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RELATIVISTIC STELLAR STABILITY: AN EMPIRICAL APPROACH*

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ABSTRACT

The "PPN formalism" — which encompasses the post-Newtonian limit of nearly every metric theory of gravity — is used to analyze stellar stability. This analysis enables one to infer, for any given gravitation theory, the extent to which post-Newtonian effects induce instabilities in white dwarfs, in neutron stars, and in supermassive stars. It also reveals the extent to which our current empirical knowledge of post-Newtonian gravity (based on solar-system experiments) actually guarantees that relativistic instabilities exist. In particular, it shows that: (i) for "conservative theories of gravity", current solar-system experiments guarantee that the critical adiabatic index, Γ_{crit} , for the stability of stars against radial pulsations exceeds the Newtonian value of $4/3$:

$$\Gamma_{\text{crit}} = 4/3 + K M/R, \quad K \text{ positive and of order unity};$$

- (ii) for "nonconservative theories", current experiments do not permit any firm conclusion about the sign of $\Gamma_{\text{crit}} - 4/3$;
(iii) in the PPN approximation to every metric theory, the standard Schwarzschild criterion for convection is valid.

I. INTRODUCTION AND SUMMARY

Relativistic corrections to Newtonian gravity should induce dynamical instabilities in stars with adiabatic indices slightly greater than $4/3$. This fact was first discovered, within the framework of General Relativity (GR), by Chandrasekhar (1964a,b) and independently by Feynman [unpublished, but quoted in Fowler (1964)]. More recently Nutku (1969) has shown that the same type of instability is predicted by the Brans-Dicke theory of gravity (BDT), but that it is slightly weaker (stars are slightly more stable) than in GR. If the dynamical relativistic instability actually exists, as predicted by GR and BDT, then it plays a fundamental role in white dwarfs, in neutron stars, and in supermassive stars [see e.g. Thorne (1967) or Zel'dovich and Novikov (1971) for a review].

But it is conceivable that neither GR nor BDT is the correct relativistic theory of gravity. If so, might the relativistic instability not exist? Is it conceivable that relativistic effects would stabilize stars rather than destabilize them? William A. Fowler has asked this question of gravitation theorists so often at Caltech, that we have felt compelled to seek a firm answer. The most firm answer possible is one which relies heavily on experimental tests of relativistic gravitational effects, while assuming nothing (or almost nothing) about which relativistic theory of gravity is correct.

Of course, one cannot work in a complete theoretical vacuum. A minimal amount of theory is required to link the relativistic instability in stars to solar-system measurements of perihelion shift, light deflection, radar time delay, etc. That the amount of theory needed is small, however, one can see heuristically by noticing that both the perihelion shift and the

relativistic instability are caused by a relativistic strengthening of Newtonian gravitational forces. [Stronger gravity than predicted by Newton when a star contracts means greater force to pull the star on inward, i.e. means less stability; stronger gravity than predicted by Newton when a planet approaches close to the sun (perihelion) means greater force to "whip" the planet around, and a resultant advance of its perihelion.]

The purpose of this paper is to derive a quantitative measure of the extent to which solar-system experiments imply the existence of the dynamical relativistic instabilities in stars. The "minimal amount of theory" to be used in the derivation is the Parametrized Post-Newtonian ("PPN") Framework of Nordtvedt and Will (Will and Nordtvedt 1972; Will 1971a; Nordtvedt 1968).

The PPN Framework is a post-Newtonian theory of gravity with adjustable parameters. In Will's fluid version, it has nine PPN parameters, γ , β , α_1 , α_2 , α_3 , ξ_1 , ξ_2 , ξ_3 , and ξ_4 . The parameter γ measures curvature of the space-geometry; β measures the non-linearity of gravity; α_1 , α_2 , and α_3 measure "preferred-frame" effects; ξ_1 , ξ_2 , ξ_3 , and ξ_4 measure the effects resulting from a breakdown of conservation laws. For theories which have no "preferred-frame" effects, all α 's vanish (Nordtvedt and Will 1972; Will 1971b). For theories which have conservation laws for energy, momentum, angular momentum, and center-of-mass motion ("conservative theories"), all α 's and ξ 's vanish (Will 1971b). The post-Newtonian limit of every "metric theory of gravity"¹

¹Metric theories of gravity are a wide class of theories including (i) every theory that satisfies the equivalence principle (laws of physics in local Lorentz frames the same as in special relativity), and (ii) every

theory that the Caltech group has thus far examined and found to be complete, self-consistent, and in agreement with experiment. See Thorne, Will, and Ni (1971); Ni (1972) and Will (1972b) for full discussions.

known to us [except Whitehead's theory which is non-viable (Will 1971c)] is a special case of the PPN Framework, corresponding to particular values of the PPN parameters. Ni (1972) has calculated the values of the parameters for a variety of theories, including general relativity, the scalar-tensor theories of Bergmann-Wagoner, Nordtvedt, and Brans-Dicke-Jordan, the conformally-flat theories of Witrow-Morduch, Littlewood-Bergmann and Nordström, and the stratified theories of Einstein, Witrow-Morduch, Page-Tupper, Yilmaz, Papapetrou and Rosen.

Experiments to date have placed the following limits on the PPN parameters [see Thorne, Will, and Ni (1971) or Will (1972b) for detailed discussion; see also Nordtvedt and Will (1972)]:

$$\gamma = 1.04 \pm 0.08 \text{ [time delay and deflection experiments except that of Sramek (1972)]} \quad (1)$$

$$\gamma = 0.88 \pm 0.12 \text{ [Sramek's (1972) deflection experiment]} \quad (2)$$

$$\beta = 1.14^{+0.2}_{-0.3} \text{ [perihelion shift plus time delay experiments]} \quad (3)$$

$$|\xi_4 - \frac{1}{3} \xi_1 - \frac{1}{2} \alpha_3| \lesssim 0.4 \text{ [Kreuzer (1966, 1968) measurement of } m_{\text{active}}/m_{\text{passive}}]^2} \quad (4)$$

$$|\xi_3| \lesssim 0.05 \text{ [Kreuzer (1966, 1968) measurement of } m_{\text{active}}/m_{\text{passive}}]^2} \quad (5)$$

$$|\alpha_1| \lesssim 0.2 \text{ [Earth rotation rate experiments (Nordtvedt and Will 1972)]} \quad (6)$$

$$|\alpha_2| \lesssim 0.03 \text{ [Earth-tide measurements (Will 1971b)]} \quad (7)$$

$$|\alpha_3| \lesssim 2 \times 10^{-5} \text{ [perihelion shift observations (Nordtvedt and Will 1972)]} \quad (8)$$

²Kreuzer's (1966) analysis of his data was completely correct, despite a recent claim to the contrary by Gilvarry and Muller (1972). Gilvarry and Muller err in making a quadratic fit to Kreuzer's data, rather than restricting themselves to a linear fit as did Kreuzer. Kreuzer measured the expansion of his liquid relative to teflon over a wide temperature range and thereby showed experimentally that the quadratic correction to the linear behavior must be negligibly small over the small temperature range used for the experiment. Moreover, the magnitude of the quadratic term obtained by Gilvarry and Muller using their least-squares fits is ridiculously large for any but pathological materials. We thank R. H. Dicke for a helpful discussion of these points.

In this paper it is shown that for conservative theories of gravity current experimental limits on the PPN parameters — based on perihelion shift, time delay, and deflection experiments — guarantee the existence of the dynamical relativistic instability in stars; while for non-conservative theories the present, experimentally undetermined state of the two PPN parameters ζ_2 and ζ_4 makes it uncertain whether relativistic effects will actually stabilize or destabilize stars. In quantitative terms, a non-rotating spherically symmetric star with adiabatic index

$$\Gamma_1 = \left(\frac{\partial}{\partial} \frac{\ln p}{\ln n} \right)_s = \frac{\rho + p}{p} \left(\frac{\partial p}{\partial \rho} \right)_s \quad (9)$$

constant throughout its interior is unstable against adiabatic radial perturbations if and only if its radius R and geometrized mass ($2M \equiv$ Schwarzschild radius) satisfy

$$\Gamma_1 - 4/3 \leq K(2M/R) . \quad (10)$$

Here K is a constant that depends on the star's structure and upon the PPN parameters. If K is positive, there is a relativistic instability. If K is negative, relativity stabilizes the star. In the Newtonian limit $K = 0$. In GR and BDT K is positive and of order unity. Values of K for polytropic gas spheres, as evaluated in §IV of this paper, are tabulated in Tables 1 and 2.

