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CLOSED-FORM SOLUTIONS FOR ATMOSPHERIC FLIGHT WITH APPLICATIONS TO SHUTTLE GUIDANCE

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16. ABSTRACT

Closed-form solutions for the motion of a rocket-powered vehicle during atmospheric ascent and closed-form solutions for unpowered atmospheric reentry are developed. These closed-form solutions are then used to develop a simplified guidance scheme and to develop a variation-of-parameters integration of more accurate equations of motion with the closed-form solutions as base solutions. The variation-of-parameters integration of the more accurate equations of motion also allows the transition partial derivative matrices associated with these equations to be easily developed. Then the partial derivative transition matrices are used to develop a guidance scheme based on the more accurate equations of motion instead of the less accurate closed-form solutions.

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INTRODUCTION

The numerical integration of any set of differential equations for the atmospheric motion of a rocket-powered vehicle is usually a time-consuming procedure even when a high-speed digital computer is used. For this reason, approximate closed-form solutions, which take much less time to evaluate, are often used to represent the atmospheric motion of a rocket-powered vehicle. A set of closed-form solutions for reentry and atmospheric ascent is developed in this report. For both cases accurate closed-form solutions for the three-dimensional Cartesian coordinates of the vehicle's position and velocity vector are obtained. Also, the partial derivative transition matrices of the final state of the vehicle with respect to the initial state are obtained as closed-form expressions. In addition, the closed-form solutions for the state and partial derivative transition matrices are used to develop a variation-of-parameters technique which can provide rapid numerical integration of the original equations of motion.

DEVELOPMENT OF CLOSED-FORM SOLUTIONS

The starting point for all the work in this report will be the equations of motion for atmospheric flight developed in Reference 1. The techniques developed could be applied to any other equations of motion, but a particular example is useful in illustrating the ideas. Thus, the equations of motion derived in Reference 1 are chosen as an example for this report. The derivations of these equations will not be repeated here, therefore, the reader may want to examine Reference 1 before starting the next section.

The equations of motion for the center of gravity of a space vehicle in an inertial three-dimensional Cartesian coordinate system as given in Reference 1 are first-order ordinary nonlinear differential equations for the position vector ($\bar{x}$ of dimension 3) and the velocity vector ($\bar{v}$ of dimension 3); i. e.,

$$\dot{\bar{x}} = \bar{v}$$

$$\dot{\bar{v}} = \frac{F}{m}\frac{\bar{p}}{|\bar{p}|} + \frac{\bar{L} + \bar{D}}{m} - \frac{GM\bar{x}}{R^3} \quad ,$$

where

$$F = F_S + A_e(P_0 - P) \quad ,$$

$$m = m_0 - \dot{m}(t - t_0) \quad ,$$

$$\overline{L} = \frac{1}{2}\rho A_r c_{L_\alpha}\left[|\overline{v}_r|^2\left(\frac{\overline{p}}{|\overline{p}|}\right) - \left(\overline{v}_r \cdot \frac{\overline{p}}{|\overline{p}|}\right)\overline{v}_r\right] \quad ,$$

$$\overline{D} = -\left(\frac{1}{2}\rho A_r\right)\left[|\overline{v}_r|\left(c_A + 2\eta c^2_{L_\alpha}\right) - 2\eta c^2_{L_\alpha}\left(\overline{v}_r \cdot \frac{\overline{p}}{|\overline{p}|}\right)\right]\overline{v}_r \quad ,$$

and

$$\overline{v}_r = \overline{v} - \overline{\omega} \times \overline{x} - \overline{W} \quad .$$

It should be noted that $\overline{p}$ is the three dimensional control vector which must be determined or specified as a function of time, and the quantities F_S, A_e, P_0, m_0, $\dot{m}$, A_r, $\overline{\omega}$, and GM are constants defined in Reference 1. The quantities ρ, P, c_A, c_{L_α}, and η are (in general) specified functions of $\overline{x}$ and $\overline{v}$. The exact form for these functions is also given in Reference 1. With the preceding statements, it can be seen that the right-hand side of the $\dot{\overline{v}}$ equation is a function of t, $\overline{x}$, and $\overline{v}$. Thus, a given time function for the control vector $\overline{p}$ and initial conditions x_0 and v_0 at some time t_0 allows numerical integration to be used to obtain values of $\overline{x}$ and $\overline{v}$ at any $t > t_0$. Usually the time function for control vector p is determined by optimization theory to maximize or minimize a quantity J defined by

$$J = \varphi(\overline{x}_f, \overline{v}_f, t_f) + cq_f$$

and to satisfy a vector function (of dimension ≤ 6) called the physical boundary condition and written in the following general form:

$$F(\overline{x}_f, \overline{v}_f, t_f) = 0 \quad .$$

In the above expression for J, c is an arbitrary positive or negative weighting factor for q_f; and q_f is a new variable defined by another first-order nonlinear ordinary differential equation with the following form:

$$\dot{q} = f(\bar{x}, \bar{v}, \bar{p}, t) \quad .$$

Then, for a given control vector time function and initial conditions $\bar{x}_0$, $\bar{v}_0$, q_0, at some time t_0, numerical integration can be used to obtain $\bar{x}_f$, $\bar{v}_f$, and q_f at the final time t_f. Thus, it can be seen that the value of J depends on the control vector time function and the initial conditions $\bar{x}_0$, $\bar{v}_0$, and q_0 at t_0. Since the initial conditions are usually assumed to be specified by the nature of the problem, the only thing left that affects the value of J is the form of the control vector time function $\bar{p}(t)$, where $t_0 \leq t \leq t_f$.

In order to use optimization theory to determine the control vector time function $\bar{p}(t)$, the time functions $\bar{\lambda}$ (a three-dimensional vector), $\bar{u}$ (another three-dimensional vector), and γ (a scalar) are introduced. These time functions are usually called Lagrangian multipliers, or simply multipliers, and are defined by their differential equations. To obtain the differential equations for $\bar{\lambda}$, $\bar{u}$, and γ, define

$$H = \bar{\lambda}^T \dot{\bar{x}} + \bar{u}^T \dot{\bar{v}} + \gamma \dot{q} \quad ,$$

where the superscript T denotes the ordinary matrix transpose.

Then

$$\dot{\bar{\lambda}}^T = - \frac{\partial H}{\partial \bar{x}} \quad ,$$

$$\dot{\bar{u}}^T = - \frac{\partial H}{\partial \bar{v}} \quad ,$$

$$\dot{\gamma} = - \frac{\partial H}{\partial q} \quad .$$

The above expressions allow the differential equations for $\bar{\lambda}$, $\bar{u}$, and γ to be obtained by performing the indicated partial differentiations of H with $\dot{\bar{x}}$ and $\dot{\bar{v}}$ replaced by their right-hand sides in the expression for H. Then values for

$\bar{\lambda}$, $\bar{u}$, and γ at any $t > t_0$ can be obtained by numerical integration of the differential equations $\dot{\bar{\lambda}}^T$, $\dot{\bar{u}}^T$, and $\dot{\gamma}$ along with a simultaneous numerical integration of the differential equations $\dot{\bar{x}}$, $\dot{\bar{v}}$, and $\dot{q}$ when a control vector time function $\bar{p}(t)$, where $t_0 \leq t \leq t_f$, is specified and the initial conditions $\bar{\lambda}_0$, $\bar{u}_0$, $\bar{\gamma}_0$, $\bar{x}_0$, $\bar{v}_0$, q_0 at t_0 are specified.

Now the results of optimization theory can be used to state that any control vector time function which minimizes J must maximize at every t, where $t_0 \leq t \leq t_f$, the quantity H, and any control vector time function which maximizes J must minimize at every t, where $t_0 \leq t \leq t_f$, the quantity H. In addition, any control vector time function which minimizes or maximizes J and satisfies the physical boundary conditions [denoted by $F(\bar{x}, \bar{v}, t_f) = 0$] must maximize or minimize H as stated previously and must be produced by initial conditions $\bar{\lambda}_0$, $\bar{u}_0$, and γ_0 that satisfy the following conditions:

$$\bar{\lambda}_f^T = -\frac{\partial \varphi}{\partial \bar{x}_f}\bigg| - \Psi^T\left(\frac{\partial F}{\partial \bar{x}_f}\right) ,$$

$$\bar{u}_f^T = -\frac{\partial \varphi}{\partial \bar{v}_f}\bigg| - \Psi^T\left(\frac{\partial F}{\partial \bar{v}_f}\right) ,$$

$$\gamma_f = -c \bigg|$$

$$H_f = 0 .$$

In the above expressions, Ψ is another vector of constant multipliers that must be introduced and has the same dimension as $F(\bar{x}, \bar{v}, t_f)$. If a control vector time function and the multipliers $\bar{\lambda}_0$, $\bar{u}_0$, γ_0, and Ψ can be found that satisfy the preceding necessary conditions, the optimization boundary valve problem has been solved. Reference 1 discusses this problem in more detail and indicates a few techniques for solving the problem when numerical integration is used on all the ordinary nonlinear differential equations involved. The purpose of this report is to make simplifications in the ordinary nonlinear differential equations involved so that closed-form solutions can be used (instead of numerical integration) to solve the optimization boundary value problem. This work will be done separately for the ascent case and the reentry case.

Ascent Case

For the ascent case, let $q = m$ and $J = -q_f$ so that it is desired to determine the control vector time function $\bar{p}(t)$ to minimize J, thus maximizing mass. Then the equations of motion become

$$\dot{\bar{x}} = \bar{v} ,$$

$$\dot{\bar{v}} = \frac{F}{q}\left(\frac{\bar{p}}{|\bar{p}|}\right) + \frac{\bar{L} + \bar{D}}{q} - \frac{GM\bar{x}}{R^3} \quad ,$$

$$\dot{q} = -\dot{m} \quad .$$

When the expressions for $\bar{L}$ and $\bar{D}$ are substituted into the $\dot{\bar{v}}$ equation, the following expression is obtained:

$$\dot{\bar{v}} = \frac{F}{q}\left(\frac{\bar{p}}{|\bar{p}|}\right) + \frac{1}{q}\left\{\left(\frac{1}{2}\rho A_r\right)c_{L_\alpha}\left[|\bar{v}_r|^2\left(\frac{\bar{p}}{|\bar{p}|}\right) - \left(\bar{v}_r \cdot \frac{\bar{p}}{|\bar{p}|}\right)\bar{v}_r\right]\right.$$

$$\left. -\left(\frac{1}{2}\rho A_r\right)\left[|\bar{v}_r|\left(c_A + 2\eta c^2_{L_\alpha}\right) - 2\eta c^2_{L_\alpha}\left(\bar{v}_r \cdot \frac{\bar{p}}{|\bar{p}|}\right)\right]\bar{v}_r\right\} - \frac{GM\bar{x}}{R^3} \quad .$$

This expression can be rewritten as follows:

$$\dot{\bar{v}} = \left(\frac{1}{q}\right)\left[F + \left(\frac{\rho A_r}{2}\right)c_{L_\alpha}|\bar{v}_r|^2\right]\left(\frac{\bar{p}}{|\bar{p}|}\right)$$

$$- \left(\frac{\rho A_r}{2q}\right)c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left(\bar{v}_r\,\bar{v}_r^T\right)\left(\frac{\bar{p}}{|\bar{p}|}\right)$$

$$- \left(\frac{\rho A_r}{2q}\right)|\bar{v}_r|\left(c_A + 2\eta c^2_{L_\alpha}\right)\bar{v}_r - \frac{GM\bar{x}}{R^3}$$

$$= \left(\frac{1}{q}\right)\left\{\left[F + \left(\frac{\rho A_r}{2}\right)|\bar{v}_r|^2\, c_{L_\alpha}\right]I\right.$$

$$\left. - \left(\frac{\rho A_r}{2}\right)c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left(\bar{v}_r\,\bar{v}_r^T\right)\right\}\frac{\bar{p}}{|\bar{p}|}$$

$$- \left(\frac{\rho A_r}{2q}\right)|\bar{v}_r|\left(c_A + 2\eta c^2_{L_\alpha}\right)\bar{v}_r - \frac{GM\bar{x}}{R^3} \quad .$$

Now define

$$[A] = \left[F + \left(\frac{\rho A_r}{2}\right)|\bar{v}_r|^2\, c_{L_\alpha}\right]I - \left(\frac{\rho A_r}{2}\right)c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left(\bar{v}_r\,\bar{v}_r^T\right) \quad ,$$

$$\bar{b} = -\left(\frac{\rho A_r}{2}\right)|\bar{v}_r|\left(c_A + 2\eta c^2_{L_\alpha}\right)\bar{v}_r \quad ,$$

$$\bar{g} = -\frac{GM\bar{x}}{R^3} \quad .$$

Then

$$\dot{\bar{v}} = \left(\frac{1}{q}\right)[A]\frac{\bar{p}}{|\bar{p}|} + \left(\frac{1}{q}\right)\bar{b} + \bar{g} \quad ,$$

so that the expression for H can be written easily as

$$H = \bar{\lambda}^T \bar{v} + \bar{u}^T \left\{\left(\frac{1}{q}\right) [A]\frac{\bar{p}}{|\bar{p}|} + \left(\frac{1}{q}\right)\bar{b} + \bar{g}\right\} - \gamma \dot{m} \quad .$$

Then

$$\dot{\bar{\lambda}}^T = -\frac{\partial H}{\partial \bar{x}} = -\left(\frac{1}{q}\right)\left\{\left(\frac{\bar{p}^T}{|\bar{p}|}\right)\frac{\partial([A]^T\bar{u})}{\partial \bar{x}} + \bar{u}^T\left(\frac{\partial \bar{b}}{\partial \bar{x}}\right)\right\} - \bar{u}^T\left(\frac{\partial \bar{g}}{\partial \bar{x}}\right) ,$$

$$\dot{\bar{u}}^T = -\frac{\partial H}{\partial \bar{v}} = -\bar{\lambda}^T - \left(\frac{1}{q}\right)\left\{\left(\frac{\bar{p}^T}{|\bar{p}|}\right)\frac{\partial([A]^T\bar{u})}{\partial \bar{v}} + \bar{u}^T\left(\frac{\partial \bar{b}}{\partial \bar{v}}\right)\right\} \quad ,$$

$$\dot{\gamma} = -\frac{\partial H}{\partial q} = \left(\frac{1}{q^2}\right)\bar{u}^T\left\{[A]\frac{\bar{p}}{|\bar{p}|} + \bar{b}\right\} \quad .$$

Since J is to be minimized, $\bar{p}$ must be chosen to maximize H at every t, where $t_0 \leq t \leq t_f$. By examining the expression for H, it can be seen that $\bar{p} = [A]^T\bar{u}$ is sufficient to assure a maximization of H with respect to $\bar{p}$ at every t such that $t_0 \leq t \leq t_f$. This result for $\bar{p}$ can be substituted back into all the differential equations, and the formulation of the optimization boundary value problem is practically complete. The entire system of differential equations that must be integrated simultaneously can be written as:

$$\dot{\bar{x}} = \bar{v} \quad ,$$

$$\dot{\bar{v}} = \left(\frac{1}{q}\right)[A]\left\{\frac{[A]^T\bar{u}}{\sqrt{\bar{u}^T[A][A]^T\bar{u}}}\right\} + \left(\frac{1}{q}\right)\bar{b} + \bar{g} \quad ,$$

$$\dot{q} = -\dot{m} \quad ,$$

$$\dot{\bar{\lambda}}^T = -\left(\frac{1}{q}\right)\left\{\left(\frac{\bar{u}^T[A]}{\sqrt{\bar{u}^T[A][A]^T\bar{u}}}\right)\frac{\partial([A]^T\bar{u})}{\partial\bar{x}} + \bar{u}^T\frac{\partial\bar{b}}{\partial\bar{x}}\right\} - \bar{u}^T\frac{\partial\bar{g}}{\partial\bar{x}} \quad ,$$

$$\dot{\bar{u}}^T = -\bar{\lambda}^T - \frac{1}{q}\left\{\left(\frac{\bar{u}^T[A]}{\sqrt{\bar{u}^T[A][A]^T\bar{u}}}\right)\frac{\partial([A]^T\bar{u})}{\partial\bar{v}} + \bar{u}^T\left(\frac{\partial\bar{b}}{\partial\bar{v}}\right)\right\} \quad ,$$

$$\dot{\gamma} = \left(\frac{1}{q^2}\right)\bar{u}^T\left\{[A]\left(\frac{[A]^T\bar{u}}{\sqrt{\bar{u}^T[A][A]^T\bar{u}}}\right) + \bar{b}\right\} \quad .$$

Given initial conditions $\bar{x}_0$, $\bar{v}_0$, q_0, $\bar{\lambda}_0$, $\bar{u}_0$, and γ_0 at t_0, this system can be integrated numerically; but a closed-form solution is to be derived here that will approximate with some degree of effectiveness the solution obtained by numerical integration. The first step in obtaining a closed-form solution for the entire system is to write $\bar{\lambda}(t)$ and $\bar{u}(t)$ as a Taylor Series expansion about the initial conditions; i. e.,

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0(t - t_0) + \frac{1}{2}\ddot{\bar{\lambda}}_0(t - t_0)^2 + \ldots$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0(t - t_0) + \frac{1}{2}\ddot{\bar{u}}_0(t - t_0)^2 + \ldots \quad .$$

The differential equations for $\dot{\bar{\lambda}}$ and $\dot{\bar{u}}$ can be used to evaluate $\dot{\bar{\lambda}}_0$, $\dot{\bar{u}}_0$, $\ddot{\bar{\lambda}}_0$, $\ddot{\bar{u}}_0$, etc. Also, the quantities $[A]$, $\bar{b}$, and $\bar{g}$ are expanded in a Taylor Series about the initial conditions; i. e.,

$$[A] = [A_0] + [\dot{A}_0](t - t_0) + \frac{1}{2}[\ddot{A}_0](t - t_0)^2 + \ldots \quad ,$$

$$\bar{b} = \bar{b}_0 + \dot{\bar{b}}_0(t - t_0) + \frac{1}{2}\ddot{\bar{b}}_0(t - t_0)^2 + \ldots \quad ,$$

$$\bar{g} = \bar{g}_0 + \dot{\bar{g}}_0(t - t_0) + \frac{1}{2}\ddot{\bar{g}}_0(t - t_0)^2 + \ldots \quad .$$

The quantities $[\dot{A}_0]$, $\dot{\bar{b}}_0$, $\dot{\bar{g}}_0$, $[\ddot{A}_0]$, $\ddot{\bar{b}}_0$, $\ddot{\bar{g}}_0$, etc., are determined by differentiating the expressions for $[A]$, $\bar{b}$, and $\bar{g}$ with respect to t and evaluating them at t_0. Then the Taylor Series expansions for $[A]$, $\bar{b}$, $\bar{g}\bar{\lambda}$, and $\bar{u}$ can be truncated at some power of $(t - t_0)$ and substituted into the expressions for $\dot{\bar{v}}$ and $\dot{\gamma}$ to give expressions for $\dot{\bar{v}}$ and $\dot{\gamma}$ whose right-hand sides are functions of only the independent variable time. Then numerical quadrature techniques can be used to find the solutions $\bar{v}(t)$, $\gamma(t)$, and $\bar{x}(t)$, where $t_0 \le t \le t_f$, much more rapidly than could be done by simultaneous numerical integration of the original system. The accuracy of this approach is limited only by the accuracy of the truncated Taylor Series expansions over the time interval of interest. To go further and obtain actual closed-form solutions requires a few more approximations. First $\overline{\lambda}(t)$ and $\overline{u}(t)$ are assumed to be represented adequately by using only the linear terms in the Taylor Series expansions. Then $[A]$ must be assumed to be constant. With these assumptions, the expressions for $\dot{\overline{v}}$ and $\dot{\gamma}$ become

$$\dot{\overline{v}} = \left(\frac{1}{q}\right)[A_0] \frac{\left[A_0\right]^T\left[\overline{u}_0 + \dot{\overline{u}}_0(t - t_0)\right]}{\sqrt{\overline{u}_0^T + \dot{\overline{u}}_0^T(t - t_0)\ \left[A_0\right]\left[A_0\right]^T\left[\overline{u}_0^T + \dot{\overline{u}}_0^T(t - t_0)\right]}}$$

$$+ \frac{1}{q}\left[\overline{b}_0 + \dot{\overline{b}}_0(t - t_0) + \frac{1}{2}\ddot{\overline{b}}_0(t - t_0)^2 + \ldots\right]$$

$$+ \left[\overline{g}_0 + \dot{\overline{g}}_0(t - t_0) + \frac{1}{2}\ddot{\overline{g}}_0(t - t_0^2) + \ldots\right] ,$$

$$\dot{\gamma} = \frac{1}{q^2}\left\{\sqrt{\left[\overline{u}_0^T + \dot{\overline{u}}_0^T\ (t - t_0)\right]\left[A_0\right]\left[A_0\right]^T\left[\overline{u}_0^T + \dot{\overline{u}}_0^T\ (t - t_0)\right]}\right.$$

$$\left. + \left[\overline{u}_0^T + \dot{\overline{u}}_0^T(t - t_0)\right]\left[\overline{b}_0 + \dot{\overline{b}}_0(t - t_0) + \frac{1}{2}\ddot{\overline{b}}_0(t - t_0)^2 + \ldots\right]\right\}$$

