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AN ADAPTIVE OBSERVER FOR SINGLE-INPUT SINGLE-OUTPUT LINEAR SYSTEMS

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Introduction

The Luenberger observer [1,2,3] allows extraction of the state of an observable linear system when given (1) the system input, (2) the system output, (3) the form of the system, and (4) the parameter values of the system. In those cases for which the system parameters are unknown the state observation is subject to error. Some previous investigators of parameter ignorance in observers [4,5] alleviate to some degree the observation error, but they are unable to guarantee the error vanishes or that their computational algorithm converges when the magnitude of parameter ignorance is large. We have previously reported [6] the basics of a full order adaptive observer which negates these disadvantages. Our present paper considerably simplifies the exposition of the previous paper and extends, both computationally and theoretically, the topic of that paper. Briefly, the full order adaptive observer for single-input single-output observable continuous stable linear differential systems in the absence of a deterministic or random disturbance vector guarantees the vanishing of observation error regardless of the size of the constant or slowly varying parameter ignorance. The observer parameters are directly changed in a Liapunov adaptive way so as to eventually yield the unknown full order Luenberger observer. The observer poles may throughout be placed freely in the stable region and no derivatives are required in the adaptive law.

The Problem

A differential system is assumed of the form

$$\dot{w} = \tilde{A}w + Br \qquad w(0) = w^0$$

$$y = [1\ 0\ 0\ \text{---}\ 0]w \equiv Cw \tag{1}$$

$\tilde{A}$ nxn

B nx1

for which only the single output $y = Cw = w_1$ is available for measurement. It is assumed that a similarity transformation has been made if necessary so that the single-input - single-output system has the form of (1). It is assumed that some or all of the elements of matrices $\tilde{A}$ and B are unknown, A is stable, w^0 may be unknown, and the pair $(C,\tilde{A})$ is completely observable. The observer is of the form

$$\dot{z} = Kz + GCw + Dr + Hu \qquad z(0) = z^0 \tag{2}$$

F nxn G nx1
D nx1 H nxn and diagonal

where K is arbitrary and u is a control vector yet to be defined but with the property that $u \to 0$ as $t \to \infty$. The problem is to adaptively form a triple (G,D,T) so that the error vector defined as $e = z - T^{-1}w$ vanishes as the system adapts. T is a non-singular square matrix with the property that CT=C. Fig. 1 illustrates the adaptation. $\hat{T}(t)$ is a matrix which varies according to the adaptation procedure so that, when the adaptation is completed $\hat{T}(t)$ becomes T (i.e. $\lim_{t \to \infty} \hat{T}(t) = T$).

Define a transformation $x = T^{-1}w$ so that $e = z - x$. Then (1) becomes

$$\dot{x} = \tilde{A}_0 x + T^{-1}Br \qquad x(0) = T^{-1}w^0$$
$$y = CTx = Cx \tag{1A}$$
$$\tilde{A}_0 = T^{-1}\tilde{A}T$$

and (2) becomes

$$\dot{z} = K_0 z + GCx + Dr + Hu$$
$$z(0) = Z^0 \tag{2A}$$

It is desired for subsequent development that $\tilde{A}_0 = T^{-1}\tilde{A}T$ be in the "output" form

$$\tilde{A}_0 = \begin{bmatrix} -a_{11} & 1 & 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & 1 & 0 & \cdots & 0 \\ -a_{31} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

wherein the first column contains the system parameters and all other elements are zero save the super diagonal elements, which are unity. It is clear that for any non-zero matrix $\tilde{A}$ with a single invarient polynomial there corresponds a similar matrix $\tilde{A}_0$, although the elements of the similarity transformation may be unknown if elements of $\tilde{A}$ are unknown. The following theorem defines the additional restriction that must be placed upon $\tilde{A}$ so that both $\tilde{A}_0 = T^{-1}\tilde{A}T$ and $CT = C$.

<u>Theorem</u>

[proof given in Appendix A]. Let A be an nxn matrix, $C = [k,0,0,\ldots 0]$ a 1xn matrix with $k \neq 0$, A_0 an nxn matrix in output form, and $T = [\,\cdot\ \begin{smallmatrix} C \\ \vdots \\ \hat{T} \end{smallmatrix}\ \cdot\,]$ an nxn nonsingluar matrix. There exists an (n-1)xn matrix $\hat{T}$ such that $A = TA_0T^{-1}$ iff the pair (C,A) is completely observable.

As a result of the theorem, any observable system (1) may be placed by similarity transformation into system (1A) with CT=C. The elements of T may be unknown since $\tilde{A}$ is unknown. The problem will be considered as defined by equation (1A) and (2A), so that $e = z - x$ must vanish. Eventually the problem of constructing w from x will be solved.

The Adaptive Law

It is now assumed, more for explanatory purpose than actual practical need, that some stable "nominal" plant matrix is either known or is chosen so that $\tilde{A}_0 = A_0 + \Delta A_0$, where A_0 has all known elements and is in output form. Consequently ΔA_0 contains all zero elements except for the left column which has elements that are to be adapted. Letting $e = z-x$, the vector error equation is

$$\dot{e} = Ke + (K+GC-A_0-\Delta A_0)x + \Delta Br + Hu$$
$$e(0) = e^0$$

where $\Delta B = D-T^{-1}B$. A theorem of Luenberger [1] allows the eigenvalues of A_0-GC to be arbitrarily placed by selection of G (with the sole exception that A_0-GC cannot have the same eigenvalues as A_0). For the above error equation, let $G=G_1 + G_2$ and $K = A_0-G_2C$. Then as a result of the theorem of Luenberger and of the special forms of A_0 and C, the vector error equation is

$$\dot{e} = K_0e + (G_1C-\Delta A_0)x + \Delta Br + Hu \tag{3}$$

where K_0 is an arbitrary stable constant matrix in output form with eigenvalues differing from A_0. The adaptive strategy is to change G_1 and ΔB to eliminate the influence of x and r in (3); since by assumption K_0 is a constant matrix, changing G_1 is equivalent to changing G and will be considered as such in the ensuing.

