DIFFUSION OF CHARGED PARTICLES IN A RANDOM MAGNETIC FIELD

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The quantity described above by equation (34) can be accurately represented by

$$3(V_{01}/V)^2 = \frac{3(5-2q)}{(4-q)^2} ,$$

which leads to a more useful expression for the diffusion coefficient

$$D = \frac{V^2}{A} \frac{2(5-2q)}{(2-q)(3-q)(4-q)^2}$$

replacing equation (35). Equation (44) becomes

$$D = \frac{6.66\text{x}10^{28}(5-2q)}{(2-q)(3-q)(4-q)^2} \left(\frac{B}{2.09\text{x}10^7 \ f_o}\right)^q \frac{V_w^{\ q-1}}{P_{xx}} \beta R^{2-q}.$$

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ABSTRACT

When charged particles move in a random magnetic field superimposed upon a relatively large constant field, their pitch angle distribution can be calculated to any desired precision by an iterative approximation proceedure. Improved knowledge of the pitch angle distribution and of the characteristic time for relaxation of anisotropy leads to an accurate expression for the coefficient of diffusion parallel to the mean field.

I. INTRODUCTION

Scattering of charged particle trajectories by random magnetic fields controls the propogation of cosmic rays in the galaxy and in the solar system. The basic microscopic theory of this phenomenon is very complete (Jokipii, 1966; Hall and Sturrock, 1967; Hasselmann and Wibberenz, 1968), but the simplest application of this theory to diffusion has met difficulties that are not resolved and the exploration of higher order transport phenomena has not yet begun. One manifestation of the difficulties is the fact that two different methods of calculating the coefficient of diffusion parallel to the average "dc" field give different answers. In the first method (Jokipii, 1966; Hasselmann and Wibberenz, 1970), diffusive anisotropies, treated as a small perturbation on the particle distribution function, are found by familiar proceedures similar to those used in quantum mechanics. The second method (Jokipii, 1968a), which involves an expansion of the distribution function as a series of Legendre polynomials, embodies a well known and often used proceedure that is extensively discussed in standard references on transport theory. Which method is correct?

To answer this question, this paper develops transport theory in terms of eigenfunctions of the operator which describes pitch angle scattering (Section II). Many aspects of this treatment which are closely analogous to the standard development in terms of Legendre polynomials will not be discussed in detail. Instead, specific references to the volume by Weinberg and Wigner (1958) will be given where appropriate.

In section III, diffusion is discussed with special emphasis upon the important role played by the relaxation time for the lowest order anisotropic eigenfunction. An exact evaluation of this parameter leads to a precise expression for the parallel diffusion coefficient. However, it will be seen that the diffusion coefficient obtained from perturbation theory represents a fairly satisfactory approximation accurate enough for many purposes. The derivation of the diffusion coefficient from a Legendre expansion leads to inaccuracies which put severe limits on the validity of this approach. Diffusion perpendicular to the average field is not affected by the considerations presented here, and, consequently, will not be discussed.

II. EIGENFUNCTIONS OF THE SCATTERING OPERATOR

Jokipii (1966) has demonstrated that pitch angle scattering can be described by a Fokker Planck coefficient of the form

$$\varphi(\mu) = \frac{\langle \Delta\mu^2 \rangle}{\Delta t} = A\ |\mu|^{q-1}(1-\mu^2) \tag{1}$$

where μ is the cosine of the pitch angle. The parameter A is given by

$$A = \frac{V}{R^2}\, Q_{xx}\ \left(\frac{Vk_o}{\omega_o}\right)^q \tag{2}$$

where ω_o is the relativistic Larmor frequency, and V and R are, respectively, particle velocity and rigidity. The energy density contained in fluctuations of magnetic-field components perpendicular to the average field of wave number k in the interval dk is assumed to be $Q_{xx}(k_o/k)^q$ dk so that Q_{xx} represents the spectral density at the reference wavenumber k_o. In an interplanetary context, where spatial inhomogeneities convected by the solar wind past a satellite observing platform give rise to temporal fluctuations of magnetic intensity characterized by a power spectrum $P(f) = P_{xx}(f_o/f)^q$, the parameter A can be expressed as,

$$A = \frac{V_w V}{R^2}\ \ P_{xx}\left(\frac{2\pi V f_o}{V_w \omega_o}\right)^q \tag{3}$$

where V_w is the velocity of the solar wind, and P_{xx} is the spectral density of perpendicular magnetic field components at the reference frequency f_o (Jokipii, 1967; Jokipii and Coleman, 1968, Jokipii, 1971).

The Boltzmann equation can be expressed in terms of the Fokker-Planck coefficient $\varphi(\mu)$ as

$$\frac{\partial f}{\partial t} - \mu V \frac{\partial f}{\partial z} - \frac{1}{2}\frac{\partial}{\partial \mu}\varphi\frac{\partial f}{\partial \mu} = 0 \tag{4}$$

where $f(\mu,z,t)$ is the distribution function, z is distance parallel to the mean field, and t is time. In the treatment that follows, the streaming term $\mu V \partial f/\partial z$ will be regarded as a perturbation which is usually but not necessarily small. The equation

$$\frac{\partial}{\partial \mu} \varphi \frac{\partial R_K}{\partial \mu} + \left(\frac{2}{\tau_K}\right) R_K = 0 \tag{5}$$

defines eigenfunctions $R_K(\mu)$ and eigenvalues $(2/\tau_K)$ which are useful in describing the time evolution of the distribution function. Physically, τ_K is the relaxation time required for an anisotropy proportional to R_K to decay by a factor of (1/e). Sturm Liouville theory ensures that the R_K exist and form an orthogonal set satisfying boundary conditions that the finite value $R_K(1) = \pm R_K(-1)$ and that $R_K(0)$ is finite.