Since $\dot{q} = -\dot{m}$, then $q = q_0 - \dot{m}(t - t_0)$ so that $\frac{dt}{dq} = -\frac{1}{\dot{m}}$ and $(t - t_0) = -\frac{1}{\dot{m}}(q - q_0)$. With these substitutions, $\frac{d\overline{v}}{dq}$ and $\frac{d\gamma}{dq}$ can be written as follows:

$$\frac{d\overline{v}}{dq} = -\frac{1}{\dot{m}}\left\{\left(\frac{1}{q}\right)\frac{\left[A_0\right]\left[A_0\right]^T\left[\overline{u}_0 - \frac{\dot{\overline{u}}_0}{\dot{m}}(q - q_0)\right]}{\sqrt{\left[\overline{u}_0^T - \frac{\dot{\overline{u}}_0}{\dot{m}}(q - q_0)\right]\left[A_0\right]\left[A_0\right]^T\left[\overline{u}_0^T - \frac{\dot{\overline{u}}_0^T}{\dot{m}}(q - q_0)\right]}}\right.$$

$$+ \frac{1}{q}\left[\overline{b}_0 - \frac{\dot{\overline{b}}_0}{\dot{m}}(q - q_0) + \frac{1}{2}\frac{\ddot{\overline{b}}_0}{\dot{m}^2}(q - q_0)^2 - \ldots\right]$$

$$\left. + \overline{g}_0 - \frac{\dot{\overline{g}}_0}{\dot{m}}(q - q_0) + \frac{1}{2}\frac{\ddot{\overline{g}}_0}{\dot{m}^2}(q - q_0)^2 - \ldots\right]\right\} ,$$

$$\frac{d\gamma}{dq} = -\left(\frac{1}{\dot{m}}\right)\left(\frac{1}{q^2}\right)\left\{\sqrt{\left[\bar{u}_0^T - \frac{\dot{\bar{u}}_0^T}{\dot{m}}(q - q_0)\right]\left[A_0\right]\left[A_0\right]^T\left[\bar{u}_0^T - \frac{\dot{\bar{u}}_0^T}{\dot{m}}(q - q_0)\right]}\right.$$

$$+\left[\bar{u}_0^T - \frac{\dot{\bar{u}}_0^T}{\dot{m}}(q - q_0)\right]\left[\bar{b}_0 - \frac{\dot{\bar{b}}_0}{\dot{m}}(q - q_0)\right.$$

$$\left.\left. + \frac{1}{2}\left(\frac{\ddot{\bar{b}}_0}{\dot{m}^2}\right)(q - q_0)^2 - \ldots\right]\right\} .$$

Now these equations can be integrated fairly easily in closed form. To illustrate this closed-form solution, only the constant terms in the Taylor Series expansion for $\bar{b}$ and $\bar{g}$ will be used, although the integration for the inclusion of any number of terms in the series for $\bar{b}$ and $\bar{g}$ is easily added. First, note that $q = m$ and $q_0 = m_0$ so that the following quantities can be defined:

$$\bar{d}_1 = [A_0][A_0]^T\left[\bar{u}_0 + \dot{\bar{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right] = [A_0][A_0]^T\bar{u}_0 - m_0\bar{d}_2 ,$$

$$\bar{d}_2 = -[A_0][A_0]^T\dot{\bar{u}}_0\left(\frac{1}{\dot{m}}\right) ,$$

$$d_7 = \left[\bar{u}_0^T + \dot{\bar{u}}_0^T\left(\frac{m_0}{\dot{m}}\right)\right][A_0][A_0]^T\left[\bar{u}_0 + \dot{\bar{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]$$

$$= \left[\bar{u}_0^T + \dot{\bar{u}}_0^T\left(\frac{m_0}{\dot{m}}\right)\right]\bar{d}_1 = \bar{u}_0^T\,\bar{d}_1 - \left(\frac{m_0}{2}\right)d_8 ,$$

$$d_8 = -\left[\bar{u}_0^T + \dot{\bar{u}}_0^T\left(\frac{m_0}{\dot{m}}\right)\right][A_0][A_0]^T\dot{\bar{u}}_0\left(\frac{1}{\dot{m}}\right)$$

$$-\left(\frac{1}{\dot{m}}\right)\dot{\bar{u}}_0^T[A_0][A_0]^T\left[\bar{u}_0 + \dot{\bar{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]$$

$$= -\bar{d}_1^T\,\dot{\bar{u}}_0\left(\frac{1}{\dot{m}}\right) - \left(\frac{1}{\dot{m}}\right)\dot{\bar{u}}_0^T\,\bar{d}_1 = -\left(\frac{2}{\dot{m}}\right)\dot{\bar{u}}_0^T\,\bar{d}_1 ,$$

$$d_9 = \left(\frac{1}{\dot{m}^2}\right)\dot{\bar{u}}_0^T[A_0][A_0]^T\dot{\bar{u}}_0 = -\left(\frac{1}{\dot{m}}\right)\dot{\bar{u}}_0^T\,\bar{d}_2 ,$$

$$d_{10} = \left[\bar{u}_0^T + \dot{\bar{u}}_0^T \left(\frac{m_0}{\dot{m}} \right) \right] \bar{b}_0 \quad ,$$

$$d_{11} = -\left(\frac{1}{\dot{m}} \right) \dot{\bar{u}}_0^T \, \bar{b}_0 \quad .$$

With these definitions, the expressions for $\frac{d\bar{v}}{dq} = \frac{d\bar{v}}{dm}$ and $\frac{d\gamma}{dq} = \frac{d\gamma}{dm}$ become

$$\frac{d\bar{v}}{dm} = -\left(\frac{1}{\dot{m}} \right) \left\{ \left(\frac{1}{m} \right) \frac{\bar{d}_1 + \bar{d}_2 m}{\sqrt{d_7 + d_8 m + d_9 m^2}} + \left(\frac{1}{m} \right) \bar{b}_0 + \bar{g}_0 \right\} \quad ,$$

$$\frac{d\gamma}{dm} = -\left(\frac{1}{\dot{m}} \right) \left(\frac{1}{m^2} \right) \left\{ \sqrt{d_7 + d_8 m + d_9 m^2} + d_{10} + d_{11} m \right\} \quad .$$

The above expressions for $\frac{d\gamma}{dm}$ and $\frac{d\bar{v}}{dm}$ can now be used along with a table of integrals to obtain expressions for $\bar{v}(m)$ and $\gamma(m)$ at any $m > m_0$. If $\bar{v}(t)$ and $\gamma(t)$ are desired, then the relation $m = m_0 - \dot{m}(t - t_0)$ can be used to calculate the m corresponding to any $t > t_0$. To see how the expressions for $\bar{v}(m)$ and $\gamma(m)$ are obtained with the aid of a table of integrals, define

$$X = d_7 + d_8 m + d_9 m^2 \quad .$$

Then

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}} \left\{ \bar{d}_1 \left[\int_{m_0}^{m} \frac{dm}{m\sqrt{X}} \right] + \bar{d}_2 \left[\int_{m_0}^{m} \frac{dm}{\sqrt{X}} \right] + \bar{b}_0 \left[\int_{m_0}^{m} \frac{dm}{m} \right] + \bar{g}_0 \left[\int_{m_0}^{m} dm \right] \right\} ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left\{ \left[\int_{m_0}^{m} \frac{\sqrt{X}\, dm}{m^2} \right] + d_{10} \left[\int_{m_0}^{m} \frac{dm}{m^2} \right] + d_{11} \left[\int_{m_0}^{m} \frac{dm}{m} \right] \right\} .$$

In order to evaluate the integrals involving $\sqrt{X}$ in the above expressions, four cases must be considered. These cases are at the top of the next page.

1. $d_7 = 0$ but $d_9 > 0$

2. $d_7 > 0$ but $d_9 = 0$

3. $d_7 > 0$ and $d_9 > 0$

4. $d_7 = 0$ and $d_9 = 0$.

It should be noted that d_7 and d_9 can never be less than zero because

$$d_7 = \left| [A_0]^T \left[\overline{u}_0 + \dot{\overline{u}}_0 \left(\frac{m_0}{\dot{m}} \right) \right] \right|$$

and

$$d_9 = \frac{-1}{\dot{m}^2} \left| [A_0]^T \dot{\overline{u}}_0 \right| \quad .$$

Also, if either d_7 or d_9 is zero, then d_8 is zero because

$$d_8 = \frac{-2}{\dot{m}} \left\{ \dot{\overline{u}}_0 [A_0] \right\} \left\{ [A_0]^T \left[\overline{u}_0 + \dot{\overline{u}}_0 \left(\frac{m_0}{\dot{m}} \right) \right] \right\} \quad ;$$

and if d_7 or d_9 is zero, then $\overline{d}_1$ is zero or $\overline{d}_2$ is zero, respectively.

For the first case, since $d_7 = 0$ and $d_9 > 0$, the expressions for $\overline{v}(m)$ and $\gamma(m)$ become

$$\overline{v}(m) = \overline{v}_0 - \frac{1}{\dot{m}} \left\{ \frac{\overline{d}_2}{\sqrt{d_9}} \left[\int_{m_0}^{m} \frac{dm}{m} \right] + \overline{b}_0 \left[\int_{m_0}^{m} \frac{dm}{m} \right] + \overline{g}_0 \left[\int_{m_0}^{m} dm \right] \right\} \quad ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left\{ \sqrt{d_9} \left[\int_{m_0}^{m} \frac{dm}{m} \right] + d_{10} \left[\int_{m_0}^{m} \frac{dm}{m^2} \right] + d_{11} \left[\int_{m_0}^{m} \frac{dm}{m} \right] \right\} .$$

The integrals appearing on the right-hand sides of the above expressions are now easily evaluated to give:

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}} \left\{ \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \ell n \left(\frac{m}{m_0} \right) + \bar{g}_0 (m - m_0) \right\} ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left\{ \left(\sqrt{d_9} + d_{11} \right) \ell n \left(\frac{m}{m_0} \right) + d_{10} \left(\frac{1}{m_0} - \frac{1}{m} \right) \right\} .$$

To complete the closed-form solution for this first case, it should be noted that

$$\frac{d\bar{x}}{dm} = \frac{d\bar{x}}{dt} \left(\frac{dt}{dm} \right) = \left(-\frac{1}{\dot{m}} \right) \frac{d\bar{x}}{dt} = \left(-\frac{1}{\dot{m}} \right) \bar{v}$$

$$= -\frac{1}{\dot{m}} \left\{ \bar{v}_0 - \frac{1}{\dot{m}} \left[\left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \ell n \left(\frac{m}{m_0} \right) + \bar{g}_0 (m - m_0) \right] \right\} .$$

This expression for $\frac{dx}{dm}$ can be integrated to give

$$\bar{x}(m) = \bar{x}_0 - \left(\frac{1}{\dot{m}} \right) \bar{v}_0 \int_{m_0}^{m} dm$$

$$+ \left(\frac{1}{\dot{m}^2} \right) \left[\left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \int_{m_0}^{m} (\ell n\, m - \ell n\, m_0) dm \right.$$

$$\left. + \bar{g}_0 \int_{m_0}^{m} (m - m_0) dm \right] = \bar{x}_0 - \bar{v}_0 \left(\frac{m - m_0}{\dot{m}} \right)$$

$$+ \left(\frac{1}{\dot{m}^2} \right) \left\{ \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \left[m\, \ell n \left(\frac{m}{m_0} \right) - \left(m - m_0 \right) \right] \right.$$

$$\left. + \bar{g}_0 \left[\frac{(m - m_0)^2}{2} \right] \right\} .$$

Thus, the closed-form solutions that approximate the entire system of ordinary nonlinear differential equations when $d_7 = 0$ and $d_9 > 0$ can be calculated for any $t > t_0$ as follows:

$$m(t) = m_0 - \dot{m}(t - t_0) \quad ,$$

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0 (t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0 (t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 - \frac{1}{\dot{m}} \left[(\sqrt{d_9} + d_{11}) \, \ell n \left(\frac{m}{m_0} \right) + d_{10} \left(\frac{1}{m_0} - \frac{1}{m} \right) \right] \quad ,$$

$$\bar{v}(t) = \bar{v}_0 - \frac{1}{\dot{m}} \left[\left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \ell n \left(\frac{m}{m_0} \right) + \bar{g}_0 (m - m_0) \right] \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0 (t - t_0) + \bar{g}_0 \left[\frac{(t - t_0)^2}{2} \right]$$

$$+ \left(\frac{1}{\dot{m}^2} \right) \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 \right) \left[m \, \ell n \left(\frac{m}{m_0} \right) - (m - m_0) \right] \quad .$$

For case 2, since $d_7 > 0$ and $d_9 = 0$, the expressions for $\bar{v}(m)$ and $\gamma(m)$ become

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}} \left\{ \frac{\bar{d}_1}{\sqrt{d_7}} \left[\int_{m_0}^{m} \frac{dm}{m} \right] + \bar{b}_0 \left[\int_{m_0}^{m} \frac{dm}{m} \right] + \bar{g}_0 \left[\int_{m_0}^{m} dm \right] \right\} \quad ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left\{ \sqrt{d_7} \left[\int_{m_0}^{m} \frac{dm}{m^2} \right] + d_{10} \left[\int_{m_0}^{m} \frac{dm}{m^2} \right] + d_{11} \left[\int_{m_0}^{m} \frac{dm}{m} \right] \right\} \quad .$$

The integrals appearing on the right-hand sides of the above expressions are also easily evaluated, and the results follow:

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}} \left[\left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 \right) \ell n \left(\frac{m}{m_0} \right) + \bar{g}_0 (m - m_0) \right] \quad ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left[(\sqrt{d_7} + d_{10}) \left(\frac{1}{m_0} - \frac{1}{m} \right) + d_{11} \, \ell n \left(\frac{m}{m_0} \right) \right] \quad .$$

Then, to complete the closed-form solutions for this case, it should be noted that

$$\frac{d\bar{x}}{dm} = \frac{d\bar{x}}{dt}\left(\frac{dt}{dm}\right) = -\left(\frac{1}{\dot{m}}\right)\frac{d\bar{x}}{dt} = \left(-\frac{1}{\dot{m}}\right)\bar{v}$$

$$= -\frac{1}{\dot{m}}\left\{\bar{v}_0 - \frac{1}{\dot{m}}\left[\left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0\right)\ell n\left(\frac{m}{m_0}\right) + \bar{g}_0(m - m_0)\right]\right\} .$$

This expression for $\frac{d\bar{x}}{dm}$ can be integrated to give

$$\bar{x}(m) = \bar{x}_0 - \left(\frac{1}{\dot{m}}\right)\bar{v}_0\int_{m_0}^{m} dm$$

$$+ \left(\frac{1}{\dot{m}^2}\right)\left[\left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0\right)\int_{m_0}^{m}(\ell n\ m - \ell n\ m_0)dm\right.$$

$$\left. + \bar{g}_0\int_{m_0}^{m}(m - m_0)dm\right]$$

$$= \bar{x}_0 - \left(\frac{1}{\dot{m}}\right)\bar{v}_0(m - m_0)$$

$$+ \left(\frac{1}{\dot{m}^2}\right)\left\{\left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0\right)\left[m\ \ell n\left(\frac{m}{m_0}\right) - (m - m_0)\right]\right.$$

$$\left. + \bar{g}_0\left[\frac{(m - m_0)^2}{2}\right]\right\} .$$

Thus, the closed-form solutions that approximate the entire system of ordinary nonlinear differential equations when $d_7 > 0$ and $d_9 = 0$ can be calculated for any $t > t_0$ as follows:

$$m(t) = m_0 - \dot{m}(t - t_0) \quad ,$$

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0 (t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0 (t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 - \frac{1}{\dot{m}} \left[(\sqrt{d_7} + d_{10}) \left(\frac{1}{m_0} - \frac{1}{m} \right) + d_{11} \ln \left(\frac{m}{m_0} \right) \right] \quad ,$$

$$\bar{v}(t) = \bar{v}_0 - \frac{1}{\dot{m}} \left[\left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 \right) \ln \left(\frac{m}{m_0} \right) + \bar{g}_0 (m - m_0) \right] \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0 (t - t_0) + \bar{g}_0 \left[\frac{(t - t_0)^2}{2} \right]$$

$$+ \left(\frac{1}{\dot{m}^2} \right) \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 \right) \left[m \ln \left(\frac{m}{m_0} \right) - (m - m_0) \right] \quad .$$

For case 3, where $d_7 > 0$ and $d_9 > 0$, the expressions for $\bar{v}(m)$ and $\gamma(m)$ remain the same as the ones given on page 11; i.e.,

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}} \left\{ \bar{d}_1 \left[\int_{m_0}^{m} \frac{dm}{m\sqrt{X}} \right] + \bar{d}_2 \int_{m_0}^{m} \frac{dm}{\sqrt{X}} + \bar{b}_0 \left[\int_{m_0}^{m} \frac{dm}{m} \right] \right.$$

$$\left. + \bar{g}_0 \left[\int_{m_0}^{m} dm \right] \right\} \quad ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}} \left\{ \left[\int_{m_0}^{m} \frac{\sqrt{X}\,dm}{m^2} \right] + d_{10} \left[\int_{m_0}^{m} \frac{dm}{m^2} \right] + d_{11} \left[\int_{m_0}^{m} \frac{dm}{m} \right] \right\} \quad .$$

The integrals $\int_{m_0}^{m} \frac{dm}{m\sqrt{X}}$, $\int_{m_0}^{m} \frac{dm}{\sqrt{X}}$, and $\int_{m_0}^{m} \frac{\sqrt{X}\,dm}{m^2}$ appearing in the above expressions are evaluated with a table of integrals. This gives

$$\int_{m_0}^{m} \frac{dm}{m\sqrt{X}} = -\frac{1}{\sqrt{d_7}}\left[\ln\left(\frac{2\sqrt{d_7}\sqrt{X} + d_8 m + 2d_7}{2\sqrt{d_7}\sqrt{X_0} + d_8 m_0 + 2d_7}\right) - \ln\left(\frac{m}{m_0}\right)\right]$$

$$\int_{m_0}^{m} \frac{dm}{\sqrt{X}} = \left(\frac{1}{\sqrt{d_9}}\right)\ln\left(\frac{2\sqrt{d_9}\sqrt{X} + 2d_9 m + d_8}{2\sqrt{d_9}\sqrt{X_0} + 2d_9 m_0 + d_8}\right)$$

$$\int_{m_0}^{m} \frac{\sqrt{X}\,dm}{m^2} = -\left(\frac{\sqrt{X}}{m} - \frac{\sqrt{X_0}}{m_0}\right) + \frac{d_8}{2}\int_{m_0}^{m}\frac{dm}{m\sqrt{X}} + d_9\int_{m_0}^{m}\frac{dm}{\sqrt{X}} \quad .$$

To facilitate writing the above expressions, let

$$\Phi = 2\sqrt{d_7}\sqrt{X} + d_8 m + 2d_7$$

$$\Omega = 2\sqrt{d_9}\sqrt{X} + 2d_9 m + d_8 \quad ,$$

and Φ_0 and Ω_0 will denote Φ and Ω evaluated at the initial conditions. Then the expressions for $\bar{v}(m)$ and $\gamma(m)$ can be written as follows:

$$\bar{v}(m) = \bar{v}_0 - \frac{1}{\dot{m}}\left\{\bar{d}_1\left(-\frac{1}{\sqrt{d_7}}\right)\left[\ln\left(\frac{\Phi}{\Phi_0}\right) - \ln\left(\frac{m}{m_0}\right)\right]\right.$$

$$\left. + \bar{d}_2\left(\frac{1}{\sqrt{d_9}}\right)\ln\left(\frac{\Omega}{\Omega_0}\right) + \bar{b}_0 \ln\left(\frac{m}{m_0}\right) + \bar{g}_0(m - m_0)\right\} \quad ,$$

$$\gamma(m) = \gamma_0 - \frac{1}{\dot{m}}\left\{-\left(\frac{\sqrt{X}}{m} - \frac{\sqrt{X_0}}{m_0}\right) - \left(\frac{d_8}{2\sqrt{d_7}}\right)\left[\ln\left(\frac{\Phi}{\Phi_0}\right) - \ln\left(\frac{m}{m_0}\right)\right]\right.$$

$$\left. + \sqrt{d_9}\ln\left(\frac{\Omega}{\Omega_0}\right) + d_{10}\left(\frac{1}{m_0} - \frac{1}{m}\right) + d_{11}\ln\left(\frac{m}{m_0}\right)\right\} \quad .$$

To complete the closed-form solutions for case 3, it should be noted again that

$$\frac{d\bar{x}}{dm} = \frac{d\bar{x}}{dt}\left(\frac{dt}{dm}\right) = -\left(\frac{1}{\dot{m}}\right)\frac{d\bar{x}}{dt} = -\left(\frac{1}{\dot{m}}\right)\bar{v} \quad .$$

Thus,

$$\frac{d\bar{x}}{dm} = -\left(\frac{1}{m}\right)\bar{v}_0 + \left(\frac{1}{m^2}\right)\left\{ \bar{d}_1\left(-\frac{1}{\sqrt{d_7}}\right)\left[\ell n\left(\frac{\Phi}{\Phi_0}\right) - \ell n\left(\frac{m}{m_0}\right)\right]\right.$$

$$\left. + \bar{d}_2\left(\frac{1}{\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right) + \bar{b}_0\,\ell n\left(\frac{m}{m_0}\right) + \bar{g}_0(m - m_0)\right\} .$$

This expression for $\frac{d\bar{x}}{dm}$ can be integrated to give

$$\bar{x}(m) = \bar{x}_0 - \left(\frac{1}{m}\right)\bar{v}_0 \int_{m_0}^{m} dm + \frac{1}{m^2}\left\{ -\left(\frac{d_1}{\sqrt{d_7}}\right)\left[\int_{m_0}^{m} (\ell n\ \Phi - \ell n\ \Phi_0)dm\right.\right.$$

$$\left. - \int_{m_0}^{m} (\ell n\ m - \ell n\ m_0)dm\right] + \frac{\bar{d}_2}{\sqrt{d_9}}\int_{m_0}^{m} (\ell n\ \Omega - \ell n\ \Omega_0)dm$$

$$\left. + \bar{b}_0 \int_{m_0}^{m} (\ell n\ m - \ell n\ m_0)dm + \bar{g}_0 \int_{m_0}^{m} (m - m_0)dm\right\} .$$

