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# AN EFFICIENT ALGORITHM FOR CALCULATION OF THE LUENBERGER CANONICAL FORM

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## Introduction

The Luenberger canonical form [1] is an extension of the control canonical form[2] for single-input or single-output controllable and observable systems to multivariable systems. The canonical form is not unique in the multivariable case. However, the controllability indicies are structural invariants of the system and correspond to the various blocks in the Luenberger canonical form.[3-7]

Consider the linear time-invariant controllable system

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \tag{1}$$

where $\underline{x}$ is an nx1 state vector and $\underline{u}$ is an mx1 input vector. In addition it is assumed that the columns of B are linearly independent. The controllability matrix $\Gamma = [B, AB, A^2B, \ldots, A^{n-1}B]$ has rank n and an nxn nonsingular matrix $P = [\underline{b}_1, A\underline{b}_1, \ldots A^{k_1-1}\underline{b}_1, \underline{b}_2, A\underline{b}_2, \ldots A\underline{b}_2^{k_2-1}, \ldots, A\underline{b}_m^{k_m-1}]$ can be selected from the columns of $\Gamma$. Let $e_1, e_2, \ldots e_m$ be the $\sigma_1, \sigma_2, \ldots \sigma_m$-th row respectively of $P^{-1}$ where $\sigma_i = \sum_{j=1}^{i} k_j$. The vectors $e_1, e_2, \ldots e_m$ are used to construct the transformation matrix [1].

$$T = \begin{bmatrix} e_1 \\ e_1A \\ \cdot \\ \cdot \\ \cdot \\ e_1A^{k_1-1} \\ e_2 \\ e_2A \\ \cdot \\ \cdot \\ \cdot \\ e_mA^{k_m-1} \end{bmatrix}$$

The transformation T reduces the system (1) to the form

$$\dot{\underline{y}} = \hat{A}\underline{y} + \hat{B}\underline{u} \tag{2}$$

where

$$\hat{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
a_{\sigma_1,1} & a_{\sigma_1,2} & \cdots & a_{\sigma_1,\sigma_1} & & & & & , & \cdots & & a_{\sigma_1,n} \\
 & & & & 0 & 1 & 0 & 0 & & & & \\
 & & & & 0 & 0 & 1 & 0 & & & & \\
 & & & & \vdots & & \ddots & & & & & \\
 & & & & & & & 1 & & & & \\
a_{\sigma_2,1} & a_{\sigma_2,2} & \cdots & \cdots & a_{\sigma_2,\sigma_1+1} & \cdots & a_{\sigma_2,\sigma_2} & & \cdots & & & a_{\sigma_2,n} \\
 & & & & & & & & \ddots & & & \\
 & & & & & & & & 0 & & 1 & 0 \\
a_{\sigma_m,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{\sigma_m,\sigma_{m-1}+1} & & \cdots & a_{\sigma_m,\sigma_m}
\end{bmatrix}$$

$$\hat{B} = \begin{bmatrix}
0 & 0 & & & 0 \\
\vdots & \vdots & & & \vdots \\
1 & b_{\sigma_1,2} & \cdots & & b_{\sigma_1,m} \\
0 & & & & 0 \\
 & 1 & b_{\sigma_2,3} & \cdots & b_{\sigma_2,m} \\
 & & & & \vdots \\
0 & \cdots & & & 1
\end{bmatrix}$$

There are several matrix computational difficulties, arising out of the need to find $P^{-1}$ and $T^{-1}$, in arriving at the canonical form following Luenbergers's construction.

Here, a new algorithm is presented which is more efficient and accurate than Luenberger's construction. Also, the canonical form is computed directly and the transformation matrix T is computed only if necessary.

## Basic Results

Let the transformation matrix

$$T = [\underline{t}_1 \vdots \underline{t}_2 \cdots \underline{t}_{\sigma_1} \vdots \underline{t}_{\sigma_1+1}, \cdots \underline{t}_{\sigma_2} \vdots \cdots \underline{t}_{\sigma_m}]. \tag{4}$$

Then the similarity transformation satisfies the conditions

$$AT = T\hat{A} \tag{5}$$

$$B = T\hat{B} \tag{6}$$

Numbering the columns of B as $[\underline{b}_1 \; \underline{b}_2 \; \cdots \; \underline{b}_m]$, equation (6) imposes the following restrictions on the columns of T:

$$\underline{b}_1 = \underline{t}_{\sigma_1}$$

$$\underline{b}_2 = \underline{t}_{\sigma_2} + b_{\sigma_1,2}\,\underline{t}_{\sigma_1}$$

$$\vdots$$

$$\underline{b}_m = \underline{t}_{\sigma_m} + b_{\sigma_1,m}\,\underline{t}_{\sigma_1} + \cdots + b_{\sigma_{m-1},m}\,\underline{t}_{\sigma_{m-1}}$$

solving for $\underline{t}_{\sigma_i}$, we can find constants $c_{ji}$ such that

$$\underline{t}_{\sigma_i} = \sum_{j=1}^{i} c_{ji}\,\underline{b}_j \quad \text{with } c_{ii} = 1, \quad i = 1,..m. \tag{7}$$