In the spirit outlined above, we write Equation 4 as

$$\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial}{\partial \mu} \varphi \frac{\partial f}{\partial \mu} = \epsilon(\mu,t,z) \tag{6}$$

where ϵ is the perturbation and express both ϵ and f as eigenfunction series

$$f = \Sigma f_K(t,z) R_K(\mu) \tag{7}$$

$$\epsilon = \Sigma \epsilon_K(t,z) R_K(\mu) \tag{8}$$

where the expression

$$\epsilon_K = \frac{\int_{-1}^{+1} \epsilon\, R_K \, d\mu}{\int_{-1}^{+1} R_K^{\,2} \, d\mu} \tag{9}$$

which follows from the orthogonality of the R_K describes the coefficient ϵ_K. A similar expression gives the coefficients f_K. Equations (7) and (8) can be inserted in equation (6) to yield, upon invoking the orthogonality of the R_K, a set of independent equations for the coefficients f_K

$$\frac{\partial f_K}{\partial t} + \left(\frac{1}{\tau_K}\right) f_K = \epsilon_K \tag{10}$$

whose solution is

$$f_K(t,z) = f_K(t=0)e^{-t/\tau_K} + \int_0^t \epsilon_K(s,z)e^{-(t-s)/\tau_K}\, ds \tag{11}$$

Equation (11) shows that the value of f_K at time t is primarily dependent upon values of ϵ within a period of width τ_K preceeding t. When $t \gg \tau_K$ and when ϵ_K does not change significantly in a time τ_K, $(1/\epsilon)(\partial\epsilon/\partial t) \ll (1/\tau_K)$, then f_K reaches an equilibrium value

$$f_K = \epsilon_K \, \tau_K \tag{12}$$

Under these conditions, equation (12), together with general theorems which require that the eigenvalues $(2/\tau_K)$ increase with K at least rapidly as K^2, guarantee that the series for f (Equation 7) converges absolutely and uniformly (Courant and Hilbert, 1953, p. 427). By the same token, the approach to equilibrium of higher order eigenfunctions is more rapid than of lower order ones. (See Equation 11.)

In the case of isotropic scattering, $\varphi = A(1-\mu^2)$, the eigenfunctions are Legendre polynomials $P_K(\mu)$ with eigenvalues $(2/\tau_K) = K(K+1)A$. When φ is complicated, the R_K are found by standard methods which also yield useful estimates of the eigenvalues. The qualitative behavior of the R_K is defined by

theorems which also apply to the P_K. Consequently, many features of the R_K can be predicted by analogy with the P_K. For example, eigenfunctions having even indices are even functions of μ while those having odd indices are odd functions, the number of zeros between $\mu = -1$ and $\mu = +1$ is equal to the index, etc. The lowest eigenfunction R_o = const. describes the isotropic component and is, of course, identical to P_o. The fact that the relaxation time τ_o is infinite means that an isotropic distribution is stable.

Because the ratio of successive terms in equation (7) depends not only upon the index K but also upon the boundary conditions and coefficients ε_K, no general statements can be made about how rapidly the series for f converges. However, for the case considered here, $\varepsilon \alpha \mu$, the properties of R_K just described imply that, for $K > 1$, the ε_K decrease rapidly with increasing K but are not, necessarily, identically zero as they are when $R_K = P_K$. This fact ensures the validity of the basic assumption of the diffusion approximation, that terms above R_1 can be neglected. In contrast, when the diffusion problem is described by another arbitrary set of orthogonal functions, f is essentially represented by the isotropic component plus a series expansion of R_1 in terms of the new basis set. In general, the convergence of this series, which depends only upon the nature of R_1 and not upon physical parameters, is not as rapid as that of the series in R_K. An example that will be discussed in detail later is the Legendre series for R_1 when $q \gtrsim 2$ in which the coefficients of P_K decrease as $(1/K)$, much less rapidly than the $(1/K^2)$ dependence implied by equation (12). These facts single out the R_K as a preferred set of orthogonal functions whose use leads to greatly simplied analysis.

To approximate R_1, we invoke the iterative method of Stodolla and Vianello (Hildebrand, 1949, Chapter 5). In the first iteration, a trial function w satisfying the boundary conditions but otherwise chosen arbitrarily is inserted in Equation 5

$$\frac{\partial}{\partial\mu}\varphi\frac{\partial x}{\partial\mu} = -\frac{2}{\tau}w\ (\mu) \tag{13}$$

to obtain, after two integrations, an improved estimate x of the eigenfunction

$$x(\mu) = \left(\frac{2}{\tau}\right)\int_0^{\mu}\frac{d\nu}{\varphi(\nu)}\int_{\nu}^{1}w(\rho)\ d\rho \tag{14}$$

In the first integration over ρ, the constant must be chosen so that, at $\nu = +1$, $\varphi(\nu)(\partial x/\partial\nu) = 0$. (See Equation 1.) To satisfy a similar restriction at $\nu = -1$, w must be an odd function of ρ. In the second integration over ν, the constant is chosen to make x an odd function of μ. An accurate (Hildebrand, 1949, Chapter 5, Eq. 89a) estimate for the eigenvalue $(2/\tau)$ is

$$\left(\frac{2}{\tau}\right) = \frac{\int_{-1}^{+1} w^2\ d\mu}{\int_{-1}^{+1} gw\ d\mu} \tag{15}$$

where the function $g = (\tau/2)x$ is the double integral which appears in equation (14). This process can be repeated by using x as the trial function, replacing w, to calculate an improved estimate y of the eigenfunction.