The only difficult integrals to evaluate in the above expression for $\bar{x}(m)$ are the integrals involving $\ell n\ \Phi$ and $\ell n\ \Omega$. The following steps will explain the integrals $\int_{m_0}^{m} \ell n\ \Phi\ dm$ and $\int_{m_0}^{m} \ell n\ \Omega\ dm$. Both integrals will be evaluated using the integration-by-parts formula; i. e.,

$$\int u\ dv = uv - \int v\ du .$$

For $\int_{m_0}^{m} \ell n\ \Phi\ dm$, let $u = \ell n\ \Phi$ and $dv = dm$. Then,

$$\int_{m_0}^{m} \ell n\ \Phi\ dm = m\ \ell n\ \Phi - m_0\ \ell n\ \Phi_0 - \int_{m_0}^{m} m\left[\frac{d(\ell n\ \Phi)}{dm}\right]dm .$$

Then note that

$$\frac{d(\ell n\ \Phi)}{dm} = \frac{1}{\Phi}\left[\sqrt{d_7}\left(\frac{1}{\sqrt{X}}\right)(d_8 + 2d_9 m) + d_8\right]$$

$$= \left(\frac{1}{\Phi}\right)\left(\frac{1}{\sqrt{d_7}}\right)\left[\frac{\sqrt{d_7}\ d_8\ \sqrt{X} + 2d_7 d_9 m + d_7 d_8}{\sqrt{X}}\right]$$

$$= \left(\frac{1}{\Phi}\right)\left(\frac{d_8}{2\sqrt{d_7}}\right)\left[\frac{2\sqrt{d_7}\ \sqrt{X} + 2d_7 + d_8\ m + \left(\frac{4d_7d_9}{d_8} - d_8\right)m}{\sqrt{X}}\right]$$

$$= \left(\frac{1}{\Phi}\right)\left(\frac{d_8}{2\sqrt{d_7}}\right)\left[\frac{\Phi + \left(\frac{4d_7d_9 - d_8^2}{d_8}\right)m}{\sqrt{X}}\right]$$

$$= \frac{d_8}{2\sqrt{d_7}}\left[\frac{1}{\sqrt{X}} + \frac{(4d_7d_9 - d_8^2)m}{d_8\sqrt{X}\ \Phi}\right]$$

$$= \frac{d_8}{2\sqrt{d_7}}\left[\frac{1}{\sqrt{X}} + \frac{(4d_7d_9 - d_8^2)(2\sqrt{d_7}\ \sqrt{X} - d_8 m - 2d_7)m}{d_8\sqrt{X}\ \Phi(2\sqrt{d_7}\ \sqrt{X} - d_8 m - 2d_7)}\right]$$

$$= \frac{d_8}{2\sqrt{d_7}}\left[\frac{1}{\sqrt{X}} + \frac{(2\sqrt{d_7}\ \sqrt{X} - d_8 m - 2d_7)}{md_8\sqrt{X}}\right]$$

$$= \frac{d_8}{2\sqrt{d_7}}\left[\frac{1}{\sqrt{X}} + \left(\frac{2\sqrt{d_7}}{md_8}\right) - \frac{}{\sqrt{X}} - \frac{2d_7}{md_8\sqrt{X}}\right]$$

$$= \frac{1}{m} - \frac{\sqrt{d_7}}{m\sqrt{X}} \quad .$$

When this result is substituted into the expression for $\int_{m_0}^{m} \ell n\ \Phi\ dm$, the result is

$$\int_{m_0}^{m} \ln \Phi \, dm = m \ln \Phi - m_0 \ln \Phi_0 - \int_{m_0}^{m} m\left[\frac{1}{m} - \frac{\sqrt{d_7}}{m\sqrt{X}}\right]dm$$

$$= m \ln \Phi - m_0 \ln \Phi_0 - \int_{m_0}^{m} dm + \sqrt{d_7} \cdot \int_{m_0}^{m} \frac{dm}{\sqrt{X}}$$

$$= m \ln \Phi - m_0 \ln \Phi_0 - (m - m_0) + \left(\frac{\sqrt{d_7}}{\sqrt{d_9}}\right) \ln \left(\frac{\Omega}{\Omega_0}\right) \quad .$$

A similar procedure is used to evaluate $\int_{m_0}^{m} \ln \Omega \, dm$; i. e.,

$$\int_{m_0}^{m} \ln \Omega \, dm = m \ln \Omega - m_0 \ln \Omega_0 - \int_{m_0}^{m} m\left[\frac{d(\ln \Omega)}{dm}\right] dm \quad .$$

Then

$$\frac{d \ln \Omega}{dm} = \frac{1}{\Omega}\left[\sqrt{d_9}\left(\frac{1}{\sqrt{X}}\right)(d_8 + 2d_9 m) + 2d_9\right]$$

$$= \left(\frac{1}{\Omega}\right)\sqrt{d_9}\left[\frac{2\sqrt{d_9}\sqrt{X} + 2d_9 m + d_8}{\sqrt{X}}\right]$$

$$= \left(\frac{1}{\Omega}\right)\sqrt{d_9}\left[\frac{\Omega}{\sqrt{X}}\right]$$

$$= \frac{\sqrt{d_9}}{\sqrt{X}} \quad .$$

Now, as before, this result is substituted into the expression for $\int_{m_0}^{m} \ln \Omega \, dm$ to give

$$\int_{m_0}^{m} \ln \Omega \, dm = m \ln \Omega - m_0 \ln \Omega_0 - \int_{m_0}^{m} m\left[\frac{\sqrt{d_9}}{\sqrt{X}}\right]dm$$

$$= m \ln \Omega - m_0 \ln \Omega_0 - \sqrt{d_9} \int_{m_0}^{m} \frac{m\,dm}{\sqrt{X}} \quad .$$

For the expression $\int_{m_0}^{m} \frac{m dm}{\sqrt{X}}$, a table of integrals is again used to give

$$\int_{m_0}^{m} \frac{m dm}{\sqrt{X}} = \frac{\sqrt{X}}{d_9} - \frac{\sqrt{X_0}}{d_9} - \left(\frac{d_8}{2d_9}\right) \int_{m_0}^{m} \frac{dm}{\sqrt{X}}$$

$$= \frac{1}{d_9}\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ln \left(\frac{\Omega}{\Omega_0}\right)\right] .$$

Thus,

$$\int_{m_0}^{m} \ln \Omega \, dm = m \ln \Omega - m_0 \ln \Omega_0 - \left(\frac{1}{\sqrt{d_9}}\right)\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ln \left(\frac{\Omega}{\Omega_0}\right)\right] .$$

With these results, the evaluation of the integrals appearing in the expression for $\bar{x}(m)$ can now be completed; i. e., for case 3, where $d_7 > 0$ and $d_9 > 0$,

$$\bar{x}(m) = \bar{x}_0 - \left(\frac{1}{\dot{m}}\right)\bar{v}_0(m - m_0) + \bar{g}_0 \left[\frac{(m - m_0)^2}{\dot{m}^2}\right]$$

$$+ \left(\frac{1}{\dot{m}^2}\right) \bar{b}_0 \left[m \ln \left(\frac{m}{m_0}\right) - (m - m_0)\right]$$

$$+ \left(\frac{1}{\dot{m}^2}\right)\left(\frac{-\bar{d}_1}{\sqrt{d_7}}\right) \left\{ m \left[\ln \left(\frac{\Phi}{\Phi_0}\right) - \ln \left(\frac{m}{m_0}\right)\right]\right.$$

$$\left. + \left(\frac{\sqrt{d_7}}{\sqrt{d_9}}\right) \ln \left(\frac{\Omega}{\Omega_0}\right)\right\} + \left(\frac{1}{\dot{m}^2}\right)\left(\frac{\bar{d}_2}{\sqrt{d_9}}\right) \left\{ m \ln \left(\frac{\Omega}{\Omega_0}\right)\right.$$

$$\left. - \frac{1}{\sqrt{d_9}} \left[\left(\sqrt{X} - \sqrt{X_0}\right) - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ln \left(\frac{\Omega}{\Omega_0}\right)\right]\right\} .$$

Now, again, the closed-form solutions that approximate the entire system of ordinary nonlinear differential equations when $d_7 >_0$ and $d_9 >_0$ can be calculated for any $t > t_0$ as follows:

$$m(t) = m_0 - \dot{m}(t - t_0) \quad ,$$

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0 (t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0 (t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 - \frac{1}{\dot{m}} \left\{ -\left(\frac{\sqrt{X}}{m} - \frac{\sqrt{X_0}}{m_0}\right) - \left(\frac{d_8}{2\sqrt{d_7}}\right) \left[\ell n \left(\frac{\Phi}{\Phi_0}\right) - \ell n \left(\frac{m}{m_0}\right)\right] \right.$$

$$\left. + \sqrt{d_9}\; \ell n \left(\frac{\Omega}{\Omega_0}\right) + d_{10}\left(\frac{1}{m_0} - \frac{1}{m}\right) + d_{11}\; \ell n \left(\frac{m}{m_0}\right) \right\} \quad ,$$

$$\bar{v}(t) = \bar{v}_0 + \bar{g}_0 (t - t_0) + \frac{1}{\dot{m}} \left\{ -\left(\frac{\bar{d}_1}{\sqrt{d_7}}\right) \left[\ell n \left(\frac{\Phi}{\Phi_0}\right) - \ell n \left(\frac{m}{m_0}\right)\right] \right.$$

$$\left. + \left(\frac{\bar{d}_2}{\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right) + \bar{b}_0\; \ell n \left(\frac{m}{m_0}\right) \right\} \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0 (t - t_0) + \bar{g}_0 \left[\frac{(t - t_0)^2}{2}\right] + \left(\frac{1}{\dot{m}^2}\right) \bar{b}_0 \left[m\; \ell n \left(\frac{m}{m_0}\right)\right.$$

$$\left. - (m - m_0)\right] - \left(\frac{\bar{d}_1}{\dot{m}^2}\right) \left\{ \left(\frac{m}{\sqrt{d_7}}\right) \left[\ell n \left[\left(\frac{\Phi}{\Phi_0}\right) - \ell n \left(\frac{m}{m_0}\right)\right]\right.\right.$$

$$\left. + \left(\frac{1}{\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right) \right\} + \left(\frac{\bar{d}_2}{\dot{m}^2}\right) \left\{ \left(\frac{m}{\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right) \right.$$

$$\left. - \left(\frac{1}{d_9}\right) \left[\left(\sqrt{X} - \sqrt{X_0}\right) - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right)\right]\right\} \quad .$$

For case 4, when $d_7 = 0$ and $d_9 = 0$, the aerodynamic forces and thrust forces are zero so that motion is two-body motion for which closed-form solutions are developed in Reference 2.

The preceding work completes the closed-form solutions for the ascent case which can be used to approximate the optimal atmospheric motion of a rocket-powered space vehicle. Some of the results will be used in the next section, which is concerned with the closed-form solutions for reentry motion.

Reentry Case

For the reentry case, F is assumed to be zero and m is assumed to be a constant equal to m_0. Also, for the reentry case, q, which is a linear combination of the instantaneous stagnation point heating rate and deceleration, is defined by the following differential equation:

$$\dot{q} = k_1\left(\frac{e}{\sqrt{\sigma}}\right)\rho^{1/2}|\bar{v}_r|^3 + k_2\left(\frac{\rho A_r}{2m}\right)^2\left[|\bar{v}_r|^4\left(c_A^2 + 4\eta c_A c_{L_\alpha}^2 + 2c_{L_\alpha}^2\right)\right.$$
$$\left. - 2\left(\bar{v}_r \cdot \frac{\bar{p}}{|\bar{p}|}\right)|\bar{v}_r|^3\left(c_{L_\alpha}^2 + 2\eta c_A c_{L_\alpha}^2\right)\right] ,$$

and the initial conditions $q_0 = 0$. To simplify the writing of the equation for $\dot{q}$, define

$$\bar{h}^T = -2k_2\left(\frac{\rho A_R}{2m}\right)^2 |\bar{v}_r|^3\left(c_{L_\alpha}^2 + 2\eta c_A c_{L_\alpha}^2\right)\bar{v}_r^T$$

and

$$h = k_1\left(\frac{e}{\sqrt{\sigma}}\right)\rho^{1/2}|\bar{v}_r|^3 + k_2\left(\frac{\rho A_r}{2m}\right)^2\left[|\bar{v}_r|^4\left(c_A^2 + 4\eta c_A c_{L_\alpha}^2 + 2c_{L_\alpha}^2\right)\right]$$

Then

$$\dot{q} = h + \bar{h}^T \frac{\bar{p}}{|\bar{p}|} .$$

Since F = 0 and m is a constant, the equations of motion for the reentry case become

$$\dot{\bar{x}} = \bar{v} \quad ,$$

$$\dot{\bar{v}} = [A] \frac{\bar{p}}{|\bar{p}|} + \bar{b} + \bar{g} \quad ,$$

where

$$[A] = \frac{1}{m} \left\{ \left[\left(\frac{\rho A_r}{2} \right) |\bar{v}_r|^2 c_{L_\alpha} \right] [I] - \left(\frac{\rho A_r}{2} \right) c_{L_\alpha} \left(1 - 2\eta c_{L_\alpha} \right) \left(\bar{v}_r \bar{v}_r^T \right) \right\}$$

$$\bar{b} = -\left(\frac{\rho A_r}{2m} \right) |\bar{v}_r| \left(c_A + 2\eta c_{L_\alpha}^2 \right) \bar{v}_r$$

$$\bar{g} = -\frac{GM\bar{x}}{R^3} \quad .$$

Now, as in the ascent case, the multipliers $\bar{\lambda}(t)$, $\bar{u}(t)$, and $\gamma(t)$ (where $t_0 \leq t \leq t_f$) are introduced to allow the following expression for H to be written:

$$H = \bar{\lambda}^T \bar{u} + \bar{u}^T \left\{ [A] \frac{\bar{p}}{|p|} + \bar{b} + \bar{g} \right\} + \gamma \left(h + \bar{h}^T \frac{\bar{p}}{|\bar{p}|} \right) \quad .$$

As in the ascent case,

$$\dot{\bar{\lambda}}^T = -\frac{\partial H}{\partial \bar{x}} \, , \quad \dot{\bar{u}}^T = -\frac{\partial H}{\partial \bar{v}} \, , \quad \text{and } \dot{\gamma} = -\frac{\partial H}{\partial q} \quad .$$

Now the quantity J, which is to be maximized or minimized by the choice of $\bar{p}(t)$ (where $t_0 \leq t \leq t_f$), is defined. As in Reference 1, the choice $J = q_f$ is made, and it is desired that J be minimized by the choice of a control function $\bar{p}(t)$. Thus, $\bar{p}(t)$ must maximize H at every t such that $t_0 \leq t \leq t_f$. An examination of the above expression for H will indicate that

$$\bar{p} = [A]^T \bar{u} + \gamma \bar{h}$$

is sufficient to maximize H at every t, where $t_0 \leq t \leq t_f$.

When the above result for $\bar{p}$ is substituted into both the equations of motions and the multiplier differential equations, the entire system of simultaneous ordinary nonlinear differential equations for the reentry problem is obtained; i. e.,

$$\dot{\bar{x}} = \bar{v} \quad ,$$

$$\dot{\bar{v}} = [A] \left\{ \frac{[A]^T\bar{u} + \gamma\bar{h}}{\sqrt{\left(\bar{u}^T[A] + \gamma\bar{h}^T\right)\left([A]^T\bar{u} + \gamma\bar{h}\right)}} \right\} + \bar{b} + \bar{g} \quad ,$$

$$\dot{q} = h + \bar{h}^T \left\{ \frac{[A]^T\bar{u} + \gamma\,\bar{h}}{\sqrt{\left(\bar{u}^T[A] + \gamma\,\bar{h}^T\right)\left([A]^T\,\bar{u} + \gamma\,\bar{h}\right)}} \right\} \quad ,$$

$$\dot{\bar{\lambda}}^T = -\left\{ \frac{\bar{u}^T[A] + \gamma\,\bar{h}^T}{\sqrt{\left(\bar{u}^T[A] + \gamma\,\bar{h}^T\right)\left([A]^T\bar{u} + \gamma\,\bar{h}\right)}} \right\} \left\{ \frac{\partial([A]^T\bar{u})}{\partial\bar{x}} + \gamma\,\frac{\partial\bar{h}}{\partial\bar{x}} \right\}$$

$$-\bar{u}^T\left(\frac{\partial\bar{b}}{\partial\bar{x}} + \frac{\partial\bar{g}}{\partial\bar{x}}\right) - \gamma\left(\frac{\partial h}{\partial\bar{x}}\right) \quad ,$$

$$\dot{\bar{u}}^T = -\left\{ \frac{\bar{u}^T[A] + \gamma\,\bar{h}^T}{\sqrt{\left(\bar{u}^T[A] + \gamma\,\bar{h}^T\right)\left([A]^T\bar{u} + \gamma\,\bar{h}\right)}} \right\} \left\{ \frac{\partial([A]^T\bar{u})}{\partial\bar{v}} + \gamma\,\frac{\partial\bar{h}}{\partial\bar{v}} \right\}$$

$$-\bar{u}^T\left(\frac{\partial\bar{b}}{\partial\bar{v}}\right) - \gamma\left(\frac{\partial h}{\partial\bar{v}}\right) \quad ,$$

$$\dot{\gamma} = 0 \quad .$$

This system of ordinary nonlinear differential equations can be solved simultaneously by numerical integration if the initial conditions $\bar{x}_0$, $\bar{v}_0$, q_0, $\bar{\lambda}_0$, $\bar{u}_0$, and γ_0 are given; or, as in the ascent case, the quantities, $\bar{\lambda}$, $\bar{u}$, h, $\bar{h}$, [A], $\bar{b}$, and $\bar{g}$ can be expanded in truncated Taylor Series about the initial conditions which will allow numerical integration by quadratures to be used.

To continue and obtain closed-form solutions for the reentry system of differential equations, the following assumptions are made:

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0(t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0(t - t_0) \quad ,$$

$$\bar{h}(t) = \bar{h}_0 + \dot{\bar{h}}_0(t - t_0) \quad ,$$

$$[A(t)] = [A_0] \quad ,$$

$$\bar{b}(t) = \bar{b}_0 + \dot{\bar{b}}_0(t - t_0) + \frac{\ddot{\bar{b}}_0}{2}(t - t_0)^2 + \ldots \quad ,$$

$$\bar{g}(t) = \bar{g}_0 + \dot{\bar{g}}_0(t - t_0) + \frac{\ddot{\bar{g}}_0}{2}(t - t_0)^2 + \ldots \quad ,$$

$$h = h_0 + \dot{h}_0(t - t_0) + \frac{\ddot{h}_0}{2}(t - t_0)^2 + \ldots \quad .$$

As in the ascent case, any number of terms can be used in the Taylor Series expansions for $\bar{b}(t)$, $\bar{g}(t)$, and $h(t)$; but only the constant terms will be used in illustrating this approach. With the preceding assumptions, the equations for $\dot{\bar{v}}$ and $\dot{q}$ become:

$$\dot{\bar{v}} = [A_0]\left\{\frac{[A_0]^T[\bar{u}_0 + \dot{\bar{u}}_0(t - t_0)] + \gamma[\bar{h}_0 + \dot{\bar{h}}_0(t - t_0)]}{\sqrt{\left(\left[\bar{u}_0^T + \dot{\bar{u}}_0(t - t_0)\right][A_0] + \gamma\left[\bar{h}_0^T + \dot{\bar{h}}_0^T(t - t_0)\right]\right)\left([A_0]^T[\bar{u}_0 + \dot{\bar{u}}_0(t - t_0)] + \gamma[\bar{h}_0 + \dot{\bar{h}}_0(t - t_0)]\right)}}\right\} + \bar{b}_0 + \bar{g}_0$$

$$\dot{q} = h_0 + \left[\bar{h}_0^T + \dot{\bar{h}}_0^T(t - t_0)\right] \{\text{same term as above in the } \dot{\bar{v}} \text{ equation}\} \quad .$$

To simplify the expressions for $\dot{\bar{v}}$ and $\dot{q}$, define

$$\bar{d}_1 = [A_0]\left\{[A_0]^T[\bar{u}_0 - \dot{\bar{u}}_0 t_0] + \gamma[\bar{h}_0 - \dot{\bar{h}}_0 t_0]\right\} \quad ,$$

$$\bar{d}_2 = [A_0]\left\{[A_0]^T\dot{\bar{u}}_0 + \gamma\dot{\bar{h}}_0\right\} \quad ,$$

$$d_7 = \left\{[\bar{u}_0^T - \dot{\bar{u}}_0^T t_0][A_0] + \gamma[\bar{h}_0^T - \dot{\bar{h}}_0^T t_0]\right\}\left\{[A_0]^T[\bar{u}_0 - \dot{\bar{u}}_0 t_0] + \gamma[\bar{h}_0 - \dot{\bar{h}}_0 t_0]\right\} \quad ,$$

$$d_8 = 2\left\{[\bar{u}_0^T - \dot{\bar{u}}_0^T t_0][A_0] + \gamma[\bar{h}_0^T - \dot{\bar{h}}_0^T t_0]\right\}\left\{[A_0]^T\dot{\bar{u}}_0 + \gamma\dot{\bar{h}}_0\right\} \quad ,$$

$$d_9 = \left\{\dot{\bar{u}}_0^T[A_0] + \gamma\dot{\bar{h}}_0^T\right\}\left\{[A_0]^T\dot{\bar{u}}_0 + \gamma\dot{\bar{h}}_0\right\} \quad ,$$

$$d_{10} = [\bar{h}_0^T - \dot{\bar{h}}_0^T t_0]\left\{[A_0]^T[\bar{u}_0 - \dot{\bar{u}}_0 t_0] + \gamma[\bar{h}_0 - \dot{\bar{h}}_0 t_0]\right\} \quad ,$$

$$d_{11} = [\bar{h}_0^T - \dot{\bar{h}}_0^T t_0]\left\{[A_0]^T\dot{\bar{u}}_0 + \gamma\dot{\bar{h}}_0\right\} + \dot{\bar{h}}_0^T\left\{[A_0]^T[\bar{u}_0 - \dot{\bar{u}}_0 t_0] + \gamma[\bar{h}_0 - \dot{\bar{h}}_0 t_0]\right\} \quad ,$$

$$d_{12} = \dot{\bar{h}}_0^T\left\{[A_0]^T\dot{\bar{u}}_0 + \gamma\dot{\bar{h}}_0\right\} \quad .$$

Then the equations for $\dot{\bar{v}}$ and $\dot{q}$ become

$$\dot{\bar{v}} = \frac{\bar{d}_1 + \bar{d}_2 t}{\sqrt{d_7 + d_8 t + d_9 t^2}} + \bar{b}_0 + \bar{g}_0$$

$$\dot{q} = h_0 + \frac{d_{10} + d_{11} t + d_{12} t^2}{\sqrt{d_7 + d_8 t + d_9 t^2}} \quad .$$

These equations can be integrated in closed-form with the aid of a table of integrals, but four cases must again be considered as in the ascent case. These cases are

1. $d_7 = 0$ but $d_9 > 0$

2. $d_7 > 0$ but $d_9 = 0$

3. $d_7 > 0$ and $d_9 > 0$

4. $d_7 = 0$ and $d_9 = 0$.

