NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1565

Large-Deformation Modal Coordinates for Nonrigid Vehicle Dynamics

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Preface

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

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Abstract

*This report documents the derivation of minimum-dimension sets of discrete-*coordinate and hybrid-coordinate equations of motion of a system consisting of an arbitrary number of hinge-connected rigid bodies assembled in tree topology. These equations are useful for the simulation of dynamical systems that can be idealized as tree-like arrangements of substructures, with each substructure consisting of either a rigid body or a collection of elastically interconnected rigid bodies restricted to small relative rotations at each connection. Thus, some of the substructures represent elastic bodies subjected to small strains or local deformations, but possibly large gross deformations; in the hybrid formulation, distributed coordinates, referred to herein as large-deformation modal coordinates, are used for the deformations of these substructures. The equations are in a form suitable for incorporation into one or more computer programs to be used as multipurpose tools in the simulation of spacecraft and other complex electromechanical systems.

Large-Deformation Modal Coordinates for Nonrigid Vehicle Dynamics

I. Introduction

A. Background and Motivation

Although the influence of vehicle nonrigidity on spacecraft attitude stability and control has been noted in singular cases since the very beginning of the space age (Refs. 1, 2), it is only in recent years that this influence has been widely recognized as a systems problem of paramount importance to the success of many missions considered crucial to the space program.

The determination of the influence of spacecraft nonrigidity on mission performance is not a single problem but a family of problems. It is therefore appropriate that the engineer have at his disposal an arsenal of analytical and computational weapons, and not just a single approach or a single computer program. This report focuses accordingly on the exposition of a particular method for simulating a certain class of spacecraft subject to large changes in attitude and configuration. The formulation provided here seems to be more comprehensive than any previously published for application to spacecraft amenable to mathematical modeling as a collection of point-connected rigid bodies assembled in a topological tree (with contiguous bodies sharing a common point and no closed rings of bodies); but not every spacecraft should be modeled in this way. In order to establish the position of this contribution in the spectrum of methods currently available for the simulation of nonrigid spacecraft, the literature in this field must be reviewed and the need for extensions of existing procedures established.

*1. **Energy sink and reaction force methods.*** Attitude anomalies observed after the launch of Explorer I (Fig. 1) were explained (Ref. 1) by means of an approximate dynamic analysis in which the vehicle nonrigidity was ignored in generating a preliminary representation of rotational motion, and then the oscillatory deformations of flexible appendages on the vehicle (the wire turnstile antennas in Fig. 1) were estimated by prescribing the appendage base motion to be that obtained from the rigid vehicle approximation. Estimates of the energy dissipation rate were then obtained from the predictions of relative motions of the appendage components, and finally, the kinetic energy of the original rigid-body model of the vehicle was reduced over time in correspondence with the estimated rate of dissipation of energy in the nonrigid portions of the vehicle. In this fashion, it was determined in Ref. 1 that nonrigidity and consequent energy dissipation would cause Explorer I to change asymptotically from initial spin about its longitudinal axis to eventual spin about a transverse axis, thus explaining the observed behavior of this satellite.

Since the nonrigidity of a spacecraft is acknowledged by the analyst who follows the practices of Ref. 1 only in the process of predicting the energy dissipation rate, the vehicle mathematical model which he adopts is often characterized by the necessarily imprecise term *quasirigid,* which implies that over a "short" time interval, the overall vehicle motion is essentially that of a rigid body. The procedure described here has been widely employed in the dynamic analysis of spinning and dual-spin spacecraft (Refs. 3–5), and it is known in its various guises as the *energy-sink* approach. In a variation of this procedure applicable also to systems of quasi-rigid bodies, the energy loss calculation is replaced by an estimate of the *reaction forces* on the spacecraft due to the relative motion of components (Refs. 6, 7).

Although the energy-sink and reaction-force methods have proven extremely useful, particularly in the interpretation of observed attitude anomalies (Refs. 1, 5) and in preliminary design (Refs. 3, 4, 6, 7), they are inherently approximate, and their predictions generally require more formal corroboration. In some industrial applications, an energy-sink or reaction-force analysis might suffice to establish, in a given case, the noncritical nature of vehicle nonrigidity, but in this study, a deeper investigation is presumed necessary. At issue is the question of how to proceed.

2. ***Discrete-coordinate methods.*** The practice of modeling a spacecraft as a collection of discrete rigid bodies (or particles and rigid bodies) is limited only by the skill with which the analyst can devise his model, the size of the computer (and computer budget) available to integrate numerically the resulting ordinary differential equations of motion in the discrete coordinates of the system, and the accuracy and computational stability of his integration program. Since there need be no limitations on the relative motions of the constituent parts of the spacecraft, this *discrete-coordinate* approach is conceptually ideal for spacecraft undergoing large changes in orientation and configuration, such as the dual-spin vehicle shown in Fig. 2 in the process of deploying solar panels while maintaining antenna pointing control. This vehicle would seem to require *at least* eleven bodies in the mathematical model, and more bodies would be required if it became necessary to accommodate the flexibility of the individual solar panels or the antenna. Additional problems might be presented by fluids or flexible components in the rotor. Practical limitations in this approach arise because the costs of computer simulation increase rapidly with the number of bodies in the system, and problems of computational inaccuracy and instability increase correspondingly.

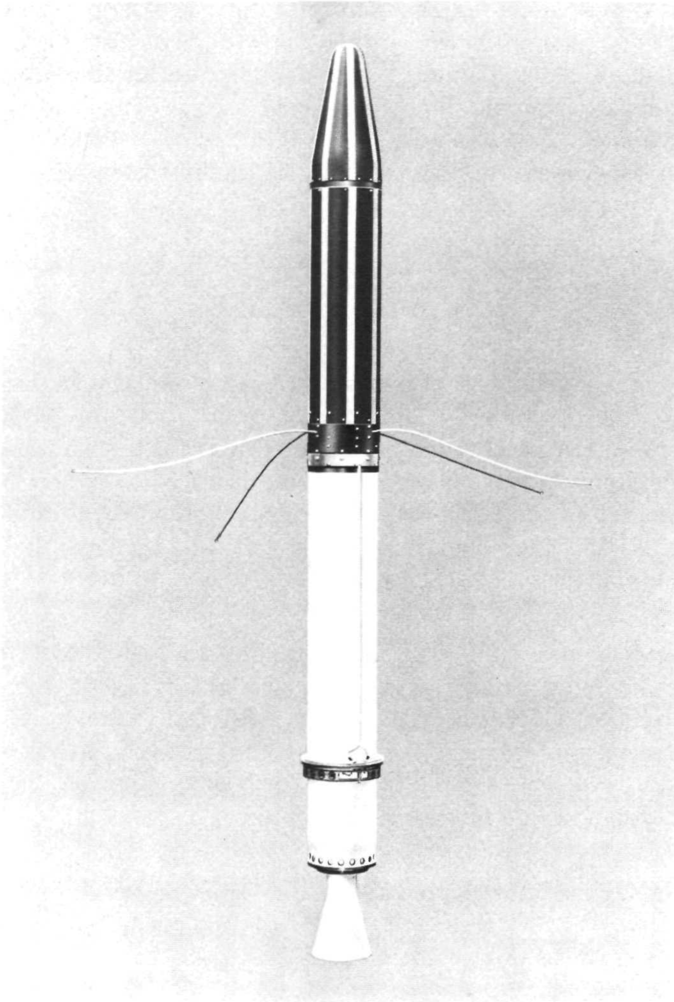

Fig. 1. Explorer I

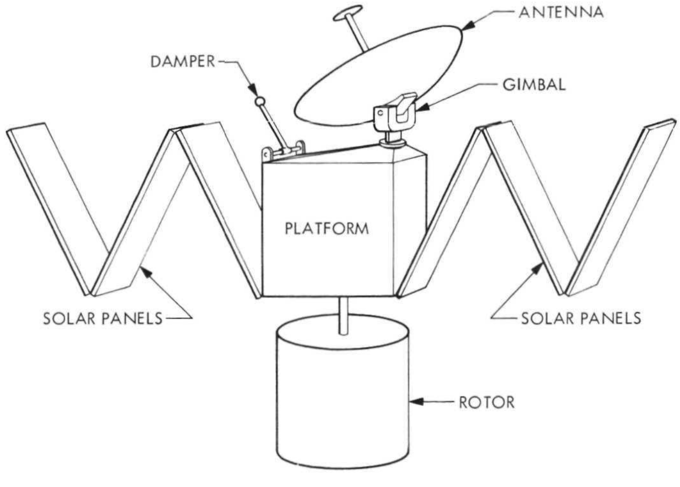

Fig. 2. Dual-spin spacecraft during solar panel deployment

The objective of formulating equations of motion in a form most conveniently obtained and most readily solved has preoccupied dynamicists for centuries. The accomplishments of Euler, Lagrange, and Hamilton in this area are well known; Lagrange found, for example, that if he *restricted* the dynamical system to one involving only *holonomic* constraints, and if all forces other than those imposed to satisfy constraints were *conservative*, then he could write scalar equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \cdots, n$$

where the Lagrangian L is the difference in kinetic and potential energies, and $q_1, \cdots, q_n$ are the generalized coordinates of the system. Similarly (but less spectacularly), modern dynamicists have discovered that if they impose other sets of restrictive conditions on the dynamical system, they too can obtain generic equations of motion in a form particularly convenient for practical application; in a modern context, the equations are often judged by their suitability for digital computer programs for numerical integration.

The specific dynamical system for which equations of motion have most successfully been formulated generically consists of a collection of rigid bodies which are *point-connected* (i.e., interconnected in such a way that each body shares at least one point in common with some other body), with no closed loops or rings of point-connected bodies in the system. This system has become known as the *set of point-connected rigid bodies in a topological tree.*

The eleven-body model suggested for the dual-spin spacecraft in Fig. 2 provides a natural example of such a system of point-connected rigid bodies in a topological tree. This is a special case because contiguous bodies in every case share a common *line* and not just a common *point*; such systems are called *hinge-connected.* If it is deemed essential that the dynamical model accommodate the flexibility of each of the six individual segments of the solar panels in Fig. 2, then this can be accomplished within the framework of the point-connected topological tree model in a variety of ways, as illustrated by Figs. 3a, b, c. In each model, a series of line hinges has been introduced, with suitable restraints on rotation (spring torques and damping torques). In model (a), only the bending flexibility of the panel is accommodated. Model (b) also permits the panel to twist about its central axis. Model (c) permits twisting and bending about two orthogonal axes. With the introduction of another family of hinges normal to the page, one could accommodate in-plane deformations of the panels.

Should it become necessary to connect rigid bodies with linear springs *in addition to* the basic point connections, this can be accomplished within the framework of the point-connected topological tree system equations, even if the linear springs provide the closure of a chain of bodies into a ring; the linear spring forces must then be included with the *external* forces in the system equations.

Note that in Figs. 2 and 3, the bodies are *hinge-connected* rather than merely *point-connected.* It is always possible to substitute a hinge-connected system for a point-connected system simply by introducing auxiliary bodies, such as the small blocks shown in Figs. 3b and 3c; such bodies can be idealized as massless and

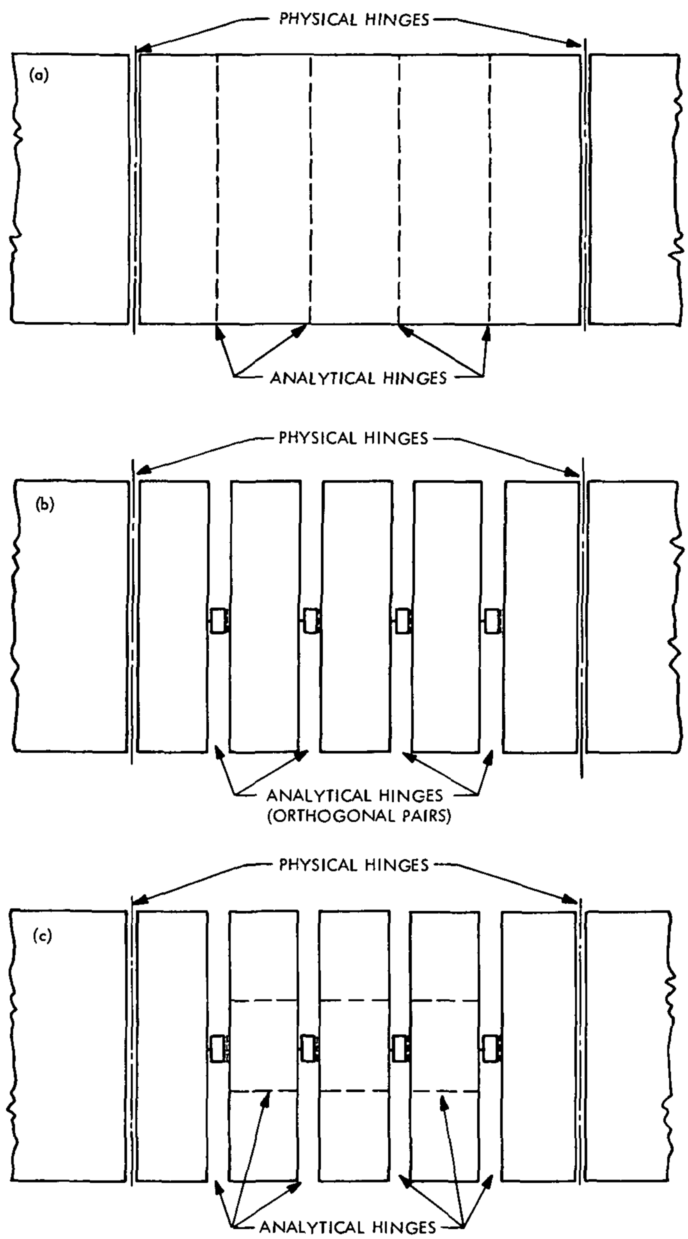

Fig. 3. Rigid-body model of flexible panel segments from Fig. 2

dimensionless, and their introduction does not alter the minimum dimension of the problem because it does not change the number of degrees of freedom in the system.

The *disadvantages* of employing discrete-coordinate equations of motion of a system of point-connected (or hinge-connected) rigid bodies in a topological tree are obvious: (1) No degree of sophistication in mathematical modeling can provide proper representation of actual *local* deformations, as required for the calculation of stresses; and (2) the number of rigid bodies in the system soon grows too large for computer simulation. (If model 3c is used for each of the six panels in Fig. 2, the vehicle model involves 95 hinge-connected bodies!)

The *advantages* of employing discrete-coordinate equations of motion of a system of point-connected (or hinge-connected) rigid bodies in a topological tree can also be clearly stated: (1) A minimum-dimension set of equations of motion can easily be formulated generically for an n-body system and programmed once and for all for reasonably efficient digital computer numerical integration; and (2) relative rotations between connected bodies can be unrestricted in magnitude, permitting the simulation of time-varying configurations or large deformations.

The contributions of the present report in the area of discrete-coordinate formulations are based upon a very substantial foundation provided by others, whose work is briefly reviewed in what follows.

The publication in 1963 by Fletcher, Rongved, and Yu (Ref. 8) of the equations of motion of two point-connected bodies representing a gravity-stabilized satellite stimulated the parallel derivations by Hooker and Margulies (Ref. 9) and by Roberson and Wittenburg (Ref. 10) of the corresponding equations for an arbitrary number of point-connected rigid bodies in a topological tree. Reference 9 provides vector-dyadic equations, and Ref. 10 contains equivalent but independently derived matrix equations. Both sets of equations involve all torques transmitted between bodies at connection points, and this can be a substantial liability if relative motions are constrained (as by a line hinge). As a consequence of their retention of unknown kinematical constraint torques and the equations necessary to provide solutions for these unknowns, neither Ref. 9 nor Ref. 10 offers a set of equations of motion of the minimum dimension required by the number of kinematical degrees of freedom in the system.

The equations provided in Refs. 9 and 10 fulfilled in a timely manner a keenly felt industrial need, and several aerospace organizations developed computer programs based on these equations or others of similar character derived independently. Procedures were proposed and implemented by Velman (Ref. 11) and by Fleischer (Ref. 12) to enable the computer to integrate the equations without actually solving for the constraint torques. Russell (Ref. 13) devised an approach to deriving the equations without ever introducing the constraint torques; he formulated vector equations of motion for subsets of rigid bodies selected so that only one joint connected each subset to a body outside of the subset, and then dot-multiplied each subset equation by unit vectors dictated by the degrees of freedom at that one joint of the subset linked to an external body. Russell provided a procedure for systematically deriving equations rather than a generic system of equations suitable for complete pre-programming. Hooker (Ref. 14) then indicated in a brief note a procedure for operating on the Hooker–Margulies equations (Ref. 9) in such a way as to obtain a minimum-dimension set of equations. Hooker's suggestion required coordinate transformations, summation of selected subsets of equations corresponding to those used by Russell, and dot-multiplying by unit vectors established by the details of the joint geometry. In his published note (Ref. 14), these operations are outlined but not presented in explicit detail.

Computer programs based on generic formulation of equations of motion of discrete-coordinate systems have been widely used in the aerospace community since 1965, and in a few cases (Refs. 11, 15, 16), public documentation is available. Although these programs have served a useful and even necessary function, they have not become a panacea for all problems of simulation of nonrigid spacecraft, because of the disadvantages noted previously. The deficiencies of the rigid-body models in representing local strains and stresses are rarely of concern to the attitude control specialists who have used this approach, but the prohibitive cost of computer simulations involving many bodies has effectively restricted the use of these programs. It soon became evident that more efficient approaches would be required.

3. ***Vehicle normal-coordinate methods.*** The problem of accurate representation of the vibrations of a complex elastic structure with a modest number of coordinates has traditionally been resolved among structural dynamicists by the use of *distributed* or *modal* coordinates. When such coordinates are chosen so that each one represents a *normal mode* of vibration, in which the structure can vibrate without the excitation of other modal coordinates, then the modal coordinate equations are uncoupled, and one can with impunity solve the scalar equations of motion one at a time. Estimates of the motion can then be based on the superposition of as many (or as few) of the modal coordinate solutions as may be desired. This approach, called the *vehicle normal-coordinate method,* is rarely appropriate for vehicles of the complexity of modern spacecraft, however, and in recent years, other alternatives have been developed which combine some of the computational efficiencies of the vehicle normal-coordinate method with some of the advantages of generality offered by the *discrete-coordinate method* by using combinations of discrete and distributed coordinates. Any approach which combines discrete coordinates describing the translations and rotations of some bodies or reference frames of the system with distributed or modal coordinates describing the *small* relative motions of other parts of the system is called a *hybrid-coordinate method.*

4. ***Hybrid-coördinate methods.*** Hybrid-coordinate methods are many and varied, being represented, for example, by Refs. 17–21. The last of these references provides an extensive discussion of each of the three approaches (discrete coordinate, vehicle normal coordinate, and hybrid coordinate) in the stage to which these methods had evolved by 1970. The application to complex spacecraft of the hybrid coordinate method described in Ref. 21 is documented in Refs. 22–24, and an extension of Ref. 21 providing for the representation of a flexible appendage with a distributed-mass finite element model is given in Ref. 25. Quantitative comparisons of simulation results and computer time for discrete-coordinate and hybrid-coordinate simulations of a Viking spacecraft may be found in Ref. 16, and similar comparisons of discrete-coordinate and vehicle normal-coordinate simulations are available for the Radio Astronomy Explorer satellite in Ref. 15.

Because distributed coordinates are used to describe the deformations of flexible substructures in the hybrid coordinate approach, the internal mathematical model of the flexible structure is obscured; it might be an elastic continuum (Ref. 18), or a collection of particles interconnected by massless elastic elements (Refs. 22–24), or a collection of interconnected elastic elements possessing mass (Ref. 25). The question might even be ignored in formulating the equations (as in Ref. 20), but when the time comes for applications, some decision must be made. In the formulations which have been published thus far, no one has suggested the adoption for an elastic substructure of a model consisting of a point-connected set of rigid bodies in a topological tree, but, as suggested by the flexible solar panel segment models in Fig. 3, this is a realistic option. The advantages associated with this choice (to be amplified later) stem from the possibility of beginning with a description in terms of discrete coordinates describing *relative* rotations of contiguous bodies, and therefore representing *local* deformations. This distinction permits the use of modal coordinates to represent large gross deformations.

The several variations of the hybrid-coordinate approach which have emerged thus far share certain limitations. (1) The critical task of modal analysis of flexible appendages has in all publications to date been based on restricted boundary conditions; in the most general formulation (Ref. 25), the flexible substructure is assumed to be attached to one rigid body with arbitrary motion, and in other formulations in which the flexible substructure is connected to more than one additional substructure, the system is assumed to be nominally inertially stationary. (2) Flexible appendages are restricted to small displacements from a reference state, implying small deformations both in a local sense and in an overall sense. (3) Except in restricted cases (e.g., Ref. 20), hybrid-coordinate formulations have not been incorporated into comprehensive generic computer programs (although such programs are under development at JPL and elsewhere).

5. The need for new procedures. When one considers the limitations of the discrete-coordinate, vehicle normal-coordinate, and hybrid-coordinate procedures in their present stages of development, it becomes clear that none of the methods described in the references is suitable for certain classes of vehicles. As an example, note that a flexible substructure experiencing small strains and large overall deformations can at present be accommodated only by a discrete-coordinate formulation, which must be based on a crude model to avoid prohibitive expense (Ref. 15). As an illustration of a different class of problem beyond the scope of present methods, note that none of the published formulations will suffice in the accurate simulation of the dual-spin vehicle shown in Fig. 2 while its solar panels are being deployed, unless the panel segments are presumed rigid. The discrete-coordinate approach is ideal for a gross approximation of the deployment operation (using rigid panel segments), and the hybrid coordinate approach in Ref. 21 is ideal for the simulation of flexible panels after deployment. But if antenna pointing control requirements during deployment demand the accommodation of panel segment flexibility in the mathematical model, then the discrete-coordinate approach becomes computationally cumbersome and perhaps prohibitively expensive, while the existing hybrid-coordinate approaches are inapplicable because of theoretical restrictions. New procedures are therefore required.

B. Scope of Study

The objectives of this report include the documentation in scalar detail not available elsewhere of a minimum-dimension set of discrete-coordinate equations of motion of a hinge-connected set of rigid bodies in a topological tree, and the introduction of a new procedure for modifying these equations by coordinate transformation and truncation, so as to obtain a hybrid-coordinate formulation of the system equations of motion, employing large deformation modal coordinates in combination with discrete coordinates.

The discrete-coordinate equations presented here (see Section II) are the results of the application of a variation of Hooker's procedure (Ref. 14) in a systematic fashion that results in generic matrix (or scalar) equations, formulated in the spirit of the Roberson–Wittenburg equations (Ref. 10) but with constraint torques removed.

The hybrid-coordinate equations presented here (see Section III) have been obtained by (1) isolating discrete-coordinate equations for subsets of rigid bodies identified collectively as a flexible appendage or flexible substructure, (2) linearizing in the variables which describe the relative angular motions of contiguous bodies within the flexible substructure, (3) transforming these substructure variables into distributed coordinates for the substructure, and (4) truncating these distributed coordinates. As noted in detail in Section III, the linearization employed here remains a useful engineering approximation as long as *contiguous* bodies within the flexible substructure experience "small" relative rotations, even if the rotations between the first and last bodies in a chain become "large" (say, 90 deg). This is not the case for traditional "small deflection theory" as applied to beams, etc., because this theory assumes small relative displacements and rotations of all differential elements. The modal coordinates in Section III are useful even when the actual elastic substructure experiences large elastic deflections from a reference state as long as the elongations and shear deformations of differential elements remain small.

The useful results in this report consist of three versions of the equations of motion of an arbitrary vehicle modeled as a collection of point-connected rigid bodies in a topological tree: (1) unrestricted discrete-coordinate equations (see Eq. 1), (2) partially linearized discrete-coordinate equations (see Eq. 30), and (3) hybrid-coordinate equations (see Eq. 118). These equations can all be incorporated as subroutines in a single computer program, or they can become members of a family of computer programs, with use of the hybrid-coordinate program or subroutine requiring additional input from a separate program designed for eigenvalue–eigenvector analysis of linear subsets of differential equations.

II. Discrete-Coordinate Equations of Motion

A. Mathematical Model

Figure 4 illustrates a set of eleven hinge-connected rigid bodies, labeled according to a convention designed to facilitate the processing of the equations of motion. This example, which portrays a system of a degree of complexity that would minimally suffice for the spacecraft shown in Fig. 2, will be useful in illustrating and interpreting the general mathematical model and the labeling conventions adopted here for its description. (In reading the lengthy list of conventions and symbol definitions, frequent reference should be made to Fig. 4.) Only after entertaining the complete cast of symbols used in what follows will consideration be given to the much smaller list of symbols to which values must be assigned by the user of the projected computer program.

The mathematical model consists of a set of $n + 1$ rigid bodies interconnected by n hinges*; the indicated numbers of bodies and hinges imply a tree topology. Any interbody connection forces in addition to those at the n hinges must be treated as forces external to the entire system. (If there were a linear spring connecting the mid-point of body $\mathcal{b}_6$ in Fig. 4 to some point on body $\mathcal{b}_0$, for example, it would be necessary to replace the spring with a pair of equal and opposite forces

*The word "hinge" as used here implies a connection which maintains a line common to both bodies; such a connection is sometimes called a *line hinge*.

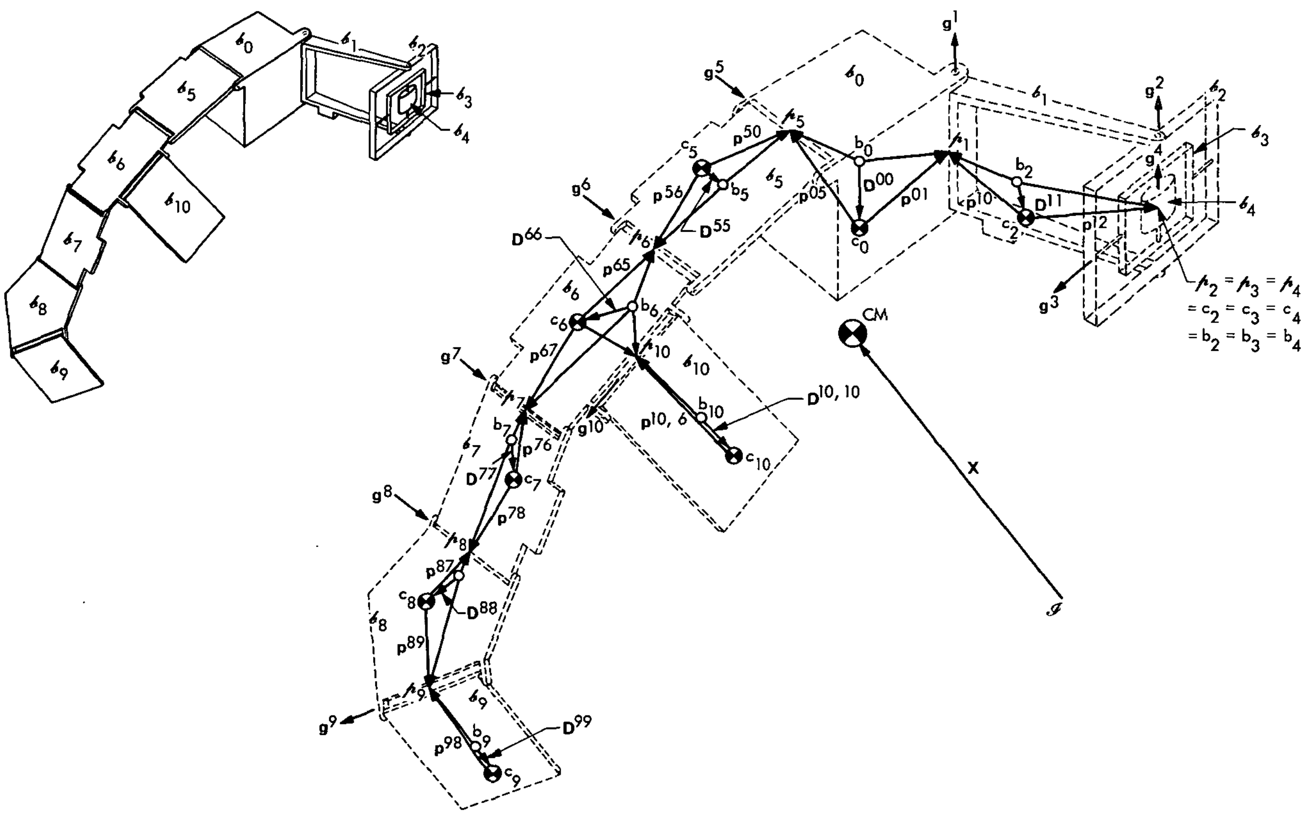

Fig. 4. Eleven rigid bodies interconnected by ten hinges

applied to the spring attachment points on $\mathscr{b}_6$ and $\mathscr{b}_0$; the magnitude and direction of these forces would have to be expressed in terms of the unknown kinematical variables of the system.)

