OPTIMAL TRAJECTORY GENERATION FOR MECHANICAL ARMS

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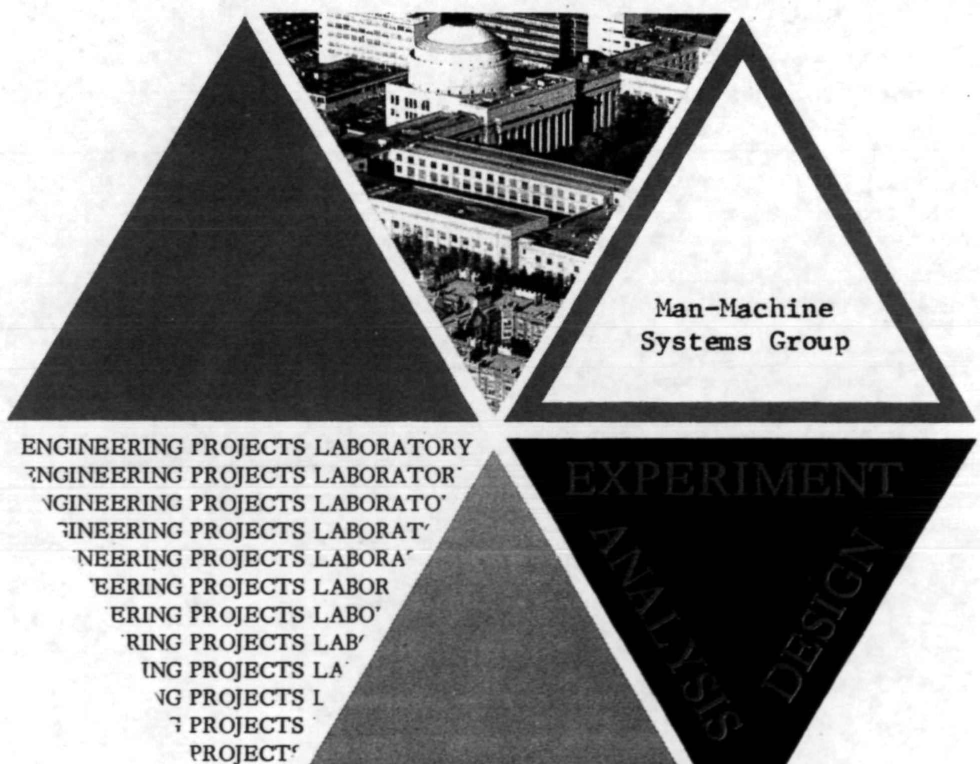

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ABSTRACT

A general method of generating optimal trajectories between an initial and a final position of an n degree of freedom manipulator arm with nonlinear equations of motion is proposed. The method is based on the assumption that the time history of each of the coordinates can be expanded in a series of simple time functions. By searching over the coefficients of the terms in the expansion, trajectories which minimize the value of a given cost function can be obtained.

The method has been applied to a planar three degree of freedom arm. The coordinates of the arm are the three joint angles. Two types of trajectories have been assumed. These are such that the time history for each joint angle is:

1. a series expansion of polynomials,
2. a series expansion of periodic functions.

Two integral type cost functions have been used:

1. the integral of the kinetic energy of the arm,
2. the integral of the magnitude of the joint torques.

The optimal values of the coefficients in the series expansion show a distinct pattern. For a particular combination of type of trajectory and cost function the optimal values of the coefficients have been approximated by rather simple functions. This results in suboptimal values of the coefficients, but they can be obtained without performing an on-line search. The difference between the optimal and suboptimal value of the cost function is of the order of 8%.

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Title: Associate Professor of Mechanical Engineering

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CHAPTER 1

INTRODUCTION AND PROBLEM STATEMENT

The design and control of mechanical manipulators which perform functions similar to those of the human has been the subject of many recent studies. A particular area of interest in these studies is the supervisory controlled manipulator. In supervisory control the operator specifies task subgoals to a remote computer which in turn executes pieces of the task through direct command of the manipulator supported by local control loops. Visual sensors enable the operator to monitor the execution of the task. This technique is well suited for a manipulator in outer space or other remote locations where the distance between the operator and the arm causes a significant time delay in the communication. Supervisory control can be applied equally well to performance of complex non-routine manipulation tasks as the routine execution of repetitive operations. Often it is required that a task is executed optimally in the sense that a particular cost function, for instance time or the expenditure of energy is minimized.

The dynamic equations of motion of a manipulator arm are non-linear and would require nonlinear control techniques to minimize a given cost function. These techniques may require a considerable amount of computer storage and real time computation. Townsend [1] investigated the possibility of controlling a nonlinear arm with feedback control computed for the linearized arm motion equations. The non-linear equations of the arm are linearized about a certain desired motion.

It is assumed that the deviations of the actual motion from the desired motion are small so that linear control laws can be used to let the arm follow the desired motion. Townsend used two types of linear system controls: a regulator and a variable gain tracking technique. If the desired motion must be optimal the problem of how to generate the optimal motion strategies arises. This thesis describes a possible solution to this problem for a particular class of tasks, namely moving a manipulator arm from one position to another.

If the manipulator arm has n degrees of freedom the position of the arm with n general coordinates $x_1, \ldots, x_n$ is described by a vector $\underline{x}$ in the coordinate space. The motion of the arm between an initial position $\underline{x}_i = [x_{1i}, \ldots, x_{ni}]$ and a final position $\underline{x}_f = [x_{1f}, \ldots, x_{nf}]$ is described by a trajectory in the coordinate space between $\underline{x}_i$ and $\underline{x}_f$. The motion is optimal if the value of the cost function is minimum along the trajectory.

CHAPTER 2

METHOD OF GENERATING OPTIMAL TRAJECTORIES

In this chapter a method of generating the optimal trajectory between an initial and a final position of a n degree of freedom mechanical arm is described.

2.1 General Approach

The method is based on the assumption that the time history of each of the elements of the position vector $x_k (k = 1, \ldots, n)$ between x_{ki} and x_{kf} can be expanded in a series of simple time functions.

$$x_k = A_k + a_{k0}f_0(t) + a_{k1}f_1(t) + a_{k2}f_2(t) + \ldots\ldots + a_{km}f_m(t) \tag{2.1}$$

where $a_{k\ell} (\ell = 0,\ldots,m)$ are coefficients independent of time and A_k is constant. If $f_0(t),\ldots,f_m(t)$ are given time functions x_k is only a function of the coefficients $a_{k1},\ldots,a_{km}$. The cost function J which must be minimized along the trajectory between the initial and final position of the arm is generally a function of x_k, its derivatives $\dot{x}_k$ and $\ddot{x}_k$, and the task performance time T. J takes the general form:

$$J = \int_0^T L(\underline{x}, \underline{\dot{x}}, \underline{\ddot{x}})dt \tag{2.2}$$

Substituting the functions for x_k, $\dot{x}_k$ and $\ddot{x}_k$ in the expression for J, J can be written as a function of the parameters $a_{k\ell}(k = 1,\ldots,n;$ $\ell = 0,\ldots\ldots,m)$ and the performance time T. For a given T, J is only a function of the a's. So:

$$J = J(\underline{a}) \tag{2.3}$$

where $\underline{a}$ is the matrix of the parameters $a_{k\ell}(k = 1,\ldots,n;\ \ell = 0,\ldots,m)$.

By this procedure the problem of finding the optimal trajectory has been reduced to a parameter optimization problem, i.e. finding the values of the parameters $a_{k\ell}$ for which the value of the cost function $J(\underline{a})$ is minimum.

To obtain the optimal values of the parameters, one can follow different procedures which can be divided into two main categories:

a. analytical method,

b. numerical methods.

The two methods are discussed briefly in the following sections.

2.2 Analytical Method

If there are no constraints on the possible values of the parameters $a_{k\ell}$ and the function $J(\underline{a})$ has first and second partial derivatives everywhere, necessary conditions for a minimum are:

$$\frac{\partial J}{\partial a_i} = 0 \tag{2.4}$$

where a_i is the i-th element of $\underline{a}$ with $i = \ell n + k$ and:

$$\frac{\partial^2 J}{\partial a_i \partial a_j} \geq 0 \qquad (2.5)$$

which means that the matrix whose components are $\frac{\partial^2 J}{\partial a_i \partial a_j}$ must be positive semidefinite. Equation (2.4) will give as many equations as there are unknown parameters. The advantage of the analytical method is that it gives all the possible solutions. However, in practice this method can present problems if the function $J(\underline{a})$ is complicated.

2.3 Numerical Methods

There are various numerical methods available. Bryson and Ho [2], Bekey [3], Sage and Melsa [4] give a survey and a description of several of these methods. In general they are based on the following principle. Make an initial guess for the values of the parameters and supply these values as part of the input to a computer program. The program changes the values of the parameters according to a certain algorithm until it has found a set of values for the parameters which minimizes the value of the cost function. The particular numerical method one uses depends on the behavior of the function $J(\underline{a})$ as a function of its argument $\underline{a}$. A disadvantage of these numerical methods is that , if $J(\underline{a})$ has several local minima, only one local minimum is found, depending on the initial guess for the values of the a's. This local minimum is not necessarily the global minimum. The numerical methods are very suitable for the cases that $J(\underline{a})$ is a complicated function.

CHAPTER 3

GENERATION OF OPTIMAL TRAJECTORIES FOR A PARTICULAR MANIPULATOR ARM

The generation of optimal trajectories as described in Chapter 2 has been applied to a planar three degree of freedom mechanical arm.

3.1 Description of the Manipulator Arm

The arm consists of two rigid straight links with lengths ℓ_1 and ℓ_2 connected to a fixed reference frame and to each other by moveable joints. As shown in Fig. 3.1 the joint with the fixed frame is considered as a double hinge with two degrees of freedom; the joint between the two links is a hinge with one degree of freedom. The mass of the arm is lumped as two point masses m_1 and m_2 at the ends of the links. The point masses have no rotational moment of inertia about the axis of the associated links. This lumping of masses simplifies the mathematics but does not affect the generality of the results.

The following joint angles have been chosen as coordinates of the arm (see also Fig. 3.2):

1. the angle θ_1 between the plane through the two links and a fixed plane through axis 1,
2. the angle θ_2 between link 1 and a line in the plane of the arm perpendicular to axis 1,
3. the relative angle θ_3 between link 1 and link 2.

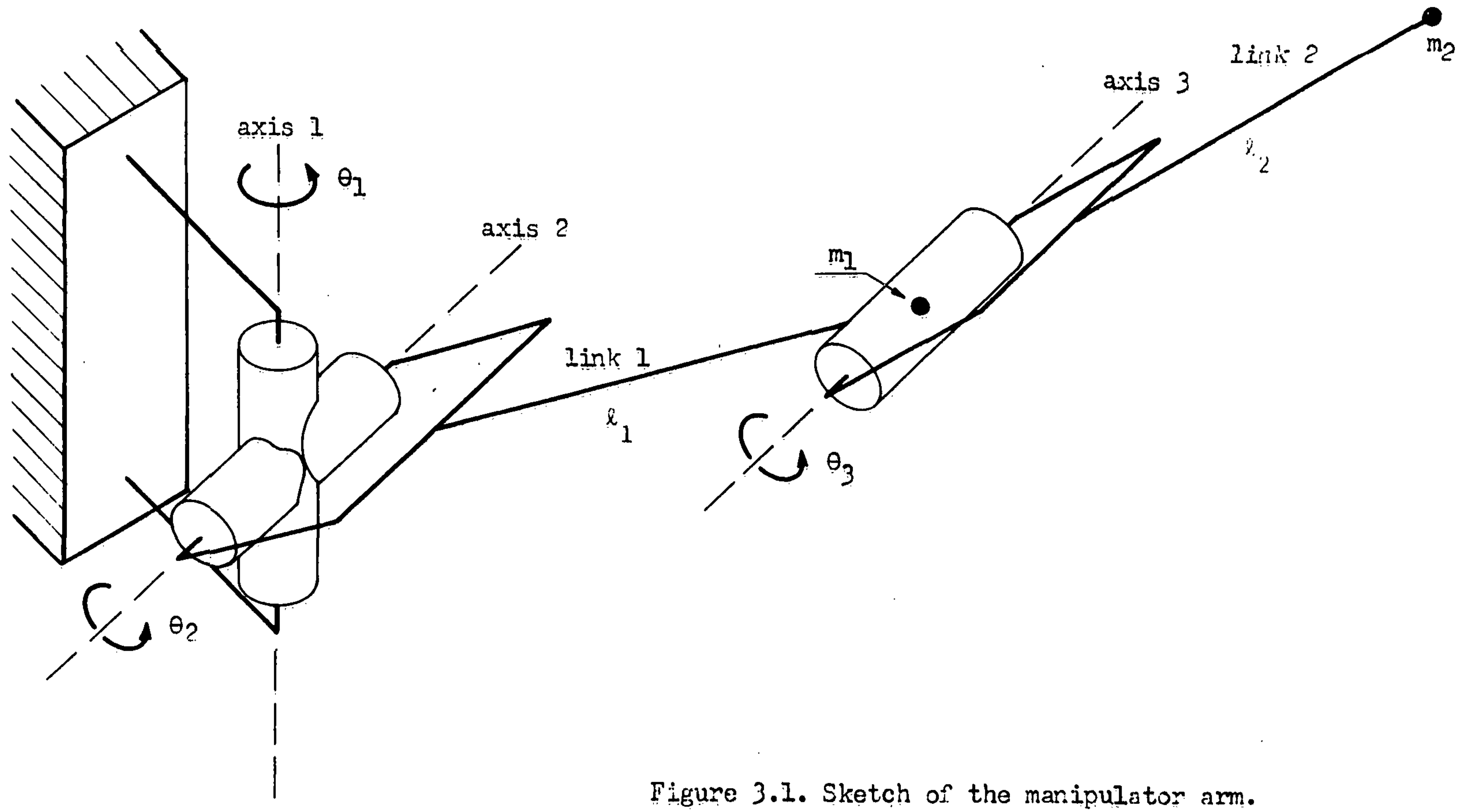


Figure 3.1. Sketch of the manipulator arm.

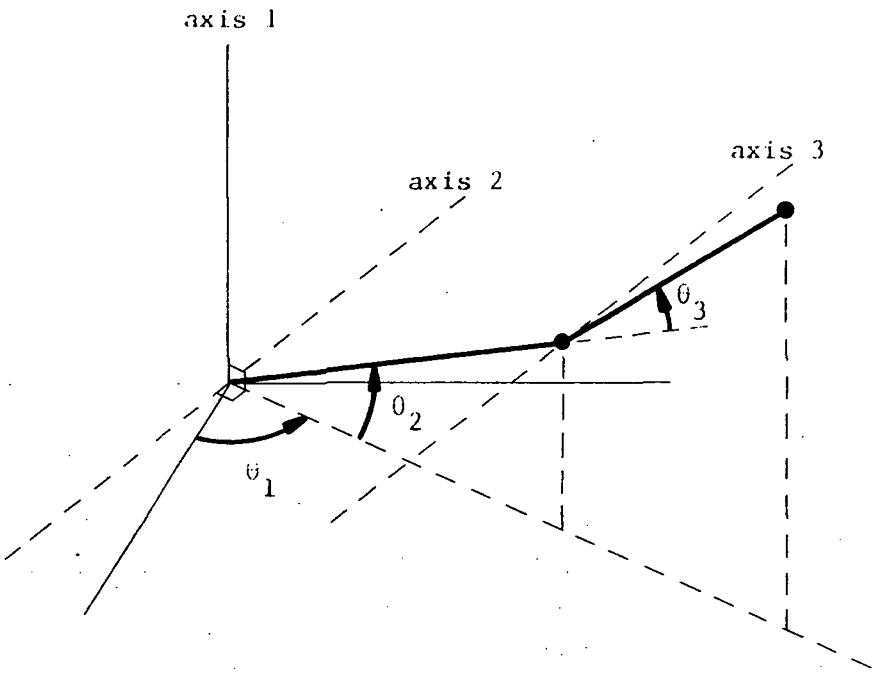


Figure 3.2. Coordinates of the manipulator arm.

The arrows in Fig. 3.2 indicate the positive direction of rotation. This coordinate system is convenient both mathematically and physically for manipulators with torque drive at the joints.

The position of the manipulator arm is described by a vector $\underline{\theta}$ with elements θ_1, θ_2, and θ_3. This $\underline{\theta}$ corresponds to $\underline{x}$ used in Chapter 2.

3.2 Trajectories

Trajectories can be categorized depending on the type of functions $f_\ell(t)(\ell = 0,\ldots,m)$ used in the expansion of the functions $\theta_k(k = 1,2,3)$:

$$\theta_k = A_k + a_{k0}f_0(t) + a_{k1}f_1(t) + a_{k2}f_2(t) + \ldots. + a_{km}f_m(t) \tag{3.1}$$

For the purpose of this study two types of trajectories have been assumed.

Type 1:

The function for each joint angle θ_k $(k = 1,2,3)$ between $t = 0$ and $t = T$ is a series expansion of polynomials.

$$f_0(t) = \frac{1}{T} t \tag{3.2a}$$

$$f_1(t) = \frac{4}{T^2} t(T-t) \tag{3.2b}$$

$$f_2(t) = \frac{64}{T^3} t\left(\frac{T}{2} - t\right)(T - t) \tag{3.2c}$$

Only the first three terms of the series have been taken into account, so $m = 2$.

For $t < 0$ and $t > T$ the value of θ_k is equal to the function value at $t=0$ and $t=T$ respectively. Figure 3.3. shows a plot of the functions $f_\ell(t)$ and their first and second derivatives.

