)P_{11} + BNHP_{12}^T + DQ_1D^T = 0 \quad (5.109)$$

$$P_{12}^T(A+BNC)^T + FP_{12}^T + GL_1D^T = 0 \quad (5.110)$$

$$P_{22}F^T + FP_{22} + GR_1G^T = 0 \quad (5.111)$$

$$(A+BNC)^TK_{11} + K_{11}(A+BNC)+Q+C^TN^TRNC = 0 \quad (5.112)$$

$$(BNH)^TK_{11} + F\ K_{12}^T(A+BNC) + H^TN^TRNC = 0 \quad (5.113)$$

$$(BNH)^TK_{12} + F^TK_{22} + K_{12}^TBNH + K_{22}F + H^TN^TRNH = 0 \quad (5.114)$$

and the optimal feedback matrix N is given by

$$N = -R^{-1}B^T[K_{11}P_{11}C^T+K_{12}P_{12}H^T+K_{12}P_{12}^TC^T+K_{12}P_{22}H^T][CP_{12}C^T$$
$$+CP_{12}H^T+HP_{12}^TC^T+HP_{22}H^T]^{-1} \quad . \quad (5.115)$$

If the measurement noise and plant disturbance are uncorrelated, then $L_1 = 0$. Thus (5.115) reduces to

$$N = -R^{-1}B^T[K_{11}P_{11}C^T+K_{12}P_{22}H^T][HP_{22}H^T]^{-1} \quad (5.116)$$

5.6 EVALUATION OF OPTIMAL COST UNDER STEADY STATE CONDITION

Note that as $t_f \to \infty$, $t_o = 0$ the integrals are in effect being dropped and criterion is

$$\frac{1}{2} E\ [\underline{x}^TQ\underline{x} + u^TRu] = c(\underline{x},\underline{u}) \quad (5.117)$$

In order to retain a finite cost, $\int_{to}^{t_f} c(\underline{x},u)dt$ as $t_f \to \infty$, Q and R can contain a factor $\frac{1}{(t_f-t_o)}$ which will not affect resulting gains.

5.7 CONCLUSION

A unified design procedure for optimal incomplete state feedback controllers for stochastic system has been presented. Various gains specifying the output feedback controllers with and without dynamics are obtained by minimizing a quadratic criterion. It has also been established that white noise observations must be filtered irrespective of the dimension of the dynamic compensator. In the absence of any controller dynamics, the optimal feedback gain turns out to be zero if the observation process is contaminated with white noise and the quadratic performance index involves both the state and control. This merely suggests an alternative problem formulation. When the measurement noise is non-white, the optimal feedback matrix satisfies a set of nonlinear algebraic equations. In the absence of measurement noise, the optimal feedback gain must satisfy a set of algebraic nonlinear equations. In all the problem formulations, it is assumed that the feedback controller with or without dynamics stabilizes the system. It should be noted that the feedback matrices result from necessary condition of optimality. Thus the solution is not necessarily globally optimal.

VI CONCLUSION

6.1 CONCLUSION AND LIMITATIONS OF THE PRESENT WORK

The present thesis has attempted to present a unified design philosophy for limited state feedback control problems with parameter uncertainty for both deterministic and stochastic problems.

Basically two different approaches have been suggested. In one approach, a linear compensator is specified, in some cases with dynamic elements. In the deterministic problem with parameter uncertainty, a minimax design has been accomplished by proposing an integral quadratic performance criterion which was maximized with respect to an uncertainty matrix and minimized by the feedback matrix. Various other integral quadratic criteria and design procedures have been examined including a sensitivity type criterion. The resulting minimax controller is linear and it has been shown that minimax designs offer better system performance than a purely nominal design under a wide range of parameter variations. The various minimax procedures assume that the nominal system can be stabilized with output feedback. The stochastic problem without parameter uncertainty has been treated in a similar way. The stochastic problem dealt with white noie plant disturbance and white and colored measurement noise. Optimal limited state feedback controllers with and without dynamics have been formuated and optimized. The criterion is the average of an energy function. Various optimization techniques for both stochastic and deterministic problems result in nonlinear algebraic equations which must be solved recursively for the compensator matrices.

The second approach to design is applicable for single input- single output systems with parameter uncertainty and uses a model of order equal to that of the system less the number of zeros. A criterion involving tracking error, control, and a signal related to parameter uncertainty was maximized with respect to the uncertainty signal and minimized with respect to the control. The resulting controller is linear and uses only partial state feedback from states of a companion form. It has been shown that the plant can be stabilized with this partial state feedback, and the tracking error can be made arbitrarily small despite arbitrary parameter uncertainty, provided sufficient control energy is available and provided the plant is minimum phase type. The results hold true for nonlinearities that do not involve control. In order to generate the control when some of the necessary states are not available, a minimax design of reduced order dynamic compensator has been accomplished. The design procedure assumes noise-free measurements. When some of the available states are contaminated with white noise, an ad hoc scheme has been suggested to estimate the necessary states to implement the controller.