Table 1 lists values of K for polytropic stars in the case of conservative theories of gravity. From the positive signs of the minimum values of K (column 3), we have the following conclusion: for conservative theories which are compatible with current solar-system experiments, relativistic corrections to Newtonian theory will always induce dynamical instabilities. It is interesting to note that γ has a positive contribution to K while β has a negative contribution; the same is true for the perihelion advance. This, together with the positivity of K , confirms the heuristic argument given at the beginning of this section.

Table 2 lists the values of K for the general PPN formalism and for several particular non-conservative theories. The third column gives minimum values of K corresponding to current experimental limits on the PPN parameters. If ξ_2 or ξ_4 (which are undetermined by experiments to date) were sufficiently negative, then K would be negative. For example, for the currently viable cases $\{\gamma = 0.76, \beta = 1.34, \xi_2 = -0.5, \xi_3 = \xi_4 = 0\}$, and $\{\gamma = \beta = 1, \xi_2 = -2.2, \xi_3 = \xi_4 = 0\}$ the value of K is negative.

Therefore we arrive at the following conclusion due to the lack of experimental information on ξ_2 and ξ_4 , it is inconclusive whether relativistic effects will actually stabilize or destabilize stars. From the last three columns, one may notice that the Vector-Metric theory (Will and Nordtvedt 1972) and the Papapetrou (1954a,b,c) theories have the same K-values as general relativity, while K-values for the Modified Yilmaz theory (Ni 1972) are all negative.

Other aspects of dynamical stellar pulsations are also investigated in this paper. The Schwarzschild criterion is found to hold for the onset of dynamical instability against non-radial oscillations (convection). Sufficient conditions for self-adjointness of the linearized pulsation equations are derived. These conditions together with the condition $\xi_1 = 0$ coincide with Will's conditions for the existence of ten post-Newtonian conserved integrals.

In §II the PPN formalism is summarized, the linearized pulsation equations are derived, and "preferred-frame terms" (which lead to vibrational-secular and other Machian-type instabilities) are separated out of the pulsation equations and reserved for study in a future paper. Section III derives a variational principle for dynamical stellar stability. Section IV derives the post-Newtonian conditions for the onset of a dynamical instability. Section V derives the Schwarzschild criterion for non-radial instabilities. Concluding remarks are made in §VI. An Appendix treats the problem of self-adjointness.

Throughout this paper, we follow closely the methods of Chandrasekhar (1965b), and we use geometrized units. The notations and conventions of this paper are the same as those of Chandrasekhar (1965b), and Will and

Nordtvedt (1972) — unless otherwise specified.

II. PPN FORMALISM AND EQUATIONS OF MOTION FOR SMALL OSCILLATIONS ABOUT EQUILIBRIUM

In the PPN formalism one describes the response of matter to gravity by the "local law of energy-momentum conservation"

$$\nabla \cdot \mathbf{T} = 0 \quad (11)$$

(where \mathbf{T} is the stress-energy tensor, and ∇ is the covariant derivative with respect to the PPN metric); and one describes the generation of gravity by matter in terms of the PPN metric (Will and Nordtvedt 1972):

$$\begin{aligned} g_{00} = & 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \xi_1)\Phi_1 + \xi_1 \mathcal{A} \\ & - 2[(3\gamma - 2\beta + 1 + \xi_2)\Phi_2 + (1 + \xi_3)\Phi_3 + 3(\gamma + \xi_4)\Phi_4] \\ & + (\alpha_1 - \alpha_2 - \alpha_3)w^2 U + \alpha_2 w^\alpha w^\beta U_{\alpha\beta} - (2\alpha_3 - \alpha_1)w^\alpha v_\alpha \quad , \quad (12) \end{aligned}$$

$$\begin{aligned} g_{0\alpha} = & \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \xi_1)v_\alpha + \frac{1}{2}(1 + \alpha_2 - \xi_1)w_\alpha \\ & + \frac{1}{2}(\alpha_1 - 2\alpha_2)w^\alpha U + \alpha_2 w^\beta U_{\alpha\beta} \quad , \end{aligned}$$

$$g_{\alpha\beta} = - (1 + 2\gamma U)\delta_{\alpha\beta} \quad .$$

Here w is the velocity of the chosen coordinate frame relative to the "preferred-frame" of the Universe (if any); and

$$U(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \quad , \quad (13)$$

$$\Phi_1(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) v^2(\underline{x}', t)}{|\underline{x} - \underline{x}'|} d\underline{x}' , \quad (14)$$

$$\Phi_2(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) U(\underline{x}', t)}{|\underline{x} - \underline{x}'|} d\underline{x}' , \quad (15)$$

$$\Phi_3(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) \Pi(\underline{x}', t)}{|\underline{x} - \underline{x}'|} d\underline{x}' , \quad (16)$$

$$\Phi_4(\underline{x}, t) = \int \frac{p(\underline{x}', t)}{|\underline{x} - \underline{x}'|} d\underline{x}' , \quad (17)$$

$$\mathcal{O}(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) [\underline{v}(\underline{x}', t) \cdot (\underline{x} - \underline{x}')]^2}{|\underline{x} - \underline{x}'|^3} d\underline{x}' , \quad (18)$$

$$V_\alpha(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) v_\alpha(\underline{x}', t)}{|\underline{x} - \underline{x}'|} d\underline{x}' , \quad (19)$$

$$W_\alpha(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) v_\beta(\underline{x}', t) (x_\beta - x'_\beta) (x_\alpha - x'_\alpha)}{|\underline{x} - \underline{x}'|^3} d\underline{x}' , \quad (20)$$

$$U_{\alpha\beta}(\underline{x}, t) = \int \frac{\rho(\underline{x}', t) (x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\underline{x} - \underline{x}'|^3} d\underline{x}' . \quad (21)$$

Here ρ is rest-mass density, p is pressure, and Π is specific internal energy all measured in the matter's rest frame, and $v_\alpha = dx^\alpha/dt$ is the matter's coordinate velocity.

The equations of hydrodynamics governing a perfect fluid follow from equations (11), (12) and the form of the stress-energy tensor (Will 1972a):

$$\begin{aligned} & \frac{\partial}{\partial t}(\sigma v^\alpha) + \frac{\partial}{\partial x^\beta}(\sigma v^\alpha v^\beta) - \rho \frac{\partial U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \left\{ p[1 + (3\gamma - 1)U] \right\} \\ & + \rho \frac{d}{dt} [(5\gamma - 1) U v^\alpha - \frac{1}{2} (4\gamma + 4 + \alpha_1) v_\alpha - \frac{1}{2} \alpha_1 U w^\alpha] \end{aligned} \quad (22)$$

$$\begin{aligned}
& - \frac{1}{2}(\alpha_2 - \xi_1 + 1) \rho \frac{\partial}{\partial t}(w_\alpha - v_\alpha) \\
& + \frac{1}{2} \rho [(4\gamma + 4 + \alpha_1) v^\beta + (\alpha_1 - 2\alpha_3) w^\beta] \frac{\partial v_\beta}{\partial x^\alpha} \\
& - \rho \frac{\partial}{\partial x^\alpha} [2\phi - \frac{1}{2} \xi_1 \sigma - \frac{1}{2} \alpha_2 w^\alpha w^\beta U_{\alpha\beta} + \alpha_2 w^\beta x_{,\beta 0}] \\
& - \rho \frac{\partial U}{\partial x^\alpha} [2\bar{\phi} - \frac{1}{2} \alpha_1 \tilde{w} \cdot \tilde{v} + \frac{1}{2} (\alpha_2 + \alpha_3 - \alpha_1) w^2] = 0 \quad (22 \text{ cont'd.})
\end{aligned}$$

and

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial(\rho^* v^\alpha)}{\partial x^\alpha} = 0 \quad (23)$$

where

$$\sigma = \rho(1 + v^2 + 2U + \Pi + p/\rho) \quad (24)$$

$$\phi = \frac{1}{4}(\alpha_3 + 2\gamma + 2 + \xi_1) v^2 + \frac{1}{2}(3\gamma + 1 + \xi_2 - 2\beta) U + \frac{1}{2}(1 + \xi_3) \Pi + \frac{3}{2}(\gamma + \xi_4) p/\rho \quad (25)$$

$$\bar{\phi} = \frac{1}{2}(\gamma + 1) v^2 + \frac{1}{2}(3\gamma - 2\beta + 1) U + \frac{1}{2} \Pi + \frac{3}{2} \gamma p/\rho \quad (26)$$

$$\rho^* = \rho(1 + \frac{1}{2} v^2 + 3\gamma U) \quad (27)$$

$$\chi(\tilde{x}, t) = \int \rho(\tilde{x}', t) |\tilde{x} - \tilde{x}'| d\tilde{x}' \quad (28)$$

$$\Phi(\tilde{x}, t) = \int \frac{\rho(\tilde{x}', t) \phi(\tilde{x}', t)}{|\tilde{x} - \tilde{x}'|} d\tilde{x}' \quad (29)$$

Consider an equilibrium spherically symmetric distribution of matter.