For notational convenience in the next sections the following definitions are made.

$$K_0 = \begin{bmatrix} -k_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ -k_{n-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -k_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$GC-\Delta A_0 = \begin{bmatrix} \alpha_{n-1} & 0 & 0 & \cdots & 0 \\ \alpha_{n-2} & 0 & 0 & \cdots & 0 \\ \alpha_{n-3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\Delta B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta_m \\ \beta_{m-1} \\ \beta_{m-2} \\ \vdots \\ \beta_0 \end{bmatrix} \qquad G = \begin{bmatrix} g_{n-1} \\ g_{n-2} \\ g_{n-3} \\ \vdots \\ g_0 \end{bmatrix}$$

$H = \text{diag}\,[0, h_{n-2}, h_{n-3}, \cdots, h_0]$ and nxn

n = order of plant

m = number of zeroes in system transfer function. (5)

The error between plant state x and observer state z may be measured only by the scalar state variable $e_1 = z_1 - y = z_1 - x_1$. To insure that only available measurements are called for in the adaptive laws, (3) is "collapsed" to yield a scalar differential equation of the form

$$\sum_{i=0}^{n} k_i e_i^{(i)} = \sum_{j=0}^{n-1} \sum_{i=0}^{n-i-j} \binom{i+j}{i} \alpha_{i+j}^{(i)} \quad x_i^{(j)}$$

$$+ \sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{i+j}{i} \beta_{i+j}^{(i)} \quad r^{(j)} \tag{5}$$

$$+ \sum_{i=0}^{n-2} h_i \, u_i^{(i)}$$

For simplicity let λ_1 be a real characteristic value of A_0-GC. Letting $p=d/dt$, the left side of (5) may be written as

$$(p + \lambda_1) \left(\sum_{i=0}^{n-1} a_i \, p^i \right) e_1$$

where the a_i, $0 \leq i \leq n-1$, are defined by equating the above expression with the left part of (5). If it is desired to have no real observer pole, an obvious modification to Eq. 6 is required. Now a reduction of order technique, similar to that of Gilbart and Monopoli [7], is applied to (5). The result is

$$(p+\lambda_1) \left(\sum_{i=0}^{n-1} a_i p^i \right) e_1 = \left(\sum_{i=0}^{n-1} a_i p^i \right) \left[\sum_{i=0}^{n+m} \phi_i v_i \right]$$

$$+ f_x + f_r + \sum_{j=0}^{n-2} h_j \, u_j^{(j)} \tag{6}$$

in which, assuming $m \leq n-2$, in (6)

$$\phi_i = \begin{cases} \alpha_i - a_i \, \alpha_{n-1} & i = 0,1,2,\ldots,n-2 \\ \alpha_{n-1} & i = n-1 \\ \beta_{i-n} & i = n, n+1, n+2, \ldots, m+n < 2n-1 \end{cases} \tag{7}$$

and, defining the "state variable filters" v_i,

$$\sum_{j=0}^{n-1} a_j v_i^{(j)} = x_1^{(i)} \qquad i = 0, 1, 2, \ldots, n-2$$

$$v_{n-1} = x_1 \tag{8}$$

$$\sum_{j=0}^{n-1} a_j v_i^{(j)} = r^{(i-n)} \qquad i = n, n+1, n+2, \ldots, 2(n-1)$$

$$v_{2n-1} = r$$

$$f_x = \sum_{k=1}^{n-2} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [x^{(k-j-1)} \dot{\phi}_k] \tag{9}$$

$$- \sum_{k=0}^{n-2} \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k]$$

$$f_r = \sum_{k=1}^{m} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [\dot{\phi}_{n+k} r^{(k-j-1)}] \tag{10}$$

$$- \sum_{k=n}^{n+m} \sum_{j=0}^{n-2} \sum_{i=0}^{n-j-2} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k]$$

Should m=n-1 then (7) should be changed to the extent that

$$\phi_i = \begin{cases} \beta_{i-n} - a_{i-n} \beta_{n-1} & i = n, n+1, n+2, \ldots, 2n-2 \\ \beta_{n-1} & i = 2n-1 \end{cases} \tag{7.a}$$

and (10) is changed to

$$f_r = \sum_{k=1}^{n-2} \sum_{j=0}^{k-1} \frac{d^j}{dt^j} [\dot{\phi}_{n+k} r^{(k-j-1)}] \tag{10.a}$$

$$- \sum_{k=n}^{2n-2} \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-j} a_{i+j+1} \frac{d^j}{dt^j} [v_k^{(i)} \dot{\phi}_k]$$

It is noted that (9) and (10) contain no other derivative of ϕ_k, $0 \le k \le n+m$, but $\dot{\phi}_k$. According to the adaptive law (16), $\dot{\phi}$ is an available measurement. By careful manipulation of the state-space generation of (8), it is possible therefore to remove the totality of terms in (9) and (10) from (6). Generation of (8), it is pointed out, requires no derivatives of x_1.