1. Definitions of fundamental symbols

Def. 1. Let n be the number of hinges interconnecting a set of $n + 1$ bodies.

Def. 2. Define the integer set $\mathcal{B} \triangleq \{0, 1, \cdots, n\}$.

Def. 3. Define the integer set $\mathcal{P} \triangleq \{1, \cdots, n\}$.

Def. 4. Let $\mathscr{b}_0$ be a label assigned to one body chosen arbitrarily as a reference body, and let $\mathscr{b}_1, \cdots, \mathscr{b}_n$ be labels assigned to the remaining bodies in such a way that if $\mathscr{b}_j$ is located between $\mathscr{b}_0$ and $\mathscr{b}_k$, then $0 < j < k$.

Def. 5. Define the kth ***neighbor set*** $\mathcal{B}_k$ for $k \in \mathcal{B}$ such that $r \in \mathcal{B}_k$ if $\mathscr{b}_r$ is attached to $\mathscr{b}_k$.

Def. 6. Define the dextral, orthogonal sets of unit vectors $\mathbf{b}_1^k, \mathbf{b}_2^k, \mathbf{b}_3^k$ so as to be imbedded in $\mathscr{b}_k$ for $k \in \mathcal{B}$, and such that in some arbitrarily selected nominal configuration of the system $\mathbf{b}_\alpha^k = \mathbf{b}_\alpha^j$ for $\alpha = 1, 2, 3$ and $k, j \in \mathcal{B}$.

Def. 7. Define $\{\mathbf{b}^k\}$ as the column array of unit vectors*

$$\{\mathbf{b}^k\} \triangleq \begin{Bmatrix} \mathbf{b}_1^k \\ \mathbf{b}_2^k \\ \mathbf{b}_3^k \end{Bmatrix}$$

for $k \in \mathcal{B}$.

Def. 8. Define $\{\mathbf{i}\}$ as a column array of inertially fixed, dextral, orthogonal unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, so that

$$\{\mathbf{i}\} \triangleq \begin{Bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{Bmatrix}$$

Def. 9. Let C be the direction cosine matrix defined by

$$\{\mathbf{b}^0\} = C\,\{\mathbf{i}\}$$

Def. 10. Let $\boldsymbol{\omega}^0 \triangleq \{\mathbf{b}^0\}^T \omega^0$ be the inertial angular velocity vector of $\mathscr{b}_0$, so that ω^0 is the corresponding 3 by 1 matrix in vector basis $\{\mathbf{b}^0\}$.

Def. 11. Let c_k be the mass center of $\mathscr{b}_k$ for $k \in \mathcal{B}$ (see Fig. 4).

Def. 12. Let $\mathscr{p}_k$ be a point on the hinge axis common to $\mathscr{b}_k$ and $\mathscr{b}_j$ for $j < k$ and $k \in \mathcal{P}$ (see Fig. 4).

Def. 13. For $j \in \mathcal{B}_k$, let $\mathbf{p}^{kj} \triangleq \{\mathbf{b}^k\}^T p^{kj}$ be the position vector with respect to c_k of the labeled point on the hinge axis (either $\mathscr{p}_j$ or $\mathscr{p}_k$) which is common to $\mathscr{b}_k$ and $\mathscr{b}_j$, so that p^{kj} is the corresponding 3 by 1 matrix in vector basis $\{\mathbf{b}^k\}$. (See Fig. 4, and note the special cases $\mathbf{p}^{21} = \mathbf{p}^{23} = \mathbf{p}^{32} = \mathbf{p}^{34} = \mathbf{p}^{43} = 0$.)

Def. 14. Let $\mathbf{g}^k \triangleq \{\mathbf{b}^k\}^T g^k$ be a unit vector parallel to the hinge axis through $\mathscr{p}_k$, so that g^k is the corresponding 3 by 1 matrix in vector basis $\{\mathbf{b}^k\}$.

Def. 15. For $k \in \mathcal{P}$, let γ_k be the angle of a $\mathbf{g}^k$-rotation of $\mathscr{b}_k$ with respect to the body attached at $\mathscr{p}_k$, that is, a rotation during which a right-handed screw fixed in $\mathscr{b}_k$ with its axis parallel to $\mathbf{g}^k$ advances in the direction of $\mathbf{g}^k$. Assign the value zero to γ_k for $k \in \mathcal{P}$ when the system is in its arbitrarily chosen nominal configuration, for which $\mathbf{b}_\alpha^k = \mathbf{b}_\alpha^j$, $\alpha = 1, 2, 3$; $k, j \in \mathcal{B}$.

Def. 16. Let $\mathbf{X} \triangleq \{\mathbf{i}\}^T X$ be the position vector of the system mass center CM with respect to an inertially fixed point $\mathcal{O}$, so that X is the corresponding 3 by 1 matrix in an inertial vector basis $\{\mathbf{i}\}$.

Def. 17. Let m_k be the mass of $\mathscr{b}_k$ for $k \in \mathcal{B}$.

Def. 18. Let $\mathbf{I}^k \triangleq \{\mathbf{b}^k\}^T I^k \{\mathbf{b}^k\}$ be the inertia dyadic of $\mathscr{b}_k$ for c_k, so that I^k is the corresponding inertia matrix in vector basis $\{\mathbf{b}^k\}$.

*Note that braces { } are used in this report both to identify column arrays of vectors and to enclose sets of integers; the distinction is apparent as soon as the objects within the braces are identified as vectors (boldface) or scalars.

Def. 19. Let $\mathbf{F}^k \triangleq \{\mathbf{b}^k\}^T F^k$ be the resultant vector of all forces applied to $\mathscr{b}_k$ *except for those due to interbody forces transmitted at hinge connections,* so that F^k is the corresponding 3 by 1 matrix in vector basis $\{\mathbf{b}^k\}$. (Thus, any interbody forces due to spring connections, etc., which are separate from the n hinge connections would contribute to $\mathbf{F}^k$, as would all forces from sources external to the system.)

Def. 20. Let $\mathbf{T}^k \triangleq \{\mathbf{b}^k\}^T T^k$ be the resultant moment vector with respect to c_k of all forces applied to $\mathscr{b}_k$ *except for those due to interbody forces transmitted at hinge connections,* so that T^k is the corresponding 3 by 1 matrix in vector basis $\{\mathbf{b}^k\}$.

Def. 21. Let τ_k be the scalar magnitude of the torque component applied to $\mathscr{b}_k$ in the direction of $\mathbf{g}^k$ by the body connected at $\mathscr{h}_k$.

Def. 22. Let the Kronecker delta symbol $\delta_{\alpha\beta}$ be defined by

$$\delta_{\alpha\beta} \triangleq 1 - \frac{1}{4}(\alpha - \beta)^2 [5 - (\alpha - \beta)^2] \qquad \alpha, \beta = 1, 2, 3$$

so that

$$\delta_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ otherwise} \end{cases}$$

Def. 23. Let the Levi-Civita density symbol $\epsilon_{\alpha\beta\theta}$ be defined by

$$\epsilon_{\alpha\beta\theta} \triangleq \frac{1}{2}(\alpha - \beta)(\beta - \theta)(\theta - \alpha) \qquad \alpha, \beta, \theta = 1, 2, 3$$

so that

$$\epsilon_{\alpha\beta\theta} = \begin{cases} +1 \text{ for } \alpha, \beta, \theta \text{ a cyclic permutation of } 1, 2, 3 \\ 0 \text{ for } \alpha = \beta \text{ or } \beta = \theta \text{ or } \theta = \alpha \\ -1 \text{ for } \alpha, \beta, \theta \text{ a cyclic permutation of } 1, 3, 2 \end{cases}$$

Def. 24. Let the tilde symbol ($\sim$) signify in application to a 3 by 1 matrix V with elements V_θ ($\theta = 1, 2, 3$) transformation to a skew-symmetric 3 by 3 matrix V, whose elements are given by*

$$\tilde{V}_{\alpha\beta} = \epsilon_{\alpha\theta\beta} V_\theta$$

Thus, the matrix $\tilde{V}$ has the expanded form

$$\tilde{V} \triangleq \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix}$$

2. *Definitions of derived symbols*

Def. 25. Let $\mathbf{F} \triangleq \{\mathbf{i}\}^T F \triangleq \sum_{k \in \mathscr{B}} \mathbf{F}^k$, so that F is the 3 by 1 matrix corresponding to $\mathbf{F}$ in vector basis $\{\mathbf{i}\}$.

*When lowercase Greek subscripts appear twice in a single term, summation over the values 1, 2, 3 is implied, so that, for example, $\epsilon_{\alpha\theta\beta} V_\theta = \epsilon_{\alpha 1 \beta} V_1 + \epsilon_{\alpha 2 \beta} V_2 + \epsilon_{\alpha 3 \beta} V_3$.

Def. 26. For $r \in (\mathcal{B} - k)$,* let N_{kr} denote the index of the body attached to b_k and on the path leading to b_r and let $N_{kk} \triangleq k$ and $N_k \triangleq N_{k0}$ (to simplify notation). The term *network elements* will be applied to the scalars typified by N_{kr}, which are $(n+1)^2$ in number.

Def. 27. Define the scalar ε_{sk} such that for $k \in \mathcal{B}$ and $s \in \mathcal{P}$,

$$\varepsilon_{sk} \triangleq \begin{cases} 1 \text{ if } h_s \text{ lies between } b_0 \text{ and } b_k \\ 0 \text{ otherwise} \end{cases}$$

The term *path elements* will be applied to the scalars ε_{sk}, which are $n(n+1)$ in number.

Def. 28. Let $\mathcal{B}_{kj}$ be the *branch set* of integers r such that $r \in \mathcal{B}_{kj}$ if $j = N_{kr}$. Thus, $\mathcal{B}_{kj}$ consists of the indices of those bodies attached to b_k on a branch which begins with b_j. Since there are n hinges, there are $2n$ branch sets appearing in pairs, such that $\mathcal{B}_{jk} = \mathcal{B} - \mathcal{B}_{kj}$.

Def. 29. Let C^{rj} be the direction cosine matrix defined by

$$\{\mathbf{b}^r\} \triangleq C^{rj}\{\mathbf{b}^j\}$$

for $r, j \in \mathcal{B}$. In the nominal state, $\gamma_k = 0$ for $k \in \mathcal{P}$, and all unit vectors of corresponding subscript are aligned, so that C^{rj} then is U, the 3 by 3 unit matrix.

Def. 30. Define $\mathcal{M} \triangleq \sum_{k \in \mathcal{B}} m_k$, so that $\mathcal{M}$ is the total system mass.

Def. 31. Define $\mathcal{M}_{kj} \triangleq \sum_{r \in \mathcal{B}_{kj}} m_r$, so that $\mathcal{M}_{kj}$ is the total mass on the branch of the system attached to b_k and commencing with b_j.

Def. 32. For $r \in \{\mathcal{B} - k\}$, define $\mathbf{L}^{kr} \triangleq \mathbf{p}^{kN_{kr}}$, and define $\mathbf{L}^{kk} \triangleq 0$. Thus, $\mathbf{L}^{kr}$ is the vector from c_k to the hinge point attached to b_k and on the path leading from b_k toward b_r. (In Fig. 4, for example, $\mathbf{L}^{56} = \mathbf{L}^{57} = \mathbf{L}^{58} = \mathbf{p}^{56}$ and $\mathbf{L}^{3r} = 0$ for $r \in \mathcal{B}$.)

Def. 33. Define $\{\mathbf{b}^k\}^T D^{kk} \triangleq \mathbf{D}^{kk} \triangleq -\sum_{j \in \mathcal{B}} \mathbf{L}^{kj} m_j / \mathcal{M}$ for $k \in \mathcal{B}$.

Def. 34. Let b_k be a point called the *barycenter* of b_k and fixed relative to b_k such that $\mathbf{D}^{kk}$ is the position vector of c_k with respect to b_k.

Def. 35. Define $\{\mathbf{b}^k\}^T D^{kj} \triangleq \mathbf{D}^{kj} \triangleq \mathbf{D}^{kk} + \mathbf{L}^{kj}$ for $k, j \in \mathcal{B}$. Thus, $\mathbf{D}^{kj}$ is the vector from b_k to the hinge point fixed in b_k and on the path leading to b_j.

Def. 36. Define the dyadic

$$\boldsymbol{\Phi}^{kk} \triangleq \mathbf{I}^k + m_k(\mathbf{D}^{kk} \cdot \mathbf{D}^{kk}\mathbf{U} - \mathbf{D}^{kk}\mathbf{D}^{kk}) + \sum_{j \in \mathcal{B}_k} \mathcal{M}_{kj}(\mathbf{D}^{kj} \cdot \mathbf{D}^{kj}\mathbf{U} - \mathbf{D}^{kj}\mathbf{D}^{kj})$$

*For notational brevity, the set $\mathcal{B} - \{\mathbf{k}\}$ is designated $\mathcal{B} - k$.

where $\mathbf{U}$ is the unit dyadic, and define the corresponding constant 3 by 3 matrix in vector basis $\{\mathbf{b}^k\}$ as

$$\Phi^{kk} \triangleq \{\mathbf{b}^k\} \cdot \mathbf{\Phi}^{kk} \cdot \{\mathbf{b}^k\}^T$$

$$= I^k + m_k (D^{kk^T} D^{kk} U - D^{kk} D^{kk^T}) + \sum_{j \in \mathcal{B}_k} \mathcal{M}_{kj} (D^{kj^T} D^{kj} U - D^{kj} D^{kj^T})$$

Def. 37. For $j \in \{\mathcal{B} - k\}$, define the dyadic

$$\mathbf{\Phi}^{kj} \triangleq -\mathcal{M} (\mathbf{D}^{jk} \cdot \mathbf{D}^{kj} \mathbf{U} - \mathbf{D}^{jk} \mathbf{D}^{kj})$$

and define a corresponding variable 3 by 3 matrix as

$$\Phi^{kj} \triangleq \{\mathbf{b}^j\} \cdot \mathbf{\Phi}^{kj} \cdot \{\mathbf{b}^k\}^T$$

$$= -\mathcal{M} [\{\mathbf{b}^j\} \cdot (D^{jk^T} \{\mathbf{b}^j\} \cdot \{\mathbf{b}^k\}^T D^{kj}) \{\mathbf{b}^k\}^T$$

$$- \{\mathbf{b}^j\} \cdot \{\mathbf{b}^j\}^T D^{jk} D^{kj^T} \{\mathbf{b}^k\} \cdot \{\mathbf{b}^k\}^T]$$

$$= -\mathcal{M} [\{\mathbf{b}^j\} \cdot (D^{jk^T} C^{jk} D^{kj}) \{\mathbf{b}^k\}^T - D^{jk} D^{kj^T}]$$

$$= -\mathcal{M} (C^{jk} D^{jk^T} C^{jk} D^{kj} - D^{jk} D^{kj^T})$$

Def. 38. Define the 3 by 3 matrix

$$a_{00} \triangleq \sum_{k \in \mathcal{B}} \sum_{j \in \mathcal{B}} C^{0j} \Phi^{kj} C^{k0} = \sum_{k \in \mathcal{B}} C^{0k} \Phi^{kk} C^{k0}$$

$$-\mathcal{M} \sum_{k \in \mathcal{B}} \sum_{j \in \mathcal{B} - k} (C^{0k} D^{jk^T} C^{jk} D^{kj} - C^{0j} D^{jk} D^{kj^T}) C^{k0}$$

Def. 39. For $k \in \mathcal{P}$, define the 3 by 1 matrices

$$a_{0k} \triangleq \sum_{r \in \mathcal{B}} \sum_{j \in \mathcal{B}} \varepsilon_{kr} C^{0j} \Phi^{rj} C^{rk} g^k = \sum_{r \in \mathcal{P}} \varepsilon_{kr} C^{0r} \Phi^{rr} C^{rk} g^k$$

$$- \mathcal{M} \sum_{r \in \mathcal{P}} \sum_{j \in \mathcal{B} - r} \varepsilon_{kr} (C^{0r} D^{jr^T} C^{jr} D^{rj} - C^{0j} D^{jr} D^{rj^T}) C^{rk} g^k$$

and let $a_{k0} \triangleq a_{0k}^T$.

Def. 40. For $j, k \in \mathcal{P}$, define the scalars

$$a_{kj} \triangleq g^{k^T} \sum_{r \in \mathcal{P}} \sum_{s \in \mathcal{P}} \varepsilon_{kr} \varepsilon_{js} C^{ks} \Phi^{rs} C^{rj} g^j$$

$$= g^{k^T} \sum_{r \in \mathcal{P}} \varepsilon_{kr} \varepsilon_{jr} C^{kr} \Phi^{rr} C^{rj} g^j$$

$$-\mathcal{M} g^{k^T} \sum_{r \in \mathcal{P}} \sum_{s \in \mathcal{P} - r} \varepsilon_{kr} \varepsilon_{js} (C^{kr} D^{sr^T} C^{sr} D^{rs} - C^{ks} D^{sr} D^{rs^T}) C^{rj} g^j$$

Def. 41. For $k \in \mathscr{B}$, define the 3 by 1 matrices*

$$
\begin{aligned}
A^k \triangleq\ & T^k + \sum_{j \in \mathscr{B}} \tilde{D}^{kj} C^{kj} F^j - \Phi^{kk} \sum_{r \in \mathscr{P}} \varepsilon_{rk} \dot{\gamma}_r \left(C^{k0} \tilde{\omega}^0 C^{0r} + \sum_{s \in \mathscr{P}} \varepsilon_{sr} \dot{\gamma}_s C^{ks} \tilde{g}^s C^{sr} \right) g^r \\
& - \left[C^{k0} \tilde{\omega}^0 C^{0k} + \sum_{j \in \mathscr{P}} \varepsilon_{jk} \dot{\gamma}_j C^{kj} \tilde{g}^j C^{jk} \right] \Phi^{kk} \left[C^{k0} \omega^0 + \sum_{j \in \mathscr{P}} \varepsilon_{jk} \dot{\gamma}_j C^{kj} g_j \right] \\
& + \mathscr{M} \sum_{j \in \mathscr{B} - k} \tilde{D}^{kj} C^{kj} \left[C^{j0} \tilde{\omega}^0 C^{0j} + \sum_{r \in \mathscr{P}} \varepsilon_{rj} \dot{\gamma}_r C^{jr} \tilde{g}^r C^{rj} \right]^2 D^{jk} \\
& - \sum_{j \in \mathscr{P}} C^{kj} \Phi^{kj} \sum_{r \in \mathscr{P}} \varepsilon_{rj} \dot{\gamma}_r \left[C^{k0} \tilde{\omega}^0 C^{0r} + \sum_{s \in \mathscr{P}} \varepsilon_{sr} \dot{\gamma}_s C^{ks} \tilde{g}^s C^{sr} \right] g^r
\end{aligned}
$$

where the superscript 2 indicates the square of the matrix in square brackets. This expression for A^k can be written in a form which reveals more explicitly its dependence on time-variable quantities by substituting in A^k the expansion of Φ^{kj} recorded in Def. 37. The resulting expression is

$$
\begin{aligned}
A^k =\ & T^k + \sum_{j \in \mathscr{B}} \tilde{D}^{kj} C^{kj} F^j - \Phi^{kk} \sum_{r \in \mathscr{P}} \varepsilon_{rk} \dot{\gamma}_r \left(C^{k0} \tilde{\omega}^0 C^{0r} + \sum_{s \in \mathscr{P}} \varepsilon_{sr} \dot{\gamma}_s C^{ks} \tilde{g}^s C^{sr} \right) g^r \\
& - \left[C^{k0} \tilde{\omega}^0 C^{0k} + \sum_{j \in \mathscr{P}} \varepsilon_{jk} \dot{\gamma}_j C^{kj} \tilde{g}^j C^{jk} \right] \Phi^{kk} \left[C^{k0} \omega^0 + \sum_{j \in \mathscr{P}} \varepsilon_{jk} \dot{\gamma}_j C^{kj} g^j \right] \\
& + \mathscr{M} \sum_{j \in \mathscr{B} - k} \Big\{ D^{kj} C^{kj} \left[C^{j0} \tilde{\omega}^0 C^{0j} + \sum_{r \in \mathscr{P}} \varepsilon_{rj} \dot{\gamma}_r C^{jr} \tilde{g}^r C^{rj} \right]^2 D^{jk} \\
& + \left(U D^{jkT} C^{jk} D^{kj} - C^{kj} D^{jk} D^{kjT} \right) \sum_{r \in \mathscr{P}} \varepsilon_{rj} \dot{\gamma}_r \left[C^{k0} \tilde{\omega}^0 C^{0r} + \sum_{s \in \mathscr{P}} \varepsilon_{sr} \dot{\gamma}_s C^{ks} \tilde{g}^s C^{sr} \right] g^r \Big\}
\end{aligned}
$$

3. Augmented bodies and barycenters. The *barycenter* b_k of body $\mathscr{b}_k$ is defined in Def. 34 in mathematical terms which admit an interesting interpretation as a physical property of an abstraction called the *augmented body*. If one imagines the body $\mathscr{b}_k$ to be augmented by the addition of a particle of mass $\mathscr{M}_{kj}$ at the connection point of $\mathscr{b}_k$ with $\mathscr{b}_j$ (for $j \in \mathscr{B}_k$), then the result is called the kth augmented body, here referred to as $\mathscr{b}'_k$. Since $\mathscr{M}_{kj}$ is the total mass of the branch of bodies attached to $\mathscr{b}_k$ commencing with $\mathscr{b}_j$, the mass of *each* augmented body is $\mathscr{M}$, the total system mass.

Definition 33 provides

$$
-\mathscr{M} \mathbf{D}^{kk} = \sum_{j \in \mathscr{B}} m_j \mathbf{L}^{kj}
$$

which reveals the fact that $-\mathbf{D}^{kk}$ is the position vector of the mass center of $\mathscr{b}'_k$ with respect to the mass center c_k of b_k. Thus, *the barycenter b_k is the mass center of the kth augmented body $\mathscr{b}'_k$.* With this interpretation comes the relationship

$$
\sum_{r \in \mathscr{B}} m_r \mathbf{D}^{kr} = m_k \mathbf{D}^{kk} + \sum_{r \in \mathscr{B} - k} m_r \mathbf{D}^{kr} = 0
$$

Since $\mathbf{D}^{kk}$ is the position vector from the barycenter b_k to the mass center c_k of $\mathscr{b}_k$, and (from Def. 35) $\mathbf{D}^{kj}$ is the position vector from b_k to a particle of mass $\mathscr{M}_{kj}$ located at the connection point of $\mathscr{b}_k$ leading to body $\mathscr{b}_j$, for $j \in \mathscr{B}_k$, one can interpret $\mathbf{\Phi}^{kk}$ from Def. 36 as the inertia dyadic of the augmented body $\mathscr{b}'_k$ with respect to the

*It may be helpful to note that identifying superscripts are never attached to the right of a scalar in this report; a superscript on a scalar denotes its *exponent* (so that a^5_{kj} means the scalar a_{kj} raised to the fifth power). In contrast, nonscalar quantities are not exponentiated unless noted explicitly (as in Def. 41), so that a superscript to the right of a matrix, vector, or dyadic is then an identifier and not an exponent.

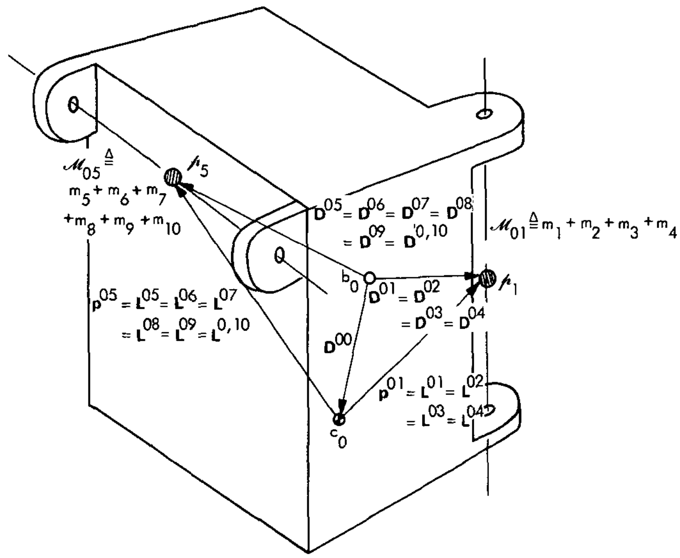

Fig. 5. Augmented $\mathscr{b}_0'$ from Fig. 4

barycenter b_k. Such physical interpretation is not possible for the dyadic $\mathbf{\Phi}^{kj}$ defined in Def. 37, since this dyadic involves vector $\mathbf{D}^{jk}$ fixed in $\mathscr{b}_j$ and vector $\mathbf{D}^{kj}$ fixed in $\mathscr{b}_k$.

In illustration of the concepts just described, Fig. 5 shows the augmented body $\mathscr{b}_0'$ associated with body $\mathscr{b}_0$ in the system of Fig. 4.

B. Matrix Dynamical Equations

In terms of the symbols defined in the preceding section, a minimum-dimension set of $n + 3$ scalar equations of rotational motion of a set of $n + 1$ rigid bodies interconnected by n line hinges can be written as:

$$
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\
a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\
a_{20} & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}^0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \\ \vdots \\ \ddot{\gamma}_n
\end{bmatrix}
=
\begin{bmatrix}
\sum_{k \in \mathcal{B}} C^{0k} A^k \\
g^{1T} \sum_{k \in \mathcal{P}} \varepsilon_{1k} C^{1k} A^k + \tau_1 \\
g^{2T} \sum_{k \in \mathcal{P}} \varepsilon_{2k} C^{2k} A^k + \tau_2 \\
\vdots \\
g^{nT} \sum_{k \in \mathcal{P}} \varepsilon_{nk} C^{nk} A^k + \tau_n
\end{bmatrix}
\tag{1}
$$

This result is proven in Appendix A.

Note that the coefficient matrix on the left side of Eq. (1) is, from Defs. 38–40, a real, nonsingular,* symmetric matrix which varies with time as the kinematical

*Singularity would imply that at least two rows (or columns) of the matrix are dependent, in which event some linear combination of the indicated scalar equations would provide a matrix equation with a zero row (or column) in the coefficient matrix of the column matrix of the most highly differentiated variables. Thus, singularity would indicate an incomplete set of equations unless the variables are constrained. Since the variables in Eq. (1) are independent, the coefficient matrix on the left must be nonsingular.

variables $\gamma_1, \cdots, \gamma_n$ change in value. If these angles and their time derivatives remain in the immediate neighborhood of their nominal zero state, and ω^0 remains in the immediate neighborhood of its nominal constant value, then upon linearization in the variational coordinates, the indicated coefficient matrix becomes a constant, symmetric, nonsingular matrix, which can be inverted in advance of numerical integration of Eq. (1).

If for preliminary analysis one wishes to replace the variable γ_j $(j \in \mathcal{P})$ by a known function of time, then one can remove $\ddot{\gamma}_j$ from the angular acceleration matrix on the left side of Eq. (1), removing also the corresponding row and column of its coefficient matrix, and then rewrite on the right side those terms just deleted by the removal of the column on the left. The result is a set of $n + 2$ scalar differential equations in which time appears explicitly.