From the end conditions at $t=0$ and $t=T$, i.e.

$$\theta_k(0) = \theta_{ki} \tag{3.3a}$$

$$\theta_k(T) = \theta_{kf} \tag{3.3b}$$

follows:

$$A_k = \theta_{ki} \tag{3.4}$$

and

$$a_{k0} = \theta_{kf} - \theta_{ki} \tag{3.5}$$

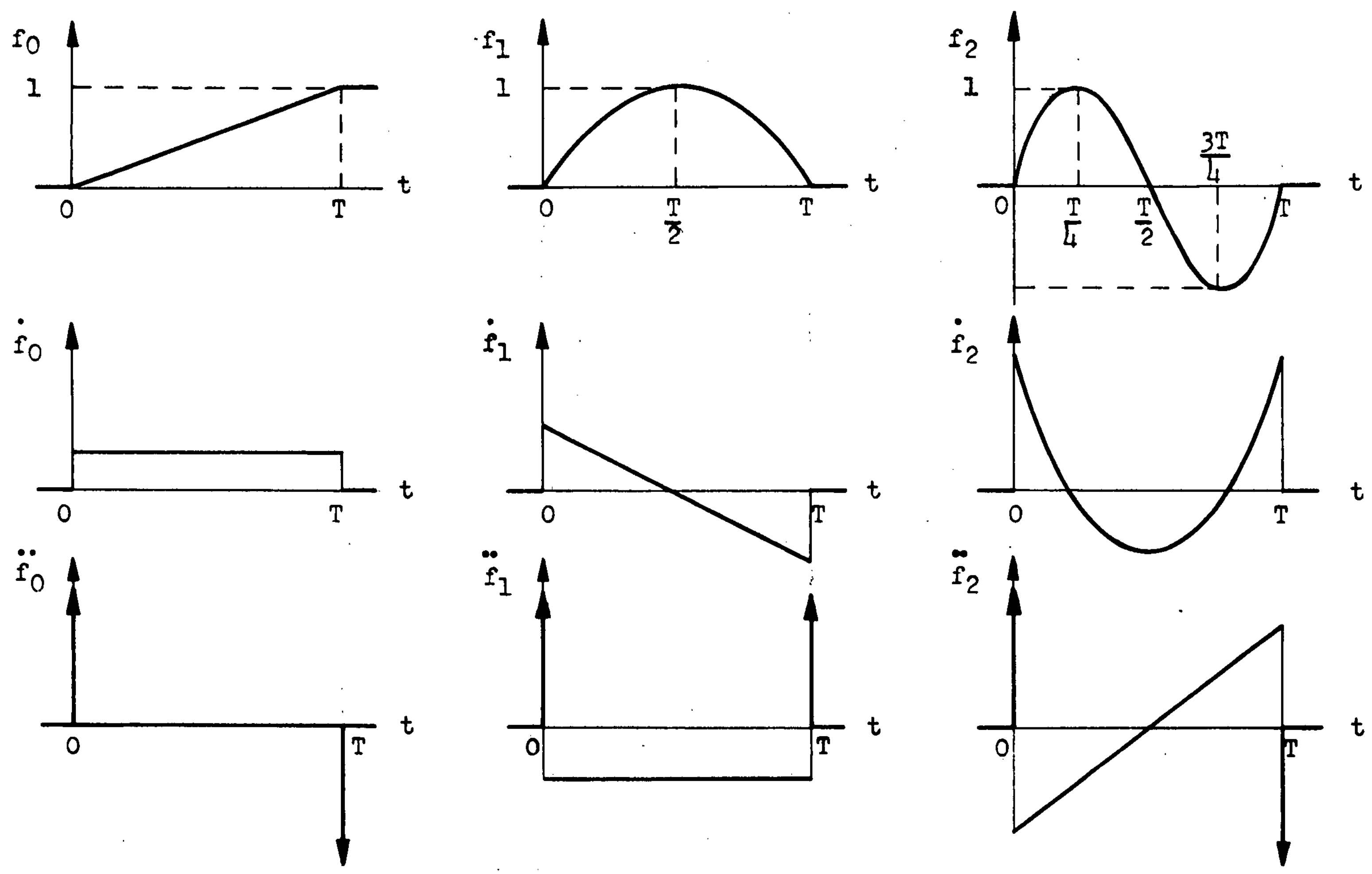


Figure 3.3. Functions $f_\ell(t)$ ($\ell=0,1,2$) and their first and second derivatives for a series expansion of polynomials.

The coefficients a_{k1} and a_{k2} are the ones to be chosen optimally. The expression for each θ_k becomes:

$$\theta_k = \theta_{ki} + (\theta_{kf} - \theta_{ki}) \frac{t}{T} + a_{k1} \frac{4}{T^2} t(T-t)$$
$$+ a_{k2} \frac{64}{T^3} t\left(\frac{T}{2} - t\right)(T-t) \tag{3.6}$$

This rather simple function does not provide a smooth start up and slow down of the manipulator arm because of the impulse singularities in the second derivatives.

Type 2:

The function for each joint angle θ_k $(k = 1,2,3)$ is a series expansion of periodic functions of the following form:

$$f_0(t) = \frac{1}{T} \left(t - \frac{1}{\omega} \sin \omega t\right) \tag{3.7a}$$

$$f_1(t) = \begin{cases} \frac{2}{T}\left(t - \frac{1}{2\omega} \sin 2\omega t\right), & 0 \leq t \leq \frac{T}{2} \\ 2 - \frac{2}{T}\left(t - \frac{1}{2\omega} \sin 2\omega t\right), & \frac{T}{2} \leq t \leq T \end{cases} \tag{3.7b}$$

$$f_2(t) = \begin{cases} \frac{4}{T}\left(t - \frac{1}{4\omega} \sin 4\omega t\right), & 0 \leq t \leq \frac{T}{4} \\ 2 - \frac{4}{T}\left(t - \frac{1}{4\omega} \sin 4\omega t\right), & \frac{T}{4} \leq t \leq \frac{3T}{4} \\ -4 + \frac{4}{T}\left(t - \frac{1}{4\omega} \sin 4\omega t\right), & \frac{3T}{4} \leq t \leq T \end{cases} \tag{3.7c}$$

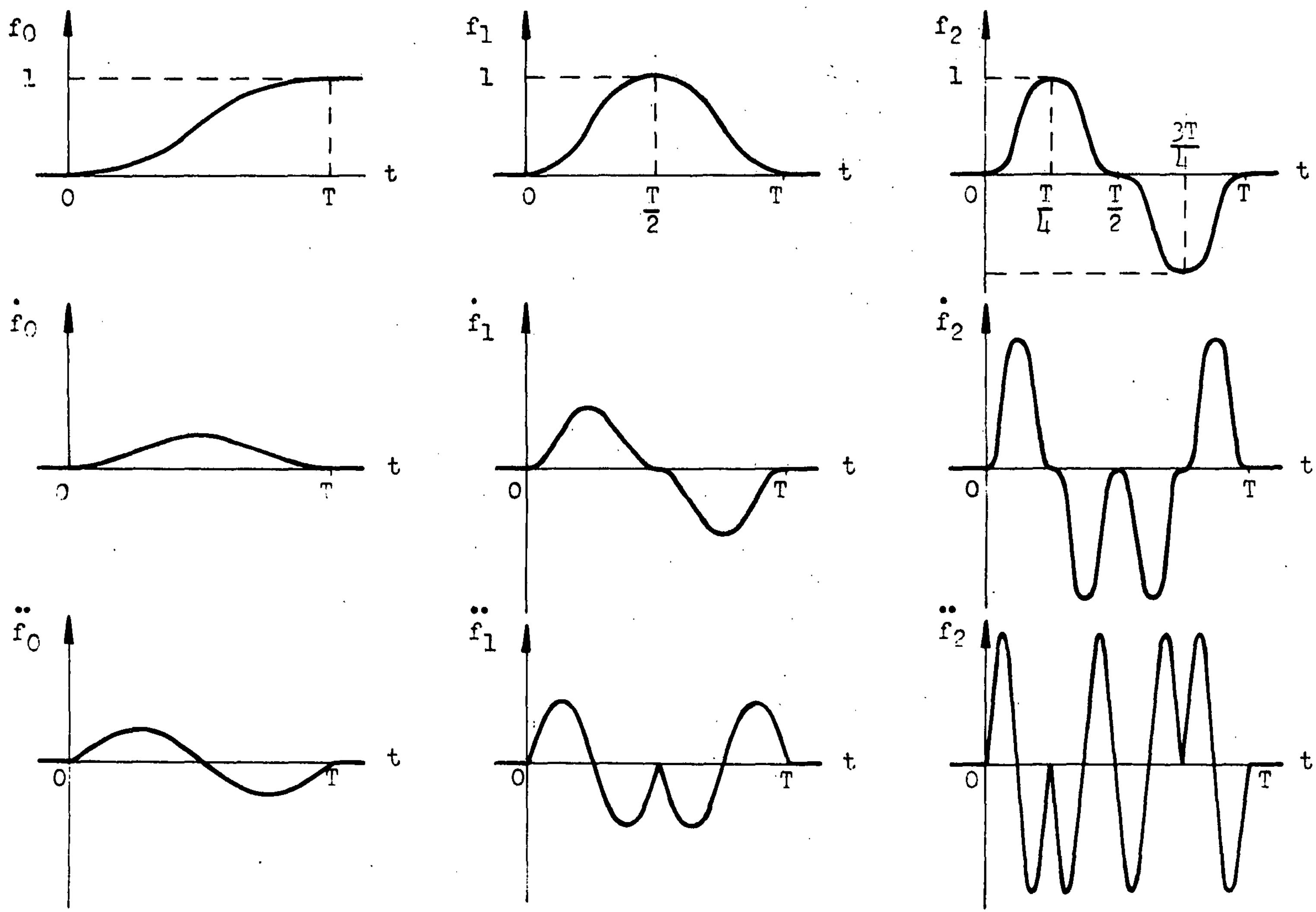


Figure 3.4. Functions $f_\ell(t)$ (ℓ=0,1,2) and their first and second derivatives for a series expansion of periodic functions.

where $\omega = 2\pi/T$. Also in this expansion only the first three terms have been taken into account.

The functions $f_\ell(t)$ and their derivatives are plotted in Fig. 3.4. From the conditions (3.3) follows:

$$A_k = \theta_{ki} \tag{3.4}$$

and

$$a_{k0} = \theta_{kf} - \theta_{ki} \tag{3.5}$$

As $\dot{\theta}_k$ and $\ddot{\theta}_k$ are zero at $t=0$ and $t=T$ this type of trajectory will give a smooth start up and slow down of the arm.

3.3 Cost Functions

Two cost functions have been used to optimize the trajectory between the initial and final position of the arm. Both cost functions are integral type functions.

Cost Function 1:

The integral between $t=0$ and $t=T$ of the kinetic energy of the manipulator arm:

$$J = \int_0^T KE \, dt \tag{3.8}$$

where KE = kinetic energy of the arm at time t .

For the particular arm studied here the expression for the kinetic energy of the arm at time t is:

$$\begin{aligned} KE = 0.5[\dot{\theta}_1^2\{(m_1 + m_2)\ell_1^2 \cos^2\theta_2 + m_2\ell_2^2 \cos^2(\theta_2 + \theta_3) \\ + 2\, m_2\ell_1\ell_2 \cos\theta_2 \cos(\theta_2 + \theta_3)\} \end{aligned} \tag{3.9}$$

CONT'D.

$$+ \dot{\theta}_2^2\{(m_1 + m_2)\ell_1^2 + m_2 \ell_2^2 + 2m_2 \ell_1\ell_2 \cos \theta_3\}$$

$$+ \dot{\theta}_3^2\{m_2\ell_2^2\} \tag{3.9}$$

$$+ \dot{\theta}_2 \dot{\theta}_3\{2m_2 \ell_2^2 + 2m_2 \ell_1\ell_2 \cos \theta_3\}$$

This expression is derived in Appendix IA.

Cost Function 2:

The integral between $t=0$ and $t=T$ of the sum of the magnitude of the joint torques:

$$J = \int_o^T \sum_{k=1}^{3} |u_k| \, dt \tag{3.10}$$

where u_k = external torque applied at the k-th axis of rotation.

This cost function is closely related to the energy consumed.

From the dynamic equations of motion:

$$\underline{T}\,\underline{\ddot{\theta}} = \underline{u} + \underline{c}$$

follows:

$$u = \underline{T}\,\underline{\ddot{\theta}} - c \tag{3.12}$$

where $\underline{u}$ = $[u_1, u_2, u_3]$, the vector representing the external torques.

$$\underline{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$, the matrix representing the moments of inertia.

For the arm studied here

$$T_{12} = T_{13} = T_{21} = T_{31} = 0.$$

$\underline{c} = [c_1, c_2, c_3]$, the vector representing the torques due to the reaction forces to centripetal and coriolis forces.

The equations for the elements of $\underline{u}$ become:

$$u_1 = T_{11}\,\ddot{\theta}_1 - c_1 \tag{3.13a}$$

$$u_2 = T_{22}\,\ddot{\theta}_2 + T_{23}\,\ddot{\theta}_3 - c_2 \tag{3.13b}$$

$$u_3 = T_{32}\,\ddot{\theta}_2 + T_{33}\,\ddot{\theta}_3 - c_3 \tag{3.13c}$$

The expressions for the elements of $\underline{T}$ and $\underline{c}$ are given in Appendix IB. A computer program which generates the equations of motion for a manipulator arm of a given configuration was available.

In both cost functions the influence of gravity has been omitted for two reasons. First, the position of a manipulator arm with respect to gravity will differ from case to case. For any particular case it will not be difficult to incorporate the influence of gravity in the cost function. Second, if a manipulator is used in outer space the influence of gravity is absent.

3.4 Computer Programs

Using the trajectories and cost functions described in the previous sections three combinations of cost function and type of trajectory are possible.

Combination I:

Minimizing $J = \int_o^T KE\,dt$ assuming trajectories of Type 1 (θ_k is a series expansion of polynomials).

Combination II:

Minimizing $J = \int_o^T KE\,dt$ assuming trajectories of Type 2 (θ_k is a series expansion of periodic functions).

Combination III:

Minimizing $J = \int_o^T \sum_{k=1}^{3} |u_k|\,dt$ assuming trajectories of Type 2.

A combination of $J = \int_o^T \sum_{k=1}^{3} |u_k|\,dt$ with trajectories of Type 1 is not possible. The joint torques $u_k(k = 1,2,3)$ are functions of $\ddot{\theta}_k$. For the trajectories of Type 1 $\ddot{\theta}_k$ is infinite at $t=0$ and $t=T$.

To obtain the optimal values of the parameters $a_i(i=1,\ldots,6)$ for a particular initial and final position of the arm a fortran coded computer program has been written for each combination of trajectory and cost function. The programs consist of:

1. main program,
2. numerical search routine,
3. subroutine to compute the value of the cost function $J(\underline{a})$.

The three parts of the programs are described next.

1. Main Program

The main program reads the input data (lengths and masses of the arm, initial and final position, performance time, and initial guess for the values of the coefficients in the expansion), calls the search routine and prints out the final (optimal) values of the coefficients and the cost function. The main program is basically the same for combination I, II and III.

2. Numerical Search Routine

The search routine used in this study is called pattern search. Pattern search is a direct search routine for minimizing a function $J(\underline{a})$ of several variables $\underline{a} = [a_1, a_2, \ldots.]$. The argument $\underline{a}$ is systematically varied until the minimum of $J(\underline{a})$ is obtained. The pattern search routine determines the sequence of values for $\underline{a}$; an independent subroutine computes the functional values of $J(\underline{a})$.

A detailed description of the pattern search routine is given by Hooke and Jeeves [5].

Figure 3.5 shows a flow diagram of the search procedure as given by Hooke and Jeeves.

3. Subroutine to Compute the Value of the Cost Function

This subroutine is called by pattern search after each change in the argument $\underline{a}$. It simulates the trajectory of a given type for the value of $\underline{a}$ supplied by pattern search and computes the value of $J(\underline{a})$ along the trajectory. The integration is carried out with Simpson's Rule. The number of intervals between t=0 and t=T is twenty. This

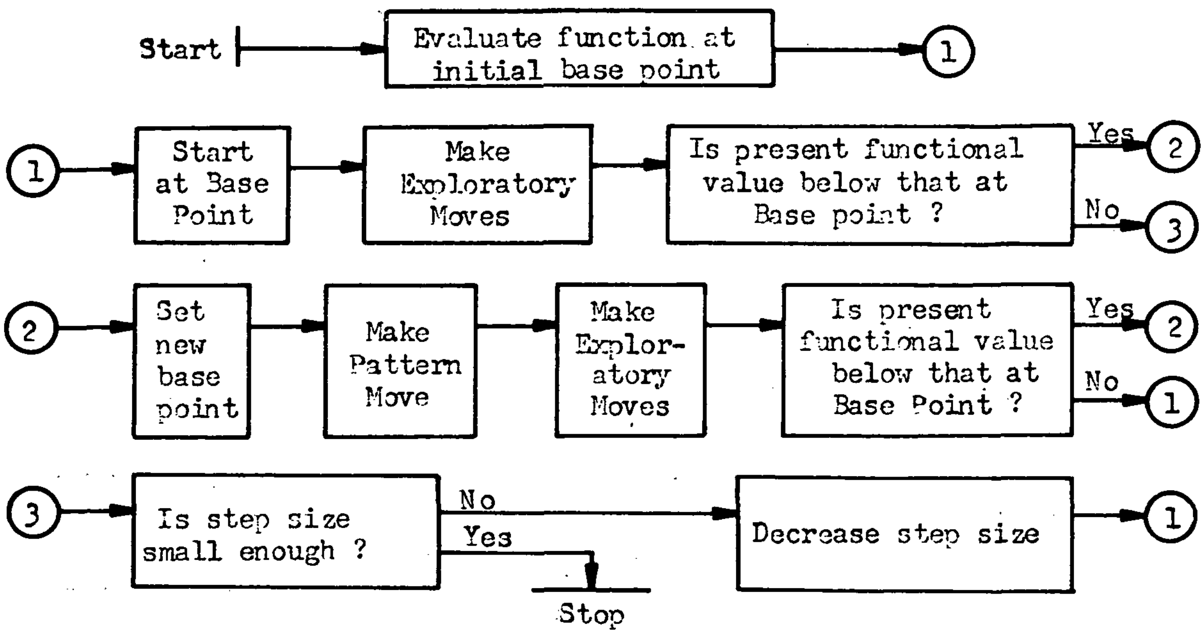


Figure 3.5a. Descriptive flow diagram of pattern search.