Further, from equation (5) the columns of T are related by the set of equations:

$$\underline{t}_{\sigma_\ell - j} = A\underline{t}_{\sigma_\ell - j+1} - \sum_{i=1}^{m} a_{\sigma_i,\, \sigma_\ell - j+1}\,\underline{t}_{\sigma_i} \qquad j=1,2,\ldots k_\ell - 1;\ \ell=1,2,\ldots m \tag{8.a}$$

$$\underline{0} = A\underline{t}_{\sigma_\ell - k_\ell + 1} \sum_{i=1}^{m} a_{\sigma_i, \sigma_\ell - k_\ell + 1}\,\underline{t}_{\sigma_i} \qquad \ell = 1,2,\ldots m \tag{8.b}$$

Examining (8) recursively, it can be seen that $\underline{t}_{\sigma_\ell - j}$ and $\underline{0}$ can be written entirely in terms of the $\underline{t}_{\sigma_i}$'s by the equations

$$\underline{t}_{\sigma_\ell - j} = A^j \underline{t}_{\sigma_\ell} - \sum_{i=1}^{m} \sum_{k=0}^{j-1} a_{\sigma_i, \sigma_\ell - j+k+1} A^k \underline{t}_{\sigma_i} \qquad j = 1,2,\ldots k_\ell - 1;\ \ell = 1,\ldots m \tag{9.a}$$

$$\underline{0} = A^{k_\ell} \underline{t}_{\sigma_\ell} - \sum_{i=1}^{m} \sum_{k=0}^{k_\ell - 1} a_{\sigma_i,\ \sigma_\ell - k_\ell + k+1} A^k \underline{t}_{\sigma_i} \qquad \ell = 1,\ldots m \tag{9.b}$$

Substituting for $\underline{t}_{\sigma_i}$ from equation (7) in equation (9.b) and rearranging the terms, we get

$$\underline{0} = A^{k_\ell} \left[ \sum_{j=1}^{\ell} c_{j\ell}\, \underline{b}_j \right] - \sum_{k=0}^{k_\ell - 1} \sum_{i=1}^{m} \sum_{j=1}^{m} (a_{\sigma_j, \sigma_\ell - k_\ell + k+1} \cdot c_{ij})\, A^k \underline{b}_i \qquad \ell = 1,\ldots m \tag{10}$$

Kalman [5] has suggested use of the decomposition form:

$$A^{k_\ell}\, \underline{b}_\ell = - \sum_{i=1}^{m} \sum_{k=0}^{k_i - 1} \alpha_{\ell i k} A^k \underline{b}_i\ , \qquad \ell = 1,2,\ldots m \tag{11}$$

where $\alpha_{\ell i k}$ are constants and $\{A^k b_i\}$ $i = 1,2,\ldots m;\ k=1, \ldots k_i$ is a basis for the n-space. Unfortunately, the accompanying change of basis suggested by Kalman[5] does not lead either to the Luenberger canonical form or to the form suggested by him, as can be seen by applying his method to the example posed herein.

Without loss of generality we can assume that $k_1 \geq k_2 \geq \ldots \geq k_m$. Then, by the selection procedure of basis vectors $A^k b_i$ we can guarantee that

$\alpha_{\ell i k} = 0$ for $k > k_\ell$ and $\alpha_{\ell i k} = 0$ for $k = k_\ell$, $i > \ell$. By defining $\alpha_{\ell \ell k_\ell} = 1$, condition (11) can be rewritten as

$$0 = A^{k_\ell} \left[ \sum_{j=1}^{\ell} \alpha_{\ell j k_\ell}\, \underline{b}_j \right] + \sum_{i=1}^{m} \sum_{k=0}^{k_\ell - 1} \alpha_{\ell i k} A^k \underline{b}_i, \quad \ell = 1,2,\ldots, m. \tag{12}$$

Comparison of equation (10) and (12) suggests the following procedure for selecting c's and a's:

(i) $c_{j\ell} = \alpha_{\ell j k_\ell}$ , $\ell = 1,2,\ldots m;\ j = 1,2,\ldots \ell$

with $c_{\ell\ell} = \alpha_{\ell\ell k_\ell} = 1$

(ii) $\sum_{j=1}^{m} c_{ij}\, a_{\sigma_j, \sigma_\ell - k_\ell + k + 1} = -\alpha_{\ell i k}$ $\quad \ell = 1,2,\ldots m;\ i = 1,2,\ldots m;$

$k = 0,1,\ldots,\ k_\ell - 1$

Given $c_{ij}$ calculated in (i) and $\alpha_{\ell i k} = 0$ for $k > k_\ell,\ i \neq \ell$

$k = k_\ell,\ i > \ell$

The a's can be obtained by backward substitution from (ii).

Also the b's can be obtained in the same manner through the equation

(iii) $b_{\sigma_{k,p}} + \sum_{j=k+1}^{p-1} c_{kj}\, b_{\sigma_{j,p}} + c_{kp} = 0,\quad k = p-1,\ p-2,\ldots 1;\ p=2,3\ldots m.$

The algorithm suggested here computes the canonical form directly and, if necessary, the transformation matrix T can be obtained from equations (7) and (8.a).