It is further noted that, since $\dot{\phi}_k$ is the change in parameters due to adaptation, as adaptation is completed $\dot{\phi}_k \to 0$, $0 \le k \le n+m$, and consequently $\lim_{t\to\infty} u_j = 0$.

Table I gives the terms in u_j for systems ranging from second to fourth order when (8) has been chosen to be generated by a normal form. The table can be extended.

The implementation of u_j as described reduces (6) to

$$(p+\lambda_1)\left(\sum_{i=0}^{n-1} a_i\, p^i\right)e_1 = \left(\sum_{i=0}^{n-1} a_i\, p^i\right)\left[\sum_{i=0}^{n+m} \phi_i\, v_i\right] \tag{11}$$

Taking Laplace transform of (11) and dividing by $\sum_{i=0}^{n-1} a_i s^i$ yields

$$(s+\lambda_1)e_1 = \left[\sum_{i=0}^{n+m} \phi_i\, v_i\right] + \frac{\mathcal{L}(\text{initial conditions})}{\sum_{i=0}^{n-1} a_i s^i} \tag{12}$$

for which follows

$$\dot{e}_1 + \lambda_1 e_1 = \sum_{i=0}^{n+m} \phi_i\, v_i + \sum_{i=2}^{n-1} \psi_i \exp\left[-\lambda_i t\right] \tag{13}$$

where ψ_i are unknown constants or time dependent functions depending upon the initial conditions and $\{\lambda_i\}$, the set of characteristic values of $\sum_{i=0}^{n-1} a_i s^i$.

A Liapunov function is now to be formed so that stability of the adaptive observer may be assured. To this end a positive definite function of the measured error e_1 and the unknown parameter errors ϕ_i is defined as

$$V = \frac{1}{2}\left(m_s e_1^2 + \sum_{i=0}^{n+m} m_i \phi_i^2\right) \tag{14}$$

Following Shackcloth [8], $\dot{V}$ can be made to be

$$\dot{V} = -m_s \lambda_1 e_1^2 + e_1 \sum_{i=0}^{n-1} \psi_i \exp[-\lambda_i t] \tag{15}$$

when $$\dot{\phi}_i = -\frac{m_s}{m_i} v_i e_1 \qquad 0 \leq i \leq n+m \tag{16}$$

Other adaptive laws can easily be chosen instead if it is desired to increase convergence speed [9,10].

Implementation of the adaptive law in (16) can be accomplished by reference to (7) and to the definitions of the variables α_i and β_i. For example

$$\dot{\phi}_{n-1} = \dot{\alpha}_{n-1} = \dot{g}_{n-1} = -\frac{m_s}{m_{n-1}} x_1 e_1$$

$$\dot{\varphi}_{n-2} = \dot{\alpha}_{n-2} - a_{n-2}\dot{\alpha}_{n-1} = \dot{g}_{n-2} + a_{n-2}\frac{m_s}{m_{n-1}} x_1 e_1$$

$$= -\frac{m_s}{m_{n-2}} v_{n-2} e_1$$

etc.

in which $\dot{g}_i$ may be ascertained.

From the form of $\dot{V}$, e_1 is stable in the sense of Lagrange with the region of attraction determined by the unknown constants ψ_i and the decaying exponential time function. Clearly the region of attraction shrinks exponentially with time and eventually vanishes; consequently e_1 is eventually asymptotically stable and $\lim_{t\to\infty} e_1 = 0$. All derivatives of e_1 must vanish in the limit as well since the scalar error equation (13) is linear and of first order and possessing finite frequencies.

However, the Liapunov function (14) is defined on a non-compact manifold. Consequently $\{\phi_i\}$ is shown to be (eventually) stable but not necessarily asymptotically stable [11]. It is evident from (3) that each ϕ_i must vanish by adaptation in order to observe the correct plant state. Theorem 2 defines the restriction placed upon r(t) in order to guarantee vector error convergence.

Theorem 2 [Proof in Appendix B]

Suppose there exists no set of real constants $\{q_i\}$, i=0,1,2,..., n+m, for which the (observable) system command input r(t) in its steady-state condition is a solution of the homogeneous differential equation

$$\sum_{i=0}^{n+m} q_i r^{(i)} = 0$$

where n and m are defined in (4).

Then $\lim_{t\to\infty} \phi_i(t) = 0$, i = 0, 1, 2, ..., n+m, and $\lim_{t\to\infty} \underline{e}(t) = 0$ is assured.

Corollary

If the steady-state command input r(t) is periodic, a sufficient condition in order for $\lim_{t\to\infty} \underline{e}(t) = 0$ in (3) is that r(t) contain at least [n+m+1]/2 distinct frequencies in its steady-state condition.

It is noted parenthetically that the corollary seems a generalization of a theorem of Lion [12] although the applicability of that theorem to the present topic appears obscure.

Reconstruction of T

Using the "nominal" matrix A_0 as initial condition, the actual value of the system parameters may be determined by integrating the change in parameters $\{\phi_i\}$, defined in (16), until adaptation is complete and combining appropriately in the form of the matrix T. Thus $\hat{T}(t)$ "drifts" toward T as adaptation progresses and $\lim_{t\to\infty} \hat{T}(t) = T$. The example makes this technique clear.

$\hat{w}$, the estimate of w, is constructed from the observer output z by forming $\hat{T}(t)z$. Consequently $\lim_{t\to\infty} \hat{w} = w$.