Also, as in the ascent case, $d_7 \geq 0$ and $d_9 \geq 0$; $d_7 = 0$ implies that both $\bar{d}_1 = 0$ and $d_8 = 0$; and $d_9 = 0$ implies that both $\bar{d}_2 = 0$ and $d_8 = 0$.

Thus, for the first case,

$$\dot{\bar{v}} = \frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + \bar{g}_0$$

$$\dot{q} = h_0 + \frac{d_{10} + d_{11}t + d_{12}t^2}{\sqrt{d_9 t}} \quad .$$

These differential equations are easily integrated to give

$$\bar{v}(t) = \bar{v}_0 + \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + \bar{g}_0\right)(t - t_0)$$

$$q(t) = q_0 + \left(h_0 + \frac{d_{11}}{\sqrt{d_9}}\right)(t - t_0) + \frac{d_{10}}{\sqrt{d_9}} \ell n \left(\frac{t}{t_0}\right) + \frac{d_{12}}{\sqrt{d_9}}\left(\frac{t^2 - t_0^2}{2}\right) \quad .$$

Then, since $\dot{\bar{x}} = \bar{v}$,

$$\dot{\bar{x}} = \bar{v}_0 + \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + g_0\right)(t - t_0)$$

which is easily integrated to yield

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + \bar{g}_0\right)\frac{(t - t_0)^2}{2} \quad .$$

Thus, for case 1, the entire closed-form solution which approximates the original system of ordinary nonlinear differential equations is given by

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0(t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0(t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 \quad ,$$

$$\bar{v}(t) = \bar{v}_0 + \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + \bar{g}_0\right)(t - t_0) \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + \left(\frac{\bar{d}_2}{\sqrt{d_9}} + \bar{b}_0 + \bar{g}_0\right)\frac{(t - t_0)^2}{2} \quad ,$$

$$q(t) = q_0 + \left(h_0 + \frac{d_{11}}{\sqrt{d_9}}\right)(t - t_0) + \frac{d_{10}}{\sqrt{d_9}} \ell n \left(\frac{t}{t_0}\right) + \frac{d_{12}}{\sqrt{d_9}}\left(\frac{t^2 - t_0^2}{2}\right) \quad .$$

For case 2, since $d_7 > 0$ and $d_9 = 0$,

$$\dot{\bar{v}} = \frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0$$

$$\dot{q} = h_0 + \frac{d_{10} + d_{11}t + d_{12}t^2}{\sqrt{d_7}} \quad .$$

These differential equations are also easily integrated to give

$$\bar{v}(t) = \bar{v}_0 + \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0\right)(t - t_0)$$

$$q(t) = q_0 + \left(h_0 + \frac{d_{10}}{\sqrt{d_7}}\right)(t - t_0) + \frac{d_{11}}{\sqrt{d_7}}\left(\frac{t^2 - t_0^2}{2}\right) + \frac{d_{12}}{\sqrt{d_7}}\left(\frac{t^3 - t_0^3}{3}\right) \quad .$$

Then, since $\dot{\bar{x}} = \bar{v}$,

$$\dot{\bar{x}} = \bar{v}_0 + \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0\right)(t - t_0)$$

which is easily integrated to give

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0\right)\frac{(t - t_0)^2}{2} \quad .$$

Then, for case 2, the entire closed-form solution which approximates the original system of ordinary nonlinear differential equations is given by

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0(t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0(t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 \quad ,$$

$$\bar{v}(t) = \bar{v}_0 + \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0\right)(t - t_0) \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + \left(\frac{\bar{d}_1}{\sqrt{d_7}} + \bar{b}_0 + \bar{g}_0\right)\frac{(t - t_0)^2}{2} ,$$

$$q(t) = q_0 + \left(h_0 + \frac{d_{10}}{\sqrt{d_7}}\right)(t - t_0) + \frac{d_{11}}{\sqrt{d_7}}\left(\frac{t^2 - t_0^2}{2}\right) + \frac{d_{12}}{\sqrt{d_7}}\left(\frac{t^3 - t_0^3}{3}\right) .$$

For case 3, since $d_7 > 0$ and $d_9 > 0$, the original form of the differential equations for $\dot{\bar{v}}$ and $\dot{q}$ are unchanged; i. e.,

$$\dot{\bar{v}} = \frac{\bar{d}_1 + \bar{d}_2 t}{\sqrt{d_7 + d_8 t + d_9 t^2}} + \bar{b}_0 + \bar{g}_0$$

$$\dot{q} = h_0 + \frac{d_{10} + d_{11}t + d_{12}t^2}{\sqrt{d_7 + d_8 t + d_9 t^2}} .$$

As was done in the ascent case, define $X = d_7 + d_8 t + d_9 t^2$.

Then the differential equations for $\dot{\bar{v}}$ and $\dot{q}$ can be integrated at least symbolically to give

$$\bar{v}(t) = \bar{v}_0 + \bar{d}_1 \int_{t_0}^{t} \frac{dt}{\sqrt{X}} + \bar{d}_2 \int_{t_0}^{t} \frac{tdt}{\sqrt{X}} + (\bar{b}_0 + \bar{g}_0)(t - t_0)$$

$$q(t) = q_0 + h_0(t - t_0) + d_{10} \int_{t_0}^{t} \frac{dt}{\sqrt{X}} + d_{11} \int_{t_0}^{t} \frac{tdt}{\sqrt{X}} + d_{12} \int_{t_0}^{t} \frac{t^2dt}{\sqrt{X}} .$$

With a table of integrals, the expressions involving $\sqrt{X}$ can be integrated to yield

$$\int_{t_0}^{t} \frac{dt}{\sqrt{X}} = \frac{1}{\sqrt{d_9}} \ell n \left(\frac{2\sqrt{d_9}\sqrt{X} + 2d_9 t + d_8}{2\sqrt{d_9}\sqrt{X_0} + 2d_9 t_0 + d_8}\right) ,$$

$$\int_{t_0}^{t} \frac{tdt}{\sqrt{X}} = \frac{1}{d_9}\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ell n \left(\frac{2\sqrt{d_9}\sqrt{X} + 2d_9 t + d_8}{2\sqrt{d_9}\sqrt{X_0} + 2d_9 t_0 + d_8}\right)\right] ,$$

$$\int_{t_0}^{t} \frac{t^2 dt}{\sqrt{X}} = \frac{1}{2d_9}\left(t - \frac{3d_8}{2d_9}\right)\sqrt{X} - \frac{1}{2d_9}\left(t_0 - \frac{3d_8}{2d_9}\right)\sqrt{X_0}$$

$$+ \left(\frac{3d_8^2 - 4d_7d_9}{8d_9^2}\right) \int_{t_0}^{t} \frac{dt}{\sqrt{X}} \quad .$$

Let $\Omega = 2\sqrt{d_9}\sqrt{X} + 2d_9t + d_8$ and Ω_0 be the value of Ω at t_0. Then the expressions for $\overline{v}(t)$ and $q(t)$ can be written as

$$\overline{v}(t) = \overline{v}_0 + \frac{\overline{d}_1}{\sqrt{d_9}} \ell n\left(\frac{\Omega}{\Omega_0}\right) + \frac{\overline{d}_2}{d_9}\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right]$$

$$+ (\overline{b}_0 + \overline{g}_0)(t - t_0)$$

$$q(t) = q_0 + h_0(t - t_0) + \frac{d_{10}}{\sqrt{d_9}} \ell n\left(\frac{\Omega}{\Omega_0}\right) + \left(\frac{d_{11}}{d_9}\right)\left[\sqrt{X} - \sqrt{X_0}\right.$$

$$\left. - \left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right] + \frac{d_{12}}{2d_9}\left[(t\sqrt{X} - t_0\sqrt{X_0})\right.$$

$$\left. - \left(\frac{3d_8}{2d_9}\right)(\sqrt{X} - \sqrt{X_0}) + \left(\frac{3d_8^2 - 4d_7d_9}{4d_9\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right] \quad .$$

Since $\dot{\overline{x}} = \overline{v}$,

$$\overline{x}(t) = \overline{x}_0 + \overline{v}_0(t - t_0) + \frac{\overline{d}_1}{\sqrt{d_9}} \int_{t_0}^{t} (\ell n\, \Omega - \ell n\, \Omega_0)dt + (\overline{b}_0 + \overline{g}_0)\frac{(t - t_0)^2}{2}$$

$$+ \frac{\overline{d}_2}{d_9}\left\{\int_{t_0}^{t} (\sqrt{X} - \sqrt{X_0})dt - \frac{d_8}{2\sqrt{d_9}} \int_{t_0}^{t} (\ell n\, \Omega - \ell n\, \Omega_0)dt\right\} \quad .$$

The integral $\int_{m_0}^{m} \ell n\, \Omega\, dm$ was evaluated for the ascent case; and when m and m_0 are replaced by t and t_0, the integral $\int_{t_0}^{t} \ell n\, \Omega\, dt$ can be evaluated. That is,

$$\int_{t_0}^{t} \ell n\ \Omega\ dt = t\ \ell n\ \Omega - t_0\ \ell n\ \Omega_0 - \left(\frac{1}{\sqrt{d_9}}\right)\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right)\right] \quad .$$

Also, using a table of integrals,

$$\int_{t_0}^{t} \sqrt{X}\ dt = \frac{(2d_9 t + d_8)\sqrt{X}}{4d_9} - \frac{(2d_9 t_0 + d_8)\sqrt{X_0}}{4d_9} + \left(\frac{4d_7 d_9 - d_8^2}{8d_9}\right) \int_{t_0}^{t} \frac{dt}{\sqrt{X}}$$

$$= \frac{t\sqrt{X} - t_0\sqrt{X_0}}{2} + \frac{d_8(\sqrt{X} - \sqrt{X_0})}{4d_9}$$

$$+ \left(\frac{4d_7 d_9 - d_8^2}{8d_9}\right)\left(\frac{1}{\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right) \quad .$$

Thus,

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + \frac{\bar{d}_1}{\sqrt{d_9}} \left\{ t\ \ell n \left(\frac{\Omega}{\Omega_0}\right) - \left(\frac{1}{\sqrt{d_9}}\right)\left[\sqrt{X} - \sqrt{X_0}\right.\right.$$

$$\left.\left. - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right)\right]\right\} + \frac{\bar{d}_2}{d_9} \left\{ \frac{t\sqrt{X} - t_0\sqrt{X_0}}{2} - \sqrt{X_0}(t - t_0)\right.$$

$$+ \frac{d_8(\sqrt{X} - \sqrt{X_0})}{4d_9} + \left(\frac{4d_7 d_9 - d_8^2}{8d_9}\right)\left(\frac{1}{\sqrt{d_9}}\right) \ell n \left(\frac{\Omega}{\Omega_0}\right)$$

$$- \left(\frac{d_8}{2\sqrt{d_9}}\right) t\ \ell n \left(\frac{\Omega}{\Omega_0}\right) + \left(\frac{d_8}{2d_9}\right) \left[\sqrt{X} - \sqrt{X_0}\right.$$

$$\left.\left. - \frac{d_8}{2\sqrt{d_9}}\ \ell n \left(\frac{\Omega}{\Omega_0}\right)\right]\right\} + (\bar{b}_0 + \bar{g}_0)\ \frac{(t - t_0)^2}{2} \quad .$$

Then, for case 3, the entire closed-form solution which approximates the original system of ordinary nonlinear differential equations is given by

$$\bar{\lambda}(t) = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0(t - t_0) \quad ,$$

$$\bar{u}(t) = \bar{u}_0 + \dot{\bar{u}}_0(t - t_0) \quad ,$$

$$\gamma(t) = \gamma_0 \quad ,$$

$$\bar{v}(t) = \bar{v}_0 + \frac{\bar{d}_1}{\sqrt{d_9}} \ln\left(\frac{\Omega}{\Omega_0}\right) + \frac{\bar{d}_2}{d_9}\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ln\left(\frac{\Omega}{\Omega_0}\right)\right]$$

$$+ (\bar{b}_0 + \bar{g}_0)(t - t_0) \quad ,$$

$$\bar{x}(t) = \bar{x}_0 + \bar{v}_0(t - t_0) + (\bar{b}_0 + \bar{g}_0)\frac{(t - t_0)^2}{2} + \frac{\bar{d}_1}{\sqrt{d_9}}\left\{t \ln\left(\frac{\Omega}{\Omega_0}\right)\right.$$

$$\left. - \left(\frac{1}{\sqrt{d_9}}\right)\left[\sqrt{X} - \sqrt{X_0} - \left(\frac{d_8}{2\sqrt{d_9}}\right) \ln\left(\frac{\Omega}{\Omega_0}\right)\right]\right\}$$

$$+ \frac{\bar{d}_2}{d_9}\left\{\left(\frac{t}{2} + \frac{3d_8}{4d_9}\right)(\sqrt{X} - \sqrt{X_0}) - \frac{\sqrt{X_0}(t - t_0)}{2}\right.$$

$$\left. + \left(\frac{4d_7d_9 - 3d_8^2}{8d_9}\right)\left(\frac{1}{\sqrt{d_9}}\right) \ln\left(\frac{\Omega}{\Omega_0}\right) - \left(\frac{d_8}{2\sqrt{d_9}}\right) t \ln\left(\frac{\Omega}{\Omega_0}\right)\right\} \quad ,$$

$$q(t) = q_0 + h_0(t - t_0) + \frac{d_{10}}{\sqrt{d_9}} \ln\left(\frac{\Omega}{\Omega_0}\right) + \frac{d_{11}}{d_9}\left[\sqrt{X} - \sqrt{X_0}\right.$$

$$\left. - \frac{d_8}{2\sqrt{d_9}} \ln\left(\frac{\Omega}{\Omega_0}\right)\right] + \frac{d_{12}}{2d_9}\left[t\sqrt{X} - t_0\sqrt{X_0} - \left(\frac{3d_8}{2d_9}\right)(\sqrt{X} - \sqrt{X_0})\right.$$

$$\left. + \left(\frac{3d_8^2 - 4d_7d_9}{4d_9\sqrt{d_9}}\right) \ln\left(\frac{\Omega}{\Omega_0}\right)\right] \quad .$$

For case 4, when $d_7 = 0$ and $d_9 = 0$, the aerodynamic forces are zero so that again the motion is two-body motion for which closed-form solutions are developed in Reference 2.

The preceding work completes the closed-form solutions for the reentry case. These closed-form solutions can now be used to approximate the reentry motion of a space vehicle. In the next section, the reentry and the ascent closed-form solutions will be used to develop expressions for the closed-form partial derivative transition matrices.

THE PARTIAL DERIVATIVE TRANSITION MATRICES

In order to define most easily the partial derivative transition matrices for the three ascent cases and the three reentry cases, a new vector of dimension 14 is defined; i. e.,

$$Z = \begin{bmatrix} \bar{x} \\ \bar{v} \\ q \\ \bar{\lambda} \\ \bar{u} \\ \gamma \end{bmatrix} .$$

Then, for each of the three ascent and reentry cases, the partial derivative transition matrix is the 14 by 14 matrix $\left[\frac{\partial Z}{\partial Z_0}\right]$. This matrix can be evaluated at any t, where $t_0 \leq t \leq t_f$, by explicit differentiation of the expressions for $\bar{x}$, $\bar{v}$, q, $\bar{\lambda}$, $\bar{u}$, and γ with respect to $\bar{x}_0$, $\bar{v}_0$, q, $\bar{\lambda}_0$, $\bar{u}_0$ and γ_0. Also of interest for later use is the vector $\frac{\partial Z}{\partial t_0}$. The transition matrix $\left[\frac{\partial Z}{\partial Z_0}\right]$ and the vector $\frac{\partial Z}{\partial t_0}$ can be written more explicitly as follows:

$$\frac{\partial Z}{\partial Z_0} = \begin{bmatrix}
\frac{\partial \bar{x}}{\partial \bar{x}_0} & \frac{\partial \bar{x}}{\partial \bar{v}_0} & \frac{\partial \bar{x}}{\partial q_0} & \frac{\partial \bar{x}}{\partial \bar{\lambda}_0} & \frac{\partial \bar{x}}{\partial \bar{u}_0} & \frac{\partial \bar{x}}{\partial \gamma_0} \\
\frac{\partial \bar{v}}{\partial \bar{x}_0} & \frac{\partial \bar{v}}{\partial \bar{v}_0} & \frac{\partial \bar{v}}{\partial q_0} & \frac{\partial \bar{v}}{\partial \bar{\lambda}_0} & \frac{\partial \bar{v}}{\partial \bar{u}_0} & \frac{\partial \bar{v}}{\partial \gamma_0} \\
\frac{\partial q}{\partial \bar{x}_0} & \frac{\partial q}{\partial \bar{v}_0} & \frac{\partial q}{\partial q_0} & \frac{\partial q}{\partial \bar{\lambda}_0} & \frac{\partial q}{\partial \bar{u}_0} & \frac{\partial q}{\partial \gamma_0} \\
\frac{\partial \bar{\lambda}}{\partial \bar{x}_0} & \frac{\partial \bar{\lambda}}{\partial \bar{v}_0} & \frac{\partial \bar{\lambda}}{\partial q_0} & \frac{\partial \bar{\lambda}}{\partial \bar{\lambda}_0} & \frac{\partial \bar{\lambda}}{\partial \bar{u}_0} & \frac{\partial \bar{\lambda}}{\partial \gamma_0} \\
\frac{\partial \bar{u}}{\partial \bar{x}_0} & \frac{\partial \bar{u}}{\partial \bar{v}_0} & \frac{\partial \bar{u}}{\partial q_0} & \frac{\partial \bar{u}}{\partial \bar{\lambda}_0} & \frac{\partial \bar{u}}{\partial \bar{u}_0} & \frac{\partial \bar{u}}{\partial \gamma_0} \\
\frac{\partial \gamma}{\partial \bar{x}_0} & \frac{\partial \gamma}{\partial \bar{v}_0} & \frac{\partial \gamma}{\partial q_0} & \frac{\partial \gamma}{\partial \bar{\lambda}_0} & \frac{\partial \gamma}{\partial \bar{u}_0} & \frac{\partial \gamma}{\partial \gamma_0}
\end{bmatrix}$$

$$\frac{\partial Z}{\partial t_0} = \begin{bmatrix} \frac{\partial \overline{x}}{\partial t_0} \\ \frac{\partial \overline{v}}{\partial t_0} \\ \frac{\partial q}{\partial t_0} \\ \frac{\partial \overline{\lambda}}{\partial t_0} \\ \frac{\partial \overline{u}}{\partial t_0} \\ \frac{\partial \gamma}{\partial t_0} \end{bmatrix} .$$

To illustrate how the different components of $\frac{\partial Z}{\partial Z_0}$ amd $\frac{\partial Z}{\partial t_0}$ are obtained, the expressions for $\frac{\partial \overline{x}}{\partial \overline{x}_0}$ and $\frac{\partial \overline{x}}{\partial t_0}$ will be derived in detail for case 3 of ascent flight. The other components (including the other cases for ascent and reentry) can be derived in a similar manner, but their derivation will not be shown in detail. To start the derivation of $\frac{\partial \overline{x}}{\partial \overline{x}_0}$ and $\frac{\partial \overline{x}}{\partial t_0}$, note that for case 3 of ascent

$$\begin{aligned} \overline{x} = \overline{x}_0 &+ \overline{v}_0(t - t_0) + \overline{g}_0 \left[\frac{(t - t_0)^2}{2}\right] \\ &+ \left(\frac{1}{\dot{m}^2}\right)\overline{b}_0 \left[m\ \ell n\left(\frac{m}{m_0}\right) - (m - m_0)\right] - \frac{\overline{d}_1}{\dot{m}^2}\left\{\frac{m}{\sqrt{d_7}}\left[\ell n\left(\frac{\Phi}{\Phi_0}\right)\right.\right. \\ &\left. - \ell n\left(\frac{m}{m_0}\right)\right] + \left(\frac{1}{\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\Bigg\} + \frac{\overline{d}_2}{\dot{m}^2}\left\{\left(\frac{m}{\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right. \\ &\left. - \left(\frac{1}{d_9}\right)\left[(\sqrt{X} - \sqrt{X_0}) - \left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right]\right\} . \end{aligned}$$

