NASA TECHNICAL MEMORANDUM

NASA TM X-62,165

NASA TM X-62,165

(NASA-TM-X-62165) A PERTURBATION SOLUTION OF HELICOPTER ROTOR FLAPPING STABILITY W. Johnson (NASA) Jul. 1972 123 p

X72-774

Unclas

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CSCL 01A

N73-14006

Unclas
G3/01 52244

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ABSTRACT

The stability of the flapping motion of a single blade of a helicopter rotor is examined using the techniques of perturbation theory. The equation of motion studied is linear, with periodic aerodynamic coefficients due to the forward speed of the rotor. Solutions are found for four cases: small and large advance ratio and small and large Lock number. The perturbation techniques appropriate to each case are discussed and illustrated in the course of the analysis. The application of perturbation techniques to other problems in rotor dynamics is discussed. It is concluded that perturbation theory is a powerful mathematical technique which should prove very useful in analyzing some of the problems of helicopter dynamics.

TABLE OF CONTENTS

Table of Contents (Continued)

SUMMARY

The stability of the flapping motion of a single blade of a helicopter rotor is examined using the techniques of perturbation theory. The equation of motion studied is linear, with periodic aerodynamic coefficients due to the forward speed of the rotor. Blade pitch feedback proportional to both flapping displacement (δ_3) and flapping rate is included. Four cases are considered: small and large advance ratio and small and large Lock number. The perturbation techniques appropriate to each case are discussed and illustrated in the course of the analysis. Analytic solutions are obtained for each case, with primary emphasis on the eigenvalues (that is, root loci) as indicators of the system stability and response. The feature of the equation which makes perturbation techniques useful is the periodicity of the aerodynamic coefficients. The applicability of the four cases considered is discussed; the small advance ratio results in particular are very useful, being valid out to an advance ratio of about 0.5. The application of perturbation techniques to problems in rotor dynamics with more degrees of freedom or better aerodynamic models is discussed. It is concluded that perturbation theory is a powerful, and yet not very sophisticated, mathematical technique which should prove very useful in analyzing some of the problems of helicopter dynamics.

INTRODUCTION

This paper considers the application of perturbation techniques to helicopter rotor dynamics. Perturbation theory has been well developed in recent years, but has not found much application to rotary wing problems. Classically helicopter engineering has made use of the same perturbation theories that fixed wing engineering has, for example lifting line theory and engineering beam theory (both require a large blade aspect ratio). Another classical example is actuator disk theory (a large number of blades is required). These theories were developed on an intuitive basis however, and the more rigorous mathematical techniques of perturbation theory have not yet found widespread use for rotary wings. The classical applications are largely for aerodynamic problems; the mathematics of these problems can be very complicated however because the equations involved are highly nonlinear partial differential equations. The treatment of dynamic problems can be more tractable since only ordinary differential equations are involved. Problems with constant coefficient linear differential equations can be solved exactly with well established methods, so for these problems the extra effort of perturbation theory may not be justified. On the other hand for problems with time varying or nonlinear differential equations the only solution procedure generally applicable is the numerical integration of the equations of motion. However, purely numerical solutions are not entirely satisfactory for obtaining an understanding of the physical character of the system, or for formulating general design rules. Furthermore, an analytic solution for the general case would be difficult to obtain (if possible at all) and would be so complex as to be hardly better than the numerical solution. The only systems that can be practicably handled analytically are those involving linear constant coefficient differential equations. Perturbation techniques are available which are methods to study time varying or nonlinear systems such that at each step in the analysis only linear constant coefficient equations must be handled. Time varying or nonlinear differential equations are characteristic features of helicopter dynamics and aerodynamics, primarily due to the rotation of the wing.

Thus the possibilities for the use of perturbation theory in rotary wing problems are very extensive.

This paper considers the stability of the flapping motion of a single blade of a helicopter rotor. This is a single degree of freedom, second order system, with analytic aerodynamic coefficients. The governing equation is linear with time varying coefficients; it is given below.

$$\ddot{\beta} + \nu^2 \beta = \gamma \left[(M_{\dot{\beta}} - K_R M_\theta) \dot{\beta} + (M_\beta - K_p M_\theta) \beta \right] \tag{1}$$

<u>Regions</u>

$$M_{\dot{\beta}} = \begin{cases} -\left(\frac{1}{8} + \frac{1}{6} \mu \sin \psi\right) & \text{(i)} \\ -\left(\frac{1}{8} + \frac{1}{6} \mu \sin \psi + \frac{1}{12} (\mu \sin \psi)^4\right) & \text{(ii)} \\ \frac{1}{8} + \frac{1}{6} \mu \sin \psi & \text{(iii)} \end{cases}$$

$$M_\beta = \begin{cases} -\mu \cos \psi \left(\frac{1}{6} + \frac{1}{4} \mu \sin \psi\right) & \text{(i)} \\ -\mu \cos \psi \left(\frac{1}{6} + \frac{1}{4} \mu \sin \psi - \frac{1}{6} (\mu \sin \psi)^3\right) & \text{(ii)} \\ \mu \cos \psi \left(\frac{1}{6} + \frac{1}{4} \mu \sin \psi\right) & \text{(iii)} \end{cases}$$

$$M_\theta = \begin{cases} \frac{1}{8} + \frac{1}{3} \mu \sin \psi + \frac{1}{4} (\mu \sin \psi)^2 & \text{(i)} \\ \frac{1}{8} + \frac{1}{3} \mu \sin \psi + \frac{1}{4} (\mu \sin \psi)^2 - \frac{1}{12} (\mu \sin \psi)^4 & \text{(ii)} \\ -\left(\frac{1}{8} + \frac{1}{3} \mu \sin \psi + \frac{1}{4} (\mu \sin \psi)^2\right) & \text{(iii)} \end{cases}$$

where the coefficients have separate definitions in three regions of the disk defined by

Region (i) $0 < \mu \sin \psi < \mu$

Region (ii) $-1 < \mu \sin \psi < 0$

Region (iii) $-\mu < \mu \sin \psi < -1$

This is the homogeneous equation for small perturbations of the flapping motion of the blade about an equilibrium state; the derivation of this equation may be found in the literature (Ref. 1). β is the degree of freedom representing the blade flapping motion perturbation. The equation is nondimensionalized with the rotor speed, so the time variable is the azimuth angle ψ. ν is the rotating natural frequency (nondimensionalized with the rotor speed) of the flapping motion, which may be greater than 1.0 for flapping hinge offset or cantilever root restraint of the blade. γ is the Lock number, defined by $\gamma = \rho\, acR^4/I_b$ (ρ is the air density, a the two-dimensional lift curve slope, c the blade chord, and R the blade radius); I_b is the equivalent mass of the flapping motion, given by the integral over the span of the square of the mode shape of the flapping motion weighted by the mass per unit length of the blade; for the rigid flapping motion of an articulated blade, the mode shape is proportional to the radial distance from the hinge, and so I_b is just the moment of inertia of the blade about the flapping hinge. K_p is the flap proportional feedback gain, better known as $\tan \delta_3$; K_R is the flap rate feedback gain. A feedback law $\Delta\theta = - K_p\beta - K_R\dot{\beta}$ has been used ($\Delta\theta$ is the blade pitch change due to flapping feedback control). μ is the rotor advance ratio (forward velocity divided by rotor tip speed). The coefficients $M_{\dot{\beta}}$, M_β, and M_θ are the aerodynamic forces on the blade, hence their multiplication by γ and their dependence on μ. The three regions for the coefficients reflect the influence of the reverse flow region of the rotor disk. In region (i) there is normal flow over the entire blade span; in region (iii) reverse flow over the entire span; and in region (ii) normal flow outboard of $r = -\mu \sin \psi$ and reverse flow inboard. Region (iii) is encountered only if $\mu > 1$. The aerodynamic coefficients were obtained using a rigid blade motion, and should properly be changed some to handle a blade with cantilever root restraint. However the major effects of a cantilever root on the dynamics of the system are due to the change in ν and γ (both are increased, ν to 1.15 say and γ to about 5/3 the Lock number based on the rigid mode inertial). Since these are free parameters in the analysis this formulation of the problem should give reasonable results for all rotors.

This equation has been studied numerically in recent literature, primarily in the context of Floquet theory (Ref. 2), which must be used because the aerodynamic coefficients are periodic in ψ if $\mu \neq 0$. The equation will be studied in this paper using the techniques of perturbation theory. The mathematical techniques are not very sophisticated actually; they are rather long, especially when the higher order solutions are sought; and there are some tricks to be learned, but the standard ones work for most systems, including this one (Refs. 3 and 4). It is not maintained that the differential equation studied is a true model of rotor dynamics; nonlinear aerodynamics and coupling with pitch and lag motions are certainly very important. The purpose of this paper is not to present a study of true flapping dynamics; rather it is intended to demonstrate what information can be obtained by the perturbation techniques, and to explore the methods which are most useful for rotor dynamics, so helicopter engineers will be able to decide whether to use these techniques with more complicated or more realistic systems. The dynamic problem considered here is the question of rotor flapping stability. The stability of the motion is determined by the roots or eigenvalues of the system (there are two for this second order equation), and so most of the results discussed will be concerned with the roots. The equation considered is linear; perturbation theory is used because the aerodynamic coefficients are time varying (specifically, periodic) for forward flight, i.e., when μ is greater than zero. A brief discussion of the characteristic behavior of the eigenvalues of a periodic system is given in Appendix I.

NOMENCLATURE

K_P	Flap proportional feedback gain
K_R	Flap rate feedback gain
M_β	Aerodynamic moment due to flapping displacement
$M_{\dot\beta}$	Aerodynamic moment due to flapping rate
M_θ	Aerodynamic moment due to blade pitch
β	Flap motion degree of freedom
γ	Blade Lock number
λ	Eigenvalue or root of the system
μ	Rotor advance ratio (forward speed/rotor tip speed)
ν	Rotating natural frequency of flap motion (centrifugal and structural stiffening), nondimensionalized with rotor rotational speed
ψ	Rotor azimuth angle, measured from downstream
o( )	"the order of"
$\overline{(\ )}$	Conjugate of a complex number

LHP	Left-hand plane
RHP	Right-hand plane
LHS	Left-hand side
RHS	Right-hand side
Re	Real part of a complex number
Im	Imaginary part of a complex number

ANALYSIS AND DISCUSSION

Introduction to Perturbation Techniques

Fundamental to the use of perturbation techniques is the existence of some parameter which is either very small or very large (how small or how large is determined during the analysis); for the moment represent the small parameter (or the inverse of the large parameter) by ϵ. In the present problem it is desired to find the roots of the motion, which means investigating a solution which is uniformly valid over long time periods. The appropriate perturbation technique is the method of multiple time scales. This method assumes that the behavior of the system may be investigated over several time scales, i.e.,

$$\psi_n = \epsilon^n \psi$$

The time scales ψ_n are all assumed to be the same order; then for $\psi_1 = \epsilon\psi$ the actual time ψ must be of order ϵ^{-1}, i.e., very large compared to the basic time scale $\psi_0 = \psi$. Next the dependent variables are expanded as a series in ϵ,

$$\beta = \beta_0(\psi_0, \psi_1, \psi_2, \ldots) + \epsilon\beta_1(\psi_0, \psi_1, \ldots) + \ldots$$

where the terms β_0, β_1, etc. are all assumed to be the same order, and depend on all the time scales now. The requirement that all the β_n be the same order for the long time scale behavior of the motion is crutial to obtaining the solution: it leads, for certain values of the free parameters, to critical regions characterized typically by a reduction of the stability of the system. The details of this method will be given below in the context of the treatment of the rotor flapping equation.

Often an equation of motion is such that in the limit $\epsilon = 0$ the order of the differential equation is reduced. Such problems are called boundary layer problems, since they are characterized by narrow regions in which the solution changes greatly. The outer solution may be found by use of a substitution of the form

$$\beta = \exp \int^{\psi} p \, d\psi$$

followed by an expansion of p as a series in ϵ:

$$p = \frac{1}{\epsilon^n} p_{-n} + \cdots + p_0 + \epsilon p_1 + \cdots$$

This main solution is not valid in certain narrow transition regions or boundary layers. A basic part of this perturbation technique is methods to obtain solutions through the transition region, so that it is possible to match one main solution to another on the other side of the transition region, or to boundary conditions at the base of the boundary layer. Again, details of the method will be given in the context of the solution of the flapping equation.

For the flapping equation there are two parameters which may be used for perturbation quantities: the advance ratio μ and the Lock number γ. Then there are four cases to be considered: small and large μ, and small and large γ. The flapping natural frequency ν is also a parameter in the problem, but it varies little and furthermore always has a value at or slightly above unity (i.e., is neither small nor large). Each of these four cases will be examined in turn in the following sections.

The Small μ Case

For the small μ case (to $0(\mu^2)$) it is possible to ignore the reverse flow region, and the aerodynamic coefficients in region (i) can be used for all ψ. The equation of motion is then (considering the case $K_R = 0$ first):

$$\ddot{\beta} + \left(\frac{\gamma}{8} + \frac{\gamma}{6}\mu\sin\psi\right)\dot{\beta}$$

$$+ \left[\nu^2 + \mu\cos\psi\left(\frac{\gamma}{6} + \frac{\gamma}{4}\mu\sin\psi\right) + K_P\left(\frac{\gamma}{8} + \frac{\gamma}{3}\mu\sin\psi + \frac{\gamma}{4}(\mu\sin\psi)^2\right)\right]\beta = 0 \qquad (2)$$

The small parameter is the advance ratio μ; the perturbation technique to be used is the method of multiple time scales. The solution will be examined to $0(\mu^2)$.

Hover

For the hover case, i.e., the limit $\mu = 0$, the equation reduces to

$$\ddot{\beta} + \frac{\gamma}{8}\dot{\beta} + \left(\nu^2 + K_P\frac{\gamma}{8}\right)\beta = 0$$

which is a constant coefficient equation now. The roots are obtained from

$$\lambda^2 + \frac{\gamma}{8}\lambda + \left(\nu^2 + K_P\frac{\gamma}{8}\right) = 0$$

as

$$\lambda = -\frac{\gamma}{16} + i\sqrt{\nu^2 + \frac{\gamma}{8}K_P - \left(\frac{\gamma}{16}\right)^2} \qquad (3)$$

(and its conjugate).

Expansion in μ

Using the method of multiple time scales, the behavior of the equation is examined for ψ of order 1, μ^{-1}, μ^{-2}, etc.; that is, let

$$\psi_0 = \psi$$
$$\psi_1 = \mu\psi$$
$$\psi_2 = \mu^2\psi$$

Next expand β as a series in μ, with each term depending on all the time scales ψ_n:

$$\beta = \beta_0(\psi_0, \psi_1, \psi_2, \ldots) + \mu\beta_1(\psi_0, \psi_1, \ldots) + \ldots$$

The time derivative now becomes

$$\frac{d}{d\psi} = \frac{\partial}{\partial\psi_0} + \mu\frac{\partial}{\partial\psi_1} + \mu^2\frac{\partial}{\partial\psi_2} + \ldots$$

So that the ordinary differential equation (Eq. 2) now becomes a partial differential equation. Furthermore, the two remaining parameters in the equation, ν and γ, are also expanded as series in μ:

$$\nu = \nu_0 + \mu\nu_1 + \mu^2\nu_2 + \ldots$$

$$\gamma = \gamma_0 + \mu\gamma_1 + \mu^2\gamma_2 + \ldots$$

(This is done because it is the characteristic of a system with periodic coefficients that for certain values of ν_0 and γ_0 there are stability degradation regions described by boundaries in ν_1 and γ_1, or ν_2 and γ_2, etc.).

Now β, $d/d\psi$, ν, and γ have all been expanded as series in μ. These expansions are substituted in the differential equation. It is assumed that all the coefficients in the expansion are of the same order; thus ψ_0, ψ_1, ψ_2, etc. must all be of order 1; and β_0, β_1, β_2, etc. must all be of the same order for the behavior over all the time scales ψ_n (how large is arbitrary since the equation is linear in β, although if β is too large the equation of motion may not be valid). The equation of motion will then contain terms of order 1, μ, μ^2, etc.; all the terms of like order are collected and separately equated to zero, to give the equation that starts the analysis at each order.

Order 1 Results

To order 1 the equation is

$$\frac{\partial^2}{\partial\psi_0^2}\beta_0 + \frac{\gamma_0}{8}\frac{\partial}{\partial\psi_0}\beta_0 + \left(\nu_0^2 + K_P\frac{\gamma_0}{8}\right)\beta_0 = 0 \tag{4}$$

The solution of this equation is

$$\beta_0 = \mathrm{Re}\left[\beta_{01}(\psi_1, \psi_2, \ldots)\, e^{\lambda_0\psi_0}\right] \tag{5}$$

where the root λ_0 is given by

$$\lambda_0^2 + \frac{\gamma_0}{8}\lambda_0 + \left(\nu_0 + K_P\frac{\gamma_0}{8}\right) = 0$$

or

$$\lambda_0 = -\frac{\gamma_0}{16} + i\sqrt{\nu_0^2 + K_P\frac{\gamma_0}{8} - \left(\frac{\gamma_0}{16}\right)^2} \tag{6}$$

and its conjugate. Since β_0 depends on all the time scales, $\beta_0 = \beta_0(\psi_0, \psi_1, \ldots)$, Eq. 4 is a partial differential equation, and only determines the behavior of β_0 as a function of ψ_0. Thus the quantity β_{01} still depends on ψ_1, ψ_2, etc.

The order 1 equation is identical with that obtained for $\mu = 0$, i.e., the hover limit, and indeed λ_0 is exactly the hover root (to order 1). The variation of these roots with γ for $\nu = 1$ and several K_P is shown in Fig. 1 (the $\mu = 0$ loci). The complex portion of the root locus is a circular arc, with center on the real axis at $\lambda = -K_P$, and radius of $\sqrt{\nu^2 + K_P^2}$. The corresponding γ for a point on the complex portion of the γ locus may be obtained from the real part of λ since $\text{Re}\,\lambda = -\gamma/16$ (no dependence on ν or K_P). For $\gamma = 0$ the locus is at $\lambda = i\nu$ (there is no effect of K_P since there are no aerodynamic terms if $\gamma = 0$). For $K_P > 0$, $\text{Im}\,\lambda$ increases as γ increases from zero; a peak in $\text{Im}\,\lambda$ is reached at $\gamma/16 = K_P$ where $\lambda = -K_P + i\sqrt{\nu^2 + K_P^2}$ (the peak occurs just over the center of the circle so the frequency is given by the circle radius). For $K_P < 0$, $\text{Im}\,\lambda$ decreases immediately as γ increases from zero. The locus intercepts the real axis at $\gamma/16 = K_P + \sqrt{\nu^2 + K_P^2}$, where $\lambda = -K_P + \sqrt{\nu^2 + K_P^2}$. Then as $\gamma \to \infty$, the roots remain on the real axis, one going to $\lambda = -\infty$ and the other to $\lambda = -K_P$ (the center of the circle). Thus one branch of the γ locus crosses into the RHP if $K_P < 0$; the crossover point occurs for $\gamma/16 = -\nu^2/2K_P$; at this γ the other branch is at $\lambda = \nu^2/K_P$ (which is less than zero since $K_P < 0$). When the solution is examined to higher order in μ (as below), special problems occur when the frequency of the hover root is at or near a multiple of $\frac{1}{2}$/rev. The order 1 root crosses $\text{Im}\,\lambda = \frac{1}{2}$ for $\gamma/16 = K_P + \sqrt{\nu^2 + K_P^2 - 1/4}$. The locus will cross $\text{Im}\,\lambda = 1$ for $\gamma/16 = K_P + \sqrt{\nu^2 + K_P^2 - 1}$. Since $\nu \geq 1$ there can be only one crossing of $\text{Im}\,\lambda = 1$ or $\frac{1}{2}$ by the locus (except when $\nu = 1$, in which case the locus starts at $\text{Im}\,\lambda = 1$ for $\gamma = 0$).

The root loci for fixed γ and varying ν or K_P would be somewhat simpler than the γ loci. What is being varied is the natural frequency of the system, $\omega_n = \sqrt{\nu^2 + K_P \frac{\gamma}{8}}$ for (λ complex, $|\lambda| = \omega_n$). For the complex portions of the loci, Re $\lambda = -\gamma/16$ is fixed, so the locus would be a vertical line in the LHP. For $\nu^2 + K_P\,(\gamma/8) = \infty$ the locus would be at Im $\lambda = \infty$; for $\nu^2 + K_P\,(\gamma/8) = (\gamma/16)^2$ the locus would intercept the real axis, i.e., would be at Im $\lambda = 0$. For smaller $\nu^2 + K_P\,(\gamma/8)$ the locus would have two branches on the real axis. For $\nu^2 + K_P\,(\gamma/8) = -\infty$ the locus would be at $\lambda = \pm\infty$. The locus would go through the origin, crossing into the RHP, at $\nu^2 + K_P\,(\gamma/8) = 0$ (the other branch would be at $\lambda = -(\gamma/8)$); since $\nu \geq 1$ this can occur only for $K_P = -(8/\gamma)\nu^2 \leq -8/\gamma$, i.e., for sufficiently negative K_P.

The preceding paragraphs have discussed the behavior of the hover root loci, i.e., the $\mu = 0$ roots. For $\mu \neq 0$, λ_0 has the same form as the hover roots, but in terms of γ_0 and ν_0; so it is not the entire root but rather only the order 1 part of it. Thus for example, if Im λ_0 is exactly at a multiple of $\frac{1}{2}$/rev for some μ, this only implies that the hover root is <u>near</u> that point; the hover root, based on $\gamma = \gamma_0 + \mu\gamma_1 + \ldots$ and $\nu = \nu_0 + \mu\nu_1 + \ldots$ must be a small distance ($0(\mu)$ if γ_1 and ν_1 are not zero) away from λ_0, which is based on γ_0 and ν_0 only. The order μ, μ^2, etc. parts of the roots for $\mu \neq 0$ will be obtained in the analysis below.

<u>Order μ Results</u>

The order μ terms in the differential equation are (dropping the common factor of μ):

$$-\left[\frac{\gamma^2}{\partial\psi_0{}^2}\beta_1+\frac{\gamma_0}{8}\frac{\partial}{\gamma\psi_0}\beta_1+\left(\nu_0{}^2+K_P\frac{\gamma_0}{8}\right)\beta_1\right]$$

$$=2\frac{\partial^2}{\partial\psi_0\partial\psi_1}\beta_0+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_1}\beta_0+\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\frac{\partial}{\partial\psi_0}\beta_0$$

$$+\left(2\,\nu_0\nu_1+\frac{\gamma_0}{6}\cos\psi_0+\frac{\gamma_1}{8}K_P+\frac{\gamma_0}{3}K_P\sin\psi_0\right)\beta_0 \tag{7}$$

This equation will be regarded as an ordinary differential equation for β_1 in terms of ψ_0. The behavior of the RHS as a function of ψ_0 is obtained by substituting the solution for β_0; this gives:

$$-\left[\frac{\partial^2}{\partial\psi_0{}^2}\beta_1+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_0}\beta_1+\left(\nu_0{}^2+K_P\frac{\gamma_0}{8}\right)\beta_1\right]$$

$$=\Bigg[\left(2\,\lambda_o+\frac{\gamma_0}{8}\right)\frac{\partial\beta_{01}}{\partial\psi_1}+\left(\frac{\gamma_1}{8}\lambda_0+2\nu_0\nu_1+\frac{\gamma_1}{8}K_P\right)\beta_{01}$$

$$+\left(\frac{\gamma_0}{6}\lambda_0+\frac{\gamma_0}{3}K_P\right)\left(\frac{e^{i\psi_0}-e^{-i\psi_0}}{2i}\right)\beta_{01}+\frac{\gamma_0}{6}\left(\frac{e^{i\psi_0}+e^{-i\psi_0}}{2}\right)\beta_{01}\Bigg]e^{\lambda_0\psi_0}$$

$$+\Bigg[\left(2\bar{\lambda}_0+\frac{\gamma_0}{8}\right)\frac{\partial\bar{\beta}_{01}}{\partial\psi_1}+\left(\frac{\gamma_1}{8}\bar{\lambda}_0+2\nu_0\nu_1+\frac{\gamma_1}{8}K_P\right)\bar{\beta}_{01}$$

$$+\left(\frac{\gamma_0}{6}\bar{\lambda}_0+\frac{\gamma_0}{3}K_P\right)\left(\frac{e^{i\psi_0}-e^{-i\psi_0}}{2i}\right)\bar{\beta}_{01}+\frac{\gamma_0}{6}\left(\frac{e^{i\psi_0}+e^{-i\psi_0}}{2}\right)\bar{\beta}_{01}\Bigg]e^{\bar{\lambda}_0\psi_0} \tag{8}$$

This is to be considered an ordinary differential equation for β_1 of the form

$$-\left[\frac{\partial^2}{\partial\psi_0{}^2}\beta_1+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_0}\beta_1+\left(\nu_0{}^2+K_P\frac{\gamma_0}{8}\right)\beta_1\right]=A_1\,e^{\lambda_0\psi_0}+\ldots$$

where A_1 is a complex constant (which really depends on ψ_1, ψ_2, etc.). The solution to this equation is

$$\beta_1 = \mathrm{Re}\left(\beta_{11}\, e^{\lambda_0 \psi_0} - \frac{A_1}{2\lambda_0 + \frac{\gamma_0}{8}}\, \psi_0\, e^{\lambda_0 \psi_0} + \ldots\right)$$

where $\beta_{11}\, e^{\lambda_0 \psi_0}$ is the homogeneous solution. The particular solution for β_1 has a term proportional to $A_1 \psi_0 e^{\lambda_0 \psi_0}$; compare this with the solution for β_0:

$$\frac{\beta_1}{\beta_0} = \frac{(\text{constant}) A_1 \psi_0 e^{\lambda_0 \psi_0}}{(\text{constant}) e^{\lambda_0 \psi_0}} = (\text{constant}) A_1 \psi_0$$

Then β_1 will become arbitrarily large compared to β_0 if ψ_0 is large enough, which violates the assumption that β_0 and β_1 are of the same order for all ψ_0. The only way such a term in β_1 can be avoided as if it is required that A_1 be zero. Recall however that the equation for β_1 is really a partial differential equation, and A_1 has terms like $\partial\beta_{01}/\partial\psi_1$ and β_{01}. Thus setting $A_1 = 0$ gives a differential equation for β_{01} in terms of ψ_1, the solution of which carries the solution for β_0 out to time scales of the order of μ^{-1}. In general, the forcing terms on the RHS of the equation come from the homogeneous solutions for the lower orders of the β expansion. It is the nature of the perturbation expansion (not of the particular equation being studied) that to each order the equation for β_n always has the same homogeneous solution (in this case $e^{\lambda_0 \psi_0}$). Thus the equation for β_n is being forced by its own homogeneous solution, which gives rise to solutions of the form ψ_0 times the homogeneous solution ($\psi_0 e^{\lambda_0 \psi_0}$ here), unless the coefficient of the homogeneous solution is set to zero. It is a fundamental feature of the method of multiple time scales that setting this coefficient to zero gives another differential equation, which may be used to find the behavior of β_{n-1} to the next time scale.

The method of multiple time scales, as outlined above and described in more detail in the literature (Ref. 3), involves then the following steps.

a) Expand β, $d/d\psi$, and all parameters as series in μ.

b) Obtain the partial differential equation for order μ^n; write it as an ordinary differential equation for β_n in terms of ψ_0; substitute the solutions obtained for $\beta_1, \beta_2, \ldots, \beta_{n-1}$ into the RHS.

c) Find in the forcing terms the coefficient of the homogeneous solution; this coefficient is called the secular term. Set the secular term to zero, thereby obtaining a differential equation for β_{n-1} in terms of ψ_1. This is done in order that the solution be uniformly valid for all time. Usually it is the behavior of the solution to longer time scales (e.g., $\beta_0(\psi_0, \psi_1)$) that is of interest, rather than the higher order corrections to the solution (e.g., $\beta_1(\psi_0)$, which is an order μ correction to β_0). So it is really the differential equation resulting from the secular term that is sought.

For higher orders, $0(\mu^2)$ and above, the solution procedure is a bit more involved (there are then secular terms in the secular terms), but this is best explained by example.

Returning now to the flap equation to order μ (Eq. 8), the secular term (the coefficient of $e^{\lambda_0\psi_0}$) is, if $\bar{\lambda}_0 \pm i \neq \lambda_0$:

$$\left(2\lambda_0 + \frac{\gamma_0}{8}\right)\frac{\partial\beta_{01}}{\partial\psi_1} + \left(\frac{\gamma_1}{8}\lambda_0 + 2\nu_0\nu_1 + \frac{\gamma_1}{8}K_P\right)\beta_{01} = 0$$

or

$$\frac{\partial \beta_{01}}{\partial \psi_1} - \lambda_1 \beta_{01} = 0 \tag{9}$$

where

$$\lambda_1 = - \frac{\frac{\gamma_1}{8}(\lambda_0 + K_P) + 2\nu_0 \nu_1}{2\lambda_0 + \frac{\gamma_0}{8}} = \frac{-\frac{\gamma_1}{16}(\lambda_0 + K_P) - \nu_0 \nu_1}{i \operatorname{Im} \lambda_0}$$

The solution of this equation is $\beta_{01} = \operatorname{Re}(\beta_{02}(\psi_2 \ldots) e^{\lambda_1 \psi_1})$; then the solution for β is

$$\beta = \operatorname{Re}\left(\beta_{02}(\psi_2 \ldots) e^{\lambda_0 \psi_0 + \lambda_1 \psi_1}\right) + 0(\mu) \tag{10}$$

The root is

$$\lambda = \lambda_0 + \mu \lambda_1 = - \frac{\gamma}{16} + i\sqrt{\nu^2 + K_P \frac{\gamma}{8} - \left(\frac{\gamma}{16}\right)^2} \tag{11}$$

to order μ. Thus to order μ the root remains the hover root; there is no effect of advance ratio or of the periodic coefficients. This is the case for most γ and ν, the exception being when γ and ν are near γ_0 and ν_0 such that $\bar{\lambda}_0 \pm i = \lambda_0$.