Practical Considerations

Reference to Table I reveals that, speaking practically of analog implementation, for high order systems a prohibitively large number of multipliers must be employed to generate the observer input $u_j(t)$. Since the magnitude of each $u_j(t)$ depends upon the magnitude of parameter change due solely to the adaptation process, it is reasonable to inquire whether $u_j(t)$ can be omitted altogether (i.e. make $H \equiv 0$), especially when the adaptation proceeds slowly by choice of constants in (16). By analysis of (14), it may be seen that omitting $u_j(t)$ tends to degrade the adaptation process due to the inclusion of disturbances f_x and f_r, eq. (9) and (10).

However a theoretical analysis of a second order system indicates that the u term may be safely omitted when the observer eigenvalues lie left of a curve passing through the left half-plane. This curve represents a trade off between frequency filtering in the adaptive law and magnitude of the adaptive gains.

Generalization of this work awaits completion.

Example

A third order plant with one zero is considered for illustration. Let the plant be described by

$$\dot{W} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0+\alpha_0) & -(a_1+\alpha_1) & -(a_2+\alpha_2) \end{bmatrix} w + \begin{bmatrix} 0 \\ c_1 \\ c_0 \end{bmatrix} r \qquad (1^*)$$

$$y = w_1$$

in which α_0, α_1, α_2, c_0 and c_1 are unknown. a_0, a_1, a_2 are the nominal values. In output form, (1*) is

$$\dot{x} = \begin{bmatrix} -(a_2+\alpha_2) & 1 & 0 \\ -(a_1+\alpha_1) & 0 & 1 \\ -(a_0+\alpha_0) & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_1-\beta_1 \\ b_0-\beta_0 \end{bmatrix} r \qquad (1A^*)$$

$$y = x_1 = w_1$$

The error equation (3) is now

$$\dot{e} = \begin{bmatrix} -k_2 & 1 & 0 \\ -k_1 & 0 & 1 \\ -k_0 & 0 & 0 \end{bmatrix} e + \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ \beta_1 \\ \beta_0 \end{bmatrix} r + \begin{bmatrix} 0 \\ u_1 \\ u_0 \end{bmatrix} \qquad (3^*)$$

and the scalar error equation (5) is now

$$\dddot{e}_1 + k_2\ddot{e}_1 + k_1\dot{e}_1 + k_0 e_1 = (\alpha_0 + \dot{\alpha}_1 + \ddot{\alpha}_2)x_1 + (\alpha_1 + 2\dot{\alpha}_2)\dot{x}_1 + {}_2\ddot{x}_1$$

$$+ \beta_1\dot{r} + (\beta_0 + \dot{\beta}_1)r + \dot{u}_1 + u_0 \qquad (5^*)$$

Employing the definitions given in (7), (8), and Table 1 when n=3 and m=1, the scalar error equation (5*) is equivalent to

$$(p+\lambda_1)\,(p^2 + a_1 p + a_0)e_1 = (p^2 + a_1\, p + a_0)\,\left(\sum_{i=0}^{4} \phi_i\, v_i\right) \qquad (11^*)$$

when u_1 and u_0 have been implemented as

$$u_0 = \dot{\phi}_0(v_0(2) + a_1\, V_0(1)) + \dot{\phi}_1(v_1(2) + a_1\, v_1(1))$$
$$+ \dot{\phi}_3(v_3(2) + a_1\, v_3(1)) + \dot{\phi}_4(v_4(2) + a_1\, V_4(1))$$
$$u_1 = \dot{\phi}_0\, V_0(1) + \dot{\phi}_1\, v_1(1) + \dot{\phi}_3 v_3(1) + \dot{\phi}_4\, v_4(1)$$

As an illustration of the generation of $v_i(j)$ appearing above, consider v_1 which is defined by

$$\ddot{v}_1 + a_1\,\dot{v}_1 + a_0\, v_1 = \dot{x}_1$$

and the generation of $v_1(j)$ in normal form.

Then, what is equivalent for v_1,

$$\begin{bmatrix} \dot{v}_1(1) \\ \dot{v}_1(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} v_1(1) \\ v_1(2) \end{bmatrix} + \begin{bmatrix} 1 \\ -a_1 \end{bmatrix} x_1$$

Consequently $v_1 \equiv v_1(1)$ and both $v_1(1)$ and $v_1(2)$ are available for measurement. Other $v_i(j)$ are generated in a similar manner.

Defining $\dot{\phi}_i$ as in (16), the observer has the form

$$\dot{z} = \begin{bmatrix} -k_2 & 1 & 0 \\ -k_1 & 0 & 1 \\ -k_0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} g_2 \\ g_1 \\ g_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b_1 \\ b_0 \end{bmatrix} r + \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix}$$

where

$$\dot{b}_1 = -\frac{m_s}{m_4} e_1 v_4$$

$$\dot{b}_0 = -\frac{m_s}{m_3} e_1 v_3$$

$$\dot{g}_2 = -\frac{m_s}{m_2} e_1 x_1$$

$$\dot{g}_1 = -e_1 \left(\frac{m_s}{m_1} v_1 + a_1 \frac{m_s}{m_2} x_1\right)$$

$$\dot{g}_0 = -e_1 \left(\frac{m_s}{m_0} v_0 + a_0 \frac{m_s}{m_2} x_1\right)$$

and

$$\hat{w} = \begin{bmatrix} 1 & 0 & 0 \\ a_2 - \int_0^t \dot{g}_2\, dt & 1 & 0 \\ a_1 - \int_0^t \dot{g}_1\, dt & a_2 - \int_0^t \dot{g}_2\, dt & 1 \end{bmatrix}^{-1} z = \hat{T}(t) z$$

$\hat{w}$ is the estimate of plant state w, and $\lim_{t\to\infty} \hat{w} = w$. Note that CT = C.