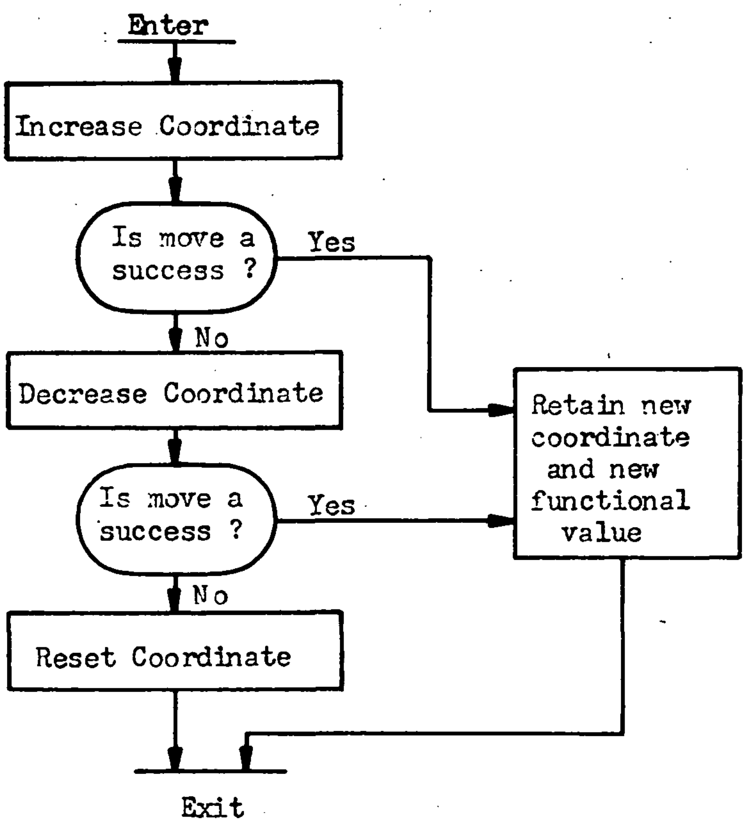


Figure 3.5b. Descriptive flow diagram for exploratory moves. This routine is carried out for each coordinate separately.

subroutine is different for each of the combinations of trajectory and cost function.

The programs are listed in Appendix II.

CHAPTER 4

DESCRIPTION AND RESULTS OF THE PROGRAM RUNS

This chapter gives a description of the program runs made, possible difficulties in the use of the search routine and an interpretation of the results of the searches. All runs were made for an arm with $\ell_1 = \ell_2 = 0.3$ m and $m_1 = m_2 = 1$ kg.

4.1 Initial and Final Positions

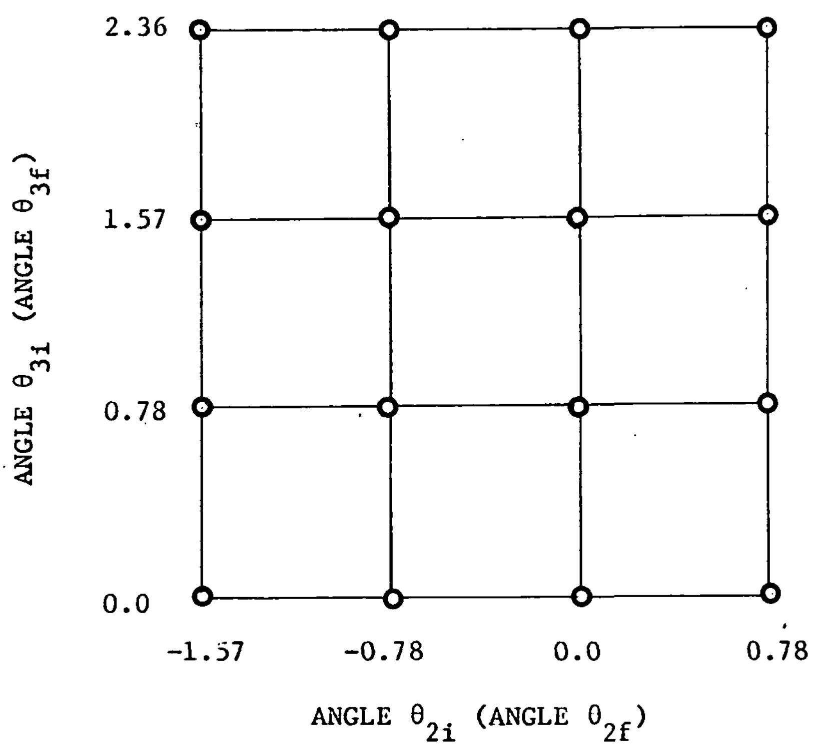


Figure 4.1. Combinations of initial (final) angles.

The initial and final positions of the arm were chosen in a certain region in θ-space at discrete points. The dots in Fig. 4.1 indicate the values of θ_{2i} and θ_{3i} or θ_{2f} and θ_{3f} . For fixed values of θ_{1i} and θ_{1f} this will give 256 possible combinations of initial and final position for each of the combinations I, II and III. To limit the number of runs a choice was made out of the 256 possible combinations.

For most runs $\theta_{1i} = -0.78$ and $\theta_{1f} = 0.78$ (angles in radians) were chosen. For combination III a number of runs were made with different values for θ_{1i} and θ_{1f} while keeping θ_{2i}, θ_{2f}, θ_{3i} and θ_{3f} constant.

4.2 Results of the Searches

The results of the individual searches for $\theta_{1i} = -0.78$ and $\theta_{1f} = 0.78$ are given in Appendices III A, B and C. The following observations can be made concerning the results for combinations I, II and III.

I. For all combinations of $\underline{\theta}_i$ and $\underline{\theta}_f$ the values of a_{12}, a_{22} and a_{32} are zero. When both $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the value of a_{11} is zero too.

II. When $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the optimal values of a_{11}, a_{21}, a_{22} and a_{32} are zero. The optimal value of a_{12} lies between 0.11 and 0.18.

III. As for I the optimal values of a_{12}, a_{22} and a_{32} are zero or very small in all cases, and a_{11} is zero when both $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$.

For the special cases that both $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the optimal values of the parameters a_{21} and a_{31} for the three combinations of trajectory and cost function are plotted as functions of $\theta_{2i} = \theta_{2f} = \theta_2$ and $\theta_{3i} = \theta_{3f} = \theta_3$ in Figs. 4.2, 4.3 and 4.4.

The results of the searches for combination III with varying θ_{1i} and θ_{1f} are listed in Appendix III D. The optimal values of a_{21}, a_{22} and a_{32} are zero in all cases. The optimal values of a_{11}, a_{21}, a_{31} and $J(\underline{a})$ are plotted as function of $\theta_{1f} - \theta_{1i}$ in Figs. 4.5, 4.6 and 4.7.

Each search gives only a local optimum. Therefore one can not be sure that the optimum found is a global optimum. However by starting the search in a proper point based on physical considerations one can increase the probability that the global optimum will be found. An example of choosing a wrong starting point is given next.

For combination III with $\underline{\theta}_i = [-0.78, -0.78, 1.57]$ and $\underline{\theta}_f = [0.78, -0.78, 1.57]$ the initial values of the a's were chosen all equal to zero. This resulted in a set of optimal values for the a's of [0.0, 0.0, 0.27, 0.0, 0.0, 0.0] and a value of the cost function J of 0.935. Starting the search at [0.0, -0.168, 0.492, 0.0, 0.0, 0.0] resulted in a set of optimal values for the a's of [0.0, -0.219, -0.626, 0.0, 0.0, 0.0] and a value of the cost function J of 0.630 which is much less than in the other case. The new starting point was the result of an interpolation between the results for the other combinations of θ_2 and θ_3 (see Fig. 4.4). This example indicates that the choice of

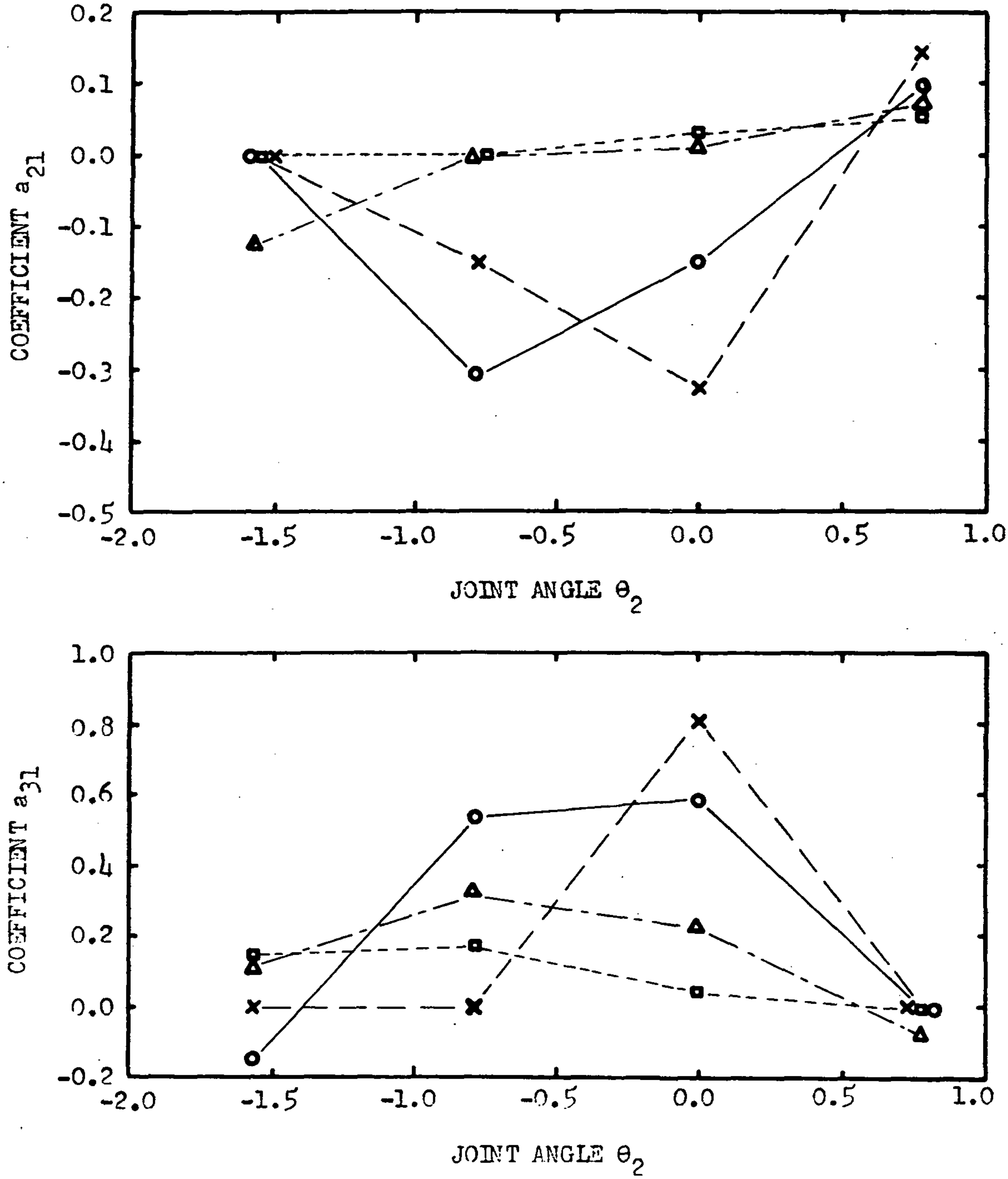


Figure 4.2. Optimal values of coefficients a_{21} and a_{31} for combination I with $\theta_{2i}=\theta_{2f}=\theta_2$ and $\theta_{3i}=\theta_{3f}=\theta_3$.

The symbols in the figures indicate the value of θ_3:

x—x $\theta_3=0.0$; o—o $\theta_3=0.78$; Δ-·-Δ $\theta_3=1.57$; □----□ $\theta_3=2.36$.

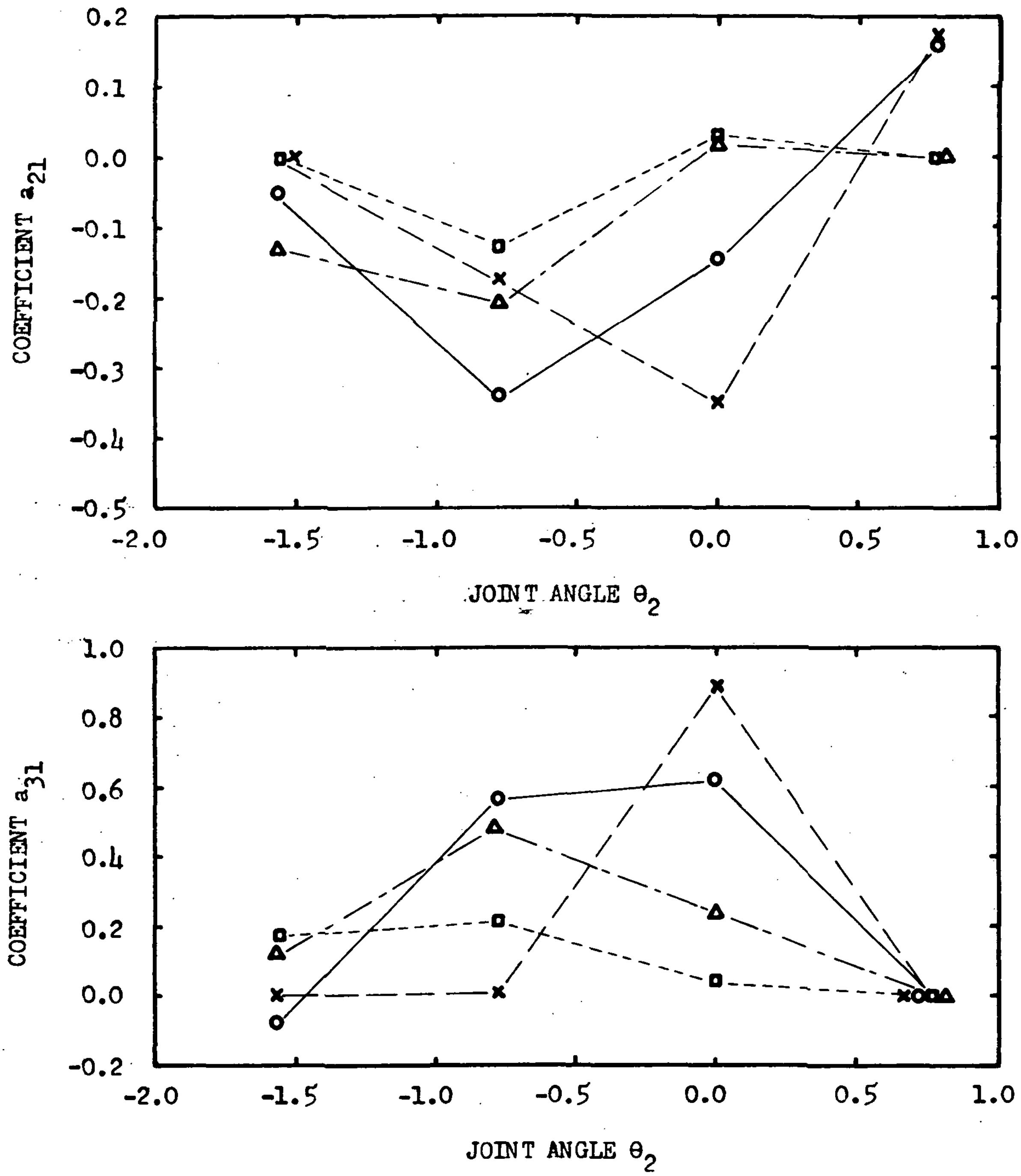


Figure 4.3. Optimal values of coefficients a_{21} and a_{31} for combination II with $\theta_{2i}=\theta_{2f}=\theta_2$ and $\theta_{3i}=\theta_{3f}=\theta_3$.

The symbols in the figures indicate the value of θ_3:

× – – × $\theta_3=0.0$; o—o $\theta_3=0.78$; △—·—△ $\theta_3=1.57$; □-----□ $\theta_3=2.36$.