To ease the problem of writing the above expression for $\bar{x}$, define the scalars:

$$c_1 = \left[m\ \ell n\left(\frac{m}{m_0}\right) - (m - m_0)\right]\left(\frac{1}{\dot{m}^2}\right) \quad ,$$

$$c_2 = -\left(\frac{1}{\dot{m}^2}\right)\left\{\frac{m}{\sqrt{d_7}}\left[\ell n\left(\frac{\dot{\Phi}}{\Phi_0}\right) - \ell n\left(\frac{m}{m_0}\right)\right] + \left(\frac{1}{\sqrt{d_9}}\right)\ell n'\left(\frac{\Omega}{\Omega_0}\right)\right\} \quad ,$$

$$c_3 = \left(\frac{1}{\dot{m}^2}\right)\left\{\left(\frac{m}{\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right) - \left(\frac{1}{d_9}\right)\left[(\sqrt{X} - \sqrt{X_0})\right.\right.$$

$$\left.\left.-\left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right]\right\} \quad .$$

Then

$$\bar{x} = \bar{x}_0 + \bar{v}_0(t - t_0) + \bar{g}_0\left[\frac{(t - t_0)^2}{2}\right] + c_1\bar{b}_0 + c_2\bar{d}_1 + c_3\bar{d}_2$$

and

$$\frac{\partial \bar{x}}{\partial \bar{x}_0} = I + \left(\frac{\partial \bar{g}_0}{\partial \bar{x}_0}\right)\left[\frac{(t - t_0)^2}{2}\right] + c_1\left(\frac{\partial \bar{b}_0}{\partial \bar{x}_0}\right) + \bar{b}_0\left(\frac{\partial c_1}{\partial \bar{x}_0}\right) + c_2\left(\frac{\partial \bar{d}_1}{\partial \bar{x}_0}\right)$$

$$+ \bar{d}_1\left(\frac{\partial c_2}{\partial \bar{x}_0}\right) + c_3\left(\frac{\partial \bar{d}_2}{\partial \bar{x}_0}\right) + \bar{d}_2\left(\frac{\partial c_3}{\partial \bar{x}_0}\right) \quad .$$

To complete the derivation of $\frac{\partial \bar{x}}{\partial \bar{x}_0}$, the unevaluated terms in the above expression are taken one at a time. Since

$$\bar{g}_0 = \frac{-GM\,\bar{x}_0}{(\bar{x}_0^T\,\bar{x}_0)^{3/2}} \, ,$$

then

$$\frac{\partial \bar{g}_0}{\partial \bar{x}_0} = \frac{-GM}{(\bar{x}_0^T\,\bar{x}_0)^{3/2}}\, I + \frac{3GM\,\bar{x}_0\,\bar{x}_0^T}{(\bar{x}_0^T\,\bar{x}_0)^{5/2}} = \frac{GM}{(\bar{x}_0^T\,\bar{x}_0)^{3/2}}\left[\frac{3\bar{x}_0\,\bar{x}_0^T}{(\bar{x}_0^T\,\bar{x}_0)} - I\right] \quad .$$

Similarly,

$$\bar{b}_0 = -\left(\frac{\rho A_r}{2}\right)|\bar{v}_r|\left(c_A + 2\eta c_{L_\alpha}^2\right)\bar{v}_r \quad ,$$

where ρ, $\bar{v}_r$, c_A, η, and c_{L_α} are all evaluated at t_0 and thus depend on $\bar{x}_0$. Then,

$$\frac{\partial \bar{b}_0}{\partial \bar{x}_0} = -\left(\frac{A_r}{2}\right) |\bar{v}_r| \left(c_A + 2\eta c_{L_\alpha}^{\;2}\right) \bar{v}_r \left(\frac{\partial \rho}{\partial \bar{x}_0}\right)$$

$$-(\rho A_r)\left(c_A + 2\eta c_{L_\alpha}^{\;2}\right)\left(\bar{v}_r \, \bar{v}_r^T\right) \frac{\partial \bar{v}_r}{\partial \bar{x}_0}$$

$$-\left(\frac{\rho A_r}{2}\right) |\bar{v}_r| \left(c_A + 2\eta c_{L_\alpha}^{\;2}\right) \frac{\partial \bar{v}_r}{\partial \bar{x}_0} - \left(\frac{\rho A_r}{2}\right) |\bar{v}_r| \, \bar{v}_r \left[\frac{\partial c_A}{\partial \bar{x}_0}\right.$$

$$\left. + 2\left(\frac{\partial \eta}{\partial \bar{x}_0} c^2_{L_\alpha} + 2\eta c_{L_\alpha} \frac{\partial c_{L_\alpha}}{\partial \bar{x}_0}\right)\right] .$$

Since $\bar{v}_r$ evaluated at t_0 is given by

$$\bar{v}_r = \bar{v}_0 - \omega \times \bar{x}_0 \quad ,$$

then

$$\frac{\partial \bar{v}_r}{\partial \bar{x}_0} = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

where ω_1, ω_2, and ω_3 are the components of the constant vector $\bar{\omega}$ explained in Reference 1. Also explained in Reference 1 are subroutines for the atmospheric functions and the aerodynamic functions. The subroutine for the atmospheric functions yields values for the density ρ, the pressure P, and the velocity of sound v_S as functions of altitude and also the derivatives $\frac{d\rho}{dalt}$, $\frac{dP}{dalt}$, and $\frac{dv_S}{dalt}$. The altitude at t_0 is given by

$$alt = (\bar{x}_0^T \, \bar{x}_0)^{1/2} - R_e$$

where R_e is assumed (for this development) to have a constant value equal to the mean value of the earth's radius. The subroutine for the aerodynamic functions yields values for the coefficients c_A, c_{L_α}, and η as functions of Mach number M and the derivatives $\frac{\partial c_A}{\partial M}$, $\frac{\partial c_{L_\alpha}}{\partial M}$, and $\frac{\partial \eta}{\partial M}$. The Mach number M is computed by the following formula:

$$M = \frac{|\bar{v}_r|}{v_S} \quad .$$

Now the quantities $\frac{\partial \rho}{\partial \bar{x}_0}$, $\frac{\partial c_A}{\partial \bar{x}_0}$, $\frac{\partial \eta}{\partial \bar{x}_0}$, and $\frac{\partial c_{L_\alpha}}{\partial \bar{x}_0}$ appearing in the expression for $\frac{\partial \bar{b}_0}{\partial \bar{x}_0}$ can be written in more detail, but first the quantites $\frac{\partial \, alt}{\partial \bar{x}_0}$ and $\frac{\partial M}{\partial \bar{x}_0}$ must be written. This gives

$$\frac{\partial \, alt}{\partial \bar{x}_0} = \frac{\bar{x}_0^T}{(\bar{x}_0^T \; \bar{x}_0)^{1/2}}$$

$$\frac{\partial M}{\partial \bar{x}_0} = \frac{\bar{v}_r^T \left(\frac{\partial \bar{v}_r}{\partial \bar{x}_0}\right)}{|\bar{v}_r| \; v_S} - \left(\frac{|\bar{v}_r|}{v_S^2}\right)\left(\frac{\partial v_S}{\partial \, alt}\right)\left(\frac{\partial \, alt}{\partial \bar{x}_0}\right) \quad .$$

Then

$$\frac{\partial \rho}{\partial \bar{x}_0} = \left(\frac{\partial \rho}{\partial \, alt}\right)\left(\frac{\partial \, alt}{\partial \bar{x}_0}\right) \quad ,$$

$$\frac{\partial c_A}{\partial \bar{x}_0} = \left(\frac{\partial c_A}{\partial M}\right)\left(\frac{\partial M}{\partial \bar{x}_0}\right) \quad ,$$

$$\frac{\partial \eta}{\partial \bar{x}_0} = \left(\frac{\partial \eta}{\partial M}\right)\left(\frac{\partial M}{\partial \bar{x}_0}\right) \quad ,$$

$$\frac{\partial c_{L_\alpha}}{\partial \bar{x}_0} = \left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\left(\frac{\partial M}{\partial \bar{x}_0}\right) \quad .$$

This completes the derivation of $\frac{\partial \overline{b}_0}{\partial \overline{x}_0}$. Now note that (in the expression for $\frac{\partial \overline{x}}{\partial \overline{x}_0}$) c_1 does not depend on $\overline{x}_0$ so that $\frac{\partial c_1}{\partial \overline{x}_0} = [0, \ 0, \ 0]$. Then to continue,

$$\overline{d}_1 = [A_0][A_0]^T\left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right] \quad .$$

Thus,

$$\frac{\partial \overline{d}_1}{\partial \overline{x}_0} = 2[A_0]\left\{\frac{\partial [A_0]^T}{\partial x_0}\left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]\right\} - \left(\frac{m_0}{\dot{m}}\right)[A_0][A_0]^T\left(\frac{\partial \dot{\overline{u}}_0}{\partial \overline{x}_0}\right) \quad .$$

To evaluate the quantity $\left\{\frac{\partial [A_0]^T}{\partial \overline{x}_0}\left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]\right\}$, note that

$$[A_0]^T = \left[F + \left(\frac{\rho A_r}{2}\right) |\overline{v}_r|^2 \ c_{L_\alpha}\right] I$$

$$- \left(\frac{\rho A_r}{2}\right) c_{L_\alpha} \left(1 - 2\eta c_{L_\alpha}\right)\left(\overline{v}_r \ \overline{v}_r^T\right) \quad .$$

Then,

$$\left\{\frac{\partial [A_0]^T}{\partial \overline{x}_0}\left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]\right\} = \left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]\left[\frac{\partial F}{\partial \overline{x}_0} + \left(\frac{A_r}{2}\right) |\overline{v}_r|^2 c_{L_\alpha}\left(\frac{\partial \rho}{\partial \overline{x}_0}\right) + (\rho A_r) \ c_{L_\alpha} \overline{v}_r^T\left(\frac{\partial \overline{v}_r}{\partial \overline{x}_0}\right) + \left(\frac{\rho A_r}{2}\right) |\overline{v}_r|^2 \frac{\partial c_{L_\alpha}}{\partial \overline{x}_0}\right]$$

$$- \left\{\overline{v}_r^T\left[\overline{u}_0 - \dot{\overline{u}}_0\left(\frac{m_0}{\dot{m}}\right)\right]\right\}\left\{\overline{v}_r\left[\left(\frac{A_r}{2}\right) c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left(\frac{\partial \rho}{\partial \overline{x}_0}\right) + \left(\frac{\rho A_r}{2}\right)\left(1 - 4\eta c_{L_\alpha}\right)\frac{\partial c_{L_\alpha}}{\partial \overline{x}_0} - (\rho A_r) \ c_{L_\alpha}^2\left(\frac{\partial \eta}{\partial \overline{x}_0}\right)\right]\right\}$$

$$-(\rho A_r) \ c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left\{\overline{v}_r^T\left[\overline{u}_0 - \dot{\overline{u}}_0 \frac{m_0}{\dot{m}}\right]\right\}\left(\frac{\partial \overline{v}_r}{\partial \overline{x}_0}\right) \quad .$$

The only quantity not already derived in the above expression is $\frac{\partial F}{\partial \overline{x}_0}$. From Reference 1,

$$F = F_S + A_e(P_0 - P)$$

where F_S, A_e, and P_0 are constants and P is obtained from the atmospheric subroutine. Thus,

$$\frac{\partial F}{\partial \overline{x}_0} = -A_e \left(\frac{\partial P}{\partial alt}\right) \left(\frac{\partial alt}{\partial x_0}\right) .$$

This completes the derivation of the expression $\left\{ \frac{\partial [A_0]^T}{\partial \overline{x}_0} \left[\overline{u}_0 - \dot{\overline{u}}_0 \left(\frac{m_0}{\dot{m}}\right)\right] \right\}$.

To complete the expression for $\frac{\partial \overline{d}_1}{\partial \overline{x}_0}$, the expression $\frac{\partial \dot{\overline{u}}_0}{\partial \overline{x}_0}$ must be evaluated. To do this, it should be noted that

$$\dot{\overline{u}}^T = -\overline{\lambda}^T - \frac{1}{m} \left\{ \frac{\overline{u}^T [A]}{|\overline{u}^T [A]|} \frac{\partial ([A]^T \overline{u})}{\partial \overline{v}} + \overline{u}^T \left(\frac{\partial \overline{b}}{\partial \overline{v}}\right) \right\} .$$

Then

$$\begin{aligned} \frac{\partial ([A]^T \overline{u})}{\partial \overline{v}} = & (\rho A_r)\, c_{L_\alpha} \left(\overline{u}\, \overline{v}_r^T\right) + \left(\frac{\rho A_r}{2}\right) |\overline{v}_r|^2 \left[\overline{u} \left(\frac{\partial c_{L_\alpha}}{\partial \overline{v}}\right)\right] \\ & - \left(\frac{\rho A_r}{2}\right) c_{L_\alpha} \left(1 - 2\eta c_{L_\alpha}\right) \left(\overline{v}_r^T\, \overline{u}\right) I \\ & - \left(\frac{\rho A_r}{2}\right) \left(\overline{v}_r^T\, \overline{u}\right) \left[\left(1 - 4\eta c_{L_\alpha}\right) \frac{\partial c_{L_\alpha}}{\partial M} \right. \\ & \left. - 2 c_{L_\alpha}^2 \left(\frac{\partial \eta}{\partial M}\right)\right] \left[\overline{v}_r \left(\frac{\partial M}{\partial \overline{v}}\right)\right] . \end{aligned}$$

Note that

$$\frac{\partial M}{\partial \overline{v}} = \frac{\overline{v}_r^T}{|\overline{v}_r| v_S}$$

and

$$\frac{\partial c_{L_\alpha}}{\partial \overline{v}} = \frac{\partial c_{L_\alpha}}{\partial M} \left(\frac{\partial M}{\partial \overline{v}}\right) .$$

Also,

$$\frac{\partial \bar{b}}{\partial \bar{v}} = -\left(\frac{\rho A_r}{2}\right) |\bar{v}_r| \left(c_A + 2\eta c^2_{L_\alpha}\right) I - (\rho A_r)\left(c_A + 2\eta c^2_{L_\alpha}\right)\left(\bar{v}_r \bar{v}_r^T\right)$$

$$-\left(\frac{\rho A_r}{2}\right) |\bar{v}_r| \left[\frac{\partial c_A}{\partial M} + 2\left(\frac{\partial \eta}{\partial M}\right) c^2_{L_\alpha}\right.$$

$$\left. + 4\eta c_{L_\alpha} \frac{\partial c_{L_\alpha}}{\partial M}\right] \left[\bar{v}_r \left(\frac{\partial M}{\partial \bar{v}}\right)\right] .$$

Thus,

$$\dot{\bar{u}} = -\bar{\lambda} - \left\{\left[\rho A_r c_{L_\alpha} + \left(\frac{\rho A_r}{2}\right) M \left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right] \left(\bar{v}_r \bar{u}^T\right)\right.$$

$$-\left(\frac{\rho A_r}{2}\right) c_{L_\alpha} \left(1 - 2\eta c_{L_\alpha}\right)\left(\bar{v}_r^T \bar{u}\right) I$$

$$-\left(\frac{\rho A_r}{2}\right)\left(\frac{\bar{v}_r^T \bar{u}}{|\bar{v}_r| v_S}\right) \left[\left(1 - 4\eta c_{L_\alpha}\right)\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right.$$

$$\left.\left. -2c^2_{L_\alpha} \left(\frac{\partial \eta}{\partial M}\right)\right]\left(\bar{v}_r \bar{v}_r^T\right)\right\} \frac{[A]^T \bar{u}}{m|[A]^T \bar{u}|}$$

$$+\left(\frac{\rho A_r}{2m}\right) |\bar{v}_r| \left(c_A + 2\eta c^2_{L_\alpha}\right)\bar{u} + \left\{(\rho A_r) \; c_A + \left(2\eta c^2_{L_\alpha}\right)\right.$$

$$\left. +\left(\frac{\rho A_r}{2}\right)\left(\frac{1}{v_S}\right)\left[\frac{\partial c_A}{\partial M} + 2\left(\frac{\partial \eta}{\partial M}\right) c^2_{L_\alpha} + 4\eta c_{L_\alpha} \frac{\partial c_{L_\alpha}}{\partial M}\right]\right\} \frac{\left(\bar{v}_r^T \bar{u}\right)}{m} \bar{v}_r .$$

Now the above expression, along with the initial conditions, can be used to evaluate $\dot{\bar{u}}_0$, and then $\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}$ can be computed; but before attempting this task, the matrix $[B]$ and the scalar c_4 will be defined as follows:

$$[B] = \frac{1}{m}\left\{\left[\rho A_r \, c_{L_\alpha} + \left(\frac{\rho A_r}{2}\right) M \left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right]\left(\bar{v}_r \, \bar{u}^T\right)\right.$$

$$-\left(\frac{\rho A_r}{2}\right) c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left(\bar{v}_r^T \bar{u}\right) I$$

$$-\left(\frac{\rho A_r}{2}\right)\left(\frac{\bar{v}_r^T \, \bar{u}}{|\bar{v}_r| \, v_S}\right)\left[\left(1 - 4\eta c_{L_\alpha}\right)\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right.$$

$$\left.\left. -2c_{L_\alpha}^2 \left(\frac{\partial \eta}{\partial M}\right)\right]\left(\bar{v}_r \, \bar{v}_r^T\right)\right\} \quad ,$$

$$c_4 = \frac{1}{m}\left\{(\rho A_r)\left(c_A + 2\eta c_{L_\alpha}^2\right) + \left(\frac{\rho A_r}{2}\right)\left(\frac{1}{v_S}\right)\left[\frac{\partial c_A}{\partial M}\right.\right.$$

$$\left.\left. + 2\left(\frac{\partial \eta}{\partial M}\right) c_{L_\alpha}^2 + 4\eta c_{L_\alpha}\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right]\right\} \quad ,$$

$$c_8 = \frac{\rho A_r}{2m}\left(c_A + 2\eta c_{L_\alpha}^2\right)|\bar{v}_r| \quad .$$

Then,

$$\dot{\bar{u}}_0 = -\bar{\lambda}_0 - [B_0] \frac{[A_0]^T \, \bar{u}_0}{|[A_0]^T \, \bar{u}_0|} + c_{4_0}\left(\bar{v}_{r_0}^T \, \bar{u}_0\right) \bar{v}_{r_0} + c_{8_0} \, \bar{u}_0$$

and

$$\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0} = - \frac{\left\{ \frac{\partial [B_0]}{\partial \bar{x}_0} \left([A_0]^T \bar{u}_0 \right) \right\}}{|[A_0]^T \bar{u}_0|} - \frac{[\dot{B}_0] \left\{ \left(\frac{\partial [A_0]^T}{\partial \bar{x}_0} \right) \bar{u}_0 \right\}}{|[A_0]^T \bar{u}_0|} + u_0 \left(\frac{\partial c_{8_0}}{\partial \bar{x}_0} \right)$$

$$+ \left(\bar{v}_{r_0}^T \bar{u}_0 \right) \bar{v}_{r_0} \left(\frac{\partial c_{4_0}}{\partial \bar{x}_0} \right) + c_{4_0} \left[\left(\bar{v}_{r_0} \bar{u}_0^T \right) + \left(\bar{v}_r^T \bar{u}_0 \right) I \right] \left(\frac{\partial \bar{v}_{r_0}}{\partial \bar{x}_0} \right)$$

In the above expression for $\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}$, the quantities $\left\{ \frac{\partial [B_0]}{\partial \bar{x}_0} \left([A_0]^T \bar{u}_0 \right) \right\}$, $\left\{ \left(\frac{\partial [A_0]^T}{\partial \bar{x}_0} \right) \bar{u}_0 \right\}$, $\frac{\partial c_{8_0}}{\partial \bar{x}_0}$, and $\frac{\partial c_{4_0}}{\partial \bar{x}_0}$ have not yet been calculated. Before attempting to calculate these quantities, some more constants will be defined; i.e.,

$$c_5 = \left(\frac{\rho A_r}{2m} \right) \left[2c_{L_\alpha} + M \left(\frac{\partial c_{L_\alpha}}{\partial M} \right) \right] ,$$

$$c_6 = \left(\frac{\rho A_r}{2m} \right) c_{L_\alpha} \left(1 - 2\eta c_{L_\alpha} \right) ,$$

$$c_7 = \left(\frac{\rho A_r}{\partial m} \right) \left[\left(1 - 4\eta c_{L_\alpha} \right) \left(\frac{\partial c_{L_\alpha}}{\partial M} \right) - 2c^2_{L_\alpha} \left(\frac{\partial \eta}{\partial M} \right) \right] .$$