The equation of hydrostatic equilibrium, which follows from (24), is

$$\frac{d}{dr} \left[1 + (3\gamma - 1) U \right] p = \rho \left[1 + \left(\frac{1}{3} \alpha_2 + \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_1 \right) w^2 \right] \frac{dU}{dr} + 2\rho \left(\frac{d\phi}{dr} + \bar{\phi} \frac{dU}{dr} \right). \quad (30)$$

Let the equilibrium configuration be slightly perturbed and describe its

perturbation by a Lagrangian displacement of the form

$$\xi(x) e^{i\Omega t} \quad (31)$$

The linearized form of the equations governing the perturbation, as derived by combining equations (22), (23), and (30), is

$$\begin{aligned} & \Omega^2 \left\{ \sigma \xi^\alpha + \frac{1}{2}(\alpha_2 - \xi_1 + 1) \rho (v_\alpha - w_\alpha) + [(5\gamma - 1) \xi^\alpha U - \frac{1}{2}(\alpha_1 + 4\gamma + 4) v_\alpha] \right\} \\ & + i\Omega \left\{ \frac{1}{2} \alpha_1 \rho \Delta U w^\alpha - \frac{1}{2} \rho (\alpha_1 - 2\alpha_3) w^\beta \frac{\partial v_\beta}{\partial x^\alpha} + \alpha_2 \rho \frac{\partial^2 (\Delta x)}{\partial x^\alpha \partial x^\beta} w^\beta - \frac{1}{2} \alpha_1 w^\beta \xi^\beta \rho \frac{\partial U}{\partial x^\alpha} \right\} \\ & = \frac{\partial}{\partial x^\alpha} \left\{ [1 + (3\gamma - 1) U] \Delta p + (3\gamma - 1) p \Delta U \right\} - \frac{\Delta \rho}{\rho} \frac{\partial}{\partial x^\alpha} \left\{ [1 + \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1) w^2 \right. \\ & \quad \left. + (3\gamma - 1) U] p \right\} \\ & - \rho [1 + (\frac{1}{3} \alpha_2 + \frac{1}{2} \alpha_3 - \frac{1}{2} \alpha_1) w^2] \frac{\partial \Delta U}{\partial x^\alpha} - 2\rho (\Delta \phi) \frac{\partial U}{\partial x^\alpha} + \bar{\phi} \frac{\partial \Delta U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \Delta \phi \\ & + \frac{1}{2} \rho \alpha_2 w^\gamma w^\delta \frac{\partial}{\partial x^\alpha} (\Delta U_{\gamma\delta} - \frac{1}{3} \delta_{\gamma\delta} \Delta U) \quad , \end{aligned} \quad (32)$$

and

$$\Delta \rho^* = - \rho^* \operatorname{div} \xi \quad (33)$$

Here and henceforth σ , ϕ , $\bar{\phi}$, and ρ^* are defined by equations (24), (25), (26), and (27), with all quadratic velocity terms (v^2 terms) omitted; and v_α and w_α are given by definitions (19) and (20) with $v_\alpha(x')$ replaced by $\xi^\alpha(x')$. [New v_α and w_α equal to $(1/i\Omega)$ times old v_α and w_α .] The symbol Δ denotes the Lagrangian change in the quantity that it qualifies.

Now we must evaluate the Lagrangian changes for various quantities explicitly in terms of ξ . From the definition of ρ^* and equation (33), it follows that

$$\Delta \rho = - \rho (\operatorname{div} \xi + 3\gamma \Delta U) \quad , \quad (34)$$

correct to post-Newtonian order. Similarly, the first law of thermodynamics and the definition of "the adiabatic index" Γ_1 lead to

$$\Delta\Pi = (p/\rho^2) \cdot \Delta\rho, \quad \Delta p = \Gamma_1 (p/\rho) \Delta\rho \quad (35)$$

respectively, Therefore,

$$\Delta\phi = \frac{1}{2}(3\gamma + 1 + \zeta_2 - 2\beta) \Delta U - \frac{1}{2} \frac{p}{\rho} [(3\Gamma_1 - 1)(\gamma + \zeta_4) + 1 + \zeta_3](\text{div } \underline{\xi} + 3\gamma\Delta U) \quad (36)$$

and

$$\overline{\Delta\phi} = \frac{1}{2}(3\gamma - 2\beta + 1) \Delta U - \frac{1}{2} \frac{p}{\rho} (3\Gamma_1\gamma - 3\gamma + 1)(\text{div } \underline{\xi} + 3\gamma\Delta U) \quad (37)$$

Finally, the expressions for ΔU , $\Delta U_{\gamma\delta}$, ΔX , and $\Delta\phi$ can be written down from equations (13), (21), (28), (29) as follows:

$$\begin{aligned} \Delta U = & \underline{\xi} \cdot \underline{\nabla} U + \int_V \rho(\underline{x}') \xi^\alpha(\underline{x}') \frac{\partial}{\partial x^{\alpha'}} \frac{1}{|\underline{x} - \underline{x}'|} d\underline{x}' \\ & - 3\gamma \int_V \frac{\rho(\underline{x}') \Delta U(\underline{x}')}{|\underline{x} - \underline{x}'|} d\underline{x}' \end{aligned} \quad (38)$$

$$\begin{aligned} \Delta U_{\gamma\delta} = & \underline{\xi} \cdot \underline{\nabla} U_{\gamma\delta} + \int_V \rho(\underline{x}') \xi^\alpha(\underline{x}') \frac{\partial}{\partial x^{\alpha'}} \cdot \frac{(x_\gamma - x_\gamma')(x_\delta - x_\delta')}{|\underline{x} - \underline{x}'|^3} d\underline{x}' \\ & - 3\gamma \int_V \rho(\underline{x}') \Delta U(\underline{x}') \frac{(x_\gamma - x_\gamma')(x_\delta - x_\delta')}{|\underline{x} - \underline{x}'|^3} d\underline{x}' \end{aligned} \quad (39)$$

$$\begin{aligned} \Delta X = & \underline{\xi} \cdot \underline{\nabla} X + \int_V \rho(\underline{x}') \xi^\alpha(\underline{x}') \frac{\partial}{\partial x^{\alpha'}} |\underline{x} - \underline{x}'| d\underline{x}' \\ & - 3\gamma \int_V \rho(\underline{x}') \Delta U(\underline{x}') |\underline{x} - \underline{x}'| d\underline{x}' \end{aligned} \quad (40)$$

and

$$\begin{aligned} \Delta\phi = & \underline{\xi} \cdot \underline{\nabla} \phi + \int_V \rho(\underline{x}') \phi(\underline{x}') \xi^\alpha(\underline{x}') \frac{\partial}{\partial x^{\alpha'}} \frac{1}{|\underline{x} - \underline{x}'|} d\underline{x}' \\ & + \int_V \frac{\rho(\underline{x}') \Delta\phi(\underline{x}')}{|\underline{x} - \underline{x}'|} d\underline{x}' \quad (41) \end{aligned}$$

The last two terms in equations (38), (39), (40), and (41) make up the Eulerian changes in U , $U_{\gamma\delta}$, χ and Φ , corresponding to their respective Lagrangian changes.