The equations of mass center translation of the total system can from Newton's second law and Defs. 3, 16, 25, and 30 be written as the vector equation

$$\{\mathrm{i}\}^T F = \mathcal{M}\,\{\mathrm{i}\}^T \ddot{X}$$

or the matrix equivalent

$$F = \mathcal{M}\ddot{X} \tag{2}$$

Equations (1) and (2) constitute a complete set of *dynamical equations*, but they are not fully descriptive of the system motion until they are augmented by control equations specifying $\tau_1, \cdots, \tau_n$ and certain external forces and torques, and augmented also by kinematical equations as provided in the next section.

C. Matrix Kinematical Equations

The kinematical variables adopted in the previous sections are as follows: γ_k for $k \in \mathcal{P}$ (Def. 15); C^{rj} for $r, j \in \mathcal{B}$ (Def. 29); C (Def. 9); $\omega^0 \triangleq \{\mathrm{b}^0\} \cdot \boldsymbol{\omega}^0$ (Def. 10); and $X \triangleq \{\mathrm{i}\} \cdot \mathbf{X}$ (Def. 16). Although the equations of motion have been expressed in terms of these quantities, the latter are not all independent. Relationships among kinematical variables developed in this section must therefore either be considered in conjunction with the dynamical equations or be substituted into them to remove redundant variables whenever a solution is sought.

The direction cosine matrix C (Def. 9) which establishes the inertial attitude of $\mathscr{b}_0$ is related to the inertial angular velocity matrix ω^0 of $\mathscr{b}_0$ (Def. 10) by

$$\dot{C} = -\tilde{\omega}^0 C \tag{3}$$

The relationship $\tilde{\omega}^0 \triangleq C\dot{C}^T$ is used here to *define* the angular velocity $\boldsymbol{\omega}^0$, as on p. 96 of Ref. 26 (where the symbol C is the transpose of that used here).

The direction cosine matrix C^{rj} (Def. 29) relates sets of orthogonal unit vectors fixed in $\mathscr{b}_r$ and $\mathscr{b}_j$, and hence depends upon those angles γ_α for which $\mathscr{h}_\alpha$ lies between $\mathscr{b}_r$ and $\mathscr{b}_j$, and also upon the corresponding unit vectors $\mathbf{g}^\alpha$ defining the intervening hinge axes. For the special case in which $\mathscr{b}_r$ and $\mathscr{b}_j$ are *contiguous*, it is always possible to identify them in one sequence or another as k and N_k (as introduced in Def. 26), and then to express C^{kN_k} and $C^{N_k k}$ in terms of the single angle γ_k and the single matrix g^k, as follows.

The orientation of $\mathscr{b}_k$ with respect to $\mathscr{b}_{N_k}$ is established by a simple right-handed rotation through angle γ_k about a hinge axis with positive direction of rotation established by $\mathbf{g}^k = \{\mathbf{b}^k\}^T g^k$. As will be demonstrated, the direction cosine matrices then become

$$C^{kN_k} = U \cos \gamma_k - \tilde{g}^k \sin \gamma_k + g^k g^{kT} (1 - \cos \gamma_k) \tag{4}$$

and

$$C^{N_k k} = U \cos \gamma_k + \tilde{g}^k \sin \gamma_k + g^k g^{kT} (1 - \cos \gamma_k) = (C^{kN_k})^T \tag{5}$$

The proof of Eqs. (4) and (5), which can be found (in different terminology) in Ref. 26, is reproduced here for convenience. Figure 6 portrays two bodies $\mathscr{b}_{N_k}$ and $\mathscr{b}_k$ in some relative orientation for which $\gamma_k \neq 0$. The unit vectors shown on the sketch are fixed in the two bodies in such a way that at all times $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = \mathbf{g}^k = \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$, and when $\gamma_k = 0$, then $\{\mathbf{b}^k\} = \{\mathbf{b}^{N_k}\}$ (as in Def. 7), and also $\boldsymbol{\alpha}_1 = \boldsymbol{\beta}_1$ and $\boldsymbol{\alpha}_2 = \boldsymbol{\beta}_2$. Thus, for some constant values of p_s, q_s, and r_s for $s = 1, 2, 3$, the equations

$$\mathbf{b}_s^{N_k} = p_s \boldsymbol{\alpha}_1 + q_s \boldsymbol{\alpha}_2 + r_s \mathbf{g}^k$$

and

$$\mathbf{b}_s^k = p_s \boldsymbol{\beta}_1 + q_s \boldsymbol{\beta}_2 + r_s \mathbf{g}^k$$

must apply. Substituting the relationships

$$\boldsymbol{\beta}_1 = \cos \gamma_k \boldsymbol{\alpha}_1 + \sin \gamma_k \boldsymbol{\alpha}_2$$

$$\boldsymbol{\beta}_2 = -\sin \gamma_k \boldsymbol{\alpha}_1 + \cos \gamma_k \boldsymbol{\alpha}_2$$

into the preceding equations, one finds that

$$\mathbf{b}_s^k = (p_s \cos \gamma_k - q_s \sin \gamma_k)\, \boldsymbol{\alpha}_1 + (p_s \sin \gamma_k + q_s \cos \gamma_k)\, \boldsymbol{\alpha}_2 + r_s \mathbf{g}^k$$

or, since $\boldsymbol{\alpha}_1 \times \mathbf{g}^k = -\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_2 \times \mathbf{g}^k = \boldsymbol{\alpha}_1$,

$$\mathbf{b}_s^k = \mathbf{b}_s^{N_k} \cos \gamma_k - \mathbf{b}_s^{N_k} \times \mathbf{g}^k \sin \gamma_k + \mathbf{b}_s^{N_k} \cdot \mathbf{g}^k \mathbf{g}^k (1 - \cos \theta) \tag{6}$$

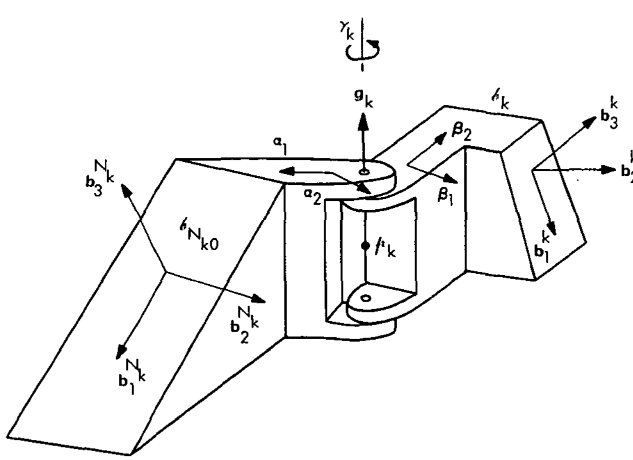

Fig. 6. Simple rotation of $\mathscr{b}_k$ relative to $\mathscr{b}_{N_k}$

From the definition of C^{rj}, the element $C^{kN_k}_{su}$ is given by

$$C^{kN_k}_{su} = \mathbf{b}^k_s \cdot \mathbf{b}^{N_k}_u \tag{7}$$

permitting, in combination with Eq. (6), the determination of the elements

$$C^{kN_k}_{11} = \cos\gamma_k + (g^k_1)^2 (1 - \cos\gamma_k) \tag{8}$$

$$C^{kN_k}_{12} = g^k_3 \sin\gamma_k + g^k_1 g^k_2 (1 - \cos\gamma_k) \tag{9}$$

$$C^{kN_k}_{13} = -g^k_2 \sin\gamma_k + g^k_1 g^k_3 (1 - \cos\gamma_k) \tag{10}$$

$$C^{kN_k}_{21} = -g^k_3 \sin\gamma_k + g^k_1 g^k_2 (1 - \cos\gamma_k) \tag{11}$$

$$C^{kN_k}_{22} = \cos\gamma_k + (g^k_2)^2 (1 - \cos\gamma_k) \tag{12}$$

$$C^{kN_k}_{23} = g^k_1 \sin\gamma_k + g^k_2 g^k_3 (1 - \cos\gamma_k) \tag{13}$$

$$C^{kN_k}_{31} = g^k_2 \sin\gamma_k + g^k_1 g^k_3 (1 - \cos\gamma_k) \tag{14}$$

$$C^{kN_k}_{32} = -g^k_1 \sin\gamma_k + g^k_2 g^k_3 (1 - \cos\gamma_k) \tag{15}$$

$$C^{kN_k}_{33} = \cos\gamma_k + (g^k_3)^2 (1 - \cos\gamma_k) \tag{16}$$

Although the validity of Eq. (4) can be established by expansion and comparison with Eqs. (8)–(16), this step may for some readers be more apparent with the intervening representation of Eqs. (8)–(16) in indicial or Cartesian tensor notation. With the definitions of the Levi-Civita symbol ϵ_{abc} (Def. 23) and the Kronecker delta symbol (Def. 22), one can record the elements of C^{kN_k} generically as

$$C^{kN_k}_{su} = \delta_{su} \cos\gamma_k + \epsilon_{suv} g^k_v \sin\gamma_k + g^k_s g^k_u (1 - \cos\gamma_k) \qquad (u, v = 1, 2, 3) \tag{17}$$

and then recognize the validity of Eq. (4). Finally, one might prefer to construct directly from Eqs. (6) and (7) the *direction cosine dyadic*

$$\mathbf{C}^{kN_k} = \mathbf{U} \cos\gamma_k - \mathbf{U} \times \mathbf{g}^k \sin\gamma_k + \mathbf{U} \cdot \mathbf{g}^k \mathbf{g}^k (1 - \cos\gamma_k) \tag{18}$$

such that

$$C^{kN_k}_{su} = \mathbf{b}^k_s \cdot \mathbf{C}^{kN_k}_{su} \cdot \mathbf{b}^{N_k}_u \tag{19}$$

Any of these approaches can be used to verify Eq. (4), and perhaps to provide useful information or perspective as well. Equation (5) is, of course, available directly by transposition of Eq. (4), using

$$C^{N_k k} = (C^{kN_k})^T \tag{20}$$

Equations (4) and (5) provide the direction cosine matrices between contiguous bodies only, and the equations of motion are more conveniently expressed in terms of the more general matrices C^{rj} $(r, j \in \mathscr{B})$ provided in Def. 29. A direction cosine matrix relating the orientations of two bodies connected by a chain of intermediate bodies can be written as the ordered product of the direction cosine matrices relating contiguous pairs of bodies in the chain. This product can be written symbolically as

$$C^{rj} = \Pi_{k=r}^{N_{jr}} C^{kN_{kj}} \tag{21}$$

where the symbol $\Pi_{k=r}^{N_{jr}}$ is understood to imply the following algorithm:

(1) Define $p \triangleq N_{rj}$ and construct C^{rp} from Eq. (4) if $r > p$ and from Eq. (5) if $r < p$.

(2) Define $q \triangleq N_{pj}$ and construct C^{pq} from Eq. (4) if $p > q$ and from Eq. (5) if $p < q$.

(3) Proceed until an integer u emerges such that $j \triangleq N_{uj}$, finally constructing C^{uj} from Eq. (4) if $u > j$ and from Eq. (5) if $u < j$.

(4) Multiply the matrices obtained in the sequence

$$C^{rj} = C^{rp}C^{pq} \cdots C^{uj} \tag{22}$$

D. System Specification

Once the appropriate equations from the three preceding sections are embodied in a computer program, the user need not deal directly with all of the symbols and concepts introduced in the formulation of these equations. He can provide the computer with the very limited body of input required to specify his system, and then concentrate on interpreting the numerical integration output. Although the specific problems of programming the digital computer to process the input are deferred to a later report, the definition of the required input is to be established immediately.

It is assumed that the engineer in his wisdom and experience has devised for his physical system a mathematical model consisting of a system of $n + 1$ bodies interconnected by n hinges (such as that illustrated in Fig. 4). He has labeled the bodies as indicated by Def. 4 (Section IIA), and fixed in his mind a nominal configuration (for which $\gamma_k = 0$ for $k \epsilon \mathcal{P}$ and $\mathbf{b}_\alpha^k = \mathbf{b}_\alpha^j$ for $j, k \epsilon \mathcal{B}$ and $\alpha = 1, 2, 3$). He must then provide the computer with the following information:

Computer input (required)

(1) The integer n (see Def. 1).

(2) The n network elements $N_k (k \epsilon \mathcal{P})$ (see Def. 26).

(3) For $k \epsilon \mathcal{B}$ and $j \epsilon \mathcal{B}_k$, the 3 by 1 matrices p^{kj} (see Def. 13).

(4) The n 3 by 1 matrices g^k for $k \epsilon \mathcal{P}$ (see Def. 14).

(5) The $n + 1$ masses m_k for $k \epsilon \mathcal{B}$ (see Def. 17).

(6) The $n + 1$ 3 by 3 inertia matrices I^k for $k \epsilon \mathcal{B}$ (see Def. 18).

(7) For $k \in \mathcal{B}$, the $2(n+1)$ 3 by 1 matrices F^k and T^k, either as explicit functions of system variables (possibly zero) or in the form of differential equations characterizing control laws (see Defs. 19 and 20).

(8) For $k \in \mathcal{P}$, the n scalar functions τ_k, either as explicit functions of system variables (possibly zero) or as differential equations representing control laws (see Def. 21).

(9) Initial values for the $2n$ scalars γ_k and $\dot{\gamma}_k$ for $k \in \mathcal{P}$ (see Def. 15), the nine elements of the direction cosine matrix C (see Def. 9) and the three elements of the angular velocity matrix ω^0 (see Def. 10).

(10) Initial values of X and $\dot{X}$, to be included if $F = \mathcal{M}\ddot{X}$ (Eq. 2) is included in the simulation program. More often, Eq. (2) will be solvable for vehicle mass center motion $X(t)$ without reference to the rotational equations (Eq. 1), and $X(t)$ will be input to the rotational dynamics program as an explicit function of time. Very often, Eq. (1) will not involve $X(t)$ or its time derivatives, and then, of course, no input regarding X is required.

Computer input (optional)

(1) A set of attitude variables (attitude angles, Euler parameters, or other kinematical quantities) to characterize the inertial attitude of $\mathscr{b}_0$; if this option is exercised, an expression must be provided in terms of these variables for the direction cosine matrix C introduced in Def. 9, and initial conditions on these variables and their derivatives (consistent with constraints) might be adopted rather than initial values of C and ω^0. Equation (3) can then be replaced by a kinematical equation appropriate for the variables selected.

(2) Any kinematically prescribed variables, such as $\gamma_j(t)$ for some $j \in \mathcal{P}$. As noted in Section IIB, it is possible to modify Eq. (1) so as to accommodate the substitution of an explicit function of time for a variable, such as $\gamma_j(t)$, which in Eq. (1) is treated as an unknown. If this option is elected, the computer program input must include the specification of variables to be prescribed and the appropriate time functions.

No additional input is required. The computer can be programmed to construct numerically all intermediate concepts (such as the neighbor sets $\mathcal{B}_k$ and the remaining network elements N_{kr} for $k, r \in \mathcal{P}$, the path elements ε_{sk} for $k \in \mathcal{B}, s \in \mathcal{P}$, the barycentric position matrices D^{kk} and D^{kj} for $k, j \in \mathcal{B}$, the augmented body inertia matrices Φ^{kk}, and the inertia-like matrices Φ^{kj} for $k, j \in \mathcal{B}$). The computer can then evaluate numerically the functions displayed in Eq. (1) and, with initial values prescribed, accomplish the numerical integrations required for Eq. (1) and any necessary kinematical equations (such as Eq. 3). This process is illustrated in general terms by example in the next section. Explicit scalar equations of motion are obtained from Eq. (1) for a simple three-body system in Appendix B.

E. Sample Problem Formulation

Consider once again the eleven-body system portrayed in Fig. 4. Imagine that equations from the preceding sections have been programmed for digital computations. In this section, an outline will be provided for the specific input required by this system, and the internal operations of the computer will be traced conceptually to the point of integrating equations of motion and printing out results.

According to the input list in the preceding section, the following system parameters would suffice for a simulation of the vehicle model illustrated in Fig. 4:

Required input

(1) $n = 10$

(2) $N_1 = 0$ $\quad N_6 = 5$

$N_2 = 1$ $\quad N_7 = 6$

$N_3 = 2$ $\quad N_8 = 7$

$N_4 = 3$ $\quad N_9 = 8$

$N_5 = 0$ $\quad N_{10} = 6$

(3) Numerical values for the 3 by 1 matrices p^{01}, p^{05}, p^{10}, p^{12}, p^{21}, p^{23}, p^{32}, p^{34}, p^{43}, p^{50}, p^{56}, p^{65}, p^{67}, $p^{6,10}$, p^{76}, p^{78}, p^{87}, p^{89}, p^{98}, $p^{10,6}$. In this case, $p^{21} = p^{23} = p^{32} = p^{34} = p^{43} = 0$, and other values are nonzero, but are not recorded here.

(4) Numerical values for the ten 3 by 1 matrices g^k for $k = 1, 2, \cdots, 10$. Here we adopt $g^3 = g^5 = g^6 = g^7 = g^8 = [0\ -1\ 0]^T$, $g^{10} = [-1\ 0\ 0]^T$, $g^9 = [-\sqrt{2}/2\ \sqrt{2}/2\ 0]^T$, and $g^1 = g^2 = g^4 = [0\ 0\ 1]^T$.

(5) Numerical values for the eleven scalars m_k for $k = 0, 1, 2, \cdots, 10$, not fully recorded here. In this example, the masses m_2 and m_3 are taken as zero in order to illustrate the possibility of simulating 3 deg of rotational freedom between $\mathscr{b}_4$ and $\mathscr{b}_1$.

(6) Numerical values for the eleven 3 by 3 matrices I^k for $k = 0, 1, 2, \cdots, 10$, not fully recorded here. In this example, I^2 and I^3 are taken as zero, and all matrices except I^9 are assumed diagonal.

(7) Eleven values for each of the 3 by 1 matrices F^k and T^k, for $k = 0, 1, \cdots, 10$. Although these are often complex functions representing environmental interactions and controller influences, in this example, the system is assumed to be free of all external forces, so that $F^k = T^k = 0$ for all $k \in \mathscr{B}$.

(8) Ten scalars $\tau_1, \cdots, \tau_{10}$, which for this example are given by $\tau_\alpha = -k_\alpha \gamma_\alpha$ for $\alpha = 5, 6, 7, 8, 9, 10$ and $\tau_\beta = 0$ for $\beta = 2, 3, 4$. The scalar τ_1 is unspecified here because γ_1 will be prescribed as an explicit function of time.

(9) Initial values for $\gamma_1, \cdots, \gamma_{10}, \dot{\gamma}_1, \cdots, \dot{\gamma}_{10}$, C and ω^0, which for this example are given by $\gamma_2 = \gamma_3 = \gamma_4 = 0$, $\gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = \gamma_{10} = \pi/8$ rad, $\dot{\gamma}_4 = 10\pi$ rad/s, $\dot{\gamma}_2 = \dot{\gamma}_3 = \dot{\gamma}_5 = \dot{\gamma}_6 = \dot{\gamma}_7 = \dot{\gamma}_8 = \dot{\gamma}_9 = \dot{\gamma}_{10} = 0$, $C = U$, and $\omega^0 = 0$. The function $\gamma_1(t)$ is prescribed, as noted below.

(10) Initial values of X and $\dot{X}$; in this example, these are irrelevant because they do not appear in Eq. (1), and it is assumed that mass center trajectory data can be obtained more efficiently from a special-purpose program for the integration of Eq. (2).

Optional input

(1) Inertial attitude variables for $\mathscr{b}_0$, here assumed to be the set of body three-axis angles $\theta_1, \theta_2, \theta_3$ so defined that $\mathbf{b}^0_\alpha = \mathbf{i}_\alpha$ when $\theta_1 = \theta_2 = \theta_3$ for $\alpha = 1, 2, 3$, and in general, the attitude of $\{\mathbf{b}^0\}$ can be obtained by rotating $\{\mathbf{b}^0\}$ from a state of alignment with $\{\mathbf{i}\}$ first through an angle θ_1 about $\mathbf{i}_1 = \mathbf{b}^0_1$, then through

an angle θ_2 about the newly displaced $\mathbf{b}_2^0$, and finally, through an angle θ_3 about $\mathbf{b}_3^0$. The resulting direction cosine matrix is given by (Ref. 26, p. 61, after transposition)

$$C = \begin{bmatrix} c_2c_3 & c_2s_3 & -s_2 \\ s_1s_2c_3 - s_3c_1 & s_1s_2s_3 + c_3c_1 & c_1s_2s_3 - c_3s_1 \\ c_1s_2c_3 + s_3s_1 & s_1c_2 & c_1c_2 \end{bmatrix} \tag{23}$$

where $c_\alpha \triangleq \cos\theta_\alpha$ and $s_\alpha \triangleq \sin\theta_\alpha$.

Although one could simply input Eq. (23) into a computer program designed to integrate the kinematical equation $\dot{C} = -\tilde{\omega}^0 C$ (Eq. 3), this option has little merit when contrasted with the alternative of replacing Eq. (3) in the program with

$$\dot{\theta} = P\omega^0 \tag{24}$$

where $\dot{\theta} \triangleq [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3]^T$ and (from Ref. 26, p. 147, with $P \triangleq M^{-1^T}$)

$$P \triangleq \frac{1}{c_2}\begin{bmatrix} c_2 & s_1s_2 & c_1s_2 \\ 0 & c_1c_2 & -s_1c_2 \\ 0 & s_1 & c_1 \end{bmatrix}$$

Equation (23) is then no longer required input in this example since it does not appear in Eq. (1).

Note that P is undefined for $c_2 = 0$, that is, for $\theta_2 = \pi/2, 3\pi/2$, etc. This singularity is characteristic of all three-variable systems for attitude description. The choice of the particular set of attitude angles adopted for this example is based on the conviction that, for the given parameters and initial conditions, the variable θ_2 will remain away from those values for which P is singular.

(2) Kinematically prescribed variables, in this case being $\gamma_1 = \sin St$ rad, with S given a numerical value not recorded here.

This completes the input required by the computer. Since, ultimately, the differential equations, (1) and (24), must be integrated, the computer must generate numerical values for all parameters appearing in these equations. Before this process is outlined, the equations to be integrated will be rewritten as a single system of first-order differential equations, in a form most suitable for computer operations.

Define the 24 by 1 matrix $x \triangleq [\theta_1\ \theta_2\ \theta_3\ \omega_1^0\ \omega_2^0\ \omega_3^0\ \gamma_2\ \dot{\gamma}_2\ \cdots\ \gamma_{10}\ \dot{\gamma}_{10}]^T$ and recast the required equations as the state equation

$$V\dot{x} = W \tag{25}$$

Note that the kinematically prescribed variables γ_1 and $\dot{\gamma}_1$ have been excluded from the state variable x. The second-order equations in Eq. (1) have been cast as first-order equations in Eq. (25) by the expedience of including $\dot{\gamma}_2 \stackrel{\Delta}{=} x_8, \dot{\gamma}_3 \stackrel{\Delta}{=} x_{10}, \cdots, \dot{\gamma}_{10} \stackrel{\Delta}{=} x_{24}$ in the state variable x. This step necessitates the inclusion in Eq. (25) of identities such as $x_8 = \dot{x}_7$ (or $\dot{\gamma}_2 = \dot{\gamma}_2$). For the particular arrangement of variables indicated for x, the matrices $V\dot{x}$ and W in Eq. (25) are, for equivalence with Eqs. (1) and (24), given by

$$V\dot{x} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
U & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & a_{00} & 0 & a_{02} & 0 & a_{03} & \cdots & 0 & a_{0,10} \\
\hline
000 & 000 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
000 & a_{20} & 0 & a_{22} & 0 & a_{23} & \cdots & 0 & a_{2,10} \\
000 & 000 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
000 & a_{30} & 0 & a_{32} & 0 & a_{33} & \cdots & 0 & a_{3,10} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
000 & 000 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
000 & a_{10,0} & 0 & a_{10,2} & 0 & a_{10,3} & \cdots & 0 & a_{10,10}
\end{array}\right]
\left[\begin{array}{c}
\dot{\theta} \\
\hline
\dot{\omega}^0 \\
\hline
\dot{\gamma}_2 \\
\ddot{\gamma}_2 \\
\dot{\gamma}_3 \\
\ddot{\gamma}_3 \\
\vdots \\
\dot{\gamma}_{10} \\
\ddot{\gamma}_{10}
\end{array}\right] \tag{26}$$

and

$$W \stackrel{\Delta}{=} \left[\begin{array}{c}
P_{\omega^0} \\
\hline
\sum\limits_{k\in\mathcal{B}} C^{0k}A^k - a_{01}\ddot{\gamma}_1 \\
\hline
\dot{\gamma}_2 \\
g^{2T} \sum\limits_{k\in\mathcal{P}} \varepsilon_{2k} C^{2k} A^k + \tau_2 - a_{21}\ddot{\gamma}_1 \\
\dot{\gamma}_3 \\
g^{3T} \sum\limits_{k\in\mathcal{P}} \varepsilon_{3k} C^{3k} A^k + \tau_3 - a_{31}\ddot{\gamma}_1 \\
\vdots \\
\dot{\gamma}_{10} \\
g^{10T} \sum\limits_{k\in\mathcal{P}} \varepsilon_{10,k} C^{10,k} A^k + \tau_{10} - a_{10,1}\ddot{\gamma}_1
\end{array}\right] \tag{27}$$

Of course, the rows of V (and of W) could be shifted around by adopting a different ordering of the elements of x. The indicated choice is governed by considerations that become evident in Section III.

The only task remaining in preparation for integration is the identification of procedures for evaluating the elements of V and W from the input. Functions such as $P, \tau_2, \tau_{10}, g^2, g^{10}$ are given explicitly as input, and need no further consideration. Direction cosine matrices appearing explicitly in W are obtained from Eq. (21) or the algorithm following that equation, and to apply that algorithm, the network elements $N_{kr}\,(k, r \in \mathcal{B})$ are required. These quantities are provided as input for $r = 0$, and are generated by the computer from knowledge of the branch sets $\mathcal{B}_{kj}$ ($k \in \mathcal{B}$, and $j \in \mathcal{B}_k$), which in turn are obtained from the neighbor sets $\mathcal{B}_k$ ($k \in \mathcal{B}$), which in turn are available from the input quantities N_k $(= N_{k0})$.

Specifically, the neighbor sets $\mathcal{B}_k$ may be obtained from the input network elements N_k by the following interpretations of Def. 5, written here both verbally and in terms of set theory symbolism: $\mathcal{B}_0$ is the set of all integers j such that $N_j = 0$ (or $\mathcal{B}_0 = \{j | N_j = 0\}$), and for $k \epsilon \mathcal{P}$, $\mathcal{B}_k$ is N_k plus the set of all integers j such that $N_j = k$ (or $\mathcal{B}_k = \{N_k + \{j | N_j = k\}\}$ for $k \epsilon \mathcal{P}$).

By scanning the input values of N_1 through N_{10} for this sample problem, the computer can readily find the following neighbor sets:

$\mathcal{B}_0 = \{1,5\}$

$\mathcal{B}_1 = \{0,2\}$

$\mathcal{B}_2 = \{1,3\}$

$\mathcal{B}_3 = \{2,4\}$

$\mathcal{B}_4 = \{3\}$

$\mathcal{B}_5 = \{0,6\}$

$\mathcal{B}_6 = \{5,7,10\}$

$\mathcal{B}_7 = \{6,8\}$

$\mathcal{B}_8 = \{7,9\}$

$\mathcal{B}_9 = \{8\}$

$\mathcal{B}_{10} = \{6\}$

The validity of these results can be established by inspection of Fig. 4, with Def. 5 in mind.

As a second major step, the elements of the $2n$ branch sets $\mathcal{B}_{kj}$ can be obtained by the following general algorithm:

(1) Set $k = 0$, and from the previous calculation identify the elements of $\mathcal{B}_k$ as $a_1, a_2, \cdots, a_{n_k}$, where n_k is the number of elements in $\mathcal{B}_k$.