If $\bar{\lambda}_0 + i = \lambda_0$ then the periodic coefficients contribute to the secular term; $\bar{\lambda}_0 + i = \lambda_0$ means that γ_0 and ν_0 are such that

$$\operatorname{Im} \lambda_0 = \frac{1}{2}$$

or

$$\nu_0^2 + K_P \frac{\gamma_0}{8} - \left(\frac{\gamma_0}{16}\right)^2 = \frac{1}{4}$$

So this case occurs when the hover root has a frequency near ½/rev. (Note that $\bar{\lambda}_0 - i = \lambda_0$ is not possible because λ_0 has been defined to have positive imaginary part.) The 0(1) root in this case is

$$\lambda_0 = -\frac{\gamma_0}{16} + \frac{i}{2}$$

The secular term is now

$$\left(2\lambda_0 + \frac{\gamma_0}{8}\right)\frac{\partial\beta_{01}}{\partial\psi_1} + \left(\frac{\gamma_1}{8}\lambda_0 + 2\nu_0\nu_1 + \frac{\gamma_1}{8}K_P\right)\beta_{01}$$

$$+\left(\frac{\gamma_0}{12i}\bar{\lambda}_0 + \frac{\gamma_0}{12} + \frac{\gamma_0}{6i}\,K_P\right)\bar{\beta}_{01} = 0$$

or

$$\frac{\partial\beta_{01}}{\partial\psi_1} - \alpha\beta_{01} + \left(\frac{\gamma_0^2}{192} - \frac{\gamma_0}{6}K_P - \frac{\gamma_0}{24}i\right)\bar{\beta}_{01} = 0 \tag{12}$$

where

$$\alpha = \alpha_r + i\,\alpha_i = -\frac{\gamma_1}{16} + i\left[2\nu_0\nu_1 - \frac{\gamma_1}{8}\left(\frac{\gamma_0}{16} - K_P\right)\right]$$

The $\bar{\beta}_{01}$ term arises due to the periodic coefficients. The solution to the equation (see Appendix II) depends on the quantity

$$D^2 = \alpha_i^2 - \left[\left(\frac{\gamma_0^2}{192} - \frac{\gamma_0}{6}K_P\right)^2 + \left(\frac{\gamma_0}{24}\right)^2\right]$$

$$= \left[2\nu_0\nu_1 - \frac{\gamma_1}{8}\left(\frac{\gamma_0}{16} - K_P\right)\right]^2 - \left(\frac{\gamma_0}{24}\right)^2\left[1 + \left(\frac{\gamma_0}{8} - 4K_P\right)^2\right] \tag{13}$$

Now if $D^2 > 0$, β_{01} has terms with time behavior like

$$e^{-(\alpha_r \pm i D)\psi_1} = e^{-\frac{\mu\gamma_1}{16}\psi \pm i D\mu\psi},$$

and then β_0 has terms like

$$\beta_{01} e^{\lambda_0\psi_0} = e^{-\frac{\gamma}{16}\psi \pm i \psi(\frac{1}{2} + \mu D)};$$

the damping is unchanged by the additional secular terms, and there is an $0(\mu)$ change in the frequency. If $D^2 < 0$, β_{01} has terms like

$$e^{-(\alpha_r \pm \mu D)\psi_1} = e^{\left(-\frac{\mu\gamma_1}{16} \pm \mu D\right)\psi},$$

so β_0 has terms like

$$\beta_{01} e^{\lambda_0\psi_0} = e^{\left(-\frac{\gamma}{16} \pm \mu D\right)\psi + \frac{i}{2}\psi};$$

there is an $0(\mu)$ change in the damping (both more and less stable), while the frequency remains fixed at $\frac{1}{2}$/rev. $D^2 = 0$ must give the boundary between the two types of behavior. Consider next the interpretation of the quantity D. Constant D^2 implies, for a given γ_0, ν_0, and K_P, that α_i is a constant, i.e.,

$$2\nu_0\nu_1 - \frac{\gamma_1}{8}\left(\frac{\gamma_0}{16} - K_P\right) = \pm\sqrt{D^2 + \left(\frac{\gamma_0}{24}\right)^2\left[1 + \left(\frac{\gamma_0}{8} - 4K_P\right)^2\right]}$$

$$= \pm\ (\text{constant}) \qquad (14)$$

This equation represents a straight line of ν versus γ with a slope of

$$\frac{\partial\nu_1}{\partial\gamma_1/16} = \frac{\gamma_0/16 - K_P}{\nu_0}$$

Compare this with the slope of the line represented by

$$\operatorname{Im} \lambda_0 = \sqrt{\nu_0^{\,2} + K_P \frac{\gamma_0}{8} - \left(\frac{\gamma_0}{16}\right)^2} = \frac{1}{2}:$$

$$\frac{\partial \nu_0}{\partial \gamma_0/16} = \frac{\gamma_0/16 - K_P}{\nu_0}$$

Thus the lines given by D^2 = constant are parallel to the line given by $\operatorname{Im} \lambda_0 = \frac{1}{2}$ (these can be considered lines of ν as a function of γ). Furthermore, in this case γ_1 and ν_1 give an $0(\mu)$ perturbation from γ_0 and ν_0, which are such that $\operatorname{Im} \lambda_0 = \frac{1}{2}$; thus it follows, since a given value of D^2 gives two lines of ν_1 versus γ_1, that D^2 = constant represents two lines an $0(\mu)$ distance either side of and parallel to the line $\operatorname{Im} \lambda_0 = \frac{1}{2}$. Two values of D^2 are of particular interest, first the boundary $D^2 = 0$, and second the maximum possible negative value of D^2. The latter is given by $\alpha_i = 0$, i.e.,

$$2\nu_0\nu_1 - \frac{\gamma_1}{8}\left(\frac{\gamma_0}{16} - K_P\right) = 0$$

This line runs through $\nu_1 = \gamma_1 = 0$ and has the same slope as the $\operatorname{Im} \lambda_0 = \frac{1}{2}$ line there; thus this line simply represents, to $0(\mu)$, the $\operatorname{Im} \lambda_0 = \frac{1}{2}$ line, and it is sufficient to use the point $\nu_1 = \gamma_1 = 0$ for the minimum D^2. Return now to the solution for β_0 in terms of D; it has the characteristics expected of a periodic system (see Appendix I). $D^2 > 0$ and very large implies ν_1 and γ_1 very large, which means a root far from $\operatorname{Im} \lambda = \frac{1}{2}$. As D^2 decreases, $\operatorname{Re} \lambda$ remains at the basic value $(-\gamma/16)$ but there is an $0(\mu)$ change in the frequency, until at $D^2 = 0$ the frequency has reached exactly $\operatorname{Im} \lambda = \frac{1}{2}$. The boundary $D^2 = 0$ occurs for nonzero values of ν_1 and γ_1, and so the root has reached the Im $\lambda = \frac{1}{2}$ line while the hover root is still an $0(\mu)$ distance away. For $D^2 < 0$, the frequency remains fixed at ½/rev while there is an $0(\mu)$ change in the damping, both positive and negative. This type of change in the stability of the system is characteristic if periodic systems; indeed it appears here due to the terms in the secular equation that come from the periodic coefficients in the equation of motion; it is not seen in the behavior of the basic root to $0(\mu)$. There is a critical region, bounded by $D^2 = 0$, inside which the change in the damping occurs. In many problems with periodic coefficients, the system is unstable inside such a region; in this case however there is the basic (hover) damping, represented by Re $\lambda = -\gamma/16$, which is large and stable. The change in the damping is $\pm\mu D$, which is $0(\mu)$ compared to the basic damping, so the critical region is a region of stability degradation rather than of instability. With this discussion as a guide, the solution of the flapping equation near Im $\lambda_0 = \frac{1}{2}$ will be considered in more detail.

The boundary of the critical region is given by $D^2 = 0$, or

$$2\nu_0\nu_1 = \left(\frac{\gamma_0}{16} - K_P\right)\frac{\gamma_1}{8} \pm \frac{\gamma_0}{24}\sqrt{1 + \left(\frac{\gamma_0}{8} - 4K_P\right)^2} \qquad (15)$$

The first term on the RHS makes the line parallel to Im $\lambda_0 = \frac{1}{2}$, and the second term gives the width of the region; the critical region is a narrow band, of width $0(\mu)$, about Im $\lambda_0 = \frac{1}{2}$. Outside the critical region there is an $0(\mu)$ change in the frequency while the real part of the root is unchanged from the hover value. Inside, the frequency is fixed at ½/rev while there is an $0(\mu)$ change in the damping. The maximum stability change occurs at the center of the critical region, where D^2 has its maximum negative value, i.e., at $\nu_1 = \gamma_1 = 0$. The root there is

$$\lambda = \frac{i}{2} - \frac{\gamma}{16} \pm \mu\, D_{max} \qquad (16)$$

So the maximum stability degradation (and enhancement) is

$$\left(\frac{\Delta\lambda}{\mathrm{Re}\lambda}\right)_{max} = \frac{\mu D_{max}}{\mathrm{Re}\lambda} = \frac{\mu \frac{\gamma}{24}\sqrt{1+\left(\frac{\gamma}{8}-4K_P\right)^2}}{\gamma/16} \approx \frac{2}{3}\mu\sqrt{\nu_0^{\,2}-\frac{\gamma_0}{8}K_P+4K_P^{\,2}} \tag{17}$$

which is an $0(\mu)$ small reduction; the system remains stable because of the large hover damping. In general the root is given by

$$\lambda = \lambda_0 + \alpha_r - i\,\mu\,D \tag{18}$$

where for fixed ν (i.e., $\nu_1 = 0$) have

$$D^2 = \left(\frac{\gamma_1}{16}\right)^2 4\left(\frac{\gamma_0}{16}-K_P\right)^2 - \left(\frac{\gamma_0}{24}\right)^2\left[1+\left(\frac{\gamma_0}{8}-4\,K_P\right)^2\right]$$

Let $\Delta\gamma/16 = \mu(\gamma_1/16)$, so $\gamma = \gamma_0 + \Delta\gamma$; recall γ_0 is given by the requirement $\mathrm{Im}\,\lambda_0 = \frac{1}{2}$; γ must be such that $\Delta\gamma$ is $0(\mu)$ small, i.e., γ must be such that the hover locus is an $0(\mu)$ distance from $\mathrm{Im}\,\lambda = \frac{1}{2}$. Then

$$\lambda = -\frac{\gamma}{16} + \frac{i}{2} - i\sqrt{\left(\frac{\Delta\gamma}{16}\right)^2 4\left(\frac{\gamma}{16}-K_P\right)^2 - \mu^2\left(\frac{\gamma}{24}\right)^2\left[1+\left(\frac{\gamma}{8}-4K_P\right)^2\right]}$$

The critical region boundary is crossed when the quantity under the square root sign is zero, that is when

$$\mu = \mu_{corner} = \frac{\frac{\Delta\gamma}{16}\,2\left(\frac{\gamma}{16}-K_P\right)}{\frac{\gamma}{24}\sqrt{1+\left(\frac{\gamma}{8}-4K_P\right)^2}} \tag{19}$$

for a fixed γ (i.e., for the μ locus), or when

$$\frac{\Delta\gamma}{16} = \left(\frac{\Delta\gamma}{16}\right)_{corner} = \frac{\mu \frac{\gamma}{24}\sqrt{1+\left(\frac{\gamma}{8}+4K_P\right)^2}}{2\left(\frac{\gamma}{16}-K_P\right)} \tag{20}$$

for a fixed μ (i.e., for the γ locus). Then the root locus is given by

$$\lambda = -\frac{\gamma}{16} + \frac{1}{2} - i\left(\frac{\Delta\gamma}{16}\right)^2 2\left(\frac{\gamma}{16} - K_P\right)\sqrt{1 - (\mu/\mu_{corner})^2} \tag{21}$$

or

$$\lambda = -\frac{\gamma}{16} + \frac{1}{2} - i\,\mu\left(\frac{\gamma}{24}\right)\sqrt{1+\left(\frac{\gamma}{8} - 4K_P\right)^2}\sqrt{(\Delta\gamma/\Delta\gamma_{corner})^2 - 1} \tag{22}$$

The $(-i)$ in the last term of λ becomes (± 1) for $\mu > \mu_{corner}$ or $\Delta\gamma < \Delta\gamma_{corner}$. For $K_P = 0$ these expressions simplify to

$$\mu_{corner} = \frac{\Delta\gamma}{16}\frac{3}{2\nu}$$

$$\left(\frac{\Delta\gamma}{16}\right)_{corner} = \mu\frac{2\nu}{3}$$

and

$$\lambda = -\frac{\gamma}{16} + \frac{1}{2} - i\left(\frac{\Delta\gamma}{16}\right)\frac{\gamma}{8}\sqrt{1 - (\mu/\mu_{corner})^2}$$

$$\lambda = -\frac{\gamma}{16} + \frac{1}{2} - i\,\mu\frac{\gamma}{12}\nu\sqrt{(\Delta\gamma/\Delta\gamma_{corner})^2 - 1}$$

Furthermore, for $\mu \ll \mu_{corner}$ or $\Delta\gamma \gg \Delta\gamma_{corner}$, i.e., far outside the critical region, the expression for the root becomes

$$\lambda = -\frac{\gamma}{16} + \frac{i}{2} - i\left(\frac{\Delta\gamma}{16}\right)2\left(\frac{\gamma}{16} - K_P\right)$$

which is just an $0(\mu)$ expansion of the hover root when $\mathrm{Im}\,\lambda$ is near $\frac{1}{2}$/rev. Finally, the solution for β_{01}, with $D^2 > 0$, is

$$\beta_{01} = e^{-\frac{\gamma_1}{16}\psi_1}\Bigg(\beta_{02}(\psi_2 \ldots)\left[-\alpha_i - D + i\left(\frac{\gamma_0}{12}\left(\frac{\gamma_0}{16} - 2K_P\right) - \frac{\gamma_0}{24}i\right)\right]e^{iD\psi_1}$$
$$+ \bar{\beta}_{02}(\psi_2 \ldots)\left[-\alpha_i + D + i\left(\frac{\gamma_0}{12}\left(\frac{\gamma_0}{16} - 2K_P\right) - \frac{\gamma_0}{24}i\right)\right]e^{-iD\psi_1}\Bigg) \qquad (23)$$

with α_i and D given above; a similar expression may be obtained for the case $D^2 < 0$ (see Appendix II).

These results have been used to plot the root loci for varying μ and γ; the results obtained so far are valid for small μ. Figure 1 shows typical root loci for varying γ, with $\mu = 0$ and $\mu = 0.1$. The behavior of the hover loci ($\mu = 0$) has been described above. The hover loci cross $\mathrm{Im}\,\lambda = \frac{1}{2}$ for $\gamma = 37.7$, 13.9, and 5.2 for $K_P = 1$, 0, and -1 respectively; for usual rotors then the $\mathrm{Im}\,\lambda = \frac{1}{2}$ critical region is likely to be encountered only if $K_P \le 0$. The point D on the hover locus ($K_P = 1$, in Fig. 1) is where the locus crosses $\mathrm{Im}\,\lambda = \frac{1}{2}$. As γ increases, since this point is at the center of the critical region ($\Delta\gamma/16 = 0$) it receives the maximum stability change, and so is pulled out to the point B. In terms of the γ locus, as γ increases and the hover locus nears $\mathrm{Im}\,\lambda = \frac{1}{2}$, the root has an $0(\mu)$ change in the frequency, pulling the locus toward $\mathrm{Im}\,\lambda = \frac{1}{2}$. When the locus crosses into the critical region the frequency has just reached $\frac{1}{2}$/rev, and the root locus is at the point A. For still larger γ the frequency remains fixed while the real part of one root decreases and that of the other increases. When γ reaches the value for which the hover root has a frequency of $\frac{1}{2}$/rev, the locus is at the center of the critical region; there the roots have their maximum stability change (which is $0(\mu)$) so the locus is at the point B. As γ increases more, the locus moves toward the other boundary of the critical region. The locus reaches that boundary at the point C, and for still larger γ the frequency is no longer fixed at $\frac{1}{2}$/rev; rather the real part of the root is the same as the hover value, while there is an $0(\mu)$ change in the frequency which decreases in size as γ increases. When $\Delta\gamma/16$ is no longer $0(\mu)$, the locus is again identical (to $0(\mu)$) to the hover locus.

Figure 2 shows typical root loci for several γ and varying μ. The circle the locus starts from is the γ locus for hover ($\mu = 0$) and the appropriate K_P. The γ for each locus may be found from $\operatorname{Re}\lambda$ at $\mu = 0$, since for the hover root $\operatorname{Re}\lambda = -\gamma/16$. As μ increases from zero, for the roots near $\operatorname{Im}\lambda = \frac{1}{2}$ there is an $0(\mu)$ change in the frequency pulling the root toward $\operatorname{Im}\lambda = \frac{1}{2}$, while the damping remains fixed at the hover value. The locus reaches $\operatorname{Im}\lambda = \frac{1}{2}$, i.e., crosses the boundary of the critical region, when $\mu = \mu_{corner}$. For larger μ, the frequency remains fixed at $\frac{1}{2}$/rev while one locus moves to the left (increased stability) and the other to the right (decreases stability). Again, this is the characteristic behavior of roots of a system with periodic coefficients.

Order μ^2 Results

In order to find the roots to $0(\mu^2)$, it is first necessary to complete the $0(\mu)$ solution. After the secular terms have been removed, the equation for β_1 becomes, for $\operatorname{Im}\lambda_0 \neq \frac{1}{2}$:

$$\frac{\partial^2}{\partial \psi_0^{\,2}} \beta_1 + \frac{\gamma_0}{8} \frac{\partial}{\partial \psi_0} \beta_1 + \left(\nu_0^{\,2} + \frac{\gamma_0}{8} K_P\right) \beta_1$$

$$= -\frac{\gamma_0}{6} \beta_{01} \left((\lambda_0 + 2K_P) \sin \psi_0 + \cos \psi_0\right) e^{\lambda_0 \psi_0} + \text{conjugate} \tag{24}$$

The solution of this is

$$\beta_1 = \text{Re}\left[\beta_{11}(\psi_1) e^{\lambda_0 \psi_0} + \frac{\gamma_0}{6} \beta_{01} (A_1 \sin \psi_0 + A_2 \cos \psi_0) e^{\lambda_0 \psi_0}\right] \tag{25}$$

where

$$A_1 = \frac{\lambda_0 + 2K_P - 2i \,\text{Im}\, \lambda_0}{1 - (2 \,\text{Im}\, \lambda_0)^2}, \quad A_2 = \frac{1 \div (\lambda_0 + 2K_P)\, 2i \,\text{Im}\, \lambda_0}{1 - (2 \,\text{Im}\, \lambda_0)^2}$$

or recalling that $\beta_{01} = \beta_{02}\, e^{\lambda_1 \psi_1}$, the solution so far is

$$\beta_0 = \text{Re}\left[\beta_{02}(\psi_2)\, e^{\lambda_1 \psi_1}\, e^{\lambda_0 \psi_0}\right]$$

$$\beta_1 = \text{Re}\left[\beta_{11}(\psi_1)\, e^{\lambda_0 \psi_0} + \frac{\gamma_0}{6} \beta_{02}(\psi_2) (A_1 \sin \psi_0 + A_2 \cos \psi_0)\, e^{\lambda_1 \psi_1}\, e^{\lambda_0 \psi_0}\right] \tag{26}$$

The order μ^2 terms in the equation of motion are (dropping the common factor of μ^2):

$$-\left[\frac{\partial^2}{\partial\psi_0{}^2}\beta_2+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_0}\beta_2+\left(\nu_0{}^2+K_P\frac{\gamma_0}{8}\right)\beta_2\right]$$

$$=2\frac{\partial^2}{\partial\psi_0\partial\psi_1}\beta_1+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_1}\beta_1+\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\frac{\partial}{\partial\psi_0}\beta_1$$

$$+\left(2\nu_0\nu_1+\frac{\gamma_0}{6}\cos\psi_0+K_P\frac{\gamma_1}{8}+K_P\frac{\gamma_0}{3}\sin\psi_0\right)\beta_1$$

$$+\frac{\partial^2}{\partial\psi_1{}^2}\beta_0+2\frac{\partial^2}{\partial\psi_0\partial\psi_2}\beta_0+\frac{\gamma_0}{8}\frac{\partial}{\partial\psi_2}\beta_0+\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\frac{\partial}{\partial\psi_1}\beta_0$$

$$+\left(\frac{\gamma_2}{8}+\frac{\gamma_1}{6}\sin\psi_0\right)\frac{\partial}{\partial\psi_0}\beta_0$$

$$+\left[\nu_1{}^2+2\nu_0\nu_2+\frac{\gamma_1}{6}\cos\psi_0+\frac{\gamma_0}{8}\sin 2\psi+K_P\left(\frac{\gamma_2}{8}+\frac{\gamma_1}{3}\sin\psi_0+\frac{\gamma_0}{4}(\sin\psi_0)^2\right)\right]\beta_0$$

$$=\left[\left(2\lambda_0+\frac{\gamma_0}{8}\right)\frac{\partial\beta_{11}}{\partial\psi_1}\right.$$

$$\left.+\left[\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\lambda_0+2\nu_0\nu_1+\frac{\gamma_0}{6}\cos\psi_0+K_P\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{3}\sin\psi_0\right)\right]\beta_{11}\right]e^{\lambda_0\psi_0}$$

$$+\left[\left(2\lambda_0+\frac{\gamma_0}{8}\right)\frac{\partial\beta_{02}}{\partial\psi_2}+\left(\lambda_1{}^2+\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\lambda_1+\left(\frac{\gamma_2}{8}+\frac{\gamma_1}{6}\sin\psi_0\right)\lambda_0\right.\right.$$

$$+\nu_1{}^2+2\nu_0\nu_2+\frac{\gamma_1}{6}\cos\psi_0+\frac{\gamma_0}{8}\sin 2\psi_0$$

$$\left.\left.+K_P\left(\frac{\gamma_2}{8}+\frac{\gamma_1}{3}\sin\psi_0+\frac{\gamma_0}{4}(\sin\psi_0)^2\right)\right)\beta_{02}\right]e^{\lambda_1\psi_1}\,e^{\lambda_0\psi_0}$$

$$+\frac{\gamma_0}{6}\beta_{02}\left[\left(2\lambda_1+\frac{\gamma_1}{8}+\frac{\gamma_0}{6}\sin\psi_0\right)\left((\lambda_0A_1-A_2)\sin\psi_0+(\lambda_0A_2+A_1)\cos\psi_0\right)\right.$$

$$\left.+\left(\frac{\gamma_0}{8}\lambda_1+2\nu_0\nu_1+\frac{\gamma_0}{6}\cos\psi_0+K_P\left(\frac{\gamma_1}{8}+\frac{\gamma_0}{3}\sin\psi_0\right)\right)(A_1\sin\psi_0+A_2\cos\psi_0)\right]e^{\lambda_1\psi_1}e^{\lambda_0\psi_0}$$

$$+\text{ conjugate} \qquad (27)$$

Considering now $\mathrm{Im}\,\lambda_0 \neq 1$ (already have assumed $\mathrm{Im}\,\lambda_0 \neq \frac{1}{2}$), the secular term is

$$-\left(\frac{\partial\beta_{11}}{\partial\psi_1} - \lambda_1\beta_{11}\right) = \left[\frac{\partial\beta_{02}}{\partial\psi_2} + \left(-\lambda_2 + \left(\frac{K_P\frac{\gamma_0}{8} + \left(\frac{\gamma_0}{6}\right)^2\frac{\lambda_0 A_1}{2} + K_P\left(\frac{\gamma_0}{6}\right)^2 A_1}{2\,i\,\mathrm{Im}\,\lambda_0}\right)\right)\beta_{02}\right] e^{\lambda_1\psi_1} \tag{28}$$

where λ_2 is the $0(\mu^2)$ term in an expansion of the hover root, i.e.,

$$\lambda_0 + \mu\lambda_1 + \mu^2\lambda_2 = -\frac{\gamma}{16} + i\sqrt{\nu^2 + K_P\frac{\gamma}{8} - \left(\frac{\gamma}{16}\right)^2} + 0(\mu^3)$$

Regarding this is an ordinary differential equation for β_{11} in terms of ψ_1, the RHS is a constant times the homogeneous solution. In order that the solution be uniformly valid, i.e., that β_{11} be no more singular than β_{01}, the secular term of <u>this</u> equation must also be set to zero. The result is a differential equation for β_{02} in terms of ψ_2, which will give the root to $0(\mu^2)$:

$$\frac{\partial\beta_{02}}{\partial\psi_2} + \left(-\lambda_2 + \left(K_P\frac{\gamma_0}{8} + \left(\frac{\gamma_0}{6}\right)^2\frac{\lambda_0 A_1}{2} + K_P\left(\frac{\gamma_0}{6}\right)^2 A_1\right)/(2\,i\,\mathrm{Im}\,\lambda_0)\right)\beta_{02} = 0 \tag{29}$$

Thus the root is

$$\lambda = \lambda_0 + \mu\lambda_1 + \mu^2\left(\lambda_2 - \left(K_P\frac{\gamma_0}{8} + \left(\frac{\gamma_0}{6}\right)^2\frac{\lambda_0 A_1}{2} + K_P\left(\frac{\gamma_0}{6}\right)^2 A_1\right)/(2\,i\,\mathrm{Im}\,\lambda_0)\right)$$

$$= \lambda_{hover} + \mu^2 i\left(\left(\frac{\gamma_0}{6}\right)^2\frac{\nu_0^2 - \frac{\gamma_0}{8}K_P + 4K_P^2}{4\,\mathrm{Im}\,\lambda_0\left(1-(2\,\mathrm{Im}\,\lambda_0)^2\right)} + K_P\frac{\gamma_0}{16}\frac{1}{\mathrm{Im}\,\lambda_0}\right)$$

To order μ^2 this result is

$$\lambda = -\frac{\gamma}{16} + i\sqrt{\nu^2 + (1+\mu^2)\frac{\gamma}{8}K_P - \left(\frac{\gamma}{16}\right)^2\left(1 - \mu^2\frac{8}{9}\frac{\nu^2 - \frac{\gamma}{8}K_P + 4K_P^2}{\left(\frac{\gamma}{16}\right)^2 - \nu^2 - \frac{\gamma}{8}K_P + \frac{1}{4}}\right)} \tag{30}$$

Thus to order μ^2 the roots for most γ_0 and ν_0 (away from $\mathrm{Im}\,\lambda = \frac{1}{2}$ or 1 that is) are just the hover roots with an $0(\mu^2)$ change in the frequency. There are two effects of μ; the first corrects the term $\gamma/8\, K_P$ in the hover frequency to properly account for the average of $K_P M_\theta$, i.e., multiplies this term by the factor $(1 + \mu^2)$. The second effect, that in the last term of the frequency, is entirely due to the periodic aerodynamic coefficients; this is the first effect of the periodic coefficients seen in the analysis, except for the critical regions near $\mathrm{Im}\,\lambda = \frac{1}{2}$. Typical root loci for varying μ, constructed from Eq. 30, are shown in Fig. 2. These are the loci that are not near $\mathrm{Im}\,\lambda = \frac{1}{2}$ or 1; the frequency change is small even at $\mu = 0.5$. Equation 30 may also be used for the branches of the root loci on the real axis when the quantity under the square root sign is negative (i.e., for γ large enough). There are two real roots then, the (+i) in the frequency becoming (±1). A point on the locus of special interest is where one branch of the locus on the real axis crosses into the RHP, i.e., becomes unstable (see Fig. 1). The criterion for this divergence boundary is that $\lambda = 0$, or

$$\nu^2 + (1 + \mu^2)\frac{\gamma}{8} K_P = -\left(\frac{\gamma}{16}\right)^2 \mu^2 \frac{8}{9} \frac{\nu^2 - \frac{\gamma}{8} K_P + 4K_P^{\,2}}{\left(\frac{\gamma}{16}\right)^2 - \nu^2 - \frac{\gamma}{8} K_P + \frac{1}{4}}$$

To order μ^2 this becomes (since $\gamma/8\, K_P$ must be an $0(\mu^2)$ distance from $-\nu^2$)

$$\nu^2 + (1 + \mu^2)\frac{\gamma}{8} K_P = -\nu^2 \mu^2 \frac{16}{9} \frac{\left(\frac{\gamma}{16}\right)^2 + \frac{\nu^2}{2}}{\left(\frac{\gamma}{16}\right)^2 + \frac{1}{4}}$$

or

$$(1 + \mu^2) \frac{\gamma}{8} K_P = - \nu^2 \left(1 + \mu^2 \frac{16}{9} \frac{\left(\frac{\gamma}{16}\right)^2 + \frac{\nu^2}{2}}{\left(\frac{\gamma}{16}\right)^2 + \frac{1}{4}}\right) \tag{31}$$

The effect of μ on the RHS (due to the periodic coefficients) dominates that on the LHS (due to the average of $K_P M_\theta$) for all values of γ and ν. Thus the critical value of negative K_P, beyond which the locus lies in the LHP, is actually increased by increasing μ. The criterion from the hover case is conservative then; this is the opposite of the conclusion that would have been reached from a consideration of the averaged coefficients only.