Notice that the linearized pulsation equation (32) is not invariant under rotations. Terms linear in ω couple " ℓ -modes" (modes with spherical-harmonic index ℓ) to $(\ell - 1)$ and $(\ell + 1)$ modes; terms quadratic in ω couple ℓ -modes to $(\ell - 2)$, $(\ell - 1)$, $(\ell + 1)$, and $(\ell + 2)$ modes; all other terms are invariant under rotation. The terms linear in ω have imaginary coefficients; therefore they (like viscosity, energy generation, and radiative transport) contribute to the vibrational-secular stability of the star, but do not affect its dynamical stability. Terms quadratic in ω contribute to the dynamical stability and couple different angular modes. We will delay until a later paper all analyses of ω -dependent terms ("preferred-frame terms") — including both the problem of vibrational-secular stability (linear in ω) and preferred-frame influence on dynamical stability (quadratic in ω). Thus, we shall set $\omega = 0$ throughout this paper.

III. THE VARIATIONAL PRINCIPLE

Equation (32), when supplemented by the expressions for the various Lagrangian changes in terms of ξ , becomes explicitly an equation for ξ . As boundary conditions, we shall demand that $\Delta p = 0$ at the surface of the star ($r = R$), and that there be no physical singularity at the star's center ($r = 0$). Equation (32) together with the boundary conditions then constitutes a characteristic value problem for Ω .

If a characteristic value problem is self-adjoint, the orthogonality relations for its characteristic functions hold and a variational base for

determining Ω can be obtained. The stellar pulsation equations in the post-Newtonian limits of general relativity and Brans-Dicke theory are self-adjoint (Chandrasekhar 1956; Nutku 1969) as is the equation for radial oscillations in the full theory of general relativity (Chandrasekhar 1964b). In the Appendix, it is shown that the characteristic value problem in the PPN formalism is self-adjoint if and only if

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = \zeta_3 = \zeta_4 = 0 . \quad (42)$$

Although, in the general case, the characteristic value problem is not self-adjoint and the orthogonality relations do not hold, a variational integral can still be constructed in the following manner:

Take equation (32) with Ω replaced by the characteristic value $\Omega^{(i)}$ for the i -th normal mode, with ξ replaced by the corresponding characteristic function $\xi^{(i)}$, and with w -terms deleted. Dot into this equation $\xi^{(j)}$, the characteristic function for the j -th mode, and integrate over the interior of the star. Thereby obtain

$$[\Omega^{(i)}]^2 Q^{(i,j)} = S^{(i,j)} + R^{(i,j)} . \quad (43)$$

Here $Q^{(i,j)}$ is expression (A.4) with 1 replaced by i , 2 replaced by j , and complex conjugations deleted; $S^{(i,j)}$ is the symmetric part of the right-hand side of (A.3), with similar replacements; and $R^{(i,j)}$ is (A.4) with similar replacements. Notice that $R^{(i,j)}$ is of post-Newtonian order:

$$R^{(i,j)} = O(2) . \quad (44)^3$$

³By $O(n)$ we mean, in Chandrasekhar's (1965a) language, $O(c^{-n})$.

From equations (43) and (44), it follows immediately that the standard orthogonality relation for characteristic functions is valid to Newtonian order, i.e.

$$Q^{(i,j)} = O(2) [\Omega^{(i)} \neq \Omega^{(j)}] . \quad (45)$$

We assume, without proof, that the characteristic functions $\left\{ \xi^{(i)} \right\}$ from a complete set; and we normalize them to give

$$Q^{(I,I)} = 1 \quad (46)$$

(no summation on capital letters).

Let $P_{\xi}^{(I)}$ be a solution which differs from $\xi^{(I)}$ by post-Newtonian order and has norm 1, i.e.

$$P_{\xi}^{(I)} = \xi^{(I)} + O(2) \quad (47)$$

$$Q\left(P_{\xi}^{(I)}, P_{\xi}^{(I)}\right) = 1 . \quad (48)$$

Expand $P_{\xi}^{(I)}$ in terms of $\xi^{(j)}$:

$$P_{\xi}^{(I)} = \sum_j C_{Ij} \xi^{(j)} ,$$

and from equations (45), (46), (47), and (48), obtain

$$C_{Ij} = 1 + O(4), \quad (j = I) \quad (49)$$

$$C_{Ij} = O(2), \quad (j \neq I) . \quad (50)$$

By combining equations (43), (45), (49), and (50), we obtain

$$\begin{aligned} S\left(P_{\xi}^{(I)}, P_{\xi}^{(I)}\right) + R\left(P_{\xi}^{(I)}, P_{\xi}^{(I)}\right) &= \sum_{j,j'} [\Omega^{(j)}]^2 C_{Ij} C_{Ij'} Q^{(j,j')} \\ &= [\Omega^{(I)}]^2 Q^{(I,I)} + O(4) ; \end{aligned} \quad (51)$$

by combining equations (48), (49), and (50), we obtain

$$Q\left(\tilde{P}_{\xi}^{(I)}, \tilde{P}_{\xi}^{(I)}\right) = \sum_{j, j'} C_{Ij} C_{Ij'} Q^{(j, j')} = Q^{(I, I)} + O(4) ; \quad (52)$$

and by combining equations (51), (42), we finally conclude that

$$[\Omega^{(I)}]^2 Q\left(\tilde{P}_{\xi}^{(I)}, \tilde{P}_{\xi}^{(I)}\right) = S\left(\tilde{P}_{\xi}^{(I)}, \tilde{P}_{\xi}^{(I)}\right) + R\left(\tilde{P}_{\xi}^{(I)}, \tilde{P}_{\xi}^{(I)}\right) + O(4) . \quad (53)$$

Therefore we can use this equation, and any functions $\tilde{P}_{\xi}^{(I)}$ that agree with $\xi^{(I)}$ only at Newtonian order, to calculate $[\Omega^{(I)}]^2$ to post-Newtonian order. The Newtonian proper solutions $\left\{N_{\xi}^{(i)}\right\}$ are one set of such functions.

By suppressing the prefix "P" and superscript labels, by inserting from Appendix the values of Q, S, and R, and by performing some reductions, we bring our variational expression (53) into the form

$$\begin{aligned} Q\Omega^2 = & \int_V \Gamma_1 p [1 + (3\gamma - 1) U] (\text{div } \xi)^2 d\tilde{x} + \int_V (\text{div } \xi) \xi^\alpha \frac{\partial}{\partial x^\alpha} \left\{ [1 + (3\gamma - 1) U] p \right\} d\tilde{x} \\ & + \int_V (3\Gamma_1 \gamma - 3\gamma + 1) p \Delta U \text{div } \xi d\tilde{x} + \int_V (2\beta - 1) \Delta U \xi^\alpha \frac{\partial p}{\partial x^\alpha} d\tilde{x} \\ & + \int_V (3\Gamma_1 \gamma - 3\gamma + 1) p \text{div } \xi \xi^\alpha \frac{\partial U}{\partial x^\alpha} d\tilde{x} - \int_V \rho \xi^\alpha \frac{\partial \Delta U}{\partial x^\alpha} d\tilde{x} \\ & - 2 \int_V \rho \bar{\phi} \xi^\alpha \frac{\partial \Delta U}{\partial x^\alpha} d\tilde{x} - 2 \int_V \rho \xi^\alpha \frac{\partial \Delta \phi}{\partial x^\alpha} d\tilde{x} . \end{aligned} \quad (54)$$

Here

$$\begin{aligned} Q = & \int_V \sigma |\xi|^2 d\tilde{x} + \frac{1}{2}(\alpha_2 - \xi_1 + 1) \int_V \int_V \rho(\tilde{x}) \rho(\tilde{x}') \frac{\xi(\tilde{x}) \cdot \xi(\tilde{x}')}{|\tilde{x} - \tilde{x}'|} d\tilde{x} d\tilde{x}' \\ & - \frac{1}{2}(\alpha_2 - \xi_1 + 1) \int_V \int_V \rho(\tilde{x}) \rho(\tilde{x}') \frac{[\xi(\tilde{x}) \cdot (\tilde{x} - \tilde{x}')] [\xi(\tilde{x}') \cdot (\tilde{x} - \tilde{x}')] }{|\tilde{x} - \tilde{x}'|^3} d\tilde{x} d\tilde{x}' \\ & + (5\gamma - 1) \int_V \int_V \rho(\tilde{x}) \rho(\tilde{x}') \frac{|\xi(\tilde{x})|^2}{|\tilde{x} - \tilde{x}'|} d\tilde{x} d\tilde{x}' - \frac{1}{2}(4\gamma + 4 + \alpha_1) \int_V \int_V \\ & \rho(\tilde{x}) \rho(\tilde{x}') \frac{\xi(\tilde{x}) \cdot \xi(\tilde{x}')}{|\tilde{x}' - \tilde{x}|} d\tilde{x} d\tilde{x}' , \end{aligned} \quad (55)$$

is a positive definite quantity in the post-Newtonian limit since the dominant, Newtonian part, $\int_V \sigma |\xi|^2 dx$, is positive definite. This equation can be used for a variational determination of Ω^2 .