## Computational Efficiency

The algorithm suggested in this paper results in a large reduction in the amount of computation necessary to obtain the canonical form. A comparison of Table 1 and Table 2 shows that, using Gaussian elimination techniques where applicable, there is a saving of at least $2n^3$ multiplications (ie about 43% reduction in the number of multiplications).

| | |
|---|---|
| $P$ | $n^2(n-m)$ |
| $P^{-1}$ | $\frac{4}{3}(n^3-n)$ |
| $T$ | $n^2(n-m)$ |
| $T^{-1}$ | $\frac{4}{3}(n^3-n)$ |
| $T^{-1}AT$ | $2n^3$ |
| $T^{-1}B$ | $nm$ |
| Total | $4\frac{2}{3}n^3 + 2n^2(n-m) - 4\frac{2}{3}n + nm$ $> 4\frac{2}{3}n^3$ |

Table I

Total number of multiplications in Luenberger's Method

| | |
|---|---|
| basis | $n^2(n-m)$ |
| $\alpha$'s | $\frac{n^3}{3} - \frac{n}{3} + m(n^2-1)$ |
| $a_{ij}$ | $mn(m-1)/2$ |
| b's | $m^3/3 - m/3$ |
| $T$ | $n^3 + \frac{3}{2}nm^2 - 2n^2m + \frac{1}{2}mn$ |
| Total | $\frac{7}{3}n^3 + \frac{m^3}{3} - 2mn(n-m) - \frac{1}{3}(4m+n)$ $< 2\frac{2}{3}n^3$ |

Table 2

Total number of multiplications in the new algorithm

If only the canonical form is necessary there is a further reduction in the number of multiplications since the canonical form is computed directly without having to compute T.

The algorithm should give better accuracy than Luenberger's method for two reasons. (1) the zeros are preserved in the canonical form and, as such, round-off errors in their computation are avoided. (2) the reduced amount of computation and Gaussian techniques lend to greater inherent accuracy and the ability to refine the solution with additional computations.

Example

Consider the system with

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Then,

$$\underline{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \quad \underline{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } A\underline{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} .$$

Thus $k_1 = 2$ and $k_2 = 1$. Further, we have

$$A^2 b_1 = \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$Ab_2 \quad \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

This gives $\alpha_{110} = 3$, $\alpha_{111} = -4$, $\alpha_{120} = 0$

$$\alpha_{210} = \frac{1}{2}, \quad \alpha_{211} = -\frac{1}{2}, \quad \alpha_{220} = -2.$$

It is assumed that $\alpha_{121}=0$ and $\alpha_{212}=0$.

Now, from (I) and (II), $c_{11}=1$, $c_{12} = \alpha_{211} = -\frac{1}{2}$ and $c_{22} = 1$.

$\ell$ i k

1 2 0 $\quad c_{22}\, a_{\sigma_2,\sigma_1-k_1+k+1} = -\alpha_{120} \rightarrow a_{31} = 0$

1 1 0 $\quad c_{11}\, a_{\sigma_1,\sigma_1-k_1+k+1} + c_{12}\, a_{\sigma_2,\sigma_1-k_1+k+1} = -\alpha_{110}$

$\rightarrow a_{21} - \frac{1}{2} a_{31} = -3$ i.e. $a_{21} = -3$.

1 2 1 $\quad c_{22}\, a_{\sigma_2,\sigma_1-k_1+k+1} = -\alpha_{121} \rightarrow a_{32} = 0.$

1 1 1 $\quad c_{11}\, a_{\sigma_1,\sigma_1-k_1+k+1} + c_{12}\, a_{\sigma_2,\sigma_1-k_1+k+1} = -\alpha_{111}$

$$\rightarrow a_{22} - \frac{1}{2} a_{32} = 4 \quad \text{i.e. } a_{22} = 4$$

2 2 0 $\quad c_{22}\, a_{\sigma_2,\sigma_2-k_2+k+1} = -\alpha_{220} \rightarrow a_{33} = 2$

2 1 0 $\quad c_{11}\, a_{\sigma_1,\sigma_2-k_2+k+1} + c_{12}\, a_{\sigma_2,\sigma_2-k_2+k+1} = -\alpha_{210}$

$$\rightarrow a_{23} - \frac{1}{2} a_{33} = 0 \quad \text{i.e. } a_{23} = \frac{1}{2}$$

B can be obtained from (111).

$$b_{\sigma_1,2} + c_{12} = 0 \quad \text{i.e. } b_{\sigma_1,2} = -c_{12} = \frac{1}{2}.$$

This gives the canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Application of (8.a) yields

$$T = \begin{bmatrix} -3 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ -1 & 1 & \frac{1}{2} \end{bmatrix}.$$

Conclusions

A new algorithm is suggested to obtain the Luenberger canonical form for multivariable systems. This method computes the canonical form directly without having to compute the transformation matrix. In addition, there is a large reduction in the number of calculations. The reduced computations along with Gaussian techniques lend to greater inherent accuracy and the ability to refine the solution with additional computations.

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