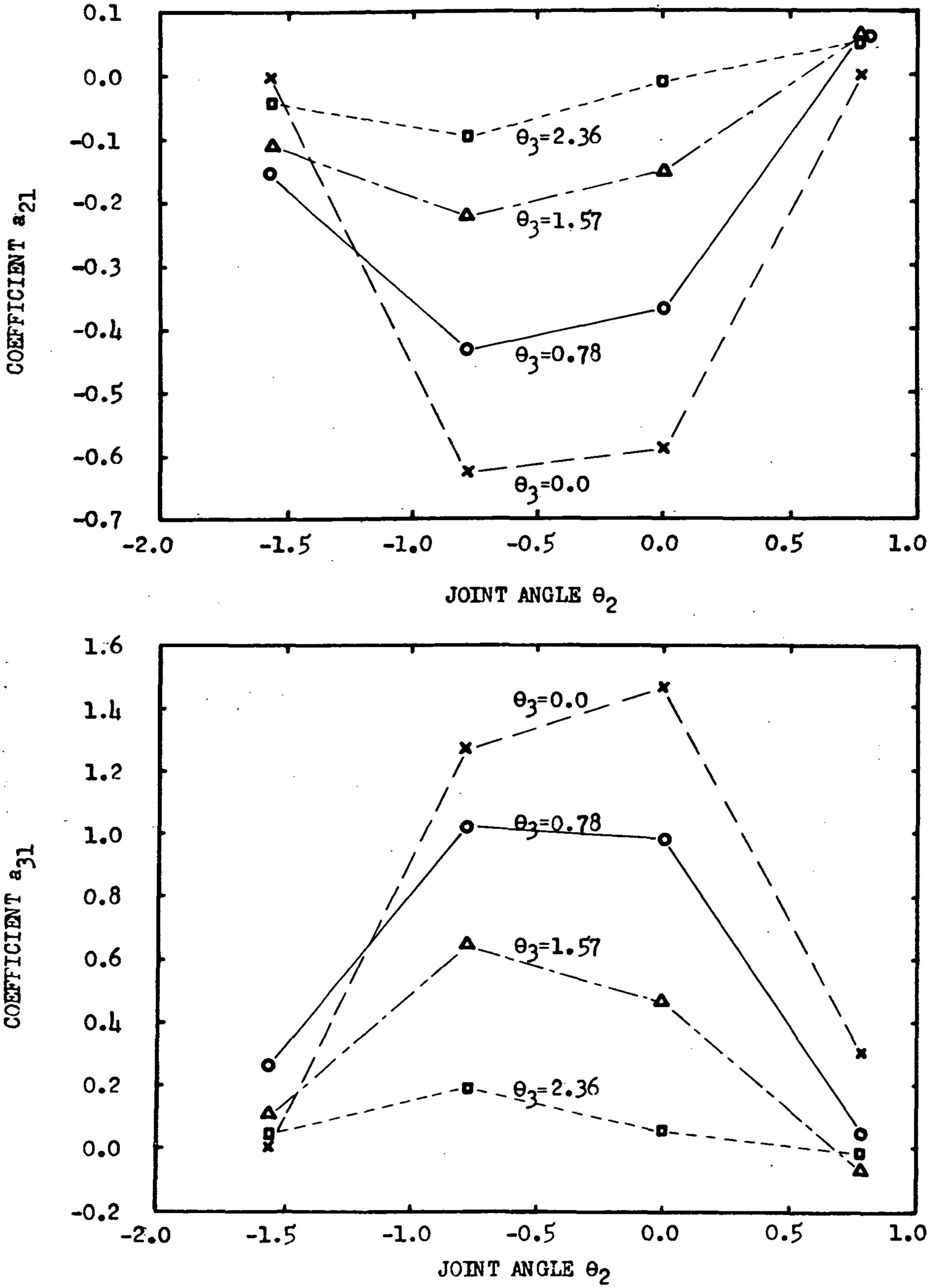


Figure 4.4. Optimal values of coefficients a_{21} and a_{31} for combination III with $\theta_{2i}=\theta_{2f}=\theta_2$ and $\theta_{3i}=\theta_{3f}=\theta_3$.

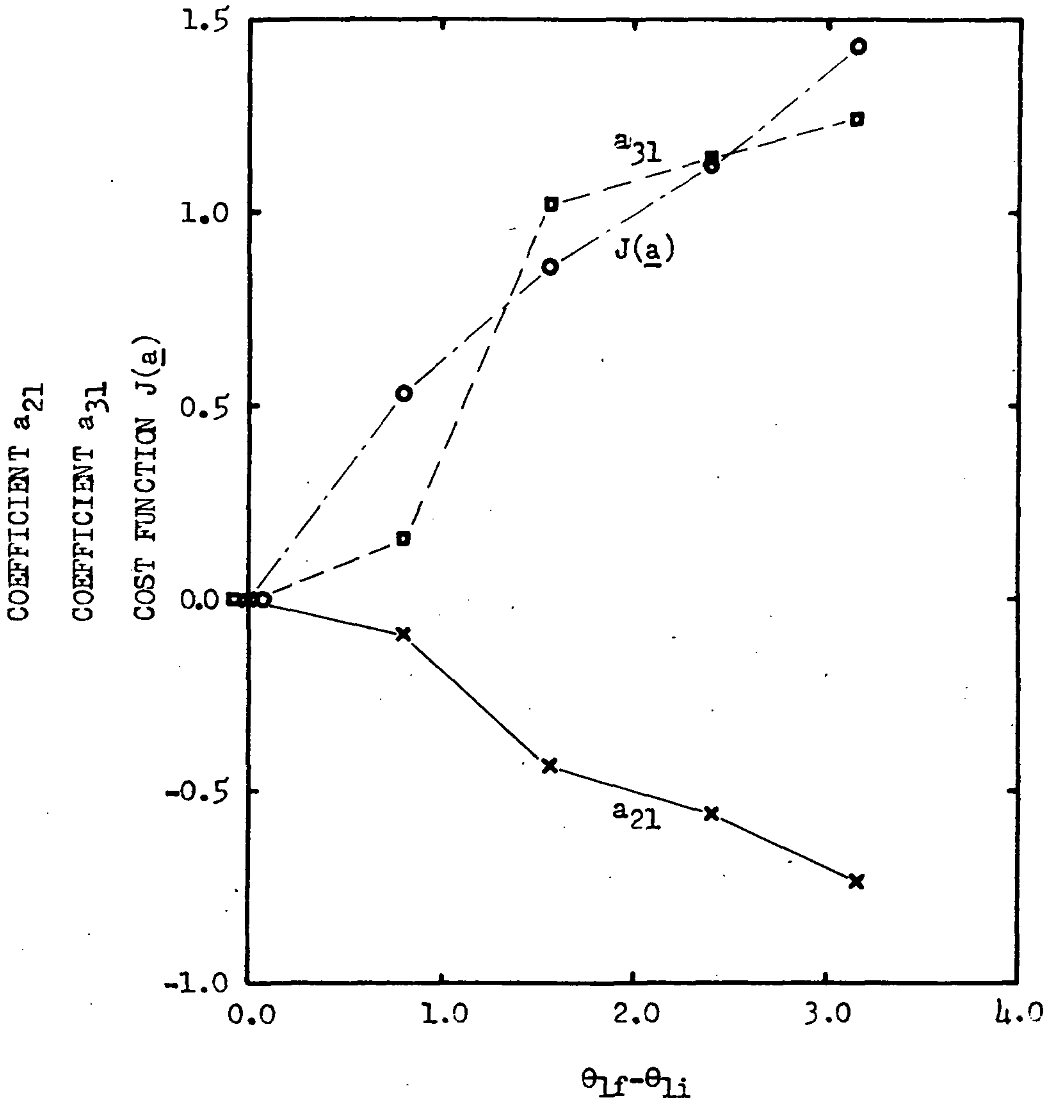


Figure 4.5. Optimal values of coefficients a_{21} and a_{31} and cost function $J(\underline{a})$ as functions of $\theta_{1f}-\theta_{1i}$ for combination III with $\theta_{2i}=\theta_{2f}=-0.78$ and $\theta_{3i}=\theta_{3f}=0.78$.

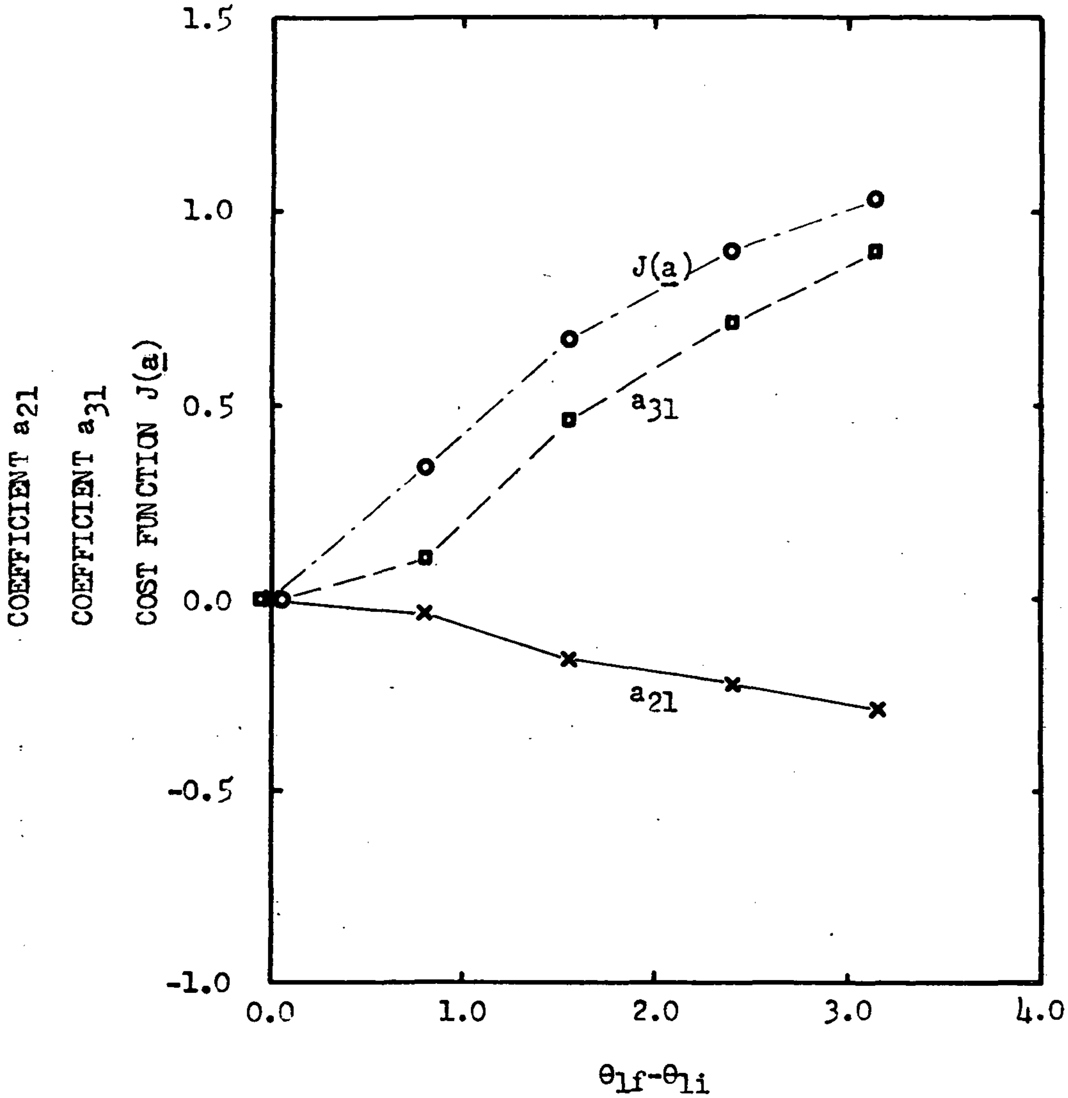


Figure 4.6. Optimal values of coefficients a_{21} and a_{31} and cost function $J(\underline{a})$ as functions of $\theta_{1f}-\theta_{1i}$ for combination III with $\theta_{2i}=\theta_{2f}=0.0$ and $\theta_{3i}=\theta_{3f}=1.57$.

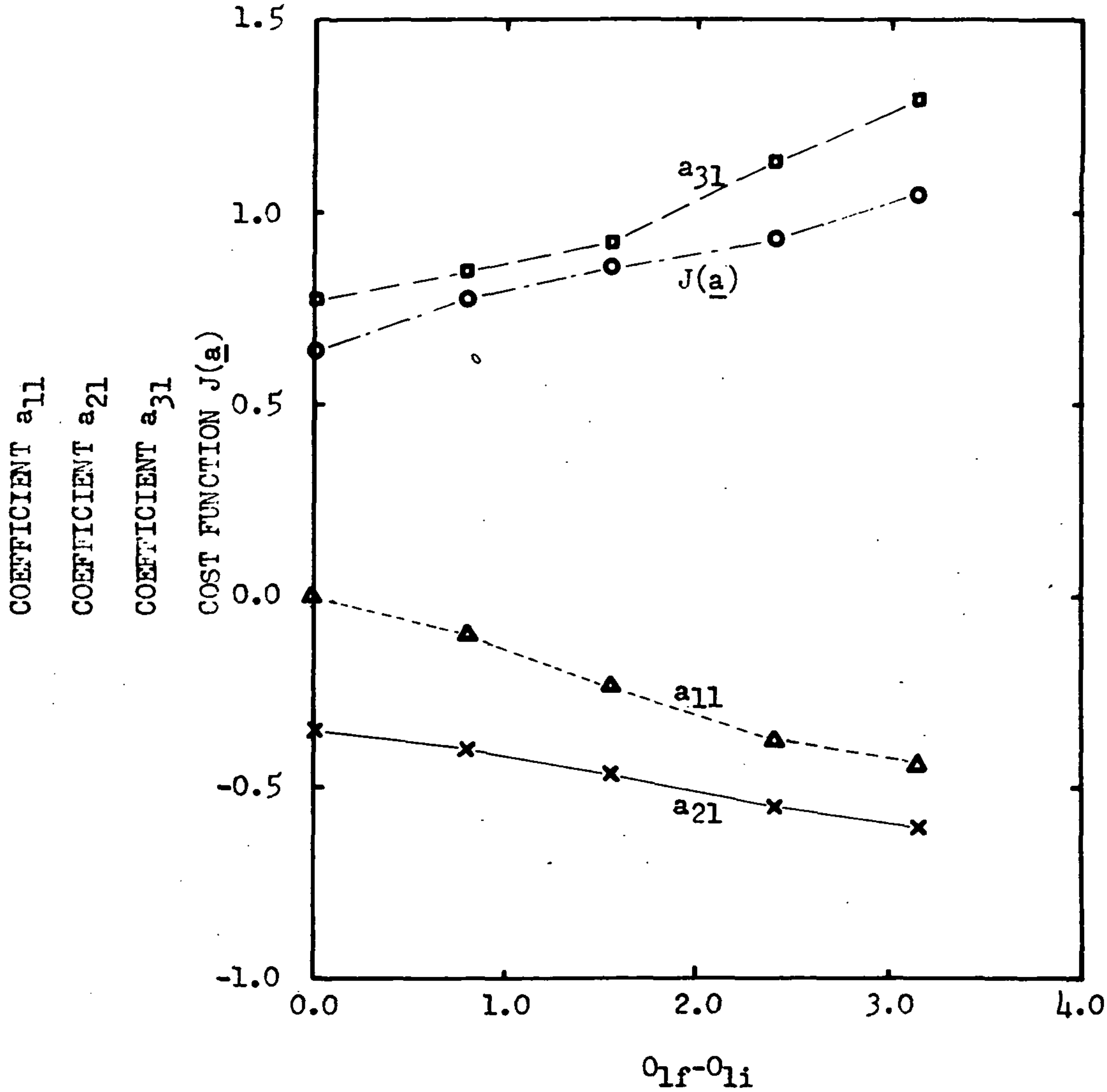


Figure 4.7. Optimal values of coefficients a_{11}, a_{21} and a_{31} and cost function $J(\underline{a})$ as functions of $\theta_{1f}-\theta_{1i}$ for combination III with $\theta_{2i}=-0.78$, $\theta_{2f}=-1.57$ and $\theta_{3i}=\theta_{3f}=0.78$.

the starting point for the search can be very important. Therefore, the starting points have been chosen carefully in accordance with the physics of the problem.

4.3 Some Examples of Optimal Trajectories

In Fig. 4.8 and 4.9 two examples of how the functions for θ_1, θ_2 and θ_3 will look like for different combinations of trajectory type and cost functions, using the optimal values of the coefficients in the series expansion. The Roman numbers in the figures indicate the combination of trajectory and cost function as mentioned in Section 3.4.

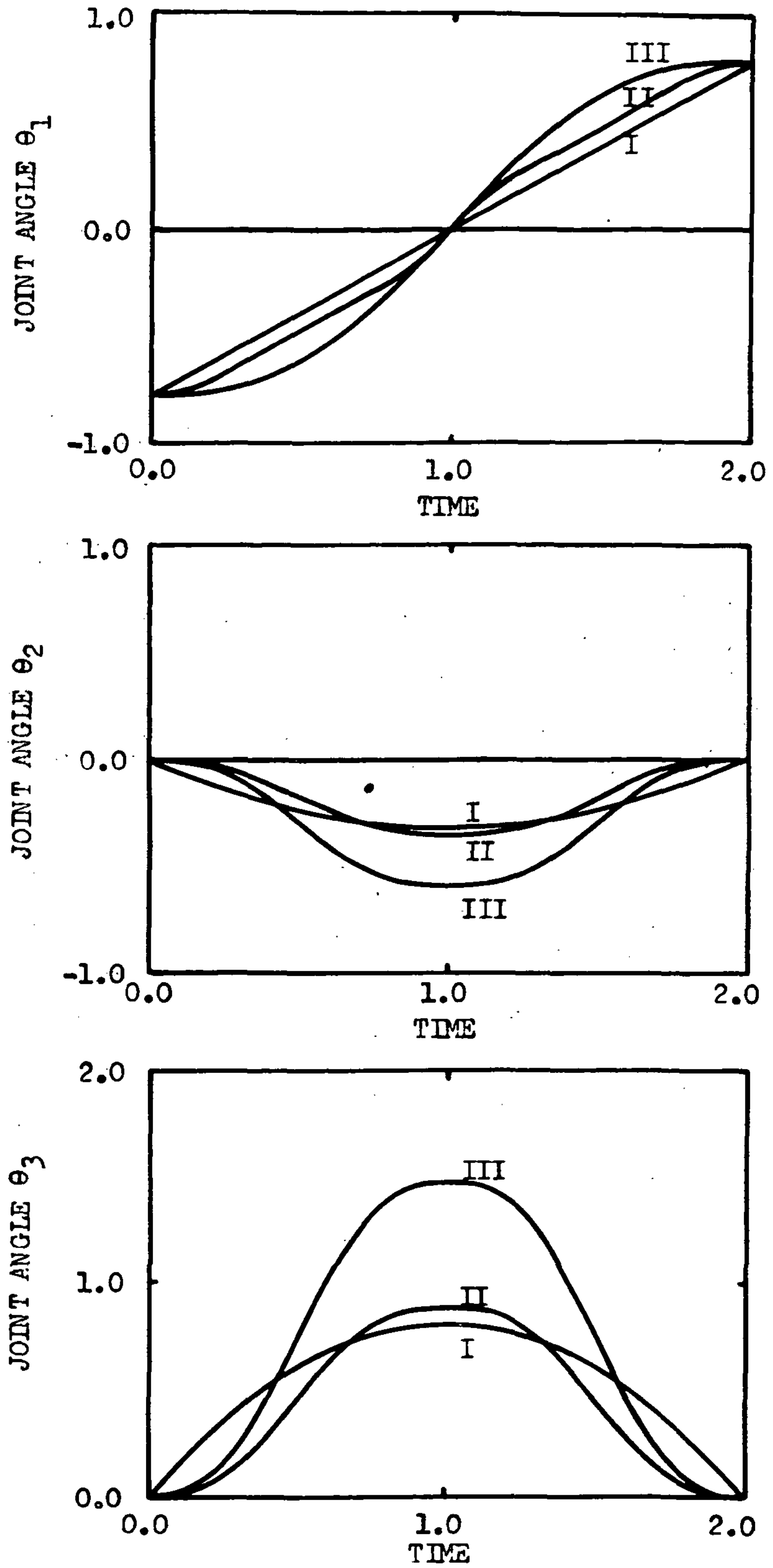


Figure 4.8. Time histories for θ_1, θ_2 and θ_3 for $\underline{\theta_i} = [-0.78, 0.0, 0.0]$ and $\underline{\theta_f} = [0.78, 0.0, 0.0]$.