We shall now analyze the Lagrangian displacement ξ into normal modes belonging to different vector spherical harmonics. (Since the pulsation equation without ω -dependent terms is invariant under rotation, this procedure is justified.) Following the procedure of Chandrasekhar (1961, 1964c) and Lebovitz (1965), we define:

$$\xi_r = \frac{\psi(\gamma)}{r^2} Y_\ell^m(\theta, \phi) , \quad (56)$$

$$\xi_\theta = \frac{1}{\ell(\ell+1)r} \frac{d\chi(r)}{dr} \frac{\partial Y_\ell^m(\theta, \phi)}{\partial \theta} , \quad (57)$$

and

$$\xi_\phi = \frac{1}{\ell(\ell+1)r \sin \theta} \frac{d\chi(r)}{dr} \frac{\partial Y_\ell^m(\theta, \phi)}{\partial \phi} \quad (58)$$

($\xi_r, \xi_\theta, \xi_\phi$ are physical components, not covariant components). After manipulations similar to those in §IV of Chandrasekhar (1965b), we obtain the following expression for the variational determination of Ω^2 :

$$\begin{aligned} \Omega^2 = & \int_0^R \Gamma_1 p [1 + (3\gamma - 1)U] \left[\frac{d}{dr} (\psi - x) \right]^2 \frac{dr}{r^2} \\ & + 2 \int_0^R \frac{d}{dr} \left\{ [1 + (3\gamma - 1)U] p \right\} \left(2 \frac{\psi^2}{r} - \psi \frac{d\chi}{dr} \right) \frac{dr}{r^2} \\ & - \frac{4\pi}{2\ell+1} \int_0^R (1 + 2\phi + 2\bar{\phi}) \left(J_\ell \frac{dK_\ell}{dr} - K_\ell \frac{dJ_\ell}{dr} \right) dr \\ & + (2\beta - 1 - \zeta_2) \int_0^R \rho [\Delta U(r)]^2 r^2 dr \\ & + (6\Gamma_1 \gamma - 6\gamma + 2 + 3\Gamma_1 \zeta_4 - 3\zeta_4 + \zeta_3) \int_0^R p \Delta U(r) \frac{d}{dr} (\psi - x) dr . \quad (59) \end{aligned}$$

where

$$J_\ell(r) = \int_0^r \rho(s) s^\ell \left[\ell \frac{\psi(s)}{s} + \frac{d\chi(s)}{ds} \right] ds, \quad (60)$$

$$K_\ell(r) = \int_r^R \frac{\rho(s)}{s^{\ell+1}} \left[(\ell + 1) \frac{\psi(s)}{s} - \frac{d\chi(s)}{ds} \right] ds, \quad (61)$$

and

$$\Delta U(r) = \frac{4\pi}{4\ell + 1} \left[\frac{J_\ell(r)}{r^{\ell+1}} - r^\ell K_\ell(r) \right] + \frac{\psi}{r^2} \frac{dU}{dr}. \quad (62)$$

IV. THE POST-NEWTONIAN CONDITION FOR THE ONSET OF DYNAMICAL INSTABILITY

Consider the case of radial pulsations, i.e. pulsations with

$$\ell = 0 \text{ and } \chi = 0. \quad (63)$$

The substitutions

$$\psi = r^3 \eta \text{ and } \xi_r = r \eta, \quad (64)$$

reduce equation (59) to the form

$$\begin{aligned} Q^2 = & \int_0^R p [1 + (3\gamma - 1) U] \left[\Gamma_1 r^4 \left(\frac{d\eta}{dr} \right)^2 + (3\Gamma_1 - 4) \frac{d}{dr} (r^3 \eta^2) \right] dr \\ & + (2\beta - 1 - \zeta_2) \int_0^R \rho [\Delta U(r)]^2 r^2 dr \\ & + (6\Gamma_1 \gamma - 6\gamma + 2 + 3\Gamma_1 \zeta_4 - 3\zeta_4 + \zeta_3) \int_0^R p \Delta U(r) \frac{d}{dr} (r^3 \eta) dr \\ & - (3\Gamma_1 \zeta_4 - 3\zeta_4 + \zeta_3) \int_0^R p \frac{dU}{dr} \eta \frac{d(\eta r^3)}{dr} r dr \\ & + \zeta_2 \int_0^R \frac{dp}{dr} \Delta U \eta r^3 dr, \end{aligned} \quad (65)$$

where

$$\Delta U(r) = - 4\pi \int_r^R \rho(s) \psi(s) \frac{ds}{s^2} + \frac{\psi}{r^2} \frac{dU}{dr} . \quad (66)$$

Recall that p and ρ are the distributions of pressure and density in the equilibrium configuration in the post-Newtonian approximation, and they therefore include terms of $O(2)$.

The condition for marginal stability (instability) follows from equation (65) by setting $\Omega^2 = 0$. In the particular case $\Gamma_1 = \text{const.}$ — which implies $\eta = \text{const.}$ at the point of onset of instability in the Newtonian limit, i.e. $\eta = \text{const.} + O(2)$ — the condition for marginal instability (eq. [65] with $\Omega = 0$) involves the structure of the equilibrium configuration in the Newtonian approximation alone. Under these conditions the criterion for marginal instability becomes $\Gamma_1 = \Gamma_{\text{crit}}$, where

$$\begin{aligned} \Gamma_{\text{crit}} = \frac{4}{3} + \frac{1}{3W} & \left\{ (2\beta - 1 - \zeta_2) \int_0^R [\Delta U(r)]^2 dM(r) + 3[2\gamma + 2 + \zeta_3 + \zeta_4] \right. \\ & \int_0^R \frac{p}{\rho} \Delta U(r) dM(r) \\ & \left. - 3(\zeta_3 + \zeta_4) \int_0^R \frac{p}{\rho} \left(\frac{dU}{dr} r \right) dM(r) + \zeta_2 \int \frac{\Delta U}{\rho} r \left(\frac{dp}{dr} \right) dM(r) \right\} . \quad (67) \end{aligned}$$

Here

$$W = - 12\pi \int_0^R p r^2 dr , \quad (68)$$

$$dM = 4\pi p r^2 dr , \quad (69)$$

and

$$\Delta U(r) = - 4\pi \int_r^R \rho s ds + r \frac{dU}{dr} . \quad (70)$$

This result agrees with those in general relativity (Chandrasekhar 1965b) and in Brans-Dicke-Jordan theory (Nutku 1969) when specialized to the corresponding PPN values.

Criterion (67) for marginal instability may be reformulated as follows. A dynamical instability will set in if and only if the following inequality is satisfied:

$$\Gamma_1 \leq \Gamma_{\text{crit}} \equiv \frac{4}{3} + K \frac{2M}{R} \quad (71)$$

Here M is the mass and R is the radius of the configuration, and K is a constant (typically of order unity), depending on the Newtonian structure of the configuration. If K is positive, there is a relativistic instability; if K is negative, then relativistic effects stabilize the star.