If $w = \mu$ and φ is described by equation (1) with $q < 2$, equation (14) gives:

$$x(\mu) = \left(\frac{2}{\tau}\right)\ \frac{1}{2}\int_0^{\mu}\frac{d\nu(1-\nu^2)}{\varphi(\nu)}$$

$$= \left(\frac{2}{\tau A}\right)\frac{1}{2(2-q)}\ \mu^{2-q} \tag{16}$$

an expression identical to that obtained by Hasselman and Wibberenz (1970) from perturbation theory. In equation (16), as well as in all other expressions for the first eigenfunction presented in this paper, only positive values of μ are considered. Since we are dealing with odd functions, corresponding results for negative μ are easily obtained by reflection. The approximate eigenvalue calculated from equation (15) is

$$\left(\frac{2}{\tau A}\right) = \frac{2}{3}\,(2-q)(4-q) \tag{17}$$

from which the diffusion coefficient first derived by Jokipii (1966) can be obtained by the procedures outlined in the next section.

In the second iteration, using equation (16) as the trial function, equation (13) is satisfied identically for $\mu \to 0$ provided that

$$\left(\frac{2}{\tau_1 A}\right) = (2-q)(3-q) \tag{18}$$

which is, therefore, an <u>exact</u> expression for the eigenvalue. The corresponding approximate eigenfunction, accurate enough for most purposes, is given by

$$y(\mu) = C_1 \int_0^\mu \nu^{1-q}\, \frac{1 - \nu^{3-q}}{1 - \nu^2}\, d\nu \tag{19}$$

where the normalization constant

$$(1/C_1) = \frac{1}{2}\left[\psi\left(\frac{5-2q}{2}\right) - \psi\left(\frac{1-q}{2}\right)\right] \tag{20}$$

can be defined in terms of the digamma function ψ in such a way that $y(1) = 1$. In the limit $\mu \to 0$,

$$y(\mu) = \frac{C_1}{(2-q)}\,\mu^{2-q} \tag{21}$$

while an accurate expression when $\mu \to 1$ is

$$y(\mu) = 1 - \frac{C_1}{4}\left[\frac{(5-q)}{(2-q)} - (1-q)\mu^{3-q} - \frac{(3-q)(1+q)}{(2-q)}\,\mu^{2-q}\right] \tag{22}$$

In Figure 1, the function y is plotted vs. μ over a range from $q = 2$, where the eigenfunction reduces to a step at the origin from -1 to +1, through $q = 1$, where the Legendre polynomial $P_1 = \mu$ is obtained, to $q = -1$, where a pronounced peaking in the forward direction (μ=1) is evident. For $q > 2$, the treatment given here breaks down and the eigenvalues diverge but for all other values of q including negative ones the method converges to well behaved eigenfunctions. For $q < 1$, the eigenfunction has zero slope at the origin but this slope is infinite when $1 < q < 2$. In the latter range, the shape of the eigenfunctionis evidently quite sensitive to the parameter q.

It is appropriate in the present context to introduce the Legendre expansion of the eigenfunctions:

$$R_K(\mu) = \sum_j a_{Kj} P_j(\mu) \tag{23}$$

and to express $y(\mu)$ in terms of its coefficients a_{1j}. This representation not only provides a useful method of evaluating certain parameters discussed later but also it makes explicit the relationship between the approach given here and the classical approach based upon Legendre expansions of the distribution function. When equation (19) is inserted into the formula for Legendre coefficients and when the order of integration over μ and ν is interchanged, the coefficients are given by,

$$\begin{aligned} a_{1j} &= C_1 \; (2j+1) \int_0^1 d\nu \; \nu^{1-q} \, \frac{1-\nu^{3-q}}{1-\nu^2} \int_\nu^1 P_j(\mu) \, d\mu \\ &= C_1 \, \frac{2j+1}{j(j+1)} \int_0^1 d\nu \; \nu^{1-q} (1-\nu^{3-q}) \, \frac{d}{d\nu} P_j(\nu) \end{aligned} \tag{24}$$

where the second integral, which involves only powers of ν, leads to the following expressions for the first few coefficients:

$$a_{11} = \frac{3}{2} \frac{(3-q)C_1}{(2-q)(5-2q)}$$

$$a_{13} = \frac{7}{4} \frac{(1-q)(3-q)\ (11-4q)\ C_1}{(2-q)(4-q)\ (5-2q)(7-2q)}$$

$$a_{15} = \frac{11}{54} \frac{(q-1)(3-q)(8-3q)(71+138q-36q^2)C_1}{(2-q)(4-q)(6-q)(5-2q)(7-2q)(9-2q)} \tag{25}$$

The coefficients for even indices are zero. The expressions in equation (22) are such that no singularities occur for $q < 2.5$. (Note that, as $q \to 2$, $C_1 \to (2-q)$.) When $q = 1$, $a_{13} = a_{15} = 0$, and the eigenfunction reduces to $P_1(\mu)$. When $q = -.4588$, a_{15} changes sign.