If $\bar{\lambda}_0 + 2i = \lambda_0$ then the periodic coefficients contribute to the secular term; $\bar{\lambda}_0 + 2i = \lambda_0$ means that ν_0 and γ_0 are such that

$$\mathrm{Im}\, \lambda_0 = 1$$

or

$$\nu_0^2 + K_P \frac{\gamma_0}{8} - \left(\frac{\gamma_0}{16}\right)^2 = 1$$

Thus there will be an $0(\mu^2)$ critical region where the hover root is near $\mathrm{Im}\, \lambda = 1$ The $0(1)$ root in this case is

$$\lambda_0 = - \frac{\gamma_0}{16} + i$$

The secular term for the equation for β_{02} is then

$$-\left(\frac{\partial\beta_{11}}{\partial\psi_1} - \lambda_1\beta_{11}\right) = \left[\frac{\partial\beta_{02}}{\partial\psi_2} + \left(-\lambda_2 + \left(\frac{\gamma_0}{6}\right)^2 \frac{i}{12}\left(\nu_0^{\,2} - \frac{\gamma_0}{8}K_P + 4K_P^{\,2}\right) - K_P\frac{\gamma_0}{16}i\right)\beta_{02}\right]e^{\lambda_1\psi_1}$$

$$+\left[-\frac{\gamma_0}{32} + \left(\frac{\gamma_0}{6}\right)^2 \frac{1}{4i}\left(\bar{A}_2 - \frac{\bar{\lambda}_0}{2}\bar{A}_1 - i\frac{\bar{\lambda}_0}{2}\bar{A}_2 - i\bar{A}_1\right) - \frac{1}{2i}\frac{\gamma_0}{16}K_P\right.$$

$$\left.+\frac{K_P}{4i}\left(\frac{\gamma_0}{6}\right)^2\left(-\bar{A}_1 - i\bar{A}_2\right)\right]\bar{\beta}_{02}\,e^{\bar{\lambda}_1\psi_1} \qquad (32)$$

The first term on the RHS is the same as the RHS of Eq. 28 (with $\mathrm{Im}\,\lambda_0 = 1$). Unless λ_1 is real, so that $\bar{\lambda}_1 = \lambda_1$, the secular equation for β_{02} will therefore be exactly the same as above (Eq. 29). Recalling that

$$\lambda_1 = \frac{-\frac{\gamma_1}{16}(\lambda_0 + K_P) - \nu_0\nu_1}{i\,\mathrm{Im}\,\lambda_0} = -\frac{\gamma_1}{16} + i\,\frac{\frac{\gamma_1}{16}\left(-\frac{\gamma_0}{16} + K_P\right) + \nu_0\nu_1}{\mathrm{Im}\,\lambda_0}$$

$\mathrm{Im}\,\lambda_1 = 0$ requires

$$2\nu_0\nu_1 - \frac{\gamma_1}{8}\left(\frac{\gamma_0}{16} - K_P\right) = 0$$

This is recognized as a line of constant $\mathrm{Im}\,\lambda$; since it goes through $\nu_1 = \gamma_1 = 0$, it is just the $\mathrm{Im}\,\lambda = 1$ line to $0(\mu)$, and it is therefore sufficient to consider only the case $\nu_1 = \gamma_1 = 0$ (so $\lambda_1 = 0$ also). Then the secular term in Eq. 32 is

$$\frac{\partial\beta_{02}}{\partial\psi_2} + \left[-\frac{i}{2}\left(\frac{\gamma_2}{8}\lambda_0 + 2\nu_0\nu_2 + K_P\frac{\gamma_2}{8}\right) + \left(\frac{\gamma_0}{6}\right)^2 \frac{i}{12}\left(\nu_0^{\,2} - \frac{\gamma_0}{8}K_P + 4K_P^{\,2}\right) - K_P\frac{\gamma_0}{16}i\right]\beta_{02}$$

$$+\left[-\frac{\gamma_0}{32} + iK_P\frac{\gamma_0}{32} + i\,\frac{1}{8}\left(\frac{\gamma_0}{6}\right)^2\left(\frac{\gamma_0}{16} - 2K_P\right)\left(-\frac{\gamma_0}{16} + 2K_P + i\right)\right]\bar{\beta}_{02} = 0 \qquad (33)$$

The solution depends on

$$D^2 = \left[\left(\frac{\gamma_0}{16} - K_P\right)\frac{\gamma_2}{16} - \nu_0\nu \;+ \left(\frac{\gamma_0}{6}\right)^2 \frac{\nu_0{}^2 - \frac{\gamma_0}{8}K_P + 4K_P{}^2}{12} - K_P\frac{\gamma_0}{16}\right]^2$$

$$- \left(\frac{\gamma_0}{32}\right)^2 \left[\left(1 + \frac{\gamma_0}{9}\left(\frac{\gamma_0}{16} - 2K_P\right)\right)^2 + \left(K_P - \frac{\gamma_0}{9}\left(\frac{\gamma_0}{16} - 2K_P\right)\right)^2\right]^2 \tag{34}$$

The behavior of the solution near the critical region is similar to that near $\mathrm{Im}\,\lambda_0 = \frac{1}{2}$. The boundary is given by $D^2 = 0$, which gives a narrow band, of width $0(\mu^2)$ here (as opposed to $0(\mu)$ for the $\mathrm{Im}\,\lambda_0 = \frac{1}{2}$ case), about $\mathrm{Im}\,\lambda = 1$. Outside the critical region $(D^2 > 0)$, the damping is the same as the hover root and there is an $0(\mu^2)$ change in the frequency; at the boundary of the critical region the frequency reaches 1/rev. Inside the critical region $(D^2 < 0)$ the frequency is fixed at 1/rev while there is an $0(\mu^2)$ change in the damping, with one root becoming less stable and the other more.

The boundary of the critical region $(D^2 = 0)$ is

$$\left(\frac{\gamma_0}{16} - K_P\right)\frac{\gamma_2}{16} - \nu_0\nu_2 = -\left(\frac{\gamma_0}{6}\right)^2 \frac{\nu_0{}^2 - \frac{\gamma_0}{8}K_P + 4K_P{}^2}{12} + K_P\frac{\gamma_0}{16}$$

$$\pm \frac{\gamma_0}{32}\sqrt{\left[1 + \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)\right]^2 + \left[K_P - \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)^2\right]^2} \tag{35}$$

This is a line parallel to $\mathrm{Im}\,\lambda = 1$. The terms on the LHS give a line parallel to $\mathrm{Im}\,\lambda_0 = 1$; the first two terms on the RHS give the correction required due to the $0(\mu^2)$ change in the frequency found above (Eq. 31), so the line is parallel to $\mathrm{Im}\,\lambda = 1$ for the basic root to $0(\mu^2)$. The last term on the RHS gives the width of the critical region; so the critical region is an $0(\mu^2)$ band around $\mathrm{Im}\,\lambda = 1$.

The maximum stability change occurs for the maximum negative value of D^2, i.e., at

$$\left(\frac{\gamma_0}{16} - K_P\right)\frac{\gamma_2}{16} - \nu_0\nu_2 = -\left(\frac{\gamma_0}{6}\right)^2 \frac{\nu_0^{\,2} - \frac{\gamma}{8}K_P + 4K_P^{\,2}}{12} + K_P\frac{\gamma_0}{16} \tag{36}$$

which is just where $\operatorname{Im}\lambda = 1$ for the basic root (the hover root plus the $0(\mu^2)$ correction to the frequency). At this point the root is

$$\lambda = i - \frac{\gamma}{16} \pm \mu^2 D_{max}$$

So the maximum stability degradation (and enhancement) is

$$\left(\frac{\Delta\lambda}{\operatorname{Re}\lambda}\right)_{max} = \frac{\mu^2 D_{max}}{\operatorname{Re}\lambda} = \frac{\mu^2 \frac{\gamma_0}{32}\sqrt{\left[1 + \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)\right]^2 + \left[K_P - \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)^2\right]^2}}{\frac{\gamma_0}{16}}$$

$$= \frac{\mu^2}{2}\sqrt{\left[1 + \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)\right]^2 + \left[K_P - \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)^2\right]^2} \tag{38}$$

which is an $0(\mu^2)$ small change; the system remains stable because of the large hover damping. In general the root is given by

$$\lambda = \lambda_0 - \frac{\gamma_2}{16} - i\,D\,\mu^2 \tag{39}$$

where for fixed ν (i.e., $\nu_2 = 0$) we have

$$D^2 = \left[\left(\frac{\gamma_0}{16} - K_P\right)\frac{\gamma_2}{16} + \left(\frac{\gamma_0}{6}\right)^2 \frac{\nu_0^{\,2} - \frac{\gamma_0}{8}K_P + 4K_P^{\,2}}{12} - K_P\frac{\gamma_0}{16}\right]^2$$

$$- \left[\frac{\gamma_0}{32}\sqrt{\left(1 + \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)\right)^2 + \left(K_P - \frac{\gamma}{9}\left(\frac{\gamma}{16} - 2K_P\right)^2\right)^2}\right]^2$$

Let $(\Delta\gamma/16) = \mu^2 (\gamma_2/16)$, so $\gamma = \gamma_0 + \Delta\gamma$; recall γ_0 is given by the requirement Im $\lambda_0 = 1$; γ must be such that $\Delta\gamma/16$ is $0(\mu^2)$ small, i.e., γ must be such that the hover locus is an $0(\mu^2)$ distance from Im $\lambda = 1$. Then

$$(\mu^2 D)^2 = \left(\frac{\Delta\gamma}{16}\left(\frac{\gamma_0}{16} - K_P\right) - \mu^2 C_1\right)^2 - (\mu^2 C_2)^2$$

where

$$C_1 = -\left(\frac{\gamma_0}{6}\right)^2 \frac{\nu^2 - \frac{\gamma_0}{8} K_P + 4K_P^{\ 2}}{12} + K_P \frac{\gamma_0}{16}$$

$$C_2 = \frac{\gamma_0}{32}\sqrt{\left[1 + \frac{\gamma_0}{9}\left(\frac{\gamma_0}{16} - 2K_P\right)\right]^2 + \left[K_P - \frac{\gamma_0}{9}\left(\frac{\gamma_0}{16} - 2K_P\right)^2\right]^2}$$

The critical region boundary is crossed when $(\mu^2 D)^2 = 0$, that is when

$$\mu^2 = \mu^2_{\text{corner}} = \frac{\frac{\Delta\gamma}{16}\left(\frac{\gamma_0}{16} - K_P\right)}{C_1 \pm C_2} = \mu_1^{\ 2},\ \mu_2^{\ 2} \tag{40}$$

for the μ locus, or when

$$\frac{\Delta\gamma}{16} = \left(\frac{\Delta\gamma}{16}\right)_{\text{corner}} = \frac{\mu^2 (C_1 \pm C_2)}{\frac{\gamma_0}{16} - K_P} = \left(\frac{\Delta\gamma}{16}\right)_1,\ \left(\frac{\Delta\gamma}{16}\right)_2 \tag{41}$$

for the γ locus. Then the root locus is given by

$$\lambda = -\frac{\gamma}{16} + i - i\left(\frac{\Delta\gamma}{16}\right)\left(\frac{\gamma}{16} - K_P\right)\sqrt{(1 - \mu^2/\mu_1^{\ 2})(1 - \mu^2/\mu_2^{\ 2})} \tag{42}$$

or

$$\lambda = -\frac{\gamma}{16} + i - i\,\mu^2\sqrt{C_1^{\,2} - C_2^{\,2}}\;\sqrt{(\Delta\gamma/\Delta\gamma_1 - 1)\,(\Delta\gamma/\Delta\gamma_2 - 1)} \tag{43}$$

The $(-i)$ in the last term in λ becomes (± 1) inside the critical region. These results have been used to plot in Figs. 1 and 2 typical root loci near $\mathrm{Im}\,\lambda = 1$ for varying γ and μ.

Flap Rate Feedback

The use of flap rate feedback, $K_R = 0$, does not change the behavior of the solution qualitatively. The hover root becomes

$$\lambda_0 = -\frac{\gamma_0}{16}(1 + K_R) + i\sqrt{\nu_0^{\,2} + \frac{\gamma_0}{8}K_P - \left[\frac{\gamma_0}{16}(1 + K_R)\right]^2} \tag{44}$$

and there are critical regions about $\mathrm{Im}\,\lambda_0 = \frac{1}{2}$ and 1 again. The critical region boundaries and stability degradation depend on K_R now. It is necessary that $K_R > -1$ for the hover root to be stable, but $K_R > 0$ will be the usual case anyway.

$\gamma - \mu$ Plane

The results of the small μ analysis may be used to plot lines of constant $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$ on the $\gamma - \mu$ plane. Typical results are shown in Figs. 3, 4, and 5 for $\nu = 1.0$ and $K_P = 0$, 0.1, and -0.1 respectively. The critical regions appear in the $\gamma - \mu$ plane as regions in which $\mathrm{Im}\,\lambda$ is constant ($\frac{1}{2}$/rev or 1/rev); they are indicated in the figures by the circled values of $\mathrm{Im}\,\lambda$ (the region where $\mathrm{Im}\,\lambda = 0$ is where there are two real roots, not a critical region). These figures are interpreted as follows. A horizontal line is a line of constant γ, and so as μ varies it gives the corresponding value of $\mathrm{Re}\,\lambda$

and Im λ as a μ root locus does. Similarly a vertical line is a constant μ line, and so gives λ as a function of γ just as a γ root locus does. For example, consider a horizontal line in Fig. 3 ($K_P = 0$) with $\gamma = 8$, i.e., $\gamma/16 = 0.5$. As μ increases, the line remains parallel to the Re λ = constant lines so Re λ remains fixed at the hover value. The Im $\lambda = \frac{1}{2}$ region comes closer to the horizontal line as μ increases, which means that Im λ moves toward $\frac{1}{2}$/rev. Eventually the constant γ line crosses into the Im $\lambda = \frac{1}{2}$ region; then Im λ is fixed at $\frac{1}{2}$/rev while for each point in the region there are two values of Re λ, giving the damping for the two branches (one more and one less stable than the hover root). This behavior is just that seen already in the μ loci (Fig. 2). Figures 3, 4, and 5 may be compared with similar ones in Ref. 2, which were constructed from numerical calculations; on the basis of this comparison, the $0(\mu^2)$ analytic results are quite accurate up to $\mu = 0.5$ or so. There is some discrepancy between the results for the Im $\lambda = 1$ region however, particularly with $K_P = -0.1$, although the change in scale (Ref. 2, shows results out to $\mu = 2.5$) exaggerate the difference. For ν exactly 1 the analytic results indicate no Im $\lambda = 1$ critical region if $K_P < 0$ (for $\nu_0 = 1$, Eq. 35 shows that the critical region at $\gamma_0/16 = 0$ has zero width unless $K_P = 0$); but only a slightly larger ν (for example $\nu = 1.01$) is necessary to get a sizable critical region with $K_P = -0.1$ (see Fig. 5). The analytic results show the $\nu = 1$ case is a very sensitive one for small γ, and it is unlikely that a numerical calculation would treat the case accurately. Of course an actual rotor will always have ν at least slightly greater than 1, so the numerical calculations would be reliable then; furthermore, the discrepancy may also be an indication that for very small γ the analytic results are not valid out to as large a μ as they are for more reasonable γ.

In any case this discussion illustrates the kinds of problems that may be hidden in a purely numerical solution; they can only be found and studied by analytic procedures (which at least tell where to look for problems).

Reduction to Mathieu's Equation

The equation for rotor flapping stability may be studied by converting it to Mathieu's equation; Mathieu's equation is an equation of the form

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)y = 0$$

It is the classic example of a differential equation with periodic coefficients, and the functions satisfying it (the purely periodic ones are called Mathieu functions) have been well studied and documented. To transform the flapping equation to Mathieu's equation it is first necessary to remove the $\dot{\beta}$ term; this is done by the substitution

$$\beta = y\, e^{-\frac{\gamma}{16}\psi + \mu\frac{\gamma}{12}\cos\psi}$$

This is equivalent to separating out the hover damping from the solution. Then the classic instability regions of Mathieu's equation (for certain values of a versus q) are just the critical regions without the large (stabilizing) hover damping. To get the flapping equation in the required form it is also necessary however to neglect all $0(\mu^2)$ terms; when this is done one obtains (with $K_P = K_R = 0$)

$$a = 4\left[\nu^2 - \left(\frac{\gamma}{16}\right)^2\right]$$

$$q = -\mu\frac{\gamma}{6}\sqrt{1 + \left(\frac{\gamma}{8}\right)^2}$$

and

$$2z = \psi + \tan^{-1}\frac{\gamma}{8}$$

Thus using Mathieu's equation implies only an $0(\mu)$ analysis. There are of course classical techniques for handling the more general equation with periodic coefficients, of the form

$$\frac{d^2y}{dz^2} + f(2z)\,y = 0$$

which is called Hill's equation; f(t) is a general periodic function with period $T = 2\pi$. These techniques could be used to study the flapping equation, to all orders in μ. However these general techniques are all unsatisfactory in that they tend to obscure the physics of the system being studied, both because some transformation is necessary to arrive at the required form of the equation, and because using standard solutions or formal calculation techniques means the great amount of information gained in the process of deriving the solution is lost. Furthermore, the classic treatments have only considered a single degree of freedom system, so that they are not immediately applicable to more general problems.

The Small γ Case

Consider for the small γ case the flapping equation with both proportional and rate feedback. Now μ is arbitrary, so the general equation is considered, of the form

$$\ddot{\beta} + \nu^2\,\beta = \gamma\,[(M_{\dot{\beta}} - K_R M_\theta)\,\dot{\beta} + (M_\beta - K_P M_\theta)\,\beta] \tag{45}$$

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where the aerodynamic coefficients are functions of μ and ψ (periodic in ψ), and include the effect of the reverse flow region (see Eq. 1). The small parameter is the Lock number γ. The perturbation technique to be used is the method of multiple time scales. The solution will be examined primarily to $0(\gamma)$.

Zero Lock Number

In the absence of aerodynamic forces, i.e., the limit $\gamma = 0$, the solution is

$$\beta = \mathrm{Re}\ (\beta_1\ e^{i\nu})$$

so the roots are

$$\lambda = i\nu \tag{46}$$

and its conjugate; i.e., the solution is an undamped oscillation at the rotating natural frequency of the flap motion. With no aerodynamics, there is of course no effect of μ. This result for the root agrees with the low μ result for $\gamma = 0$ (see Fig. 1).

Expansion in γ

Using the method of multiple times scales, write

$$\begin{aligned}\psi_0 &= \psi \\ \psi_1 &= \gamma\psi \\ \psi_2 &= \gamma^2\ \psi \\ &\vdots\end{aligned}$$

and then expand β and $\varphi/\varphi\psi$ as series in γ:

$$\beta = \beta_0 (\psi_0, \psi_1, \ldots) + \gamma \beta_1 (\psi_0, \psi_1, \ldots) + \ldots$$

$$\frac{d}{d\psi} = \frac{\partial}{\partial\psi_0} + \gamma \frac{\partial}{\partial\psi_1} + \gamma^2 \frac{\partial}{\partial\psi_2} + \ldots$$

Also expand the free parameter ν as a series:

$$\nu = \nu_0 + \gamma \nu_1 + \ldots$$

<u>Order 1 Results</u>

To order 1 the equation is

$$\frac{\partial^2}{\partial\psi_0^2} \beta_0 + \nu_0^2 \beta_0 = 0 \tag{47}$$

The solution of this equation is

$$\beta_0 = \mathrm{Re} \left[\beta_{01} (\psi_1, \psi_2 \ldots) e^{i \nu_0 \psi_0} \right]$$

So the roots are

$$\lambda_0 = i \nu_0 \tag{48}$$

and its conjugate. This solution is of course just the $\gamma = 0$ limit.

<u>Order γ Results</u>

The order γ terms in the differential equation give (dropping the common factor of γ):

$$\frac{\partial^2}{\partial\psi_0^2}\beta_1 + \nu_0^2\,\beta_1 = -2\frac{\partial^2}{\partial\psi_0\partial\psi_1}\beta_0 - 2\nu_0\nu_1\beta_0$$

$$+ (M_{\dot\beta} - K_R M_\theta)\frac{\partial\beta_0}{\partial\psi_0} + (M_\beta - K_P M_\theta)\,\beta_0$$

$$= -2\,i\,\nu_0\frac{\partial\beta_{01}}{\partial\psi_1}e^{i\nu_0\psi_0}$$

$$+ \left[-2\nu_0\nu_1 + (M_{\dot\beta} - K_R M_\theta)\,i\,\nu_0 + (M_\beta - K_P M_\theta)\right]\beta_{01}\,e^{i\nu_0\psi_0}$$

$$+ \text{conjugate} \qquad (49)$$

Now expand the aerodynamic coefficients as complex Fourier series:

$$M_{\dot\beta} = \sum_{n=-\infty}^{\infty} M_{\dot\beta}^{n}\,e^{in\psi}$$

where

$$M_{\dot\beta}^{n} = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\psi}\,M_{\dot\beta}\,\varphi\psi$$

and similarly for M_β and M_θ.

If $-\nu_0 + n \neq \nu_0$, i.e., $\nu_0 \neq n/2$ for any integer n (i.e., Im λ_0 not equal to a multiple of ½/rev), then the secular term has no contributions from any harmonics of the aerodynamic coefficients except the zeroth harmonic (the average over the azimuth); setting the secular term to zero gives

$$\frac{\partial\beta_{01}}{\partial\psi_1} + \left[-\,i\,\nu_1 - \tfrac{1}{2}\left(M_{\dot\beta}^{0} - K_R M_\theta^{0}\right) + \frac{i}{2\nu_0}\left(M_\beta^{0} - K_P M_\theta^{0}\right)\right]\beta_{01} = 0 \qquad (50)$$

The solution of this equation is

$$\beta_{01} = \beta_{02}(\psi_2 \ldots)\, e^{\left[i\nu_1 + \frac{1}{2}\left(M^0_{\dot\beta} - K_R M^0_\theta\right) - \frac{i}{2\nu_0}\left(M^0_\beta - K_P M^0_\theta\right)\right]\psi_1} \tag{51}$$

and the solution for β is

$$\beta = \mathrm{Re}\left\{\beta_{02}\, e^{i\nu_0\psi + \gamma\left[i\nu_1 + \frac{1}{2}\left(M^0_{\dot\beta} - K_R M^0_\theta\right) - \frac{i}{2\nu_0}\left(M^0_\beta - K_P M^0_\theta\right)\right]\psi}\right\} + 0(\gamma) \tag{52}$$

The roots are

$$\lambda = i\nu_0 + \gamma\left[i\nu_1 + \frac{1}{2}\left(M^0_{\dot\beta} - K_R M^0_\theta\right) - \frac{i}{2\nu_0}\left(M^0_\beta - K_P M^0_\theta\right)\right]$$

and its conjugate. Before proceeding further, it is noted that for the particular aerodynamic coefficients considered here many of the harmonics are zero. Actual calculation of the harmonics or symmetry arguments can demonstrate that

$$\begin{aligned} M^{1c}_{\dot\beta} &= M^{2s}_{\dot\beta} = M^{3c}_{\dot\beta} = \ldots = 0 \\ M^{0}_{\beta} &= M^{1s}_{\beta} = M^{2c}_{\beta} = \ldots = 0 \\ M^{1c}_{\theta} &= M^{2s}_{\theta} = M^{3c}_{\theta} = \ldots = 0 \end{aligned} \tag{53}$$

(the superscripts relate to a cosine/sine Fourier series). Using the fact that $M^0_\beta = 0$, and also that to $0(\gamma)$ $\nu = \nu_0 + \gamma\nu_1$, the root becomes

$$\lambda = \frac{\gamma}{2}\left(M^0_{\dot\beta} - K_R M_\theta\right) + \quad \left(1 + \frac{\gamma}{2\nu^2} K_P M^0_\theta\right) \tag{54}$$

Flap rate feedback only effects (to $0(\gamma)$) the damping; the real part of λ may be written

$$\mathrm{Re}\,\lambda = \frac{\gamma}{2} M_{\dot\beta}^0 \left[1 + K_R / \left(-\frac{M_{\dot\beta}^0}{M_\theta^0}\right)\right]$$

The ratio $-M_{\dot\beta}^0/M_\theta^0$ determines the relative effect of K_R; this ratio is shown in Fig. 6 as a function of μ. This parameter is a positive number, which varies little with μ (from 1 at $\mu = 0$ to $\frac{1}{2}$ at $\mu = \infty$, with most of the change below $\mu = 1$). The negative of this ratio gives a critical K_R, since it is necessary that $K_R > K_{R_{crit}} = M_{\dot\beta}^0/M_\theta^0$ in order that the system be stable for small γ. $K_R > -\frac{1}{2}$ insures stability for all μ (to $0(\gamma)$ and with $\nu_0 \neq n/2$); for very small μ the criterion is $K_R > -1$, which agrees with the result from the hover root (Eq. 44). For $K_R = 0$ the root is

$$\lambda = -\frac{\gamma}{16}\left(-8M_{\dot\beta}^0\right) + i\nu\left[1 + \frac{\gamma}{16}\frac{K_P}{\nu^2}\left(8M_\theta^0\right)\right] \tag{55}$$

The aerodynamic coefficients $(-8M_{\dot\beta}^0)$ and $(8M_\theta^0)$ are always positive; they have the value 1 for $\mu \to 0$ and are asymptotic to $8/3\pi\,\mu$ and $16/3\pi\,\mu$ respectively for large μ; these coefficients are shown in Fig. 7. To order μ^2 the root is

$$\lambda = -\frac{\gamma}{16} + i\nu\left[1 + \frac{\gamma}{16}\frac{K_P}{\nu^2}\left(1 + \mu^2\right)\right]$$

which agrees with the $0(\gamma)$ expansion of the small μ results for λ (to $0(\mu^2)$, but with $\mathrm{Im}\,\lambda_0 \neq \frac{1}{2}$ or 1).

The root loci for varying γ and varying μ are shown in Fig. 8 for $K_P = 1$, $K_R = 0$, and $\nu = 1$; the locus for $K_P = -1$ is obtained by reflecting this locus about the

$\mathrm{Im}\,\lambda = \nu$ line. The locus for $K_P = 0$ is difficult to plot since $\mathrm{Im}\,\lambda = \nu$ for all γ and μ (to order γ); it may be visualized by projecting the $K_P = 1$ loci onto the line $\mathrm{Im}\,\lambda = \nu$. These loci should be compared with the small γ portions of the curves in Fig. 1, which are for small μ. In Fig. 8, the γ locus for a given μ starts out at $\lambda = i\nu$ always, and is a straight line with slope

$$\frac{\partial\,\mathrm{Im}\,\lambda}{\partial\,\mathrm{Re}\,\lambda} = -\frac{K_P}{\nu}\,\frac{1}{-M^0_{\dot\beta}/M^0_\theta}$$

which varies from $-K_P/\nu$ to $-2K_P/\nu$ for μ from 0 to ∞. The step size on the γ locus, for a unit change in $\gamma/16$, is

$$\sqrt{\left(8M^0_{\dot\beta}\right)^2 + (K_P/\nu)^2\left(8M^0_\theta\right)^2}\,,$$

which varies from

$$\sqrt{1 + (K_P/\nu)^2} \text{ to } \frac{9}{8}\sqrt{1 + \frac{25}{9}(K_P/\nu)^2} \text{ to } \frac{8}{3\pi}\mu\sqrt{1 + 4(K_P/\nu)^2}$$

for μ from 0 to 1 to ∞ respectively. The μ locus for a given γ starts out vertically from the $\mu = 0$ line, and is asymptotic to the $\mu = \infty$ line, with the step size on the locus for a unit change in μ increasing as μ increases. For reasonable μ the locus does not vary much from the small μ results. An $0(\gamma)$ analysis can only obtain the slope of the γ locus at $\gamma = 0$, so the locus is a straight line, as found above; to find the curvature effect it is necessary to go to order γ^2. The significance of the curvature $(0(\gamma^2))$ may be judged from a comparison of the γ loci of Fig. 8 (all μ, small γ) and Fig. 1 (small μ, all γ); on this basis the small γ results should be limited to $\gamma/16 = 0.2$

or less. On the basis of neglect of the curvature effects alone the results might be accepted to higher γ, but the $0(\gamma)$ results will also be limited by the effects of the critical region, which will be examined next.

The discussion of the $0(\gamma)$ analysis has so far only been concerned with the basic roots, meaning the roots away from the influence of a critical region. In this problem, the criterion for being away from a critical region is that $\nu_0 \neq n/2$ for any integer n; this may be written $\nu \neq (n/2) + 0(\gamma)$, i.e., the rotating natural frequency may not be a distance of order $0(\gamma)$ from (n/2)/rev. Since ν is almost always just slightly above 1/rev ($\nu = 1.2$ would be very large for a rotor; it would require very stiff blades and thus also mean high blade loads) this criterion is seldom fulfilled, and the critical regions may be expected to dominate the root loci behavior for small γ. Furthermore, if K_P is large enough positive or negative, the basic locus will also cross Im $\lambda = 3/2$ or $\frac{1}{2}$ for $\gamma/16$ still small (see Fig. 8), so these critical regions may affect the loci even if the Im $\lambda = 1$ region does not.