Then

$$[\dot{B}_0] = c_5 \left(\bar{v}_r \bar{u}_0^T \right) - c_6 \left(\bar{v}_r^T \bar{u}_0 \right) I - \left[\frac{\left(\bar{v}_r^T u_0 \right)}{|\bar{v}_r| \, v_S} \right] c_7 \left(\bar{v}_r \bar{v}_r^T \right) ,$$

where c_5, c_6, c_7, $\bar{v}_r$, and v_S are all evaluated at t_0 and thus depend on $\bar{x}_0$. Now the quantity $\left\{ \frac{\partial [B_0]}{\partial \bar{x}_0} \left([A_0]^T \bar{u}_0 \right) \right\}$ can be written; i.e.,

$$\left\{\frac{\partial[B_0]}{\partial \bar{x}_0}\left([A_0]^T\bar{u}_0\right)\right\} = \left(\bar{v}_r\bar{u}_0^T\right)\left([A_0]^T\bar{u}_0\right)\left(\frac{\partial c_5}{\partial \bar{x}_0}\right) + c_5\left(\bar{u}_0^T[A_0]^T\bar{u}_0\right)\left(\frac{\partial \bar{v}_r}{\partial \bar{x}_0}\right)$$

$$- \left(\bar{v}_r^T\bar{u}_0\right)\left([A_0]^T\bar{u}_0\right)\left(\frac{\partial c_6}{\partial \bar{x}_0}\right) - c_6\left([A_0]^T\bar{u}_0\right)\bar{u}_0^T\left(\frac{\partial \bar{v}_r}{\partial \bar{x}_0}\right)$$

$$-\left[\frac{\left(\bar{v}_r^T\bar{u}_0\right)}{|\bar{v}_r|v_S}\right]\left[\bar{v}_r^T\left([A_0]^T\bar{u}_0\right)\right]\left[2c_7\left(\frac{\partial \bar{v}_r}{\partial \bar{x}_0}\right) + \bar{v}_r\left(\frac{\partial c_7}{\partial \bar{x}_0}\right)\right]$$

$$-\left(\frac{1}{|\bar{v}_r|v_S}\right)c_7\left[\bar{v}_r^T\left([A_0]^T\bar{u}_0\right)\right]\left(\bar{v}_r\bar{u}_0^T\right)\left(\frac{\partial \bar{v}_r}{\partial \bar{x}_0}\right)$$

$$+\left(\frac{\bar{v}_r^T\bar{u}_0}{|\bar{v}_r|v_S^2}\right)c_7\left[\bar{v}_r^T\left([A_0]^T\bar{u}_0\right)\right]\left[\bar{v}_r\left(\frac{\partial v_S}{\partial \bar{x}_0}\right)\right]$$

$$+\left(\frac{\bar{v}_r^T\bar{u}_0}{|\bar{v}_r|^3v_S}\right)c_7\left[\bar{v}_r^T\left([A_0]^T\bar{u}_0\right)\right]\left(\bar{v}_r\bar{v}_r^T\right)\left(\frac{\partial \bar{v}_r}{\partial x_0}\right).$$

To complete the evaluation of the above expression, $\frac{\partial c_5}{\partial \bar{x}_0}$, $\frac{\partial c_6}{\partial \bar{x}_0}$, and $\frac{\partial c_7}{\partial \bar{x}_0}$ must be written. This gives

$$\frac{\partial c_5}{\partial \bar{x}_0} = \left(\frac{c_5}{\rho}\right)\left(\frac{\partial \rho}{\partial \bar{x}_0}\right) + \left(\frac{\rho A_r}{2m}\right)\left[3\left(\frac{\partial c_{L_\alpha}}{\partial M}\right) + M\left(\frac{\partial^2 c_{L_\alpha}}{\partial M^2}\right)\right]\frac{\partial M}{\partial \bar{x}_0} ,$$

$$\frac{\partial c_6}{\partial \bar{x}_0} = \left(\frac{c_6}{\rho}\right)\left(\frac{\partial \rho}{\partial \bar{x}_0}\right) + \left\{\left(\frac{c_6}{c_{L_\alpha}}\right)\left(\frac{\partial c_{L_\alpha}}{\partial M}\right) - \left(\frac{\rho A_r}{m}\right)c_{L_\alpha}\left[\eta\left(\frac{\partial c_{L_\alpha}}{\partial M}\right) + c_{L_\alpha}\left(\frac{\partial \eta}{\partial M}\right)\right]\right\}\frac{\partial M}{\partial \bar{x}_0} ,$$

$$\frac{\partial c_7}{\partial \overline{x}_0} = \frac{c_7}{\rho}\left(\frac{\partial \rho}{\partial \overline{x}_0}\right) + \frac{\rho A_r}{2m}\left\{\left(1 - 4\eta c_{L_\alpha}\right)\left(\frac{\partial^2 c_{L_\alpha}}{\partial M^2}\right)\right.$$

$$\left. -4\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\left[\eta\left(\frac{\partial c_{L_\alpha}}{\partial M}\right) + 2c_{L_\alpha}\left(\frac{\partial \eta}{\partial M}\right)\right] - 2c_{L_\alpha}^2\left(\frac{\partial^2 \eta}{\partial M^2}\right)\right\}\frac{\partial M}{\partial \overline{x}_0} \quad .$$

Note that $\frac{\partial^2 \eta}{\partial M^2}$ and $\frac{\partial^2 c_{L_\alpha}}{\partial M^2}$ are assumed to be available from the aerodynamic subroutine of Reference 1. To continue evaluating $\left(\frac{\partial \dot{\overline{u}}_0}{\partial \overline{x}_0}\right)$, the expression $\left\{\left(\frac{\partial [A_0]^T}{\partial \overline{x}_0}\right)\overline{u}_0\right\}$ will now be evaluated. Since

$$[A_0]^T = \left[F + \left(\frac{\rho A_r}{2}\right) |\overline{v}_r|^2 \, c_{L_\alpha}\right] I - \left(\frac{\rho A_r}{2}\right) c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\overline{v}_r \, \overline{v}_r^T \quad ,$$

then

$$\left\{\left(\frac{\partial [A_0]^T}{\partial \overline{x}_0}\right)\overline{u}_0\right\} = \overline{u}_0\left[\frac{\partial F}{\partial \overline{x}_0} + \left(\frac{A_r}{2}\right) |\overline{v}_r|^2 \, c_{L_\alpha}\left(\frac{\partial \rho}{\partial \overline{x}_0}\right) + (\rho A_r) \, c_{L_\alpha} \, \overline{v}_r^T\left(\frac{\partial \overline{v}_r}{\partial \overline{x}_0}\right)\right.$$

$$\left. + \left(\frac{\rho A_r}{2}\right) |\overline{v}_r|^2 \, \frac{\partial c_{L_\alpha}}{\partial \overline{x}_0}\right] - \left(\overline{v}_r^T \, \overline{u}_0\right)\left\{\left(\rho A_r \, c_{L_\alpha}\right)\left(1 - 2\eta c_{L_\alpha}\right)\left(\frac{\partial \overline{v}_r}{\partial \overline{x}_0}\right)\right.$$

$$- \left(\frac{A_r}{2}\right) c_{L_\alpha}\left(1 - 2\eta c_{L_\alpha}\right)\left[\overline{v}_r\left(\frac{\partial \rho}{\partial \overline{x}_0}\right)\right] - \left(\frac{\rho A_r}{2}\right)\left(1 - 2\eta c_{L_\alpha}\right)\left[\overline{v}_r\left(\frac{\partial c_{L_\alpha}}{\partial \overline{x}_0}\right)\right]$$

$$\left. + \rho A_r \, c_{L_\alpha}\left[\eta \overline{v}_r\left(\frac{\partial c_{L_\alpha}}{\partial \overline{x}_0}\right) + c_{L_\alpha} \, \overline{v}_r\right] + \rho A_r \, c_{L_\alpha}\left[\eta \, \frac{c_{L_\alpha}}{\partial M} + c_{L_\alpha} \, \frac{\partial \eta}{\partial M}\right]\left[\overline{v}_r\left(\frac{\partial M}{\partial \overline{x}_0}\right)\right]\right\}$$

To complete the evaluation of $\frac{\partial \dot{\overline{u}}_0}{\partial \overline{x}_0}$,

$$\frac{\partial c_{40}}{\partial \overline{x}_0} = \left(\frac{c_{40}}{\rho}\right) \frac{\partial \rho}{\partial \overline{x}_0} + \left(\frac{\partial c_{40}}{\partial M}\right)\left(\frac{\partial M}{\partial \overline{x}_0}\right) + \left(\frac{\partial c_{40}}{\partial v_S}\right)\left(\frac{\partial v_S}{\partial \overline{x}_0}\right) ,$$

where

$$\frac{\partial c_{40}}{\partial M} = \frac{1}{m} \left\{ (\rho A_r) \left[\frac{\partial c_A}{\partial M} + 2c_{L_\alpha}^2\left(\frac{\partial \eta}{\partial M}\right) + 4\eta c_{L_\alpha}\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right] \right.$$

$$+ \left(\frac{\rho A_r}{2}\right)\left(\frac{1}{v_S}\right)\left[\frac{\partial^2 c_A}{\partial M^2} + 2\left(\frac{\partial^2 \eta}{\partial M^2}\right) c_{L_\alpha}^2 + 8c_{L_\alpha}\left(\frac{\partial \eta}{\partial M}\right)\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right.$$

$$\left.\left. + 4\eta\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)^2 + 4\eta c_{L_\alpha}\left(\frac{\partial c_{L_\alpha}^2}{\partial M^2}\right)\right] \right\}$$

and

$$\frac{\partial c_{40}}{\partial v_S} = -\left(\frac{\rho A_r}{2}\right)\left(\frac{1}{v_S^2}\right)\left[\left(\frac{\partial c_A}{\partial M}\right) + 2\left(\frac{\partial \eta}{\partial M}\right) c_{L_\alpha}^2 + 4\eta c_{L_\alpha}\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right] .$$

Also,

$$\frac{\partial c_{80}}{\partial \overline{x}_0} = \left(\frac{c_8}{\rho}\right)\left(\frac{\partial \rho}{\partial \overline{x}_0}\right) + \left(\frac{\rho A_r}{2m}\right)\left(c_A + 2\eta c_{L_\alpha}^2\right)\frac{\overline{v}_r^T \left(\frac{\partial \overline{v}_r}{\partial \overline{x}_0}\right)}{|\overline{v}_r|}$$

$$+\left(\frac{\rho A_r}{2m}\right) |\overline{v}_r| \left[\frac{\partial c_A}{\partial M} + 2\left(\frac{\partial \eta}{\partial M}\right) c_{L_\alpha}^2 + 4\eta c_{L_\alpha}\left(\frac{\partial c_{L_\alpha}}{\partial M}\right)\right] .$$

Again note that $\frac{\partial^2 c_A}{\partial M^2}$ needed to compute $\frac{\partial c_{A_0}}{\partial M}$ is assumed to be available from the aerodynamic subroutine of Reference 1. This completes the evaluation of $\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}$ and thus also $\frac{\partial \bar{d}_1}{\partial \bar{x}_0}$. To continue the evaluation of $\frac{\partial \bar{x}}{\partial \bar{x}_0}$, the next term to be evaluated is $\frac{\partial c_2}{\partial \bar{x}_0}$. Since

$$c_2 = -\frac{1}{\dot{m}^2}\left\{\frac{m}{\sqrt{d_7}}\left[\ell n\left(\frac{\Phi}{\Phi_0}\right) - \ell n\left(\frac{m}{m_0}\right)\right] + \left(\frac{1}{\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right\} \quad ,$$

then

$$\frac{\partial c_2}{\partial \bar{x}_0} = -\frac{1}{\dot{m}^2}\left\{-\frac{m}{2(\sqrt{d_7})^3}\left(\frac{\partial d_7}{\partial \bar{x}_0}\right)\left[\ell n\left(\frac{\Phi}{\Phi_0}\right) - \ell n\left(\frac{m}{m_0}\right)\right]\right.$$

$$\left. -\frac{1}{2(\sqrt{d_9})^3}\left(\frac{\partial d_9}{\partial \bar{x}_0}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right\} - \frac{1}{\dot{m}^2}\left\{\frac{m}{\sqrt{d_7}}\left(\frac{\Phi_0}{\Phi}\right)\left[\frac{\partial\left(\frac{\Phi}{\Phi_0}\right)}{\partial \bar{x}_0}\right]\right.$$

$$\left. + \frac{1}{\sqrt{d_9}}\left(\frac{\Omega_0}{\Omega}\right)\left[\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{x}_0}\right]\right\} \quad .$$

To complete the evaluation of $\frac{\partial c_2}{\partial \bar{x}_0}$, the terms $\frac{\partial d_7}{\partial \bar{x}_0}$, $\frac{\partial d_9}{\partial \bar{x}_0}$, $\frac{\partial\left(\frac{\Phi}{\Phi_0}\right)}{\partial \bar{x}_0}$, and $\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{x}_0}$ must be evaluated. First,

$$d_7 = \left[\bar{u}_0^T + \dot{\bar{u}}_0^T\left(\frac{m_0}{\dot{m}}\right)\right]\bar{d}_1$$

and thus

$$\frac{\partial d_7}{\partial \bar{x}_0} = \left[\bar{u}_0^T + \dot{\bar{u}}_0^T\left(\frac{m_0}{\dot{m}}\right)\right]\frac{\partial \bar{d}_1}{\partial \bar{x}_0} + \left(\frac{m_0}{\dot{m}}\right)\bar{d}_1^T\left(\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}\right) \quad .$$

Then, since

$$d_9 = \left(\frac{1}{\dot{m}}\right)\dot{\bar{u}}_0 + \bar{d}_2 \quad ,$$

$$\frac{\partial d_9}{\partial \bar{x}_0} = \left(\frac{1}{\dot{m}}\right)\left[\dot{\bar{u}}_0^T\left(\frac{\partial \bar{d}_2}{\partial \bar{x}_0}\right) + \bar{d}_0^T\left(\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}\right)\right] \quad .$$

To evaluate $\frac{\partial \bar{d}_2}{\partial \bar{x}_0}$, note that

$$\bar{d}_2 = -[A_0][A_0]^T\dot{\bar{u}}_0\left(\frac{1}{\dot{m}}\right) \quad .$$

Thus,

$$\frac{\partial \bar{d}_2}{\partial \bar{x}_0} = -\left(\frac{2}{\dot{m}}\right)[A_0]\left\{\left(\frac{\partial [A_0]^T}{\partial \bar{x}_0}\right)\dot{\bar{u}}_0\right\} - \frac{1}{\dot{m}}[A_0][A_0]^T\left(\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}\right) \quad .$$

Note that the expression for $\left\{\left(\frac{\partial [A_0]^T}{\partial \bar{x}_0}\right)\dot{\bar{u}}_0\right\}$ is the same as the expression for $\left\{\frac{\partial [A_0]^T}{\partial \bar{x}_0}\,\bar{u}_0\right\}$ with the exception that $\bar{u}_0$ is replaced by $\dot{\bar{u}}_0$. This completes the expressions for $\frac{\partial d_7}{\partial \bar{x}_0}$ and $\frac{\partial d_9}{\partial \bar{x}_0}$. Also, since

$$d_8 = -\left(\frac{2}{\dot{m}}\right)\dot{\bar{u}}_0^T\,\bar{d}_1 \quad ,$$

then

$$\frac{\partial d_8}{\partial \bar{x}_0} = -\left(\frac{2}{\dot{m}}\right)\left[\dot{\bar{u}}_0^T\left(\frac{\partial \bar{d}_1}{\partial \bar{x}_0}\right) + \bar{d}_1^T\left(\frac{\partial \dot{\bar{u}}_0}{\partial \bar{x}_0}\right)\right]$$

which will be needed in the expressions for $\frac{\partial\left(\frac{\Phi}{\Phi_0}\right)}{\partial \bar{x}_0}$ and $\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{x}_0}$. Since

$$\Phi = 2\sqrt{d_7}\,\sqrt{X} + d_8 m + 2d_7$$

and

$$\Omega = 2\sqrt{d_9}\,\sqrt{X} + 2d_9 m + d_8 \quad ,$$

then

$$\frac{\partial\left(\frac{\Phi}{\Phi_0}\right)}{\partial \bar{X}_0} = \frac{1}{\Phi_0}\left(\frac{\partial \Phi}{\partial \bar{X}_0}\right) - \frac{\Phi}{(\Phi_0)^2}\left(\frac{\partial \Phi_0}{\partial \bar{X}_0}\right)$$

$$= \frac{1}{\Phi_0}\left[\frac{\sqrt{d_7}}{\sqrt{X}}\left(\frac{\partial X}{\partial \bar{X}_0}\right) + \left(\frac{\sqrt{X}}{\sqrt{d_7}}\right)\left(\frac{\partial d_7}{\partial \bar{X}_0}\right) + m\left(\frac{\partial d_8}{\partial \bar{X}_0}\right) + 2\left(\frac{\partial d_7}{\partial \bar{X}_0}\right)\right]$$

$$- \frac{\Phi}{(\Phi_0)^2}\left[\frac{\sqrt{d_7}}{\sqrt{X_0}}\left(\frac{\partial X_0}{\partial \bar{X}_0}\right) + \frac{\sqrt{X_0}}{\sqrt{d_7}}\left(\frac{\partial d_7}{\partial \bar{X}_0}\right) + m_0\left(\frac{\partial d_8}{\partial \bar{X}_0}\right) + 2\left(\frac{\partial d_7}{\partial \bar{X}_0}\right)\right]$$

and

$$\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{X}_0} = \left(\frac{1}{\Omega_0}\right)\left(\frac{\partial \Omega}{\partial \bar{X}_0}\right) - \frac{\Omega}{(\Omega_0)^2}\left(\frac{\partial \Omega_0}{\partial \bar{X}_0}\right)$$

$$= \frac{1}{\Omega_0}\left[\frac{\sqrt{d_9}}{\sqrt{X}}\left(\frac{\partial X}{\partial \bar{X}_0}\right) + \frac{\sqrt{X}}{\sqrt{d_9}}\left(\frac{\partial d_9}{\partial \bar{X}_0}\right) + 2m\left(\frac{\partial d_9}{\partial \bar{X}_0}\right) + \left(\frac{\partial d_8}{\partial \bar{X}_0}\right)\right]$$

$$- \frac{\Omega}{(\Omega_0)^2}\left[\frac{\sqrt{d_9}}{\sqrt{X_0}}\left(\frac{\partial X_0}{\partial \bar{X}_0}\right) + \frac{\sqrt{X_0}}{\sqrt{d_9}}\left(\frac{\partial d_9}{\partial \bar{X}_0}\right) + 2m_0\left(\frac{\partial d_9}{\partial \bar{X}_0}\right) + \left(\frac{\partial d_8}{\partial \bar{X}_0}\right)\right].$$

In the preceding expressions, the terms $\frac{\partial X}{\partial \bar{X}_0}$ amd $\frac{\partial X_0}{\partial \bar{X}_0}$ are evaluated as follows:

$$\frac{\partial X}{\partial \bar{X}_0} = \frac{\partial d_7}{\partial \bar{X}_0} + m\left(\frac{\partial d_8}{\partial \bar{X}_0}\right) + m^2\left(\frac{\partial d_9}{\partial \bar{X}_0}\right)$$

$$\frac{\partial X_0}{\partial \bar{X}_0} = \frac{\partial d_7}{\partial \bar{X}_0} + m_0\left(\frac{\partial d_8}{\partial \bar{X}_0}\right) + m^2\left(\frac{\partial d_9}{\partial \bar{X}_0}\right) \quad .$$

This completes the evaluation of the term $\frac{\partial c_2}{\partial \bar{X}_0}$. The next term to be evaluated in the expression for $\frac{\partial \bar{X}}{\partial \bar{X}_0}$ is $\frac{\partial \bar{d}_2}{\partial \bar{X}_0}$ which was evaluated already in order to obtain $\frac{\partial c_2}{\partial \bar{x}_0}$. Thus, the only term left unevaluated in the expression for $\frac{\partial \bar{x}}{\partial \bar{x}_0}$ is the term $\frac{\partial c_3}{\partial \bar{x}_0}$. To evaluate $\frac{\partial c_3}{\partial \bar{x}_0}$, note that

$$c_3 = \left(\frac{1}{m^2}\right)\left[\frac{m}{\sqrt{d_9}}\,\ell n\left(\frac{\Omega}{\Omega_0}\right) - \left(\frac{1}{d_9}\right)(\sqrt{X} - \sqrt{X_0}) - \left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right] .$$

Then

$$\frac{\partial c_3}{\partial \bar{x}_0} = \left(\frac{1}{m^2}\right)\Bigg\{-\frac{m}{2(\sqrt{d_9})^3}\left(\frac{\partial d_9}{\partial \bar{x}_0}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right) + \left(\frac{m}{\sqrt{d_9}}\right)\left(\frac{\Omega_0}{\Omega}\right)\left[\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{x}_0}\right]$$

$$+\left(\frac{1}{d_9^2}\right)\left(\frac{\partial d_9}{\partial \bar{x}_0}\right)(\sqrt{X} - \sqrt{X_0}) - \frac{1}{d_9}\left(\frac{1}{2\sqrt{X}}\right)\left(\frac{\partial X}{\partial \bar{x}_0}\right)$$

$$-\frac{1}{2\sqrt{X_0}}\left(\frac{\partial X_0}{\partial \bar{x}_0}\right)\Bigg] - \ell n\left(\frac{\Omega}{\Omega_0}\right)\left[\frac{1}{2\sqrt{d_9}}\left(\frac{\partial d_8}{\partial \bar{x}_0}\right) - \frac{d_8}{4(\sqrt{d_9})^3}\left(\frac{\partial d_9}{\partial \bar{x}_0}\right)\right]$$

$$-\left(\frac{d_8}{2\sqrt{d_9}}\right)\left(\frac{\Omega_0}{\Omega}\right)\left[\frac{\partial\left(\frac{\Omega}{\Omega_0}\right)}{\partial \bar{x}_0}\right]\Bigg\} .$$

This completes the detailed derivation of $\frac{\partial \bar{x}}{\partial \bar{x}_0}$. A similar approach is used to derive an expression for $\frac{\partial \bar{x}}{\partial t_0}$. That is, since

$$\bar{x} = \bar{x}_0 + \bar{v}_0\,(t - t_0) + \bar{g}_0\left[\frac{(t - t_0)^2}{2}\right] + c_1\bar{b}_0 + c_2\bar{d}_1 + c_3\bar{d}_2 \quad ,$$

then

$$\frac{\partial \bar{x}}{\partial t_0} = -\bar{v}_0 + \bar{g}_0\,(t - t_0) + \left(\frac{\partial c_1}{\partial t_0}\right)\bar{b}_0 + \left(\frac{\partial c_2}{\partial t_0}\right)\bar{d}_2 + \left(\frac{\partial c_3}{\partial t_0}\right)\bar{d}_3 \quad .$$

Note that $\frac{\partial \bar{b}_0}{\partial t_0}$, $\frac{\partial \bar{d}_1}{\partial t_0}$, and $\frac{\partial \bar{d}_2}{\partial t_0}$ are zero because $\bar{b}_0$, $\bar{d}_1$, and $\bar{d}_2$ are constants.