(2) Set $q = 1$, and record the calculated elements in $\mathcal{B}_{a_q}$; then identify the elements of $\{\mathcal{B}_{a_q} - k\}$ as $b_1, b_2, \cdots, b_{n_{a_q}}$, where n_{a_q} is the number of elements in $\{\mathcal{B}_{a_q} - k\}$.

(3) Set $r = 1$, and obtain the neighbor set $\mathcal{B}_{b_r}$ from the calculations; then identify the elements of $\{\mathcal{B}_{b_r} - a_q\}$ as $c_1, c_2, \cdots, c_{n_{b_r}}$, where n_{b_r} is the number of elements in $\{\mathcal{B}_{b_r} - a_q\}$.

(4) Proceed as in (2) and (3), finding $\mathcal{B}_{c_s}$ for $s = 1$ and identifying the elements of $\{\mathcal{B}_{c_s} - b_r\}$ as $d_1, d_2, \cdots, d_{n_{c_s}}$, where n_{c_s} is the number of elements in $\{\mathcal{B}_{c_s} - b_r\}$, proceeding in this manner, identifying elements $e_1, \cdots, e_{n_{d_t}}$ and $f_1, \cdots, f_{n_{e_u}}$, etc., until an empty set is encountered. (This sequence of operations cannot involve more than n steps.)

(5) Upon obtaining an empty set, say the set $\{\mathcal{B}_{f_v} - e_u\}$ for $v = 1$, change the most recently generated index (here v) from 1 to 2, and proceed with the indicated sequence of operations until an empty set is again obtained. Then set the most recent index (here v) to 3 and proceed again to an empty set. Continue this process until $v = n_{e_u}$ has been considered.

(6) Change u from 1 to 2, and repeat the process from the point at which u was previously given value 1. Continue until $u = n_{d_t}$ has been considered.

(7) Change t from 1 to 2 and repeat, continuing in this manner until the index r introduced in step (3) exhausts the n_{a_q} elements of the set $\{\mathcal{B}_{a_q} - k\}$, for $q = k = 1$.

(8) Populate the set $\mathcal{B}_{ka_q}$ with the elements of the following sets:

$$\{a_q\}$$

$$\{\mathcal{B}_{a_q} - k\}$$

$$\sum_{r=1}^{n_{a_q}} \{\mathcal{B}_{b_r} - a_q\}$$

$$\sum_{r=1}^{n_{a_q}} \sum_{s=1}^{n_{b_r}} \{\mathcal{B}_{c_s} - b_r\}$$

$$\vdots$$

$$\sum_{r=1}^{n_{a_q}} \sum_{s=1}^{n_{b_r}} \cdots \sum_{v=1}^{n_{e_u}} \{\mathcal{B}_{f_v} - e_u\}$$

At this point, the branch set $\mathcal{B}_{0a_1}$ has been obtained.

(9) Return to step (2), change q from 1 to 2, and repeat the process to find $\mathcal{B}_{0a_2}$, repeating again until $\mathcal{B}_{0a_{n_0}}$ is obtained.

(10) Return to step (1), change k from 0 to 1, and repeat the entire process until the final branch set $\mathcal{B}_{na_{n_n}}$ is obtained.

Although the indicated algorithm for finding the $2n$ branch sets may appear to be lengthy and arduous, it is exactly this process that a man accomplishes in a moment's time when he examines a sketch such as Fig. 4 and records the branch sets "by inspection." If you will close your eyes to the sketch in Fig. 4 and consider only the neighbor sets generated for that system from item (2) of the computer input, you will be able to obtain the following branch sets from the algorithm provided:

$\mathcal{B}_{01} = \{1, 2, 3, 4\}$

$\mathcal{B}_{05} = \{5, 6, 7, 8, 9, 10\}$

$\mathcal{B}_{1,0} = \{0, 5, 6, 7, 8, 9, 10\}$

$\mathcal{B}_{12} = \{2, 3, 4\}$

$\mathcal{B}_{21} = \{1, 0, 5, 6, 7, 8, 9, 10\}$

$\mathcal{B}_{23} = \{3, 4\}$

$\mathcal{B}_{32} = \{2, 1, 0, 5, 6, 7, 8, 9, 10\}$

$\mathcal{B}_{34} = \{4\}$

$\mathcal{B}_{43} = \{3, 2, 1, 0, 5, 6, 7, 8, 9, 10\}$

$\mathcal{B}_{50} = \{0, 1, 2, 3, 4\}$

$\mathcal{B}_{56} = \{6, 7, 8, 9, 10\}$

$\mathcal{B}_{65} = \{5, 0, 1, 2, 3, 4\}$

$\mathcal{B}_{67} = \{7, 8, 9\}$

$\mathcal{B}_{6,10} = \{10\}$

$\mathcal{B}_{76} = \{6, 5, 0, 1, 2, 3, 4\}$

$\mathcal{B}_{78} = \{8, 9\}$

$\mathcal{B}_{87} = \{7, 6, 5, 0, 1, 2, 3, 4\}$

$\mathcal{B}_{89} = \{9\}$

$\mathcal{B}_{98} = \{8, 7, 6, 5, 0, 1, 2, 3, 4, 10\}$

$\mathcal{B}_{10,6} = \{6, 5, 0, 1, 2, 3, 4, 7, 8, 9\}$

These results are easily confirmed by examination of Fig. 4. By having the computer generate this information internally, we minimize user input and user error.

The next step is construction of the network elements N_{kr} for $r \epsilon \mathcal{P}$ and $k \epsilon \mathcal{B}$; these elements were input for $r = 0$, since $N_k \triangleq N_{k0}$. This is a simple matter since, from the definition, $N_{kk} \triangleq k$ and, for $r \neq k$, $N_{kr} = j$ for $r \epsilon \mathcal{B}_{kj}$. Hence, for a given k and r, one merely searches $\mathcal{B}_{ks}$ for $s = 0, 1, \cdots$ until one finds a value $s = j$ for which $\mathcal{B}_{ks}$ includes r, and then assigns this value j of s to N_{kr}. In application to Fig. 4, the result is as shown in Table 1, which includes the input N_{k0}.

Table 1. Network elements N_{kr} for Fig. 4, $k, r \epsilon \mathcal{B}$

$\downarrow k$ \ $\vec{r}$	0 (input)	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	5	5	5	5	5	5
1	0	1	2	2	2	0	0	0	0	0	0
2	1	1	2	3	3	1	1	1	1	1	1
3	2	2	2	3	4	2	2	2	2	2	2
4	3	3	3	3	4	3	3	3	3	3	3
5	0	0	0	0	0	5	6	6	6	6	6
6	5	5	5	5	5	5	6	7	7	7	10
7	6	6	6	6	6	6	6	7	8	8	6
8	7	7	7	7	7	7	7	7	8	9	7
9	8	8	8	8	8	8	8	8	8	9	8
10	6	6	6	6	6	6	6	6	6	6	6

With the network elements available, the computer can, by means of the algorithm culminating in Eq. (22), construct any direction cosine matrix appearing in Eqs. (25)–(27) as a product of matrices whose numerical values are available in terms of input parameters g^k and kinematical variables γ_k ($k \epsilon \mathcal{B}$) from Eq. (4), Eq. (5), or Eqs. (8)–(16). For example, Eq. (22) provides

$$C^{27} = C^{21}C^{10}C^{05}C^{56}C^{67}$$

$$C^{5,10} = C^{56}C^{6,10}$$

$$C^{40} = C^{43}C^{32}C^{21}C^{10}$$

and each of the matrices on the right sides is available from Eq. (4) or Eq. (5) (or its elements can be written directly from Eqs. 8–16). For example, from Eq. (4),

$$C^{21} = U \cos \gamma_2 - \tilde{g}^2 \sin \gamma_2 + g^2 g^{2T} (1 - \cos \gamma_2)$$

and from Eq. (5),

$$C^{05} = U \cos \gamma_5 + \tilde{g}^5 \sin \gamma_5 + g^5 g^{5T} (1 - \cos \gamma_5)$$

In addition to the direction cosines, Eqs. (25)–(27) involve many other functions to be generated internally by the computer. Among these are the path elements ε_{sk} for $s \in \mathscr{P}$ and $k \in \mathscr{B}$. These are readily determined from the branch sets $\mathscr{B}_{rs}$ $(r \in \mathscr{B}, s \in \mathscr{P})$ and the network elements N_s (for $s \in \mathscr{P}$) as follows: If $k \in \mathscr{B}_{N_s s}$, then $\varepsilon_{sk} = 1$; otherwise, $\varepsilon_{sk} = 0$. In application to the system of Fig. 4, this algorithm provides the results shown in Table 2.

Table 2. Path elements ε_{sk} for Fig. 4, $s \in \mathscr{P}$, $k \in \mathscr{B}$

$\downarrow s$ \ $\vec{k}$	0	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	0	0	0
3	0	0	0	1	1	0	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0	0	0
5	0	0	0	0	0	1	1	1	1	1	1
6	0	0	0	0	0	0	1	1	1	1	1
7	0	0	0	0	0	0	0	1	1	1	0
8	0	0	0	0	0	0	0	0	1	1	0
9	0	0	0	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	0	0	1

All that remains for the computation of the terms in Eqs. (25)–(27) is the evaluation of the matrices A^k $(k \in \mathscr{B})$ shown in Eq. (27), and the evaluation of the symbols appearing in the upper left partition of V in Eq. (26). All of these symbols represent functions for which explicit expressions are provided in Defs. 38–41 (Section IIA). These definitions, in turn, involve symbols requiring reference to the preceding Defs. 33 and 35–37, and these require additional reference to Defs. 30–32; but finally, all quantities are available in terms of those provided in the input. Thus, the coefficient matrices V and W in Eq. (25) can be evaluated numerically for time zero, and the numerical integration process can begin.

The discussion of this example is continued in Section IIIE, where partially linearized equations are recorded and coordinate transformations and truncations are considered.

III. Hybrid-Coordinate Equations of Motion

A. Rationale

In many applications, it can be anticipated that some of the kinematical variables appearing in Eq. (1) will remain in some sense "small." It may be, for example, that some of the angles $\gamma_1, \cdots, \gamma_n$ represent relative rotations of contiguous bodies connected by an *analytical hinge*, rather than a *physical hinge* (see Fig. 3, for example). Then, the characteristics of the hinge (stiffness, etc.) may be based upon an idealization which retains its validity only for small structural strains. Nonlinear terms in the angle of relative rotation then become meaningless and should be dismissed, with the understanding that solutions indicating large values of such variables are also meaningless. Further argument must be advanced in order to justify the additional assumption that nonlinear terms in the time derivatives of these small angles are also negligible; this step would be justified, for example, if it could be anticipated that the angle would experience only low-frequency oscillations.

Linearization of kinematical variables and their time derivatives might alternatively be justified in an entirely different way, relying upon mathematical theorems rather than physically based arguments. If an exact solution of Eq. (1) can be found, then this solution can be adopted as a nominal motion. By transforming to a new set of kinematical variables which describe the deviation from the nominal motion, and linearizing in these variables and their time derivatives, one can obtain a set of linear differential equations which, in many cases, quite rigorously establish the Liapunov stability properties of the nominal motion. The new variables are often called *variational coordinates,* and the transformed equations are then known as the *linearized variational equations.*

In many spacecraft applications, an exact solution of Eq. (1) will not be available, but still there will exist a desired motion which can informally take the place of the nominal solution in the preceding paragraph. The resulting linear equations may have considerable engineering significance, even though they are no longer rigorously indicative of motion stability properties.

Linearization based on mathematical arguments is at best formally indicative of *local* stability properties of solutions; thus, all variational coordinates and their time derivatives are taken to be *arbitrarily* small. Since the sum of a finite number of arbitrarily small quantities is still arbitrarily small, a formal interpretation of linearized variational equations representing the deviation from an exact solution of Eq. (1) does not permit large relative motions of even physically separated bodies of the system.

In engineering applications, however, linearization of a variable is generally considered to be an acceptable practice as long as the linear term in the variable is "substantially larger" than additive terms of higher degree. In this sense, linearization is a process whose range of validity is somewhat ill-defined. For example, if we anticipate that γ_2 will have a solution approximating $\gamma_2 = 0.2 \cos 2t$ rad, then it is quite reasonable to replace $\sin \gamma_2 = \gamma_2 - \gamma_2^3/3! + \gamma_2^5/5! - \cdots$ by γ_2 (since $\sin 0.2 = 0.199 \cong 0.2$) and to replace $\cos \gamma_2 = 1 - \gamma_2^2/2! + \gamma_2^4/4! - \cdots$ by 1 (since $\cos 0.2 = 0.98 \cong 1$); but one might hesitate to replace $\ddot{\gamma}_2 - \dot{\gamma}_2^2 = -0.8 \cos 2t - 0.16 \sin^2 2t$ by $\ddot{\gamma}_2 = -0.8 \cos 2t$.

Evidently subjective judgments are involved in the engineering interpretation of linearized equations. Within this framework, however, one can interpret linear differential equations obtained from Eq. (1) as descriptive of *large gross deformations* of a vehicle experiencing small strains as manifested by small angles of relative motion of contiguous pairs of bodies. If, for example, for the eleven-body system illustrated in Fig. 4, it could be assumed that angles $\gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9$, and γ_{10} remain "small," then equations of motion linearized in these variables would still permit the dynamical description of "large" relative rotations of physically separated bodies such as $\mathscr{b}_9$ and $\mathscr{b}_0$, which might experience rotations on the order of magnitude of 1 rad (0.2 rad for each of five hinge angles), within the reasonable limits of the equations. If there were fifty bodies in the chain rather than five as in the example cited, large gross deformations could be accommodated even if every relative rotation angle remained well below 0.1 rad.

When any justification can be found for the linearization of any of the kinematical variables appearing in Eq. (1), this step should be taken, since the mathematical and computational benefits of linearization are quite substantial. The primary immediate benefit is the elimination of variable terms from the coefficient matrix on the left side of Eq. (1) (see Defs. 38–40). If ω^0 and *all* angles $\gamma_1, \cdots, \gamma_n$ and their time derivatives can be assumed to remain *arbitrarily* small, then the indicated coefficient matrix is formally constant, and much computer simulation time can be gained by computing its inverse only once, in advance of numerical integration. When the angles are expected to be small in an engineering sense, less rigorous arguments can be marshalled to gain the same computational advantage.

Potentially more significant advantages can be gained when coordinate transformations can be found which permit the substitution for the linearized variables of new coordinates, some of which may with impunity be deleted entirely from the dynamical description. In order for this coordinate truncation to be justifiable, the new coordinates must be *distributed* or *modal* coordinates which are to some degree uncoupled, and some of these coordinates must be demonstrably inconsequential to the dynamic response. The subject of coordinate transformation and truncation has been explored extensively (see Refs. 21 and 25, for example), and previous results can be applied directly once some or all of the scalar equations implied by Eq. (1) have been written in linearized form.

If all of the kinematical variables in Eq. (1) are linearized, then the transformed coordinates are *vehicle normal mode coordinates*, while if only a subset of the kinematical variables in Eq. (1) are linearized, then a *hybrid-coordinate* formulation results.

B. Partial Linearization of Discrete-Coordinate Equations

The immediate objective is to isolate a group of unknown kinematical variables appearing in the rotational equations of motion (Eq. 1), and to linearize these equations in those variables and their time derivatives.

In the special case in which all angles of relative motion between contiguous bodies remain small, it becomes advantageous to introduce a chain of three imaginary, massless bodies connected at one end to one of the rigid bodies of the system. This rigid body is then labeled $\mathscr{b}_3$, and the chain of imaginary bodies is labeled sequentially $\mathscr{b}_2$, $\mathscr{b}_1$ and $\mathscr{b}_0$. All other rigid bodies of the system are labeled according to previously established conventions, which apply now to both real and imaginary bodies. Unit vectors $\mathbf{g}^3$, $\mathbf{g}^2$, and $\mathbf{g}^1$ are fixed along hinge axes in the chain of imaginary bodies in such a way that in the nominal configuration they are mutually orthogonal and $\mathbf{g}^1 \times \mathbf{g}^2 = \mathbf{g}^3$. The hinge torques $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$, and $\boldsymbol{\tau}^3$ are all set equal to zero. With these interpretations, Eq. (1) continues to apply without change, with the understanding that the "$n + 1$ rigid bodies" to which it applies now include the three imaginary bodies. In order to minimize computations, it is convenient to specify these imaginary bodies as having their mass centers all coincident with the mass center c_3 of $\mathscr{b}_3$, and to let the hinge axes parallel to $\mathbf{g}^3$, $\mathbf{g}^2$, and $\mathbf{g}^1$ all pass through c_3. Figure 7 provides an illustration of a system for which imaginary bodies

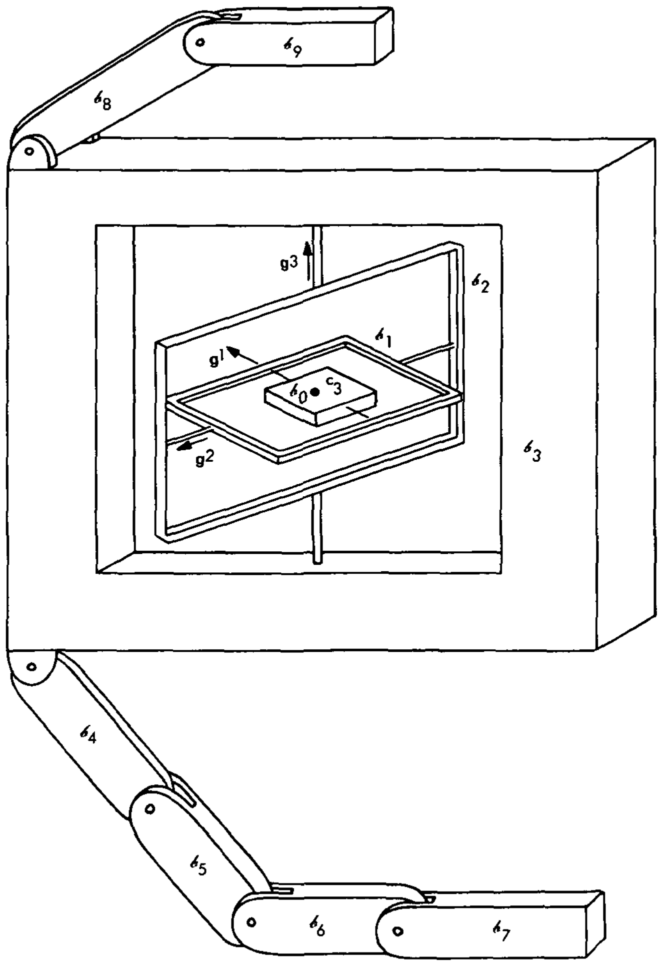

Fig. 7. System with imaginary bodies $\mathscr{b}_0$, $\mathscr{b}_1$, $\mathscr{b}_2$

have been introduced. The new (imaginary) reference body $\mathscr{b}_0$ now serves as a floating reference frame which accommodates the gross motion or mean motion of the collection of bodies in a sense to be established by the coordinate transformations in the sections to follow. The coordinate truncation which follows the transformation eliminates the coordinate redundancy introduced with the creation of three imaginary bodies.

In the more general case, not all of the angles of relative rotation can be assumed (with their time derivatives) to remain small. Then, $\mathscr{b}_0$ is selected as some rigid body of the system which is attached to some other body by a hinge where large-angle rotations are not precluded.

In every case, it will be possible to identify some number of angles (say, ν) for which linearization is in some sense justified, and to collect the indices of these angles in the set $\mathscr{A}$. If all angles of the system defy linearization, then $\nu = 0$ and $\mathscr{A}$ is empty. There is then no recourse but to integrate Eq. (1) directly (numerically, of course). If all of the angles of the system permit linearization (so that "the system" has been augmented to include imaginary bodies), then $\nu = n$ and $\mathscr{A} = \{1, 2, \cdots, n\} = \mathscr{P}$. More generally, we must expect $0 \leq \nu \leq n$.

In any case, Eqs. (1) can be rewritten in the form

$$a_{00}\dot{\omega}^0 + \sum_{k\epsilon\mathcal{P}} a_{0k}\ddot{\gamma}_k = \sum_{k\epsilon\mathcal{B}} C^{0k}A^k \tag{28a}$$

and

$$a_{i0}\dot{\omega}^0 + \sum_{k\epsilon\mathcal{P}} a_{ik}\ddot{\gamma}_k = g^{iT}\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik}C^{ik}A^k + \tau_i \qquad (i\,\epsilon\,\mathcal{P}) \tag{28b}$$

It should be noted that, for a wide class of systems, it can be recognized in advance that $\varepsilon_{ik} = 0$ for $i\,\epsilon\,\mathcal{A}$ and $k\,\epsilon\,\mathcal{P} - \mathcal{A}$. This is the case whenever none of the hinge points whose indices comprise $\mathcal{A}$ lies on a path from $\mathscr{b}_0$ to any hinge point whose index is not in $\mathcal{A}$. In physical terms, this means that the small-angle rotations are confined to *terminal appendages*, each of which is attached to only one body whose index lies outside of $\mathcal{A}$. In programming, it may be convenient to have the option to declare an appendage terminal and reduce the sum in Eq. (28) from the range $k\,\epsilon\,\mathcal{P}$ to the range $k\,\epsilon\,\mathcal{A}$.

If in the example depicted as Fig. 4, the angles $\gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9$, and γ_{10} remain small enough to justify linearization, then $\nu = 6$ and (since the flexible substructure is a terminal appendage), whenever $i = 5, \cdots, 10$, the summations over the set $k\,\epsilon\,\mathcal{P}$ in Eq. (28) could be replaced by sums over $\mathcal{A}$.

If, on the other hand, it became necessary to permit nonlinear terms in γ_{10} (still keeping $\gamma_5, \cdots, \gamma_9$ small), then one would have $\nu = 5$, and nonzero summations over the set $\mathcal{P} - \mathcal{A}$. (The path elements $\varepsilon_{6,10}$ and $\varepsilon_{5,10}$ would be nonzero, since $\mathscr{h}_6$ and $\mathscr{h}_5$ lie on the path between $\mathscr{b}_0$ and $\mathscr{b}_{10}$.)

The next step is the linearization of Eq. (28) in the ν variables γ_k and their time derivatives for $k\,\epsilon\,\mathcal{A}$. To this end, we expand each symbol in Eq. (28) which may involve such variables into three parts, the first being free of these variables (indicated by overbar), the second being linear in the variables (indicated by overcaret), and the third containing terms above the first degree in the variables (indicated by three dots). In particular, we write (for any i, k)

$$C^{ik} = \overline{C}^{ik} + \hat{C}^{ik} + \cdots \tag{29a}$$

$$\tau_\alpha = \bar{\tau}_\alpha + \hat{\tau}_\alpha + \cdots \tag{29b}$$

$$a_{ik} = \overline{a}_{ik} + \hat{a}_{ik} + \cdots \tag{29c}$$

$$A^k = \overline{A}^k + \hat{A}^k + \cdots \tag{29d}$$

and then determine explicit expressions for the new barred and careted symbols from the definitions of a_{ik} (Defs. 38–40), A^k (Def. 41), and the expansion for C^{ik} (Eq. 21). In terms of the symbols introduced in Eq. (29), the linearized form of Eq. (28) becomes

$$(\overline{a}_{00} + \hat{a}_{00})\,\dot{\omega}^0 + \sum_{k\epsilon\mathcal{P}-\mathcal{A}} (\overline{a}_{0k} + \hat{a}_{0k})\,\ddot{\gamma}_k + \sum_{k\epsilon\mathcal{A}} \overline{a}_{0k}\ddot{\gamma}_k = \sum_{k\epsilon\mathcal{B}} [(\overline{C}^{0k} + \hat{C}^{0k})\,\overline{A}^k + \overline{C}^{0k}\hat{A}^k] \tag{30a}$$

$$(\bar{a}_{i0} + \hat{a}_{i0})\,\dot{\omega}^0 + \sum_{k\in\mathcal{B}-\mathcal{A}} (\bar{a}_{ik} + \hat{a}_{ik})\,\ddot{\gamma}_k + \sum_{k\in\mathcal{A}} \bar{a}_{ik}\ddot{\gamma}_k =$$
$$g^{iT} \sum_{k\in\mathcal{P}} \varepsilon_{ik}\,[\bar{C}^{ik}\hat{A}^k + (\bar{C}^{ik} + \hat{C}^{ik})\,\bar{A}^k] + \bar{\tau}_i + \hat{\tau}_i \qquad (i\in\mathcal{P}) \tag{30b}$$

If the flexible substructure is one or more terminal appendages, then in Eq. (30b), whenever $i \in \mathcal{A}$, the sum over the range $k \in \mathcal{P}$ can be reduced to the sum over $k \in \mathcal{A}$, since $\varepsilon_{ik} = 0$ for $k \in \mathcal{P} - \mathcal{A}$.

It is convenient both for immediate computations and for later coordinate transformations to have Eqs. (30) written in matrix form. To this end, we can define column matrices $\gamma^2, \gamma^4, \cdots, \gamma^{N+1}$ (with N an odd number) to consist of angles with sequentially numbered indices within the appendage set $\mathcal{A}$, so that all angles γ_j for $j \in \mathcal{A}$ appear in some matrix γ^k for some even integer k. Similarly, we define $\gamma^1, \gamma^3, \gamma^5, \cdots, \gamma^N$ (for N odd) to accommodate those angles of the system not assumed small, so that all angles γ_j for $j \in \mathcal{P} - \mathcal{A}$ appear in some matrix γ^k for k odd. As a general convention in what follows, an even index on a matrix establishes an identification with a flexible substructure having small angles of relative rotation, and an odd number is identified with a set of unrestricted angles. Equations (30a, b) then take the matrix form shown in Eq. (30c).

$$\left[\begin{array}{c|c|c|c|c|c|c}
\bar{a}^{00} + \hat{a}^{00} & \bar{a}^{01} + \hat{a}^{01} & \bar{a}^{02} & \bar{a}^{03} + \hat{a}^{03} & \cdots & \bar{a}^{0N} + \hat{a}^{0N} & \bar{a}^{0,N+1} \\ \hline
\bar{a}^{10} + \hat{a}^{10} & \bar{a}^{11} + \hat{a}^{11} & \bar{a}^{12} & \bar{a}^{13} + \hat{a}^{13} & \cdots & \bar{a}^{1N} + \hat{a}^{1N} & \bar{a}^{1,N+1} \\ \hline
\bar{a}^{20} + \hat{a}^{20} & \bar{a}^{21} + \hat{a}^{21} & \bar{a}^{22} & \bar{a}^{23} + \hat{a}^{23} & \cdots & \bar{a}^{2N} + \hat{a}^{2N} & \bar{a}^{2,N+1} \\ \hline
\bar{a}^{30} + \hat{a}^{30} & \bar{a}^{31} + \hat{a}^{31} & \bar{a}^{32} & \bar{a}^{33} + \hat{a}^{33} & \cdots & \bar{a}^{3N} + \hat{a}^{3N} & \bar{a}^{3,N+1} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
\bar{a}^{N0} + \hat{a}^{N0} & \bar{a}^{N1} + \hat{a}^{N1} & \bar{a}^{N2} & \bar{a}^{N3} + \hat{a}^{N3} & \cdots & \bar{a}^{NN} + \hat{a}^{NN} & \bar{a}^{N,N+1} \\ \hline
\bar{a}^{N+1,0} + \hat{a}^{N+1,0} & \bar{a}^{N+1,1} + \hat{a}^{N+1,1} & \bar{a}^{N+1,2} & \bar{a}^{N+1,3} + \hat{a}^{N+1,3} & \cdots & \bar{a}^{N+1,N} + \hat{a}^{N+1,N} & \bar{a}^{N+1,N+1}
\end{array}\right]
\left[\begin{array}{c}
\dot{\omega}^0 \\ \hline \ddot{\gamma}^1 \\ \hline \ddot{\gamma}^2 \\ \hline \ddot{\gamma}^3 \\ \hline \vdots \\ \hline \ddot{\gamma}^N \\ \hline \ddot{\gamma}^{N+1}
\end{array}\right] =$$

$$\left[\begin{array}{c}
\sum_{k\in\mathcal{B}} [(\bar{C}^{0k} + \hat{C}^{0k})\,\bar{A}^k + \bar{C}^{0k}\hat{A}^k] - \sum_{k\in\mathcal{G}} a_{0k}\ddot{\gamma}_k \\ \hline
R^1(\omega^0, \gamma^1, \cdots, \gamma^{N+1}, \dot{\gamma}^1, \cdots, \dot{\gamma}^{N+1}, t) \\ \hline
R^2(\omega^0, \gamma^1, \cdots, \gamma^{N+1}, \dot{\gamma}^1, \cdots, \dot{\gamma}^{N+1}, t) \\ \hline
R^3(\omega^0, \gamma^1, \cdots, \gamma^{N+1}, \dot{\gamma}^1, \cdots, \dot{\gamma}^{N+1}, t) \\ \hline
\vdots \\ \hline
R^N(\omega^0, \gamma^1, \cdots, \gamma^{N+1}, \dot{\gamma}^1, \cdots, \dot{\gamma}^{N+1}, t) \\ \hline
R^{N+1}(\omega^0, \gamma^1, \cdots, \gamma^{N+1}, \dot{\gamma}^1, \cdots, \dot{\gamma}^{N+1}, t)
\end{array}\right] \tag{30c}$$

The many previously undefined symbols appearing in Eq. (30c) can be constructed by comparison with Eq. (30b), with the set $k \in \mathcal{G}$ defining given or prescribed variables. The purpose of writing the coefficient matrix on the left side in partitioned form is to facilitate the repeated inversion that is required for numerical integration. Any terminal appendages in the system can always be labeled so that the appendage bodies bear the highest indices in the system; this puts the corresponding angles in the matrix γ^{N+1}. (If only *internal* flexible subsystems are present, there is nothing in γ^{N+1}, and if *all* angles are small then γ^N is empty.) Inspection of the matrix $\bar{a}_{N+1,N+1}$ then reveals that it consists solely of scalar elements $\bar{a}_{ik}$ for $i, k \in \mathcal{A}_{N+1}$ (the terminal appendage set), and it can be shown that these quantities are all constants. The right sides of Eq. (30c) will depend explicitly on t when either torques or kinematical variables are prescribed functions of time.