If $-\nu_0 + n = \nu_0$ for some integer n, then the higher harmonics of the aerodynamic coefficients contribute to the secular equation, and there arise critical regions with behavior of the root loci similar to that encountered already in the small μ case. The criterion $-\nu_0 + n = \nu_0$ means $\nu_0 = n/2$, i.e., $\nu = \nu_0 + \gamma\nu_1 + \ldots$ is $0(\gamma)$ from a multiple of $\frac{1}{2}$/rev; the only cases likely to be encounteref for rotors are $\nu_0 = \frac{1}{2}$, 1, and 3/2 (n = 1, 2, and 3). Only the case $K_R = 0$ will be considered for now, and use will also be made of the fact that $M_\beta^0 = 0$. Setting the secular term in Eq. 49 to zero gives then

$$\frac{\partial \beta_{01}}{\partial \psi_1} + \left(-i\nu_1 - \tfrac{1}{2} M_{\dot{\beta}}^0 - \frac{i}{2\nu_0} K_P M_\theta^0\right) \beta_{01}$$

$$+ \left[\tfrac{1}{2} M_{\dot{\beta}}^n + \frac{i}{2\nu_0}\left(M_\beta^n - K_P M_\theta^n\right)\right] \bar{\beta}_{01} = 0 \tag{56}$$

where recall that M^n is the nth harmonic in the complex Fourier series expansion of the aerodynamic coefficient, and n is here given by $n = 2\nu_0$. For this equation

$$D^2 = \left(\nu_1 + \frac{K_P}{2\nu_0} M_\theta^0\right)^2 - \frac{1}{\left(2\nu_0\right)^2} \left| M_\beta^n - K_P M_\theta^n - i\nu_0 M_{\dot{\beta}}^n \right|^2 \tag{57}$$

Recall that for the aerodynamic coefficients considered here, either the cosine or sine term for each harmonic is zero (Eq. 53), so this parameter may be written

$$D^2 = \left(\nu_1 + \frac{K_P}{2\nu_0} M_\theta^0\right)^2 - \frac{1}{\left(2\nu_0\right)^2}\left(\left|M_\beta^n - i\nu_0 M_{\dot{\beta}}^n\right|^2 + K_P{}^2 \left|M_\theta^n\right|^2\right) \tag{58}$$

For n odd, $M_\beta^n - i\nu_0 M_{\dot{\beta}}^n = \frac{1}{2}\left(M_\beta^{nc} - \nu_0 M_{\dot{\beta}}^{ns}\right)$ and $M_\theta^n = -\frac{i}{2} M_\theta^{ns}$; for n even, $M_\beta^n - i\nu_0 M_{\dot{\beta}}^n = -\frac{i}{2}\left(M_\beta^{ns} + \nu_0 M_{\dot{\beta}}^{nc}\right)$ and $M_\theta^n = \frac{1}{2} M_\theta^{nc}$. The boundary of the critical region is given by $D^2 = 0$; outside the region ($D^2 > 0$) the real part of λ remains at the basic root value ($\gamma/2\ M_{\dot{\beta}}^0$) while there is an $0(\gamma)$ change in the frequency; inside ($D^2 < 0$) there is an $0(\gamma)$ change (both positive and negative) in the real part of λ while the frequency remains fixed at (n/2)/rev. Constant D^2 means $(2\nu_0\nu_1 + K_P M_\theta^0)$ = constant, or

$$\frac{\partial \nu}{\partial \gamma} = \nu_1 = \frac{-K_P}{2\nu_0} M_\theta^0$$

Compare this with the slope of the basic root (Eq. 55) when Im λ = constant:

$$\mathrm{Im}\,\lambda = \nu + \frac{\gamma}{2}\frac{K_P}{\nu} M_\theta^0 = \text{constant}$$

so

$$\left.\frac{\partial\nu}{\partial\gamma}\right|_{\gamma=0} = \frac{-K_P}{2\nu_0} M_\theta^0$$

So the critical region is a narrow band, of width $0(\gamma)$, around Im $\lambda = (n/2)$/rev. The boundary $(D^2 = 0)$ is

$$\nu_1 = \frac{-K_P}{2\nu_0} M_\theta^0 \pm \frac{1}{2\nu_0}\sqrt{\left|M_\beta^n - i\nu_0 M_{\dot\beta}^n\right|^2 + K_P{}^2\left|M_\theta^n\right|^2} \qquad (59)$$

The maximum stability change occurs at the center of the critical region:

$$\nu_1 = -\frac{K_P}{2\nu_0} M_\theta^0$$

which is where the basic root would cross Im $\lambda = \nu_0 = n/2$; there the root is

$$\begin{aligned}\lambda &= i\nu_0 + \frac{\gamma}{2} M_{\dot\beta}^0 \pm \gamma D_{max} \\ &= i\nu_0 + \frac{\gamma}{2}\left(M_{\dot\beta}^0 \pm \frac{1}{\nu_0}\sqrt{\left|M_\beta^n - i\nu_0 M_{\dot\beta}^n\right|^2 + K_P{}^2\left|M_\theta^n\right|^2}\right)\end{aligned} \qquad (61)$$

Inside the critical region the frequency is fixed at $\nu_0 = n/2$ while there is an $0(\gamma)$ change in the damping. The damping of the basic root is itself $0(\gamma)$ however, so in contrast to the small μ case, the critical region can here lead to actual instability, not just stability degradation. In general the root is given by

$$\lambda = i\nu_0 + \frac{\gamma}{2} M^0_{\dot\beta} + i\gamma D \tag{62}$$

Let $\Delta\nu = \gamma\nu_1$, so $\nu = \nu_0 + \Delta\nu = n/2 + \Delta\nu$ and ν must be such that $\Delta\nu$ is $0(\gamma)$ small, i.e., an $0(\gamma)$ distance from (n/2)/rev. Then

$$\lambda = i\nu_0 + \frac{\gamma}{2} M^0_{\dot\beta}$$
$$+ i\sqrt{\left(\Delta\nu + \frac{K_P\gamma}{2\nu_0} M^0_\theta\right)^2 - \left(\frac{\gamma}{2\nu_0}\right)^2 \left(\left|M^n_\beta - i\nu_0 M^n_{\dot\beta}\right|^2 + K_P{}^2 \left|M^n_\theta\right|^2\right)} \tag{63}$$

Far outside the critical region this result approaches the basic root (Eq. 55); however for the small γ case it is difficult to get far away from all critical regions, since $|\Delta\nu| = |\nu - n/2|$ is at most $\frac{1}{4}$, which is not very small. The critical region is crossed when $\gamma D = 0$, i.e., when

$$\gamma = \gamma_{corner} = \frac{2\nu_0\Delta\nu}{-K_P M^0_\theta \pm \sqrt{\left|M^n_\beta - i\nu_o M^n_{\dot\beta}\right|^2 + K_P{}^2\left|M^n_\theta\right|^2}} = \gamma_1, \gamma_2 \tag{64}$$

So the γ locus is given by

$$\lambda = i\nu_0 + \frac{\gamma}{2} M^0_{\dot\beta} + i\Delta\nu\sqrt{(1 - \gamma/\gamma_1)(1 - \gamma/\gamma_2)} \tag{65}$$

Inside the critical region the $(+i)$ in the last term in λ becomes (± 1). The μ locus is best found from Eq. 63 directly, since the harmonics of the aerodynamic coefficients are rather complex functions of μ.

The μ loci show the behavior characteristic of periodic systems, and familiar from the discussion of the loci for small μ. For small μ, Re λ is fixed at the basic root value while the frequency moves toward (n/2)/rev. For ν near 1 and small γ, the locus moves toward Im $\lambda = 1$ for small K_P, toward Im $\lambda = 3/2$ for $K_P = 1$ or so, and toward Im $\lambda = \frac{1}{2}$ for $K_P = 1$ or so. For some μ the locus crosses the boundary of the critical region; at this point the frequency has reached (n/2)/rev. For larger μ Im λ remains at n/2 while the effect of the critical region is to decrease the stability of one root and increase that of the other. The maximum stability change occurs at the center of the critical region. The center is reached when (see Eq. 60)

$$\Delta\nu = \nu - \frac{n}{2} = -\frac{K_P}{n/2}\frac{\gamma}{16}(8M_\theta^0) \tag{66}$$

The aerodynamic coefficient $(8M_\theta^0)$ is a positive number greater than 1 and monotonically increasing with μ; it is shown in Fig. 7. If $K_P \neq 0$, this criterion will always be satisfied for some $\gamma/16$, so the γ locus always reaches the center (just as the γ locus for small μ always goes through the center — see Fig. 1; however in this case the value of $\gamma/16$ required may be outside the range of validity of the solution if K_P is too small). For the μ locus, again if $K_P \neq 0$, the center will be reached for some μ provided $\Delta\nu$ has the same sign as $-K_P$, and $|\Delta\nu| > |K_P|/(n/2)\,\gamma/16$. However, if $K_P = 0$ the center is never reached for either the γ or μ loci unless $\Delta\nu = 0$, i.e., ν is exactly n/2; in that case the locus is always at the center, since the frequency of the basic locus is fixed at $\nu = n/2$ then. This discussion has just been a more quantitative examination of the criterion that the center of the critical region is reached when the basic root would have crossed Im $\lambda = n/2$; it is illustrated graphically in Fig. 8 for $K_P = 1$ and $n/2 = 3/2$.

As such, the criterion is limited by the fact only an $0(\gamma)$ analysis has been used; the curvature of the root loci due to $0(\gamma^2)$ effects can be quite important, particularly for small K_P. For example, for $K_P = 0$ the basic locus has a slope of zero to $0(\gamma)$, i.e., Im λ is equal to ν for all γ and μ, and in general for small K_P the $0(\gamma^2)$ change in the frequency will be more important than the $0(\gamma)$ change. An $0(\gamma^2)$ analysis would change significantly the conclusions about whether the center of the critical region is reached under certain conditions; for example, Fig. 1 indicates that with ν slightly greater than 1 the γ locus (for small μ) would never cross Im $\lambda = 3/2$ for $K_P = 1$ while it would always cross Im $\lambda = 1$ for $K_P = 0$, just the opposite of the conclusions indicated by the $0(\gamma)$ results (Fig. 8). In any case, since the maximum stability change occurs at the center of the critical region, it is useful to examine it as a worst possible case, which may perhaps be approached but never reached for certain values of ν.

Returning now to the behavior of the μ locus, the maximum change in the damping from the value of the basic root is (Eq. 61)

$$\mathrm{Re}\,\lambda = \frac{\gamma}{2}\left(M^0_{\dot\beta} \pm \frac{1}{\nu_0}\sqrt{\left|M^n_\beta - i\,\nu_0 M^n_{\dot\beta}\right|^2 + K_P{}^2\left|M^n_\theta\right|^2}\right) \tag{67}$$

The contribution from the basic root damping, $(\gamma/2)\,M^0_{\dot\beta}$, is always negative (Fig. 7); as for the small μ case it is $0(\gamma)$, but here that means the basic damping is small. In fact it is the same order as the critical region contribution, so the destabilized root may be actually unstable, rather than just a small perturbation from the basic damping as for the small μ case. The behavior of the locus depends on the relative effects of

$M_{\dot{\beta}}^{0}$ and the nth harmonics of the aerodynamic coefficients under the square root in Eq. 67. For most cases the critical region effect dominates, so that as μ increases it eventually reaches a critical value, at which point (for the case of the maximum stability change) one root crosses into the RHP, i.e., becomes unstable. From Eq. 67, it follows that increasing $(K_P)^2$ (K_P either positive or negative) always increases the effect of the critical regions, which means decreasing the critical μ for which the root becomes unstable. Thus the critical μ is a function of $|K_P|$, for each of the critical regions; this function may be found from Eq. 67 by setting Re $\lambda = 0$ (the requirement for crossing the Im λ axis). Since the aerodynamic coefficients are rather complex functions of μ, it is more convenient to find the critical $|K_P|$ as a function of μ:

$$|K_P|_{crit} = \sqrt{\frac{\left(\nu_0 M_{\dot{\beta}}^{0}\right)^2 - \left|M_{\beta}^{n} - i\nu_0 M_{\dot{\beta}}^{n}\right|^2}{\left|M_{\theta}^{n}\right|^2}} \tag{68}$$

This may be regarded as a maximum $|K_P|$ for a given μ; for large $|K_P|$ the locus is in the RHP at that μ. These boundaries of $|K_P|$ versus μ are shown in Fig. 9 for Im λ near $\frac{1}{2}$, 1, and 3/2. With the exception of roots near Im $\lambda = \frac{1}{2}$ (which requires $K_P < 0$ since ν is near 1) with μ above 0.5 or so, Fig. 9 shows the criterion on $|K_P|$ is not very stringent; a value for $|K_P|$ of 2.0 for example is quite large, corresponding to $|\delta_3| = 63.4$ degrees. Figure 9 shows also that near Im $\lambda = 1$ the roots are always stable, regardless of μ, if $|K_P| < \sqrt{2}$; the loci may be expected to be near Im $\lambda = 1$ for zero or small $|K_P|$. In terms of the μ locus, this means that as μ increases the locus does not cross into the RHP. Just after the locus crosses the critical region boundary, the effect of the critical region is seen and one branch moves to the right and the other to the left (as do the loci in Fig. 2). As μ increases further however, the damping of the basic root (which is always stable, and increases with μ) eventually dominates the effect of the critical region, and the root which was becoming less stable turns around before reaching the RHP. So for larger μ both branches of the locus will be moving to the left, i.e., becoming more stable as μ increases. For the root loci near Im = $\frac{1}{2}$ or 3/2, the effect of the critical region remains dominant, and so one root eventually crosses into the RHP as μ is increased. The critical μ is considerably lower for Im = $\frac{1}{2}$ than for Im $\lambda = 3/2$. This points out an undesirable feature of negative pitch flap coupling, $K_P < 0$: not so much that it reduces the critical μ, but rather that it moves the basic root nearer to Im $\lambda = \frac{1}{2}$.

Flap Rate Feedback

The use of flap rate feedback, $K_R \neq 0$, results in no qualitative changes in the behavior of the loci. K_R is however a useful design parameter; it may be used for example to raise the critical μ or $|K_P|$.

Evaluation of the Order γ Results

Numerical calculations were made of the μ root loci for moderate and small values of γ. On the basis of a comparison of the numerical and analytic results, it is concluded that the small γ analysis to order $0(\gamma)$ is useful only for truly small γ, e.g., $\gamma = 2$ or 3 ($\gamma/16 = 0.2$ or so). Problems are encountered with both the basic roots and the effects of the critical region. The basic root to order γ neglects the curvature of the γ locus, which is especially important for zero or small K_P, since then the change of Im λ for small γ is due more to $0(\gamma^2)$ terms than to the $0(\gamma)$ term.

For example, a μ locus based on the $0(\gamma)$ analysis would start out ($\mu = 0$) at the wrong point, the error being the difference between the circle giving the exact γ locus at $\mu = 0$ and a line tangent to the circle at $\gamma = 0$ (see Figs. 2 and 8). The damping of the basic root is $0(\gamma)$ always, no matter what order the analysis is carried to; e.g., the $0(\mu^2)$ results give $\mathrm{Re}\,\lambda = -\gamma/16$ for all γ. But while the basic damping is $0(\gamma)$, the contribution to the damping due to the critical region will have terms that are $0(\gamma^2)$. Thus for large enough γ the conclusions in the discussion above of the effects of the critical region on the μ root loci will not be valid, since they depend on the basic and critical region damping being of the same order in γ. In particular, the behavior of the locus in which the root being destabilized by the critical region turns around and becomes more stable due to the eventual dominance of the basic damping is not possible except for very small γ, for which $0(\gamma^2)$ effects are in fact negligible. Indeed, it was found in the numerical calculations that with $\gamma = 6$ ($\gamma/16 = 0.375$, i.e., not very small), ν near 1, and K_P zero or small so the root is near 1/rev, that the μ locus does not turn around but rather eventually crosses into the RHP. The stability boundaries given in Fig. 9 are only valid then for truly small values of $\gamma/16$.

Order γ^2 Results

To carry the solution to order γ^2, it is first necessary to finish the order γ solution. Considering only the case $K_R = 0$ and $\nu_0 \neq n/2$ for any n (i.e., the basic root), removal of the secular term from the equation for β_1 leaves

$$\frac{\partial^2}{\partial\psi_0^2}\beta_1 + \nu_0^2\,\beta_1 = \sum_{n\neq 0}\left(M^n_{\dot\beta}\, i\nu_0 + M^n_\beta - K_P M^n_\theta\right)\beta_{01}\, e^{i\psi_0(\nu_0 + n)}$$

$$+ \text{ conjugate} \qquad (69)$$

The solution of this equation is

$$\beta_1 = \mathrm{Re}\left[\beta_{11}(\psi_1 \ldots)\, e^{i\nu_0\psi_0} - \beta_{01} \sum_{n\neq 0} \frac{M_{\dot\beta}^n\, i\nu_0 + M_\beta^n - K_P M_\theta^n}{n^2 + 2\nu_0 n}\, e^{i\psi_0(\nu_0 + n)}\right] \tag{70}$$

and with

$$\beta_0 = \mathrm{Re}\left(\beta_{01}\, e^{i\nu_0\psi_0}\right) \tag{71}$$

$$\beta_{01} = \beta_{02}\,(\psi_2 \ldots)\, e^{\lambda_1\psi_1}\,, \quad \lambda_1 = i\nu_1 + \frac{1}{2} M_{\dot\beta}^0 + i\,\frac{K_P}{2\nu_0} M_\theta^0 \tag{72}$$

this completes the solution to γ.

The order γ^2 terms in the equation of motion give (dropping the common factor of γ^2)

$$\begin{aligned}
\frac{\partial^2}{\partial\psi_0^2}\beta_2 + \nu_0^2\,\beta_2 &= -\frac{\partial^2}{\partial\psi_1^2}\beta_0 - 2\,\frac{\partial^2}{\partial\psi_0\partial\psi_2}\beta_0 - 2\,\frac{\partial^2}{\partial\psi_0\partial\psi_1}\beta_1 - 2\nu_0\nu_1\beta_1 \\
&\quad - (\nu_1^2 + 2\nu_0\nu_2)\,\beta_0 + (M_\beta - K_P M_\theta)\,\beta_1 + M_{\dot\beta}\,\frac{\partial\beta_1}{\partial\psi_0} + M_{\dot\beta}\,\frac{\partial\beta_0}{\partial\psi_1} \\
&= \Bigg\{\left[-2i\nu_0\,\frac{\partial\beta_{02}}{\partial\psi_2} + (-\lambda_1^2 - \nu_1^2 - 2\nu_0\nu_2 + M_{\dot\beta}\lambda_1)\,\beta_{02}\right] e^{\lambda_1\psi_1} \\
&\quad + \left[(-2\nu_0\nu_1 + M_\beta - K_P M_\theta + i\nu_0 M_{\dot\beta})\,\beta_{11} - 2i\nu_0\,\frac{\partial\beta_{11}}{\partial\psi_1}\right] \\
&\quad - \beta_{02}\sum_{n\neq 0}\frac{M_{\dot\beta}^n\, i\nu_0 + M_\beta^n - K_P M_\theta^n}{n^2 + 2\nu_0 n}\, e^{in\psi_0}\,[-2\lambda_1 i(\nu_0 + n) - 2\nu_0\nu_1 \\
&\quad + M_\beta - K_P M_\theta + M_{\dot\beta}\, i\,(\nu_0 + n)]\, e^{\lambda_1\psi_1}\Bigg\}\, e^{i\nu_0\psi_0} + \text{conjugate}
\end{aligned} \tag{73}$$

The secular term is, since $\nu_0 \neq n/2$:

$$\frac{\partial\beta_{11}}{\partial\psi_1} - \lambda_1\beta_{11} = \frac{e^{\lambda_1\psi_1}}{-2i\nu_0}\left[2i\nu_0\frac{\partial\beta_{02}}{\partial\psi_2} + \left(\lambda_1^{\ 2} + \nu_1^{\ 2} + 2\nu_0\nu_2 - M_{\dot\beta}^0\lambda_1\right)\beta_{02}\right] \tag{74}$$

Regarding this as a differential equation for β_{11} in terms of ψ_1, its secular term is

$$\frac{\partial\beta_{02}}{\partial\psi_2} - \lambda_2\,\beta_{02} = 0 \tag{75}$$

with solution

$$\beta_{02} = \beta_{03}(\psi_3 \ldots)\, e^{\lambda_2\psi_2} \tag{76}$$

where

$$\lambda_2 = \frac{\lambda_1^{\ 2} + \nu_1^{\ 2} + 2\nu_0\nu_2 - M_{\dot\beta}^0\,\lambda_1}{-2i\nu_0}$$

$$= \frac{i}{2\nu_o}\left[2\nu_0\nu_2 - \frac{\nu_1}{\nu_0}K_P\,M_\theta^0 - \left(\frac{K_P}{2\nu_0}M_\theta^0\right)^2 - \left(\frac{M_{\dot\beta}^0}{2}\right)^2\right]$$

Thus the basic root, to $0(\gamma^2)$, is

$$\lambda = \lambda_0 + \gamma\lambda_1 + \gamma^2\lambda_2$$

$$= i\nu_0 + \gamma\left[i\nu_1 + \frac{1}{2}M_{\dot\beta}^0 + i\frac{K_P}{2\nu_0}M_\theta^0\right]$$

$$+ \gamma^2\frac{i}{2\nu_0}\left[2\nu_0\nu_2 - \frac{\nu_1}{\nu_0}K_P M_\theta^0 - \left(\frac{K_P}{2\nu_0}M_\theta^0\right)^2 - \left(\frac{M_{\dot\beta}^0}{2}\right)^2\right]$$

$$= \frac{\gamma}{2}M_{\dot\beta}^0 + i\nu\left[1 + \frac{\gamma}{2\nu^2}K_P M_\theta^0 - \frac{\gamma^2}{8\nu^2}\left(M_{\dot\beta}^{0\,2} + (K_P/\nu)^2\,M_\theta^{0\,2}\right)\right] \tag{77}$$

As reported above, there is no $0(\gamma^2)$ change in Re λ, and the $0(\gamma^2)$ change in the frequency is dominant for small K_P, indeed for $K_P = 0$ the only change in the frequency is $0(\gamma^2)$. To order μ this root is

$$\lambda = -\frac{\gamma}{16} + i\nu\left\{1 + \frac{\gamma}{16}\frac{K_P}{\nu^2} - \left[\frac{\gamma}{16}\right]^2 \frac{1}{2\nu^2}\left[1 + \left(\frac{K_P}{\nu}\right)^2\right]\right\}$$

which checks with the expansion to $0(\gamma^2)$ of the root from the small μ analysis (the hover root for $0(\mu)$).

The order γ^2 results would significantly alter the plots of the basic root loci shown in Fig. 8. Extending the results for the critical region to $0(\gamma^2)$ would be much more involved because of the greater complexity of the solution for $\beta_{01}(\psi_1)$ when $\nu_0 = n/2$.

The Large γ Case

For the large γ case, consider the general equation of motion, of the form:

$$\ddot{\beta} + \nu^2 \beta = \gamma\left[(M_{\dot{\beta}} - K_R M_\theta)\dot{\beta} + (M_\beta - K_P M_\theta)\beta\right] \tag{78}$$

The small parameter in this case is the inverse of the Lock number. For γ very large, the aerodynamics dominate the system. For $\gamma = \infty$ the inertia and centrifugal spring terms (the LHS of Eq. 78) are negligible, leaving a first order system which does not depend on γ, namely

$$(M_{\dot{\beta}} - K_R M_\theta)\dot{\beta} + (M_\beta - K_P M_\theta)\beta = 0 \tag{79}$$

Reduction of the order of the equation of motion when the small parameter $(1/\gamma)$ is set equal to zero is a characteristic of a boundary layer type of problem. The solution of the reduced equation is valid over most of the range in ψ. As a first order equation however, its solution can involve only one free constant; thus it is not possible to start the solution from the two general initial conditions allowed for the original second order system. Furthermore, at certain points the solution of the reduced equation will exhibit singular behavior, indicating that the assumptions used to derive it must be reexamined. In general, there must be narrow regions in which the higher time derivatives are very large, so that inside the region the inertia terms are of the same order as the aerodynamic terms and may not be neglected (other simplifications of the equation of motion are often possible though). If such a narrow region is used to connect a solution of the reduced equation to two initial conditions it is called a boundary layer; if it is used to connect a solution of the reduced equation to another such solution on the other side of the layer, it is called a transition region. The solution of the reduced equation is called the main solution. More general terminology is inner and outer regions, and inner and outer solutions. Because the procedure for connecting the solutions in the inner and outer regions is central to the analysis of boundary layer problems, the entire analysis technique has been named the method of matched asymptotic expansions; the name properly refers to the process of connecting the inner and outer solutions, but is usually used to include the entire analysis. Many techniques may be used to find the main solutions. Usually the technique used is a direct expansion of the dependent variable (β) as a series in the small parameter $(1/\gamma)$. This technique is not satisfactory for the present problem because it does not yield a solution which is uniformly valid for all ψ

(which is required in order to find the roots); details of the application of this technique to problems are given in Ref. 3. It is also possible to use the method of multiple time scales to find the main solution. This technique is not entirely satisfactory either however, since the equation obtained to lowest order is just the reduced equation given by setting $1/\gamma = 0$ (Eq. 79). Since this is a first order equation, the solution gives only one root, which to lowest order (i.e., for $\gamma = \infty$) is independent of γ. The reduction of the order of the equation means that one of the roots goes to $-\infty$ as γ goes to ∞; i.e., the solution corresponding to this root is exponentially small compared with the solution of the reduced equation. There is no way that the method of multiple time scales (as described here anyway) can find this root. The perturbation technique useful for finding both solutions in the outer regions is the use of a substitution of the form $\beta = \exp \int^{\psi} p d\psi$, followed by an expansion of $p(\psi)$ as a series in the small parameter.

The large γ case is a boundary layer type of problem, which means that in general there will be several outer regions around the azimuth with a separate expression obtained for the solution in each region. Thus it will not in general be possible to find a single solution, uniformly valid for all ψ, from which the eigenvalues of the system may be found by inspection, as was possible for the case of small μ or small γ. Instead it will be necessary to use the general techniques of the analysis of a system with periodic coefficients (as outlined in Appendix I). This entails obtaining the solution for one revolution of the rotor (one period), in pieces if necessary, each piece valid in a particular inner or outer region. First the main solutions must be obtained in the outer regions, but here it is necessary to find both main solutions rather than just the solution of the reduced equation. Next the method of matched asymptotic expansions is used to connect the main solutions across the transition regions, or across boundary layers to initial or final conditions. Finally with the solution constructed over a complete period, the results of Floquet theory (see Appendix I) may be used to find the eigenvalues of the system from the initial and final values of the solution.

Expansion in γ

Consider first the equation with $K_R = 0$; write

$$\beta = \exp \int^{\psi} p\, d\psi \tag{80}$$

and

$$p = \gamma p_{-1} + p_0 + \frac{1}{\gamma} p_1 + \ldots$$

so

$$\dot{\beta} = p \exp \int^{\psi} p\, d\psi$$

$$\ddot{\beta} = (\dot{p} + p^2) \exp \int^{\psi} p\, d\psi$$

Substituting these expressions for β, $\dot{\beta}$, and $\ddot{\beta}$ into the equation of motion, and collecting all terms of like order in γ gives ($\exp \int^{\psi} p\, d\psi$ is a common factor in the entire equation, so drops out; the common factor of γ^n has also been dropped from the following equations):

$$\begin{aligned} 0(\gamma^2):&\quad p_{-1}^2 = M_{\dot{\beta}}\, p_{-1} \\ 0(\gamma):&\quad 2p_{-1}\, p_0 + \dot{p}_{-1} = M_{\dot{\beta}}\, p_o + M_{\beta} - K_P M_{\theta} \\ 0(1):&\quad p_0^{\,2} + 2p_{-1}\, p_1 + \dot{p}_0 + \nu^2 = M_{\dot{\beta}}\, p_1 \end{aligned} \tag{81}$$

etc.

The order γ^2 equation gives $p_{-1} = 0$ or $p_{-1} = M_{\dot{\beta}}$, which give the two main solutions.

First Solution

The order γ^2 equation gives $p_{-1} = 0$ so to order γ find

$$p_0 = -\frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}$$

and to order 1

$$p_1 = \frac{p_0{}^2 + \dot{p}_0 + \nu^2}{M_{\dot{\beta}}} = \frac{\nu^2 - \left(\frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}\right)^{\cdot} + \left(\frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}\right)^2}{M_{\dot{\beta}}}$$

Then the solution for β is

$$\beta = \beta_1 \exp\left[-\int^{\psi} \frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}\, d\psi \right.$$

$$\left. + \frac{1}{\gamma}\int^{\psi} \frac{\nu^2 - \left(\frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}\right)^{\cdot} + \left(\frac{M_\beta - K_P M_\theta}{M_{\dot{\beta}}}\right)^2}{M_{\dot{\beta}}}\, d\psi + 0(\gamma^{-2})\right] \tag{82}$$

where β_1 is a constant.

Second Solution

The order γ^2 equation gives $p_{-1} = M_{\dot{\beta}}$ so to order γ find

$$p_o = \frac{M_\beta - K_P M_\theta - (M_{\dot{\beta}})^{\cdot}}{M_{\dot{\beta}}}$$

Then noting that

$$\exp\left[-\int^{\psi} \frac{(M_{\dot\beta})^{\cdot}}{M_{\dot\beta}}\,d\psi\right] = \exp\left(-\ln M_{\dot\beta}\right) = \left(M_{\dot\beta}\right)^{-1}$$

the solution for β is

$$\beta = \beta_2 \frac{1}{M_{\dot\beta}} \exp\left[\gamma \int^{\psi} M_{\dot\beta}\,d\psi + \int^{\psi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\,d\psi + 0(\gamma^{-1})\right] \tag{83}$$

where β_2 is a constant.

Eigenvalues

From the continual appearance of $M_{\dot\beta}$ in the denominator, it is evident that a transition region occurs where $M_{\dot\beta} = 0$. This criterion means a transition region occurs where the damping goes through zero. Alternatively, if $M_{\dot\beta}$ is near zero, the $\dot\beta$ term in the reduced equation (Eq. 79) is much smaller than the β term, which implies that the inertia ($\ddot\beta$) terms must be included in order to obtain a differential equation with all terms of the same order; that is, there must be a transition region about the point where $M_{\dot\beta} = 0$. As it happens however, $M_{\dot\beta}(\psi, \mu)$ is a negative quantity which never reaches zero; in fact, $8M_{\dot\beta} < -(1 - 2^{-1/3}) = -0.206$ and even that value is never reached unless $\mu > 0.795$; for $\mu = 0$, $8M_{\dot\beta} = -1$ for all ψ. Thus for the case considered ($K_R = 0$), the main solutions are uniformly valid over the whole azimuth, and it is not necessary to deal with transition regions and boundary layers to find the complete solution.