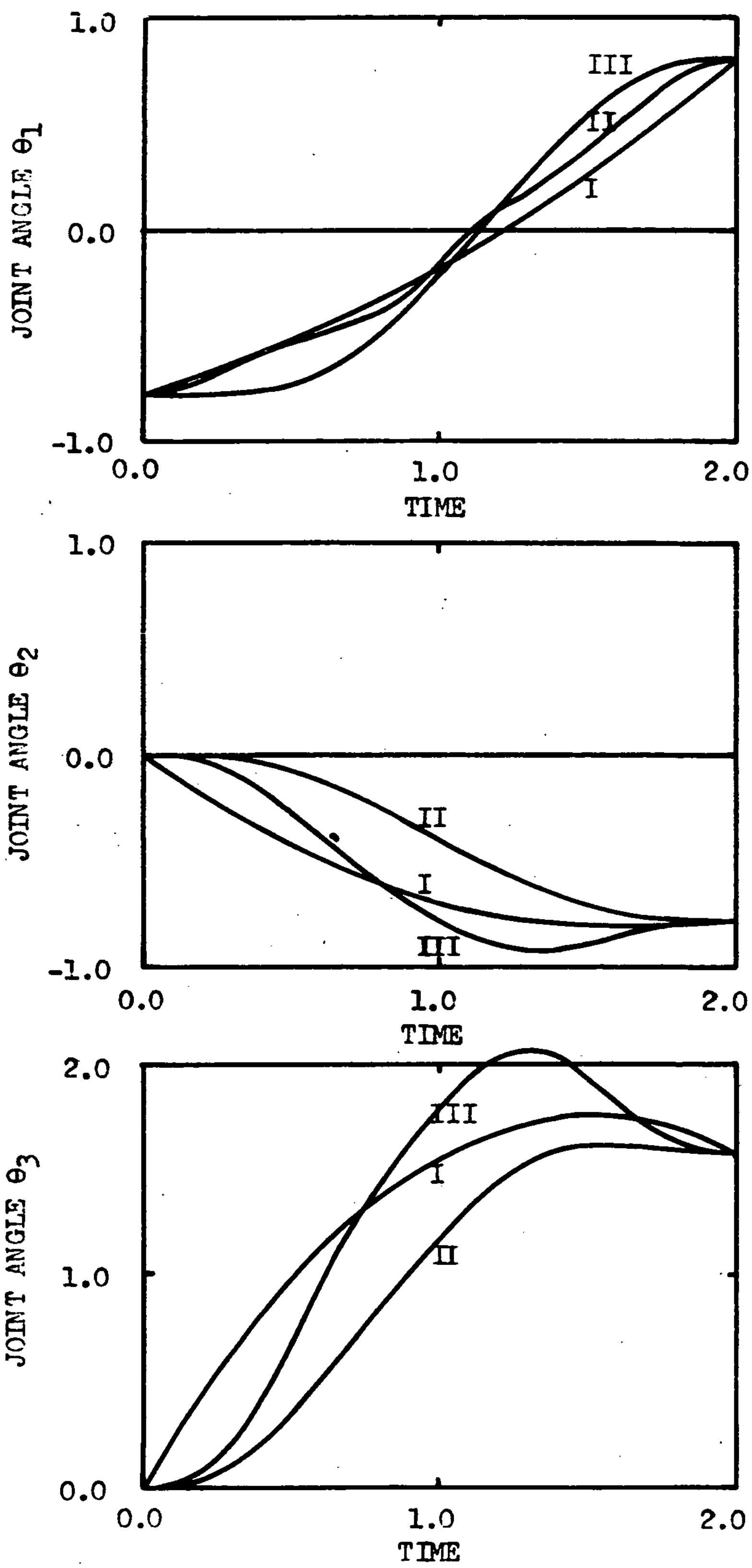


Figure 4.9. Time histories for θ_1, θ_2 and θ_3 for $\underline{\theta}_i=[-0.78,0.0,0.0]$ and $\underline{\theta}_f=[0.78,-0.78,1.57]$.

CHAPTER 5

GENERALIZATION OF THE RESULTS

This chapter discusses a method of generalizing the results of the individual searches for specific combinations of trajectory and cost function. This method will lead to suboptimal values of the coefficients $a_{k\ell}$. However, these values can be obtained by on-line computation in a short time. It is not necessary anymore to perform an on-line search.

The method is worked out for combination III where $J = \int_o^T \sum_{k=1}^{3} |u_k| \, dt$ and the trajectories are of Type 2.

The values of θ_{1i} and θ_{1f} are - 0.78 and 0.78 respectively.

For combination III the optimal values of a_{12}, a_{22} and a_{32} are zero or at least very small. Therefore the suboptimal values of these parameters have been chosen zero for all combinations of $\underline{\theta}_i$ and $\underline{\theta}_f$. To obtain general expressions for determining the values of a_{11}, a_{21} and a_{31} four categories of combinations of $\underline{\theta}_i$ and $\underline{\theta}_f$ have been considered separately. The assumptions made and the results obtained for each of the categories are described in the following sections.

5.1 $\theta_{1i} = -0.78$, $\theta_{1f} = 0.78$, $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$.

For this case the optimal values of a_{11} are zero. The optimal values of a_{21} and a_{31} are plotted in Fig. 4.4.

It is assumed that for each value of θ_3 the values of a_{21} and a_{31} as functions of θ_2 can be approximated by parabolic functions of the form:

$$a_{21} = p(\theta_2 - a)^2 + b \tag{5.1}$$

$$a_{31} = q(\theta_2 - c)^2 + d \tag{5.2}$$

Using a linear regression technique the best fitting parabolas through the four values of a_{21} and a_{31} for a constant θ_3 were determined. The formula used for this purpose is derived in Appendix IV. This gives four expressions for a_{21} of the form (5.1) and four for a_{31} of the form (5.2). Consideration of the values of p, a, b, q, c and d as functions of θ_3 indicated that a good approximation for a,b,q,c and d as a function of θ_3 is a straight line and for p as a function of θ_3 is a parabola. Fitting the curves through the points the following expressions were obtained:

$$p = 0.4936 - 0.3079\,\theta_3 + 0.0504\,\theta_3^2 \tag{5.3}$$

$$a = -0.3929 - 0.2002\,\theta_3 \tag{5.4}$$

$$b = -0.6628 + 0.2619\,\theta_3 \tag{5.5}$$

$$q = -0.9985 + 0.3903\,\theta_3 \tag{5.6}$$

$$c = -0.3370 - 0.1106\,\theta_3 \tag{5.7}$$

$$d = 1.5442 - 0.5772\,\theta_3 \tag{5.8}$$

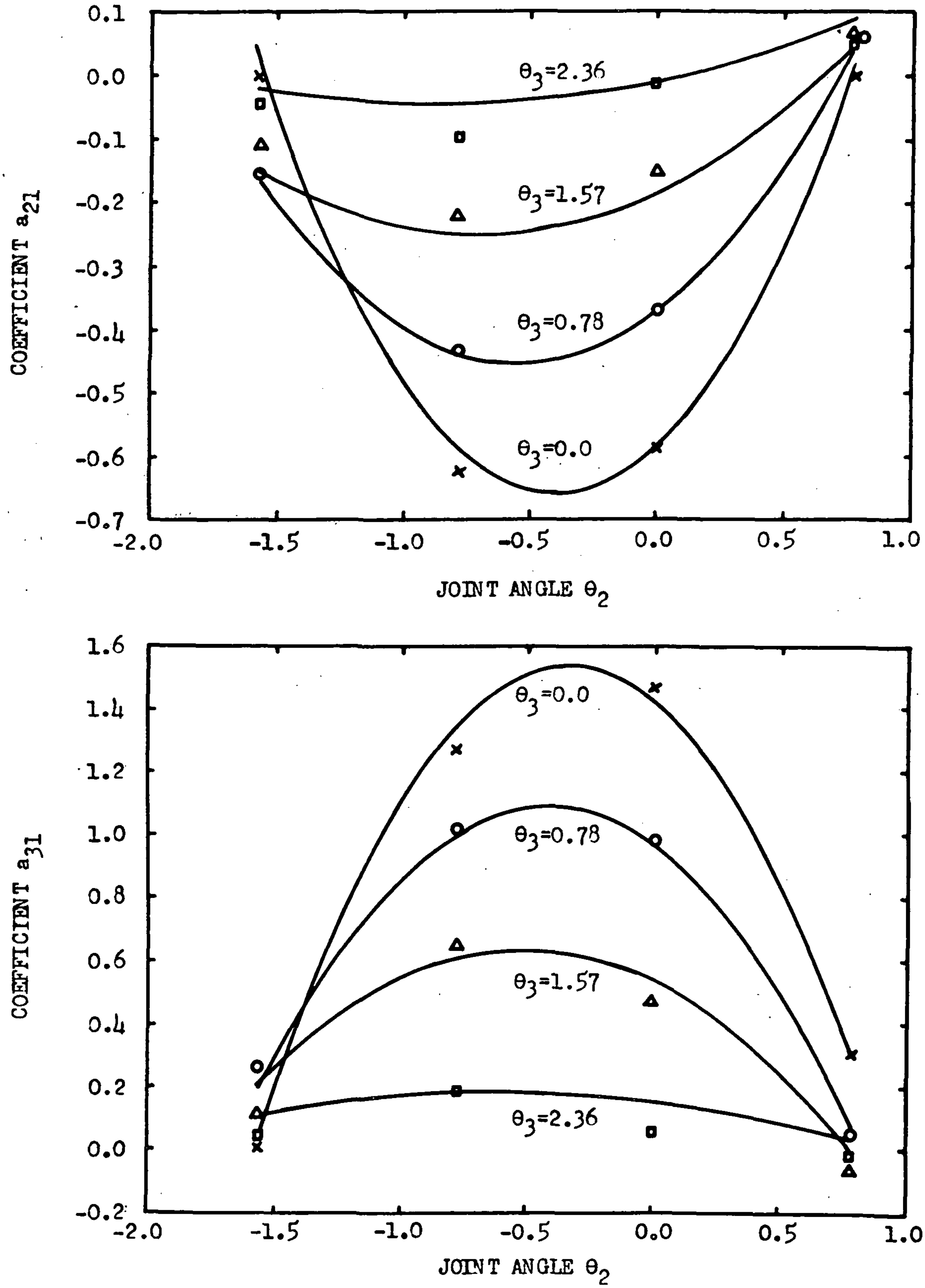


Figure 5.1. Functions for suboptimal values of coefficients a_{21} and a_{31} for combination III with $\theta_{2i}=\theta_{2f}=\theta_2$ and $\theta_{3i}=\theta_{3f}=\theta_3$.

Substituting (5.3) through (5.8) in Eqs. (5.1) and (5.2) a_{21} and a_{31} can be written as functions of $\theta_2 = \theta_{2i} = \theta_{2f}$ and $\theta_3 = \theta_{3i} = \theta_{3f}$. In Figure 5.1 the functions for a_{21} and a_{31} have been plotted for four values of θ_3. For comparison the optimal values of a_{21} and a_{31} obtained from the searches have been plotted too.

The value of the cost function $J(\underline{a})$ has been computed for the suboptimal values of the parameters and for all parameters equal to zero. This showed that on the average the difference between the suboptimal value and the optimal value of $J(\underline{a})$ is 7.4% of the optimal value. The difference between the value of $J(\underline{a})$ for all a's equal to zero and the optimal value is 27.5% of the optimal value.

5.2 $\underline{\theta_{1i} = -0.78,\ \theta_{1f} = 0.78,\ \theta_{2i} = \theta_{2f} \text{ and } \theta_{3i} \neq \theta_{3f}}$

For this category the expressions for a_{21} and a_{31} mentioned in 5.1 have been used to determine the suboptimal values of a_{21} and a_{31} as follows. Setting $\theta_2 = \theta_{2i} = \theta_{2f}$ and $\theta_3 = \theta_{3i}$ in Eqs. (5.1) through (5.8) gives certain values for a_{21} and a_{31}, say a'_{21} and a'_{31}. Setting $\theta_2 = \theta_{2i} = \theta_{2f}$ and $\theta_3 = \theta_{3f}$ the values of a_{21} and a_{31} are a''_{21} and a''_{31}. Using the following expressions a reasonable fit to the data for $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} \neq \theta_{3f}$ was obtained.

$$a_{21} = \min\{a'_{21}, a''_{21}\} \tag{5.9}$$

$$a_{31} = \min\{a'_{31}, a''_{31}\} + r|a'_{31} - a''_{31}| \tag{5.10}$$

where

$$r = 1.3(\theta_2 + 0.4)^4 + 0.6 \tag{5.11}$$

For the suboptimal value of parameter a_{11} the expression

$$a_{11} = -0.20 \operatorname{sgn}\{2 \cos \theta_{2i} + \cos(\theta_{2i} + \theta_{3i}) - 2 \cos \theta_{2f} - \cos(\theta_{2f} + \theta_{3f})\} \tag{5.12}$$

gave satisfactory results.

5.3 $\underline{\theta_{1i} = -0.78,\ \theta_{1f} = 0.78,\ \theta_{2i} \neq \theta_{2f} \text{ and } \theta_{3i} = \theta_{3f}}$.

For this category the suboptimal values for a_{21} and a_{31} were obtained from (again using Eqs. (5.1) through (5.8)):

$$a_{21} = \min\{\text{all } a_{21} \text{ for } \theta_3 = \theta_{3i} = \theta_{3f} \text{ and } \theta_2 \text{ between } \theta_{2i} \text{ and } \theta_{2f}\} \tag{5.13}$$

$$a_{31} = \max\{\text{all } a_{31} \text{ for } \theta_3 = \theta_{3i} = \theta_{3f} \text{ and } \theta_2 \text{ between } \theta_{2i} \text{ and } \theta_{2f}\} \tag{5.14}$$

The value for a_{11} followed from (5.12).

5.4 $\underline{\theta_{1i} = -0.78,\ \theta_{1f} = 0.78,\ \theta_{2i} \neq \theta_{2f} \text{ and } \theta_{3i} \neq \theta_{3f}}$.

Satisfactory values for the cost function $J(\underline{a})$ were obtained by choosing for the values of a_{21} and a_{31} (using (5.1) through (5.8)):

$$a_{21} = \min\{\text{all } a_{21} \text{ for } \theta_2 \text{ between } \theta_{2i} \text{ and } \theta_{2f} \text{ and } \theta_3 \text{ between } \theta_{3i} \text{ and } \theta_{3f}\} \tag{5.15}$$

$$a_{31} = \max\{\text{all } a_{31} \text{ for}$$
$$\theta_2 \text{ between } \theta_{2i} \text{ and } \theta_{2f} \text{ and}$$
$$\theta_3 \text{ between } \theta_{3i} \text{ and } \theta_{3f}\} \qquad (5.16)$$

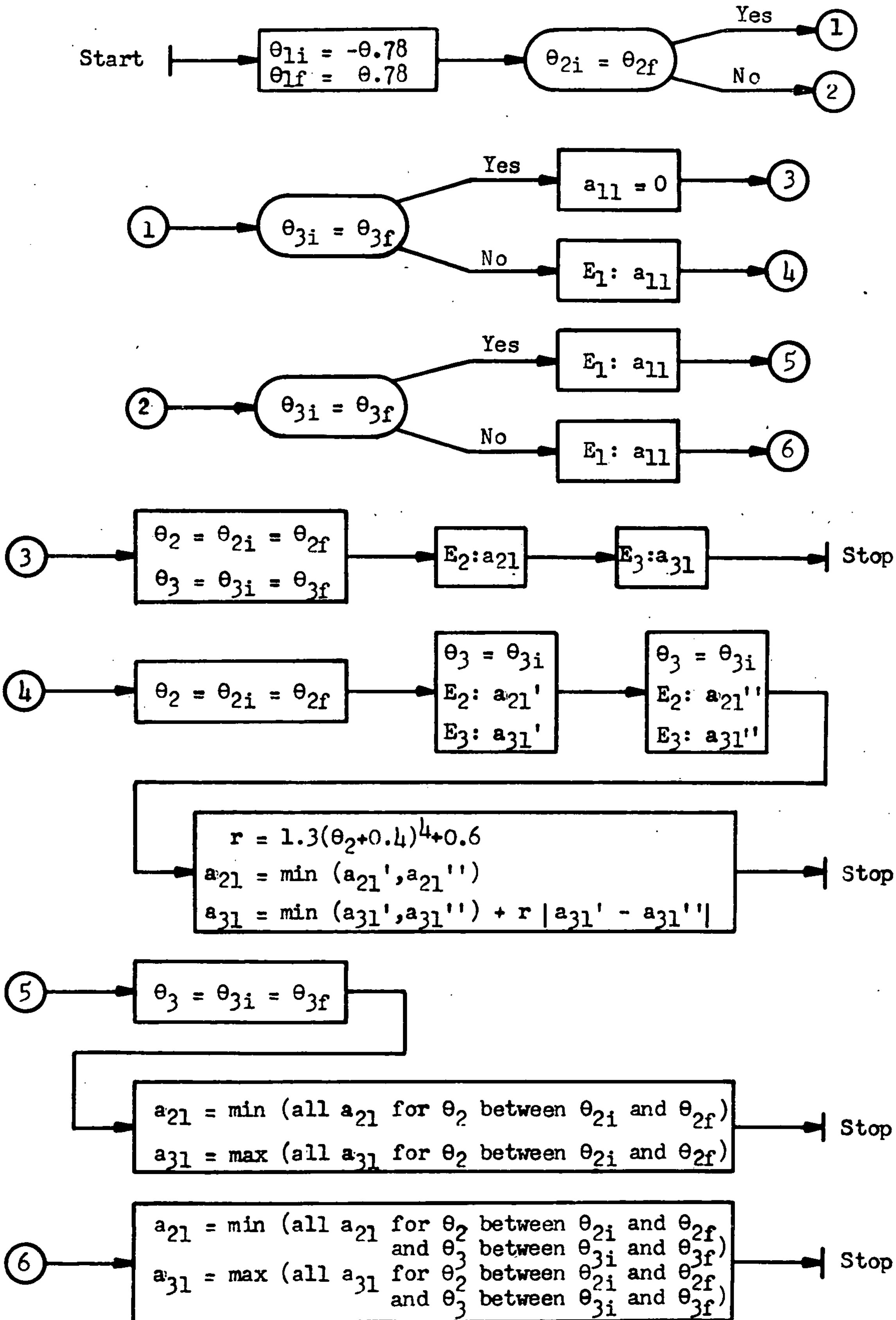
The value of a_{11} was chosen from Eq. (5.12).