Both for computational reasons and for purposes of coordinate transformation, it is necessary to replace Eq. (30c) by a first-order matrix differential equation. This can be accomplished in a variety of ways, and the optimum arrangement for computations is often inappropriate for the transformations to be discussed in sections following. Guided primarily by the latter consideration, we define (for $j, k = 1, \cdots, N+1$)

$$\Gamma^j \triangleq \left[\begin{array}{c} \gamma^j \\ \hline \dot{\gamma}^j \end{array}\right]$$

$$A_{00} \triangleq \bar{a}^{00} + \hat{a}^{00}$$

$$A_{0j} \triangleq [0 \mid \bar{a}^{0j} + \hat{a}^{0j}] \; (j \text{ odd}); \qquad A_{j0} \triangleq A^T_{0j}$$

$$A_{0j} \triangleq [0 \mid \bar{a}^{0j}] \; (j \text{ even}); \qquad A_{j0} \triangleq A^T_{0j}$$

$$A_{jk} \triangleq \left[\begin{array}{c|c} \delta_{jk} U & 0 \\ \hline 0 & \bar{a}^{jk} + \hat{a}^{jk} \end{array}\right] (j, k \text{ odd})$$

$$A_{jk} \triangleq \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{a}^{jk} \end{array}\right] (j \text{ odd}, k \text{ even}); \qquad A_{kj} \triangleq A^T_{jk}$$

$$A_{jk} \triangleq \left[\begin{array}{c|c} \delta_{jk} U & 0 \\ \hline 0 & \bar{a}^{jk} \end{array}\right] (j, k \text{ even})$$

and rewrite Eq. (30c) as

$$\left[\begin{array}{c|c|c|c|c} A_{00} & A_{01} & A_{02} & \cdots & A_{0,N+1} \\ \hline A_{10} & A_{11} & A_{12} & \cdots & A_{1,N+1} \\ \hline A_{20} & A_{21} & A_{22} & \cdots & A_{2,N+1} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline A_{N+1,0} & A_{N+1,1} & A_{N+1,2} & \cdots & A_{N+1,N+1} \end{array}\right] \left[\begin{array}{c} \dot{\omega}^0 \\ \hline \dot{\Gamma}^1 \\ \hline \dot{\Gamma}^2 \\ \hline \vdots \\ \hline \dot{\Gamma}^{N+1} \end{array}\right] = \left[\begin{array}{c} \sum_{k \in \mathcal{B}} [(\bar{C}^{0k} + \hat{C}^{0k}) \bar{A}^k + \bar{C}^{0k} \hat{A}^k] - \sum_{k \in \mathcal{G}} a_{0k} \ddot{\gamma}_k \\ \hline \dot{\gamma}^1 \\ \hline R^1 \\ \hline \dot{\gamma}^2 \\ \hline R^2 \\ \hline \vdots \\ \hline \dot{\gamma}^{N+1} \\ \hline R^{N+1} \end{array}\right] \tag{30d}$$

Whether we choose to work with Eq. (30d), or its equivalent, Eq. (30c), or the more explicit equations in Eqs. (30a, b), any progress requires explicit expressions for the symbols $\bar{C}^{ik}$, $\hat{C}^{ik}$, $\bar{A}^k$, $\hat{A}^k$, $\bar{a}_{ik}$, and $\hat{a}_{ik}$ for all permissible i and k. (We also need $\overline{\tau}_i$ and $\hat{\tau}_i$, but these can be written only for specific systems.) The required expressions can be obtained by combining Eq. (29) with the definitions of a_{ik} and A^k in Section IIA and the expression for C^{ik} in Section IIC.

Specifically, from Eq. (21) we have an expression which provides C^{rj} (for $r, j \in \mathcal{B}$) in terms of direction cosine matrices relating contiguous bodies, and they, in turn, are always available in expanded form from Eq. (4) or Eq. (5). The basic ingredients of $\overline{C}^{rj}$ and $\hat{C}^{rj}$ are therefore available from Eqs. (4) and (5) as

$$\overline{C}^{k,N_k} = C^{k,N_k} \quad \text{for } k \in \mathcal{P} - \mathcal{A} \tag{31a}$$

$$\overline{C}^{k,N_k} = U \quad \text{for } k \in \mathcal{A} \tag{31b}$$

$$\hat{C}^{k,N_k} = 0 \quad \text{for } k \in \mathcal{P} - \mathcal{A} \tag{31c}$$

$$\hat{C}^{k,N_k} = -\gamma_k \tilde{g}^k \quad \text{for } k \in \mathcal{A} \tag{31d}$$

When C^{rj} in the symbolic form of Eq. (21) is expanded according to the algorithm culminating in Eq. (22), this latter expression then becomes

$$\begin{aligned} C^{rj} &= C^{rp}C^{pq} \cdots C^{uj} \cong (\overline{C}^{rp} + \hat{C}^{rp})(\overline{C}^{pq} + \hat{C}^{pq}) \cdots (\overline{C}^{uj} + \hat{C}^{uj}) \\ &\cong \overline{C}^{rp}\overline{C}^{pq} \cdots \overline{C}^{uj} + \overline{C}^{rp}\overline{C}^{pq} \cdots \overline{C}^{uj} \sum_{k \in \mathcal{A}} (\varepsilon_{kj} - \varepsilon_{kr}) \tilde{g}^k \gamma_k \end{aligned} \tag{32}$$

so that, with indices generated according to the algorithm following Eq. (21),

$$\overline{C}^{rj} = \overline{C}^{rp}\overline{C}^{pq} \cdots \overline{C}^{uj} \tag{33a}$$

$$\hat{C}^{rj} = \overline{C}^{rp}\overline{C}^{pq} \cdots \overline{C}^{uj} \sum_{s \in \mathcal{A}} (\varepsilon_{sj} - \varepsilon_{sr}) \tilde{g}^s \gamma_s \tag{33b}$$

In substituting Eqs. (33) into Eq. (30), simplifications can often be realized by recognizing Eqs. (31). If, for example, the flexible substructure consists of a single terminal appendage, then, in Eq. (30), the direction cosine terms for $r, j \in \mathcal{A}$ simplify to

$$\overline{C}^{rj} = U \qquad \hat{C}^{rj} = \sum_{s \in \mathcal{A}} (\varepsilon_{sj} - \varepsilon_{sr}) \tilde{g}^s \gamma_s \tag{34}$$

Equation (30) then becomes

$$(\overline{a}_{00} + \hat{a}_{00})\dot{\omega}^0 + \sum_{k \in \mathcal{A}} \overline{a}_{0k}\ddot{\gamma}_k + \sum_{k \in \mathcal{P} - \mathcal{A}} (\overline{a}_{0k} + \hat{a}_{0k})\ddot{\gamma}_k = \sum_{k \in \mathcal{B}} [(\overline{C}^{0k} + \hat{C}^{0k})\overline{A}^k + \overline{C}^{0k}\hat{A}^k] \tag{35a}$$

$$\begin{aligned} &(\overline{a}_{i0} + \hat{a}_{i0})\dot{\omega}^0 + \sum_{k \in \mathcal{A}} \overline{a}_{ik}\ddot{\gamma} + \sum_{k \in \mathcal{B} - \mathcal{A}} (\overline{a}_{ik} + \hat{a}_{ik})\ddot{\gamma}_k = \\ &\qquad g^{iT} \sum_{k \in \mathcal{A}} \varepsilon_{ik} \{\hat{A}^k + [U + \sum_{s \in \mathcal{A}} (\varepsilon_{sk} - \varepsilon_{si}) \tilde{g}^s \gamma_s] \overline{A}^k\} + \overline{\tau}_i + \hat{\tau}_i \qquad (i \in \mathcal{P}) \end{aligned} \tag{35b}$$

Whether we retain the general result in Eq. (30) or only the special equations in Eqs. (35), we require expressions for $\hat{A}^k$, $\overline{A}^k$, $\hat{a}_{ik}$ and $\overline{a}_{ik}$. These symbols can be expressed in terms of more basic quantities by expanding Defs. 38–41 in the manner indicated by Eqs. (29c) and (29d). In terms of the symbols $\overline{C}^{rj}$ and $\hat{C}^{rj}$ available from Eqs. (33), these definitions lead to the following, in which we have substituted

$T^k = \overline{T}^k + \hat{T}^k + \cdots$ and $F^k = \overline{F}^k + \hat{F}^k + \cdots$ in the manner established by Eq. (29):

$$\overline{a}_{00} = \sum_{k\in\mathcal{B}} \overline{C}^{0k}\Phi^{kk}\overline{C}^{k0} - \mathcal{M} \sum_{k\in\mathcal{B}} \sum_{j\in\mathcal{B}-k} (\overline{C}^{0k}D^{jk^T}\overline{C}^{jk}D^{kj} - \overline{C}^{0j}D^{jk}D^{kj^T})\,\overline{C}^{k0} \tag{36a}$$

For $k \in \mathcal{P}$,

$$\overline{a}_{0k} = \sum_{r\in\mathcal{P}} \varepsilon_{kr}\overline{C}^{0r}\Phi^{rr}\overline{C}^{rk}g^k - \mathcal{M} \sum_{r\in\mathcal{P}} \sum_{j\in\mathcal{B}-r} \varepsilon_{kr}(\overline{C}^{0r}D^{jr^T}\overline{C}^{jr}D^{rj} - \overline{C}^{0j}D^{jr}D^{rj^T})\,\overline{C}^{rk}g^k \tag{36b}$$

$$\overline{a}_{k0} = \overline{a}^T_{0k} \tag{36c}$$

For $i, k \in \mathcal{P}$,

$$\begin{aligned}\overline{a}_{ik} = {} & g^{i^T} \sum_{r\in\mathcal{P}} \varepsilon_{ir}\varepsilon_{kr}\overline{C}^{ir}\Phi^{rr}\overline{C}^{rk}g^k \\ & - \mathcal{M}\, g^{i^T} \sum_{r\in\mathcal{P}} \sum_{s\in\mathcal{P}-r} \varepsilon_{ir}\varepsilon_{ks}(\overline{C}^{ir}D^{sr^T}\overline{C}^{sr}D^{rs} - \overline{C}^{is}D^{sr}D^{rs^T})\,\overline{C}^{rk}g^k\end{aligned} \tag{36d}$$

For $k \in \mathcal{B}$,

$$\begin{aligned}\overline{A}^k = {} & \overline{T}^k + \sum_{j\in\mathcal{B}} \tilde{D}^{kj}\overline{C}^{kj}\overline{F}^j - \Phi^{kk} \sum_{r\in\mathcal{P}-\mathcal{A}} \varepsilon_{rk}\dot{\gamma}_r(\overline{C}^{k0}\tilde{\omega}^0\overline{C}^{0r} + \sum_{s\in\mathcal{P}-\mathcal{A}} \varepsilon_{sr}\dot{\gamma}_s\overline{C}^{ks}\tilde{g}^s\overline{C}^{sr})\,g^r \\ & - [\overline{C}^{k0}\tilde{\omega}^0\overline{C}^{0k} + \sum_{j\in\mathcal{P}-\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j\overline{C}^{kj}\tilde{g}^j\overline{C}^{jk}]\,\Phi^{kk}\,[\overline{C}^{k0}\omega^0 + \sum_{j\in\mathcal{P}-\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j\overline{C}^{kj}g^j] \\ & + \mathcal{M} \sum_{j\in\mathcal{B}-k} \{\tilde{D}^{kj}\overline{C}^{kj}\,[\overline{C}^{j0}\tilde{\omega}^0\overline{C}^{0j} + \sum_{r\in\mathcal{P}-\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\overline{C}^{jr}\tilde{g}^r\overline{C}^{rj}]^2\,D^{jk} \\ & + (UD^{jk^T}\overline{C}^{jk}D^{kj} - \overline{C}^{kj}D^{jk}D^{kj^T}) \sum_{r\in\mathcal{P}-\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\,[\overline{C}^{k0}\tilde{\omega}^0\overline{C}^{0r} + \sum_{s\in\mathcal{P}-\mathcal{A}} \varepsilon_{sr}\dot{\gamma}_s\overline{C}^{ks}\tilde{g}^s\overline{C}^{sr}]\,g^r\}\end{aligned} \tag{36e}$$

$$\begin{aligned}\hat{a}_{00} = {} & \sum_{k\in\mathcal{B}} (\hat{C}^{0k}\Phi^{kk}\overline{C}^{k0} + \overline{C}^{0k}\Phi^{kk}\hat{C}^{k0}) - \mathcal{M} \sum_{k\in\mathcal{B}} \sum_{j\in\mathcal{B}-k} [(\overline{C}^{0k}D^{jk^T}\overline{C}^{jk}D^{kj} - \overline{C}^{0j}D^{jk}D^{kj^T})\,\hat{C}^{k0} \\ & + (\hat{C}^{0k}D^{jk^T}\overline{C}^{jk}D^{kj} + \overline{C}^{0k}D^{jk^T}\hat{C}^{jk}D^{kj} - \hat{C}^{0j}D^{jk}D^{kj^T})\,\overline{C}^{k0}]\end{aligned} \tag{37a}$$

For $k \in \mathcal{P}$,

$$\begin{aligned}\hat{a}_{0k} = {} & \sum_{r\in\mathcal{P}} \varepsilon_{kr}(\hat{C}^{0r}\Phi^{rr}\overline{C}^{rk} + \overline{C}^{0r}\Phi^{rr}\hat{C}^{rk})\,g^k \\ & - \mathcal{M} \sum_{r\in\mathcal{B}} \sum_{j\in\mathcal{B}-r} \varepsilon_{kr}\,[(\overline{C}^{0r}D^{jr^T}\overline{C}^{jr}D^{rj} - \overline{C}^{0j}D^{jr}D^{rj^T})\,\hat{C}^{rk}g^k \\ & + (\hat{C}^{0r}D^{jr^T}\overline{C}^{jr}D^{rj} + \overline{C}^{0r}D^{jr^T}\hat{C}^{jr}D^{rj} - \hat{C}^{0j}D^{jr}D^{rj^T})\,\overline{C}^{rk}g^k]\end{aligned} \tag{37b}$$

$$\hat{a}_{k0} = \hat{a}^T_{0k} \tag{37c}$$

For $i, k \in \mathcal{P}$,

$$\begin{aligned}\hat{a}_{ik} = {} & g^{i^T} \sum_{r\in\mathcal{P}} \varepsilon_{ir}\varepsilon_{kr}(\hat{C}^{ir}\Phi^{rr}\overline{C}^{rk} + \overline{C}^{ir}\Phi^{rr}\hat{C}^{rk})\,g^k \\ & - \mathcal{M}\, g^{i^T} \sum_{r\in\mathcal{P}} \sum_{s\in\mathcal{P}-r} \varepsilon_{ir}\varepsilon_{ks}\,[(\overline{C}^{ir}D^{sr^T}\overline{C}^{sr}D^{rs} - \overline{C}^{is}D^{ki}D^{ik^T})\,\hat{C}^{rk}g^k \\ & + (\hat{C}^{ir}D^{sr^T}\overline{C}^{sr}D^{rs} + \overline{C}^{ir}D^{sr^T}\hat{C}^{sr}D^{rs} - \hat{C}^{is}D^{ki}D^{ik^T})\,\overline{C}^{rk}g^k]\end{aligned} \tag{37d}$$

For $k \in \mathcal{B}$,

$$\begin{aligned}
\hat{A}^k = {} & \hat{T}^k + \sum_{j\in\mathcal{B}} \tilde{D}^{kj}(\bar{C}^{kj}\hat{F}^j + \hat{C}^{kj}\bar{F}^j) - \Phi^{kk}\sum_{r\in\mathcal{P}-\mathcal{A}} \varepsilon_{rk}\dot{\gamma}_r [\hat{C}^{ko}\tilde{\omega}^0\bar{C}^{0r} + \bar{C}^{ko}\tilde{\omega}^0\hat{C}^{0r} \\
& + \sum_{s\in\mathcal{A}} \varepsilon_{sr}\dot{\gamma}_s \bar{C}^{ks}\tilde{g}^s\bar{C}^{sr} + \sum_{s\in\mathcal{P}-\mathcal{A}} \varepsilon_{sr}\dot{\gamma}_s(\hat{C}^{ks}\tilde{g}^s\bar{C}^{sr} + \bar{C}^{ks}\tilde{g}^s\hat{C}^{sr})]\, g^r \\
& - \Phi^{kk}\sum_{r\in\mathcal{A}} \varepsilon_{rk}\dot{\gamma}_r(\bar{C}^{ko}\tilde{\omega}^0\bar{C}^{0r} + \sum_{s\in\mathcal{P}-\mathcal{A}}\varepsilon_{sr}\dot{\gamma}_s\bar{C}^{ks}\tilde{g}^s\bar{C}^{sr})\, g^r \\
& - [\hat{C}^{ko}\tilde{\omega}^0\bar{C}^{ok} + \bar{C}^{ko}\tilde{\omega}^0\hat{C}^{ok} + \sum_{j\in\mathcal{P}-\mathcal{A}}\varepsilon_{jk}\dot{\gamma}_j(\hat{C}^{kj}\tilde{g}^j\bar{C}^{jk} + \bar{C}^{kj}\tilde{g}^j\hat{C}^{jk}) \\
& + \sum_{j\in\mathcal{A}}\varepsilon_{jk}\dot{\gamma}_j\bar{C}^{kj}\tilde{g}^j\bar{C}^{jk}]\, \Phi^{kk}[\bar{C}^{ko}\omega^0 + \sum_{j\in\mathcal{P}-\mathcal{A}}\varepsilon_{jk}\dot{\gamma}_j\bar{C}^{kj}g^j] \\
& - [\bar{C}^{ko}\tilde{\omega}^0\bar{C}^{ok} + \sum_{j\in\mathcal{P}-\mathcal{A}}\varepsilon_{jk}\dot{\gamma}_j\bar{C}^{kj}\tilde{g}^j\bar{C}^{jk}]\, \Phi^{kk}[\hat{C}^{ko}\omega^0 + \sum_{j\in\mathcal{P}-\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j \hat{C}^{kj}g^j + \sum_{j\in\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j\bar{C}^{kj}g^j] \\
& + \mathcal{M}\sum_{j\in\mathcal{B}-k}\Big\{\tilde{D}^{kj}\hat{C}^{kj}\,[\bar{C}^{jo}\tilde{\omega}^{0}\bar{C}^{oj} + \sum_{r\in\mathcal{P}-\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\bar{C}^{jr}\tilde{g}^r\bar{C}^{rj}]^2 D^{jk} \\
& + \tilde{D}^{kj}\bar{C}^{kj}\{[\hat{C}^{jo}\tilde{\omega}^0\bar{C}^{oj} + \bar{C}^{jo}\tilde{\omega}^0\hat{C}^{oj} + \sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r(\hat{C}^{jr}\tilde{g}^r\bar{C}^{rj} + \bar{C}^{jr}\tilde{g}^r\hat{C}^{rj}) \\
& + \sum_{r\in\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r\bar{C}^{jr}\tilde{g}^r\bar{C}^{rj}]\,[\bar{C}^{jo}\tilde{\omega}^0\bar{C}^{oj} + \sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r\bar{C}^{jr}\tilde{g}^r\bar{C}^{rj}] \\
& + [\bar{C}^{jo}\tilde{\omega}^0\bar{C}^{oj} + \sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\gamma_r\bar{C}^{jr}\tilde{g}^r\bar{C}^{rj}]\,[\hat{C}^{jo}\tilde{\omega}^0\bar{C}^{oj} + \bar{C}^{jo}\tilde{\omega}^0\hat{C}^{oj} \\
& + \sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r(\hat{C}^{jr}\tilde{g}^r\bar{C}^{rj} + \bar{C}^{jr}\tilde{g}^r\hat{C}^{rj}) + \sum_{r\in\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r\bar{C}^{jr}\tilde{g}^r\bar{C}^{rj}]\}\, D^{jk}\Big\} \\
& + \mathcal{M}\sum_{j\in\mathcal{B}-k}\Big\{(UD^{jk^T}\hat{C}^{jk}D^{kj} - \hat{C}^{kj}D^{jk}D^{kj^T})\sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r[\bar{C}^{ko}\tilde{\omega}^0\bar{C}^{or} \\
& + \sum_{s\in\mathcal{P}-\mathcal{A}}\varepsilon_{sr}\dot{\gamma}_s\bar{C}^{ks}\tilde{g}^s\bar{C}^{sr}]\, g^r + (UD^{jk^T}\bar{C}^{jk}D^{kj} - \bar{C}^{kj}D^{jk}D^{kj^T})\{\sum_{r\in\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r[\bar{C}^{ko}\tilde{\omega}^0\bar{C}^{or} \\
& + \sum_{s\in\mathcal{P}-\mathcal{A}}\varepsilon_{sr}\dot{\gamma}_s\bar{C}^{ks}\tilde{g}^s\bar{C}^{sr}]\, g^r + \sum_{r\in\mathcal{P}-\mathcal{A}}\varepsilon_{rj}\dot{\gamma}_r[\hat{C}^{ko}\tilde{\omega}^0\bar{C}^{or} + \bar{C}^{ko}\tilde{\omega}^0\hat{C}^{or} \\
& + \sum_{s\in\mathcal{P}-\mathcal{A}}\varepsilon_{sr}\dot{\gamma}_s(\hat{C}^{ks}\tilde{g}^s\bar{C}^{sr} + \bar{C}^{ks}\tilde{g}^s\hat{C}^{sr}) + \sum_{s\in\mathcal{A}}\varepsilon_{sr}\dot{\gamma}_s\bar{C}^{ks}\tilde{g}^s\bar{C}^{sr}]\, g^r\}\Big\}
\end{aligned} \tag{37e}$$

(The superscript 2 in Eq. 36e indicates the square of the preceding bracketed matrix.)

Upon substituting Eqs. (31), (33), (36), and (37) into Eq. (30), one can obtain the most general form of the partially linearized set of differential equations stemming from Eq. (1). These equations (to be assembled in detail by computer program) are then in a form whose dependence upon the small angles $\gamma_\alpha, \cdots, \gamma_\mu$ and their derivatives is fully explicit; by examining the structure of an appropriate subset of these equations, we can determine those circumstances for which transformation to distributed coordinates and subsequent coordinate transformation is feasible. It should be noted that in the special case of Eq. (30) recorded as Eq. (35), correspondingly simplified versions of Eq. (36) and (37) can be used. For example, for the special case in which the flexible substructure consists of a single terminal appendage, one can simplify Eqs. (36) and (37) by noting that if α is the appendage hinge closest to $\mathscr{b}_0$, then, for $k \in \mathscr{A}$ and $i \in \mathscr{P} - \mathscr{A}$, $\varepsilon_{ik} \equiv \varepsilon_{i\alpha}$ and $\overline{C}^{ik} \equiv \overline{C}^{i\alpha}$. Since $\overline{A}^k$ and $\hat{A}^k$ appear in Eq. (35b) only for $k \in \mathscr{A}$, this simplification is computationally quite significant.

Before proceeding with the search for coordinate transformations for the small variables and their first time-derivatives, it should be noted that the partially linearized discrete-coordinate equations are themselves of some computational value. Depending upon the extent of the partial linearization, Eq. (30) (or for terminal appendages, Eq. 35) may be much more amenable to numerical integration than the unrestricted counterpart, Eq. (1). If Eqs. (30) are to be integrated directly (without coordinate transformation), then the suggested imaginary bodies should *not* be introduced, even if all angles remain small.

C. Linear, Constant-Coefficient Differential Equations for Coordinate Transformations

Equation (30b) includes the set of ν equations

$$\sum_{k\in\mathscr{A}} \overline{a}_{ik}\ddot{\gamma}_k + \hat{a}_{i0}\dot{\omega}^0 + \sum_{k\in\mathscr{B}-\mathscr{A}} \hat{a}_{ik}\ddot{\gamma}_k - g^{iT} \sum_{k\in\mathscr{P}} \varepsilon_{ik}\left[\overline{C}^{ik}\hat{A}^k + \hat{C}^{ik}\overline{A}^k\right] - \hat{\tau}_i =$$

$$\overline{a}_{i0}\dot{\omega}^0 + \sum_{k\in\mathscr{B}-\mathscr{A}} \overline{a}_{ik}\ddot{\gamma}_k + g^{iT} \sum_{k\in\mathscr{P}} \varepsilon_{ik}\overline{C}^{ik}\overline{A}^k + \overline{\tau}_i \qquad (i \in \mathscr{A}) \tag{38}$$

and these will provide the basis for the desired coordinate transformation. The hybrid-coordinate procedure entails the transformation of the set of variables defining the substructure deformation (here γ_i and $\dot{\gamma}_i$ for $i \in \mathscr{A}$) into a set of distributed coordinates for which coordinate truncation can be accomplished without jeopardizing the salient features of the dynamic response. Truncation is never a rigorous mathematical process, and its justification must be based on rather subjective judgments of the acceptability of certain engineering approximations. Great caution is therefore necessary in adopting a truncated set of coordinates, and all possible effort should be made to select the coordinate transformation for which maximum truncation of coordinates can be accomplished with minimum risk. In order to permit the systematic evaluation of the consequences of truncation, it is important that coupling among the scalar equations of motion in the transformed coordinates be minimized. (If coupling were entirely eliminated, one could solve each of the transformed scalar equations in turn, and obtain the true solution by superposition; then, irrelevant coordinates could be discarded with no risk of adverse consequences.)

In the quest for a coordinate transformation which minimizes the coupling among distributed coordinates, the theory of linear, constant-coefficient, homogeneous, differential equations provides our only guidance. Although in fact, Eqs. (38) are not in this category, in application, it is often reasonable to approximate slowly varying coefficients or coefficients with small magnitude variations as constants, and to ignore the indirect coupling of Eqs. (38) through kinematical variables external to the flexible substructure, focusing only on the homogeneous counterparts to Eqs. (38). These approximations will therefore be adopted in this section for the purpose of finding a coordinate transformation, after which the inhomogeneous equations (30) will be transformed and retained for simulation.