A Simulation

The third order system of the example was simulated on a digital computer using the following parameters

$a_0 = 24$	$\alpha_0 = 0$	$C_1 = 30$	$k_0 = 24$	$m_0/m_3 = 8000$
$a_1 = 26$	$\alpha_1 = 74$	$C_2 = 195$	$k_1 = 26$	$m_0/m_5 = 2000$
$a_2 = 9$	$\alpha_2 = 0$	$b_1 = 30$	$k_2 = 9$	$g_0 = g_2 = 0$

The eigenvalues of the observer (determined by $\{k_i\}$) were $\lambda_1 = -4$, $\lambda_2 = -2$, $\lambda_3 = -3$. The input to the plant was a square wave of magnitude 1 and frequency 6t. Two parameters, b_0 and g_1, were adjusted by the adaptive law. These were initially set at $b_0 = 73$, $g_1 = -5$ corresponding to a correct value of $b_0 = 75$, $g_1 = -74$. Fig. 2 illustrates the behavior of b_0, g_1, e_2, and e_3 as a function of time.

Remark

As has been previously stated, $\hat{w} = \hat{T}z$ and $\lim_{t\to\infty} \hat{w} = w$. In the general case of an arbitrary plant matrix $\hat{A}$, the determinant of $\hat{T}$ may vanish for some instances of time. These monentary occurrences of course, have no deterimental effect on $\hat{w}$ since convergence of $\hat{w}$ to w is guaranteed. In the important particular case of the preceding example, however, advantage has been taken of the fact that det T is constant by writing equation (*) as $\hat{w} = (\hat{T}^{-1})^{-1} z$. Since for the case of phase variable plant of high order the literal form of T^{-1} is easily produced, it is surmised that writing $(\hat{T}^{-1})^{-1} = \hat{T}$ allows a particular simple construction of $\hat{w}$ when digital computation, rather than analog, is desired.

Conclusion

An adaptive observer has been demonstrated for single-input single-output systems with constant or slowly varying parameters. Work is currently underway to extend the observer to multivariable systems as well as systems with rapidly varying parameters and systems with noise. It is hoped that the adaptive observer will be eventually used not only for observing the state of an unknown system but in model reference parobIems and pole placement problems as well.

APPENDIX A

In order to complete the proof of the theorem when A has repeated eigenvalues (and the number of eigenvectors of A is less than n), the following lemma is needed.

Lemma

Let A be an nxn matrix, $C = [k, 0, 0, \ldots, 0]$, $k \neq 0$, and $\psi(A) = \{\text{all nonsingluar matrices } P | J = P^{-1}AP\}$ where J is an nxn matrix of Jordan form. Then there exists a $P \varepsilon \psi(A)$ such that $CP = [(p_1, \underline{0}), (p_2, \underline{0}), \ldots, (p_j, \underline{0})]$ where each vector $(p_i, \underline{0}) \equiv [p_i, 0, 0, \ldots, 0]$ has a dimension equal to the order of the corresponding $i^{\underline{th}}$ Jordan block in J.

Proof of theorem

Since the similarity transformation matrix that transforms the matrix A_0 (in output form) into the normal form A_n,

$$A_n = \begin{bmatrix} 0 & \vdots & & \\ 0 & \vdots & I & \\ \vdots & \vdots & & \\ & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

is triangular, it suffices to show that the theorem is true with A_0 replaced by A_n.

(a) Let A_n be partitioned into

$$A_n^i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}$$

where A_{11} scalar and A_{12} is $1 \times (n-1)$ row vector, A_{21} and A_{22} appropriately formed. Then

$$A^i = \begin{bmatrix} k & \underline{0} \\ T_1 & T_2 \end{bmatrix} \begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix} \begin{bmatrix} k & \underline{0} \\ -T_2^{-1}T_1 & T_2^{-1} \end{bmatrix}$$

for $i \geq 1$, and

$$CA^i = CTA_n^i T^{-1} = CA^i T^{-1} = [-A_{12}^i \, T_2^{-1} T_1 \; \vdots \; A_{12}^i T_2^{-1}]$$

since $A_{11}^i \equiv 0$ for $i \leq n-1$. The term $-A_{12}^i \, T_2^{-1} T_1$ is a scalar which may be any value.

$$A_{12}^i = [\delta_{1i} \; \delta_{2i} \; \delta_{3i} \; \ldots \delta_{(n-1)i}] \text{ where}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} . \qquad \text{Let } T_2^{-1} = \begin{bmatrix} \underline{t}_1^T \\ \cdots\cdots \\ \underline{t}_2^T \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ \underline{t}_{n-1}^T \end{bmatrix} ,$$

where $\underline{t}_i^T$ are row vectors. Then

$$Q \equiv [C^T \; \vdots \; A^T C^T \vdots \; A^{2T} C^T \vdots \ldots \vdots \; (CA^{n-1})^T] =$$

$$\begin{bmatrix} k & \vdots & k_1 & \vdots & k_2 & \vdots & & \vdots & k_{n-1} \\ 0 & \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & \vdots & \underline{t}_1 & \vdots & \underline{t}_2 & \vdots & \cdots\cdots & \vdots & \underline{t}_{n-1} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix}$$

Since T_2 nonsingular, $\underline{t}_1$ mutually independent as well as independent of C. Therefore, observability matrix Q is of rank n. Therefore, A observable.