In Table I, numerical values of C_1, given by equation (20), of $C_1/(2-q)$ and of a_{11}, a_{13} and a_{15}, given by equation (25), are tabulated for several values of q. The fact that $C_1/(2-q)$ deviates from one by no more than 10% demonstrates that equation (21) is a surprisingly accurate representation of the eigenfunction. An equivalent statement is that even the first approximation described by equation (16) is fairly accurate. In Figure 2, which presents values of y given by equations (21) (dotted line) and (22) (dashed line) for $q = 1.9$, the Legendre series including terms up to P_5 (solid line) displays large oscillations around the correct eigenfunction. This manifestation of the Gibbs phenomenon is to be expected in a range of q where the eigenfunctions are virtually equivalent to the step function reached in the limit $q \to 2$. In fact the coefficients for $q = 1.9$ are nearly equal to those of the Legendre expansion of a step function: $a_{11} = (3/2)$, $a_{13} = -(7/8)$ and $a_{15} = (11/16)$. In contrast, at $q = 1.5$ (Figure 3), the Legendre representation is relatively accurate. The accuracy of the Legendre expansion has an important bearing on the applicability of classical transport theory, a point that will be discussed further in the next section.

III. DIFFUSION EQUATIONS

When τ_1 is finite, the approximate solution of the Boltzmann equation obtained by keeping only the R_0 and R_1 terms in equations (7) and (8) and by neglecting the time derivative $(\partial f_1/\partial t)$ is the diffusion equation. (See Weinberg and Wigner, 1958, p.231). In this case, equation (10) gives:

$$\frac{\partial f_o}{\partial t} = \epsilon_0 = -V_{01}\frac{\partial f_1}{\partial z} \tag{26}$$

$$\frac{\partial f_1}{\partial t} + \left(\frac{1}{\tau_1}\right) f_1 = \epsilon_1 = -V_{01}\frac{\partial f_o}{\partial z} \tag{27}$$

where the quantity V_{01} is a "matrix element" in the eigenfunction expansion of the perturbing term,

$$\epsilon = -V\mu\frac{\partial f}{\partial z} = -\Sigma R_J V_{KJ}\frac{\partial f_K}{\partial z} \tag{28}$$

$$V_{KJ} = V_{JK} = V\frac{\int_{-1}^{+1} R_J \mu R_K d\mu}{\left[\int_{-1}^{+1} R_J^2 d\mu \int_{-1}^{+1} R_K^2 d\mu\right]^{\frac{1}{2}}} \tag{29}$$

Note that, because μ is an odd function, $V_{KK} = V_{K,K+2} = V_{K,K+4} = \ldots = 0$, while $V_{K,K+1}$, $V_{K,K+3}$ etc are finite except for unusual cases when μR_K happens to be orthogonal to R_{K+1}, etc. This coupling, through μ, between odd and even eigenfunctions is a ubiquitous feature of transport theory (Weinberg and Wigner, 1968, p.246). The diffusion equation results from inserting f_1 calculated from equation (27), neglecting $(\partial f_1/\partial t)$, into equation (26)

$$\frac{\partial f_o}{\partial t} = \frac{\partial}{\partial z}\tau_1 V_{01}^2\frac{\partial f_o}{\partial z} = \frac{\partial}{\partial z} D\frac{\partial f_o}{\partial z} \tag{30}$$

with the diffusion coefficient D given by

$$D = \tau_1 V_{01}^2 = \tfrac{1}{2}\tau_1 V^2 \frac{\left[\int_{-1}^{+1} \mu R_1 d\mu\right]^2}{\int_{-1}^{+1} R_1^2 d\mu} \tag{31}$$

where the fact that $R_0 = 1$ has been used in applying equation (29). The integrals in equation (31) can be expressed in terms of the Legendre coefficients given by equation (25)

$$\int_{-1}^{+1} \mu R_1 d\mu = (2/3)\, a_{11}$$

$$(1/C_2)^2 = \int_{-1}^{+1} R_1^2 d\mu = (2/3)\, a_{11}^2 + (2/7)\, a_{13}^2 + (2/11)\, a_{15}^2 + \ldots \tag{32}$$

where C_2 is evidently the constant multiplier required to convert R_1 into a normalized eigenfunction. Numerical values of C_2 calculated from equation (32) appear in Table I. The diffusion coefficient is given by:

$$D = (1/3)\tau_1 V^2 \; [3(V_{01}/V)^2] \approx (1/3)\tau_1 V^2 \tag{33}$$

where the second approximate equality follows from the fact that the quantity

$$3(V_{01}/V)^2 = [\tfrac{2}{3}(a_{11})^2 (C_2)^2] = [1 + \tfrac{3}{7}(a_{13}/a_{11})^2 + \tfrac{3}{11}(a_{15}/a_{11})^2 + \ldots]^{-1} \tag{34}$$

is virtually equal to one. (See Table II.)

Equations (33) and (34) make evident the fact that D is critically dependent upon τ_1 but is quite insensitive to the exact shape of the eigenfunction. When $\tau_1 V$ is defined as the mean free path λ the elementary formula $D \approx \lambda V/3$ is recovered from equation (33).