For the categories described in Sections 5.2, 5.3 and 5.4 the values of the cost function $J(\underline{a})$ for all a's equal to zero are on the average 70% bigger than the optimal values. For the suboptimal values of the a's the difference between the values of $J(\underline{a})$ and the optimal values was on the average 10%. This justifies the use of suboptimal values for the parameters $a_{k\ell}$ very well.

5.5 Flow Diagram for Determining the Suboptimal Values of the Coefficients in the Series Expansion

The results of the previous sections can be summarized in a flow diagrm as presented in Fig. 5.2. The flow diagram is for the cases that $\theta_{1i} = -0.78$, $\theta_{1f} = 0.78$, θ_{2i} and θ_{2f} between -1.57 and 0.78, and θ_{3i} and θ_{3f} between 2.36 and 0 . The flow diagram is easy to transform into a computer program.

The generalization as described in this chapter can also be done for each of the combinations I and II. The main problem will be finding a function which fits the data for the cases that $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$.


Start
$\theta_{1i} = -\theta.78$
$\theta_{1f} = \theta.78$
$\theta_{2i} = \theta_{2f}$
Yes
No
1
2
1
$\theta_{3i} = \theta_{3f}$
Yes
$a_{11} = 0$
3
No
E_1: a_{11}
4
2
$\theta_{3i} = \theta_{3f}$
Yes
E_1: a_{11}
5
No
E_1: a_{11}
6
3
$\theta_2 = \theta_{2i} = \theta_{2f}$
$\theta_3 = \theta_{3i} = \theta_{3f}$
E_2:a_{21}
E_3:a_{31}
Stop
4
$\theta_2 = \theta_{2i} = \theta_{2f}$
$\theta_3 = \theta_{3i}$
E_2: a_{21}'
E_3: a_{31}'
$\theta_3 = \theta_{3i}$
E_2: a_{21}''
E_3: a_{31}''
$r = 1.3(\theta_2+0.4)^4+0.6$
$a_{21} = \min(a_{21}', a_{21}'')$
$a_{31} = \min(a_{31}', a_{31}'') + r\,|a_{31}' - a_{31}''|$
Stop
5
$\theta_3 = \theta_{3i} = \theta_{3f}$
a_{21} = min (all a_{21} for θ_2 between θ_{2i} and θ_{2f})
a_{31} = max (all a_{31} for θ_2 between θ_{2i} and θ_{2f})
Stop
6
a_{21} = min (all a_{21} for θ_2 between θ_{2i} and θ_{2f} and θ_3 between θ_{3i} and θ_{3f})
a_{31} = max (all a_{31} for θ_2 between θ_{2i} and θ_{2f} and θ_3 between θ_{3i} and θ_{3f})
Stop

CONT'D

$$E_1: \; a_{11} = -0.20 \,\mathrm{sgn}\{\, 2\cos\theta_{2i} + \cos(\theta_{2i} + \theta_{3i}) - 2\cos\theta_{2f} - \cos(\theta_{2f} + \theta_{3f})\}$$

$$\begin{aligned} E_2: \quad p &= 0.4936 - 0.3079\,\theta_3 + 0.0504\,\theta_3^2 \\ a &= -0.3929 - 0.2002\,\theta_3 \\ b &= -0.6628 + 0.2619\,\theta_3 \\ a_{21} &= p\,(\theta_2 - a)^2 + b \end{aligned}$$

$$\begin{aligned} E_3: \quad q &= -0.9985 + 0.3903\,\theta_3 \\ c &= -0.3370 - 0.1106\,\theta_3 \\ d &= 1.5442 - 0.5772\,\theta_3 \\ a_{31} &= q\,(\theta_2 - c)^2 + d \end{aligned}$$

Figure 5.2. Flow diagram for determining the sub-optimal values of the parameters a_{11}, a_{21} and a_{31} for combination III with $\theta_{1i} = -0.78$ and $\theta_{1f} = 0.78$.

CHAPTER 6

CONCLUSIONS

The proposed method of generating optimal trajectories worked successfully for a three degree of freedom mechanical arm. However, the method is general and it can be expected that it is also applicable to more complicated arm models.

The consistency of the results with the physical understanding of the problem indicates that the pattern search routine was suitable for this problem.

The results of the searches indicate that only the first two terms of the series expansion are important, except for joint angle θ_1 in combination II where the third term has a significant influence on the shape of the function for θ_1.

The optimal value of the cost function $J = \int_o^T KE\ dt$ for combination I is lower than for combination II. From this it can be concluded that in order to minimize the integral of the kinetic energy a series expansion of polynomials is more suitable than a series expansion of periodic functions. To minimize the integral of the torque magnitude only the series expansion of periodic functions is applicable.

In the special cases that $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the optimal values of the parameters a_{21} and a_{31} showed a certain pattern, especially for combination III. For this combination the values of

a_{21} and a_{31} could be summarized by paraboloid functions with $\theta_{2i} = \theta_{2f} = \theta_2$ and $\theta_{3i} = \theta_{3f} = \theta_3$ as independent variables (see Fig. 5.1). The values of a_{21} and a_{31} obtained in this way are suboptimal. The difference between the suboptimal and the optimal value of the cost function is on the average 7.4%. From the suboptimal values of a_{21} and a_{31} in the case that $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the suboptimal values of a_{21} and a_{31} in the other cases of combination III could be derived quite easily. The suboptimal value of the cost function in these cases is on the average 10% bigger than the optimal value, which is satisfactory.

Using the algorithm for the suboptimal values the coefficients in the series expansion can be obtained on-line without search. This saves a significant amount of real time computation.

APPENDIX I

A. Derivation of the expression for the kinetic energy of the arm.

For a system of two point masses, what the arm essentially is, the kinetic energy is:

$$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{I.1}$$

where v_1 and v_2 are the velocities of the masses m_1 and m_2. Furthermore (see figure I.1):

$$v_1^2 = v_{1\theta_1}^2 + v_{1\theta_2}^2 \tag{I.2}$$

where $v_{1\theta_1}$ = component of v_1 due to rotation about axis 1,

$v_{1\theta_1}$ = component of v_1 due to rotation about axis 2.

$$v_2^2 = v_{2\theta_1}^2 + v_{2\theta_2}^2 + v_{2\theta_3}^2 + 2v_{2\theta_2}v_{2\theta_3}\cos\alpha \tag{I.3}$$

where $v_{2\theta_1}$ = component of v_2 due to rotation about axis 1,

$v_{2\theta_2}$ = component of v_2 due to rotation about axis 2,

$v_{2\theta_3}$ = component of v_2 due to rotation about axis 3,

α = angle between $v_{2\theta_2}$ and $v_{2\theta_3}$.

The expressions for the components of v_1 and v_2 are:

$$v_{1\theta_1} = \ell_1\dot{\theta}_1\cos\theta_2 \tag{I.4}$$

$$v_{1\theta_2} = \ell_1\dot{\theta}_2 \tag{I.5}$$

$$v_{2\theta_1} = \{\ell_1\cos\theta_2 + \ell_2\cos(\theta_2+\theta_3)\}\dot{\theta}_1 \tag{I.6}$$

$$v_{2\theta_2} = \sqrt{\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\theta_3}\,\dot{\theta}_2 \tag{I.7}$$

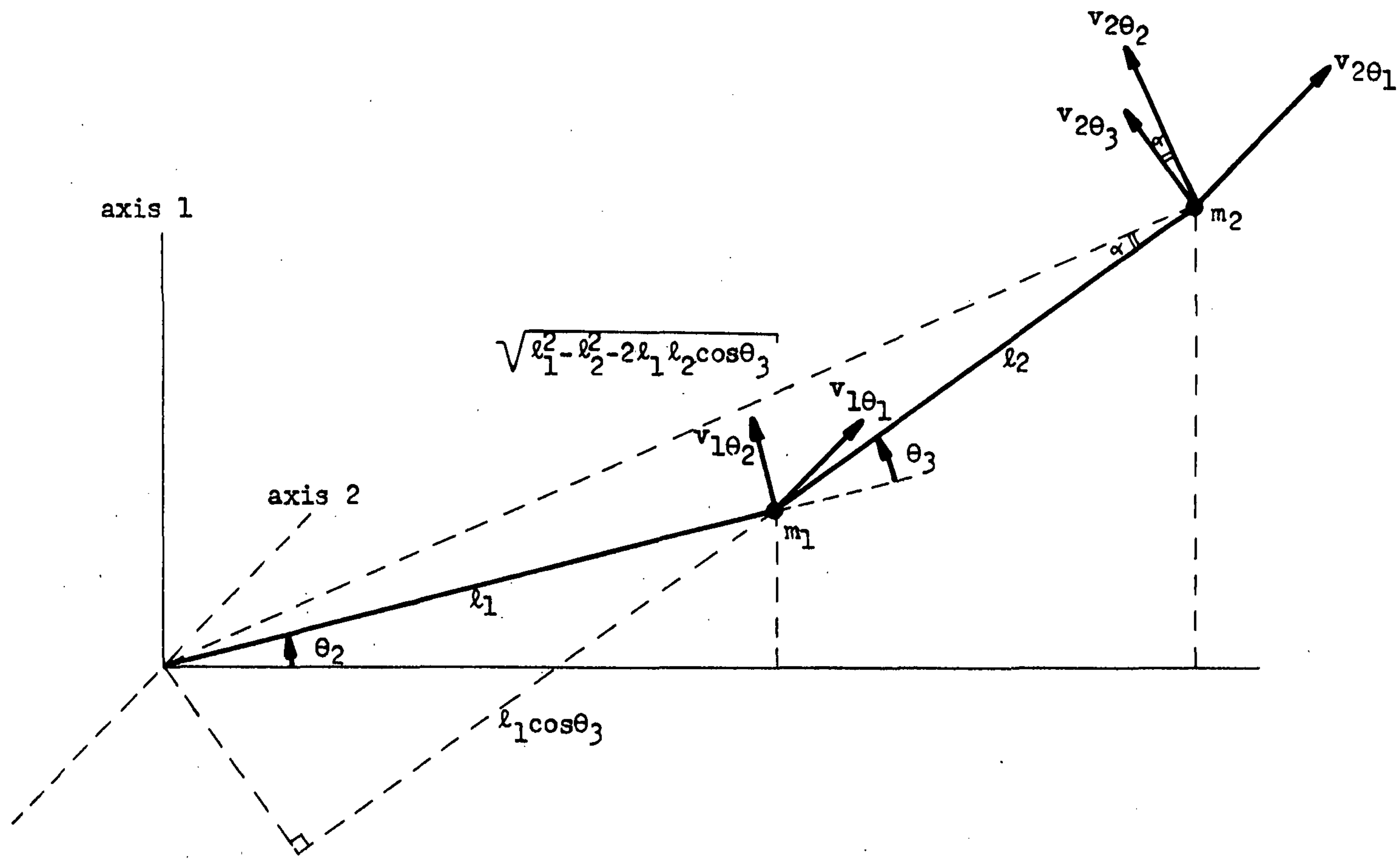


Figure I.1. Model of the arm with velocities and distances. (Figure is drawn for θ_1=0.0)

$$v_{2\theta_3} = \ell_2\dot{\theta}_3 \tag{I.8}$$

Substituting (I.2) through (I.8) in (I.1) gives the following expression for the kinetic energy:

$$\begin{aligned} KE = {} & \tfrac{1}{2}m_1[\ell_1^2\dot{\theta}_1^2\cos^2\theta_2 + \ell_1^2\dot{\theta}_2^2] \\ & + \tfrac{1}{2}m_2[\{\ell_1\cos\theta_2 + \ell_2\cos(\theta_2+\theta_3)\}^2\dot{\theta}_1^2 \\ & \quad + \sqrt{\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\theta_3}^{\,2}\dot{\theta}_2^2 \\ & \quad + \ell_2^2\dot{\theta}_3^2 \\ & \quad + 2\sqrt{\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\theta_3}\,\dot{\theta}_2\ell_2\dot{\theta}_3\cos\alpha] \end{aligned} \tag{I.9}$$

Substituting:

$$\cos\alpha = \frac{\ell_2 + \ell_1\cos\theta_3}{\sqrt{\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\theta_3}} \tag{I.10}$$

in Eqn. (I.9) and changing the order of the terms results in:

$$\begin{aligned} KE = {} & \tfrac{1}{2}[\dot{\theta}_1^2\{(m_1 + m_2)\ell_1^2\cos^2\theta_2 + m_2\ell_2^2\cos^2(\theta_2+\theta_3) \\ & \quad + 2m_2\ell_1\ell_2\cos\theta_2\cos(\theta_2+\theta_3)\} \\ & + \dot{\theta}_2^2\{(m_1 + m_2)\ell_1^2 + m_2\ell_2^2 + 2m_2\ell_1\ell_2\cos\theta_3\} \\ & + \dot{\theta}_3^2\{m_2\ell_2^2\} \\ & + \dot{\theta}_2\dot{\theta}_3\{2m_2\ell_2^2 + 2m_2\ell_1\ell_2\cos\theta_3\}] \end{aligned} \tag{3.9}$$

B. Expressions for the elements of $\underline{T}$ and $\underline{c}$ in Eqn. (3.13).

$$T_{11} = \tfrac{1}{2}(m_1 + m_2)\ell_1^2 + \tfrac{1}{2}m_2\ell_2^2 + \tfrac{1}{2}(m_1 + m_2)\ell_1^2\cos 2\theta_2$$

$$+ \tfrac{1}{2}m_2\ell_2^2\cos 2(\theta_2+\theta_3) + 2m_2\ell_1\ell_2\cos\theta_2\cos(\theta_2+\theta_3) \qquad \text{(I.11)}$$

$$T_{22} = m_1\ell_1^2 + m_2\ell_1^2 + m_2\ell_2^2 + 2m_2\ell_1\ell_2\cos\theta_3 \qquad \text{(I.12)}$$

$$T_{23} = m_2\ell_2^2 + m_2\ell_1\ell_2\cos\theta_3 \qquad \text{(I.13)}$$

$$T_{32} = T_{23} \qquad \text{(I.14)}$$

$$T_{33} = m_2\ell_2^2 \qquad \text{(I.15)}$$

$$c_1 = \dot\theta_1\dot\theta_2\{(m_1 + m_2)\ell_1^2\sin 2\theta_2 + m_2\ell_2^2\sin 2(\theta_2+\theta_3)$$

$$+ 2m_2\ell_1\ell_2\sin(\theta_3+2\theta_2)\}$$

$$+ \dot\theta_1\dot\theta_3\{2m_2\ell_1\ell_2\cos\theta_2\sin(\theta_2+\theta_3) + m_2\ell_2^2\sin 2(\theta_2+\theta_3)\} \qquad \text{(I.16)}$$

$$c_2 = -\dot\theta_1^2\{\tfrac{1}{2}(m_1 + m_2)\ell_1^2\sin 2\theta_2 + \tfrac{1}{2}m_2\ell_2^2\sin 2(\theta_2+\theta_3)$$

$$+ m_2\ell_1\ell_2\sin(\theta_3+2\theta_2)\}$$

$$+ \dot\theta_3^2\{m_2\ell_1\ell_2\sin\theta_3\}$$

$$+ \dot\theta_2\dot\theta_3\{2m_2\ell_1\ell_2\sin\theta_3\} \qquad \text{(I.17)}$$

$$c_3 = -\dot\theta_1^2\{\tfrac{1}{2}m_2\ell_2^2\sin 2(\theta_2+\theta_3) + m_2\ell_1\ell_2\cos\theta_2\sin(\theta_2+\theta_3)\}$$