With the solution over all ψ, Floquet theory may now be used to find the eigenvalues. Consider the first solution, to order 1.

$$\beta = \beta_1 e^{-\int_0^{\psi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}} d\psi}$$

This may be written

$$\beta = \beta_1 e^{-\psi \frac{1}{2\pi} \int_0^{2\pi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}} d\psi + f(\psi)} \tag{84}$$

where

$$f(\psi) = -\int_0^{\psi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}} d\psi + \psi \frac{1}{2\pi} \int_0^{2\pi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}} d\psi$$

Using the fact that the aerodynamic coefficients M_β, $M_{\dot\beta}$, and M_θ are periodic, it may be established that f is also periodic in ψ. Now Floquet theory states that the solution to a differential equation with periodic coefficients may be written in the form

$$\beta = \beta_1 e^{\lambda\psi} u(\psi) \tag{85}$$

where β_1 is a constant, λ is the eigenvalue, and $u(\psi)$ is a periodic function (see Appendix I). Comparison of Eqs. 84 and 85 shows that the eigenvalues must be

$$\lambda = -\frac{1}{2\pi} \int_0^{2\pi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}} d\psi$$

This result may be easily extended for both solutions to any order in γ . Then the two main solutions give two roots:

$$\lambda_1 = -\frac{1}{2\pi}\int_0^{2\pi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\, d\psi$$

$$+\frac{1}{\gamma}\frac{1}{2\pi}\int_0^{2\pi} \frac{\nu^2 - \left(\frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\right)^{\cdot} + \left(\frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\right)^2}{M_{\dot\beta}}\, d\psi + 0\,(\gamma^{-2})$$

$$\lambda_2 = \gamma\frac{1}{2\pi}\int_0^{2\pi} M_{\dot\beta}\, d\psi + \frac{1}{2\pi}\int_0^{2\pi} \frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\, d\psi + 0(\gamma^{-1})$$

The symmetry of M_β and $M_{\dot\beta}$ means that

$$\int_0^{2\pi} \frac{M_\beta}{M_{\dot\beta}}\, d\psi = 0$$

With this relation, the roots simplify to

$$\lambda_1 = -K_P\frac{1}{2\pi}\int_0^{2\pi} \frac{M_\theta}{-M_{\dot\beta}}\, d\psi$$

$$+\frac{1}{\gamma}\frac{1}{2\pi}\int_0^{2\pi} \frac{\nu^2 - \left(\frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\right)^{\cdot} + \left(\frac{M_\beta - K_P M_\theta}{M_{\dot\beta}}\right)^2}{M_{\dot\beta}}\, d\psi + 0(\gamma^{-2}) \qquad (86)$$

and

$$\lambda_2 = \gamma \frac{1}{2\pi} \int_0^{2\pi} M_{\dot{\beta}}\, d\psi + K_P \frac{1}{2\pi} \int_0^{2\pi} \frac{M_\theta}{-M_{\dot{\beta}}}\, d\psi + 0(\gamma^{-1}) \tag{87}$$

Thus there are two real roots, one (λ_2) approaching $-\infty$ as γ increases to ∞ ($M_{\dot{\beta}}$ is negative), the other (λ_1) approaching a constant; λ_1 is the root from the reduced equation. This behavior of the γ root loci is expected from the small μ results; Fig. 1 shows that for large enough γ the locus is on the real axis, i.e., there are two real roots, one approaching $-\infty$ and the other $-K_P$ for $\gamma \to \infty$. To lowest order λ_1 does not depend on γ, because it represents the balance of the aerodynamic damping and the aerodynamic spring only. The value of $\lambda_1/(-K_P)$ for varying μ, and $\gamma = \infty$, is shown in Fig. 10; the movement shown takes place entirely on the real axis in the λ plane. As for the small μ case (Fig. 1) the root is on the real axis, in the LHP if $K_P > 0$ and in the RHP — unstable — if $K_P < 0$. The value of $\lambda_1/(-K_P)$ varies from 1 to 7/8 for $\mu = 0$ to ∞, with most of the change between $\mu = 0.5$ and $\mu = 1$; thus there is little variation of the root with μ (to 0(1)). The size of the $0(\gamma^{-1})$ term in λ_1 is indicated by the result for $\mu = 0$, which is easily obtained (since the aerodynamic coefficients are constant then) as

$$\lambda_1 = -K_P - \frac{1}{\gamma/16} \frac{\nu^2 + K_P^{\,2}}{2} + 0(\gamma^{-2})$$

This result agrees with an $0(\gamma^{-1})$ expansion of the hover root from the small analysis. To lowest order λ_2 is:

$$\lambda_2 = \gamma \frac{1}{2\pi} \int_0^{2\pi} M_{\dot{\beta}}\, d\psi + 0(1) = \gamma M_{\dot{\beta}}^0 = -\frac{\gamma}{16}\, 2\, (-8 M_{\dot{\beta}}^0)$$

The aerodynamic coefficient $-8M^0_{\dot\beta}$ is given in Fig. 7. For $\mu \le 1$ it has the value $-8M^0_{\dot\beta} = 1 + \mu^4/8$; for large μ it is asymptotic to $8/3\pi\,\mu$. This root becomes increasingly negative as γ increases, and also as μ increases. The order 1 term in λ_2 is the negative of the lowest order (also $0(1)$) term in λ_2; thus the behavior of this term is also given by Fig. 10.

Flap Rate Feedback

When $K_R = 0$, there are no transition regions because $M_{\dot\beta} < 0$ always. With flap rate feedback, $K_R \neq 0$, the same expressions for the main solutions are obtained except that $M_{\dot\beta}$ is replaced by $M_{\dot\beta} - K_R M_\theta$. The aerodynamic coefficient $-(M_{\dot\beta} - K_R M_\theta)$ can become negative over regions of the disk for certain combinations of μ and K_R; i.e., there may be negative damping over part of the azimuth range. When such regions of negative damping exist it means there must be transition regions about the points where the damping goes through zero. The main solutions obtained above are valid still in the outer regions, but in each region there are two constants, which must be matched through the inner region to the two constants of the next main solution.

The criterion for the existence of transition regions is that there be negative damping on some portion of the disk, i.e., $-(M_{\dot\beta} - K_R M_\theta) < 0$. $M_{\dot\beta}$ is always negative; M_θ is usually positive, but may be negative on the retreating side for large enough μ ($\mu > 0.641$). If K_R is too large positive, the negative values of M_θ on the retreating side eventually dominate $M_{\dot\beta}$ as μ is increased, so there will be negative damping on the retreating side; if K_R is too large negative, $K_R M_\theta$ eventually dominates $M_{\dot\beta}$ on the advancing side and there will be negative damping there if μ is large enough.

Quantitative values of maximum and minimum K_R as a function of μ are given in Fig. 11. For the cases with negative damping there will be transition regions (of width $0(\gamma^{-2/3})$) near where $M_{\dot\beta} - K_R M_\theta = 0$, which greatly complicates the analysis. For these cases it is also expected that there will be other problems, including material computation problems, physical control problems, and large flapping amplitudes. The situation may be compared with stall flutter of a rotor in hover or forward flight, where a limit cycle oscillation is reached with the negative pitch damping in stall balanced by the positive damping below stall, resulting in high amplitude pitch motions and large control loads. Thus while a region of negative damping does not necessarily mean there is a flapping instability, it does mean that there are many problems — analytical, computational, and physical — so requiring $-(M_{\dot\beta} - K_R M_\theta) > 0$ is a reasonable design criterion. This criterion provides a maximum and minimum K_R for a given μ. The limits of K_R from this rule are much easier to obtain than actual stability boundaries; and Fig. 11 shows that although conservative, it is not a serious restriction for μ less than 1 or 2. For large μ it is a serious limitation (for large μ, $K_{R_{max}} \cong 2/3\mu\ (1 + 7/12\mu)$ and $K_{R_{min}} \cong -2/3\mu\ (1 - 7/12\mu)$), indicating that M_θ (blade pitch) is not very good for flapping rate feedback then. Time varying K_R might work better but it would have to be programmed with μ probably. Although the derivation of this rule has been based on the large γ case, the criterion of no negative damping has nothing to do with γ, and so should be a reasonable criterion for all γ. Indeed, the criterion $K_R > -1$ for $\mu = 0$ is the same as from the small μ case, where it is a true stability criterion, and valid for all γ.

The Large μ Case

For the large μ case the general flapping equation of motion, as given in Eq. 1, is used; the small parameter in this case is the inverse of the advance ratio. For μ very large, the aerodynamics again dominate the system. When μ goes to infinity, the β and $\dot{\beta}$ terms are arbitrarily large compared with the $\ddot{\beta}$ term because of the influence of μ in the aerodynamic coefficients. Thus this problem is also of boundary layer type, and the solution is sought as for the large γ case in terms of outer solutions and transition regions. When $K_R \neq 0$, the aerodynamic damping, $K_R M_\theta$, is the same order in μ as the aerodynamic spring term, $M_\beta - K_P M_\theta$, (namely $0(\mu^2)$, see Eq. 1), so setting $1/\mu$ to zero reduces the order of the system. The reduced equation gives one main solution, and the other will be exponentially smaller (or larger). When $K_R = 0$ however, the aerodynamic damping, $M_{\dot{\beta}}$, is of $0(\mu)$ while the aerodynamic spring, $M_\beta - K_P M_\theta$ is $0(\mu^2)$ (Eq. 1), so in order to obtain an equation for the outer solution with the proper ordering of terms it is necessary to include the inertia term $(\ddot{\beta})$ even in the equation for the outer region. That is, for $\mu \rightarrow \infty$ the aerodynamic spring must be balanced by the inertial forces, which leads to an equation of the form

$$\ddot{\beta} + \mu\, C(\psi)\, \dot{\beta} + \mu^2 K(\psi)\, \beta = 0 \tag{88}$$

The solution of this equation is either a rapid sinusoidal oscillation with frequency of $0(\mu)$, or a sum of exponentials with time constants of $0(\mu^{-1})$, depending on whether the aerodynamic spring is negative or positive (the criterion is a bit more complicated really, but that statement will do for the present discussion). Now the aerodynamic spring changes sign in the middle of the advancing side and again in the middle of the retreating side, and at each point there is a transition region (of width $0(\mu^{-2/3})$) across which the solutions must be matched. There are also transition regions (of width $0(\mu^{-2/3})$) between the advancing and retreating sides of the disk (i.e., around $\psi = 0$ and 180°), through which the main solutions must be matched. Thus in contrast with the large γ case, for large μ it is not possible to find an outer solution uniformly valid for all ψ. Rather it is necessary to go through the entire procedure of matching the main solutions through the transition regions and by that process construct a solution over one rotor revolution. Then the results of Floquet theory (Appendix I) may be used to obtain the eigenvalues from the initial and final values of the solution. The procedure for finding the main solutions will again be based on a substitution of the form $\beta = \exp \int^{\psi} p\, d\psi$ with p now expanded as a series in μ^{-1}. The matching techniques of the method of matched asymptotic expansions will be illustrated in the treatment of the transition regions. The procedures required here are reasonably straight forward, but in general the matching techniques can be quite complicated, particularly when solutions are sought to higher order. The reader is directed to Ref. 3 for more details of the method of matched asymptotic expansions.

In regions (i) and (iii) of the rotor disk the differential equation has the form

$$\ddot{\beta} + \nu^2 \beta = -r\gamma\left\{\left[\left(\frac{1}{8} + \frac{1}{6}\mu\sin\psi\right) + K_R\left(\frac{1}{8} + \frac{1}{3}\mu\sin\psi + \frac{1}{4}(\mu\sin\psi)^2\right)\right]\dot{\beta} + \left[\mu\cos\psi\left(\frac{1}{6} + \frac{1}{4}\mu\sin\psi\right) + K_P\left(\frac{1}{8} + \frac{1}{3}\mu\sin\psi + \frac{1}{4}(\mu\sin\psi)^2\right)\right]\beta\right\} \tag{89}$$

where r is a constant with the value +1 in region (i) and -1 in region (iii). Recall from the discussion of Eq. 1 that region (i) is the advancing side of the disk, where the blade has normal flow over its entire span, and region (iii) is the range of ψ on the retreating side where the blade has reverse flow over its entire span. As $\mu \to \infty$, region (iii) occupies nearly all the retreating side, with the exception of $0(\mu^{-1})$ bands near $\psi = 180°$ and $\psi = 360°$. In Eq. 89, μ appears in the aerodynamic coefficients nearly always in the combination $\mu \sin\psi$; any assumptions made about the order of terms based on the order of μ will be violated then if $\sin\psi$ is small enough; this is the origin of the transition regions near $\psi = 0$ and $180°$. When $K_R = 0$, there are also transition regions in the middle of the advancing and retreating sides around the point where the aerodynamic spring goes through zero. These regions arise in the analysis because the aerodynamic spring being zero or very small will again violate the assumptions made about the order of the terms; physically they arise because there must be a transition between the solutions on either side since they have very different behavior, namely sinusoidal oscillation and exponential decay or growth. If $K_R \neq 0$, such transition regions are not required; only the transition regions between th[e a]dvancing and retreating sides are needed. The case with $K_R = 0$ will be examined first.

<u>Expansion in μ</u>

Consider the outer regions, where $\mu \sin\psi$ is of order μ and all time derivatives of β are of the same order; this means the regions (i) and (iii), away from the boundaries near $\psi = 0$ and $180°$. The equation of motion in the outer region is then Eq. 89 with $K_R = 0$. For $K_R = 0$, the analysis is simplified if first the $0(\mu)$ aerodynamic damping is removed from the equation of motion (which is of the form of Eq. 88 to the lowest order); this is accomplished by the following substitution

$$\beta = e^{r\mu\frac{\gamma}{12}\cos\psi} y \qquad (90)$$

With this substitution, Eq. 89 for $K_R = 0$ becomes

$$\ddot{y} + \nu^2 y = - r\frac{\gamma}{8}\dot{y} - r\gamma y\left[-\mu\frac{\gamma}{12}|\sin\psi|\left(\frac{1}{8}+\frac{1}{12}\mu\sin\psi\right)\right.$$
$$\left. + \mu\cos\psi\left(\frac{1}{12}+\frac{1}{4}\mu\sin\psi\right) + K_P\left(\frac{1}{8}+\frac{1}{3}\mu\sin\psi+\frac{1}{4}(\mu\sin\psi)^2\right)\right] \tag{91}$$

Then the main solutions are found by use of the substitution

$$y = \exp\int^{\psi} p\,d\psi \tag{92}$$

with p expanded in a series in μ^{-1}:

$$p = \mu p_{-1} + p_0 + \frac{1}{\mu}p_1 + \ldots$$

Main Solutions

Substituting for y and collecting terms of like order, the equation of motion gives

$$0(\mu^2):\quad p_{-1}^2 = -\frac{\gamma}{4}|\sin\psi|\left(-\frac{\gamma}{36}|\sin\psi| + \cos\psi + K_P\sin\psi\right)$$

$$0(\mu):\quad \dot{p}_{-1} + 2p_{-1}p_o = -r\frac{\gamma}{8}p_{-1} - r\gamma\left(-\frac{\gamma}{96}|\sin\psi| + \frac{1}{12}\cos\psi + \frac{1}{3}K_P\sin\psi\right)$$

From the order μ^2 equation, one obtains:

$$p_{-1} = \pm\sqrt{\frac{\gamma}{4}|\sin\psi|\left(\frac{\gamma}{36}|\sin\psi| - \cos\psi - K_P\sin\psi\right)} \tag{93}$$

The solution for β has a factor of the form $\exp\mu\int^{\psi}p_{-1}d\psi$; the double sign in p_{-1} gives the two main solutions. When the quantity under the square root is positive, p_{-1} is real and there results main solutions with exponential decay or growth (with time constants of $0(\mu^{-1})$); and when the quantity under the square root is negative, p_{-1} is imaginary and there results main solutions with sinusoidal oscillatory behavior (with frequency of $0(\mu)$). When the quantity under the square root is zero, there is a transition region. On the advancing side (region (i)), p_{-1} is zero at

$$\psi_{t_1} = \tan^{-1} \frac{1}{\frac{\gamma}{36} - K_P} \tag{94}$$

and on the retreating side (region (iii)), p_{-1} is zero at

$$\psi_{t_3} = \tan^{-1} \frac{1}{-\left(\frac{\gamma}{36} + K_P\right)} \tag{95}$$

So there is a high frequency oscillation on the rear of the disk, exponential solutions on the front, and transition regions between the two types of behavior. p_{-1} is also zero when $\sin\psi = 0$, i.e., at $\psi = 0$ or 180°, so transition regions will also be required near the edges of regions (i) and (iii). In addition there is region (ii), which for large μ is an order μ^{-1} small band on the retreating side near $\psi = 0$ and 180°; in connecting the main solution from the advancing side to the retreating side it is necessary to go through this region as well as through the transition regions. In general, whenever p_{-1} is small the assumptions made about the order of the terms in obtaining Eq. 92 are violated, so the main solution can no longer be valid there. This criterion gives the four transition regions.

The order μ equation gives

$$p_0 = \frac{-\dot{p}_{-1} - r\frac{\gamma}{8}p_{-1} - r\gamma\left(-\frac{\gamma}{96}|\sin\psi| + \frac{1}{12}\cos\psi + \frac{1}{3}K_P\sin\psi\right)}{2p_{-1}}$$

from which, using the relation

$$\int^{\psi} \frac{\dot{p}_{-1}}{p_{-1}} d\psi = \ell n \left| p_{-1} \right|$$

one obtains

$$\int^{\psi} p_0 d\psi = -\frac{1}{2} \ell n \left| p_{-1} \right| - r \frac{\gamma}{16} \psi - r \frac{\gamma}{2} \int^{\psi} \frac{-\frac{\gamma}{96} \left| \sin\psi \right| + \frac{1}{12} \cos\psi + \frac{1}{3} K_P \sin\psi}{p_{-1}} d\psi$$

With the above expressions for p_{-1} and p_0, the solution for β (to 0(1) in p) is obtained by using the substitutions for β and y, i.e., Eqs. 90 and 92. There are main solutions in four regions, which will be called quadrants (although they are not really so, since ψ_{t_1} and ψ_{t_3} are not equal to 90° and 270°). Each quadrant is bounded by transition regions; the ranges of the four quadrants and the main solution valid in each are as follows.

1st quadrant: $0 < \psi < \psi_{t_1}$

$$\beta = e^{\mu \frac{\gamma}{12} \cos\psi - \frac{\gamma}{16} \psi} (-f)^{-1/4} \left[(C_1 + i\, C_2)\, e^{i\left(\mu \int_{\psi_1}^{\psi} \sqrt{-f}\, d\psi - \int_{\psi_1}^{\psi} \frac{g}{\sqrt{-f}} d\psi\right)} + \text{conjugate} \right] \tag{96}$$

2nd quadrant: $\psi_{t_1} < \psi < \pi$

$$\beta = e^{\mu \frac{\gamma}{12} \cos\psi - \frac{\gamma}{16} \psi} (f)^{-1/4} \left[C_3\, e^{\mu \int_{\psi_2}^{\psi} \sqrt{f}\, d\psi - \int_{\psi_2}^{\psi} \frac{g}{\sqrt{f}} d\psi} + C_4\, e^{-\mu \int_{\psi_2}^{\psi} \sqrt{f}\, d\psi + \int_{\psi_2}^{\psi} \frac{g}{\sqrt{f}} d\psi} \right] \tag{97}$$

3rd quadrant: $\pi < \psi < \psi_{t_3}$

$$\beta = e^{-\mu \frac{\gamma}{12} \cos \psi + \frac{\gamma}{16} \psi} (f)^{-1/4} \left[C_5 \, e^{\mu \int_{\psi_3}^{\psi} \sqrt{f}\, d\psi + \int_{\psi_3}^{\psi} \frac{g}{\sqrt{f}} d\psi} + C_6 \, e^{-\mu \int_{\psi_3}^{\psi} \sqrt{f}\, d\psi - \int_{\psi_3}^{\psi} \frac{g}{\sqrt{f}} d\psi} \right] \tag{98}$$

4th quadrant: $\psi_{t_3} < \psi < 2\pi$

$$\beta = e^{-\mu \frac{\gamma}{12} \cos \psi + \frac{\gamma}{16} \psi} (-f)^{-1/4} \left[(C_7 + i\, C_8)\, e^{i\left(\mu \int_{\psi_4}^{\psi} \sqrt{-f}\, d\psi + \int_{\psi_4}^{\psi} \frac{g}{\sqrt{-f}}\, d\psi\right)} + \text{conjugate} \right] \tag{99}$$

where

$$f = \frac{\gamma}{4} |\sin \psi| \left(\frac{\gamma}{36} |\sin \psi| - \cos \psi - K_P \sin \psi \right)$$

$$g = \frac{\gamma}{2} \left(\frac{1}{12} \cos \psi + \frac{1}{3} K_P \sin \psi - \frac{\gamma}{96} |\sin \psi| \right)$$

and C_1, C_2, ..., C_8 are constants. The matching procedures will result in connection formulas through the transition regions, which will give the two constants of one main solution in terms of the two constants of the main solution on the other side of the transition region. In addition, the constants will be matched to arbitrary initial conditions at a certain point on the disk. The quantities ψ_1, ψ_2, ψ_3, and ψ_4 in Eqs. 96 to 99 are also constants, which must be in the appropriate quadrant; they are not free constants since a change in them must be accompanied by a change in the value of the C's. These angles may be given any value convenient to the analysis; it is most convenient here to leave them arbitrary since they will drop out of the final result anyway.

Transition Regions

Consider the transition region near ψ_t (meaning ψ_{t_1} in region (i) or ψ_{t_3} in region (iii)). This transition region has a width of $0(\mu^{-2/3})$. The reader is directed to Refs. 3 and 4 for illustrations of methods for finding the proper width of a transition region or boundary layer. The technique involves assuming $d\psi = 0(\mu^{-n})$; then $\ddot{\beta}$ is of order μ^{-2n}, and similarly the order of all terms in the equation of motion may be found in terms of n. The exponent n is determined from the criteria that the resulting equation, to lowest order in μ, must a) include the highest time derivative ($\ddot{\beta}$) and b) must produce solutions capable of being matched to the outer solutions. It also helps to know what to expect of certain types of problems; for example, a width of $0(\mu^{-2/3})$ is typical of transition regions for equations of the form of Eq. 88.

Assuming $d\psi = 0(\mu^{-2/3})$, Eq. 91 becomes to lowest order $(0(\mu^{4/3}))$:

$$\ddot{y} = y\,(\psi - \psi_t)\,\mu^2 \left.\frac{\partial f}{\partial \psi}\right|_{\psi=\psi_t} \qquad (100)$$

where

$$\left.\frac{\partial f}{\partial \psi}\right|_{\psi=\psi_t} = r\,\frac{\gamma}{4}$$

Alternatively, write $z = (r\,(\gamma/4)\,\mu^2)^{1/3}(\psi - \psi_t)$, substitute for ψ in the differential equation for y, and then obtain to lowest order the equation

$$\frac{d^2y}{dz^2} - z\,y = 0 \tag{101}$$

This equation is a standard form, the solutions of which are called Airy function. There are two independent solutions, denoted Ai(z) and Bi(z). These functions may be written in terms of Bessel functions; however, the general behavior of the solution in the transition region is not of interest here. Rather the solution in the transition region is only to be used to find a connection formula between the neighboring main solutions. For this purpose all that is required is the behavior of the solution for very large z. Writing the solution of Eq. 101 as

$$y = 2\sqrt{\pi}\,a\,\mathrm{Ai}(z) + \sqrt{\pi}\,b\,\mathrm{Bi}(z)$$

where a and b are constants, then the behavior for large z is:

$$z \to \infty: \quad y \sim z^{-1/4}\,(a\,e^{-\zeta} + b\,e^{\zeta})$$

$$z \to -\infty: \quad y \sim |z|^{-1/4}\left[\left(\frac{b}{2} - i\,a\right)e^{i\frac{\pi}{4}}\,e^{i\zeta} + \text{conjugate}\right]$$

where

$$\zeta = \frac{2}{3}\,|z|^{3/2} = \frac{\sqrt{\gamma}}{3}\,\mu\,|\psi - \psi_t|^{3/2}$$

The matching procedure consists of finding the limit of the outer solution as $\psi \to \psi_t$, and the limit of the inner solution as $z \to \pm\infty$, and requiring that the two limits have identical behavior. This criterion gives the constants in the inner region in terms of the constants in the outer region. The matching procedure is considerably more complex if higher order solutions are involved. Consider first matching from the first quadrant to the second quadrant, through the transition region at $\psi = \psi_{t_1}$. The outer solution in the first quadrant is given in Eq. 96. As $\psi \to \psi_{t_1}$,

$$\int_{\psi_1}^{\psi} \sqrt{-f}\, d\psi = \int_{\psi_1}^{\psi_t} \sqrt{-f}\, d\psi + \int_{\psi_t}^{\psi} \sqrt{-f}\, d\psi$$

$$= \int_{\psi_1}^{\psi_t} \sqrt{-f} - \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2}$$

so the main solution for $\psi \to \psi_{t_1}$ is

$$y_{outer} \to \left|\frac{\gamma}{4}\, \psi - \psi_t\right|^{-1/4} (C_1 + i\, C_2)\, e^{i\left(\mu \int_{\psi_1}^{\psi_t} \sqrt{-f}\, d\psi - \int_{\psi_1}^{\psi_t} \frac{g}{\sqrt{-f}}\, d\psi\right)}\, e^{-i \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2} \mu}$$

+ conjugate

To the inner solution, the outer region in the first quadrant appears as the limit $z \to -\infty$, and in this limit

$$y_{inner} \to \left(\frac{\gamma}{4} \mu^2\right)^{-1/12} |\psi - \psi_t|^{-1/4} \left(\frac{b}{2} + i\, a\right) e^{-i \frac{\pi}{4}}\, e^{-i \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2} \mu}$$

+ conjugate

Then requiring $(y_{outer})_{\psi = \psi_t} = (y_{inner})_{z = -\infty}$ gives

$$(C_1 + i\, C_2)\, e^{i\left(\mu \int_{\psi_1}^{\psi_t} \sqrt{-f}\, d\psi - \int_{\psi_1}^{\psi_t} \frac{g}{\sqrt{-f}}\, d\psi\right)} = \left(\frac{\gamma}{4\mu}\right)^{1/6} e^{-i \frac{\pi}{4}} \left(\frac{b}{2} + i\, a\right) \qquad (102)$$

Similarly, the main solution in the second quadrant (Eq. 97) becomes, for $\psi \to \psi_t$

$$y_{outer} \to \left|\frac{\gamma}{4}\left(\psi - \psi_t\right)\right|^{-1/4} \left(C_3\, e^{\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi - \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}}\, d\psi + \mu \frac{\sqrt{\gamma}}{3} |\psi - \psi_2|^{3/2}} + C_4\, e^{-\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi + \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}}\, d\psi - \mu \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2}} \right)$$

and the inner solution becomes, for $z \to \infty$

$$y_{inner} \to \left(\frac{\gamma}{4}\mu^2\right)^{-1/12} |\psi - \psi_t|^{-1/4} \left(a\, e^{-\mu \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2}} + b\, e^{\mu \frac{\sqrt{\gamma}}{3} |\psi - \psi_t|^{3/2}} \right)$$

and the matching criterion gives

$$\begin{aligned} C_3\, e^{\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi - \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}}\, d\psi} &= \left(\frac{\gamma}{4\mu}\right)^{1/6} b \\ C_4\, e^{-\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi + \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}}\, d\psi} &= \left(\frac{\gamma}{4\mu}\right)^{1/6} a \end{aligned} \qquad (103)$$

Next combining the results of matching between the first quadrant and the inner solution (Eq. 102), and between the inner solution and the second quadrant (Eq. 103), gives the connection formula between the first and second quadrants:

$$(C_1 + i\,C_2)\, e^{i\left(\mu \int_{\psi_1}^{\psi_t} \sqrt{-f}\, d\psi - \int_{\psi_1}^{\psi_t} \frac{g}{\sqrt{-f}}\, d\psi\right)} = e^{-i\frac{\pi}{4}} \left[\frac{1}{2} C_3\, e^{\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi - \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}} d\psi} + i\,C_4\, e^{-\mu \int_{\psi_2}^{\psi_t} \sqrt{f}\, d\psi + \int_{\psi_2}^{\psi_t} \frac{g}{\sqrt{f}} d\psi} \right] \quad (104)$$

Similarly, the matching procedure on the retreating side around ψ_{t_3} gives the connection formula between the third and fourth quadrants.

$$(C_7 + i\,C_8)\, e^{i\left(\mu \int_{\psi_4}^{\psi_t} \sqrt{-f}\, d\psi + \int_{\psi_4}^{\psi_t} \frac{g}{\sqrt{-f}}\, d\psi\right)} = e^{i\frac{\pi}{4}} \left[\frac{1}{2} C_6\, e^{-\mu \int_{\psi_3}^{\psi_t} \sqrt{f}\, d\psi - \int_{\psi_3}^{\psi_t} \frac{g}{\sqrt{f}} d\psi} - i\,C_5\, e^{\mu \int_{\psi_3}^{\psi_t} \sqrt{f}\, d\psi + \int_{\psi_3}^{\psi_t} \frac{g}{\sqrt{f}} d\psi} \right] \quad (105)$$

Now consider the transition regions near $\psi = 0$ or 180°. The main solutions indicate that there must be transition regions at the edges of regions (i) and (iii), but there is also region (ii) in between regions (i) and (iii). The extent of the three regions is defined by (see Eq. 1):

region (i) $\quad \mu \sin\psi > 0$

region (ii) $\quad -1 < \mu \sin\psi < 0$

region (iii) $\quad \mu \sin\psi < -1$

The transition region has a width of order $\mu^{-2/3}$; thus region (ii), which has a width of only order μ^{-1}, lies entirely within the transition region. Region (ii) appears then as an interior region which has no effect on the solution to lowest order. This means it is not necessary to use the more complicated aerodynamic coefficients of region (ii). It also means that the appearance of μ^4 terms in the aerodynamic coefficients is deceptive, since these do not appear except in region (ii), in which region $(\mu \sin \psi)^n$ is of $0(1)$ or smaller for all n. The true order in μ is given by the aerodynamic coefficients in regions (i) and (iii), and these are of $0(\mu^2)$ at most, as expected of aerodynamic forces.