(b) Suppose (A, C) completely observable but $A_n \neq T^{-1}AT$ for any $\hat{T}$. Let P be a modal matrix satisfying the lemma condition so that $J = P^{-1}AP$ and J is in Jordan canonical form. Let V be a Van der Monde matrix corresponding to J such that $A_n = VP^{-1}APV^{-1}$. Since (A,C) observable, A possess but one invariant polynomial under supposition of the form of C [13]; therefore V exists. By assumption, $T \neq PV^{-1}$. Since $\hat{T}$ is arbitrary, this implies that $CT \neq CPV^{-1}$, or that

$$CTV = CV \neq CP$$

Because of the lemma, the P chosen in such that

$$CP = [p_1, 0, 0, \ldots, 0].$$

A necessary condition on CP for observability is that p_1 be non-zero [14].

By the form of the Van der Monde matrix, $CV = [q_1, 0, 0, \ldots, 0]$ with $q_1 \neq 0$ a constant.

$CV \neq CP$ implies $q_1 \neq p_1$. But q_1 is any non-zero constant; thus, $q_1 \neq p_1$ implies $p_1 = 0$ which implies (A,C) not observable. Thus a contradiction is reached.

APPENDIX B

Proof of Theorem 2

It has been shown that

$$\lim_{t \to \infty} e_1(t) = 0$$

$$\lim_{t\to\infty} u_j(t) = 0 \text{ for each } j \tag{B1}$$

$$\lim_{t\to\infty} \phi_i(t) = \text{constant for each } i$$

Therefore from (7) and (7.a)

$$\lim_{t\to\infty} \alpha_i(t) = \text{constant} \quad \text{for each } i$$

$$\lim_{t\to\infty} \beta_i(t) = \text{constant} \quad \text{for each } i \tag{B2}$$

Referring to n equations (3), each equation may be differentiated in a manner so as to form the vector $\underline{e}_s(t) = [e_1^{(n)}, e_2^{(n-1)}, e_3^{(n-2)}, \ldots, \dot{e}_n]^T$. Employing (B1) and (B2) in determining $\lim_{t\to\infty} \underline{e}_s(t)$ and letting $\beta_i \equiv 0$ for $i > m$, equations (B3) result.

$$0 = e_2^{(n-1)} + \alpha_{n-1}\, y^{(n-1)} + \beta_{n-1}\, r^{(n-1)}$$

$$e_i^{(n+1-i)} = e_{i+1}^{(n-i)} + \alpha_{n-i}\, y^{(n-i)} + \beta_{n-i}\, r^{(n-i)} \qquad i = 2,3,4,\ldots n-1 \tag{B3}$$

$$\dot{e}_n = \alpha_0 y + \beta_0 r$$

All e's may be easily eliminated from (B3) yielding

$$0 = \left(\sum_{i=0}^{n-1} \alpha_i s^i \right) y + \left(\sum_{i=0}^{m} \beta_i s^i \right) r \tag{B4}$$

Let the stable (but unknown) plant transfer function be

$$\left(\sum_{i=0}^{n} a_i s^i \right) y = \left(\sum_{i=0}^{m} b_i s^i \right) r \tag{B5}$$

$$a_n \equiv 1$$

Combining (B4) and (B5) yields

$$0 = [(\sum_{i=0}^{n-1} \alpha_i s^i)(\sum_{i=0}^{m} b_i s^i) + (\sum_{i=0}^{m} \beta_i s^i)(\sum_{i=0}^{n} a_i s^i)]r \qquad (B6)$$

(B6) represents a condition upon r(t) which is assured in the limit, by (B1) and (B2) that is to say, after adaptation has forced e_1 to vanish. Two distinct possibilities exist regarding the solution of the (n+m)-th order linear homogeneous differential equation (B6): (a) either the steady-state system command input r(t) obeys (B6) for some values α_i and β_i, or (b) the n+m+1 coefficients of polynomial in brackets are in the limit each zero. By supposition of the theorem, (a) cannot occur; consequently (b) must be true.

Using the assumption of observability to insure that (B5) is relatively prime, it is easy to show by mathematical induction that condition (b) implies that the constants α_i and β_i are each zero, which was to be proved.

Corollary

The corollary is a direct result of placing the characteristic values of (B6) along the imaginary axis. The least number of distinct poles required of r so that it is not a solution of (B6) is exactly one more than the order of (B6), or n+m+1. Therefore r must contain at least $\frac{n+m+1}{2}$ distinct frequencies in its steady-state condition.

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TABLE I

Notation for generation of v_i in normal form. In $v_i(j)$, j denotes the state variable, i denotes the function in (8). Example: for n=4, v_1 is defined by

$$\dddot{v}_1 + a_2 \ddot{v}_1 + a_1 \dot{v}_1 + a_0 v_0 = \dot{x}_1$$

and generated by

$$\begin{bmatrix} \dot{v}_1(1) \\ \dot{v}_1(2) \\ \dot{v}_1(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} v_1(1) \\ v_1(2) \\ v_1(3) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -a_1 \end{bmatrix} x_1$$