With the exact value of τ_1 given by equation (18) the diffusion coefficient calculated from equation (33) is,

$$D = \frac{V^2}{A} \frac{2\,[3(V_{01}/V)^2]}{3(2-q)(3-q)} \approx \frac{(2/3)}{(2-q)(3-q)} \frac{V^2}{A} \tag{35}$$

an expression which can be compared to the result calculated with τ_1 given by equation (17) (This result is identical to that obtained from the perturbation approach discussed by Jokipii (1966) and by Hasselman and Wibberenz (1970).)

$$D_p = \frac{1}{(2-q)(4-q)} \frac{V^2}{A} \tag{36}$$

and to the expression derived from a Legendre expansion neglecting all terms beyond P_1 (Jokipii, 1968) (Weinberg and Wigner, 1958, p.233),

$$D_L = \frac{q(q+2)}{9} \frac{V^2}{A} \tag{37}$$

To illustrate the relationship between these three expressions for the diffusion coefficient, Table II gives the ratios (D_p/D) and (D_L/D) as a function of q. Also given in Table II is the quantity $[\ 3(V_{01}/V)^2\ ]$ which is the ratio between the exact and approximate forms of equation (35). The exact value of D is a little smaller than that given by equation (36) for $1 < q < 2$ but deviates by no more than 25% for $0<q<2$. This good accuracy reflects the fact that equation (36) gives the first approximation in a rapidly converging sequence. The result derived from a Legendre expansion is not very useful since equation (37) does not predict the divergence at q = 2 and incorrectly gives zero diffusion for q = 0. These discrepancies occur because the Legendre derivation arbitrarily sets equal to zero the coefficients a_{13}, a_{15}, etc. When these terms are taken into consideration, the diffusion coefficient can be represented by an infinite series in which equation (37) appears as the first term but this term is not necessarily the largest one. On the other hand, when $q = 1$, equations (35), (36) and (37) give identical values of D. In this special case, all three methods of calculating the distribution function are exact.

The streaming flux S is given by the integral

$$S = (1/2)V \int_{-1}^{+1} \mu f \, d\mu \tag{38}$$

which reduces to

$$S = (1/3)a_{11}Vf_1 + (1/3)\ a_{31}Vf_3 + \ldots \tag{39}$$

an expression which involves only the coefficients a_{K1} of the first Legendre polynomial $P_1 = \mu$. In contrast to the situation in classical transport theory where μ is orthogonal to all other Legendre functions, all odd R_K contribute to the flux. However, for reasons given earlier, the terms above f_1 will almost always be negligible.

The use of equations (27), (29) and (31) shows that the familiar expression for the diffusive flux.

$$S = -D \frac{\partial f_o}{\partial z} \tag{40}$$

retains its rigorous validity. On the other hand, an often used relationship between flux and anisotropy δ which is based upon a Legendre expansion neglecting terms above P_1

$$\delta_L = \frac{f_{max} - f_{min}}{f_{max} + f_{min}} = 3\ \frac{S}{Vf_o} \tag{41}$$

must be replaced by the correct expression

$$\delta = (3/a_{11})\ \frac{S}{Vf_o} \tag{42}$$

valid when $f_1 \gg f_3$, f_5, etc. For $q = 1$ equations (41) and (42) are identical but they give different results for $q \neq 1$. For $q \lesssim 2$, the anisotropy given by equation (42) for a given flux is 50% smaller than would be expected from equation (41). (See the numerical values of a_{11} appearing in Table I.) These remarks apply to diffusive anisotropies only. Convective anisotropies, such as those discussed by Forman (1970), result from a term in the distribution function accurately proportional to μ and, consequently, are described by equation (41). These difficulties could be avoided by defining the anisotropy in terms of the coefficient of P_1. This definition lacks the direct relationship with observed intensities embodied in the first equality of equation (41) but it is entirely workable and, in fact, corresponds to what is measured by the usual Fourier analysis of periodic variations in the rate recorded by a rotating detector.

When equations (26) and (27) are solved retaining the derivative $(\partial f_1/\partial t)$ and assuming that τ_1 does not depend upon z, we obtain

$$\frac{\partial^2 f_o}{\partial z^2} - \frac{1}{(V_{01})^2} \frac{\partial^2 f_o}{\partial t^2} = \frac{1}{D} \frac{\partial f_o}{\partial t} \tag{43}$$

a version of the well known "Telegraphers equation" which embodies both diffusive phenomena characterized by D and wavelike propogation at velocity V_{01}. Equation (43) is analogous to a result of classical transport theory (Weinberg and Wigner, 1958, p.235) which has been discussed in the context of interplanetary propogation of solar cosmic rays by Axford (1965) and by Fibich and Abraham (1965). These treatments obtain for the propogation velocity $(V/\sqrt{3})$ which coincides with V_{01} only for $q = 1$. However, the deviations of V_{01} from $(V/\sqrt{3})$ are quite small as can be seen from the values of $(\sqrt{3}V_{01}/V)^2$ given in Table II. Note that, for $q \neq 1$, the velocity V_{01} is always a little smaller than $(V/\sqrt{3})$.