The proper matching procedure is to find the solutions in the transition regions at the edges of regions (i) and (iii); in addition, the solution is found in the interior region including region (ii) and the neighboring parts of regions (i) and (iii) where $\mu \sin \psi$ is of $0(1)$. Then the matching process proceeds from the main solution in region (i), to the transition region at the edge of region (i), to the interior region, to the second transition region in the edge of region (ii), and finally to the main solution in region (iii); by this process the connection formula between regions (i) and (iii) is established. With the substitution $x = \mu \sin \psi$ (x is assumed to be of $0(1)$ so $0\psi = 0(\mu^{-1})$) the equation of motion for the interior region is found to be, to lowest order $(0(\mu^2))$:

$$\frac{d^2\beta}{dx^2} = 0 \tag{106}$$

the solution of which is

$$\beta = \beta_1 + \beta_2 x$$

where β_1 and β_2 are constants. Thus matching through the interior region is just a matter of matching the displacement and slope of the neighboring transition regions.

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To the transition region (of width $0(\mu^{-2/3})$) the interior region appears $0(\mu^{-1/3})$ small; then in terms of the transition region variable $z = \mu^{-1/3} x$, the matching process is to be carried out at the limit $z \to 0$.

With the substitution $z = \mu^{2/3} \sin\psi$ (which implies an $0(\mu^{-2/3})$ region around $\psi = 0$ or 180°), the equation of motion for regions (i) and (iii) (Eq. 89) becomes, to lowest order in μ $(0(\mu^{4/3}))$:

$$\frac{d^2\beta}{dz^2} \pm \frac{\gamma}{4} |z| \beta = 0 \tag{107}$$

The plus sign applies on the back of the disk and the minus sign on the front. In terms of the variable z, the advancing side is given by $z > 0$; and the retreating side by $z < 0$; on this scale region (ii) appears as a negligibly small $(0(\mu^{-1/3}))$ area at $z = 0$.

Consider the back of the disk, i.e., from the fourth to the first quadrant. The differential equation for the transition region is (Eq. 107)

$$\frac{d^2\beta}{dz^2} + \frac{\gamma}{4} |z| \beta = 0$$

The solution again involves Airy functions, and may be written

$$\beta = \begin{cases} a\,\mathrm{Ai}\left[-\left(\frac{\gamma}{4}\right)^{1/3} z\right] + b\,\mathrm{Bi}\left[-\left(\frac{\gamma}{4}\right)^{1/3} z\right] & z > 0 \\ a^*\,\mathrm{Ai}\left[\left(\frac{\gamma}{4}\right)^{1/3} z\right] + b^*\,\mathrm{Bi}\left[\left(\frac{\gamma}{4}\right)^{1/3} z\right] & z < 0 \end{cases}$$

where a, b, a*, and b* are constants. Matching the displacement and slope at $z = 0$ gives $a^* = \sqrt{3}\, b$ and $b^* = a/\sqrt{3}$, so

$$\beta = \sqrt{3}\, b \, \mathrm{Ai}\left[\left(\frac{\gamma}{4}\right)^{1/3} z\right] + \frac{a}{\sqrt{3}} \mathrm{Bi}\left[\left(\frac{\gamma}{4}\right)^{1/3} z\right] \qquad z < 0$$

The asymptotic behavior of the solution in the transition region is then

$$z \to \infty: \ \beta \sim \left[\left(\frac{\gamma}{4}\right)^{1/3} z\right]^{-1/4} \pi^{-1/2} \left[\left(\frac{b - i\, a}{2}\right) e^{i\frac{\pi}{4}} e^{i\zeta} + \text{conjugate}\right]$$

$$z \to -\infty: \ \beta \sim \left[\left(\frac{\gamma}{4}\right)^{1/3} |z|\right]^{-1/4} \pi^{-1/2} \left[\left(\frac{a - i\, 3\, b}{2\sqrt{3}}\right) e^{i\frac{\pi}{4}} e^{i\zeta} + \text{conjugate}\right]$$

where $\zeta = \sqrt{\gamma}/3 \ \ z^{3/2}$, note that the solution has oscillatory behavior, which is the proper behavior for matching the main solutions on the rear of the disk. Now the main solution in the fourth quadrant (Eq. 99) becomes, as $\psi \to 2\pi$

$$\beta_{\text{outer}} \to e^{-\left(\mu \frac{\gamma}{12} - \frac{\gamma}{16} 2\pi\right)} \left(\frac{\gamma}{4} \mu^{-2/3} \ |z|\right)^{-1/4}$$

$$\left[(C_7 + i\, C_8)\, e^{i\left(\mu \int_{\psi_4}^{2\pi} \sqrt{-f}\, d\psi + \int_{\psi_4}^{2\pi} \frac{g}{\sqrt{-f}}\, d\psi\right) - i \frac{\sqrt{2}}{3} |z|^{3/2}} + \text{conjugate}\right]$$

and the main solution in the first quadrant (Eq. 96) becomes, as $\psi \to 0$

$$\beta_{\text{outer}} \to e^{\mu \frac{\gamma}{12} + \frac{\gamma}{16} 2\pi} \left(\frac{\gamma}{4} \mu^{-2/3} \ |z|\right)^{-1/4}$$

$$\left[(C_1 + i\, C_2)\, e^{i\left(\mu \int_{\psi_1}^{0} \sqrt{-f}\, d\psi - \int_{\psi_1}^{0} \frac{g}{\sqrt{-f}}\, d\psi\right) + i \frac{\sqrt{\gamma}}{3} |z|^{3/2}} + \text{conjugate}\right]$$

Then the matching process, requiring $(\beta_{outer})_{\psi=2\pi} = (\beta_{inner})_{z=-\infty}$ in the fourth quadrant and $(\beta_{inner})_{z=\infty} = (\beta_{outer})_{\psi=0}$ in the first quadrant, gives the connection formula:

$$\sqrt{3}\,(C_1 + i\,C_2)\, e^{i\left(\mu \int_{\psi_1}^{0} \sqrt{-f}\, d\psi - \int_{\psi_1}^{0} \frac{g}{\sqrt{-f}} d\psi\right)}\, e^{\mu \frac{\gamma}{6} - \frac{\gamma}{16} 4\pi}$$

$$= 2\,(C_7 + i\,C_8)\, e^{i\left(\mu \int_{\psi_4}^{2\pi} \sqrt{-f}\, d\psi + \int_{\psi_4}^{2\pi} \frac{g}{\sqrt{-f}} d\psi\right)}$$

$$- i\,(C_7 - i\,C_8)\, e^{-i\left(\mu \int_{\psi_4}^{2\pi} \sqrt{-f}\, d\psi + \int_{\psi_4}^{2\pi} \frac{g}{\sqrt{-f}} d\psi\right)} \qquad (108)$$

Finally consider the front of the disk, i.e., from the second to the third quadrant. The differential equation is (Eq. 107):

$$\frac{d^2\beta}{dz^2} - \frac{\gamma}{4} |z|\, \beta = 0 \qquad (109)$$

The solution is

$$\beta = \begin{cases} a\, Ai\left(\left(\frac{\gamma}{4}\right)^{1/3} z\right) + b\, Bi\left(\left(\frac{\gamma}{4}\right)^{1/3} z\right) & z > 0 \\ a^*\, Ai\left(-\left(\frac{\gamma}{4}\right)^{1/3} z\right) + b^*\, Bi\left(-\left(\frac{\gamma}{4}\right)^{1/3} z\right) & z < 0 \end{cases}$$

Matching the displacement and slope at $z = 0$ gives $a^* = \sqrt{3}\, b$ and $b^* = a/\sqrt{3}$, so

$$\beta = \sqrt{3}\, b\, Ai\left(-\left(\frac{\gamma}{4}\right)^{1/3} z\right) + \frac{a}{\sqrt{3}}\, Bi\left(-\left(\frac{\gamma}{4}\right)^{1/3} z\right) \qquad z < 0$$

The asymptotic behavior of this solution is

$$z \to \infty: \quad \beta \sim \left(\left(\frac{\gamma}{4}\right)^{1/3} z\right)^{-1/4} \pi^{-1/2}\left[\frac{a}{2}\, e^{-\zeta} + b\, e^{\zeta}\right]$$

$$z \to -\infty: \quad \beta \sim \left(\left(\frac{\gamma}{4}\right)^{1/3} |z|\right)^{-1/4} \pi^{-1/2}\left[\frac{\sqrt{3}}{2}\, b\, e^{-\zeta} + \frac{a}{\sqrt{3}}\, e^{\zeta}\right]$$

where $\zeta = \sqrt{\gamma}/3\ |z|^{3/2}$; this solution has the exponential behavior required for matching on the front of the disk. The main solution in the second quadrant (Eq. 97) becomes, as $\psi \to \pi$

$$\beta_{outer} \to e^{-\mu\frac{\gamma}{12} - \frac{\gamma}{16}3\pi}\left(\frac{\gamma}{4}\mu^{-2/3}\ |z|\right)^{-1/4}$$

$$\left[C_3\, e^{\mu\int_{\psi_2}^{\pi}\sqrt{f}\,d\psi - \int_{\psi_2}^{\pi}\frac{g}{\sqrt{f}}\,d\psi - \frac{\sqrt{\gamma}}{3}|z|^{3/2}}\right.$$

$$\left. + C_4\, e^{-\mu\int_{\psi_2}^{\pi}\sqrt{f}\,d\psi + \int_{\psi_2}^{\pi}\frac{g}{\sqrt{f}}\,d\psi + \frac{\sqrt{\gamma}}{3}|z|^{3/2}}\right.$$

Matching this to the inner solution, i.e., requiring $(\beta_{outer})_{\psi=\pi} = (\beta_{inner})_{z=\infty}$, gives

$$\begin{aligned} C_3\, e^{\mu\int_{\psi_2}^{\pi}\sqrt{f}\,d\psi - \int_{\psi_2}^{\pi}\frac{g}{\sqrt{f}}\,d\psi}\; e^{-\mu\frac{\gamma}{12} - \frac{\gamma}{16}3\pi} &= \frac{1}{\sqrt{\pi}}\left(\frac{\gamma}{4\mu}\right)^{1/6}\frac{a}{2} \\ C_4\, e^{-\mu\int_{\psi_2}^{\pi}\sqrt{f}\,d\psi + \int_{\psi_2}^{\pi}\frac{g}{\sqrt{f}}\,d\psi}\; e^{-\mu\frac{\gamma}{12} - \frac{\gamma}{16}3\pi} &= \frac{1}{\sqrt{\pi}}\left(\frac{\gamma}{4\mu}\right)^{1/6} b \end{aligned} \qquad (110)$$

The main solution in the third quadrant (Eq. 98) becomes, as $\psi \to \pi$

$$\beta_{outer} \to e^{\mu \frac{\gamma}{12} + \frac{\gamma}{16}\pi} \left(\frac{\gamma}{4}\mu^{-2/3}|z|\right)^{-1/4}$$

$$\left[C_5\, e^{\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi + \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi + \frac{\sqrt{\gamma}}{3}|z|^{3/2}} \right.$$

$$\left. + C_6\, e^{-\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi - \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi - \frac{\sqrt{\gamma}}{3}|z|^{3/2}} \right]$$

Matching this to the inner solution, i.e., requiring $(\beta_{inner})_{z=-\infty} = (\beta_{outer})_{\psi=\pi}$, gives

$$C_5\, e^{\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi + \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi}\; e^{\mu \frac{\gamma}{12} + \frac{\gamma}{16}\pi} = \frac{1}{\sqrt{\pi}}\left(\frac{\gamma}{4\mu}\right)^{1/6} \frac{a}{\sqrt{3}}$$

$$C_6\, e^{-\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi - \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi}\; e^{\mu \frac{\gamma}{12} + \frac{\gamma}{16}\pi} = \frac{1}{\sqrt{\pi}}\left(\frac{\gamma}{4\mu}\right)^{1/6} \frac{\sqrt{3}}{2}\, b \qquad (111)$$

Then combining Eq. 110 and Eq. 111 gives the connection formulae:

$$C_3\, e^{\mu \int_{\psi_2}^{\pi} \sqrt{f}\, d\psi - \int_{\psi_2}^{\pi} \frac{g}{\sqrt{f}}\, d\psi}\; e^{-\mu \frac{\gamma}{6} - \frac{\gamma}{16}4\pi} = \frac{\sqrt{3}}{2}\, C_5\, e^{\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi + \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi}$$

$$C_4\, e^{-\mu \int_{\psi_2}^{\pi} \sqrt{f}\, d\psi + \int_{\psi_2}^{\pi} \frac{g}{\sqrt{f}}\, d\psi}\; e^{-\mu \frac{\gamma}{6} - \frac{\gamma}{16}4\pi} = \frac{2}{\sqrt{3}}\, C_6\, e^{-\mu \int_{\psi_3}^{\pi} \sqrt{f}\, d\psi - \int_{\psi_3}^{\pi} \frac{g}{\sqrt{f}}\, d\psi} \qquad (112)$$

The derivation of the connection formulae across the four transition regions completes the construction of the solution around the disk. It is also necessary however to start and finish the solution with initial and final values of β and $\dot{\beta}$ at some point. It is most convenient to start and finish the solution at $\psi = \pi$; this is the middle of the interior region, where it is easier to match the two constants in the solution to arbitrary values of β and $\dot{\beta}$ than at any other point on the disk (to lowest order, the solution to Eq. 106 in the interior region is linear in x, so the two constants are easily related to β and $\dot{\beta}$). In terms of the transition region, the slope and magnitude of the solution are to be matched to β and $\dot{\beta}$ at z = 0. Now the solution of Eq. 109 becomes, for small z

$$\beta \cong d_1 \, (a + \sqrt{3}\, b) + \left(\frac{\gamma}{4}\right)^{1/3} d_2 \, (- a + \sqrt{3}\, b) \, z$$

from which

$$\beta\,(\pi) = d_1 \, (a + \sqrt{3}\, b)$$

$$\dot{\beta}\,(\pi) = \left(\frac{\gamma}{4}\mu^2\right)^{1/3} d_2 \, (a - \sqrt{3}\, b)$$

or

$$2a = \frac{\beta(\pi)}{d_1} + \frac{\dot{\beta}(\pi)}{d_2}\left(\frac{\gamma\mu^2}{4}\right)^{-1/3}$$

$$2\sqrt{3}\, b = \frac{\beta(\pi)}{d_1} - \frac{\dot{\beta}(\pi)}{d_2}\left(\frac{\gamma\mu^2}{4}\right)^{-1/3} \tag{113}$$

where d_1 and d_2 are constants associated with the Airy functions ($d_1 \cong 0.355$, $d_2 \cong 0.259$). Then for initial conditions, Eq. 111 relating C_5 and C_6 to a and b gives the main solution in the third quadrant in terms of initial values of $\beta(\pi)$ and $\dot{\beta}(\pi)$.

For ending the solution, Eq. 110 relating C_3 and C_4 to a and b gives the final values of $\beta(\pi)$ and $\dot{\beta}(\pi)$ in terms of the main solution in the second quadrant.

Eigenvalues

With the solution in each of the four outer regions, and the connection formulas between them, the complete solution may be constructed over one rotor revolution. Floquet theory for a single degree of freedom, second order equation (see Appendix I) shows that the eigenvalues are given by the quadratic equation

$$(e^{\lambda 2\pi})^2 - (\dot{\beta}_R + \beta_P)(e^{\lambda 2\pi}) + \dot{\beta}_R\beta_P - \beta_R\dot{\beta}_P = 0 \tag{114}$$

where

$$\beta_P = \beta(\pi + T) \text{ due to } \beta(\pi) = 1,\ \dot{\beta}(\pi) = 0$$

$$\dot{\beta}_P = \dot{\beta}(\pi + T) \text{ due to } \beta(\pi) = 1,\ \dot{\beta}(\pi) = 0$$

$$\beta_R = \beta(\pi + T) \text{ due to } \beta(\pi) = 0,\ \dot{\beta}(\pi) = 1$$

$$\dot{\beta}_R = \dot{\beta}(\pi + T) \text{ due to } \beta(\pi) = 0,\ \dot{\beta}(\pi) = 1$$

and T is the period of the system ($T = 2\pi$ here).

Combining the connection formulae (Eq. 104, 105, 108, and 112) and the initial and final value formulae (Eqs. 110, 111. and 113) results in the following expression for β_f and $\dot{\beta}_f$ at $\psi = \pi + T$ in terms of β_i and $\dot{\beta}_i$ at $\psi = \pi$:

$$\frac{1}{8}\left(\frac{\beta_f}{d_1}+\frac{\dot{\beta}_f}{d_2}\left(\frac{\gamma}{4}\mu^2\right)^{-1/3}\right)e^{-F_2}+\frac{i}{2\sqrt{3}}\left(\frac{\beta_f}{d_1}-\frac{\dot{\beta}_f}{d_2}\left(\frac{\gamma}{4}\mu^2\right)^{-1/3}\right)e^{F_2}$$

$$=e^{iF_1}e^{-\mu\frac{\gamma}{3}}\left[\frac{i}{8\sqrt{3}}\left(\frac{\beta_i}{d_1}-\frac{\dot{\beta}_i}{d_2}\left(\frac{\gamma}{4}\mu^2\right)^{-1/3}\right)e^{-F_3}\left(2e^{iF_4}-e^{-iF_4}\right)\right.$$

$$\left.+\frac{1}{6}\left(\frac{\beta_i}{d_1}+\frac{\dot{\beta}_i}{d_2}\left(\frac{\gamma}{4}\mu^2\right)^{-1/3}\right)e^{F_3}\left(2e^{iF_4}+e^{-iF_4}\right)\right] \tag{115}$$

where

$$F_1=\mu\int_0^{\psi_{t_1}}\sqrt{-f}\,d\psi-\int_0^{\psi_{t_1}}\frac{g}{\sqrt{-f}}\,d\psi$$

$$F_2=\mu\int_{\psi_{t_1}}^{\pi}\sqrt{f}\,d\psi-\int_{\psi_{t_1}}^{\pi}\frac{g}{\sqrt{f}}\,d\psi \tag{116}$$

$$F_3=\mu\int_{\pi}^{\psi_{t_3}}\sqrt{f}\,d\psi+\int_{\pi}^{\psi_{t_3}}\frac{g}{\sqrt{f}}\,d\psi$$

$$F_4=\mu\int_{\psi_{t_3}}^{2\pi}\sqrt{-f}\,d\psi+\int_{\psi_{t_3}}^{2\pi}\frac{g}{\sqrt{-f}}\,d\psi$$

Obtaining β_P, $\dot{\beta}_P$, β_R, and $\dot{\beta}_R$ from Eq. 115 and substituting into Eq. 114, there results (after some manipulation) the following equation for the eigenvalues:

$$\left(e^{\lambda 2\pi+\mu\frac{\gamma}{3}}\right)^2-2b\left(e^{\lambda 2\pi+\mu\frac{\gamma}{3}}\right)+1 \tag{117}$$

where

$$b=\frac{1}{2}\left\{\frac{1}{4}e^{-(F_2+F_3)}\left[2\cos(F_1+F_4)-\cos(F_1-F_4)\right]\right.$$

$$\left.+\frac{4}{3}e^{(F_2+F_3)}\left[2\cos(F_1+F_4)+\cos(F_1-F_4)\right]\right\} \tag{118}$$

So b is a function of μ, γ, and K_P. The solution for the roots is then

$$\lambda = \begin{cases} -\mu\dfrac{\gamma}{6\pi} \pm \dfrac{1}{2\pi}\cosh^{-1} b + ni & \text{for } b > 1 \\ -\mu\dfrac{\gamma}{6\pi} \pm i\dfrac{1}{2\pi}\cos^{-1} b + ni & \text{for } -1 < b < 1 \\ -\mu\dfrac{\gamma}{6\pi} \pm \dfrac{1}{2\pi}\cosh^{-1}\left|b\right| + \dfrac{i}{2} + ni & \text{for } b < -1 \end{cases} \tag{119}$$

where n is some integer. This result shows the typical behavior of the roots of periodic systems. For $b < 1$ the damping is fixed at $-\mu(\gamma/6\pi)$ with a change due to b in the frequency; for $b > 1$ the frequency is fixed at n/rev with a positive and negative change due to b in the damping; for $b < -1$ the frequency is fixed at n $\frac{1}{2}$/rev with a positive and negative change in the damping. The critical region boundaries are given by $b = 1$ and $b = -1$.

The general character of the critical regions and instability boundaries in the $\gamma - \mu$ plane, as obtained from the solution of Eq. 117, is sketched in Fig. 12. Because μ is large, it happens that $|b|$ is much greater than 1 almost always, so the critical regions dominate the behavior of the roots. Because of the cosine terms in b, the sign of b changes regularly; b must of course go through zero then, but it does so very quickly, so there is only a very narrow band between the Im λ = n/rev and the Im $\lambda = n + \frac{1}{2}$/rev regions in which $|b| < 1$. When $|b| < 1$, the real part of λ is $-\mu(\gamma/6\pi)$, i.e., the root is stable for all μ and γ; thus there must always be a band of stability surrounding the transition from n/rev to n $\frac{1}{2}$/rev. These characteristics are illustrated in Fig. 12. The locus between the critical regions has a rather fine structure which would be difficult to obtain numerically. A root locus for varying μ or γ

(a vertical or horizontal section in Fig. 12) in the vicinity of a critical region boundary would in quick succession move from the RHP (unstable) to the LHP (stable) with frequency fixed at n/rev, rapidly move from Im λ = n/rev to Im $\lambda = n + \frac{1}{2}$/rev in the RHP with damping given by $-\mu\ (\gamma/6\pi)$ (which would be nearly constant because the critical region boundaries are so close) and then moves from the LHP into the RHP with frequency fixed at $n + \frac{1}{2}$/rev.

Figure 12 shows that for a given μ the system is stable for a large enough γ. Positive K_P is stabilizing, tending to decrease the size of the instability regions; negative K_P is destabilizing in this sense. The rotating natural frequency of the flap motion, ν, does not enter the high μ case to order p_0 (the aerodynamic spring dominates the centrifugal spring until order p_1); this is consistent with the fact that the critical regions dominate the high behavior, so the frequency of the motion is fixed at a multiple of $\frac{1}{2}$/rev.

A comparison of these analytical results with the results of numerical calculations indicates that the high μ solution is good down to $\mu = 2.5$ or so. Thus numerical calculations are required to join the loci from $\mu = 0.5$ to 2.5 say (for γ neither small nor large). The behavior theoretically predicted for the locus at large μ (in particular the rapid movements between Im λ = n/rev and n+$\frac{1}{2}$/rev, and perhaps — for γ not too large — between the RHP and the LHP) actually does show up in the numerical calculations of the stability (above $\mu = 3.0$ say); such behavior of a numerical solution might be questioned without the perturbation solution to provide a guide to what to expect. It is unfortunate that the boundary of the instability region for $\gamma/16$ of order 1 is first encountered at moderate μ (around $\mu = 2.25$ for small K_P; see Fig. 12) and so cannot be obtained by perturbation techniques (to the order explored anyway). Because of the small time constant in the main solutions ($0(\mu^{-1})$) and the four transition regions (of width $0(\mu^{-2/3})$), a numerical calculation of the roots for truly large would be difficult; the perturbation theory handles these singular problems analytically, and the calculations that remain are nonsingular, short, and simple.

Flap Rate Feedback

The use of flap rate feedback ($K_R \neq 0$) changes the solution for large μ fundamentally, because the aerodynamic damping ($-K_R M_\theta$) is then the same order as the aerodynamic spring ($M_\beta - K_P M_\theta$). The derivation of the main solutions is simpler then, and only the transition regions near $\psi = 0$ and 180° are required. The main solutions are obtained using the substitution

$$\beta = \exp \int^{\psi} p \, d\psi$$

with p expanded as a series in μ:

$$p = \mu^2 p_{-2} + \mu p_{-1} + p_0 + \frac{1}{\mu} p_1 + \ldots$$

Making this substitution in Eq. 89, the terms of like order in give

$$0(\mu^4): \; p_{-2}^2 = - r\gamma \left[K_R \frac{1}{4} (\sin\psi)^2 p_{-2} \right]$$

$$0(\mu^3): \; 2p_{-2}p_{-1} = - r\gamma \left[\frac{1}{3} \sin\psi \left(\frac{1}{2} + K_R \right) p_{-2} + K_R \frac{1}{4} (\sin\psi)^2 p_{-1} \right]$$

$$0(\mu^2): \; p_{-1}^2 + 2p_{-2} p_0 + \dot{p}_{-2} = - r\gamma \left[\frac{1}{8} (1 + K_R) p_{-2} + \frac{1}{3} \sin\psi \left(\frac{1}{2} + K_R \right) p_{-1} \right.$$

$$\left. + K_R \frac{1}{4} (\sin\psi)^2 p_0 + \frac{1}{4} \sin\psi (\cos\psi + K_P \sin\psi) \right]$$

$$0(\mu):\ 2p_{-1}\,p_0 + 2p_{-2}\,p_1 + \dot{p}_{-1} = -\,r\,\gamma\left[\frac{1}{8}(1+K_R)\,p_{-1} + \frac{1}{3}\sin\psi\left(\frac{1}{2}+K_R\right)p_0 + K_R\,\frac{1}{4}(\sin\psi)^2\,p_1 + \left(\frac{1}{6}\cos\psi + \frac{1}{3}K_P\sin\psi\right)\right]$$

The first solution is given by $p_{-2} = 0$. Then the order μ^3 equation gives also $p_{-1} = 0$, and the order μ^2 equation gives

$$p_0 = -\frac{1}{K_R}(\cot\psi + K_P)$$

or

$$\int^{\psi} p_0\,d\psi = -\frac{1}{K_R}\,\ell n\sin\psi - \frac{K_P}{K_R}\,\psi$$

The order μ equation gives

$$p_1 = \frac{2}{3K_R}\left(1+\frac{1}{K_R}\right)\frac{\cos\psi}{(\sin\psi)^2} + \frac{2K_P}{3K_R^{\ 2}}\,\frac{1}{\sin\psi}$$

or

$$\int^{\psi} p_1\,d\psi = -\frac{2}{3K_R}\left(1+\frac{1}{K_R}\right)\frac{1}{\sin\psi} + \frac{2K_P}{3K_R^{\ 2}}\,\ell n\tan\frac{\psi}{2}$$

Then the solution for β is

$$\beta = \beta_1\,(\sin\psi)^{-\frac{1}{K_R}}\,e^{-\frac{K_P}{K_R}\psi}\,\left(\tan\frac{\psi}{2}\right)^{\frac{1}{\mu}\frac{3K_P}{2K_R^{\ 2}}}\,e^{-\frac{1}{\mu\sin\psi}\frac{2}{3K_R}\left(1+\frac{1}{K_R}\right) + 0(\mu^{-2})} \qquad (12$$

where β_1 is a constant. This is the solution of the reduced equation, and so is independent of γ to this order, i.e., is the result of a balance of aerodynamic damping and aerodynamic spring terms only. The solution is not valid when $\sin\psi$ is too small; and it is exponentially growing (unstable, at least in the outer region) if $K_P/K_R < 0$.

The second solution is given by $p_{-2} = -r(\gamma/4)K_R(\sin\psi)^2$ or

$$\int^{\psi} p_{-2}\,d\psi = -r\frac{\gamma}{4}K_R\left(\frac{1}{2}\psi - \frac{1}{4}\sin 2\psi\right)$$

The order μ^3 equation gives

$$p_{-1} = -r\frac{\gamma}{6}(1 + 2K_R)\sin\psi$$

or

$$\int^{\psi} p_{-1}\,d\psi = r\frac{\gamma}{6}(1 + 2K_R)\cos\psi$$

and the order μ^2 equation gives

$$p_0 = -r\frac{\gamma}{8}(1 + K_R) + \frac{1 - 2K_R}{K_R}\cot\psi + \frac{K_P}{K_R}$$

or

$$\int^{\psi} p_0\,d\psi = \frac{1 - 2K_R}{K_R}\ln\sin\psi + \left[\frac{K_P}{K_R} - r\frac{\gamma}{8}(1 + K_R)\right]\psi$$

Then the solution for β is

$$\beta = \beta_2(\sin\psi)^{\frac{1}{K_R} - 2}\exp\left\{-r\frac{\gamma}{16}\left[\mu^2 K_R(2\psi - \sin 2\psi) - \mu\frac{8}{3}(1 + 2K_R)\cos\psi\right.\right.$$
$$\left.\left. + 2(1 + K_R)\psi\right] + \frac{K_P}{K_R}\psi + 0(\mu^{-1})\right\} \qquad (121)$$

where β_2 is a constant. This solution does involve γ, i.e., it involves the inertia terms in the equation of motion; this solution is also not valid when $\sin\psi$ is too small.

There are transition regions between the advancing and retreating sides, near $\psi = 0$ and 180°. As for the $K_R = 0$ case, region (ii) is an interior region lying entirely within the $0(\mu^{-2/3})$ transition region; the solution is matched through this interior region again by matching the displacement and slope of the transition region solutions at $z = 0$. With the substitution $z = \mu^{2/3} \sin\psi$, the equation of motion in the transition region becomes, to lowest order in μ $(0(\mu^{4/3}))$

$$\frac{d^2\beta}{dz^2} \pm \frac{\gamma}{4} \left|z\right| \left(K_R\, z \frac{d\beta}{dz} + \beta\right) = 0 \tag{122}$$

where the plus sign applies on the back of the disk and the minus sign on the front. It can be shown that the solutions of this equation will have the proper asymptotic behavior for matching to the two main solutions. Unfortunately the solution of this equation is not available in terms of classical functions. The behavior of the solutions for large and small z would have to be found, probably largely by numerical methods, before the matching procedure could be carried out.