$$- \dot\theta_2^2\{m_2\ell_1\ell_2\sin\theta_3\} \qquad \text{(I.18)}$$

APPENDIX II

FORTRAN CODED COMPUTER PROGRAMS

```
C-----MAIN PROGRAM
C-----FOR COMPUTING THE OPTIMAL VALUES OF THE COEFFICIENTS
C-----IN THE SERIES EXPANSION
C-----TORQ IS THE SURROUTINE FOR COMPUTING THE COST FUNCTION
C-----TOINT IS THE VALUE OF THE COST FUNCTION
      REAL L1,L2,M1,M2
      EXTERNAL TORQ
      DIMENSION BA(9)
      COMMON TFIN,F4,F5,F6,T9,ANG1I,ANG1F,ANG2I,ANG2F,
     1ANG3I,ANG3F
C-----READ INPUT DATA
      READ(5,20)TFIN,L1,L2,M1,M2
 20   FORMAT(E15.6/4E15.6)
      READ(5,37)N
 37   FORMAT(I1)
      READ(5,10)DELIN,DLMIN
      READ (5,5) M
  5   FORMAT (I2)
      WRITE(6,4)
  4   FORMAT('1OPTIMAL PARAMETERS FOR MIN INT TORQUE')
      WRITE(6,25)
 25   FORMAT(/,' TFIN,L1,L2,M1,M2',/)
      WRITE(6,26)TFIN,L1,L2,M1,M2
 26   FORMAT(5E15.6)
      WRITE(6,38) DELIN,DLMIN
 38   FORMAT(/,' DELIN= ',E15.6,'  DLMIN=',E15.6,/)
      F1=L1**2
      F2=L2**2
      F3=L1*L2
      F4=M1*F1
      F5=M2*F1
      F6=M2*F3
      T9=M2*F2
      NCSET=1
C-----READ INITIAL AND FINAL POSTION OF THE ARM
  6   READ(5,10)ANG1I,ANG2I,ANG3I,ANG1F,ANG2F,ANG3F
 10   FORMAT(E15.6)
C-----READ INITIAL GUESS FOR THE PARAMETERS
      READ(5,10)(BA(I),I=1,6)
      WRITE(6,35)
 35   FORMAT(/,' *** INITIAL AND FINAL ANGLES',/)
      WRITE(6,36)ANG1I,ANG2I,ANG3I,ANG1F,ANG2F,ANG3F
 36   FORMAT(3E15.6/3E15.6,/)
      DEL=DELIN
      CALL PATSH (BA,TOINT,N,DEL,DLMIN,TORQ)
      WRITE(6,40)
 40   FORMAT(/,' FINAL RESULTS',/)
      WRITE(6,50)(BA(I),I=1,6)
 50   FORMAT(6E15.6)
      WRITE(6,60) TOINT
 60   FORMAT(E15.6)
```

```
      IF (NOSET.EQ.M) GO TO 70
      NOSET=NOSET+1
      GO TO 6
   70 CONTINUE
      END
```

```
      SUBRCUTINE PATSH(PSI,SSI,N,DEL,DLMIN,MRIT4)
C PSI IS THE CURRENT BASEPT
C THT IS THE PREVIOUS BASEPT
C PHI IS THE TRIAL PT
C S IS THE CBJECTIVE FCT
C
      DIMENSION PSI(9), PHI(9), THT(9), EPS(9)
      WRITE(6,603)
 6C3  FCRMAT(' CURRENT POINT, OBJ FCT AND STEPSIZE')
      ALFA=1.02
C EVALUATE AT INIT BASEPT
 1C   CALL MRIT4(PSI,SSI)
C START AT BASEPT
 1C0  S=SSI
      DC 1C1 I=1,N
 1C1  PHI(I)=PSI(I)
      ICALL=1
      WRITE(6,599)
 599  FCRMAT( ' ***')
      WRITE(6,600) (PSI(J),J=1,N)
      WRITE(6,601) S, DEL
C MAKE EXPLCRATCRY MCVES
      GC TC 150
C IS PRESENT VALUE ) BASEPT VALUE
 160  IF(S.LT.SSI) GC TC 20C
      GC TC 30C
C SET NEW BASEPT
 2C0  SSI=S
      DC 2C1 I=1,N
      THT(I)=PSI(I)
      PSI(I)=PHI(I)
C MAKE PATTERN MCVE
 2C1  PHI(I)=PHI(I)+ALFA*(PHI(I)-THT(I))
      CALL MRIT4(PHI,SPI )
      S=SPI
      WRITE(6,599)
      WRITE(6,599)
      WRITE(6,600) ( PHI(I), I=1,N)
 6C0  FCRMAT(8E15.6)
      WRITE(6,601) SPI, DEL
 6C1  FCRMAT(2E15.6)
      ICALL=2
C MAKE EXPL MCVES
      GC TC 150
C IS PRESENT VALUE ) BASEPT VALUE
 260  IF(S.LT.SSI) GO TC 20C
      GC TC 10C
 3C0  IF(DEL.LT. DLMIN) RETURN
      DEL=CEL/2.
      GC TC 10C
```

```
C MAKE EXPL MOVES
 150  DO 180 K=1,N
      EPS(K)=.05*PHI(K)
      IF(EPS(K) .EQ. 0.) EPS(K)=.05
      PHI(K)=PHI(K)+EPS(K)*DEL
      CALL MRIT4(PHI,SPI)
 155  IF(SPI.LT.S) GO TO 179
      PHI(K)=PHI(K)-2.*EPS(K)*DEL
      CALL MRIT4(PHI,SPI)
 165  IF(SPI.LT.S) GO TO 179
      PHI(K)=PHI(K)+EPS(K)*DEL
      GO TO 180
 179  S=SPI
 180  CONTINUE
      GO TO (160,260),ICALL
      END
```

```
      SUBROUTINE KNET(BA,KEINT)
C-----FOR COMPUTING THE INTEGRAL OF THE KINETIC ENERGY
C-----WITH TRAJECTORIES OF TYPE 1
C-----(SERIES EXPANSION OF POLYNOMIALS)
      REAL MAL1,MAL2,MAL4,KE(21),KEINT
      DIMENSION BA(9)
      COMMON TFIN,F4,F5,F6,T9,ANG1I,ANG1F,ANG2I,ANG2F,
     1ANG3I,ANG3F
      B1=BA(1)
      B2=BA(2)
      B3=BA(3)
      B4=BA(4)
      B5=BA(5)
      B6=BA(6)
      TF2=TFIN**2
C-----COMPUTE KINETIC ENERGY AT EACH INTERVAL POINT
      DO 2000 J=1,21
      T=TFIN*(FLOAT(J)-1.)/20.
      ANG2=ANG2I+T*(ANG2F-ANG2I)/TFIN+4.*B2*T*(TFIN-T)/TF2
      ANG2=ANG2+64.*B5*T*(TFIN/2.-T)*(TFIN-T)/(3.*TF2*TFIN)
      ANG3=ANG3I+T*(ANG3F-ANG3I)/TFIN+4.*B3*T*(TFIN-T)/TF2
      ANG3=ANG3+64.*B6*T*(TFIN/2.-T)*(TFIN-T)/(3.*TF2*TFIN)
      VEL1=(ANG1F-ANG1I)/TFIN+4.*B1*(TFIN-2.*T)/TF2
      VEL1=VEL1+64.*B4*(0.5*TF2-3.*TFIN*T+3.*T**2)/
     1(3.*TF2*TFIN)
      VEL2=(ANG2F-ANG2I)/TFIN+4.*B2*(TFIN-2.*T)/TF2
      VEL2=VEL2+64.*B5*(0.5*TF2-3.*TFIN*T+3.*T**2)/
     1(3.*TF2*TFIN)
      VEL3=(ANG3F-ANG3I)/TFIN+4.*B3*(TFIN-2.*T)/TF2
      VEL3=VEL3+64.*B6*(0.5*TF2-3.*TFIN*T+3.*T**2)/
     1(3.*TF2*TFIN)
C-----SUM OF ANGLES
      FAC1=ANG2+ANG3
C-----SINES AND COSINES
      COS1=COS(ANG2)
      COS2=COS(ANG3)
      COS4=COS(FAC1)
C-----COMBINE TERMS
      G1=F4+F5
      G6=F6*COS2
C-----COMPUTE 2*KINETIC ENERGY
      MAL1=G1*COS1**2+T9*COS4**2+2.*F6*COS1*COS4
      MAL2=G1+T9+2.*G6
      MAL4=2.*(T9+G6)
      KE(J)=VEL1**2*MAL1+VEL2**2*MAL2+VEL3**2*T9+
     1VEL2*VEL3*MAL4
 2000 CONTINUE
```

```
C-----COMPUTE INTEGRAL OF KINETIC ENERGY
      KEINT=KE(1)+4.*KE(2)+KE(21)
      DO 2050 L=2,10
 2050 KEINT=KEINT+2.*KE(2*L-1)+4.*KE(2*L)
      KEINT=(TFIN/60.)*KEINT/2.
      RETURN
      END
```

```
      SUBROUTINE KINET (BA,KEINT)
C-----FOR COMPUTING THE INTEGRAL OF THE KINETIC ENERGY
C-----WITH TRAJECTORIES OF TYPE 2
C-----(SERIES EXPANSION OF PERIODIC FUNCTIONS)
      REAL MAL1,MAL2,MAL4,KE(21),KEINT
      DIMENSION BA(9)
      COMMON TFIN,F4,F5,F6,T9,ANG1I,ANG1F,ANG2I,ANG2F,
     1ANG3I,ANG3F
      TC1=BA(1)*2.
      TC2=BA(2)*2.
      TC3=BA(3)*2.
      TC5=BA(5)*2.
      TC6=BA(6)*2.
      FC4=BA(4)*4.
      FC5=BA(5)*4.
      FC6=BA(6)*4.
      OM=6.28319/TFIN
      TOM=2.*OM
      FOM=4.*OM
C-----COMPUTE KINETIC ENERGY AT EACH INTERVAL POINT
      DO 2000 J=1,21
      T=TFIN*(FLOAT(J)-1.)/20.
      OMT=OM*T
      TOMT=2.*OMT
      FOMT=4.*OMT
      SOMT=SIN(OMT)
      COMT=COS(OMT)
      STOMT=2.*SOMT*COMT
      CTOMT=1.-2.*SOMT**2
      SFOMT=2.*STOMT*CTOMT
      CFOMT=1.-2.*STOMT**2
      PONE=(T-SOMT/OM)/TFIN
      QONE=(1.-COMT)/TFIN
      PTWO=(T-STOMT/TOM)/TFIN
      QTWO=(1.-CTOMT)/TFIN
      PFOU=(T-SFOMT/FOM)/TFIN
      QFOU=(1.-CFOMT)/TFIN
C-----FIRST APPROXIMATION  T BETWEEN 0 AND TFIN
      ANG2=ANG2I+(ANG2F-ANG2I)*PONE
      ANG3=ANG3I+(ANG3F-ANG3I)*PONE
      VEL1=(ANG1F-ANG1I)*QONE
      VEL2=(ANG2F-ANG2I)*QONE
      VEL3=(ANG3F-ANG3I)*QONE
      IF(T-TFIN/2.) 200,200,300
C-----SECOND APPROXIMATION  T BETWEEN 0 AND TFIN/2
  200 ANG2=ANG2+TC2*PTWO
      ANG3=ANG3+TC3*PTWO
      VEL1=VEL1+TC1*QTWO
      VEL2=VEL2+TC2*QTWO
      VEL3=VEL3+TC3*QTWO
      IF(T-TFIN/4.) 600,600,700
```

```
C-----SECOND APPROXIMATION  T BETWEEN TFIN/2 AND TFIN
  300 ANG2=ANG2+TC2-TC2*PTWO
      ANG3=ANG3+TC3-TC3*PTWO
      VEL1=VEL1-TC1*QTWO
      VEL2=VEL2-TC2*QTWO
      VEL3=VEL3-TC3*QTWO
      IF(T-.75*TFIN) 700,700,800
C-----THIRD APPROXIMATION  T BETWEEN 0 AND TFIN/4
  600 ANG2=ANG2+FC5*PFOU
      ANG3=ANG3+FC6*PFOU
      VEL1=VEL1+FC4*QFOU
      VEL2=VEL2+FC5*QFOU
      VEL3=VEL3+FC6*QFOU
      GO TO 1000
C-----THIRD APPROXIMATION  T BETWEEN TFIN/4 AND 3*TFIN/4
  700 ANG2=ANG2+TC5-FC5*PFOU
      ANG3=ANG3+TC6-FC6*PFOU
      VEL1=VEL1-FC4*QFOU
      VEL2=VEL2-FC5*QFOU
      VEL3=VEL3-FC6*QFOU
      GO TO 1000
C-----THIRD APPROXIMATION  T BETWEEN 3*TFIN/4 AND TFIN
  800 ANG2=ANG2-FC5+FC5*PFOU
      ANG3=ANG3-FC6+FC6*PFOU
      VEL1=VEL1+FC4*QFOU
      VEL2=VEL2+FC5*QFOU
      VEL3=VEL3+FC6*QFOU
C-----SUM OF ANGLES
 1000 FAC1=ANG2+ANG3
C-----SINES AND COSINES
      COS1=COS(ANG2)
      COS2=COS(ANG3)
      COS4=COS(FAC1)
C-----COMBINE TERMS
      G1=F4+F5
      G6=F6*COS2
C-----COMPUTE 2*KINETIC ENERGY
      MAL1=G1*COS1**2+T9*COS4**2+2.*F6*COS1*COS4
      MAL2=G1+T9+2.*G6
      MAL4=2.*(T9+G6)
      KE(J)=VEL1**2*MAL1+VEL2**2*MAL2+VEL3**2*T9+
     1VEL2*VEL3*MAL4
 2000 CONTINUE
C-----COMPUTE INTEGRAL OF KINETIC ENERGY
      KEINT=KE(1)+4.*KE(2)+KE(21)
      DO 2050 L=2,10
 2050 KEINT=KEINT+2.*KE(2*L-1)+4.*KE(2*L)
      KEINT=(TFIN/60.)*KEINT/2.
      RETURN
      END
```

```
      SUBROUTINE TORQ (BA,TCINT)
C-----FOR COMPUTING THE INTEGRAL OF THE SUM OF THE
C-----ABS. VALUES OF THE JOINT TORQUES
C-----WITH TRAJECTORIES OF TYPE 2
C-----(SERIES EXPANSION OF PERIODIC FUNCTIONS)
      DIMENSION BA(9),AU(21)
      COMMON TFIN,F4,F5,F6,T9,ANG1I,ANG1F,ANG2I,ANG2F,
     1ANG3I,ANG3F
      TC1=BA(1)*2.
      TC2=BA(2)*2.
      TC3=BA(3)*2.
      TC5=BA(5)*2.
      TC6=BA(6)*2.
      FC4=BA(4)*4.
      FC5=BA(5)*4.
      FC6=BA(6)*4.
      OM=6.28319/TFIN
      TOM=2.*OM
      FOM=4.*OM
C-----COMPUTE JOINT TORQUES AT EACH INTERVAL POINT
      DO 2000 J=1,21
      T=TFIN*(FLOAT(J)-1.)/20.
      OMT=OM*T
      TOMT=2.*OMT
      FOMT=4.*OMT
      SOMT=SIN(OMT)
      COMT=COS(OMT)
      STOMT=2.*SOMT*COMT
      CTOMT=1.-2.*SOMT**2
      SFOMT=2.*STOMT*CTOMT
      CFOMT=1.-2.*STOMT**2
      PONE=(T-SOMT/OM)/TFIN
      QONE=(1.-COMT)/TFIN
      RONE=OM*SOMT/TFIN
      PTWO=(T-STOMT/TOM)/TFIN
      QTWO=(1.-CTOMT)/TFIN
      RTWO=TOM*STOMT/TFIN
      PFOU=(T-SFOMT/FOM)/TFIN
      QFOU=(1.-CFOMT)/TFIN
      RFOU=FOM*SFOMT/TFIN
C-----FIRST APPROXIMATION  T BETWEEN 0 AND TFIN
      ANG2=ANG2I+(ANG2F-ANG2I)*PONE
      ANG3=ANG3I+(ANG3F-ANG3I)*PONE
      VEL1=(ANG1F-ANG1I)*QONE
      VEL2=(ANG2F-ANG2I)*QONE
      VEL3=(ANG3F-ANG3I)*QONE
      ACC1=(ANG1F-ANG1I)*RONE
      ACC2=(ANG2F-ANG2I)*RONE
      ACC3=(ANG3F-ANG3I)*RONE
      IF(T-TFIN/2.) 200,200,300
```

```
C-----SECOND APPROXIMATION  T BETWEEN 0 AND TFIN/2
  200 ANG2=ANG2+TC2*PTWO
      ANG3=ANG3+TC3*PTWO
      VEL1=VEL1+TC1*QTWO
      VEL2=VEL2+TC2*QTWO
      VEL3=VEL3+TC3*QTWO
      ACC1=ACC1+TC1*RTWO
      ACC2=ACC2+TC2*RTWO
      ACC3=ACC3+TC3*RTWO
      IF(T-TFIN/4.) 600,600,700
C-----SECOND APPROXIMATION  T BETWEEN TFIN/2 AND TFIN
  300 ANG2=ANG2+TC2-TC2*PTWO
      ANG3=ANG3+TC3-TC3*PTWO
      VEL1=VEL1-TC1*QTWO
      VEL2=VEL2-TC2*QTWO
      VEL3=VEL3-TC3*QTWO
      ACC1=ACC1-TC1*RTWO
      ACC2=ACC2-TC2*RTWO
      ACC3=ACC3-TC3*RTWO
      IF(T-.75*TFIN) 700,700,800
C-----THIRD APPROXIMATION  T BETWEEN 0 AND TFIN/4
  600 ANG2=ANG2+FC5*PFOU
      ANG3=ANG3+FC6*PFOU
      VEL1=VEL1+FC4*QFOU
      VEL2=VEL2+FC5*QFOU
      VEL3=VEL3+FC6*QFOU
      ACC1=ACC1+FC4*RFOU
      ACC2=ACC2+FC5*RFOU
      ACC3=ACC3+FC6*RFOU
      GO TO 1000
C-----THIRD APPROXIMATION  T BETWEEN TFIN/4 AND 3*TFIN/4
  700 ANG2=ANG2+TC5-FC5*PFOU
      ANG3=ANG3+TC6-FC6*PFOU
      VEL1=VEL1-FC4*QFOU
      VEL2=VEL2-FC5*QFOU
      VEL3=VEL3-FC6*QFOU
      ACC1=ACC1-FC4*RFOU
      ACC2=ACC2-FC5*RFOU
      ACC3=ACC3-FC6*RFOU
      GO TO 1000
C-----THIRD APPROXIMATION  T BETWEEN 3*TFIN/4 AND TFIN
  800 ANG2=ANG2-FC5+FC5*PFOU
      ANG3=ANG3-FC6+FC6*PFOU
      VEL1=VEL1+FC4*QFOU
      VEL2=VEL2+FC5*QFOU
      VEL3=VEL3+FC6*QFOU
      ACC1=ACC1+FC4*RFOU
      ACC2=ACC2+FC5*RFOU
      ACC3=ACC3+FC6*RFOU
```

```
C-----SUM OF ANGLES
 1000 FAC1=ANG2+ANG3
      FAC2=2.*FAC1
      FAC3=2.*ANG2
      FAC4=ANG3+FAC3
C-----SINES AND COSINES
      SIN2=SIN(ANG3)
      COS1=COS(ANG2)
      COS2=COS(ANG3)
      SIN3=SIN(FAC3)
      SIN4=SIN(FAC1)
      SIN5=SIN(FAC2)
      SIN6=SIN(FAC4)
      COS3=COS(FAC3)
      COS4=COS(FAC1)
      COS5=COS(FAC2)
C-----COMBINE TERMS
      G1=F4+F5
      G2=T9*SIN5
      G3=F6*SIN6
      G4=F6*COS1
      G5=F6*SIN2
      G6=F6*COS2
      G7=2.*G4*SIN4+G2
      G8=G1*SIN3+G2+2.*G3
C-----COMPUTE PRODUCTS OF VELOCITIES
      V1=.5*VEL1**2
      V2=VEL2**2
      V3=VEL3**2
      V4=VEL1*VEL2
      V5=VEL1*VEL3
      V6=VEL2*VEL3
C-----COMPUTE C-TERMS
      C1=V4*G8+V5*G7
      C2=-V1*G8+(V3+2.*V6)*G5
      C3=-V1*G7-V2*G5
C-----COMPUTE T-TERMS
      T6=T9+G6
      T5=T9+2.*G6+G1
      T8=T6
      T1=.5*(G1*(1+COS3)+T9*(1+COS5))+2.*G4*COS4
C-----COMPUTE JOINT TORQUES
      U1=T1*ACC1-C1
      U2=T5*ACC2+T6*ACC3-C2
      U3=T8*ACC2+T9*ACC3-C3
C-----COMPUTE SUM OF ABS. VALUES OF JOINT TORQUES
      AU(J)=ABS(U1)+ABS(U2)+ABS(U3)
 2000 CONTINUE
```

```
C-----COMPUTE INTEGRAL OF ABS. JOINT TORQUES
      TOINT=AU(1)+4.*AU(2)+AU(21)
      DO 2050 L=2,10
 2050 TOINT=TOINT+2.*AU(2*L-1)+4.*AU(2*L)
      TOINT=(TFIN/60.)*TOINT
      RETURN
      END
```