The homogeneous counterpart to Eq. (38) is

$$\sum_{k\in\mathcal{A}} \bar{a}_{ik}\ddot{\gamma}_k + \hat{a}_{i0}\dot{\omega}^0 + \sum_{k\in\mathcal{B}-\mathcal{A}} \hat{a}_{ik}\ddot{\gamma}_k - g^{iT}\sum_{k\in\mathcal{P}} \varepsilon_{ik}\left(\hat{C}^{ik}\bar{A}^k + \bar{C}^{ik}\hat{A}^k\right) - \hat{\tau}_i = 0 \qquad (i\in\mathcal{A}) \tag{39}$$

All careted symbols in Eq. (39) are linear in the small-angle variables of the substructure. From Eq. (36e), $\bar{A}^k$ depends on ω^0, so that the coefficients of the small variables in Eq. (39) are not constant unless $\dot{\omega}^0 = 0$. This restriction also eliminates the second term in Eq. (39). Moreover, the dependence of these coefficients on direction cosines which involve all of the angles of rotation external to the appendage requires that these angles be constant if Eq. (39) is to provide a coordinate transformation based on constant-coefficient differential equation theory. This restriction removes the third term from Eq. (39) and greatly simplifies Eqs. (36e) and (37e) for $\bar{A}^k$ and $\hat{A}^k$. As the basis for coordinate transformation, we adopt instead of Eq. (39) the following approximation of the equation:

$$\sum_{k\in\mathcal{A}} \bar{\bar{a}}_{ik}\ddot{\gamma}_k - g^{iT}\sum_{k\in\mathcal{P}} \varepsilon_{ik}\left(\bar{\hat{C}}^{ik}\bar{\bar{A}}^k + \bar{\bar{C}}^{ik}\bar{\hat{A}}^k\right) - \bar{\hat{\tau}}_i = 0 \qquad (i\in\mathcal{A}) \tag{40}$$

where $\bar{\bar{a}}_{ik}$ is a constant defined by $\bar{a}_{ik}$ in some nominal state (see Eq. 36d), and where, from Eqs. (36e) and (37e), $\bar{\bar{A}}^k$ and $\bar{\hat{A}}^k$ are given by

$$\bar{\bar{A}}^k \triangleq \bar{\bar{T}}^k + \sum_{j\in\mathcal{B}} \tilde{D}^{kj}\bar{\bar{C}}^{kj}\bar{\bar{F}}^j - \bar{\bar{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0k}\Phi^{kk}\bar{\bar{C}}^{k0}\bar{\omega}^0 + \mathcal{M}\sum_{j\in\mathcal{B}-k} \tilde{D}^{kj}\bar{\bar{C}}^{kj}\left(\bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j}\right)^2 D^{jk} \tag{41a}$$

$$\begin{aligned}
\bar{\hat{A}}^k \triangleq\ & \bar{\hat{T}}^k + \sum_{j\in\mathcal{B}} \tilde{D}^{kj}\left(\bar{\bar{C}}^{kj}\bar{\hat{F}}^j + \bar{\hat{C}}^{kj}\bar{\bar{F}}^j\right) - \Phi^{kk}\sum_{r\in\mathcal{A}} \varepsilon_{rk}\dot{\gamma}_r\bar{\bar{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0r}g^r \\
& - \Big[\bar{\hat{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0k} + \bar{\bar{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\hat{C}}^{0k} + \sum_{j\in\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j\bar{\bar{C}}^{kj}\tilde{g}^j\bar{\bar{C}}^{jk}\Big]\Phi^{kk}\bar{\bar{C}}^{k0}\bar{\omega}^0 \\
& - \bar{\bar{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0k}\Phi^{kk}\Big[\bar{\hat{C}}^{k0}\bar{\omega}^0 + \sum_{j\in\mathcal{A}} \varepsilon_{jk}\dot{\gamma}_j\bar{\bar{C}}^{kj}g^j\Big] + \mathcal{M}\sum_{j\in\mathcal{B}-k}\Big\{\tilde{D}^{kj}\bar{\hat{C}}^{kj}\left(\bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j}\right)^2 D^{jk} \\
& + \tilde{D}^{kj}\bar{\bar{C}}^{kj}\Big\{\Big[\bar{\hat{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j} + \bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\hat{C}}^{0j} + \sum_{r\in\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\bar{\bar{C}}^{jr}\tilde{g}^r\bar{\bar{C}}^{rj}\Big]\bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j} \\
& + \bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j}\Big[\bar{\hat{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0j} + \bar{\bar{C}}^{j0}\bar{\tilde{\omega}}^0\bar{\hat{C}}^{0j} + \sum_{r\in\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\bar{\bar{C}}^{jr}\tilde{g}^r\bar{\bar{C}}^{rj}\Big]\Big\} D^{jk} \\
& + \left(UD^{jkT}\bar{\bar{C}}^{jk}D^{kj} - \bar{\bar{C}}^{kj}D^{jk}D^{kjT}\right)\Big[\sum_{r\in\mathcal{A}} \varepsilon_{rj}\dot{\gamma}_r\bar{\bar{C}}^{k0}\bar{\tilde{\omega}}^0\bar{\bar{C}}^{0r}g^r\Big]\Big\}
\end{aligned} \tag{41b}$$

It is understood without the introduction of more notation that each symbol in Eqs. (40) and (41) is constant, except for those depending on γ_j or its time derivatives, for $j\in\mathcal{A}$.

Equation (40) is typical of ν scalar equations of motion, which can be written as a single matrix equation of the form

$$I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0 \tag{42}$$

where, if the ν indices in $\mathcal{A}$ range from α to μ,

$$\gamma \triangleq [\gamma_\alpha \cdots \gamma_\mu]^T$$

and where I is the ν by ν symmetric matrix whose elements are $\overline{\overline{a}}_{ij}$ for $i, j \in \mathcal{A}$ (see Eq. 36 for $\overline{a}_{ij}$). As indicated in the footnote following Eq. (1), the variable coefficient matrix with elements a_{ik} is nonsingular, so that the constant matrix I is also nonsingular.

The ν by ν matrix J comprises the coefficients of $\dot{\gamma}_j$ in terms involving $\overline{\hat{A}}^k$, and perhaps $\overline{\hat{\tau}}_i$, for $i, j \in \mathcal{A}$ and $k \in \mathcal{P}$. More specifically, a typical element of J may be written for $i, j \in \mathcal{A}$ as

$$J_{ij} = -g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \overline{\overline{C}}^{ik} \frac{\partial \overline{\hat{A}}^k}{\partial \dot{\gamma}_j} - \frac{\partial \overline{\hat{\tau}}_i}{\partial \dot{\gamma}_j} \tag{43}$$

The elements of the ν by ν matrix K are similarly available as

$$K_{ij} = -g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \left[\frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} \overline{\overline{A}}^k + \overline{\overline{C}}^{ik} \frac{\partial \overline{\hat{A}}^k}{\partial \gamma_j} \right] - \frac{\partial \overline{\hat{\tau}}_i}{\partial \gamma_j} \tag{44}$$

Since coordinate transformations which may usefully be applied to Eq. (42) are more readily calculated when the matrices J and K have certain properties of symmetry (or skew symmetry), we might profitably examine the structure of these matrices.

The hinge torques $\overline{\hat{\tau}}_i$ for $i \in \mathcal{A}$ will in most cases be representative of structural connections, in which case they will usually represent either a linear elastic spring, for which

$$\overline{\hat{\tau}}_i = -k_i \gamma_i \tag{45}$$

or a viscoelastic connection, for which

$$\overline{\hat{\tau}}_i = -k_i \gamma_i - d_i \dot{\gamma}_i \tag{46}$$

In both cases, the contribution of the hinge torque to K is symmetric, and the contribution to J is zero for the elastic spring and symmetric for the viscoelastic connection. Only in the unusual case in which $\overline{\hat{\tau}}_i$ is established by a control law that depends on appendage body relative rotations other than γ_i will the hinge torques contribute any but diagonal terms to the matrices J and K.

In order to assemble the matrix equation (42), we require the partial derivatives $\partial \overline{\hat{C}}^{ik}/\partial \gamma_j$, $\partial \overline{\hat{A}}^k/\partial \gamma_j$, and $\partial \overline{\hat{A}}^k/\partial \dot{\gamma}_j$, for $i, j \in \mathcal{A}$ and $k \in \mathcal{P}$. From Eq. (33b), we have

$$\frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} = \overline{\overline{C}}^{ip} \overline{\overline{C}}^{pq} \cdots \overline{\overline{C}}^{uk} (\varepsilon_{jk} - \varepsilon_{ji}) \tilde{g}^j \tag{47}$$

where again the algorithm following Eq. (21) is used to generate the unspecified indices. When the flexible substructure is an external appendage and α is the index of the hinge point closest to $\mathscr{b}_0$, Eq. (47) becomes

$$\frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} = \overline{\overline{C}}^{\alpha p} \overline{\overline{C}}^{pq} \cdots \overline{\overline{C}}^{uk} (\varepsilon_{jk} - \varepsilon_{ji}) \tilde{g}^j \tag{48a}$$

When $k \epsilon \mathscr{A}$, both of these expressions reduce to

$$\frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} = (\varepsilon_{jk} - \varepsilon_{ji})\, \tilde{g}^j \tag{48b}$$

Equation (41b) provides, for $k \epsilon \mathscr{P}$ and $j \epsilon \mathscr{A}$,

$$\begin{aligned}
\frac{\partial \overline{\hat{A}}^k}{\partial \dot{\gamma}_j} = {} & \frac{\partial \overline{\hat{T}}^k}{\partial \dot{\gamma}_j} + \sum_{s \epsilon \mathscr{B}} \tilde{D}^{ks} \overline{\overline{C}}^{ks} \frac{\partial \overline{\hat{F}}^s}{\partial \dot{\gamma}_j} - \Phi^{kk} \varepsilon_{jk} \overline{\overline{C}}^{k0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0j} g^j - \varepsilon_{jk} \overline{\overline{C}}^{kj} \tilde{g}^j \overline{\overline{C}}^{jk} \Phi^{kk} \overline{\overline{C}}^{k0} \overline{\omega}^0 \\
& - \overline{\overline{C}}^{k0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0k} \Phi^{kk} \varepsilon_{jk} \overline{\overline{C}}^{kj} g^j + \mathfrak{M} \sum_{s \epsilon \mathscr{B}-k} \varepsilon_{js} [\tilde{D}^{ks} \overline{\overline{C}}^{ks} (\overline{\overline{C}}^{sj} \tilde{g}^j \overline{\overline{C}}^{js} \overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} \\
& + \overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} \overline{\overline{C}}^{sj} \tilde{g}^j \overline{\overline{C}}^{js}) D^{sk} + (U D^{skT} \overline{\overline{C}}^{sk} D^{ks} - \overline{\overline{C}}^{ks} D^{sk} D^{ksT}) \overline{\overline{C}}^{k0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0j} g^j]
\end{aligned} \tag{49}$$

Also available from Eq. (41b) is the expression

$$\begin{aligned}
\frac{\partial \overline{\hat{A}}^k}{\partial \gamma_j} = {} & \frac{\partial \overline{\hat{T}}^k}{\partial \gamma_j} + \sum_{s \epsilon \mathscr{B}} \tilde{D}^{ks} \left(\overline{\overline{C}}^{ks} \frac{\partial \overline{\hat{F}}^s}{\partial \gamma_j} + \frac{\partial \overline{\hat{C}}^{ks}}{\partial \gamma_j} \overline{\overline{F}}^s \right) \\
& - \left[\frac{\partial \overline{\hat{C}}^{k0}}{\partial \gamma_j} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0k} + \overline{\overline{C}}^{k0} \tilde{\overline{\omega}}^0 \frac{\partial \overline{\hat{C}}^{0k}}{\partial \gamma_j} \right] \Phi^{kk} \overline{\overline{C}}^{k0} \overline{\omega}^0 - \overline{\overline{C}}^{k0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0k} \Phi^{kk} \frac{\partial \overline{\hat{C}}^{k0}}{\partial \gamma_j} \overline{\omega}^0 \\
& + \mathfrak{M} \sum_{s \epsilon \mathscr{B}-k} \Bigg\{ \tilde{D}^{ks} \frac{\partial \overline{\hat{C}}^{ks}}{\partial \gamma_j} [\overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s}]^2 D^{sk} \\
& + \tilde{D}^{ks} \overline{\overline{C}}^{ks} \Bigg\{ \left[\frac{\partial \overline{\hat{C}}^{s0}}{\partial \gamma_j} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} + \overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \frac{\partial \overline{\hat{C}}^{0s}}{\partial \gamma_j} \right] \overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} \\
& + \overline{\overline{C}}^{s0} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} \left[\frac{\partial \overline{\hat{C}}^{s0}}{\partial \gamma_j} \tilde{\overline{\omega}}^0 \overline{\overline{C}}^{0s} + \overline{\overline{C}}^{0s} \tilde{\overline{\omega}}^0 \frac{\partial \overline{\hat{C}}^{0s}}{\partial \gamma_j} \right] \Bigg\} D^{sk} \Bigg\}
\end{aligned} \tag{50}$$

For the special case in which there are no external forces and body $\mathscr{b}_0$ is nominally at rest in inertial space so that $T^k = F^k = \omega^0 = 0$, Eqs. (49) and (50) reduce to

$$\frac{\partial \overline{\hat{A}}^k}{\partial \dot{\gamma}_j} = \frac{\partial \overline{\hat{A}}^k}{\partial \gamma_j} = 0$$

as might be expected. If, in addition to these simplifications, we have linearly elastic spring torques at the joints, so that $\hat{\tau}_i = -k_i \gamma_i$ for $i \epsilon \mathscr{A}$, then the equation to be examined for coordinate transformation (Eq. 42) is simply

$$I\ddot{\gamma} + K\gamma = 0 \tag{51}$$

where K is the diagonal ν by ν matrix with nonzero elements $K_{ii} = K_i$, and the elements of the symmetric ν by ν matrix I are given by Eq. (36d)

A second special case of interest is one for which all hinge lines are parallel, so that for all $i \epsilon \mathscr{P}$, we can replace $\mathbf{g}_i$ by the new unit vector $\mathbf{g}$; in addition, the angular velocity vector $\boldsymbol{\omega}^0$ has the same orientation, so that $\boldsymbol{\omega}^0 = \Omega \mathbf{g}$ for some scalar spin rate Ω. This special case will be called the *transverse deformation case* for brevity; it should be understood that this term is applicable to three-dimensional systems such as that illustrated in Fig. 8. Substantial simplification of the preceding equations then follows from the vector identities (for $i, j \in \mathcal{P}$)

$$\boldsymbol{\omega}^0 \times \mathbf{g}^j = 0 \tag{52a}$$

$$\mathbf{g}^i \cdot \boldsymbol{\omega}^0 \times (\text{any vector}) = 0 \tag{52b}$$

$$\mathbf{g}^i \cdot \mathbf{g}^j \times (\text{any vector}) = 0 \tag{52c}$$

In matrix terms, these identities can take various forms, depending on the vector basis chosen for matrix representation of the various vectors involved. Since we have chosen to employ the matrices defined for $i \in \mathcal{P}$ by

$$\boldsymbol{\omega}^0 \triangleq \{\mathbf{b}^0\}^T \omega^0 \qquad \mathbf{g}^i \triangleq \{\mathbf{b}^i\}^T g^i \tag{53a}$$

with vector bases related for $i, j \in \mathcal{B}$ by

$$\{\mathbf{b}^i\} = C^{ij}\{\mathbf{b}^j\} \tag{53b}$$

we might expect to find Eqs. (52) reflected in our matrix equations in the form

$$\tilde{\omega}^0 C^{0j} g^j = 0 \tag{54a}$$

$$g^{iT} C^{i0} \tilde{\omega}^0 (\quad) = 0 \tag{54b}$$

$$g^{iT} C^{ij} \tilde{g}^j (\quad) = 0 \tag{54c}$$

Because more than one change in vector basis might intervene, we might find instead of Eqs. (54b) and (54c) equivalent expressions requiring substitution of the identities (for $i, j \in \mathcal{B}$)

$$C^{ij} = C^{ik} C^{kj} \tag{54d}$$

and

$$C^{ij} = C^{ik} C^{ks} C^{sj} \tag{54e}$$

In the transverse deformation case, the final sum in Eq. (49) for $\partial \overline{\hat{A}}^k / \partial \dot{\gamma}_j$ vanishes by virtue of Eq. (54a), as does the third term on the right side. Moreover, most of the terms remaining in Eq. (49) vanish when it is substituted into the expression for J_{ij} in Eq. (43), because of identities (54b)–(54e). All that remains in Eq. (43) is

$$\begin{aligned} J_{ij} = & -g^T \sum_{k \in \mathcal{P}} \varepsilon_{ik} \overline{\overline{C}}^{ik} \left[\frac{\partial \overline{\hat{T}}^k}{\partial \dot{\gamma}_j} + \sum_{s \in \mathcal{B}} \tilde{D}^{ks} \overline{\overline{C}}^{ks} \frac{\partial \overline{\hat{F}}^s}{\partial \dot{\gamma}_j} \right] - \frac{\partial \overline{\hat{\tau}}_i}{\partial \dot{\gamma}_j} \\ & - g^T \sum_{k \in \mathcal{P}} \varepsilon_{ik} \overline{\overline{C}}^{ik} \mathcal{M} \sum_{s \in \mathcal{B} - k} \varepsilon_{js} \tilde{D}^{ks} \left(\overline{\overline{C}}^{kj} \tilde{g} \overline{\overline{C}}^{j0} \Omega \tilde{g} \overline{\overline{C}}^{0s} + \overline{\overline{C}}^{k0} \Omega \tilde{g} \overline{\overline{C}}^{0j} \tilde{g} \overline{\overline{C}}^{js} \right) D^{sk} \end{aligned} \tag{55a}$$

If, in addition, any external forces (F^s) or torques (T^k) on the bodies are independent of the relative rotation rates ($\dot{\gamma}_j$), then the first summation over $k \in \mathcal{P}$ disappears from J_{ij}. The interbody hinge torques ($\hat{\tau}_i$) will contribute to J_{ii} if "damping" is included in the joint, or the hinge rotation is subject to automatic control with rate feedback, but only in extraordinary cases will $\hat{\tau}_i$ contribute to J_{ij} for $i \neq j$. For simplicity in what follows, we ignore *any* contribution of hinge torques to J_{ij}. Moreover, we imagine that in the nominal steady state, all direction-cosine matrices equal the unit matrix, with no loss in generality. Then we have

$$J_{ij} = -2\mathfrak{M}\Omega g^T \sum_{k \in \mathcal{P}} \varepsilon_{ik} \sum_{s \in \mathcal{B} - k} \varepsilon_{js} \tilde{D}^{ks} \tilde{g} \tilde{g} D^{sk} \tag{55b}$$

or

$$J_{ij} = 2\mathfrak{M}\Omega g^T \sum_{k \in \mathcal{P}} \sum_{s \in \mathcal{P} - k} \varepsilon_{ik} \varepsilon_{js} \tilde{D}^{ks} \tilde{g} \tilde{D}^{sk} g \tag{55c}$$

From Eq. (55b), we see that $J_{ij} = 0$ if all D^{qr} for $q, r \in \mathcal{B}$ are parallel. This restricted transverse deformation case will be explored further in what follows. Before pursuing this course, we should note that in the more general transverse deformation case represented by Eq. (55c), the matrix J is skew symmetric, so that $J_{ij} = -J_{ji}$. To establish this result, note that the scalar J_{ij} must equal its transpose. Thus, Eq. (55c) provides

$$J_{ij} = -2\mathfrak{M}\Omega g^T \sum_{k \in \mathcal{P}} \sum_{s \in \mathcal{P} - k} \varepsilon_{ik} \varepsilon_{js} \tilde{D}^{sk} \tilde{g} \tilde{D}^{ks} g$$

Exchanging i and j in this expression, we find

$$J_{ji} = -2\mathfrak{M}\Omega g^T \sum_{k \in \mathcal{P}} \sum_{s \in \mathcal{P} - k} \varepsilon_{jk} \varepsilon_{is} \tilde{D}^{sk} \tilde{g} \tilde{D}^{ks} g$$

After exchanging the dummy indices s and k, we obtain

$$J_{ji} = -2\mathfrak{M}\Omega g^T \sum_{s \in \mathcal{P}} \sum_{k \in \mathcal{P} - s} \varepsilon_{js} \varepsilon_{ik} \tilde{D}^{ks} \tilde{g} \tilde{D}^{sk} g \tag{56}$$

Comparison of this expression with Eq. (55c) establishes the claimed relationship, $J_{ji} = -J_{ij}$, since in the course of each of the two indicated summations, both s and k assume all values in the set $\mathcal{P}$.

The next question of symmetry in the transverse deformation case concerns the matrix K, whose elements are given by Eq. (44). Because the external forces and torques represent a variety of influences (environmental interactions, control forces, etc.), their contribution to K cannot be described in comprehensive general terms, and they will be ignored in the discussion of the symmetry of K. The hinge torques τ_i are also ignored. With these omissions, and the vector-dyadic identities

$$\mathbf{g}^i \cdot \boldsymbol{\omega}^0 \times \boldsymbol{\Phi}^{kk} \cdot \boldsymbol{\omega}^0 = \mathbf{g} \cdot \Omega^2 \mathbf{g} \times \boldsymbol{\Phi}^{kk} \cdot \mathbf{g} = 0 \tag{57}$$

which apply to the transverse deformation case, Eqs. (44), (41a), and (50) combine to provide

$$
\begin{aligned}
K_{ij} = -g^T \sum_{k\in\mathcal{P}} \varepsilon_{ik} \Bigg\{ & \frac{\partial \bar{\hat{C}}^{ik}}{\partial\gamma_j} \mathcal{M} \sum_{r\in\mathcal{B}-k} \Omega^2 \tilde{D}^{kr}\bar{\bar{C}}^{kr}(\bar{\bar{C}}^{ro}\tilde{g}\bar{\bar{C}}^{or})^2 D^{rk} \\
& + \bar{\bar{C}}^{ik}\mathcal{M}\sum_{s\in\mathcal{B}-k}\Omega^2\tilde{D}^{ks}\Bigg[\frac{\partial\bar{\hat{C}}^{ks}}{\partial\gamma_j}(\bar{\bar{C}}^{so}\tilde{g}\bar{\bar{C}}^{os})^2 \\
& + \bar{\bar{C}}^{ks}\left(\frac{\partial \bar{\hat{C}}^{so}}{\partial\gamma_j}\tilde{g}\bar{\bar{C}}^{os} + \bar{\bar{C}}^{so}\tilde{g}\frac{\partial\bar{\hat{C}}^{os}}{\partial\gamma_j}\right)\bar{\bar{C}}^{so}\tilde{g}\bar{\bar{C}}^{os} \\
& + \bar{\bar{C}}^{ko}\tilde{g}\bar{\bar{C}}^{os}\left(\frac{\partial\bar{\hat{C}}^{so}}{\partial\gamma_j}\tilde{g}\bar{\bar{C}}^{os}+\bar{\bar{C}}^{os}\tilde{g}\frac{\partial\bar{\hat{C}}^{os}}{\partial\gamma_j}\right)\Bigg] D^{sk}\Bigg\} \qquad (58a)
\end{aligned}
$$

The structure of the matrix K is more apparent when in the nominal steady-state motion all direction-cosine matrices are the unit matrix. Then we have

$$
\begin{aligned}
K_{ij} = -\mathcal{M}\Omega^2 g^T \sum_{k\in\mathcal{P}}\varepsilon_{ik}\Bigg\{ & \frac{\partial\bar{\hat{C}}^{ik}}{\partial\gamma_j}\sum_{r\in\mathcal{B}-k}\tilde{D}^{kr}\tilde{g}\tilde{g}D^{rk} \\
& + \sum_{s\in\mathcal{B}-k}\tilde{D}^{ks}\left[\frac{\partial\bar{\hat{C}}^{ks}}{\partial\gamma_j}\tilde{g}\tilde{g} + \frac{\partial\bar{\hat{C}}^{so}}{\partial\gamma_j}\tilde{g}\tilde{g} + \tilde{g}\frac{\partial \bar{\hat{C}}^{os}}{\partial\gamma_j}\tilde{g} + \tilde{g}\frac{\partial\bar{\hat{C}}^{so}}{\partial\gamma_j}\tilde{g} + \tilde{g}\tilde{g}\frac{\partial\bar{\hat{C}}^{os}}{\partial\gamma_j}\right]D^{sk}\Bigg\}
\end{aligned}
$$

From Eq. (33b), it follows that

$$
\bar{\hat{C}}^{so}+\bar{\hat{C}}^{os} = \sum_{p\in\mathcal{A}}(\varepsilon_{po}-\varepsilon_{ps})\tilde{g}\gamma_p\tilde{g} + \sum_{p\in\mathcal{A}}(\varepsilon_{ps}-\varepsilon_{po})\tilde{g}\gamma_p\tilde{g} = 0
$$

so that K_{ij} simplifies to

$$
\begin{aligned}
K_{ij} &= -\mathcal{M}\Omega^2 g^T \sum_{k\in\mathcal{P}}\varepsilon_{ik}\left\{\frac{\partial\bar{\hat{C}}^{ik}}{\partial\gamma_j}\sum_{s\in\mathcal{B}-k}\tilde{D}^{ks}\tilde{g}\tilde{g}D^{sk} + \sum_{s\in\mathcal{B}-k}\tilde{D}^{ks}\frac{\partial\bar{\hat{C}}^{ks}}{\partial\gamma_j}\tilde{g}\tilde{g}D^{sk}\right\} \\
&= -\mathcal{M}\Omega^2 g^T \sum_{k\in\mathcal{P}}\varepsilon_{ik}\left\{\sum_{s\in\mathcal{B}-k}\left[\frac{\partial\bar{\hat{C}}^{ik}}{\partial\gamma_j}\tilde{D}^{ks} + \tilde{D}^{ks}\frac{\partial\bar{\hat{C}}^{ks}}{\partial\gamma_j}\right]\tilde{g}\tilde{g}D^{sk}\right\} \\
&= -\mathcal{M}\Omega^2 g^T \sum_{k\in\mathcal{P}}\varepsilon_{ik}\Big\{\sum_{s\in\mathcal{B}-k}\left[(\varepsilon_{jk}-\varepsilon_{ji})\tilde{g}\tilde{D}^{ks} + \tilde{D}^{ks}\tilde{g}(\varepsilon_{js}-\varepsilon_{jk})\right]\tilde{g}\tilde{g}D^{sk}\Big\} \\
&= -\mathcal{M}\Omega^2 \sum_{k\in\mathcal{P}}\varepsilon_{ik}\sum_{s\in\mathcal{B}-k}(\varepsilon_{js}-\varepsilon_{jk})\, g^T\tilde{D}^{ks}\tilde{g}\tilde{g}\tilde{g}D^{sk} \qquad (58b)
\end{aligned}
$$

since $g^T\tilde{g} = 0$. Because $\varepsilon_{jo} = 0$, the range of s can be modified to obtain

$$
K_{ij} = -\mathcal{M}\Omega^2 \sum_{k\in\mathcal{P}}\sum_{s\in\mathcal{P}-k}\varepsilon_{ik}\varepsilon_{js}g^T\tilde{D}^{ks}\tilde{g}\tilde{g}\tilde{g}D^{sk} + \mathcal{M}\Omega^2\sum_{k\in\mathcal{P}}\sum_{s\in\mathcal{B}-k}\varepsilon_{ik}\varepsilon_{jk}g^T\tilde{D}^{ks}\tilde{g}\tilde{g}\tilde{g}D^{sk} \qquad (58c)
$$

By exchanging i and j in this expression, one finds K_{ji} and discovers that, since s and k both cover the range of $\mathcal{P}$ in the first double sum in Eq. (58c), we have $K_{ji} - K_{ij} = 0$.