The above two basic approaches have certain limitations. The problem of designing a dynamic compensator for stochastic system assumes that the system does not involve any parameter uncertainty, although a minimax compensator design for stochastic system can be carried out in a manner similar to that reported in Chapter 3. Another limitation is the unavailability of efficient computational schemes for solving the simultaneous algebraic equations which result from application of necessary conditions for an optimum controller. The minimax technique for single input - single output system has some drawbacks. First, it is difficult to extend the approach to multivariable system. The difficulty seems to be due to the fact that a suitable canonical form for multivariable system is not currently available. Second, the uncertainty signal is related to the system parameters in a complicated way. Thus an ultimate bound on parameter variation to insure system stability is difficult to ascertain. These problems, along with other limitations, and possible extensions of the techniques will be discussed in the next section.

In spite of the various limitations of the present work, the design philosophy presented in this thesis makes a considerable inroad in handling parameter uncertainty in deterministic systems, and plant disturbance and measurement noise in stochastic system. It embraces a very challenging field in system theory- control of systems with parameter uncertainty and disturbances using available measurements. Certain basic investigations have been carried out in this thesis and some basic results have been obtained. The contribution of the present work will, the author hopes, stimulate further research in this field.

6.2 OUTSTANDING PROBLEMS, POSSIBLE EXTENSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

Although some basic results have been obtained, the investigation is far from complete; however, some of the outstanding problems and suggestions for further research in this direction will be outlined below:

(i) The minimax design for single input - single output system as presented in Chapter 2 is applicable to time-invariant system. The present formulation can be, at best, extended to include some time-varying parameters. For time-varying systems, the success of the present technique depends much on the results of stability theory of time-varying systems. The present status of stability theory is not sufficient to readily establish a general result.

(ii) Another limitation of minimax technique of Chapter 2 is the assumption that the system should be minimum-phase type. One way of approaching the problem will be to constrain the control amplitude leading to a saturation-type controller. The immediate question that arises is whether or not the tracking error can be made arbitrarily small with the available control amplitude. Another problem in this direction is to ascertain a priori the control amplitude, when very little is assumed to be known about the system. This, in turn, requires some more information regarding the system.

(iii) It has been established in Chapter 3 that reduced order dynamic compensator can be designed using the available measurements. When some of the states (or output) are noisy, a reduced order estimator has been designed to estimate the necessary states to implement the control. Further research is required to establish a) what order dynamic compensator is necessary to stabilize the overall system, b) whether the reduced order compensator can give performance comparable to that of a compensator having dimension equal to that of the system.

(iv) Possibly the greatest effort should be directed to extend the basic concepts presented in Chapter 2 to general multivariable system.

This will reveal the conditions for output stabilizibility. This is important since the derivation of optimal output feedback controller with or without dynamics requires the closed-loop system to be stable. The difficulty in extending the present approach is due to the fact that no suitable canonical form for multivariable system has been found.

(v) Obvious factors regarding minimax output feedback controller as presented in Chapter 4, which require further study include:

(a) Computational feasibility and convergence properties of various algorithms.

(b) Existence and uniqueness of the solutions

(c) Stability properties of the nominal system with output feedback and conditions for stability if all the parameters in the system are allowed to vary

(d) Extension of the minimax design analysis to more general sensitivity criteria.

Some basic questions regarding the design of dynamic compensator for stochastic systems include

(1) When the dimension of the dynamic compensator is less than that of the controlled system, how much does the performance degrade?

(2) Is it possible to achieve separation in design of estimator (or dynamic elements) and the controller, when the compensator dimension is less than that of the system?

(3) Can the parameter uncertainty be effectively treated for stochastic systems in the same manner used for deterministic problems?

(4) Is it possible to stabilize the system if the dimension of the dynamic compensator is not equal to the system order?

(5) How should the mathematical optimization problem be reformulated to obtain a nonzero optimal feedback gain matrix when the observation is contaminated with white noise? A possible approach would be to reformulate it as singular problem (integral quadratic criterion penalizing the state only).

(6) Under what conditions does the algorithm presented in Chapter 5 converge?

The answer to some of these questions and investigation of certain of these factors is essential before a truly practical engineering design approach can be obtained.

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