The terms $\frac{\partial c_1}{\partial t_0}$, $\frac{\partial c_2}{\partial t_0}$, and $\frac{\partial c_3}{\partial t_0}$ must now be evaluated. Since

$$c_1 = \left[m \,\ell n\left(\frac{m}{m_0}\right) - (m - m_0)\right]\left(\frac{1}{\dot{m}^2}\right) \quad ,$$

then

$$\frac{\partial c_1}{\partial t_0} = \frac{1}{\dot{m}^2}\left[\left(\frac{\partial m}{\partial t_0}\right) \ell n \left(\frac{m}{m_0}\right) + \left(\frac{1}{m^2}\right)\left(\frac{\partial m}{\partial t_0}\right) - \left(\frac{\partial m}{\partial t_0}\right)\right] \quad .$$

Note that $m = m_0 - \dot{m}(t - t_0)$ so that $\frac{\partial m}{\partial t_0} = \dot{m}$.

Thus,

$$\frac{\partial c_1}{\partial t_0} = \frac{1}{\dot{m}}\left[\ell n \left(\frac{m}{m_0}\right) + \frac{1}{\dot{m}^2} - 1\right] \quad .$$

Next, since

$$c_2 = -\left(\frac{1}{\dot{m}^2}\right)\left\{\frac{m}{\sqrt{d_7}}\left[\ell n \left(\frac{\Phi}{\Phi_0}\right) - \ell n \left(\frac{m}{m_0}\right)\right] + \frac{1}{\sqrt{d_9}} \ell n \left(\frac{\Omega}{\Omega_0}\right)\right\} \quad ,$$

then

$$\frac{\partial c_2}{\partial t_0} = -\frac{1}{\dot{m}^2}\left\{\frac{\frac{\partial m}{\partial t_0}}{\sqrt{d_7}}\left[\ell n \left(\frac{\Phi}{\Phi_0}\right) - \ell n \left(\frac{m}{m_0}\right)\right] + \frac{m}{\sqrt{d_7}}\left(\frac{1}{\Phi}\right)\left(\frac{\partial \Phi}{\partial t_0}\right)\right.$$
$$\left. - \frac{1}{m}\left(\frac{\partial m}{\partial t_0}\right)\right] + \frac{1}{\Omega\sqrt{d_9}}\left(\frac{\partial \Omega}{\partial t_0}\right)\right\} \quad .$$

Note that

$$\Phi = 2\sqrt{d_7}\sqrt{X} + d_8 m + 2d_7 \quad ,$$

$$\Omega = 2\sqrt{d_9}\sqrt{X} + 2d_9 m + d_8 \quad ,$$

and

$$X = d_7 + d_8 m + d_9 m^2 \quad .$$

Thus,

$$\frac{\partial \Phi}{\partial t_0} = \left(\frac{\sqrt{d_7}}{\sqrt{X}}\right)\left(\frac{\partial X}{\partial t_0}\right) + d_8\left(\frac{\partial m}{\partial t_0}\right) \quad ,$$

$$\frac{\partial \Omega}{\partial t_0} = \frac{\sqrt{d_9}}{\sqrt{X}}\left(\frac{\partial X}{\partial t_0}\right) + 2d_9\left(\frac{\partial m}{\partial t_0}\right) \quad ,$$

$$\frac{\partial X}{\partial t_0} = d_8\left(\frac{\partial m}{\partial t_0}\right) + 2d_9 m\left(\frac{\partial m}{\partial t_0}\right) \quad .$$

Then, since

$$c_3 = \frac{1}{m^2}\left[\frac{m}{\sqrt{d_9}}\ \ell n\left(\frac{\Omega}{\Omega_0}\right) - \left(\frac{1}{d_9}\right)(\sqrt{X} - \sqrt{X_0}) - \left(\frac{d_8}{2\sqrt{d_9}}\right)\ell n\left(\frac{\Omega}{\Omega_0}\right)\right] \quad ,$$

$$\frac{\partial c_3}{\partial t_0} = \frac{1}{m^2}\left[\frac{\frac{\partial m}{\partial t_0}}{\sqrt{d_9}}\ \ell n\left(\frac{\Omega}{\Omega_0}\right) + \left(\frac{m}{\sqrt{d_9}}\right)\left(\frac{1}{\Omega}\right)\left(\frac{\partial \Omega}{\partial t_0}\right) - \left(\frac{1}{d_9}\right)\left(\frac{1}{\sqrt{X}}\right)\left(\frac{\partial X}{\partial t_0}\right) - \left(\frac{d_8}{2\sqrt{d_9}}\right)\left(\frac{1}{\Omega}\right)\left(\frac{\partial \Omega}{\partial t_0}\right)\right] \quad .$$

This completes the expression for $\frac{\partial \overline{x}}{\partial t_0}$.

The preceding work has shown in detail how to compute $\frac{\partial \overline{x}}{\partial \overline{x}_0}$ and $\frac{\partial \overline{x}}{\partial t_0}$. The other terms in the partial derivative transition matrix $\frac{\partial Z}{\partial Z_0}$ and the vector $\frac{\partial Z}{\partial t_0}$ can be written in a similar manner and, in fact, most of these terms are much easier to write than the ones used for illustration purposes. In the next section, the partial derivative transition matrix $\frac{\partial Z}{\partial Z_0}$ will be used in deriving a variation-of-parameters integration scheme.

VARIATION OF PARAMETERS INTEGRATION

For more information about this approach to variation of parameters integration, see References 2, 3, and 4. Variation of parameters integration is to be used to obtain $Z_a(t)$ where $Z_a(t)$ is the vector

$$Z_a(t) = \begin{bmatrix} \bar{x}_a(t) \\ \bar{v}_a(t) \\ q_a(t) \\ \bar{\lambda}_a(t) \\ \bar{u}_a(t) \\ \gamma_a(t) \end{bmatrix}$$

defined by the differential equations as given in Reference 1 (also listed in a previous section, Development of Closed-Form Solutions) and the initial conditions $Z_a(t_0) = Z_0$.

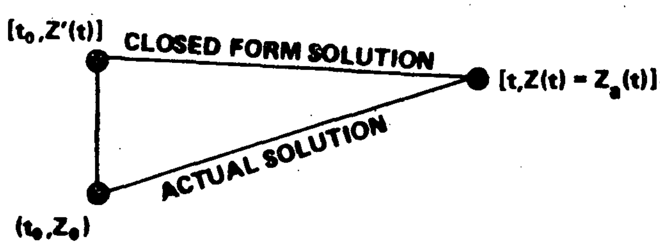
Now define $\dot{Z}_P = \dot{Z}_a - \dot{Z}$, where $\dot{Z}$ is the approximate vector differential equation used to derive either the ascent or the reentry closed-form solution and $\dot{Z}_a$ is the actual vector differential equation mentioned above. For example, in the ascent case

$$\dot{Z} = \begin{bmatrix} \bar{v} \\ \left(\frac{1}{q}\right) \dfrac{[A_0][A_0]^T[\bar{u}_0 + \dot{\bar{u}}_0(t - t_0)]}{|[A_0]^T[\bar{u}_0 + \dot{\bar{u}}_0(t - t_0)]|} + \left(\frac{1}{q}\right)\bar{b}_0 + \bar{g}_0 \\ -\dot{m} \\ \dot{\bar{\lambda}}_0 \\ \dot{\bar{u}}_0 \\ \frac{1}{q^2}\ |[A_0]^T[\bar{u}_0 + \dot{\bar{u}}_0(t - t_0)]| + [\bar{u}_0^T + \dot{\bar{u}}_0^T(t - t_0)]\bar{b}_0 \end{bmatrix}$$

and

$$\dot{Z}_a = \begin{bmatrix} \bar{v} \\ \left(\frac{1}{q}\right)[A]\left(\frac{[A]^T\bar{u}}{|[A]^T\bar{u}|}\right) + \left(\frac{1}{q}\right)\bar{b} + \bar{g} \\ -\dot{m} \\ \left\{-\left(\frac{1}{q}\right)\left[\left(\frac{\bar{u}^T[A]}{|\bar{u}^T[A]|}\right)\frac{\partial([A]^T\bar{u})}{\partial\bar{x}} + \bar{u}^T\left(\frac{\partial\bar{b}}{\partial\bar{x}}\right)\right] - \bar{u}^T\left(\frac{\partial\bar{g}}{\partial\bar{x}}\right)\right\}^T \\ \left\{-\bar{\lambda}^T - \left(\frac{1}{q}\right)\left[\left(\frac{\bar{u}^T[A]}{|\bar{u}^T[A]|}\right)\frac{\partial([A]^T\bar{u})}{\partial\bar{v}} + \bar{u}^T\left(\frac{\partial\bar{b}}{\partial\bar{v}}\right)\right]\right\}^T \\ \left(\frac{1}{q^2}\right)\bar{u}^T\frac{[A][A]^T\bar{u}}{|[A]^T\bar{u}|} + \bar{b} \end{bmatrix}.$$

Then $\dot{Z}_a = \dot{Z} + \dot{Z}_p$. Now a vector $Z'(t)$ is defined to be a set of values which, when used as initial conditions in the closed-form solution, $Z(t)$ will make $Z(t) = Z_a(t)$. The following diagram might help to understand the definition of $Z'(t)$.



The diagram illustrates that $Z_a(t)$ is obtained by numerical integration of the actual differential equations with the initial conditions (t_0, Z_0). Also, a value $Z(t) = Z_a(t)$ is obtained using the closed-form solutions and the initial conditions $(t_0, Z'(t))$ because this is the definition of $Z'(t)$. Given a value of $Z_a(t)$ a value for $Z'(t)$ is obtained from the closed-form solutions by letting $Z_a(t) = Z_0$ and interchanging the role of t and t_0. To be more specific about the definition of $Z'(t)$ let the closed form solution $Z(t)$ be denoted more generally as

$$Z(t) = g(t, t_0, Z_0) \quad .$$

Then from the definition of $Z'(t)$

$$Z_a(t) = g[t, t_0, Z'(t)]$$

or

$$Z'(t) = g[t_0, t, Z_a(t)]$$

since the closed-form solution works backwards or forwards when t is replaced by t_0. From the expression

$$Z'(t) = g[t_0, t, Z_a(t)] \quad ,$$

a differential equation for $Z'(t)$ can be written. This differential equation is the variation-of-parameters differential equation, and from the diagram the initial conditions for the variation of parameters differential equation can be seen to be $Z'(t_0) = Z_0$ because Z_0 used in the closed-form solution over a zero length of time will still be Z_0, which is also $Z_a(t_0)$. To write out the differential equation for $Z'(t)$ explicitly, note that

$$\frac{d}{dt}[Z'(t)] = \frac{\partial g[t_0, t, Z_a(t)]}{\partial t_0} + \frac{\partial g[t_0, t, Z_a(t)]}{\partial t} + \left\{ \frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)} \right\} \dot{Z}_a \quad .$$

Now

$$\dot{Z}_a = \dot{Z} + \dot{Z}_P \quad .$$

Thus,

$$\frac{d}{dt}\left(Z'(t)\right)=\left[\frac{\partial g[t_0, t, Z_a(t)]}{\partial t_0}+\frac{\partial g[t_0, t, Z_a(t)]}{\partial t}+\left\{\frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)}\right\}\dot{Z}\right]+\left\{\frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)}\right\}\dot{Z}_P .$$

Note that $Z_a(t)$ is used to evaluate $\dot{Z}$ and $\dot{Z}_P$. Now it will be shown that the term

$$\left[\frac{\partial g(t_0, t, Z_a(t))}{\partial t_0}+\frac{\partial g[t_0, t, Z_a(t)]}{\partial t}+\left\{\frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)}\right\}\dot{Z}\right]=0,$$

so that

$$\frac{d}{dt}[Z'(t)]=\left\{\frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)}\right\}\dot{Z}_P .$$

To see that the term just mentioned is zero, note that the initial conditions for the closed-form solution are constants. That is,

$$Z_0 = g[t_0, t, Z(t)]$$

for all $[t, Z(t)]$ on a particular closed-form trajectory. Thus,

$$\frac{d Z_0}{dt}=0 ,$$

and using the above expression

$$\frac{dZ_0}{dt}=\left[\frac{\partial g(t_0, t, Z(t))}{\partial t_0}+\frac{\partial g[t_0, t, Z(t)]}{\partial t}+\left\{\frac{\partial g[t_0, t, Z(t)]}{\partial Z(t)}\right\}\dot{Z}\right] .$$

Now if a particular $Z_a(t)$ is considered to define a closed-form solution, then $Z_a(t)$ substituted in the right-hand side of the above expression will give the result that

$$\left\{ \frac{\partial g(t_0, t, Z_a(t))}{\partial t_0} + \frac{\partial g[t_0, t, Z_a(t)]}{\partial t} + \left\{ \frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)} \right\} \dot{Z} \right\} = 0 \quad .$$

Thus, the variation-of-parameters differential equation can be written as

$$\frac{d}{dt}(Z'(t)) = \left\{ \frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)} \right\} \dot{Z}_P \quad .$$

Now note that since

$$Z_a(t) = g[t, t_0, Z'(t)] \text{ and } Z'(t) = g[t_0, t, Z_a(t)] \quad ,$$

then

$$\frac{\partial g[t_0, t, Z_a(t)]}{\partial Z_a(t)} = \left\{ \frac{\partial g[t, t_0, Z'(t)]}{\partial Z'(t)} \right\}^{-1} \quad ,$$

and $\left\{ \frac{\partial g[t, t_0, Z'(t)]}{\partial Z'(t)} \right\}$ is the partial derivative transition matrix $\left(\frac{\partial Z}{\partial Z_0} \right)$ evaluated with $Z_0 = Z'(t)$. Thus, the final form of the variation-of-parameters differential equation is

$$\frac{d}{dt}[Z'(t)] = \left[\frac{\partial Z(t)}{\partial Z_0} \right]^{-1}_{Z_0 = Z'(t)} \dot{Z}_P \quad ,$$

where $\dot{Z}_P$ is calculated using $Z_a(t)$ given by the closed-form solution applied to $Z'(t)$. That is,

$$Z_a(t) = g[t, t_0, Z'(t)]$$

so that the right-hand side of the variation-of-parameters differential equation can be seen to depend only on $[t, Z'(t)]$. Thus, numerical integration of this differential equation will yield $Z'(t)$ at any $t > t_0$. Then the closed-form solution can be used to determine $Z_a(t) = g[t, t_0, Z'(t)]$ at any $t > t_0$. Since $\dot{Z}_P$ is usually very small, the differential equation for $Z'(t)$ is usually integrated much more rapidly than the differential equation for $Z_a(t)$. An additional benefit of variation-of-parameters integration as explained here is that the transition partial derivative matrix $\left[\dfrac{\partial Z_a(t)}{\partial Z_0}\right]$ can be approximated very easily. To see how this is done, suppose a value of $\dfrac{\partial Z_a(t_f)}{\partial Z_0}$ is desired. Then the interval (t_0, t_f) can be divided into as many subintervals (t_0, t_1), (t_1, t_2), $\cdots$, (t_n, t_f) as desired. If this division is fine enough,

$$\frac{\partial Z_a(t_1)}{\partial Z_0} \approx \left(\frac{\partial Z(t_1)}{\partial Z'(t_1)}\right)$$

$$\frac{\partial Z_a(t_2)}{\partial Z_a(t_1)} \approx \frac{\partial Z(t_2)}{\partial Z'(t_2)}$$

$$\vdots$$

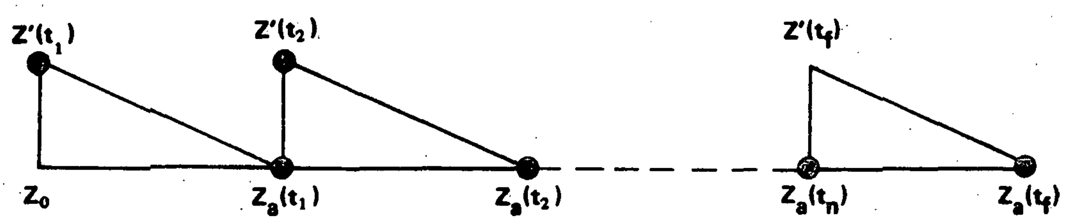
$$\frac{\partial Z_a(t_f)}{\partial Z_a(t_n)} \approx \frac{\partial Z(t_f)}{\partial Z'(t_f)} \quad .$$

Then

$$\frac{\partial Z_a(t_f)}{\partial Z_0} \approx \left[\frac{\partial Z(t_f)}{\partial Z'(t_f)}\right] \cdots\cdot \left[\frac{\partial Z(t_2)}{\partial Z'(t_2)}\right] \left[\frac{\partial Z(t_1)}{\partial Z'(t_1)}\right] \quad .$$

Note that the quantities $\left[\dfrac{\partial Z(t_1)}{\partial Z'(t_1)}\right]$, $\left[\dfrac{\partial Z(t_2)}{\partial Z'(t_2)}\right]$, $\cdots$, $\left[\dfrac{\partial Z(t_f)}{\partial Z'(t_f)}\right]$ are evaluated using the closed-form partial derivative transition matrix $\left(\frac{\partial Z}{\partial Z_0}\right)$ with Z_0 equal $Z'(t_1)$, $Z'(t_2)$, $\cdots$, $Z'(t_f)$, respectively.

If a numerical integration routine with step-size control is used to integrate $\frac{d}{dt}[Z'(t)]$, then the subinterval boundaries t_1, t_2, $\cdots$, t_n will be selected automatically to assure that across each interval the approximation in the partial derivatives is accurate. This is because the requirement of a specified accuracy on the integration of the variation-of-parameters differential equation will not allow $\dot{Z}_P$ to become very large in any particular interval of time. Then the beginning of a new interval can just be considered as a new set of initial conditions. The following diagram might help to visualize this approach:



The diagram exaggerates the differences $Z'(t_1)$ and Z_0, $Z'(t_2)$ and $Z_a(t_1)$, $\cdots$, $Z'(t_f)$ and $Z_a(t_n)$; but if the subinterval boundaries $(t_1, t_2, \cdots , t_n)$ are selected by a numerical integration routine with a step-size control (based on accuracy requirements), the differences will be small and the approximations $\left[\frac{\partial Z_a(t_1)}{\partial Z_0}\right] \approx \left[\frac{\partial Z(t_1)}{\partial Z'(t_1)}\right]$, $\left[\frac{\partial Z_a(t_2)}{\partial Z_a(t_1)}\right] \approx \left[\frac{\partial Z(t_2)}{\partial Z'(t_2)}\right]$, $\cdots$ $\left[\frac{\partial Z_a(t_f)}{\partial Z_a(t_n)}\right] \approx \left[\frac{\partial Z(t_f)}{\partial Z'(t_f)}\right]$ will be accurate.

This completes the explanation of the derivation of a variation-of-parameters numerical integration technique to obtain $Z_a(t)$ and the partial derivative transition matrix $\frac{\partial Z_a(t)}{\partial Z_0}$. In the next section, this information will be used to solve what is normally called the optimization boundary value problem. An algorithm that can solve the optimization boundary value problem rapidly enough is, in effect, a guidance scheme.

GUIDANCE SCHEME DEVELOPMENT

The problem of guiding a space vehicle consists of determining an optimal trajectory from a measured set of initial state conditions to a specified set of boundary conditions. To formulate this problem more precisely, define the state vector denoted by $X(t)$ as follows:

$$X(t) = \begin{bmatrix} \overline{x}(t) \\ \overline{v}(t) \\ q(t) \end{bmatrix}$$

where $t_0 \leq t \leq t_f$.