Applicability of the Four Cases

This section will consider the ranges of μ and γ over which the four cases investigated above are useful. Perturbation theory is based on the expansion of quantities in terms of a very small or very large parameter. In many problems however, the results are useful, even quite accurate, far beyond the limits for which they are theoretically valid. It is these perturbation solutions, extendable up or down from truly small or large values of the perturbation parameter, which are of most value. They may be found only by comparison with exact solutions, usually obtained by numerical methods, for moderate values of the perturbation parameter. Another question about the range of validity of the solutions arises in this problem because there are two parameters, μ and γ, which are available as perturbation parameters. In the perturbation analysis based on one parameter, say γ small or large, the solution is derived under the assumption that the other parameter is of order 1, e.g., $\mu = 0(1)$. This raises the question of the validity of the solution when the other parameter is itself very small or very large. It may happen that the results are still valid when the other parameter is outside its assumed range, but this must be checked in each case. The ordering process of perturbation techniques provides a quantitative framework for making this check. It is usually quite simple to determine what range the other parameter must have so that the assumptions made about the order of terms are still valid.

Extending the analysis into ranges of the other parameter where the ordering assumptions were violated is another matter; it of course means an entirely new case to be considered and analyzed by perturbation techniques. The results of the analytic solutions obtained above were compared with numerical calculations (performed by the author) of the roots of the flapping equation, primarily for moderate and small γ , over a wide range of μ ; these calculations were similar to those reported in Ref. 2.

The small μ results are good out to $\mu = 0.5$, which is a very useful range. These results are valid for all γ , since the order of γ does not change the terms retained in the small μ analysis. The large μ results are valid above $\mu = 2.5$ or so, which is also a good range. Here however γ either very small or very large will violate the assumptions made about the order of the terms in deriving the large μ solutions and so the results may not be valid in these corners of the γ-μ plane.

The small and large γ results, to the order investigated, are really useful only for truly small or truly large γ , although the results are quite informative. The small γ results are accurate up to $\gamma = 2$ or 3. The limitations of the $0(\gamma)$ solution, as discussed in the small γ analysis, prevent the accurate use of the solution for moderate or even reasonably small γ ($\gamma = 6$ say). The large γ results give two real roots, so are obviously limited in usefulness. It is unlikely that γ would be large enough to require this solution. The large γ solution is good down to $\gamma/16 = 3$ or so, which is actually a very good range in the perturbation parameter; it just happens that for practical rotors γ falls far below this limit. Letting μ go to zero does not change the order of the aerodynamic coefficients (because of the constant terms) so the results for large and small γ should be good for all μ of order 1 or smaller. Letting μ go to infinity does change the order of terms in the analysis, so both the small γ and large γ results may be invalid for very large μ (above $\mu = 10$ say).

It would be very desirable to be able to use the small μ and small γ results to construct composite root loci which are reasonably accurate for all values of μ and γ likely to be encountered in helicopter rotors. The small μ results would be used up to about $\mu = 0.5$; then the small γ results would be used up to $\mu = 5$ or so. The major obstacle to this is the lack of an $0(\gamma^2)$ analysis for the critical regions; the small γ analysis presented here, which was carried only to $0(\gamma)$ in the critical regions, is not adequate for the accurate construction of loci for any except very small γ.

Possible extensions of the solutions described here include the following:

a) Extend the small μ results to $0(\mu^2)$ for $\text{Im}\lambda = \frac{1}{2}$, and to $0(\mu^3)$ or $0(\mu^4)$ to handle the Im = 3/2 critical region.

b) Extend the small γ results to $0(\gamma^2)$ in the critical regions.

c) Extend the large γ results to $0(\gamma^{-1})$.

d) Extend the large μ results to $0(\mu^{-1})$.

The most useful would be the $0(\gamma^2)$ and $0(\mu^{-1})$ results. A small γ solution reliable to $\gamma/16 = \frac{1}{2}$ or so (which an $0(\gamma^2$ solution should accomplish) could be combined with the small μ solution to construct accurate composite loci. Together these results would then cover most of the range of μ and γ of conventional rotors. The large μ case extended to $0(\mu^{-1})$ should be able to predict accurately the first instability boundary of the μ loci, which occurs at $\mu = 2$ to 2.5. These two cases are however also the ones involving the most work.

APPLICATION OF PERTURBATION TECHNIQUES TO HELICOPTER DYNAMICS

This section returns to the question of whether perturbation techniques might be profitably applied to more complicated or more realistic dynamic systems than the one considered here. As part of the answer, consider what these techniques will not do: obviously they can not give results for cases where there is no parameter that is either small or large, for example when $\gamma = 16$ and $\mu = 1$. However, the four cases considered together cover a good deal of the ranges of μ and γ, and with primarily analytical results. For many helicopters the small μ case will be quite satisfactory alone. What the techniques can do also includes:

a) Since they give analytic solutions they provide more insight into the problem, as well as specific design criteria for the system; this feature is particularly important for nonlinear or time-varying systems, which have properties much different from those of constant coefficient, linear systems.

b) Perturbation methods can find, and handle, cases that are very sensitive to the parameters, or that are difficult to solve accurately by numerical methods.

c) The methods provide more insight into the rather unusual behavior of the solution of periodic systems, by showing explicitly how the periodic coefficients modify the transient solutions and why they give the root loci their characteristic behavior in the critical regions.

d) Finally, even if the techniques are not used to find the complete solution, it only takes a little work to find out where the problems are (e.g., critical regions and transition regions) and what the order of things is, which information would be of invaluable help in the numerical analysis of a system.

The extension to more degrees of freedom or more realistic aerodynamic coefficients would certainly make the analysis more complicated. In general however any study — analytic, computational, or experimental —

of a system becomes more complicated as the accuracy of the modelling of the true system increases, and perturbation techniques are not expected to be an exception to this rule. Regardless of the system being studied, the position perturbation techniques occupy between simple linear analyses and complex nonlinear numerical calculations makes them a very powerful tool for providing both exact solutions and increased understanding of problems in rotor dynamics.

The problems in rotor dynamics to which perturbation techniques might profitably be applied amount to all those involving nonlinear or periodic coefficients, and there are many of those. There is some additional work that might be done with the flapping dynamics problem (one degree of freedom), including for example

a) teetering rotor;

b) cantilever blade, with correct aerodynamic coefficients;

c) inclusion of stall and compressibility in the aerodynamics.

The solutions found in this paper might be extended to $0(\gamma^2)$ for the small γ case and to $0(\mu^{-1})$ for the large μ case. These extensions might prove very useful, or only slightly more so than the solutions to the order presented here; but they should be examined for the simple single degree of freedom problem before being considered for more complicated systems.

Even the small μ solution could be taken a little farther — for example, to $0(\mu^2)$ in the $\mathrm{Im}\lambda = \frac{1}{2}$ critical region. Starting with two degrees of freedom, possible problems in coupled dynamics include

a) flap dynamics of a gimballed rotor;

b) pitch/flap dynamics;

c) flap/lag dynamics.

The pitch/flap system is the problem of rotor flutter. The flap/lag system is particularly rich in possible variations; the problem has periodic coefficients if $\mu > 0$ of course, but it is also nonlinear (even in hover) due to the inertial coupling of the degrees of freedom. It is moreover very sensitive to blade root geometry, so that an articulated and a cantilever blade have quite different dynamic characteristics. Problems in coupled dynamics with more degrees of freedom include

a) pitch/flap/lag dynamics (three degrees of freedom);

b) flap dynamics of an N-bladed rotor ($N \geq 3$) with flapping feedback control in the fixed system (at least four degrees of freedom).

While for these problems the advance ratio μ would probably be of most value as a perturbation parameter, there will likely arise problems where other parameters are also useful. As long as a reasonable model is chosen for the system, and as much effort is given to the interpretation of the solution as to its derivation, perturbation techniques should prove quite useful in providing information about these problems, and many others in rotor dynamics and aerodynamics.

This paper has demonstrated the methods of perturbation theory and has provided examples of the information about dynamic systems which may be obtained using them. The techniques have proved very useful for the problem studied. It should not be concluded however that the techniques presented are all there is to perturbation theory; there are many more methods that have not been touched on here. Perturbation theory is a powerful, and yet not very sophisticated, mathematical technique which should prove very useful in analyzing some of the problems of helicopter dynamics.

Appendix I. Eigenvalues of a Periodic System

Consider a general system of differential equations with periodic coefficients; this may be reduced to a system of first order equations, and may therefore be written (in matrix notation) as

$$\dot{\vec{x}} = A\vec{x}$$

where A(t) is periodic: A(t+T) = A(t). It may be shown that the solution to this differential equation can be obtained in the form

$$\vec{x}(t) = \Sigma_i \, q_i(0) \, e^{\lambda_i t} \, \vec{u}_i(t)$$

The λ_i are the eigenvalues; the eigenvectors $\vec{u}_i$ are periodic, $\vec{u}_i(t + T) = \vec{u}_i(t)$; and the numbers $q_i(0)$ are constants obtained from the initial conditions. The theory that shows this is called Floquet theory. The solution in this form is a direct extension of the normal solution for a constant coefficient differential equation, which is characterized by constant eigenvectors.

The eigenvalues λ_i may be obtained by the following procedure. The equation

$$\dot{\phi} = A\phi$$

where $\phi(t)$ is a matrix, is integrated over one period, from t = 0 to t = T, with initial conditions $\phi(0) = I$ (the unit matrix). Then if λ_{ci} are the eigenvalues of the matrix

$$C = \phi(T)$$

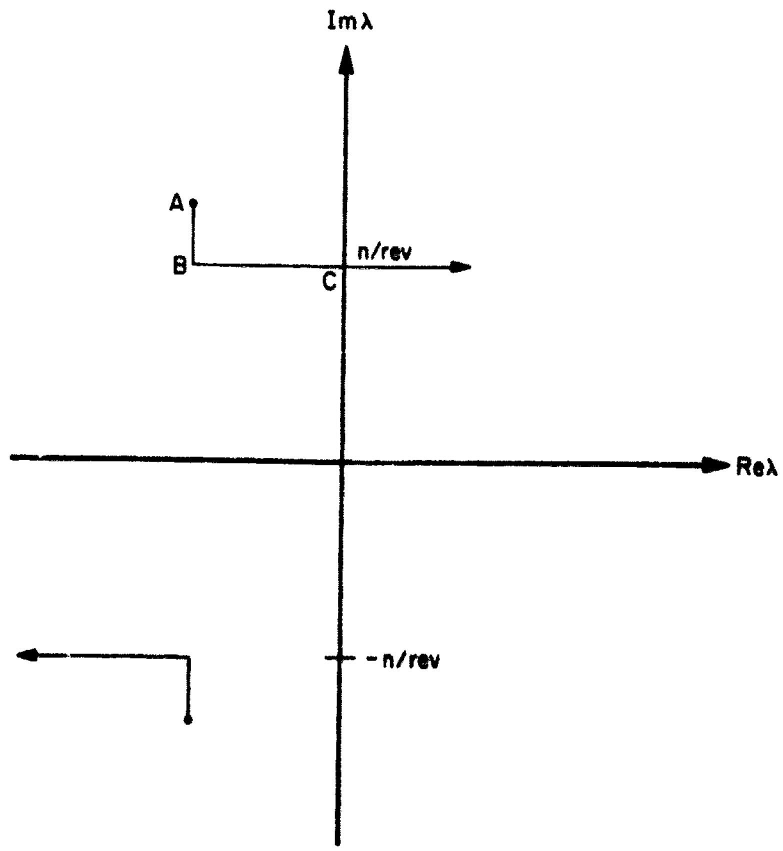
the roots λ_i are given by

$$\lambda_C = e^{\lambda T}$$

or

$$\lambda = \frac{1}{T} \ln \lambda_c$$

While the roots λ_{ci} (as eigenvalues of a real matrix C) must appear as real numbers or complex conjugate pairs, the eigenvalues λ_i are under no such restriction. The root loci of periodic systems are thus characterized by the type of behavior sketched below.



The horizontal portions of the loci can only appear at $\mathrm{Im}\,\lambda = n/\mathrm{rev}$ or $n + \frac{1}{2}/\mathrm{rev}$. If the parameter being varied, for example the advance ratio μ, is such that at $\mu = 0$ the system is not periodic, then the roots at point A are complex conjugates. As μ increases the periodicity of the system increases, and the roots start moving toward n/rev (or $n + \frac{1}{2}/\mathrm{rev}$) lines remaining complex conjugate pairs though). At some critical μ (the point B on the locus) the loci reach $\mathrm{Im}\,\lambda = n/\mathrm{rev}$, and then for still larger μ the frequency remains fixed while the real part of one root is decreased and that of the other is increased This behavior should be compared with that of two roots of a constant coefficient system which start out as complex conjugates, meet at the real axis, and then proceed in opposite directions along the real axis. The existence of periodic coefficients in the equations of motion generalizes this behavior so that it can occur at any $\mathrm{Im}\,\lambda = n/\mathrm{rev}$ or $n + \frac{1}{2}/\mathrm{rev}$, not just $\mathrm{Im}\,\lambda = 0$. The property of the solution that allows this behavior is the fact that the eigenvalues $\vec{u}_i(t)$ are themselves periodic.

For a single degree of freedom, second order system, let x_R be the solution obtained from integrating the equation with initial conditions $\dot{x}(0) = 1$, $x(0) = 0$; and let x_P be the solution with initial conditions $\dot{x}(0) = 0$, $x(0) = 0$. Then the roots λ_c are given by the quandratic equation

$$\lambda_C^2 - [(\dot{x}_R(T) + x_P(T)]\,\lambda_C + \dot{x}_R(T)\,x_P(T) - x_P(T)\,\dot{x}_P(T) = 0$$

Appendix II. Solution of the Secular Equation

The method of multiple time scales often leads to an ordinary differential equation of the form

$$\frac{d\beta}{d\psi} + (a + id)\beta + (b + ic)\bar{\beta} = 0$$

where β is a complex quantity, and the constants a, b, c, and d are real. letting

$$D^2 = d^2 - (b^2 + c^2) = d^2 - |(b = ic)|^2$$

$$D = \sqrt{|D^2|}$$

it may be verified that the solution of the above differential equation is

$D^2 > 0$:

$$\beta - e^{-a\psi}[A(d - D + i(b + ic))e^{iD\psi}$$
$$+ \bar{A}(d + D + i(b + ic))e^{-iD\psi}]$$

where A is a complex constant

$D^2 = 0$:

$$\beta = e^{-a\psi}\ [A((d + i(b + ic))\psi + 1)$$
$$+ B(d + i(b + ic))]$$

where A and B are real constants

$D^2 < 0$:

$$\beta = e^{-a\psi}[A(d + iD + i(b + ic))e^{D\psi}$$
$$+ B\ (d - iD + i(b + ic))e^{-D\psi}]$$

where A and B are real constants

The limiting case $b = c = 0$ gives $D = d$ so the solution is

$$\beta = Ae^{-(a + id)\psi}$$

where A is a complex constant

The region of decreased stability, i.e., the region where the real part of the eigenvalue becomes more positive (the critical region), is given by $D^2<0$. The boundary of the critical region is $D^2 = 0$. One root in the critical region becomes less stable, but the other becomes more stable. Furthermore, inside the critical region ($D^2<0$) there is a change in the real part of the root but no change in the imaginary part, i.e., the frequency; while outside the region ($D^2>0$) there is a change due to D in the frequency, but no change due to D in the real part. This behavior follows that expected of the eigenvalues of periodic systems (see Appendix I). Indeed, D^2 is a measure of the relative effects of the β and $\bar{\beta}$ terms in the differential equation; the former usually comes from the constant coefficients in the equation of motion and the latter from the periodic coefficients.

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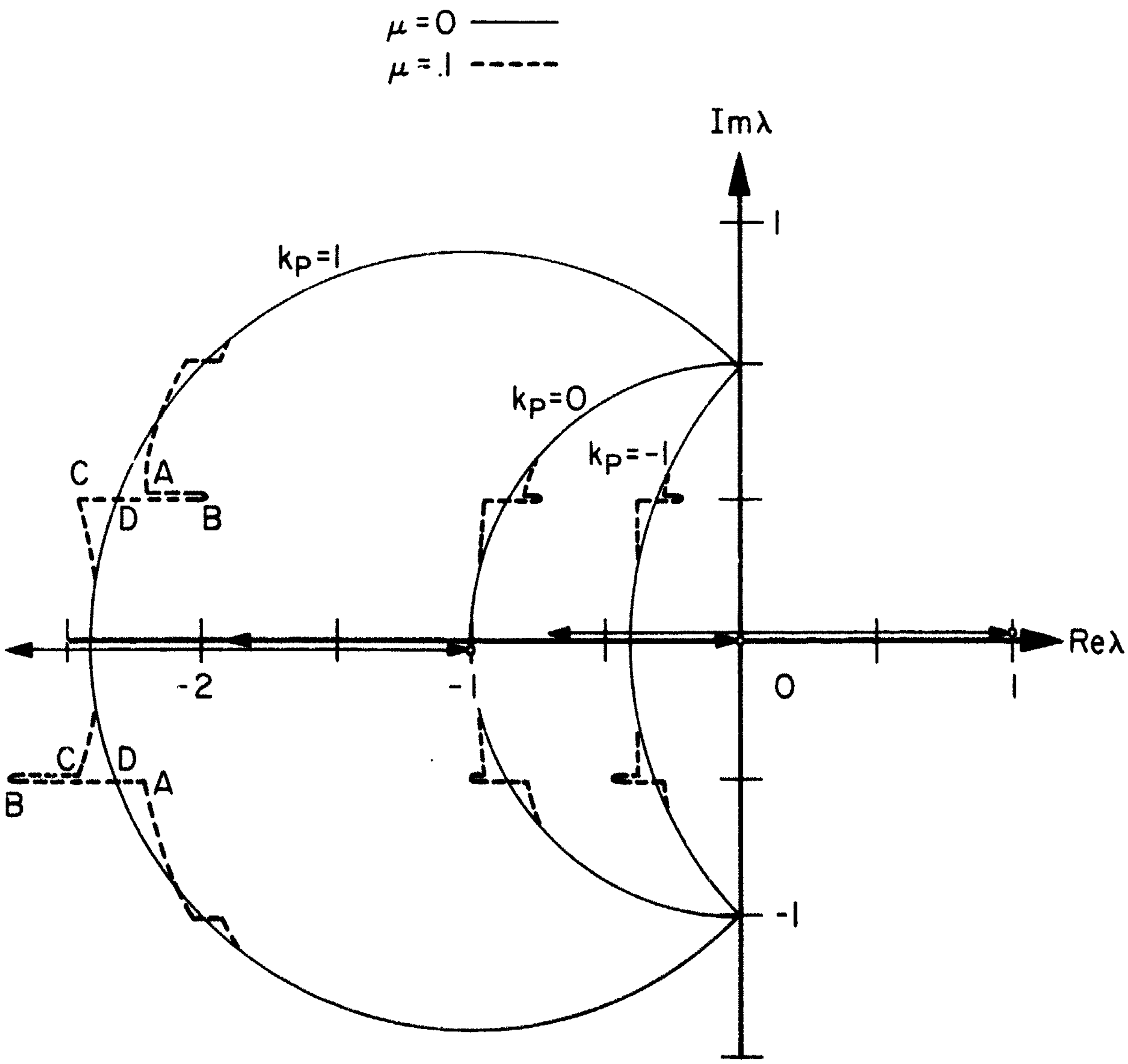

Fig. 1. Root loci for varying γ, based on the small μ results (to order μ^2); $\nu = 1$ and $\mu = 0$ and 0.1

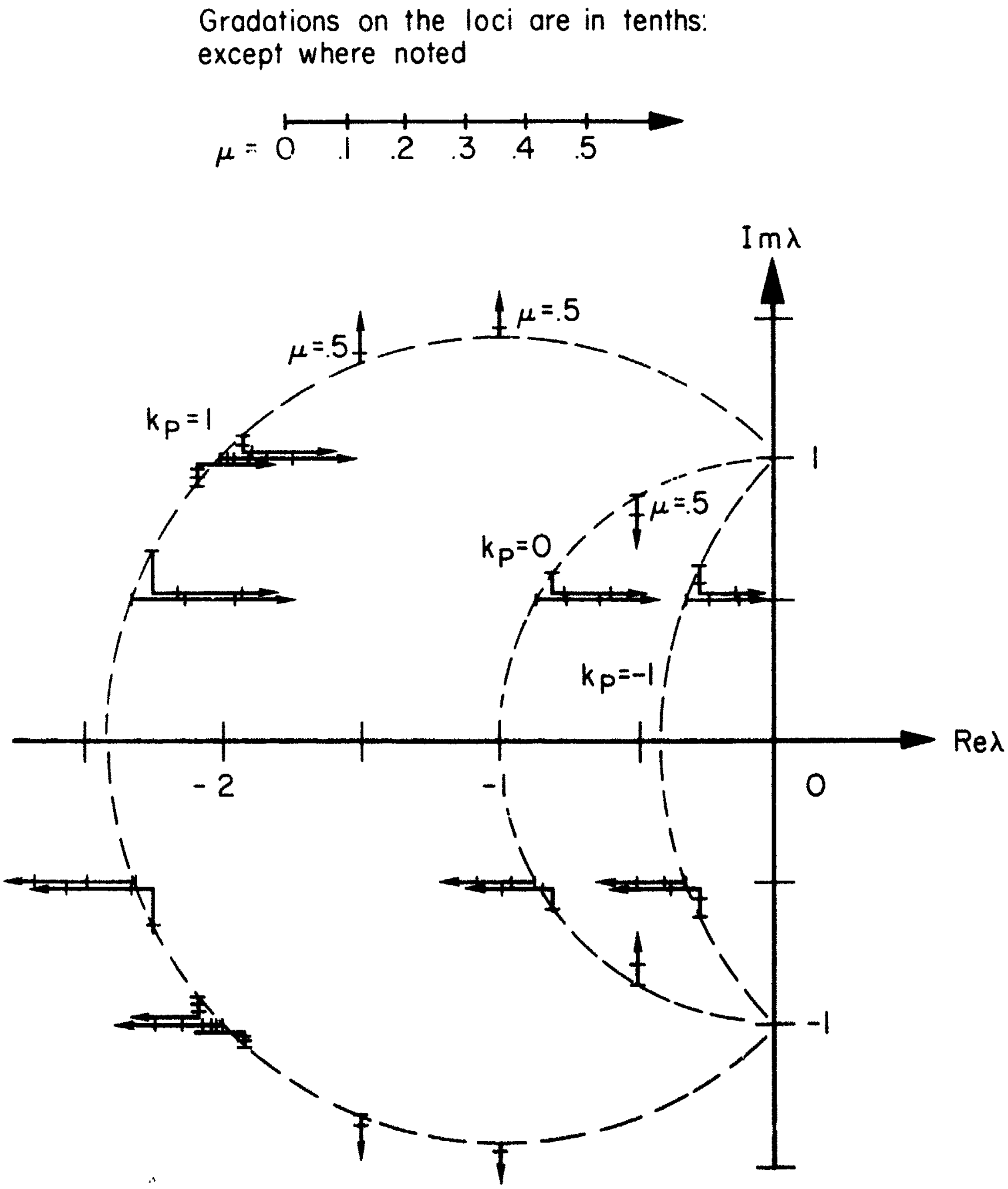

Fig. 2. **Root loci for varying μ, based on the small μ results (to order μ^2); fixed for each locus ($\mathrm{Re}\,\lambda = -\gamma/16$ for $\mu = 0$)**

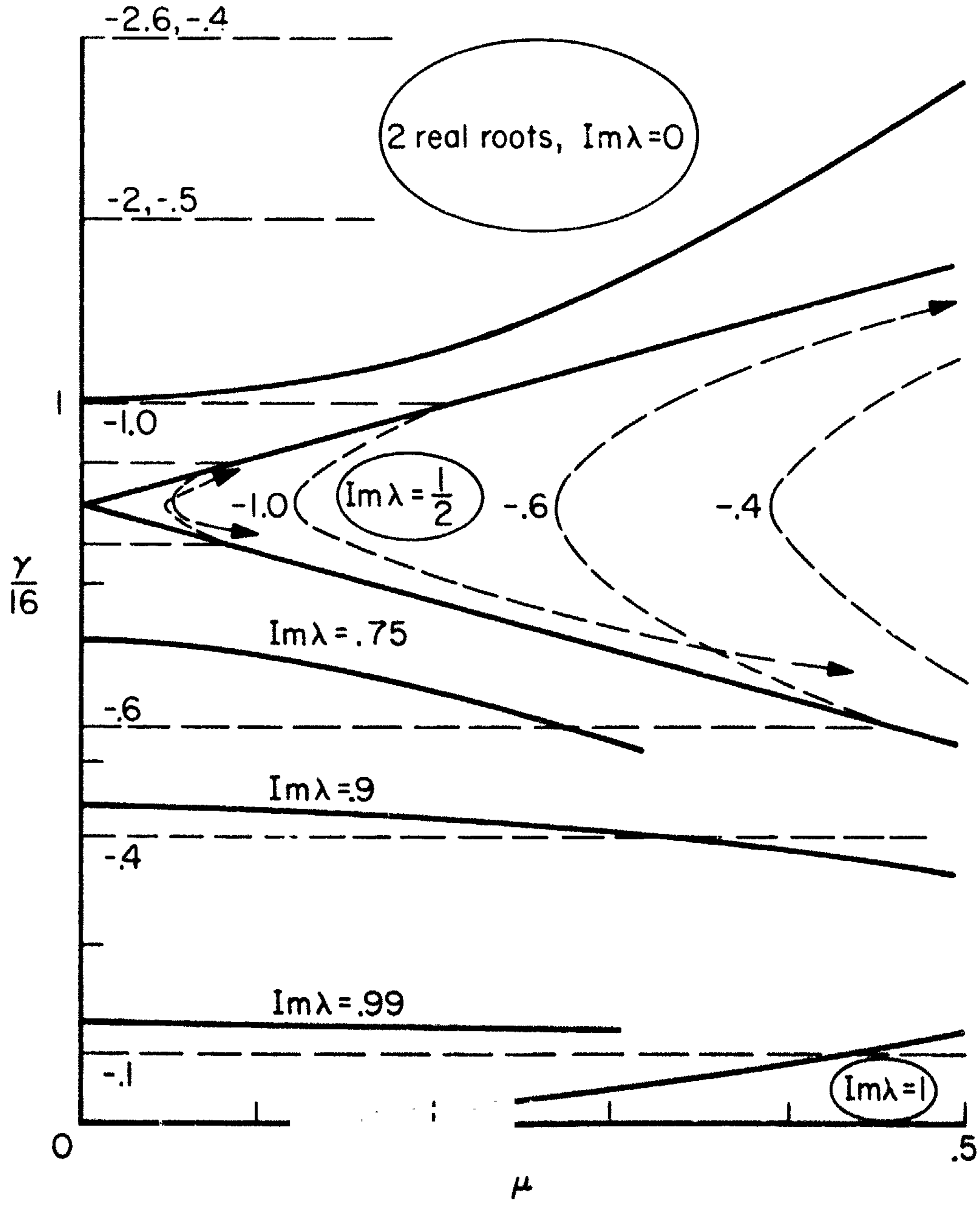

Fig. 3. Lines of constant Imλ and Reλ, based on the small μ results (to order μ^2); $\nu = 1$, $K_p = 0$; —— Imλ, – – – Reλ, circled values of Imλ indicate areas in which Imλ is constant.

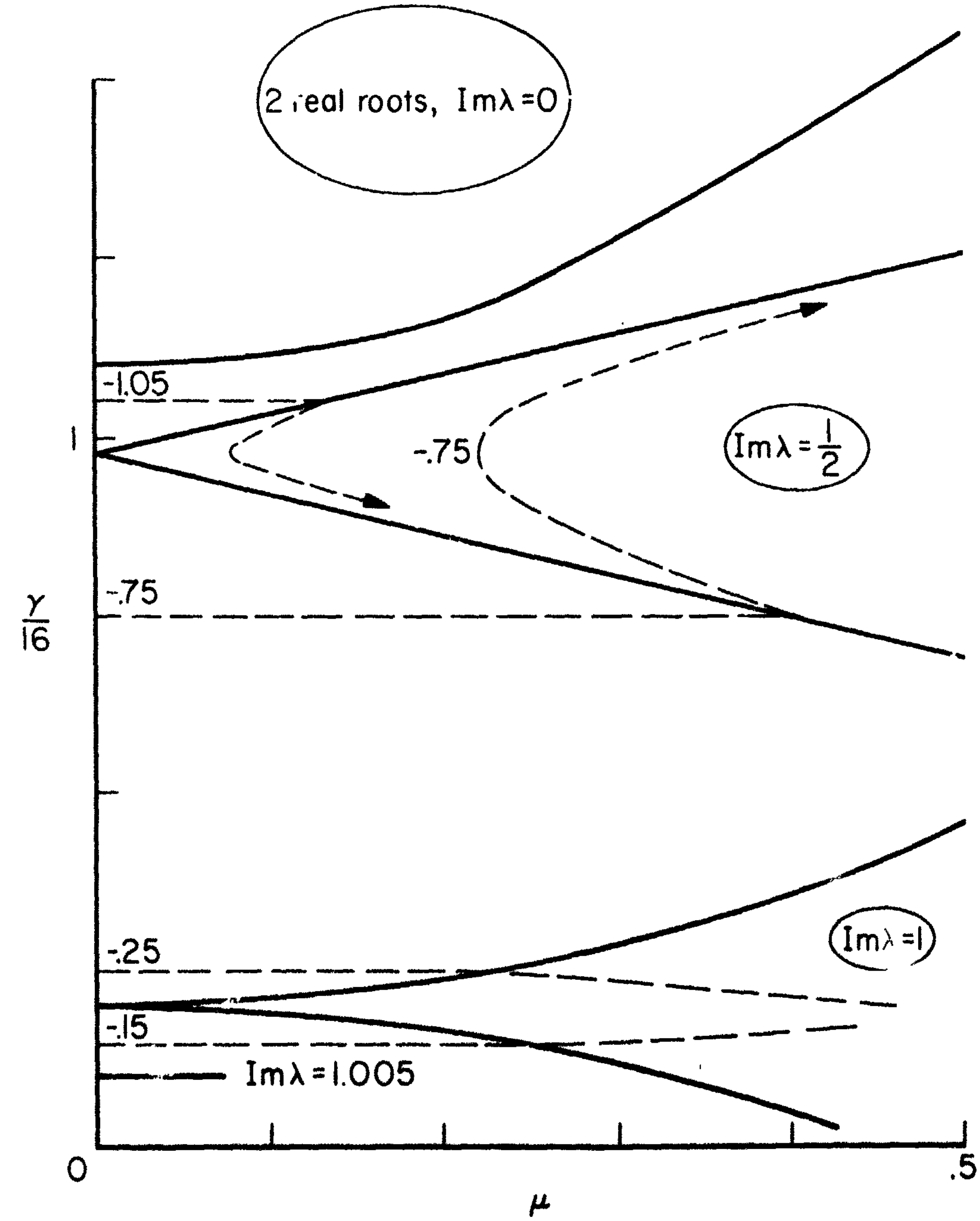

Fig. 4. Lines of constant Im λ and Re λ, based on the small μ results (to order μ^2); $\nu = 1$, $K_p = 0.1$; ——— Im λ, — — — — Re λ, circled values of Im λ indicate areas in which Im λ is constant.