IV DISCUSSION

The above analysis has treated in detail only the special case of a power law spectrum of magnetic irregularities described by the two parameters A and q. More complicated cases involving additional parameters could also be analyzed by the methods presented here. However, it is evident from equations (14) and (15) that both the first eigenfunction and the relaxation time τ_1 are given by integrals over μ which are rather insensitive to the exact behavior of φ. This insensitivity implies that the above results can be used with confidence in describing the interplanetary propagation of cosmic rays whose diffusion is controlled by a portion of the irregularity spectrum characterized by a power law with index $q \lesssim 1.5$ (Jokipii and Coleman, 1968). On the other hand, caution should be exercised in applying these results to situations in which propagation is controlled by a steeper power law (Sari and Ness, 1969) with an index close to the critical value q = 2 where divergences appear. Hasselman and Wibberenz (1970) have suggested that, for q > 2, the diffusion picture breaks down and convective effects dominate. The observed steepening of the interplanetary magnetic power spectrum implies that diffusive anisotropies at high energies are more concentrated around the average field direction than at low energies. Because the eigenfunction is quite sensitive to the value of q (See Figure 1.), this difference might be observable.

Jokipii (1968b) has already pointed out that the small amount of scattering near $\mu = 0$ implied by a Fokker-Planck coefficient described by equation (1) leads to a much larger relaxation time τ_1 than that predicted by setting τ_1 equal to the collision time as is suggested by a result of elementary kinetic theory. Equation (18), which predicts that $\tau_1 \to \infty$ as $q \to 2$, provides a rigorous basis for these ideas.

The derivation of a nearly exact expression (Equation (35).) removes any conceptual uncertainty about the value of the diffusion coefficient but the numerical results are not very different from the first order approximation. (Equation (36).). The following formula, based upon equations (3) and (35), gives the parallel diffusion coefficient D in cm^2/sec:

$$D = \frac{2.22\times10^{28}}{(2-q)(3-q)} \left[3 \left(\frac{V_{01}}{V}\right)^2\right] \left(\frac{B}{2.09\times10^7 f_o}\right)^q \frac{V_w^{\,q-1}}{P_{xx}} \beta R^{2-q} \tag{44}$$

where $\beta = (V/c)$, R is particle rigidity in GV, V_w is solar wind velocity in km/sec, B is the average field in gamma, and P_{xx} is the spectral density in $(\text{gamma})^2$/Hz of perpendicular fluctuations at a reference frequency of f_o Hz. Interpolation between the values given in Table II will yield an adequate estimate of $3(V_{01}/V)^2$. This expression is comparable to that given by Jokipii (1971, Equation (62).). In using equation (44), note that P_{xx} refers to a frequency spectrum defined, in terms of the Fourier transform of an auto-correlation function, for negative as well as positive frequencies. Observations are often given with the contributions of positive and negative frequencies combined. In this case, the reported spectrum is 2 P_{xx}.

Of greater significance, perhaps, than any of the above points is that the discussion of eigenfunctions given here provides a starting point for the investigation of higher order diffusive effects and of phenomena encountered when $q > 2$.

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FIGURE CAPTIONS

Figure 1: The shape of the function y, obtained by numerical integration of equation (19), is strongly dependent upon the parameter q.

Figure 2: For $q = 1.9$, the function y is accurately given near $\mu = 1$ by equation (22) (dashed line) and near $\mu = 0$ by equation (21) (dotted line). The Legendre expansion (solid line), including terms up to P_5, is not a very accurate representation of y.

Figure 3: For $q = 1.5$, all three representations of y give fairly accurate results.

Table I

Numerical values of some parameters

q	C_1	C_2	$C_1/(2-q)$	a_{11}	a_{13}	a_{15}
0	1.562	1.668	.781	.703	.322	-.036
0.5	1.326	1.458	.885	.829	.207	-.011
1.0	1.000	1.225	1.000	1.000	.000	.000
1.5	.548	.972	1.096	1.233	-.359	.213
1.8	.217	.826	1.086	1.397	-.662	.471
1.9	.105	.782	1.055	1.450	-.771	.577
2.0	-----	.744	1.000	1.500	-.875	.687
Equation	(20)	(32)	(21)	(25)	(25)	(25)

Table II

Comparison of Diffusion Parameters

q	(D_P/D)	(D_L/D)	$[3(V_{01}/V)^2]$
0	1.23	0	.917
0.5	1.10	.80	.974
1.0	1.00	1.00	1.000
1.5	.94	.68	.957
1.8	.92	.31	.887
1.9	.91	.16	.857
2.0	.90	0	.831