n=2
m=0

$$u_0 = \dot{\phi}_2 v_2 + \dot{\phi}_0 v_0$$

n=2
m=1

$$u_0 \quad \dot{\phi}_0 v_0 + \dot{\phi}_2 v_2$$

n=3
m=0

$$u_0 = \dot{\phi}_1 v_1(2) + a_1 \dot{\phi}_1 v_1(1) + \dot{\phi}_0 v_0(2) + a_1 \dot{\phi}_0 v_0(1) + a_1 \dot{\phi}_3 v_3(1) + \dot{\phi}_3 v_3(2)$$

$$= \dot{\phi}_1 (v_1(2) + a_1 v_1(1)) + \dot{\phi}_0 (v_0(2) + a_1 v_0(1)) + \dot{\phi}_3 (v_3(2) + a_1 v_3(1))$$

$$u_1 = \dot{\phi}_0 v_0(1) + \dot{\phi}_1 v_1(1) + \dot{\phi}_3 v_3(1)$$

n=3
m=1

$$u_0 = \dot{\phi}_0 (v_0(2) + a_1 v_0(1)) + \dot{\phi}_1 (v_1(2) + a_1 v_1(1)) + \dot{\phi}_3 (v_3(2) + a_1 v_3(1)) + \dot{\phi}_4 (v_4(2) + a_1 v_4(1))$$

$$u_1 = \sum_{\substack{i=0 \\ i\neq 2}}^{4} \dot{\phi}_i v_i(1)$$

n=3 m=2 — same as n=3 m=1 with $\{\phi_i\}$ define in (7a)

n=4 m=0

$$u_0 = \sum_{i=0}^{2} \sum_{j=1}^{3} \dot{\phi}_i \, a_j \, v_i(j) + \dot{\phi}_4 \sum_{j=1}^{3} a_j \, v_4(j)$$

$$u_1 = \sum_{i=0}^{2} \sum_{j=1}^{2} \dot{\phi}_i \, a_{j+1} v_i(j) + \dot{\phi}_4 \sum_{j=1}^{2} a_{j+1} \, v_4(j)$$

$$u_2 = \sum_{i=0}^{2} \dot{\phi}_i \, v_i(1) + \dot{\phi}_4 \, v_4(1)$$

n=4 m=1

$$u_0 = \sum_{\substack{i=0 \\ i\neq 3}}^{5} \sum^{3} \dot{\phi}_i \, a_j \, v_i(j)$$

$$u_1 = \sum_{\substack{i=0 \\ i\neq 3}}^{5} \sum_{j=1}^{2} \dot{\phi}_i \, a_{j+1} \, v_i(j)$$

$$u_2 = \sum_{\substack{i=0 \\ i\neq 3}}^{5} \dot{\phi}_i \, v_i(1)$$

n=4 m=2

$$u_0 = \sum_{\substack{i=0 \\ i\neq 3}}^{6} \sum_{j=1}^{3} \dot{\phi}_i \, a_j \, v_i(j)$$

$$u_1 = \sum_{\substack{i=0 \\ i\neq 3}}^{6} \sum_{j=1}^{2} \dot{\phi}_i \, a_{j+1} \, v_i(j)$$

$$u_2 = \sum_{\substack{i=0 \\ i\neq 3}}^{6} \dot{\phi}_i \, v_i(1)$$

n=4 m=3 — same as n=4 m=2 with $\{\phi_i\}$ defined in (7a)