APPENDIX III

RESULTS OF THE SEARCHES

Optimal values of the coefficients in the series expansion and the cost function J(a)

III A.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	-1.57	-1.57	2.36	2.36	0	0	0.147	0	0	0	0.024
-0.78	0.78	-0.78	-0.78	2.36	2.36	0	0	0.170	0	0	0	0.050
-0.78	0.78	0	0	2.36	2.36	0	0.031	0.043	0	0	0	0.059
-0.78	0.78	0.78	0.78	2.36	2.36	0	0.055	-0.013	0	0	0	0.032
-0.78	0.78	-1.57	-1.57	1.57	1.57	0	-0.122	0.115	0	0	0	0.050
-0.78	0.78	-0.78	-0.78	1.57	1.57	0	0	0.317	0	0	0	0.125
-0.78	0.78	0	0	1.57	1.57	0	0.014	0.229	0	0	0	0.101
-0.78	0.78	0.78	0.78	1.57	1.57	0	0.073	-0.082	0	0	0	0.026
-0.78	0.78	-1.57	-1.57	0.78	0.78	0	0	-0.147	0	0	0	0.024
-0.78	0.78	-0.78	-0.78	0.78	0.78	0	-0.309	0.532	0	0	0	0.166
-0.78	0.78	0	0	0.78	0.78	0	-0.145	0.585	0	0	0	0.193
-0.78	0.78	0.78	0.78	0.78	0.78	0	0.101	0.006	0	0	0	0.050
-0.78	0.78	-1.57	-1.57	0	0	0	0	0	0	0	0	0
-0.78	0.78	-0.78	-0.78	0	0	0	-0.150	0.005	0	0	0	0.125
-0.78	0.78	0	0	0	0	0	-0.322	0.809	0	0	0	0.270
-0.78	0.78	0.78	0.78	0	0	0	0.144	0.010	0	0	0	0.125

$J(\underline{a}) = {}_0\int^T KE\, dt$; trajectories of type 1 (series expansion of polynomials).

III A cont'd.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	0	0	0	1.57	-0.206	-0.270	0.703	0	0	0	0.218
-0.78	0.78	0	-0.78	0	1.57	-0.182	-0.311	0.768	0	0	0	0.203
-0.78	0.78	-0.78	0	0	1.57	0	-0.547	0.947	0	0	0	0.320
-0.78	0.78	-0.78	0	1.57	2.36	-0.150	-0.199	0.334	0	0	0	0.135

$J(\underline{a}) = \int_0^T KE\, dt$; trajectories of type 1 (series expansion of polynomials).

III B.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	-1.57	-1.57	2.36	2.36	0	0	0.168	0.137	0	0	0.029
-0.78	0.78	-0.78	-0.78	2.36	2.36	0	-0.124	0.215	0.135	0	0	0.057
-0.78	0.78	0	0	2.36	2.36	0	0.033	0.043	0.175	0	0	0.070
-0.78	0.78	0.78	0.78	2.36	2.36	0	0	0	0.178	0	0	0.038
-0.78	0.78	-1.57	-1.57	1.57	1.57	0	-0.131	0.122	0.152	0	0	0.060
-0.78	0.78	-0.78	-0.78	1.57	1.57	0	-0.207	0.476	0.119	0	0	0.136
-0.78	0.78	0	0	1.57	1.57	0	0.020	0.241	0.152	0	0	0.119
-0.78	0.78	0.78	0.78	1.57	1.57	0	0	0	0.118	0	0	0.033
-0.78	0.78	-1.57	-1.57	0.78	0.78	0	-0.052	-0.070	0.133	0	0	0.028
-0.78	0.78	-0.78	-0.78	0.78	0.78	0	-0.339	0.570	0.132	0	0	0.195
-0.78	0.78	0	0	0.78	0.78	0	-0.144	0.619	0.139	0	0	0.227
-0.78	0.78	0.78	0.78	0.78	0.78	0	0.157	0	0	0	0	0.065
-0.78	0.78	-1.57		0	0	0	0	0	0	0	0	0
-0.78	0.78	-0.78		0	0	0	-0.175	0.014	0.138	0	0	0.146
-0.78	0.78	0		0	0	0	-0.352	0.886	0.157	0	0	0.319
-0.78	0.78	0.78		0	0	0	0.169	0	0.138	0	0	0.146

$J(\underline{a}) = {}_0\int^T KE\,dt$; trajectories of type 2 (series expansion of periodic functions).

III B cont'd.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	0	0	0	1.57	-0.182	-0.253	0.706	0.152	0	-0.130	0.276
-0.78	0.78	0	-0.78	0	1.57	-0.156	0	0.371	0.174	0	0	0.277
-0.78	0.78	-0.78	0	0	1.57	0	-0.563	0.968	0.154	0.095	0	0.389
-0.78	0.78	-0.78	0	1.57	2.36	-0.109	-0.187	0.396	0.137	0	0.117	0.163

$J(\underline{a}) = {}_0\int^T KE\, dt$; trajectories of type 2 (series expansion of periodic functions).

III C.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	-1.57	-1.57	2.36	2.36	0	-0.041	0.144	0	0	0	0.194
-0.78	0.78	-0.78	-0.78	2.36	2.36	0	-0.094	0.194	0	0	0	0.354
-0.78	0.78	0	0	2.36	2.36	0	-0.010	0.057	-0.031	0	0	0.353
-0.78	0.78	0.78	0.78	2.36	2.36	0	0.055	-0.016	-0.031	0	0	0.203
-0.78	0.78	-1.57	-1.57	1.57	1.57	0	-0.107	0.106	0	0	0	0.366
-0.78	0.78	-0.78	-0.78	1.57	1.57	0	-0.219	0.624	0	0	0	0.630
-0.78	0.78	0	0	1.57	1.57	0	-0.150	0.468	0	0	0	0.674
-0.78	0.78	0.78	0.78	1.57	1.57	0	0.062	-0.069	0	0	0	0.204
-0.78	0.78	-1.57	-1.57	0.78	0.78	0	-0.157	0.266	0	0	0	0.334
-0.78	0.78	-0.78	-0.78	0.78	0.78	0	-0.432	1.023	0	0	0	0.865
-0.78	0.78	0	0	0.78	0.78	0	-0.366	0.985	0	0	0	0.968
-0.78	0.78	0.78	0.78	0.78	0.78	0	0.063	0.052	-0.010	0	0	0.556
-0.78	0.78	-1.57	-1.57	0	0	0	0	0	0	0	0	0
-0.78	0.78	-0.78	-0.78	0	0	0	-0.623	1.274	0	0	0	0.876
-0.78	0.78	0	0	0	0	0	-0.587	1.469	0	0	0	1.141
-0.78	0.78	0.78	0.78	0	0	0	0	0.309	0	0	0	1.106

$J(\underline{a}) = {}_0\int^T \sum_{k=1}^{3} |u_k|\, dt$; trajectories of type 2 (series expansion of periodic functions).

III C cont'd.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	-1.57	-1.57	0.78	2.36	0.377	-0.243	0.388	0	0	0	0.586
-0.78	0.78	-0.78	-0.78	0.78	2.36	-0.205	-0.368	0.718	0	0	0	0.881
-0.78	0.78	-0.78	-0.78	0	1.57	-0.028	-0.534	1.027	0	0	0	1.057
-0.78	0.78	0	0	0	2.36	-0.286	-0.595	1.181	0.018	-0.013	0.066	1.346
-0.78	0.78	0	0	0	1.57	-0.206	-0.588	1.264	0	0	0	1.252
-0.78	0.78	0	0	0.78	1.57	-0.204	-0.209	0.671	0	0	0	0.911
-0.78	0.78	0.78	0.78	0.78	1.57	-0.297	-0.013	0.158	0.014	0	0	0.483
-0.78	0.78	-0.78	-1.57	2.36	2.36	-0.154	-0.165	0.252	0	0	0	0.419
-0.78	0.78	0	-1.57	1.57	1.57	-0.205	-0.340	0.760	0	0	0	0.986
-0.78	0.78	0.78	-0.78	1.57	1.57	-0.205	-0.256	0.735	0	0	0	1.074
-0.78	0.78	0	-0.78	1.57	1.57	-0.064	-0.263	0.675	-0.022	0	0	0.830
-0.78	0.78	-0.78	-1.57	0.78	0.78	-0.240	-0.457	0.927	0	0	0	0.863
-0.78	0.78	0.78	-0.78	0.78	0.78	-0.026	-0.510	1.339	0	0	0	1.377
-0.78	0.78	0	-0.78	0.78	0.78	-0.063	-0.519	1.127	-0.015	0	0	1.081
-0.78	0.78	0	-0.78	0	0	-0.179	-0.754	1.650	0	0	0	1.214

$J(\underline{a}) = \int_0^T \sum_{k=1}^{3} |u_k| dt$; trajectories of type 2 (series expansion of periodic functions).

III C cont'd.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-0.78	0.78	-0.78	0	1.57	2.36	-0.172	-0.329	0.527	0	0	0	0.777
-0.78	0.78	-1.57	-0.78	0.78	2.36	0.205	-0.520	0.750	0	-0.009	0.049	0.914
-0.78	0.78	-0.78	-1.57	0.78	1.57	-0.242	-0.296	0.567	0	0	0	0.758
-0.78	0.78	-0.78	0.78	0.78	1.57	-0.117	-0.710	1.206	0	0	0	1.328
-0.78	0.78	0	-1.57	0.78	1.57	-0.205	0	0.679	0	0	0	1.542
-0.78	0.78	0.78	-0.78	0.78	1.57	0.205	-0.297	0.999	0	0	0	1.171
-0.78	0.78	-0.78	0	0	1.57	0.009	-0.754	1.312	0.007	0.007	0	1.370
-0.78	0.78	0	-0.78	0	1.57	-0.205	-0.380	1.000	0	0	0	1.101

$J(\underline{a}) = {}_0\int^T \sum_{k=1}^{3} |u_k| dt$; trajectories of type 2 (series expansion of periodic functions).

III D.

θ_{1i}	θ_{1f}	θ_{2i}	θ_{2f}	θ_{3i}	θ_{3f}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	$J(\underline{a})$
-1.57	1.57	-0.78	-0.78	0.78	0.78	0	-0.734	1.251	-0.064	0	0	1.437
-1.20	1.20	-0.78	-0.78	0.78	0.78	0	-0.557	1.148	0	0	0	1.139
-0.78	0.78	-0.78	-0.78	0.78	0.78	0	-0.432	1.023	0	0	0	0.865
-0.40	0.40	-0.78	-0.78	0.78	0.78	0	-0.091	0.165	0	0	0	0.529
0	0	-0.78	-0.78	0.78	0.78	0	0	0	0	0	0	0
-1.57	1.57	0	0	1.57	1.57	0	-0.290	0.904	-0.049	0	0	1.041
-1.20	1.20	0	0	1.57	1.57	0	-0.216	0.720	-0.040	0	0	0.902
-0.78	0.78	0	0	1.57	1.57	0	-0.150	0.468	0	0	0	0.674
-0.40	0.40	0	0	1.57	1.57	0	-0.037	0.112	-0.016	0	0	0.345
0	0	0	0	1.57	1.57	0	0	0	0	0	0	0
-1.57	1.57	-0.78	-1.57	0.78	0.78	-0.440	-0.604	1.300	0.004	0	0	1.051
-1.20	1.20	-0.78	-1.57	0.78	0.78	-0.377	-0.548	1.143	0.010	0	0	0.933
-0.78	0.78	-0.78	-1.57	0.78	0.78	-0.240	-0.457	0.927	0	0	0	0.863
-0.40	0.40	-0.78	-1.57	0.78	0.78	-0.105	-0.402	0.850	0	0	0	0.781
0	0	-0.78	-1.57	0.78	0.78	0	-0.351	0.775	0	0		0.647

$J(\underline{a}) = {}_0\int^T \sum_{k=1}^{3} |u_k| dt$; trajectories of type 2 (series expansion of periodic functions).

APPENDIX IV

DERIVATION OF THE LINEAR REGRESSION FORMULA

If the result of an experiment is a number of data points $(x_1, y_1), \ldots, (x_N, y_N)$ and the model for the experiment is assumed:

$$y = c_0 + c_1 x + c_2 x^2 + \ldots + c_K x^K \qquad \text{(IV.1)}$$

one can write the following N equations for the estimates of the y's:

$$\begin{aligned} \hat{y}_1 &= c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_K x_1^K \\ &\vdots \\ \hat{y}_N &= c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_K x_N^K \end{aligned} \qquad \text{(IV.2)}$$

or in matrix form:

$$\underline{\hat{y}} = \underline{X}\,\underline{c} \qquad \text{(IV.3)}$$

where $\underline{\hat{y}}^T = [\, y_1, y_2, \ldots, y_N \,]$

$$\underline{X} = \begin{bmatrix} 1 & x_1 & \ldots & x_1^K \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & x_N & \ldots & x_N^K \end{bmatrix}$$

$$\underline{c}^T = [\, c_1, c_2, \ldots, c_K \,]$$

The error between the actual $\underline{y}$ and the estimate of it $\hat{\underline{y}}$ is:

$$\underline{e} = \underline{y} - \underline{Xc} \tag{IV.4}$$

The squared error becomes:

$$\underline{e}^T\underline{e} = (\underline{y} - \underline{Xc})^T(\underline{y} - \underline{Xc}) \tag{IV.5}$$

The $\underline{c}$ that will minimize the squared error is $\hat{\underline{c}}$, the least square estimate of $\underline{c}$, and follows from the necessary condition for a minimum squared error:

$$\frac{\partial \underline{e}^T\underline{e}}{\partial \underline{c}} = 0 \tag{IV.6}$$

or:

$$- \underline{X}^T\underline{y} + \underline{X}^T\underline{Xc} = 0 \tag{IV.7}$$

So that:

$$\hat{\underline{c}} = [\underline{X}^T\underline{X}]^{-1}\underline{X}^T\underline{y} \tag{IV.8}$$

REFERENCES

[1] A. L. Townsend,

Linear Control Theory Applied to a Mechanical Manipulator,

Report T-558, Charles Stark Draper Laboratory,

Cambridge, Mass., January 1972.

[2] A. E. Bryson and Y. C. Ho,

Applied Optimal Control - Optimization, Estimation and Control,

Ginn and Company, Waltham, Mass., 1969.

[3] G. A. Bekey,

System Identification - an Introduction and a Survey,

Simulation, October 1970, pp. 151-166.

[4] A. P. Sage,

System Identification,

Academic Press, New York and London, 1971.

[5] R. Hooke and T. A. Jeeves,

"Direct Search" Solution of Numerical Statistical Problems,

Journal of Association for Computing Machinery,

Vol. 8, 2, April 1961, pp. 212-229.