We may conclude that, at least for a vehicle of transverse deformation configuration with linearly elastic hinge torques and no external forces or torques, the vibration equation $I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0$ found in Eq. (42) offers the following symmetry properties: I and K are symmetric and J is skew-symmetric. Such a case was illustrated previously in Fig. 8. Moreover, if all vectors $\mathbf{D}^{qr}$ are parallel in the undeformed configuration for $q, r \in \mathcal{B}$, then (from Eq. 55b) the matrix J is zero. We may call such configurations *rectilinear transverse deformation cases.* Figures 9 and 10 illustrate examples of this kind, identified respectively as *polar* and *equatorial rectilinear transverse deformation cases.*

In the quest for special cases with demonstrable symmetry properties, we might next consider the configuration for which $\mathbf{g}^i = \mathbf{g}$ for all $i \in \mathcal{P}$; $\mathbf{g} \cdot \boldsymbol{\omega}^0 = 0$, and in the nominal state, $\mathbf{g} \cdot \mathbf{D}^{rk} = 0$ and $\boldsymbol{\omega}^0 \cdot \mathbf{D}^{rk} = 0$ for all $r, k \in \mathcal{B}$. Figure 11 illustrates this configuration, which is here referred to as the *meridional deformation case.*

In order to examine the structure of the equations of motion in this case, we choose $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ such that

$$\boldsymbol{\omega}^0 = \Omega \mathbf{b}_1^0 \tag{59a}$$

$$\mathbf{g}^i = \mathbf{b}_2^0 \text{ for } i \in \mathcal{P} \tag{59b}$$

$$\mathbf{D}^{rk} = D_{rk} \mathbf{b}_3^r \text{ for } r, k \in \mathcal{B} \tag{59c}$$

assuming moreover that in the nominal state, $\mathbf{b}_3^r = \mathbf{b}_3^0$ for all $r \in \mathcal{P}$. With these substitutions, the matrices J and K which first appeared in Eq. (42) take the form of Eqs. (60a, b) (see Eqs. 43 and 44, and note that

$$\mathbf{b}_2^k = \{\mathbf{b}^k\}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \triangleq \{\mathbf{b}^k\}^T b_2$$

for any $k \in \mathcal{B}$):

$$J_{ij} = -b_2^T \sum_{k \in \mathcal{P}} \varepsilon_{ik} \frac{\partial \bar{\hat{A}}^k}{\partial \dot{\gamma}_j} - \frac{\partial \vec{\hat{\tau}}_i}{\partial \dot{\gamma}_j} \tag{60a}$$

and

$$K_{ij} = -b_2^T \sum_{k \in \mathcal{P}} \varepsilon_{ik} \left[\frac{\partial \bar{\hat{C}}^{ik}}{\partial \gamma_j} \bar{\bar{A}}^k + \frac{\partial \bar{\hat{A}}^k}{\partial \gamma_j} \right] - \frac{\partial \bar{\hat{\tau}}_i}{\partial \gamma_j} \tag{60b}$$

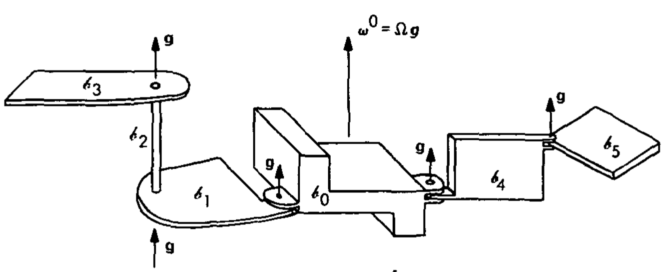

Fig. 8. Example of "transverse deformation case"

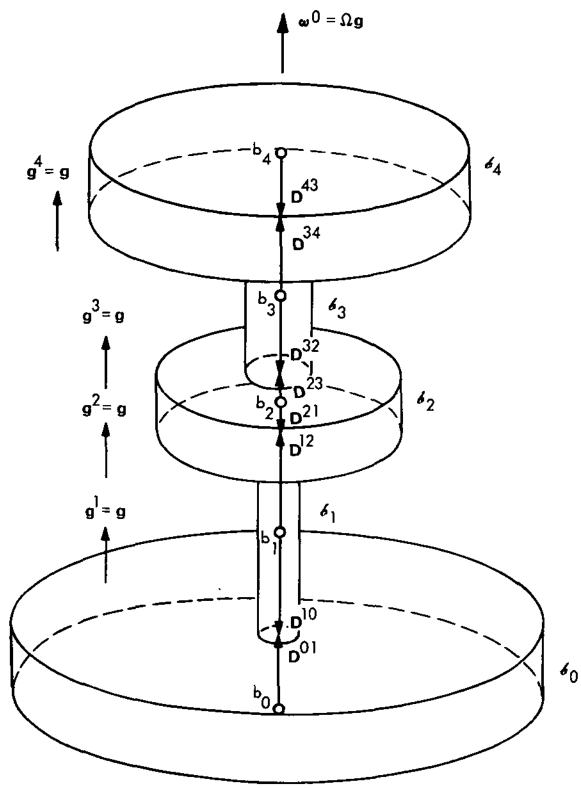

Fig. 9. Example of "polar transverse deformation case"

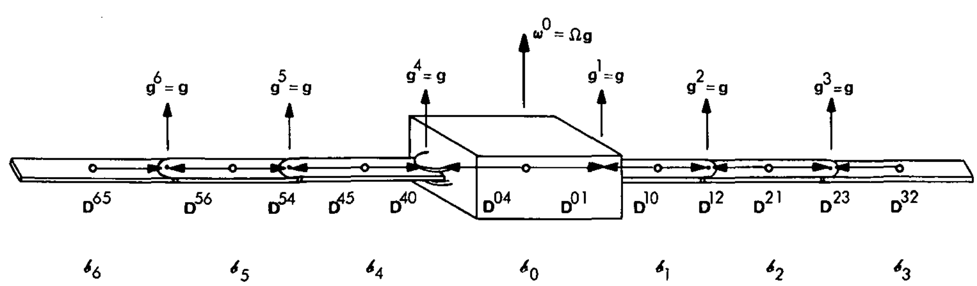

Fig. 10. Example of "equatorial transverse deformation case" in nominal configuration

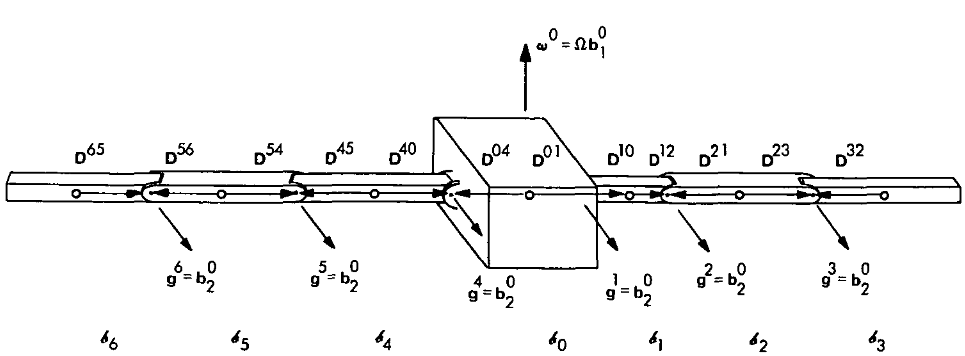

Fig. 11. Example of "meridional deformation case" in nominal configuration

These equations simplify most substantially when, as a further restriction, it is assumed that in the nominal state, the principal axes of the augmented body barycentric inertia dyadic $\boldsymbol{\Phi}^{kk}$ parallel $\mathbf{b}_1^0, \mathbf{b}_2^0$, and $\mathbf{b}_3^0$ for all $k \epsilon \mathcal{B}$. (Such a case is illustrated in Fig. 11.) In this case, Eqs. (59), (60a), and (49) can be combined to provide

$$
\begin{aligned}
J_{ij} = &-\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \Bigg\{ \frac{\partial \overline{\hat{T}}^k}{\partial \dot{\gamma}_j} + D_{ks}\tilde{b}_3 \frac{\partial \overline{\hat{F}}^s}{\partial \dot{\gamma}_j} - \varepsilon_{jk}\Phi_{33}^{kk} b_3 - \varepsilon_{jk}\Omega\,(\tilde{b}_2\Phi^{kk}b_1 - \tilde{b}_1\Phi^{kk}b_2) \\
&+ \mathcal{M} \sum_{s\epsilon\mathcal{B}-k} \Omega D_{ks}\tilde{b}_3 \varepsilon_{js} (\tilde{b}_2 b_1 + \tilde{b}_1 b_2)\, b_3 D_{sk} \\
&+ \mathcal{M} \sum_{r\epsilon\mathcal{B}-k} \varepsilon_{jr} D_{rk} D_{kr} (U b_3^T b_3 - b_3 b_3^T)\, \tilde{b}_1 b_2 \Bigg\} - \frac{\partial \overline{\hat{\mathfrak{F}}}_i}{\partial \dot{\gamma}_j} \\
= &-\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \left\{ \frac{\partial \overline{\hat{T}}^k}{\partial \dot{\gamma}_j} + D_{ks}\tilde{b}_3 \frac{\partial \overline{\hat{F}}^s}{\partial \dot{\gamma}_j} \right\} - \frac{\partial \overline{\hat{\mathfrak{F}}}_i}{\partial \dot{\gamma}_j}
\end{aligned}
\tag{61}
$$

This special case is comparable in its simplification of J to the transverse deformation case (see Eq. 55).

The matrix K in the meridional deformation case, with the indicated restrictions on the inertia dyadics, has elements given by Eqs. (59), (60b), (50), and (41a) in combination. The result is

$$
\begin{aligned}
K_{ij} = &-\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \Bigg\{ \frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} [\overline{\overline{T}}^k + \sum_{r\epsilon\mathcal{B}} D_{kr}\tilde{b}_3 \overline{\overline{F}}^r - \Omega^2 \tilde{b}_1 \Phi^{kk} b_1 \\
&+ \mathcal{M} \sum_{r\epsilon\mathcal{B}-k} \Omega^2 D_{kr}\tilde{b}_3\tilde{b}_1\tilde{b}_1 b_3 D_{rk}] + \frac{\partial \overline{\hat{T}}^k}{\partial \gamma_j} + \sum_{s\epsilon\mathcal{B}} D_{ks}\tilde{b}_3 \left(\frac{\partial \overline{\hat{F}}^s}{\partial \gamma_j} + \frac{\partial \overline{\hat{C}}^{ks}}{\partial \gamma_j} \overline{\overline{F}}^s \right) \\
&- \Omega^2 \left[\frac{\partial \overline{\hat{C}}^{ko}}{\partial \gamma_j} \tilde{b}_1 + \tilde{b}_1 \frac{\partial \overline{\hat{C}}^{ok}}{\partial \gamma_j} \right] \Phi^{kk} b_1 - \Omega^2 \tilde{b}_1 \Phi^{kk} \frac{\partial \overline{\hat{C}}^{ko}}{\partial \gamma_j} b_1 \\
&+ \mathcal{M} \sum_{s\epsilon\mathcal{B}-k} \Omega^2 D_{kj}\tilde{b}_3 \Bigg\{ \frac{\partial \overline{\hat{C}}^{ks}}{\partial \gamma_j} \tilde{b}_1\tilde{b}_1 b_3 + \Bigg\{ \left[\frac{\partial \overline{\hat{C}}^{so}}{\partial \gamma_j} \tilde{b}_1 + \tilde{b}_1 \frac{\partial \overline{\hat{C}}^{os}}{\partial \gamma_j} \right] \tilde{b}_1 \\
&+ \tilde{b}_1 \left[\frac{\partial \overline{\hat{C}}^{so}}{\partial \gamma_j} \tilde{b}_1 + \tilde{b}_1 \frac{\partial \overline{\hat{C}}^{os}}{\partial \gamma_j} \right] \Bigg\} b_3 D_{sk} \Bigg\} - \frac{\partial \overline{\hat{\mathfrak{F}}}_i}{\partial \gamma_j} \\
= &-\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \Bigg\{ \frac{\partial \overline{\hat{C}}^{ik}}{\partial \gamma_j} [\overline{\overline{T}}^k + \sum_{r\epsilon\mathcal{B}} D_{kr}\tilde{b}_3 \overline{\overline{F}}^r] + \frac{\partial \overline{\hat{T}}^k}{\partial \gamma_j} + \sum_{s\epsilon\mathcal{B}} D_{ks}\tilde{b}_3 \left(\frac{\partial \overline{\hat{F}}^s}{\partial \gamma_j} + \frac{\partial \overline{\hat{C}}^{ks}}{\partial \gamma_j} \overline{\overline{F}}^s \right) \\
&- \Omega^2 \tilde{b}_1 \left[\frac{\partial \overline{\hat{C}}^{ok}}{\partial \gamma_j} \Phi_1^{kk} + \Phi^{kk} \frac{\partial \overline{\hat{C}}^{ko}}{\partial \gamma_j} \right] b_1 \\
&+ \mathcal{M} \sum_{s\epsilon\mathcal{B}-k} \Omega^2 D_{ks} D_{sk} \tilde{b}_3 \tilde{b}_1 \left[-\frac{\partial \overline{\hat{C}}^{so}}{\partial \gamma_j} b_2 + \tilde{b}_1 \frac{\partial \overline{\hat{C}}^{os}}{\partial \gamma_j} b_3 \right] \Bigg\} - \frac{\partial \overline{\hat{\mathfrak{F}}}_i}{\partial \gamma_j}
\end{aligned}
\tag{62}
$$

By returning to Eq. (34) and comparing with Eq. (59b), we can determine that

$$
\frac{\partial \overline{\hat{C}}^{rm}}{\partial \gamma_j} = (\varepsilon_{jm} - \varepsilon_{jr})\, \tilde{g}^j = (\varepsilon_{jm} - \varepsilon_{jr})\, \tilde{b}_2 \tag{63}
$$

so that all partial derivatives of direction cosine matrices in Eq. (62) are proportional to $\tilde{b}_2$. With the substitution of Eq. (63) into Eq. (62), we find

$$
\begin{aligned}
K_{ij} = & -\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \Bigg\{ (\varepsilon_{jk} - \varepsilon_{ji}) \tilde{b}_2 \left[\bar{\bar{T}}^k + \sum_{r\epsilon\mathcal{B}} D_{kr} \tilde{b}_3 \bar{\bar{F}}^r + \frac{\partial \bar{\hat{T}}^k}{\partial \gamma_j} \right] \\
& + \sum_{s\epsilon\mathcal{B}} D_{ks} \tilde{b}_3 \left[\frac{\partial \bar{\hat{F}}^s}{\partial \gamma_j} + (\varepsilon_{js} - \varepsilon_{jk}) \tilde{b}_2 \bar{\bar{F}}^s \right] \\
& - \Omega^2 \tilde{b}_1 \left[(\varepsilon_{jk} - \varepsilon_{j0}) \, \tilde{b}_2 b_1 \Phi_1^{kk} + \Phi^{kk} \, (\varepsilon_{j0} - \varepsilon_{jk}) \tilde{b}_2 b_1 \right] \Bigg\} - \frac{\partial \hat{\bar{T}}_i}{\partial \gamma_j} \\
= & -\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \Bigg\{ (\varepsilon_{jk} - \varepsilon_{ji}) \tilde{b}_2 \left[\bar{\bar{T}}^k + \sum_{r\epsilon\mathcal{B}} D_{kr} \tilde{b}_3 \bar{\bar{F}}^r + \frac{\partial \bar{\hat{T}}^k}{\partial \gamma_j} \right] \\
& + \sum_{s\epsilon\mathcal{B}} D_{ks} \tilde{b}_3 \left[\frac{\partial \bar{\hat{F}}^s}{\partial \gamma_j} + (\varepsilon_{js} - \varepsilon_{jk}) \tilde{b}_2 \bar{\bar{F}}^s \right] \\
& - \Omega^2 b_2 \left[(\varepsilon_{jk} - \varepsilon_{j0}) \, (\Phi_1^{kk} - \Phi_3^{kk}) \right] \Bigg\} - \frac{\partial \hat{\bar{T}}_i}{\partial \gamma_j}
\end{aligned}
$$

Since $\varepsilon_{j0} = 0$ for all j, and $b_2^T \tilde{b}_2$ (any vector) $= 0$, and $b_2^T b_2 = 1$, this expression reduces to

$$
\begin{aligned}
K_{ij} = & -\sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} b_2^T \tilde{b}_3 \sum_{s\epsilon\mathcal{B}} D_{ks} \left[\frac{\partial \bar{\hat{F}}^s}{\partial \gamma_j} + (\varepsilon_{js} - \varepsilon_{jk}) \tilde{b}_2 \bar{\bar{F}}^s \right] \\
& + \Omega^2 \sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} \varepsilon_{jk} \, (\Phi_1^{kk} - \Phi_3^{kk}) - \frac{\partial \hat{\bar{T}}_i}{\partial \gamma_j}
\end{aligned}
\tag{64}
$$

Finally, we have found, in this meridional deformation case, with restricted inertias and all local deformations small, a class of system for which a symmetric K matrix is a realistic possibility. In particular, if external forces are zero and the hinges are elastically constrained as in Eq. (45), we can write Eq. (42) as

$$
I\ddot{\gamma} + K\gamma = 0 \tag{65}
$$

where K and I are symmetric matrices with elements given respectively for $i, j \,\epsilon\, \mathcal{P}$ by

$$
K_{ij} = \Omega^2 \sum_{k\epsilon\mathcal{P}} \varepsilon_{ik} \varepsilon_{jk} \, (\Phi_1^{kk} - \Phi_3^{kk}) + \delta_{ij} k_i \tag{66}
$$

and, from Eq. (36d),

$$
\begin{aligned}
I_{ij} = \bar{\bar{a}}_{ij} = & \; b^{2T} \sum_{r\epsilon\mathcal{P}} \varepsilon_{ir} \varepsilon_{jr} \Phi^{rr} b_2 \\
& - \mathcal{M} b_2^T \sum_{r\epsilon\mathcal{P}} \sum_{s\epsilon\mathcal{P}-r} \varepsilon_{ir} \varepsilon_{js} D_{sr} D_{rs} (U - b_3 b_3^T) \, b_2 \\
= & \sum_{r\epsilon\mathcal{P}} \varepsilon_{ir} \varepsilon_{jr} \Phi_2^{rr} - \mathcal{M} \sum_{r\epsilon\mathcal{P}} \sum_{s\epsilon\mathcal{P}-r} \varepsilon_{ir} \varepsilon_{js} D_{sr} D_{rs}
\end{aligned}
\tag{67}
$$

In summary, we observe that in every case the linear, constant-coefficient equations of motion which are to provide the basis for coordinate transformations have the form (see Eq. 42)

$$I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0$$

where I is a nonsingular symmetric matrix.

Moreover, $J = 0$ and K is symmetric in the following special cases:

(1) Hinge torques are elastic, and

(2) External forces and external torques are zero, and

(3) Either
  (a) The nominal spin rate is zero, or
  (b) Only rectilinear transverse deformations occur, or
  (c) Only meridional deformations occur and all augmented body principal axes are suitably aligned.

Furthermore, J is skew-symmetric and K is symmetric for the general transverse deformation case under conditions (1) and (2) above.

It should perhaps be emphasized that the conditions outlined have been shown only to be *sufficient* for the noted symmetry properties; they are *not*, in general, *necessary* conditions.

From the basic relationships of analytical dynamics, it can be shown that for conservative, holonomic systems, the linearized matrix variational equations obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \cdots, n) \tag{68}$$

(where L is the Lagrangian and q_i a generalized coordinate) always have the structure

$$M\ddot{q} + G\dot{q} + Nq = 0 \tag{69}$$

where $q \triangleq [q_1 \cdots q_n]^T$, M and N are symmetric, and G is skew-symmetric. It may be anticipated that these symmetry properties are also present for Eq. (42) for holonomic, conservative systems, despite the preference given in this report to an Eulerian rather than Lagrangian formulation of the equations of motion.

In the following section, it will be demonstrated that the indicated symmetry properties of Eq. (69) or Eq. (42) can provide computational advantages in the determination of coordinate transformations which will permit these equations to be replaced by a set of uncoupled scalar equations.

D. Transformation to Large-Deformation Modal Coordinates

As established in the preceding section, the equations to be adopted as a basis for coordinate transformation for a given substructure can be written as the matrix second-order differential equation (Eq. 42)

$$I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0$$

where γ is of dimension ν by 1, I is nonsingular and symmetric, and, in particular cases, J and K have special properties. This equation can also be written in the form

$$P\dot{\Gamma} = Q\Gamma \tag{70}$$

where, in terms of matrix partitions, we define

$$\Gamma \triangleq \left[\begin{array}{c} \gamma \\ \hline \dot{\gamma} \end{array}\right] \qquad P \triangleq \left[\begin{array}{c|c} K & 0 \\ \hline 0 & I \end{array}\right] \qquad Q \triangleq \left[\begin{array}{c|c} 0 & K \\ \hline -K & -J \end{array}\right]$$

Since I^{-1} exists, it is also possible to write Eq. (42) in the standard form

$$\dot{\Gamma} = B\Gamma \tag{71}$$

where

$$B \triangleq \left[\begin{array}{c|c} 0 & U \\ \hline -I^{-1}K & -I^{-1}J \end{array}\right]$$

with U the ν by ν unit matrix.

It is our objective in this section to find a new set of coordinates to replace γ, $\dot{\gamma}$, and $\ddot{\gamma}$ in the partially linearized equations of motion. [See Eq. 30c, where $\gamma^2, \gamma^4, \cdots, \gamma^{N+1}$ (for odd N) each represent column matrices containing flexible substructure deformation variables represented generically in this section by γ.] The rationale for coordinate transformation is to minimize coupling among the new coordinates in the equations of motion, in order to make coordinate truncation feasible. As a practical compromise with this goal, we might instead seek transformations which uncouple the second-order differential equations in Eq. (42); but even this goal is not attainable unless the matrices J and K have special properties. We must therefore often be satisfied with a transformation which uncouples the first-order scalar differential equations in Eq. (70) or its equivalent, Eq. (71). Although it may be more convenient computationally to work with Eq. (70), thereby avoiding the matrix inversion of I in Eq. (71), the latter provides a simpler basis for discussion of the properties of the system.

If the Jordan canonical form of the matrix B in Eq. (71) were nondiagonal, it would be impossible to find a transformation which would uncouple the associated scalar equations. Such a situation would, however, imply powers of t in the general solution of Eqs. (71) and (70), and this would in itself tell us that the dynamic response of the spacecraft is unacceptable. We therefore *assume* in what follows that the Jordan form of B is diagonal. In other words, we assume that there exists a similarity transformation matrix Φ such that

$$\Phi^{-1}B\Phi = \lambda \tag{72}$$

where λ is a diagonal matrix with elements $\lambda_1, \cdots, \lambda_{2\nu}$. Pre-multiplication of Eq. (72) by Φ produces

$$B\Phi = \Phi\lambda$$

indicating that the columns $\Phi^1, \cdots, \Phi^{2\nu}$ of Φ satisfy

$$B\Phi^j = \Phi^j\lambda_j = \lambda_j\Phi^j \qquad j = 1, \cdots, 2\nu \tag{73}$$

so that λ_j is an eigenvalue of B and Φ^j is the corresponding eigenvector. The existence of Φ^{-1} implied by Eq. (72) indicates that the eigenvectors of B must be independent for B to be diagonalizable.

Post-multiplication of Eq. (72) by Φ^{-1} produces

$$\Phi^{-1}B = \lambda\Phi^{-1}$$

so that the rows of Φ^{-1} are left eigenvectors of B; that is, if $\Psi^{1^T}, \cdots, \Psi^{2\nu^T}$ are the rows of Φ^{-1}, then

$$\Psi^{j^T}B = \lambda_j\Psi^{j^T} = \Psi^{j^T}\lambda_j \qquad j = 1, \cdots, 2\nu$$

With this interpretation, it is convenient to replace Φ^{-1} by the matrix Ψ^T, where the columns of Ψ are left eigenvectors $\Psi^1, \cdots, \Psi^{2\nu}$, normalized so that $\Psi^T\Phi = U$. It may be more convenient computationally to construct Ψ by assembling left eigenvectors rather than by inverting Φ.

We can now diagonalize Eq. (71) with the transformation

$$\Gamma = \Phi Y \tag{74}$$

followed by pre-multiplication by Ψ^T. The result is

$$\Psi^T\Phi\dot{Y} = \Psi^T B\Phi Y$$

or equivalently,

$$\Phi^{-1}\Phi\dot{Y} = \Phi^{-1}B\Phi Y$$

or

$$\dot{Y} = \lambda Y \tag{75}$$

Having satisfied ourselves that Eq. (71), and therefore Eq. (70), can be diagonalized if the eigenvectors of B are independent (so that Φ^{-1} exists), we should note that for this independence, it is sufficient (but not necessary) that we have distinct eigenvalues. For if the smallest set of dependent eigenvectors of B is designated $\Phi^1, \cdots, \Phi^r$, then they must be related by

$$\sum_{j=1}^{r} \alpha_j\Phi^j = 0 \tag{76}$$

in which none of the α_j is zero. Pre-multiplication by B then produces (using Eq. 73)

$$\sum_{j=1}^{r} \alpha_j B\Phi^j = \sum_{j=1}^{r} \alpha_j\lambda_j\Phi^j = 0 \tag{77}$$

Multiplying Eq. (76) by λ_r and subtracting Eq. (77) gives

$$\sum_{j=1}^{r-1} \alpha_j (\lambda_j - \lambda_r) \Phi^j = 0$$

If $(\lambda_j - \lambda_r) \neq 0$, then this equation contradicts Eq. (76), which asserted that the smallest set of dependent eigenvectors has dimension r. Therefore, the hypothesis of Eq. (76) is invalid unless the eigenvalues λ_j and λ_r are identical; that is to say, if the eigenvalues are distinct (nonrepeated), then the eigenvectors must be independent. The converse is not true; that is, eigenvectors can be independent even when eigenvalues are repeated. Nonetheless, we shall in what follows find it convenient to concentrate on the special (and most common) case in which the eigenvalues of the system of Eq. (70) or Eq. (71) are distinct.

Now we can direct our attention to Eq. (70), which we may prefer to the equivalent Eq. (71) for computational reasons. Equation (70) admits a solution

$$\Gamma = \Phi^j e^{\lambda_j t} \tag{78}$$

where Φ^j and λ_j satisfy

$$(Q - \lambda_j P) \Phi^j = 0 \tag{79}$$

as may be confirmed by substituting Eq. (78) into Eq. (70). Equation (79) requires

$$|Q - \lambda_j P| = 0 \tag{80}$$

which produces 2ν values of the scalar λ_j $(j = 1, \cdots, 2\nu)$. Because Q and P are real, complex roots of λ_j appear as complex conjugate pairs. We refer to these scalars $\lambda_1, \cdots, \lambda_{2\nu}$ as the eigenvalues of the differential operator in Eq. (70), and $\Phi^1, \cdots, \Phi^{2\nu}$ are the corresponding eigenvectors. Since Eqs. (70) and (71) are equivalent, they both admit the solution in Eq. (78), and these eigenvalues and eigenvectors also belong to the matrix B in Eq. (71). In any case, the eigenvectors can be solved (from Eq. 79 or Eq. 73) only to within a multiplicative constant, and we might select this constant (normalizing eigenvectors) differently for Eq. (70) than for Eq. (71).

If we introduce the transformation $\Gamma = \Phi Y$ (as in Eq. 74) into Eq. (70), we find

$$P\Phi\dot{Y} = Q\Phi Y \tag{81}$$

We now require a matrix pre-multiplier to take Eq. (81) into the uncoupled form of Eq. (75).