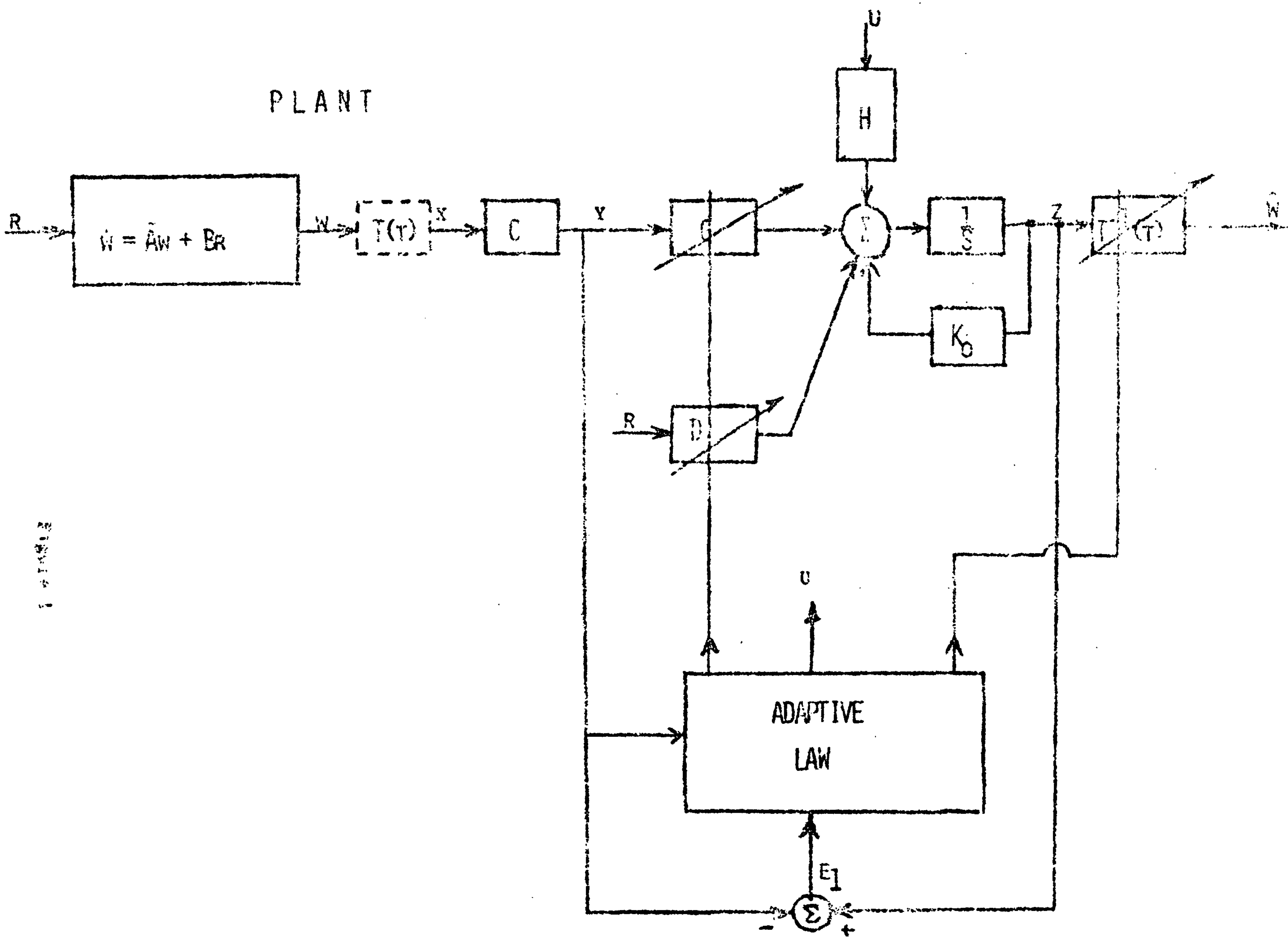

Figure 1

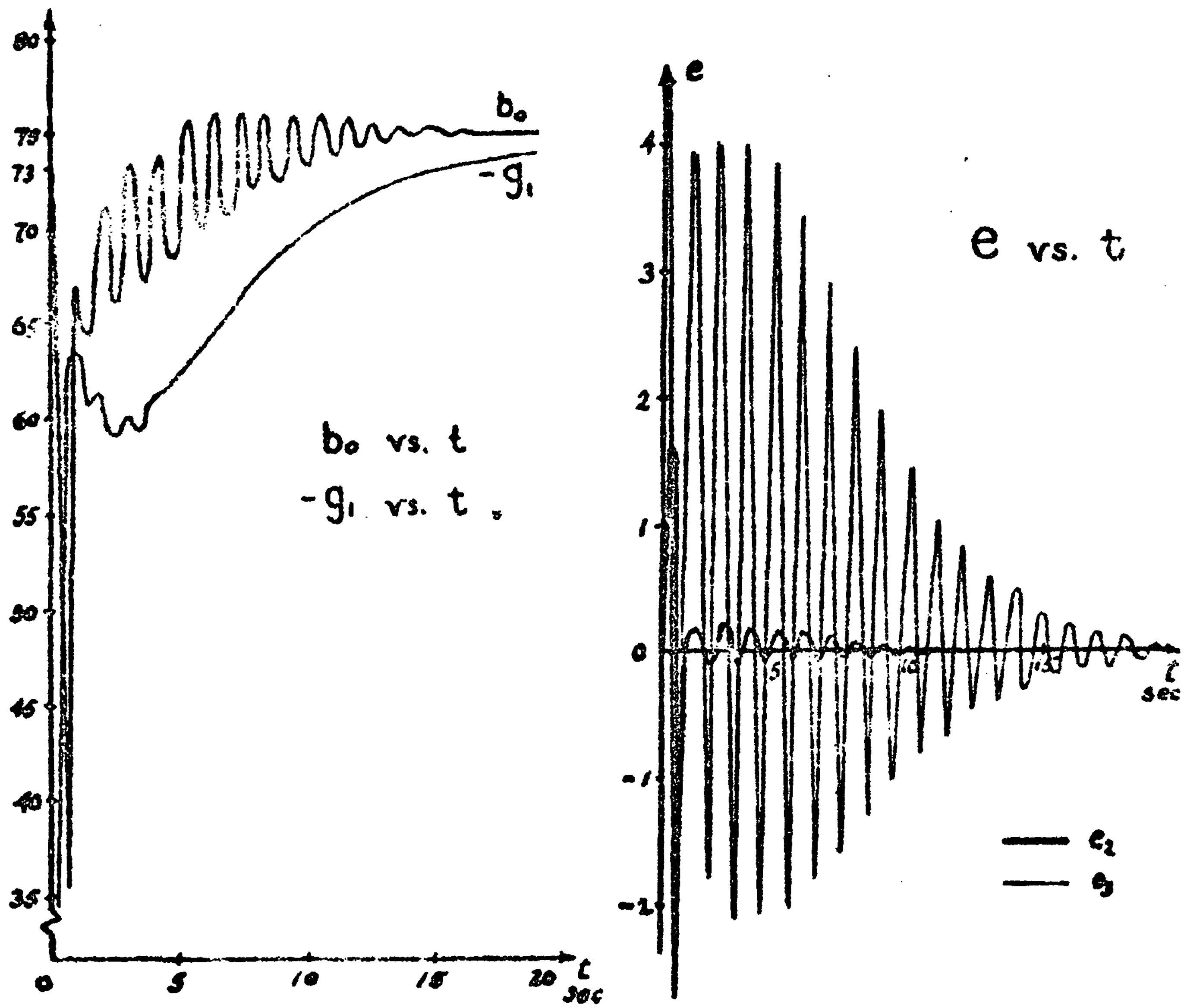

Figure 2