For polytropes, an explicit expression for K can be obtained, from equations (67)-(71), in terms of Lane-Emden functions:

$$K = \frac{1}{2} (1 + \gamma) K_1 - (2\beta - 1) K_2 + (\xi_3 + \xi_4) \left(\frac{1}{4} K_1 - 3K_3 \right) + \xi_2 (K_2 - K_4) \quad (72)$$

where

$$K_1 = \frac{12(5 - n)}{18(n + 1) \xi_1^4 |\theta_1'|^3} \cdot \int_0^{\xi_1} \left[\theta + \xi_1 |\theta_1'| \right] \theta^{n+1} \xi^2 d\xi \quad (73)$$

$$K_2 = \frac{(5 - n)}{18 \xi_1^4 |\theta_1'|^3} \cdot \int_0^{\xi_1} \left[\theta + \xi_1 |\theta_1'| \right]^2 \theta^n \xi^2 d\xi \quad (74)$$

$$K_3 = \frac{5 - n}{18(n + 1) \xi_1^4 |\theta_1'|^3} \cdot \int_0^{\xi_1} \theta^{n+1} \xi^3 \left| \frac{d\theta}{d\xi} \right| d\xi \quad (75)$$

$$K_4 = \frac{5 - n}{18 \xi_1^4 |\theta_1'|^3} \cdot \int_0^{\xi_1} \left[\theta + \xi_1 |\theta_1'| \right] \theta^n \left| \frac{d\theta}{d\xi} \right| \xi^3 d\xi \quad (76)$$

and where n is the polytropic index, ξ_1 is the first zero of the Lane-Emden function θ_n , and θ_1' is the value of the derivative of θ_n at ξ_1 . The values of the constant K , evaluated with the aid of the foregoing formula for various values of n , are listed in Table 1 for conservative theories

and in Table 2 for the general PPN formalism and for non-conservative theories. See §I for discussions of these tables and for the conclusions inferred from these tables.

V. NON-RADIAL OSCILLATIONS AND THE SCHWARZSCHILD

CRITERION FOR CONVECTION

We shall now obtain the condition for the occurrence of a neutral mode of non-radial oscillation belonging to $l \geq 1$ in the general PPN framework. As in the last two sections, all "preferred-frame" effects (ω -dependent terms) will be ignored. By setting $\Omega^2 = 0$ in equation (32), by following an analysis parallel to §VI of Chandrasekhar (1965b), and by using the result of §VIII of Chandrasekhar (1965b), one can show that, to $O(2)$, a necessary and sufficient condition for the occurrence of a neutral mode of non-radial oscillation is that

$$S(r) \left[1 + \Pi \frac{\Gamma_3 - \Gamma_4}{\Gamma_3 - 1} \frac{\rho}{p} \frac{dp/dr}{d\rho/dr} \right] = 0 ; \quad (77)$$

i.e. that

$$S(r) = 0 \quad (78)$$

over some finite interval of r . Here

$$S(r) \equiv \frac{dp}{dr} - \Gamma_1 \frac{p}{\rho} \frac{d\rho}{dr} \quad (79)$$

is the "Schwarzschild discriminant", and Γ_3 and Γ_4 are defined by

$$\Gamma_3 = 1 + \left[\frac{\partial(\log \dot{T})}{\partial(\log \rho)} \right]_s , \quad (80)$$

and

$$\Pi = \frac{1}{\Gamma_4 - 1} \frac{p}{\rho} . \quad (81)$$

By following a procedure similar to §VII of Chandrasekhar (1965b), one can also derive this condition from the variational principle (61).

The proportionality of the Newtonian and the post-Newtonian discriminants implies that the physical condition for convective instability remains the same in the PPN formalism as in general relativity and in Newtonian theory. Although for some PPN values, the characteristic value problem is not self-adjoint, the Schwarzschild criterion is still valid, and no new dynamical instabilities occur.

VI. CONCLUSIONS

In this paper, stability criteria for stellar pulsations were found using the general PPN formalism. These criteria are valid for almost all metric theories of gravity in the post-Newtonian approximation, when one ignores preferred-frame effects. As in general relativity, so also for conservative theories (conservative theories do not have preferred-frame effects), the relativistic corrections do actually induce dynamical instabilities in stars. But in the general case the present experimental uncertainty in the PPN parameters ξ_2 , ξ_4 makes it inconclusive whether relativistic effects will actually stabilize or destabilize stars. As experimental tests are performed to higher precision, the answer may become definite. The differences in the dynamical stability criterion for various theories may affect the evolution of white dwarfs and supermassive stars; such effects are worth exploration. The relationship of non-self-adjointness to the non-existence of conservation laws is intriguing and should be examined further.

A subsequent paper will deal with the problem of Machian instabilities

due to preferred-frame effects (ω -dependent terms). Those instabilities, when combined with astronomical observations on white-dwarf pulsations, may lead to tight experimental limits on the "preferred-frame parameters" α_1 , α_2 , and α_3 .

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APPENDIX

SELF-ADJOINTNESS OF THE CHARACTERISTIC VALUE PROBLEM

We shall here derive the constraints which the PPN parameters must satisfy for the characteristic value problem [eq. (32)] to be self-adjoint. For this purpose we do not delete the ω -dependent terms ab initio (cf. end of §II).

By bringing the right-hand side over to the left, write equation (32) in the form $\mathcal{L}\xi = 0$, where \mathcal{L} is a linear operator. This equation is self-adjoint [or can be made so by multiplication with some weighting function $\mathcal{W}(\underline{x})$] if and only if

$$\int_V \xi^{(1)*} \cdot \mathcal{W} \mathcal{L} \xi^{(2)} d\underline{x} = \int_V \left[\xi^{(2)*} \cdot \mathcal{W} \mathcal{L} \xi^{(1)} \right]^* d\underline{x} \quad , \quad (\text{A.1})$$

where the complex conjugation "*" is not permitted to act on the eigenvalue Ω (which is contained in \mathcal{L}). In this equation $\xi^{(1)}$ and $\xi^{(2)}$ are arbitrary functions satisfying the boundary conditions at $r = 0$ and $r = R$ (but not necessarily satisfying $\mathcal{L} \xi^{(1)} = 0$ or $\mathcal{L} \xi^{(2)} = 0$); V is the interior of the star; "*" denotes complex conjugation; and $d\underline{x}$ denotes $dx dy dz = dx^1 dx^2 dx^3$. From this definition one readily verifies that (i) if the weighting function \mathcal{W} is chosen real, then the $i\Omega$ -terms prevent self-adjointness; (ii) if \mathcal{W} has any imaginary part, then the Ω^2 -terms prevent self-adjointness. It is possible to get rid of the $i\Omega$ -terms by demanding $\alpha_1 = \alpha_2 = \alpha_3 = 0$; but it is not possible to get rid of the Ω^2 -terms. Therefore, to have any hope of self-adjointness one must choose $\mathcal{W}(\underline{x})$ real and

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \quad . \quad (\text{A.2})$$

Insist, then, that $\alpha_1 = \alpha_2 = \alpha_3 = 0$; and try, for the moment to prove

self-adjointness with the trivial weighting function $\eta(\underline{x}) = 1$. Then the terms on the left-hand side of equation (32) give, when integrated,

$$\begin{aligned}
[\Omega]^2 & \left\{ \int_V \sigma \xi^{(1)\alpha*} \xi^{(2)\alpha} d\underline{x} + \frac{1}{2} \int_V \int_V \rho(\underline{x}) \rho(\underline{x}') \frac{\xi^{(1)\alpha*}(\underline{x}) \xi^{(2)\alpha}(\underline{x}')}{|\underline{x} - \underline{x}'|} d\underline{x} d\underline{x}' \right. \\
& - \frac{1}{2}(\alpha_2 - \xi_1 + 1) \int_V \int_V \rho(\underline{x}) \rho(\underline{x}') \xi^{(1)\alpha*}(\underline{x})(x^\alpha - x'^\alpha) \xi^{(2)\beta}(\underline{x}')(x^\beta - x'^\beta) \\
& \quad \frac{d\underline{x} d\underline{x}'}{|\underline{x} - \underline{x}'|^3} \\
& + (5\gamma - 1) \int_V \int_V \rho(\underline{x}) \rho(\underline{x}') \xi^{(1)\alpha*}(\underline{x}) \xi^{(2)\alpha}(\underline{x}) \frac{d\underline{x} d\underline{x}'}{|\underline{x} - \underline{x}'|} \\
& \left. - \frac{1}{2}(\alpha_1 + 4\gamma + 4) \int_V \int_V \rho(\underline{x}) \rho(\underline{x}') \xi^{(1)\alpha*}(\underline{x}) \xi^{(2)\alpha}(\underline{x}') \frac{d\underline{x} d\underline{x}'}{|\underline{x} - \underline{x}'|} \right\} \equiv \Omega^2 Q^{(1,2)}. \quad (A.3)
\end{aligned}$$