Then $\dot{X}(t)$ can be given by the closed-form solutions or obtained from numerical integration of the actual nonlinear differential equations. Also define the multiplier vector $P(t)$ as follows:

$$P(t) = \begin{bmatrix} \overline{\lambda}(t) \\ \overline{u}(t) \\ \gamma(t) \end{bmatrix} .$$

Closed-form solutions for $X(t)$ and $P(t)$ can be used or the nonlinear differential equations for $\dot{X}$ and $\dot{P}$ can be integrated numerically by the variation-of-parameters technique to yield $X(t)$ and $P(t)$. In either case the matrices $\left[\frac{\partial X(t)}{\partial P_0}\right]$ and $\left[\frac{\partial P(t)}{\partial P_0}\right]$ are available because $\left[\frac{\partial X(t)}{\partial P_0}\right]$ and $\left[\frac{\partial P(t)}{\partial P_0}\right]$ are a part of the partial derivative transition matrices $\left[\frac{\partial Z(t)}{\partial Z_0}\right]$ and $\left[\frac{\partial Z_a(t)}{\partial Z_0}\right]$ developed in the previous sections. The matrices $\left[\frac{\partial X(t)}{\partial P_0}\right]$ and $\left[\frac{\partial P(t)}{\partial P_0}\right]$ are used to set up a Newton's method iteration to solve the guidance boundary value problem. To see how this is done, define a set of physical boundary conditions to be satisfied by a space vehicle as follows:

$$F[X(t_f), t_f] = 0 .$$

Then the optimization boundary value problem consists of determining an optimal trajectory $X(t)$ from a measured set of initial conditions $X(t_0)$ such that $X(t_f)$ satisfies the physical boundary conditions given above. In order to determine an optimal trajectory $X(t)$ satisfying the above conditions, the vector $P(t)$ must be introduced so that the control vector $\overline{p}$ shown in a previous section, Development of Closed-Form Solutions, can be defined in terms of $X(t)$ and $P(t)$. Also, from the necessary conditions of optimization theory, the following transversality conditions associated with the physical boundary condition $F[X(t_f), t_f]$ must be satisfied:

1. $$P(t_f)^T = -\frac{\partial J(t_f)}{\partial X(t_f)} - \rho^T \left\{ \frac{\partial F[X(t_f), t_f]}{\partial X(t_f)} \right\} ,$$

where $J(t_f)$ is the scalar function to be minimized and ρ^T is a new vector of constant multipliers with the same dimension as the vector function $F[X(t_f), t_f]$

2. $$\dot{J}(t_f) + \rho^T \dot{F}[X(t_f), t_f] = 0 .$$

Now the physical boundary conditions and the above transversality conditions can be combined to form the total set of boundary conditions denoted by

$$G[X(t_f), P(t_f), \rho, t_f] = 0 .$$

Note that the dimension of the vector function G is equal to the dimension of the vector $X(t)$ plus the dimension of the vector $F[X(t_f), t_f]$ plus one. Then the optimization boundary value problem consists of determining $P(t_0)$, ρ, and t_f such that the boundary conditions

$$G[X(t_f), P(t_f), \rho, t_f] = 0$$

are satisfied by an optimal trajectory $X(t)$ originating from a set of measured initial conditions $X(t_0)$. The following is a reiteration of the previous statement of the optimization boundary value problem.

1. The term $X(t_0)$ is measured by the space vehicle.

2. Values of $P(t_0)$, ρ, and t_f must be determined so that when $X(t_0)$ and $P(t_0)$ are used as initial conditions for the closed-form solutions or the variation-of-parameters integration, the resulting solution $X(t)$ and $P(t)$ when evaluated at t_f will satisfy the total set of boundary conditions

$$G[X(t_f), P(t_f), \rho, t_f] = 0 .$$

To solve the optimization boundary problem as stated by the preceding sentences, note that $X(t_f)$ and $P(t_f)$ depend on the initial conditions $X(t_0)$ and $P(t_0)$. Thus, a Taylor Series expansion of the vector function $G[X(t_f), P(t_f), \rho, t_f]$ can be made about a set of guessed values (denoted by $P(t_0)^*$, ρ^*, and t_f^*) for the parameters $P(t_0)$, ρ, and t_f. That is, assume that the initial conditions $X(t_0)$, $P^*(t_0)$ yield $X^*(t_f^*)$, $P^*(t_f^*)$.

Then

$$G[X(t_f), P(t_f), \rho, t_f] = G[X^*(t_f^*), P^*(t_f^*), \rho^*, t_f^*$$
$$+ \left\{\frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial P(t_0)}\right\}_* \Delta P(t_0)$$
$$+ \left\{\frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial \rho}\right\}_* \Delta \rho$$
$$+ \left\{\frac{d(G[X(t_f), P(t_f), \rho, t_f]}{dt}\right\}_* \Delta t_f + \cdots ,$$

where the subscript $*$ of the braces means that the entire term in the braces is evaluated with $*$ values. The terms $\left\{\frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial \rho}\right\}_*$ and $\left\{\frac{d(G[X(t_f), P(t_f), \rho, t_f])}{dt}\right\}_*$ can be evaluated explicitly, but the term $\left\{\frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial P(t_0)}\right\}_*$ can only be evaluated using the matrices $\left[\frac{\partial X(t_f)}{\partial P(t_0)}\right]_*$ and $\left[\frac{\partial P(t_f)}{\partial P(t_0)}\right]_*$.

That is,

$$\left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial P(t_0)}\right\}_* = \left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f}{\partial X(t_f)}\right\}_* \left[\frac{\partial X(t_f)}{\partial P(t_0)}\right]_*$$

$$+ \left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial P(t_f)}\right\}_* \left[\frac{\partial P(t_f)}{\partial P(t_0)}\right]_* .$$

Now a modified Newton's iteration formula can be developed from the above Taylor Series expansion truncated after the first-order terms. To do this, note that the desired value for $G[X(t_f),\ P(t_f),\ \rho,\ t_f]$ is zero so that

$$G[X(t_f),\ P(t_f),\ \rho,\ t_f] = \alpha\ G[X^*(t_f^*),\ P^*(t_f^*),\ \rho^*,\ t_f^*] \quad ,$$

where $0 \leq \alpha < 1$, can be substituted into the truncated Taylor Series expansion for $G[X(t_f),\ P(t_f),\ \rho,\ t_f]$ to give corrections $\Delta P(t_0)$, $\Delta\rho$, and Δt_f which, when added to $P^*(t_0)$, ρ^*, and t_f^*, will produce a value of $G[X(t_f),$ $P(t_f),\ \rho,\ t_f]$ nearer zero. That is,

$$\alpha\ G[X^*(t_f^*),\ P^*(t_f^*),\ \rho^*,\ t_f^*] = G[X^*(t_f^*),\ P^*(t_f^*),\ \rho^*,\ t_f^*]$$

$$+ \left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial P(t_0)}\right\}_* \Delta P(t_0)$$

$$+ \left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \rho}\right\}_* \Delta\rho$$

$$+ \left\{\frac{d(G[X(t_f),\ P(t_f),\ \rho,\ t_f])}{dt}\right\}_* \Delta t_f \quad ,$$

or

$$\begin{bmatrix} \Delta P(t_0) \\ \Delta\rho \\ \Delta t_f \end{bmatrix} = (\alpha - 1)[E]^{-1} \left\{ G[X^*(t_f^*),\ P^*(t_f^*),\ \rho^*,\ t_f^*] \right\} \quad ,$$

where

$$[E] = \left[\left\{ \frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial P(t_0)} \right\}_*, \left\{ \frac{\partial G[X(t_f), P(t_f), \rho, t_f]}{\partial \rho} \right\}_*, \left\{ \frac{d(G[X(t_f), P(t_f), \rho, t_f])}{dt} \right\}_* \right]$$

The above expression for the correction vector $\begin{bmatrix} \Delta P(t_0) \\ \Delta\rho \\ \Delta t_f \end{bmatrix}$ is used repetitively until the boundary conditions $G[X(t_f), P(t_f), \rho, t_f]$ are zero to a desired tolerance.

The preceding discussion completes the general explanation of the solution of the optimization boundary value problem.

For illustration purposes, this development will now be applied to a specific set of physical boundary values. That is, let

$$F[X(t_f), t_f] = \begin{bmatrix} \bar{x}_f^T \bar{x}_f - R_d^2 \\ \bar{v}_f^T \bar{v}_f - v_d^2 \\ \bar{x}_f^T \bar{v}_f - R_d v_d \cos \vartheta_d \end{bmatrix}$$

Then $F[X(t_f), t_f]$ is a three-dimensional vector and the quantities R_d, v_d, and ϑ_d are constant desired values for the position vector, velocity vector, and the angle between the two, respectively. Assume that $J = -q_f$ is to be minimized. Then the transversality conditions become

$$\bar{\lambda}^T(t_f) = -\rho^T \begin{bmatrix} 2\bar{x}_f^T \\ 0, 0, 0 \\ \bar{v}_f^T \end{bmatrix} ,$$

$$\bar{u}^T(t_f) = -\rho^T \begin{bmatrix} 0, \ 0, \ 0 \\ 2\,\bar{v}_f^T \\ \bar{x}_f^T \end{bmatrix} ,$$

$$\gamma(t_f) = 1 \quad ,$$

$$\bar{\lambda}^T(t_f)\ \dot{\bar{x}}(t_f) + \bar{u}^T(t_f)\ \dot{\bar{x}}(t_f) + \dot{q} = H(t_f) = 0 \quad .$$

When the transversality conditions are combined with the physical boundary condition, the following form for $G[X(t_f),\ P(t_f),\ \rho,\ t_f]$ results:

$$G[X(t_f),\ P(t_f),\ \rho,\ t_f] = \begin{bmatrix} \bar{x}_f^T\,\bar{x}_f - R_d^2 \\ \bar{v}_f^T\,\bar{v}_f - v_d^2 \\ \bar{x}_f^T\,\bar{v}_f - R_d v_d \cos\vartheta_d \\ \left\{ 2\,\bar{x}_f,\ \begin{bmatrix}0\\0\\0\end{bmatrix},\ \bar{v}_f \right\} \bar{\rho} + \bar{\lambda}_f \\ \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix},\ 2\,\bar{v}_f,\ \bar{x}_f \right\} \bar{\rho} + \bar{u}_f \\ \bar{\lambda}_f^T\,\dot{\bar{x}}_f + \bar{u}_f^T\,\dot{\bar{v}}_f + \dot{q}_f \end{bmatrix} .$$

From the above expression for the vector $G[X(t_f),\ P(t_f),\ \rho,\ t_f]$, the expression for the components of the [E] matrix can be obtained. That is

$$[E] = \left[\left\{ \frac{\partial G[X(t_0),\ P(t_f),\ \rho,\ t_f]}{\partial \bar{\lambda}_0} \right\}_*,\ \left\{ \frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \bar{u}_0} \right\}_*,\ \left\{ \frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \rho} \right\}_*,\ \left\{ \frac{dG[X(t_f),\ P(t_f),\ \rho,\ t_f]}{dt} \right\}_* \right]$$

where

$$
\left\{ \frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \bar{\lambda}_0} \right\}_* = \begin{Bmatrix} 2\,\bar{x}_f^T \\ [0,\ 0,\ 0] \\ \bar{v}_f^T \\ \rho_1 [I]_{3\times 3} \\ \rho_3 [I]_{3\times 3} \\ -\dot{\bar{\lambda}}_f^T \end{Bmatrix}_* \left(\frac{\partial \bar{x}_f}{\partial \bar{\lambda}_0} \right)_*
$$

$$
+ \begin{Bmatrix} [0,\ 0,\ 0] \\ 2\bar{v}_f^T \\ \bar{x}_f^T \\ \rho_3 [I]_{3\times 3} \\ \rho_2 [I]_{3\times 3} \\ -\dot{\bar{u}}_f^T \end{Bmatrix}_* \left(\frac{\partial \bar{v}_f}{\partial \bar{\lambda}_0} \right)_*
$$

$$
+ \begin{Bmatrix} [0]_{3\times 3} \\ [I]_{3\times 3} \\ [0]_{3\times 3} \\ \dot{\bar{x}}_f^T \end{Bmatrix}_* \left(\frac{\partial \bar{\lambda}_f}{\partial \bar{\lambda}_0} \right)_* + \begin{Bmatrix} [0]_{3\times 3} \\ [0]_{3\times 3} \\ [I]_{3\times 3} \\ \dot{\bar{v}}_f^T \end{Bmatrix}_* \left(\frac{\partial \bar{u}_f}{\partial \bar{\lambda}_0} \right)_*
$$

$$\left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \bar{u}_0}\right\}_* = \text{Same as} \left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \bar{\lambda}_0}\right\}$$

with $\left(\frac{\partial \bar{x}_f}{\partial \bar{\lambda}_0}\right)_*$, $\left(\frac{\partial v_f}{\partial \bar{\lambda}_0}\right)_*$, $\left(\frac{\partial \lambda_f}{\partial \bar{\lambda}_0}\right)_*$, and $\left(\frac{\partial \bar{u}_f}{\partial \bar{\lambda}_0}\right)_*$ replaced by $\left(\frac{\partial \bar{x}_f}{\partial \bar{u}_0}\right)_*$, $\left(\frac{\partial \bar{v}_f}{\partial \bar{u}_0}\right)_*$, $\left(\frac{\partial \bar{\lambda}_f}{\partial \bar{u}_0}\right)_*$, $\left(\frac{\partial \bar{u}_f}{\partial \bar{u}_0}\right)_*$, respectively

$$\left\{\frac{\partial G[X(t_f),\ P(t_f),\ \rho,\ t_f]}{\partial \rho}\right\}_* = \begin{bmatrix} [0]_{3\times 3} \\ \left\{2x_f,\ \begin{bmatrix}0\\0\\0\end{bmatrix}\ \bar{v}_f\right\} \\ \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},2\bar{v}_f,\ \bar{x}_f\right\} \\ 0 \end{bmatrix}_*$$

$$\left\{\frac{d(G[X(t_f),\ P(t_f),\ \rho,\ t_f])}{dt}\right\}_* = \begin{bmatrix} 2\bar{x}_f^T\ \dot{\bar{x}}_f \\ 2\bar{v}_f^T\ \dot{\bar{v}}_f \\ \dot{\bar{x}}_f^T\ \bar{v}_f + \bar{v}_f^T\ \bar{x}_f \\ \left\{2\dot{\bar{x}}_f,\ \begin{bmatrix}0\\0\\0\end{bmatrix},\ \dot{\bar{v}}_f\right\}\rho + \dot{\bar{\lambda}}_f \\ \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\ 2\,\bar{v}_f,\ \dot{\bar{x}}_f\right\}\rho + \bar{u}_f \\ 0 \end{bmatrix}_*$$

The preceding work completes the explanation of a Newton's iteration procedure for solving the optimization boundary value problem. Note that this Newton's iteration procedure can be used with either the closed-form solutions or the variation-of-parameters solutions.

CONCLUSIONS

Closed-form solutions which approximate the optimal motion of a space vehicle during powered ascent flight and unpowered reentry flight have been developed. Also, the entire system of partial derivative transition matrices for these closed-form solutions have been developed. This work allows a variation-of-parameters integration technique to be used to obtain more accurate representations of the motion of the space vehicle in both powered ascent and unpowered reentry. Also, the optimal guidance boundary value problem is formulated, and a Newton's method algorithm developed for solving the guidance boundary value problem is explained using either the closed-form solutions or the more accurate variation-of-parameters solution.

Peter Leung of Northrop Space Laboratories has checked and developed a computer program which evaluates the ascent closed-form solutions. Table 1 shows how these results compare with the numerical integration of the more accurate equations of motion given in Reference 1. The computer programming of the rest of the results developed in this report is still in progress. When these computer programs are complete, more numerical results will be published.

TABLE 1. COMPARISON OF THE CLOSED-FORM SOLUTIONS WITH NUMERICAL INTEGRATION

	Closed Form			Integrated		
Time	X	Y	Z	X	Y	Z
22	0.63738747 + 7	0.18121454 + 5	0.85866896 + 4	0.63738747 + 7	0.18121454 + 5	-0.85866896 + 4
32	0.63745911 + 7	0.21337892 + 5	0.60443498 + 4	0.63745912 + 7	0.21337813 + 5	-0.60445188 + 4
42	0.63756431 + 7	0.24550894 + 5	0.33760306 + 4	0.63756429 + 7	0.24551021 + 5	-0.33742134 + 4
52	0.63770691 + 7	0.27759840 + 5	0.47793139 + 3	0.63770583 + 7	0.27760691 + 5	-0.43855118 + 3
62	0.63789303 + 7	0.30964589 + 5	0.27074054 + 4	0.63788549 + 7	0.30966492 + 5	0.29080094 + 4
72	0.63812037 + 7	0.34165602 + 5	0.61665552 + 4	0.63810252 + 7	0.34168231 + 5	0.68085122 + 4
82	0.63831400 + 7	0.37366812 + 5	0.94994361 + 4	0.63835064 + 7	0.37366038 + 5	0.11373455 + 5
92	0.63857204 + 7	0.40554535 + 5	0.13476222 + 5	0.63862412 + 7	0.40559817 + 5	0.16713725 + 5
102	0.63894056 + 7	0.43729538 + 5	0.18676276 + 5	0.63892262 + 7	0.43748812 + 5	0.22996022 + 5
112	0.63948008 + 7	0.46869919 + 5	0.26809333 + 5	0.63924649 + 7	0.46932034 + 5	0.30416618 + 5
122	0.64011926 + 7	0.49979512 + 5	0.38033105 + 5	0.63959483 + 7	0.50108495 + 5	0.39179462 + 5
132	0.64076463 + 7	0.53069318 + 5	0.51212514 + 5	0.63996550 + 7	0.53277267 + 5	0.49488624 + 5
142	0.64138176 + 7	0.56148146 + 5	0.65572488 + 5	0.64035538 + 7	0.56437501 + 5	0.61544426 + 5
152	0.64198688 + 7	0.59213492 + 5	0.81333299 + 5	0.6407606 + 7	0.59588429 + 5	0.75542579 + 5
162	0.64259480 + 7	0.62261394 + 5	0.98900421 + 5	0.64117725 + 7	0.62729350 + 5	0.91676542 + 5
172	0.64321014 + 7	0.65289268 + 5	0.11856847 + 6	0.64160090 + 7	0.65859617 + 5	0.11014212 + 6
182	0.64383263 + 7	0.68295253 + 5	0.14057935 + 6	0.64202754 + 7	0.68978612 + 5	0.13114317 + 6
192	0.64446034 + 7	0.71277881 + 5	0.16515746 + 6	0.64245332 + 7	0.72085734 + 5	0.15489807 + 6
202	0.64509097 + 7	0.74236029 + 5	0.19251695 + 6	0.64287468 + 7	0.75180388 + 5	0.18164613 + 6
212	0.64572357 + 7	0.77168434 + 5	0.22293114 + 6	0.64328839 + 7	0.78261975 + 5	0.21165510 + 6
217	0.64606446 + 7	0.78598231 + 5	0.23907382 + 6	0.64349297 + 7	0.79809055 + 5	0.22810101 + 6

TABLE 1. COMPARISON OF THE CLOSED-FORM SOLUTIONS WITH NUMERICAL INTEGRATION (Concluded)

	Closed Form			Integrated		
Time	DX	DY	DZ	DX	DY	DZ
22	0.56048389 + 2	0.32177915 + 3	0.25210962 + 3	0.56048389 + 2	0.32177915 + 3	0.25210962 + 3
32	0.87846396 + 2	0.32150735 + 3	0.32150735 + 3	0.87859367 + 2	0.32148528 + 3	0.25842334 + 3
42	0.12289574 + 3	0.32116034 + 3	0.27805594 + 3	0.12296807 + 3	0.32114990 + 3	0.27792957 + 3
52	0.16065035 + 3	0.32077072 + 3	0.31038646 + 3	0.16041884 + 3	0.32077854 + 3	0.31163871 + 3
62	0.20134651 + 3	0.32036494 + 3	0.35410576 + 3	0.19886025 + 3	0.32037786 + 3	0.36012848 + 3
72	0.24070592 + 3	0.31997632 + 3	0.40650781 + 3	0.23401907 + 3	0.31997316 + 3	0.42176564 + 3
82	0.25170141 + 3	0.31973783 + 3	0.94994361 + 3	0.26116143 + 3	0.31958755 + 3	0.49293030 + 3
92	0.27439934 + 3	0.31918283 + 3	0.51707913 + 3	0.28585984 + 3	0.31915457 + 3	0.57798703 + 3
102	0.31527824 + 3	0.31845226 + 3	0.61075750 + 3	0.31122489 + 3	0.31862765 + 3	0.68177390 + 3
112	0.38274696 + 3	0.31707888 + 3	0.76397618 + 3	0.33636540 + 3	0.31800002 + 3	0.80575490 + 3
122	0.45419445 + 3	0.31532521 + 3	0.96365678 + 3	0.35995159 + 3	0.31727650 + 3	0.95022179 + 3
132	0.50929255 + 3	0.31350852 + 3	0.11785169 + 3	0.38086184 + 3	0.31646375 + 3	0.11149508 + 4
142	0.54487177 + 3	0.31180873 + 3	0.13929982 + 4	0.39826015 + 3	0.31557025 + 3	0.12994672 + 4
152	0.56774594 + 3	0.31016108 + 3	0.16119764 + 4	0.41164087 + 3	0.31460362 + 3	0.15033797 + 4
162	0.58285366 + 3	0.30849439 + 3	0.18421694 + 4	0.42080986 + 3	0.31356988 + 3	0.17266667 + 4
172	0.59243625 + 3	0.30677542 + 3	0.20874917 + 4	0.42580765 + 3	0.31247319 + 3	0.19698356 + 4
182	0.59758633 + 3	0.30498838 + 3	0.23507534 + 4	0.42682393 + 3	0.31131586 + 3	0.22340033 + 4
192	0.59897326 + 3	0.30312573 + 3	0.26345644 + 4	0.42413577 + 3	0.31009874 + 3	0.25209546 + 4
202	0.59709891 + 3	0.30118542 + 3	0.29416555 + 4	0.41804705 + 3	0.30882207 + 3	0.28331003 + 4
212	0.59255474 + 3	0.29916346 + 3	0.32759376 + 4	0.40890438 + 3	0.30748497 + 3	0.31737904 + 4
217	0.59165778 + 3	0.29773193 + 3	0.34516889 + 4	0.40326003 + 3	0.30678803 + 3	0.33575837 + 4

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APPROVAL

CLOSED-FORM SOLUTIONS FOR ATMOSPHERIC FLIGHT WITH APPLICATIONS TO SHUTTLE GUIDANCE

By Hugo L. Ingram

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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