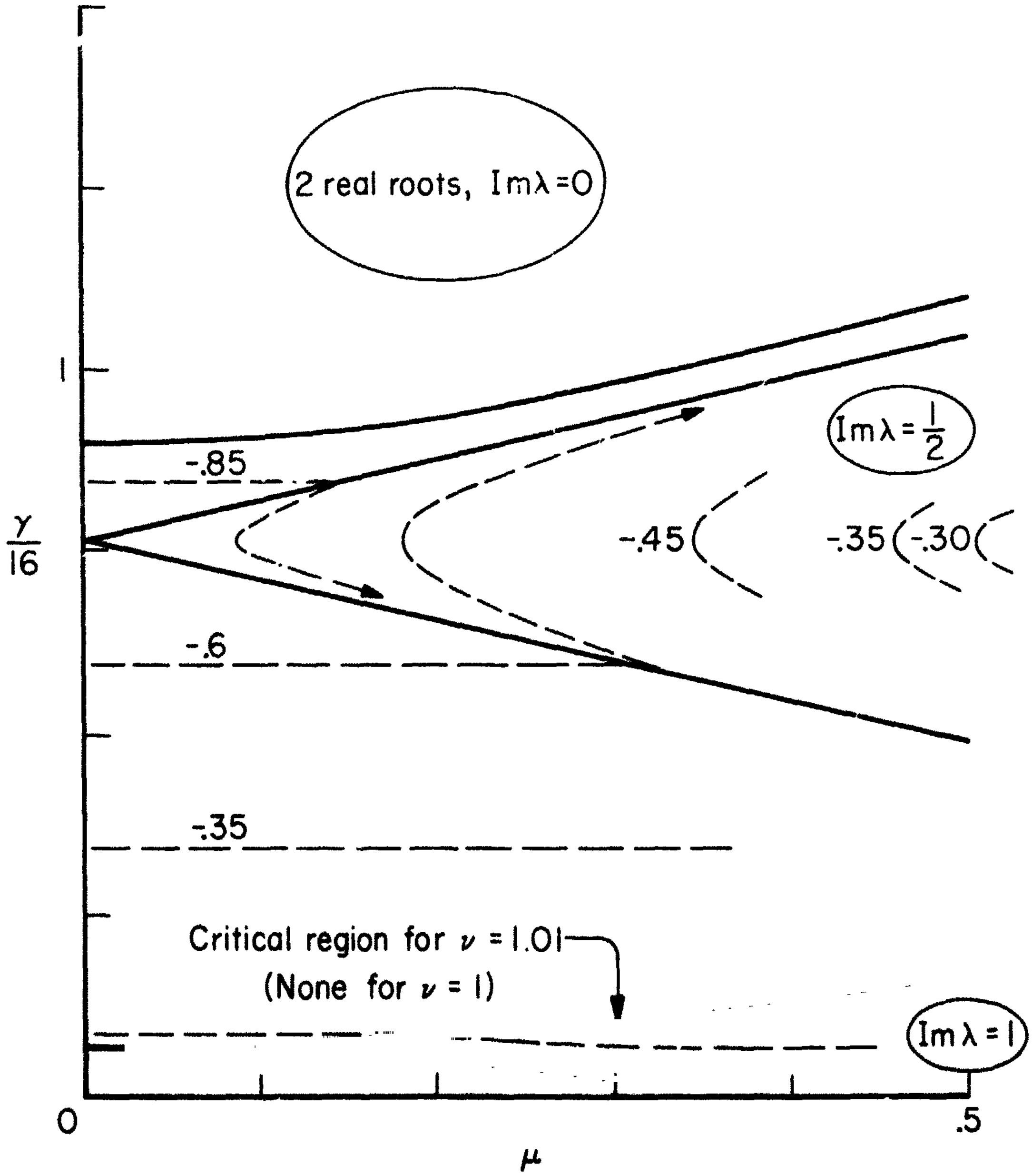

Fig. 5. Lines of constant Imλ and Reλ, based on the small μ results (to order μ^2); $\nu = 1$, $K_p = -0.1$; ——— Imλ, — — — — Reλ, circled values of Imλ indicate areas in which Imλ is constant.

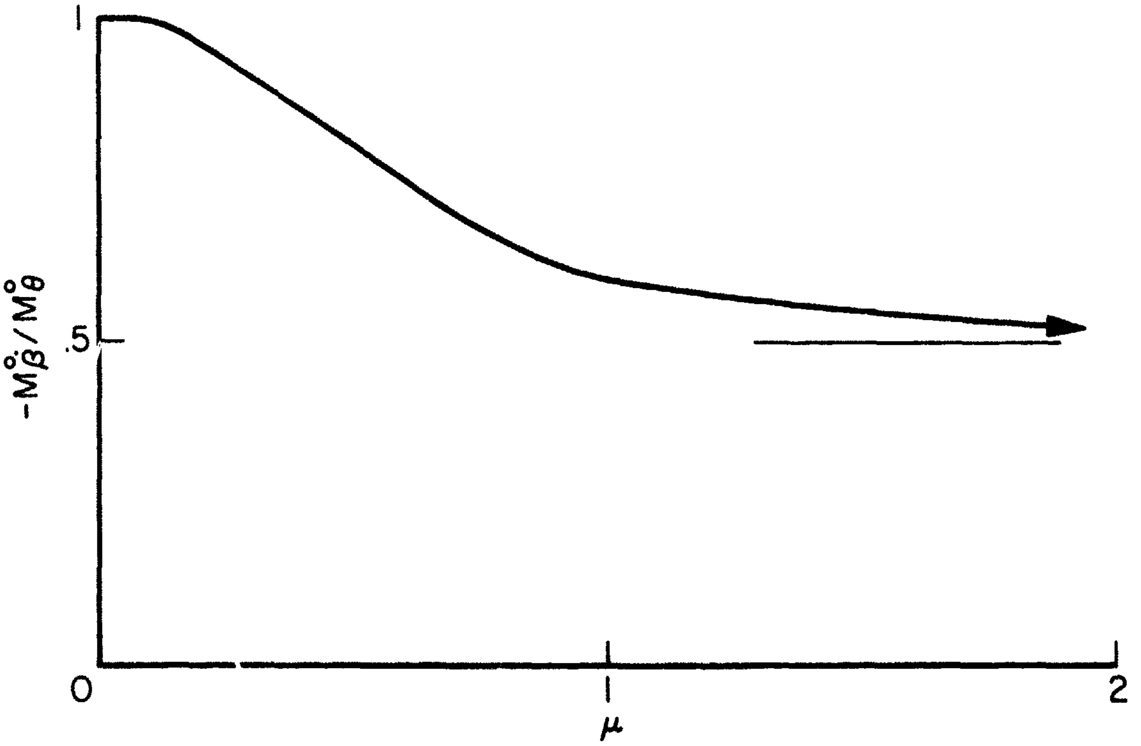

Fig. 6. The ratio $(-M^{\circ}_{\beta}/M^{\circ}_{\theta})$, which governs the effect of K_R and K_P for small γ .

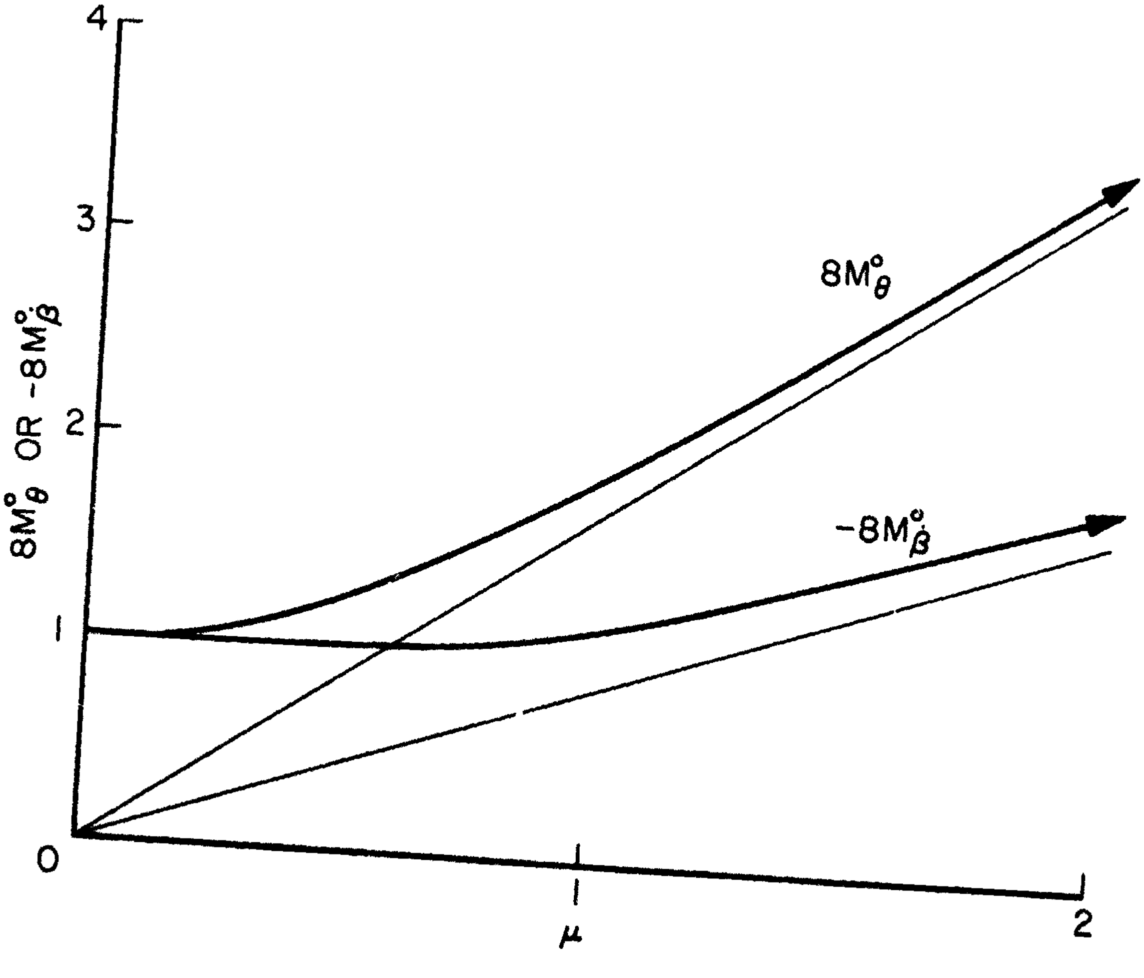

Fig. 7. The averages of the aerodynamic coefficients, which give the roots for small γ .

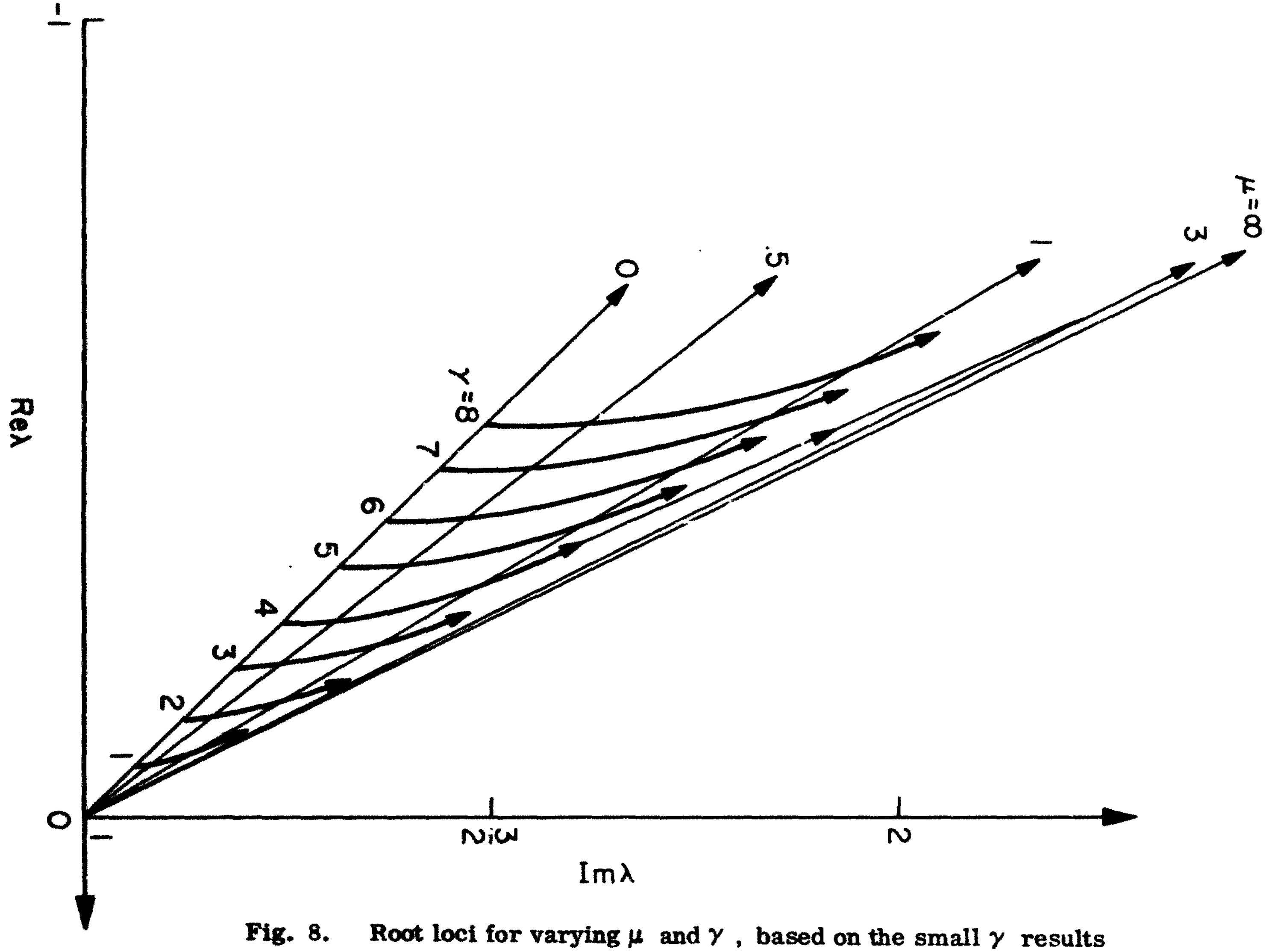

Fig. 8. Root loci for varying μ and γ, based on the small γ results (to order γ); $\nu = 1$ and $K_p = 1$; the effect of the critical regions is not shown.

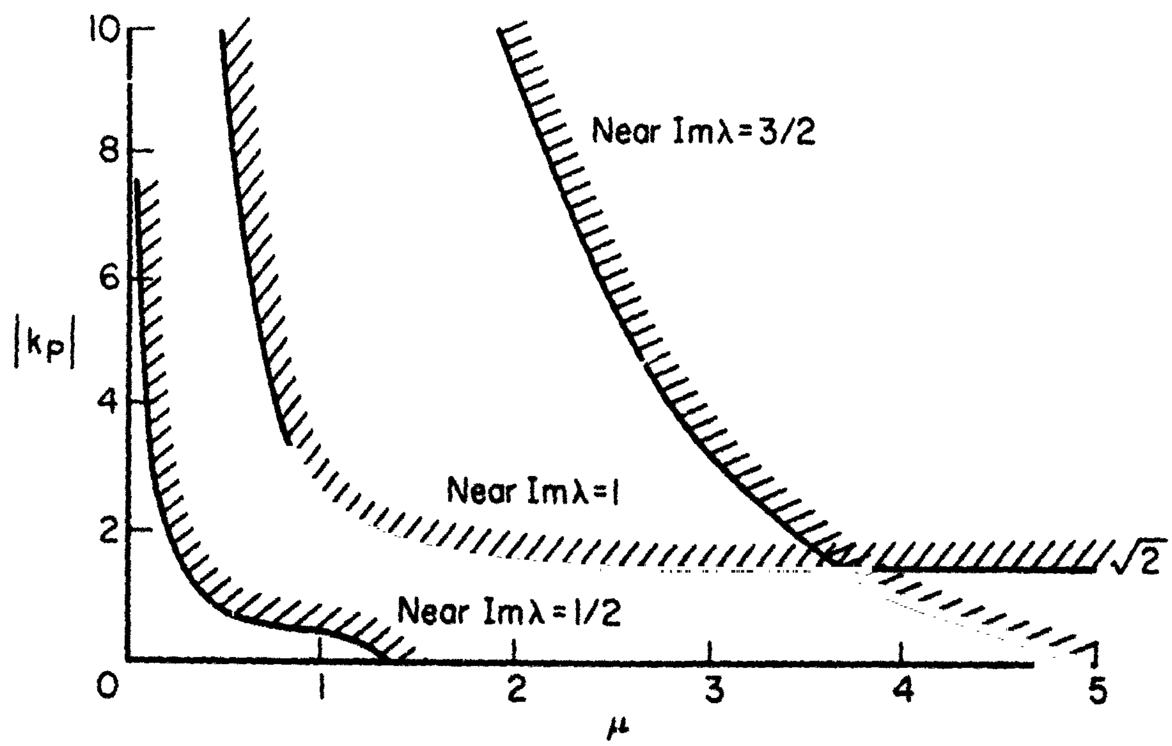

Fig. 9. The K_p vs. μ boundaries for stability in the center of the critical region, based on the small γ results (to order γ); for roots near $\mathrm{Im}\lambda \approx 1/2$, 1, and 3/2.

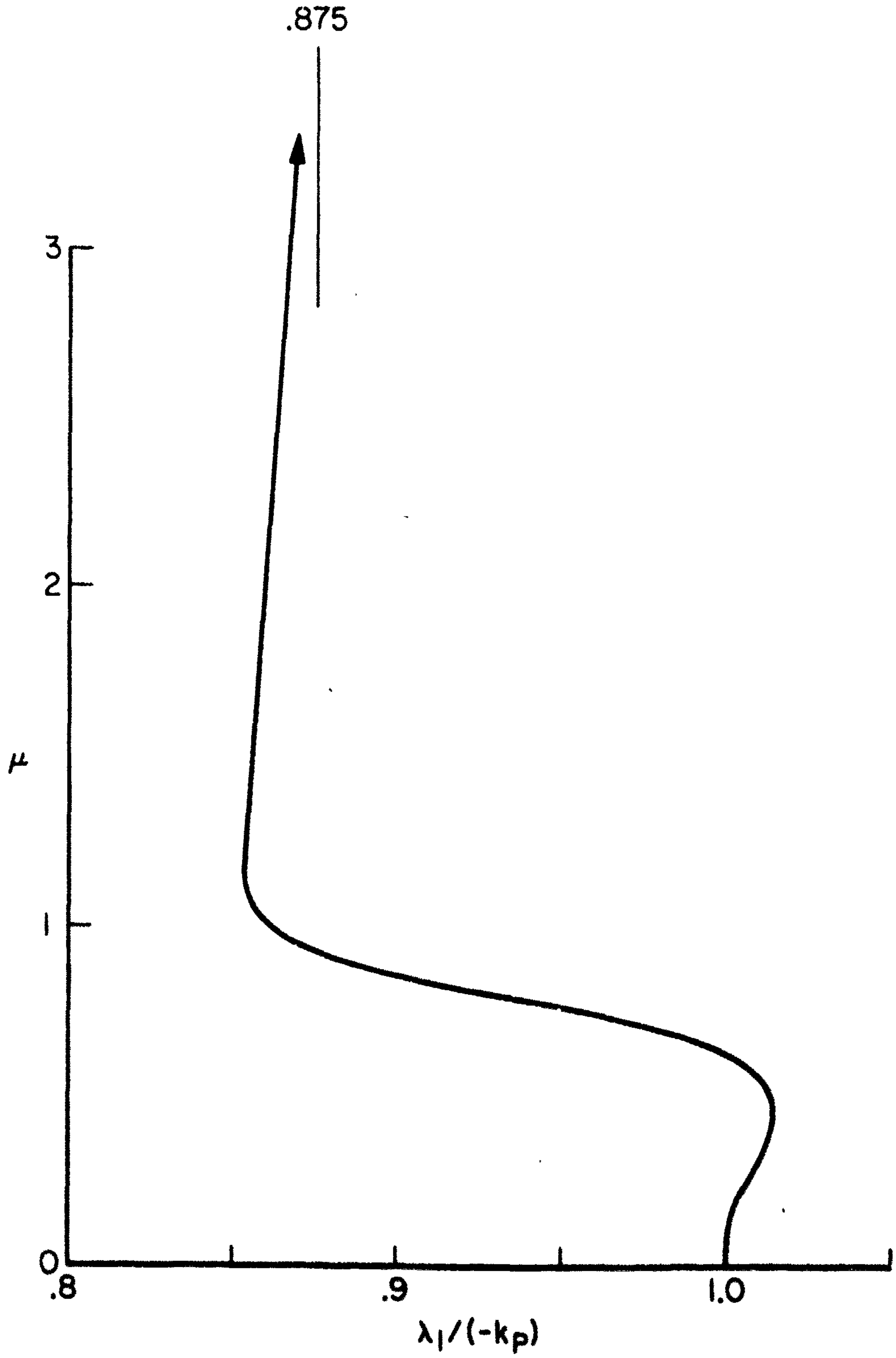

Fig. 10. Motion of the root on the real axis for varying μ , $\gamma = \infty$. For large γ there are two real roots; λ_1 is the finite real root; the other root is at $\lambda = -\infty$ for $\gamma = \infty$.

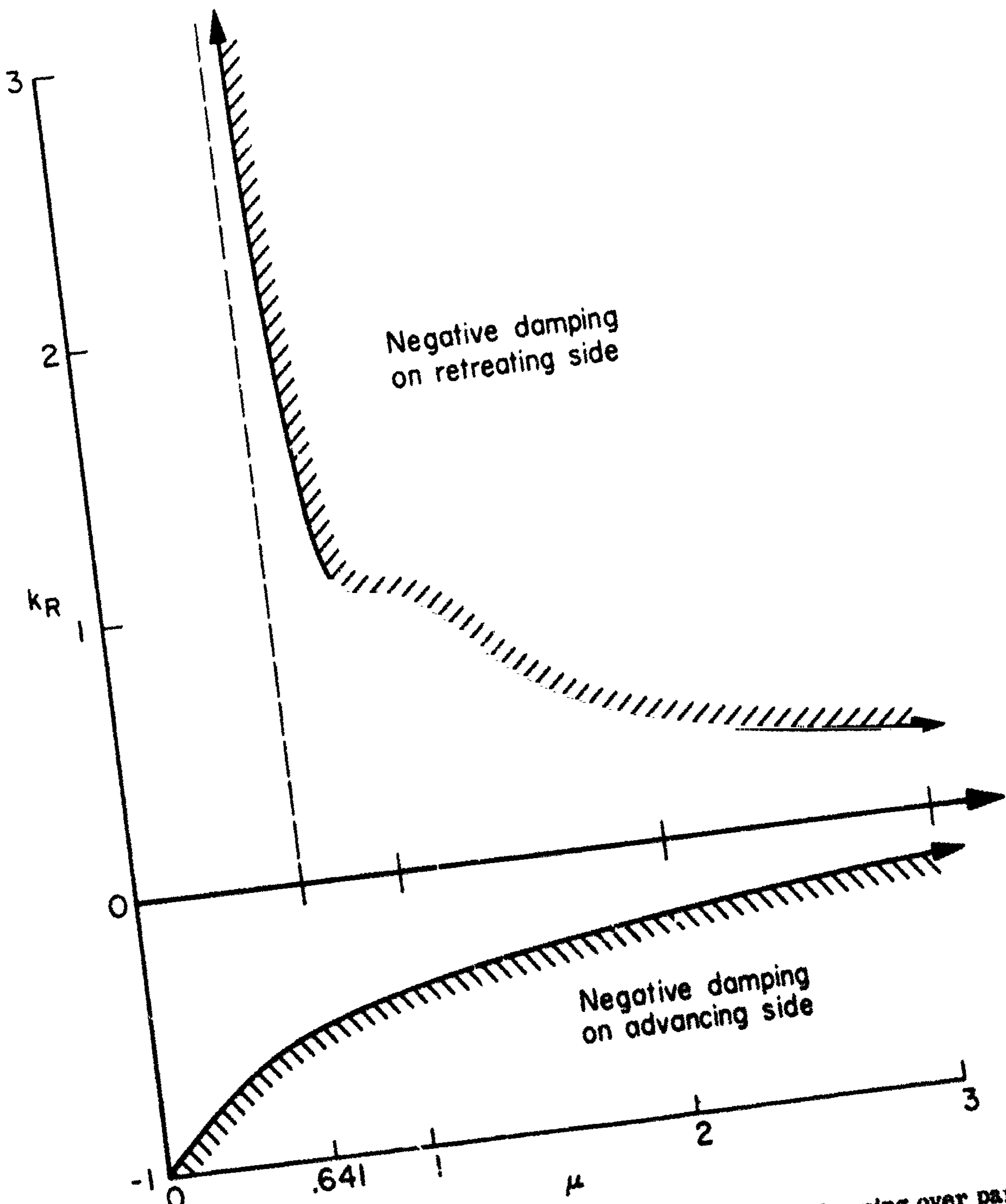

Fig. 11. Flap rate feedback required for negative damping over part of the rotor disk: $-(M_{\dot\beta} - K_R M_\theta)<0$ (enters into the large γ case).

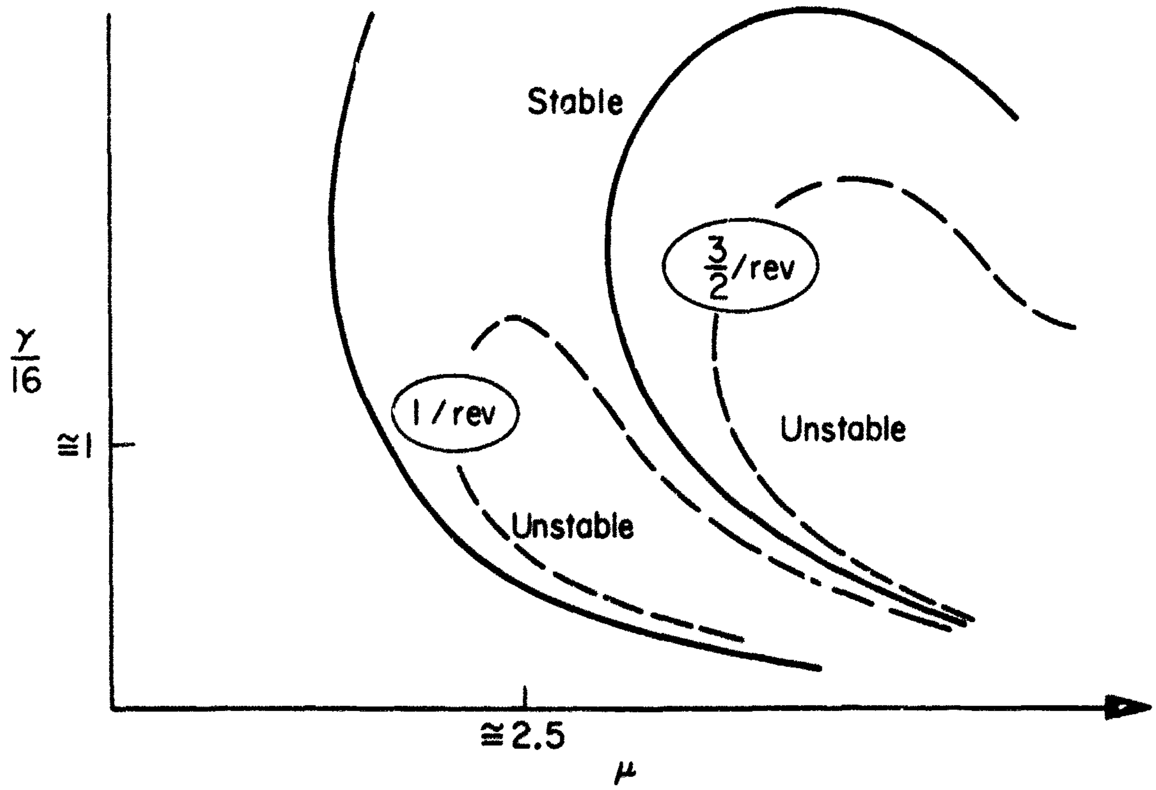

Fig. 12. Sketch of the characteristic behavior of the critical region boundaries and stability boundaries for large μ ; ——— boundary of region in which Im. is fixed at n/rev or n + ½/rev; — — — — boundary of region in which the real part of one root is positive, i.e., unstable.

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