[For use in the text of the paper we retain all α terms except those that depend on \underline{w} ; but we keep in mind that henceforth in this appendix the α 's vanish.] Notice that $Q^{(1,2)}$ is manifestly self-adjoint in the superscripts 1 and 2, i.e. $Q^{(1,2)} = Q^{(2,1)*}$. Reduction of the right-hand side of the integrated equation is less straightforward; it requires integrations by parts, followed by substitutions for various Lagrangian changes, simplifications, and rearrangements. After some work, the right-hand side is brought into the form

$$\begin{aligned}
\Omega^2 Q^{(1,2)} & = \int_V \Gamma_1 [1 + (3\gamma - 1)U] p \operatorname{div} \xi^{(1)*} \operatorname{div} \xi^{(2)} d\underline{x} \\
& + \int_V \frac{1}{\rho} \frac{d\rho}{dr} \frac{d}{dr} \left[1 + (3\gamma - 1)U \right] p \left\{ \frac{[\underline{x} \cdot \xi^{(1)*}][\underline{x} \cdot \xi^{(2)}]}{r^2} \right\} d\underline{x} \quad (A.4)
\end{aligned}$$

$$\begin{aligned}
& + \int_V [1 + (3\gamma - 1)U] \frac{d\rho}{dr} \left[\frac{\underline{x} \cdot \underline{\xi}^{(1)*}}{r} \operatorname{div} \underline{\xi}^{(2)} + \frac{\underline{x} \cdot \underline{\xi}^{(2)}}{r} \operatorname{div} \underline{\xi}^{(1)*} \right] d\underline{x} \\
& - \int_V \int_V \left\{ 1 + 2[\bar{\phi}(\underline{x}) + \bar{\phi}(\underline{x}')] \right\} \\
& \quad \frac{[\Delta\rho^{(1)*}(\underline{x}) - \underline{\xi}^{(1)*}(\underline{x}) \cdot \nabla\rho(\underline{x})][\Delta\rho^{(1)}(\underline{x}') - \underline{\xi}^{(2)}(\underline{x}') \cdot \nabla\rho(\underline{x}')] }{|\underline{x} - \underline{x}'|} d\underline{x} d\underline{x}' \\
& + \int_V \rho \frac{d}{dr} (3\gamma U + 2\phi) \left[\frac{\underline{x} \cdot \underline{\xi}^{(1)*}}{r} \Delta U^{(2)} + \frac{\underline{x} \cdot \underline{\xi}^{(2)}}{r} \Delta U^{(1)*} \right] d\underline{x} \\
& - (3\gamma - 1) \int_V \rho \frac{dU}{dr} \left[\frac{\underline{x} \cdot \underline{\xi}^{(1)*}}{r} \Delta\Pi^{(2)} + \frac{\underline{x} \cdot \underline{\xi}^{(2)}}{r} \Delta\Pi^{(1)*} \right] d\underline{x} \\
& - (3\gamma\Gamma_1 - 3\gamma + 1) \int_V \rho [\Delta U^{(1)*} \Delta\Pi^{(2)} + \Delta U^{(2)} \Delta\Pi^{(1)*}] d\underline{x} \\
& - 2 \int_V \left[\frac{dU}{dr} \frac{d\rho}{dr} (\phi - \bar{\phi}) + \frac{dU}{dr} \frac{d\phi}{dr} \rho \right] \frac{[\underline{x} \cdot \underline{\xi}^{(1)*}][\underline{x} \cdot \underline{\xi}^{(2)}]}{r^2} d\underline{x} \\
& - (6\gamma + 1 + \xi_2 - 2\beta) \int_V \rho \Delta U^{(1)*} \Delta U^{(2)} d\underline{x} + R^{(1,2)} \tag{A.4 cont'd.}
\end{aligned}$$

where

$$\begin{aligned}
R^{(1,2)} = & [-3\xi_4(\Gamma_1 - 1) - \xi_3] \int_V \rho \Delta\Pi^{(1)*} \Delta U^{(2)} d\underline{x} \\
& + 2 \int_V \rho \left(\Delta\phi^{(1)*} - \bar{\Delta\phi}^{(1)*} \frac{dU}{dr} \right) \frac{[\underline{x} \cdot \underline{\xi}^{(2)}]}{r} d\underline{x} \\
& + 2 \int_V \rho \frac{d}{dr} (\bar{\phi} - \phi) \frac{[\underline{x} \cdot \underline{\xi}^{(2)}]}{r} \Delta U^{(1)*} d\underline{x} \\
& + 2 \int_V (\phi - \bar{\phi}) \left[-\Delta U^{(1)*} \Delta\rho^{(2)} + \Delta U^{(1)*} \frac{d\rho}{dr} \frac{[\underline{x} \cdot \underline{\xi}^{(2)}]}{r} \right. \\
& \quad \left. + \Delta\rho^{(2)} \frac{dU}{dr} \frac{[\underline{x} \cdot \underline{\xi}^{(1)}]}{r} \right] d\underline{x} . \tag{A.5}
\end{aligned}$$

Equation (A.4) reduces to equation (A.6) of Chandrasekhar (1965b), if we substitute in the PPN parameter values for general relativity. Aside from $R^{(1,2)}$, the terms on the right-hand side of equation (A.4), like $Q^{(1,2)}$ of the left-hand side, are manifestly self-adjoint in the superscripts 1 and 2. Therefore, the condition for equation (32) to be self-adjoint with weighting function $\mu(\underline{x}) = 1$ is

$$R^{(1,2)} = R^{(2,1)*} \quad (\text{A.6})$$

or, equivalently (since one insists on this equality for all choices of $\xi^{(1)}$ and $\xi^{(2)}$):

$$\xi_2 = \xi_{\bar{2}} = \xi_4 = 0 \quad (\text{A.7})$$

Might some other choice of weighting function aside from constant permit one to relax these constraints and still retain self-adjointness? No; because any other choice of $\mu(\underline{x})$ will destroy the self-adjointness of the left-hand side ($\Omega^2 Q^{(1,2)}$), and the arbitrariness of Ω^2 will prevent the non-self-adjoint terms thus created from always cancelling non-self-adjoint terms on the right-hand side.

In summary, equations (A.2) and (A.7) — i.e. $\alpha_1 = \alpha_2 = \alpha_3 = \xi_2 = \xi_3 = \xi_4 = 0$ — are necessary and sufficient conditions for the self-adjointness of the linearized pulsation problem in the PPN formalism. These conditions, together with the condition that $\xi_1 = 0$, are just Will's (1971b) conditions for theories of gravity to have post-Newtonian conserved integrals for energy, momentum, angular momentum, and center-of-mass motion.

TABLE 1

VALUES OF THE CONSTANT K FOR POLYTROPIC STARS
IN CONSERVATIVE THEORIES OF GRAVITY^a

n	K				
	General Expression	Minimum Possible value ^b	General Relativity ^c	Brans-Dicke Theory ^d ($w = 6$)	Nordström's (1913) Second Theory
0	$0.4286(1 + \gamma) - 0.4047(2\beta - 1)$	0.0743	0.4524(0.452381)	0.3949(0.44047)	0
0.5	$0.4779(1 + \gamma) - 0.4527(2\beta - 1)$	0.0804	0.5030	0.4433	0
1.0	$0.5392(1 + \gamma) - 0.5131(2\beta - 1)$	0.0871	0.5654(0.565382)	0.4980(0.53964)	0
1.5	$0.6179(1 + \gamma) - 0.5908(2\beta - 1)$	0.0949	0.6450(0.645063)	0.5678	0
2.0	$0.7232(1 + \gamma) - 0.6949(2\beta - 1)$	0.1052	0.7514(0.751296)	0.6610(0.70257)	0
2.5	$0.8709(1 + \gamma) - 0.8414(2\beta - 1)$	0.1192	0.9004(0.900302)	0.7916	0
2.75	$0.9696(1 + \gamma) - 0.9394(2\beta - 1)$	0.1284	0.9999	0.8786	0
3.0	$1.0940(1 + \gamma) - 1.0628(2\beta - 1)$	0.1399	1.1252(1.12447)	0.9885(1.0294)	0
3.25	$1.2531(1 + \gamma) - 1.2210(2\beta - 1)$	0.1541	1.2852(1.28503)	1.1285	0
3.5	$1.4664(1 + \gamma) - 1.4332(2\beta - 1)$	0.1732	1.4997(1.49954)	1.3164	0
4.0	$2.2235(1 + \gamma) - 2.1869(2\beta - 1)$	0.2394	2.2602	1.9822	0
4.5	$4.5246(1 + \gamma) - 4.4826(2\beta - 1)$	0.4325	4.5666	4.0010	0

^aBy "conservative theory" we mean a theory that possesses conserved post-Newtonian integrals for energy, momentum angular momentum, and center-of-mass motion — i.e. a theory with $\alpha_1 = \alpha_2 = \alpha_3 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$.