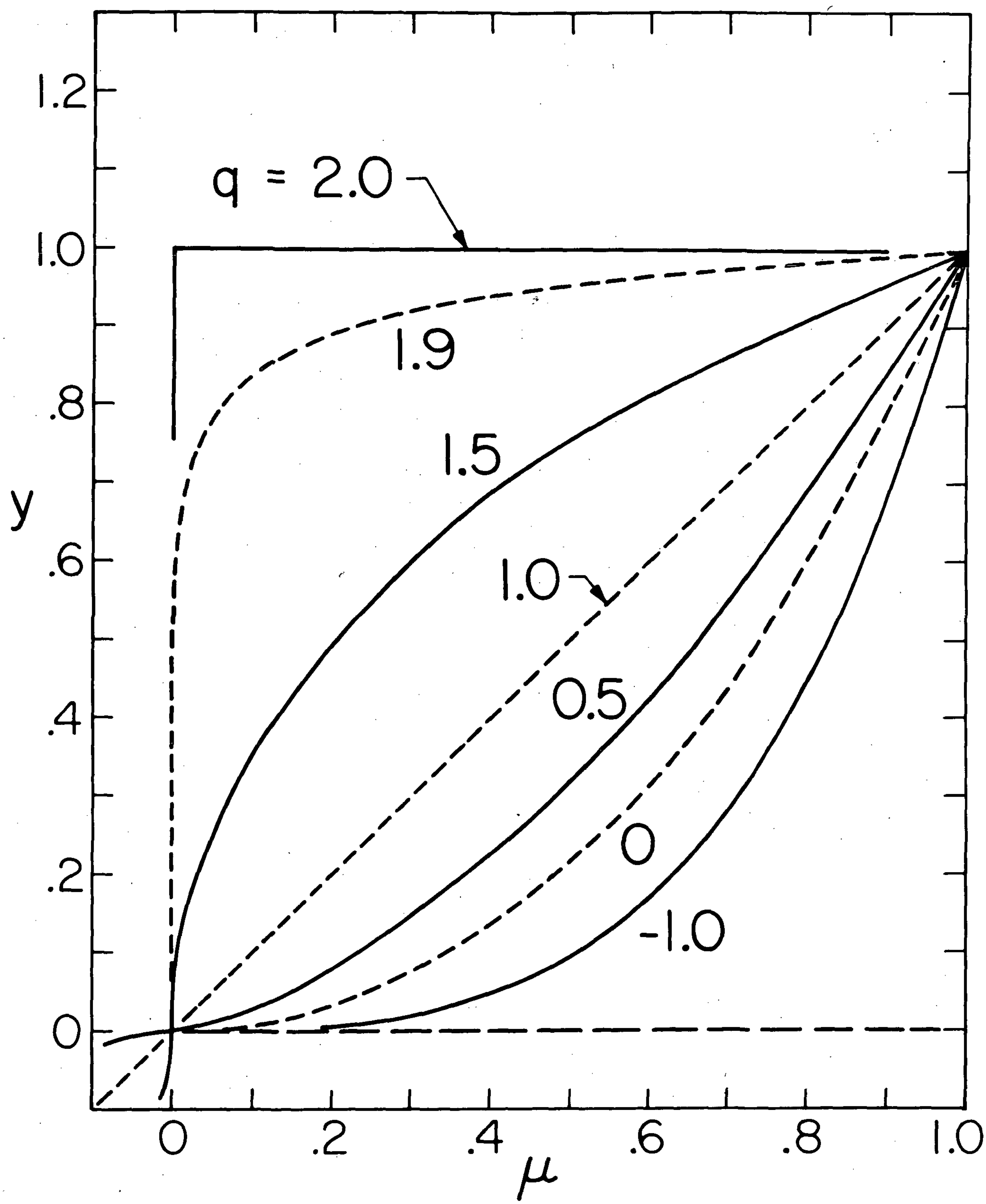

Figure 1

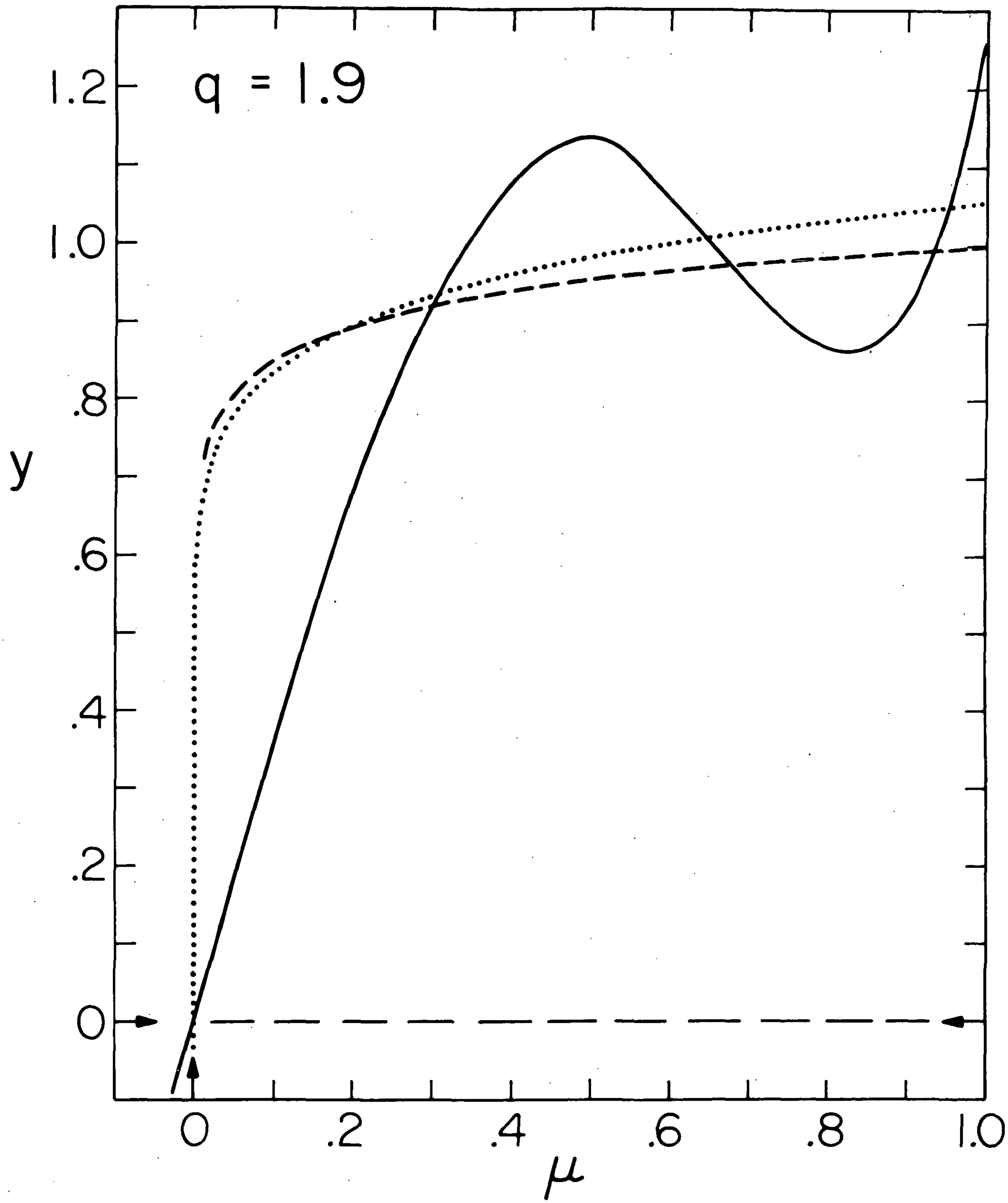