^bThese are minimum values compatible with current experimental limits on β and γ : $\beta < 1.34$, $\gamma > 0.76$.

^cThe values in parentheses are those derived by Chandrasekhar (1965).

^dThe values in parentheses are those derived by Nutku (1969); the discrepancies are presumably due to errors in Nutku's numerical calculations.

TABLE 2
VALUES OF THE CONSTANT K FOR POLYTROPIC STARS
IN THE GENERAL PPN FORMALISM

n	K				Modified Yilmaz Theory ^d
	PPN Formalism	Minimum Possible Value ^a	Vector-Metric Theories ^b and Papapetrou Theories ^c		
0	$0.4286(1 + \gamma) - 0.4047(2\beta - 1) + 0.1429(\zeta_3 + \zeta_4) + 0.2143 \zeta_2$	$0.0743 + 0.2143 \zeta_2$ $+ 0.1429(\zeta_4 - 0.05)$	SAME	- 0.1191 or - 0.2620	
0.5	$0.4779(1 + \gamma) - 0.4527(2\beta - 1) + 0.1522(\zeta_3 + \zeta_4) + 0.2427 \zeta_2$	$0.0804 + 0.2427 \zeta_2$ $+ 0.1522(\zeta_4 - 0.05)$		- 0.1346 or - 0.2868	
1.0	$0.5392(1 + \gamma) - 0.5131(2\beta - 1) + 0.1667(\zeta_3 + \zeta_4) + 0.2778 \zeta_2$	$0.0871 + 0.2778 \zeta_2$ $+ 0.1667(\zeta_4 - 0.05)$		- 0.1569 or - 0.3236	
1.5	$0.6179(1 + \gamma) - 0.5908(2\beta - 1) + 0.1871(\zeta_3 + \zeta_4) + 0.3225 \zeta_2$	$0.0949 + 0.3225 \zeta_2$ $+ 0.1871(\zeta_4 - 0.05)$	as	- 0.1871 or - 0.3742	
2.0	$0.7232(1 + \gamma) - 0.6949(2\beta - 1) + 0.2155(\zeta_3 + \zeta_4) + 0.3821 \zeta_2$	$0.1052 + 0.3821 \zeta_2$ $+ 0.2155(\zeta_4 - 0.05)$		- 0.2283 or - 0.4438	
2.5	$0.8709(1 + \gamma) - 0.8414(2\beta - 1) + 0.2583(\zeta_3 + \zeta_4) + 0.4657 \zeta_2$	$0.1192 + 0.4657 \zeta_2$ $+ 0.2583(\zeta_4 - 0.05)$	in	- 0.2893 or - 0.5476	
2.75	$0.9696(1 + \gamma) - 0.9394(2\beta - 1) + 0.2841(\zeta_3 + \zeta_4) + 0.5216 \zeta_2$	$0.1284 + 0.5216 \zeta_2$ $+ 0.2841(\zeta_4 - 0.05)$		- 0.3274 or - 0.6115	
3.0	$1.0940(1 + \gamma) - 1.0628(2\beta - 1) + 0.3190(\zeta_3 + \zeta_4) + 0.5917 \zeta_2$	$0.1399 + 0.5917 \zeta_2$ $+ 0.3190(\zeta_4 - 0.05)$	General	- 0.3772 or - 0.6962	
3.25	$1.2531(1 + \gamma) - 1.2210(2\beta - 1) + 0.3637(\zeta_3 + \zeta_4) + 0.6821 \zeta_2$	$0.1541 + 0.6821 \zeta_2$ $+ 0.3637(\zeta_4 - 0.05)$		- 0.4427 or - 0.8064	
3.5	$1.4664(1 + \gamma) - 1.4332(2\beta - 1) + 0.4242(\zeta_3 + \zeta_4) + 0.8030 \zeta_2$	$0.1732 + 0.8030 \zeta_2$ $+ 0.4242(\zeta_4 - 0.05)$	Relativity	- 0.5309 or - 0.9551	
4.0	$2.2235(1 + \gamma) - 2.1869(2\beta - 1) + 0.6395(\zeta_3 + \zeta_4) + 1.2326 \zeta_2$	$0.2394 + 1.2326 \zeta_2$ $+ 0.6395(\zeta_4 - 0.05)$		- 0.8445 or - 1.4840	
4.5	$4.5246(1 + \gamma) - 4.4826(2\beta - 1) + 1.2957(\zeta_3 + \zeta_4) + 2.5426 \zeta_2$	$0.4325 + 2.5426 \zeta_2$ $+ 1.2957(\zeta_4 - 0.05)$		- 1.8143 or - 3.1100	

FOOTNOTES FOR TABLE 2

^aThese are minimum values compatible with current experimental limits on β , γ , and ξ_3 : $\beta < 1.34$, $\gamma > 0.76$, $\xi_3 > -0.05$.

^bWill and Nordtvedt (1972).

^cPapapetrou (1954a,b,c).

^dNi (1972). The values of K depend on which "matter density" one chooses as source for the gravitational field: $\rho = T_{ij} u^i u^j$ = component of stress-energy tensor along four-velocity of matter [upper values]; or $\rho = \text{trace } (T_{ij})$ [lower values]. (cf. Ni 1972.)

REFERENCES

- Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford: Clarendon Press).
- _____ 1964a, Phys. Rev. Letters, 12, 114 and 437.
- _____ 1964b, Ap. J., 140, 417.
- _____ 1964c, ibid., 139, 664.
- _____ 1965a, ibid., 142, 1488.
- _____ 1965b, ibid., 142, 1519.
- Fowler, W. A. 1964, Rev. Mod. Phys., 36, 545.
- Gilvarry, J. J., and Muller, P. M. 1972, Phys. Rev. Letters, 28, 1665.
- Kreuzer, L. B. 1966, Ph.D. thesis, Department of Physics, Princeton University (unpublished).
- _____ 1968, Phys. Rev., 169, 1007.
- Lebovitz, N. R. 1965, Ap. J., 142, 1257.
- Ni. W.-T. 1972, Ap. J., 000, 000.
- Nordström, G. 1913, Ann. Physik, 42, 533.
- Nordtvedt, K., Jr. 1968, Phys. Rev., 169, 1017.
- Nordtvedt, K., Jr., and Will, C. M. 1972, Ap. J., in press.
- Nutku, Y. 1969, Ap. J., 155, 999.
- Papapetron, A. 1954a, Math. Nachr., 12, 129.
- _____ 1954b, Math. Nachr., 12, 143.
- _____ 1954c, Z. Phys., 139, 518.
- Sramek, R. A. 1972, reported in the Fourth Cambridge Conference on Experimental Relativity.
- Thorne, K. S. 1967, in High Energy Astrophysics, vol. 3, eds. C. DeWitt, P. Véron, E. Schatzman (Gordon and Breach: New York).

Thorne, K. S., Will, C. M., and Ni, W.-T. 1971, in Proceedings of the Conference on Experimental Tests of Gravitational Theories, ed.

R. W. Davies (NASA-JPL Technical Memorandum 33-499).

Will, C. M. 1971a, Ap. J., 163, 611.

_____ 1971b, Ap. J., 169, 125.

_____ 1971c, Ap. J., 169, 141.

_____ 1972a, private communication.

_____ 1972b, Lectures, International School of Physics "Enrico Fermi"

Varennna, Italy, July 17-July 29, 1972.

Will, C. M., and Nordtvedt, K., Jr. 1972, Ap. J., in press.

Zel'dovich, Ya. B., and Novikov, I. D. 1971, Relativistic Astrophysics

Volume 1 (Chicago: University of Chicago Press).