Figure 2

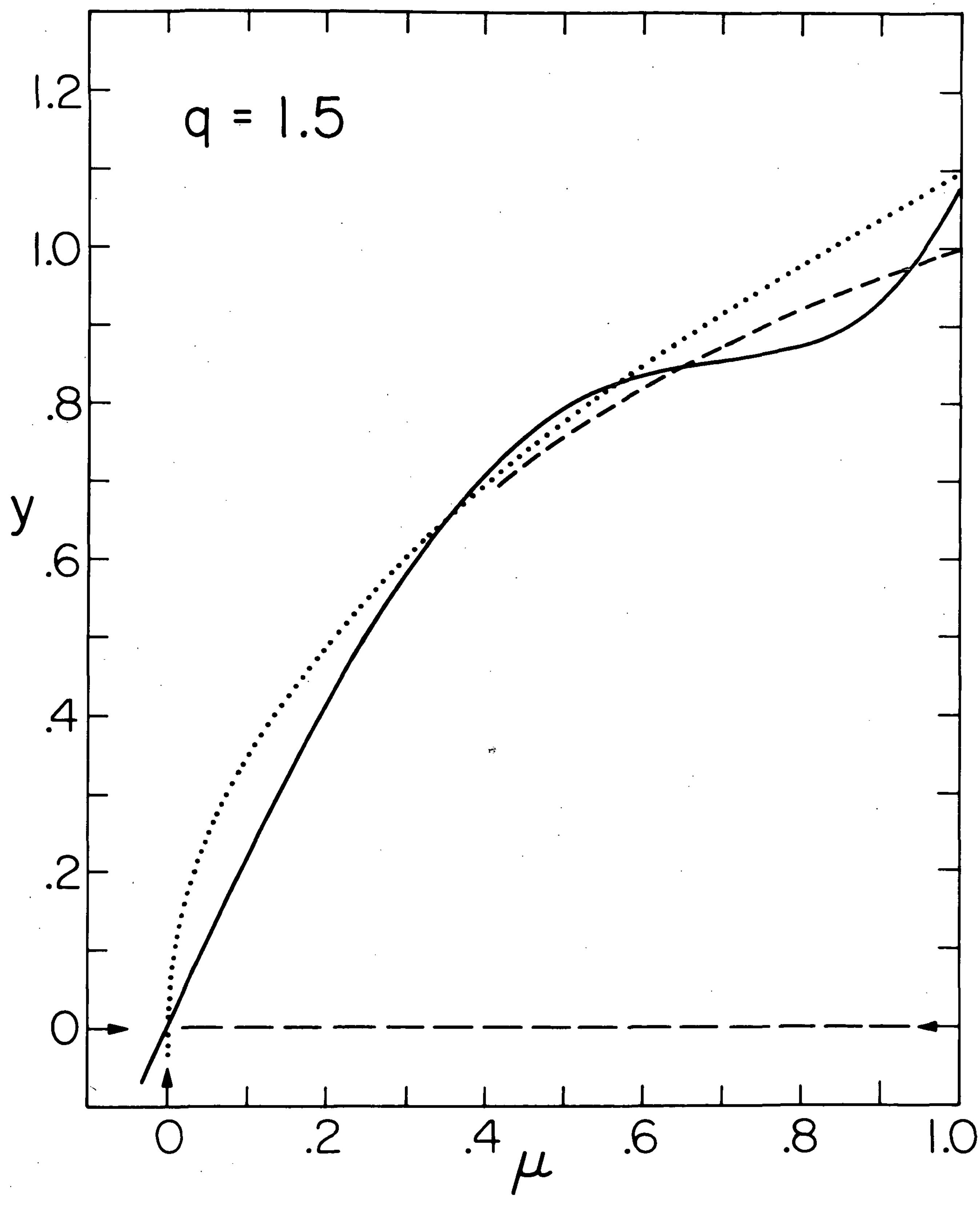

Figure 3