We can obtain the necessary matrix formally by considering the eigenvectors of

$$P^T\dot{\Gamma}' = Q^T\Gamma' \tag{82}$$

which is sometimes called the *adjoint* of Eq. (70). The eigenvalues λ'_j and eigenvectors $\Phi^{j'}$ of Eq. (82) must satisfy

$$(Q^T - \lambda'_j P^T)\,\Phi^{j'} = 0 \tag{83}$$

and

$$|Q^T - \lambda'_j P^T| = 0 \tag{84}$$

in parallel with Eqs. (79) and (80). Since the value of a determinant is unchanged by transposition, Eqs. (80) and (84) yield the same roots, and the eigenvalues of Eq. (70) and its adjoint, Eq. (82), are identical. Transposition of Eq. (83) produces (after changing the arbitrary index j to r and replacing λ'_r by λ_r)

$$\Phi^{r'T}(Q - \lambda_r P) = 0 \tag{85}$$

so that the eigenvectors $\Phi^{r'}$ of the adjoint equation are the *left eigenvectors* of the original Eq. (70)

The significance of the left eigenvectors depends upon an orthogonality property which we can establish by pre-multiplying Eq. (79) by $\Phi^{r'T}$, to find

$$\Phi^{r'T} Q \Phi^j = \Phi^{r'T} P \Phi^j \lambda_j \tag{86}$$

Post-multiplying Eq. (85) by Φ^j produces, for comparison,

$$\Phi^{r'T} Q \Phi^j = \Phi^{r'T} P \Phi^j \lambda_r \tag{87}$$

Subtracting Eq. (86) from Eq. (87) yields

$$0 = \Phi^{r'T} P \Phi^j (\lambda_r - \lambda_j)$$

which, for $\lambda_r \neq \lambda_j$, requires the orthogonality relationship

$$\Phi^{r'T} P \Phi^j = 0 \qquad r \neq j \tag{88a}$$

Equations (86) and (88a) now combine to produce

$$\Phi^{r'T} Q \Phi^j = 0 \qquad r \neq j \tag{88b}$$

If now we construct the matrix Φ' whose columns are $\Phi^{1'}, \Phi^{2'}, \cdots, \Phi^{2\nu'}$, then the orthogonality conditions in Eq. (88) indicate that, if all eigenvalues are distinct, $\Phi'^T P \Phi$ and $\Phi'^T Q \Phi$ are diagonal matrices. This means, of course, that Φ'^T is the pre-multiplier we need in order to diagonalize Eq. (81). We choose to normalize the left and right eigenvectors in such a way that

$$\Phi'^T P \Phi = U \tag{89}$$

Then, the equation (from Eq. 81)

$$\Phi'^T P \Phi \dot{Y} = \Phi'^T Q \Phi Y$$

must (by comparison with Eq. 75) become

$$\dot{Y} = \lambda Y \tag{90}$$

In summary of the general case, we have demonstrated formally that Eq. (71) can be transformed to the uncoupled form $\dot{Y} = \lambda Y$ provided only that the eigenvectors of B are independent, and we have demonstrated directly that the original Eq. (70) can also be transformed into $\dot{Y} = \lambda Y$ if its eigenvalues are distinct. For a formal proof that Eq. (70) can be transformed into $\dot{Y} = \lambda Y$ even when eigenvalues are repeated as long as eigenvectors are independent, we can rely upon the equivalence of Eqs. (70) and (71).

Let us now direct our attention to special cases for which transformation is simplified. If, in the original second-order equation (Eq. 42), the matrix K (as well as I) is symmetric and J is skew-symmetric, then, in Eq. (70), the matrix P is symmetric and Q is skew-symmetric. Equation (83) then becomes

$$(-Q - \lambda_j' P)\,\Phi^{j'} = 0$$

or (since $\lambda_j' = \lambda_j$)

$$(Q + \lambda_j P)\,\Phi^{j'} = 0 \tag{91}$$

In comparing this result with Eq. (75), we first recall that the complex eigenvalues occur in complex conjugate pairs (since Q and P are real). Next, we observe that the equation (Eq. 42)

$$I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0$$

possesses the first integral

$$\dot{\gamma}^T I \dot{\gamma} + \gamma^T K \gamma = \text{constant} \tag{92}$$

as may be confirmed by pre-multiplying Eq. (42) by $\dot{\gamma}^T$ and observing that the scalar $\dot{\gamma}^T J \dot{\gamma}$ must be zero, since its transpose is $\dot{\gamma}^T J^T \dot{\gamma} = -\dot{\gamma}^T J \dot{\gamma}$ and any scalar must equal its transpose. The first integral in Eq. (92) is a reflection of the absence of nonconservative forces in the dynamical system represented by Eq. (42) in this restricted case.

Equation (78) provides us with a solution of Eq. (42) when the former is written in partitioned form as

$$\Gamma \triangleq \left[\frac{\gamma}{\dot{\gamma}}\right] = \left[\frac{\phi^j}{\lambda_j \phi^j}\right] e^{\lambda_j t} \triangleq \Phi^j e^{\lambda_j t}$$

that is to say, γ has the solution

$$\gamma = \phi^j e^{\lambda_j t} \tag{93}$$

Since γ is real, and ϕ^j and λ_j are generally complex, the conjugate pairs of eigenvalues and eigenvectors must be so combined that pairs of solutions such as Eq. (93) appear as

$$\gamma = a_j \phi^j e^{\lambda_j t} + a_j^* \phi^{j*} e^{\lambda_j^* t} \tag{94}$$

where a_j is an arbitrary scalar multiplier and * denotes complex conjugate. To appreciate the reality of γ in Eq. (94), and the possible complexity of γ in Eq. (93), let $\lambda_j \triangleq \alpha_j + i\sigma_j$ and $a_j = c_j + id_j$, and write Eq. (94) in terms of its real and imaginary parts:

$$\begin{aligned}
\gamma = &\ e^{\alpha_j t} \{(c_j + id_j)(Re\phi^j + i\mathcal{I}m\phi^j)(\cos\sigma_j t + i\sin\sigma_j t) \\
&+ (c_j - id_j)(Re\phi^j - i\mathcal{I}m\phi^j)(\cos\sigma_j t - i\sin\sigma_j t)\} \\
= &\ e^{\alpha_j t} \{Re\phi^j [c_j \cos\sigma_j t + d_j \sin\sigma_j t] \\
&+ \mathcal{I}m\phi^j [d_j \cos\sigma_j t - c_j \sin\sigma_j t]\}
\end{aligned} \tag{95}$$

Here $i \triangleq (-1)^{1/2}$.

The solution for γ in Eq. (95) must satisfy the first integral in Eq. (92). If I and K are both positive semi-definite (or both negative semi-definite), then $\alpha_j = 0$ in Eq. (95), and the eigenvalue λ_j is purely imaginary for all j. If I and K are not both positive semi-definite (or both negative semi-definite), then (by Liapunov's theorem) the null solution of Eq. (42) is unstable. As previously, we reject the latter possibility (which would indicate unacceptable spacecraft performance). Now we can proceed with the comparison of Eqs. (91) and (75), knowing that for stable systems with the assumed symmetry properties of I, J, and K, we have purely imaginary eigenvalues. Then, Eq. (91) becomes

$$(Q - \lambda_j^* P)\,\Phi^{j\prime} = 0$$

indicating that $\Phi^{j\prime} = \Phi^{j*}$, and

$$\Phi' = \Phi^* \tag{96}$$

Thus, in this special case, the left eigenvectors need not be calculated as an extra computational task, but may be recorded by inspection of the (right) eigenvectors.

The second special case of Eq. (42) which is of practical interest requires $J \equiv 0$ and I, K symmetric. In this case, we can work directly with the second-order differential equations, constructing the ν by ν matrix ϕ whose columns are $\phi^1, \cdots, \phi^\nu$ (see Eq. 93) and introducing the transformation

$$\gamma = \phi\eta \tag{97}$$

to obtain

$$I\phi\ddot{\eta} + K\phi\eta = 0 \tag{98}$$

Again we seek a pre-multiplier which uncouples these equations, and again we will turn to the transpose of the matrix of left eigenvectors. In this case, however, we can discover upon substitution of Eq. (93) into the restricted version of Eq. (42) that

$$(\lambda_j^2 I + K)\,\phi^j = 0 \tag{99}$$

so that an imaginary eigenvalue λ_j corresponds to a real eigenvector ϕ^j. Moreover, ϕ^j is a left eigenvector as well as a right eigenvector, since, because of the symmetry of I and K, we have (changing j to r for later convenience)

$$\phi^{r^T}(\lambda_r^2 I + K) = 0 \tag{100}$$

If now we pre-multiply Eq. (99) by ϕ^{r^T} and subtract the result from the post-product of Eq. (100) by ϕ^j, we get

$$\phi^{r^T}K\phi^j - \phi^{r^T}K\phi^j + \phi^{r^T}I\phi^j(\lambda_r^2 - \lambda_j^2) = 0$$

For $\lambda_r^2 \neq \lambda_j^2$, we thus have the new orthogonality conditions

$$\phi^{r^T}I\phi^j = 0 \qquad r \neq j \tag{101a}$$

and

$$\phi^{r^T}K\phi^j = 0 \qquad r \neq j \tag{101b}$$

We further observe that pre-multiplying Eq. (99) by ϕ^{j^T} produces

$$\phi^{j^T}K\phi^j = -\lambda_j^2\phi^{j^T}I\phi^j$$

which, by virtue of the imaginary character of $\lambda_j = i\sigma_j$, becomes

$$\phi^{j^T}K\phi^j = \sigma_j^2\phi^{j^T}I\phi^j \tag{102}$$

If we choose to normalize the eigenvectors in ϕ such that $\phi^{j^T}I\phi^j = 1$ for all j, then pre-multiplication of Eq. (98) by ϕ^T yields

$$\ddot{\eta} + \sigma^2\eta = 0 \tag{103}$$

where

$$\sigma^2 \triangleq \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_\nu^2 \end{bmatrix}$$

We have proven this result formally only for the case of distinct (nonrepeated) eigenvalues, but the general argument preceding it applies whenever the eigenvectors are independent. Although there are other special cases for which special transformations can replace the general procedure developed here for Eq. (70), these have been examined previously in this series (Ref. 21), and will not be re-examined here. It should perhaps only be mentioned that it is a common practice among structural dynamicists to accommodate structural damping not by generating explicitly a symmetric matrix J for Eq. (42), but instead by modifying Eq. (103) to the form

$$\ddot{\eta} + 2\zeta\sigma\dot{\eta} + \sigma^2\eta = 0 \tag{104}$$

where ζ is a diagonal matrix whose nonzero elements (the "modal damping ratios") are established by experiment or by the judgment of an experienced analyst. This practice amounts to assuming the matrix J in Eq. (42) to be a linear combination of I and K, which seems to be a reasonable supposition for a wide class of structures.

E. Hybrid-Coordinate Equations

The matrix equation for unrestricted motion (Eq. 1) has been written as Eq. (30) for the special case in which certain of the angles of relative rotation and their time derivatives remain small enough to justify the neglect of terms above the first degree. In Eqs. (30a) and (30b), all linearized angles are collected in the single set $\mathcal{A}$. In Eq. (30c), these equations are repeated in the form of a single matrix equation, and the variables in $\mathcal{A}$ are subdivided into the column matrices $\gamma^2, \cdots, \gamma^{N+1}$, for which in each case, the hinge labels are sequential. When all of the flexible appendages are classified as terminal appendages, the corresponding bodies should be given sequential indices, including the highest index in the system, so that in Eq. (30c) all small variables are in γ^2 and all unrestricted angles are in γ^1. Finally, the partially linearized equations appear in Eq. (30d) as a first-order matrix equation, with the same grouping of linearized and unrestricted angles.

In what follows, we let γ^k designate a typical submatrix of linearized variables, so $k = 2, \cdots, N + 1$. Corresponding to each value of k, there is a linear, constant-coefficient equation of the form of Eq. (42), which we now write as

$$I^k\ddot{\gamma}^k + J^k\dot{\gamma}^k + K^k\gamma^k = 0 \tag{105}$$

We write the equivalent first-order equations (obtained either from Eq. 30 or from Eq. 105 directly) in the form of Eqs. (70) and (71), i.e.,

$$P^k\dot{\Gamma}^k = Q^k\Gamma^k \tag{106}$$

and

$$\dot{\Gamma}^k = B^k\Gamma^k \tag{107}$$

where

$$\Gamma^k \triangleq \begin{bmatrix} \gamma^k \\ \hline \dot{\gamma}^k \end{bmatrix} \tag{108}$$

In the general case, we introduce the transformation

$$\Gamma^k = \Phi^k Y^k \tag{109}$$

as in Eq. (74). The corresponding transformations for γ^k and $\dot{\gamma}^k$ individually are apparent from Eqs. (108) and (109).

In terms of the ν by ν unit matrix U and the ν by ν null matrix 0, we define the matrix operators

$$\Sigma_{U0}^T \triangleq [U \,|\, 0]; \qquad \Sigma_{0U}^T \triangleq [0 \,|\, U] \tag{110}$$

From Eqs. (108)–(110), we have

$$\gamma^k = \Sigma_{U0}^T \Gamma^k = \Sigma_{U0}^T \Phi^k Y^k \tag{111}$$

and

$$\dot{\gamma}^k = \Sigma_{0U}^T \Gamma^k = \Sigma_{0U}^T \Phi^k Y^k \tag{112}$$

These transformations can be substituted into the right-hand side of Eq. (30d), while the corresponding transformation in Eq. (109) is substituted on the left side. At the same time, we might substitute

$$\gamma^k = \Sigma_{U0}^T \Gamma^k \qquad (k \text{ odd}) \tag{113}$$

and

$$\dot{\gamma}^k = \Sigma_{0U}^T \Gamma^k \qquad (k \text{ odd}) \tag{114}$$

into the right side of Eq. (30d) in order to achieve a more consistent notation.

Eq. (30d) then becomes

$$\left[\begin{array}{c|c|c|c|c|c}
A_{00} & A_{01} & A_{02}\Phi^2 & \cdots & A_{0N} & A_{0,N+1}\Phi^{N+1} \\ \hline
A_{10} & A_{11} & A_{12}\Phi^2 & \cdots & A_{1N} & A_{1,N+1}\Phi^{N+1} \\ \hline
A_{20} & A_{21} & A_{22}\Phi^2 & \cdots & A_{2N} & A_{2,N+1}\Phi^{N+1} \\ \hline
A_{30} & A_{31} & A_{32}\Phi^2 & \cdots & A_{3N} & A_{3,N+1}\Phi^{N+1} \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
A_{N+1,0} & A_{N+1,1} & A_{N+1,2}\Phi^2 & \cdots & A_{N+1,N} & A_{N+1,N+1}\Phi^{N+1}
\end{array}\right]
\left[\begin{array}{c}
\dot{\omega}^0 \\ \hline \dot{\Gamma}^1 \\ \hline \dot{Y}^2 \\ \hline \dot{\Gamma}^3 \\ \hline \vdots \\ \hline \dot{Y}^{N+1}
\end{array}\right] =$$

$$\left[\begin{array}{c}
\sum\limits_{k\in\mathcal{B}} \left[\left(\overline{C}^{0k} + \hat{C}^{0k}\right)\overline{A}^k + \overline{C}^{0k}\hat{A}^k\right] - \sum\limits_{k\in\mathcal{G}} a_{0k}\ddot{\gamma}_k \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^T \Gamma^1 \\ \hline R^1 \end{array}\right] \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^T \Phi^2 Y^2 \\ \hline R^2 \end{array}\right] \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^T \Gamma^3 \\ \hline R^3 \end{array}\right] \\ \hline
\vdots \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^T \Phi^{N+1} Y^{N+1} \\ \hline R^{N+1} \end{array}\right]
\end{array}\right] \tag{115}$$

In anticipation of coordinate truncation, we would like to accomplish all possible uncoupling of these equations of motion. We have no hope of uncoupling the entire set of equations, but we would like to come as close as possible to the point of uncoupling from each other the coordinates which define the oscillatory deformations of an individual flexible substructure. We know from Eq. (89) that

$$\Phi'^{k^T} P^k \Phi^k = U \tag{116}$$

and we can readily identify P^k as the nominal value of A_{kk} appearing in Eq. (115) for k even. Thus, we can obtain the equations of motion most nearly suitable for coordinate truncation by replacing Eq. (115) by an equivalent equation in which the row partition corresponding to Y^k is multiplied by Φ'^{k^T}, the transposed matrix of adjoint system eigenvectors.

The result is

$$\left[\begin{array}{c|c|c|c|c}
A_{00} & A_{01} & A_{02}\Phi^2 & \cdots & A_{0,N+1}\Phi^{N+1} \\ \hline
A_{10} & A_{11} & A_{12}\Phi^2 & \cdots & A_{1,N+1}\Phi^{N+1} \\ \hline
\Phi'^{2^T}A_{20} & \Phi'^{2^T}A_{21} & \Phi'^{2^T}A_{22}\Phi^2 & \cdots & \Phi'^{2^T}A_{2,N+1}\Phi^{N+1} \\ \hline
\vdots & \vdots & \vdots & \cdot & \vdots \\ \hline
\Phi'^{N+1^T}A_{N+1,0} & \Phi'^{N+1^T}A_{N+1,1} & \Phi'^{N+1^T}A_{N+1,2}\Phi^2 & \cdots & \Phi'^{N+1^T}A_{N+1,N+1}\Phi^{N+1}
\end{array}\right]
\left[\begin{array}{c}
\dot{\omega}^0 \\ \hline \dot{\Gamma}^1 \\ \hline \dot{Y}^2 \\ \hline \vdots \\ \hline \dot{Y}^{N+1}
\end{array}\right] =$$

$$\left[\begin{array}{c}
\sum\limits_{k\in\mathcal{B}} [(\overline{C}^{0k} + \hat{C}^{0k})\overline{A}^k + \overline{C}^{0k}\hat{A}^k] - \sum\limits_{k\in\mathcal{G}} a_{0k}\ddot{\gamma}_k \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^{T}\Gamma^1 \\ \hline \overline{R}^1 \end{array}\right] \\ \hline
\Phi'^{2^T}\left[\begin{array}{c} \Sigma_{0U}^{T}\Phi^2 Y^2 \\ \hline \overline{R}^2 \end{array}\right] \\ \hline
\vdots \\ \hline
\Phi'^{N+1^T}\left[\begin{array}{c} \Sigma_{0U}^{T}\Phi^{N+1} Y^{N+1} \\ \hline \overline{R}^{N+1} \end{array}\right]
\end{array}\right] \tag{117}$$

The matrices $\Phi'^{j^T} A_{jj} \Phi^j$ are diagonal (j even) when the corresponding matrix A_{jj} assumes its nominal value, P^j. If its deviation from this value is small, and if it can be established that indirect coupling among the coordinate complex conjugate pairs in Y^j is small, then, for some purposes of dynamic analysis, it may be appropriate to truncate the matrix Y^j to the smaller matrix $\overline{Y}^j$, preserving only those conjugate pairs of coordinates with frequencies in the domain of interest and with significant influence on those aspects of the dynamic response which are of interest in a particular case. The truncation of Y^j is accompanied by truncation of the corresponding 2ν by 2ν eigenvector matrices Φ^j and Φ'^j to the rectangular matrices

$\overline{\Phi}^j$ and $\overline{\Phi}'^j$. The result is a reduced-dimension set of equations for which the solution preserves the salient dynamic characteristics of Eq. (117). These equations are recorded as Eq. (118).

$$
\left[\begin{array}{c|c|c|c|c}
A_{00} & A_{01} & A_{02}\overline{\Phi}^2 & \cdots & A_{0,N+1}\overline{\Phi}^{N+1} \\ \hline
A_{10} & A_{11} & A_{12}\overline{\Phi}^2 & \cdots & A_{1,N+1}\overline{\Phi}^{N+1} \\ \hline
\overline{\Phi}'^{2T}A_{20} & \overline{\Phi}'^{2T}A_{21} & \overline{\Phi}'^{2T}A_{22}\overline{\Phi}^2 & \cdots & \overline{\Phi}'^{2T}A_{2,N+1}\overline{\Phi}^{N+1} \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
\overline{\Phi}'^{N+1^T}A_{N+1,0} & \overline{\Phi}'^{N+1^T}A_{N+1,1} & \overline{\Phi}'^{N+1^T}A_{N+1,2}\overline{\Phi}^2 & \cdots & \overline{\Phi}'^{N+1^T}A_{N+1,N+1}\overline{\Phi}^{N+1}
\end{array}\right]
\left[\begin{array}{c}
\dot{\omega}^0 \\ \hline \dot{\Gamma}^1 \\ \hline \dot{\overline{Y}}^2 \\ \hline \vdots \\ \hline \dot{\overline{Y}}^{N+1}
\end{array}\right] =
$$

$$
\left[\begin{array}{c}
\sum_{k\epsilon\mathcal{B}} [(\overline{C}^{0k} + \hat{C}^{0k})\overline{A}^k + \overline{C}^{0k}\hat{A}^k] - \sum_{k\epsilon\mathcal{G}} a_{0k}\ddot{\gamma}_k \\ \hline
\left[\begin{array}{c} \Sigma_{0U}^T\Gamma^1 \\ \hline R^1 \end{array}\right] \\ \hline
\overline{\Phi}'^{2T}\left[\begin{array}{c} \Sigma_{0U}^T\overline{\Phi}^2\overline{Y}^2 \\ \hline R^2 \end{array}\right] \\ \hline
\vdots \\ \hline
\overline{\Phi}'^{N+1^T}\left[\begin{array}{c} \Sigma_{0U}^T\overline{\Phi}^{N+1}\overline{Y}^{N+1} \\ \hline R^{N+1} \end{array}\right]
\end{array}\right] \tag{118}
$$

A special case of Eq. (118) which is of particular interest is the case for which all angles and their derivatives may be assumed small. Recall that, in this case, it has been suggested that a chain of three imaginary massless bodies be introduced into the mathematical model, with the last of these (labeled $\mathscr{b}_0$) representing the "mean motion" reference frame of the system. In Eq. (117), the only remaining variables in this case are ω^0 and Y^2, and the first of these represents the inertial angular velocity of the mean motion frame. If ω^0 is retained as a system variable, then it is important that the "rigid-body mode" (zero-frequency mode) coordinates that arise in the transformed variable matrix Y^2 be discarded in the truncation to $\overline{Y}^2$ because ω^0 and the rigid-body modes are redundant.

A second special case of practical interest arises when in Eq. (105), the matrix J is zero and K is symmetric. As established in Section IIID, in this case, the transformation in Eq. (109) can be replaced by the simpler transformation

$$\gamma^k = \phi^k\eta^k \tag{119}$$

in parallel with Eq. (97). Differentiated terms in Eq. (30c) then become

$$\dot{\gamma}^k = \phi^k\dot{\eta}^k \qquad \ddot{\gamma}^k = \phi^k\ddot{\eta}^k \tag{120}$$

and the transformed equations remain second-order differential equations. If such transformations are applied to all of the small variables in Eq. (30c), and appropriate row partitions are multiplied by $\bar{\phi}^{2^T}, \cdots, \bar{\phi}^{N+1^T}$, the transformed equations have the structure shown in Eq. (121).

$$
\left[\begin{array}{c|c|c|c|c|c}
\bar{a}^{00}+\hat{a}^{00} & \bar{a}^{01}+\hat{a}^{01} & \bar{a}^{02}\bar{\phi}^{2} & \cdots & \bar{a}^{0N}+\hat{a}^{0N} & \bar{a}^{0,N+1}\bar{\phi}^{N+1} \\ \hline
\bar{a}^{10}+\hat{a}^{10} & \bar{a}^{11}+\hat{a}^{11} & \bar{a}^{12}\bar{\phi}^{2} & \cdots & \bar{a}^{1N}+\hat{a}^{1N} & \bar{a}^{1,N+1}\bar{\phi}^{N+1} \\ \hline
\bar{\phi}^{2T}(\bar{a}^{20}+\hat{a}^{20}) & \bar{\phi}^{2T}(\bar{a}^{21}+\hat{a}^{21}) & \bar{\phi}^{2T}\bar{a}^{22}\bar{\phi}^{2} & \cdots & \bar{\phi}^{2T}(\bar{a}^{2N}+\hat{a}^{2N}) & \bar{\phi}^{2T}\bar{a}^{2,N+1}\bar{\phi}^{N+1} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
\bar{a}^{N0}+\hat{a}^{N0} & \bar{a}^{N1}+\hat{a}^{N1} & \bar{a}^{N2}\bar{\phi}^{2} & \cdots & \bar{a}^{NN}+\hat{a}^{NN} & \bar{a}^{N,N+1}\bar{\phi}^{N+1} \\ \hline
\bar{\phi}^{N+1^T}(\bar{a}^{N+1,0}+\hat{a}^{N+1,0}) & \bar{\phi}^{N+1^T}(\bar{a}^{N+1,1}+\hat{a}^{N+1,1}) & \bar{\phi}^{N+1^T}\bar{a}^{N+1,2}\bar{\phi}^{2} & \cdots & \bar{\phi}^{N+1^T}(\bar{a}^{N+1,N}+\hat{a}^{N+1,N}) & \bar{\phi}^{N+1^T}\bar{a}^{N+1,N+1}\bar{\phi}^{N+1}
\end{array}\right]
\left[\begin{array}{c}
\dot{\omega}^{0} \\ \hline \ddot{\gamma}^{1} \\ \hline \ddot{\bar{\eta}}^{2} \\ \hline \vdots \\ \hline \ddot{\gamma}^{N} \\ \hline \ddot{\bar{\eta}}^{N+1}
\end{array}\right] =
$$

$$
\left[\begin{array}{c}
\sum_{k\in\mathcal{B}} [(\bar{C}^{0k}+\hat{C}^{0k})\bar{A}^{k}+\bar{C}^{0k}\hat{A}^{k}] - \sum_{k\in\mathcal{G}} a_{0k}\ddot{\gamma}_{k} \\ \hline
R^{1}(\omega^{0},\gamma^{1},\cdots,\gamma^{N},\dot{\gamma}^{1},\cdots,\dot{\gamma}^{N},\bar{\eta}^{1},\cdots,\bar{\eta}^{N+1},\dot{\bar{\eta}}^{1},\cdots,\dot{\bar{\eta}}^{N+1},t) \\ \hline
\bar{\phi}^{2^T}R^{2}(\omega^{0},\gamma^{1},\cdots,\gamma^{N},\dot{\gamma}^{1},\cdots,\dot{\gamma}^{N},\bar{\eta}^{1},\cdots,\bar{\eta}^{N+1},\dot{\bar{\eta}}^{1},\cdots,\dot{\bar{\eta}}^{N+1},t) \\ \hline
\vdots \\ \hline
R^{N}(\omega^{0},\gamma^{1},\cdots,\gamma^{N},\dot{\gamma}^{1},\cdots,\dot{\gamma}^{N},\bar{\eta}^{1},\cdots,\bar{\eta}^{N+1},\dot{\bar{\eta}}^{1},\cdots,\dot{\bar{\eta}}^{N+1},t) \\ \hline
\bar{\phi}^{N+1^T}R^{N+1}(\omega^{0},\gamma^{1},\cdots,\gamma^{N},\dot{\gamma}^{1},\cdots,\dot{\gamma}^{N},\bar{\eta}^{1},\cdots,\bar{\eta}^{N+1},\dot{\bar{\eta}}^{1},\cdots,\dot{\bar{\eta}}^{N+1},t)
\end{array}\right] \tag{121}
$$

This result can be compared to the more general alternative displayed in Eq. (118). For computational convenience, Eq. (121) may be rewritten in first-order form, providing once again a set of equations having the structure of Eq. (118). In this special case, however, we can define the new variables as

$$
\bar{Y}^{k} \triangleq \left[\begin{array}{c}\bar{\eta}^{k} \\ \hline \dot{\bar{\eta}}^{k}\end{array}\right] (k \text{ even}) \text{ and } \Gamma^{k} \triangleq \left[\begin{array}{c}\gamma^{k} \\ \hline \dot{\gamma}^{k}\end{array}\right] (k \text{ odd}) \tag{122}
$$

so that the transformation matrices become

$$
\bar{\Phi}^{k} \triangleq \left[\begin{array}{c|c} U & 0 \\ \hline 0 & \bar{\phi}^{k}\end{array}\right]; \qquad \bar{\Phi}'^{kT} \triangleq \left[\begin{array}{c|c} U & 0 \\ \hline 0 & \bar{\phi}^{kT}\end{array}\right] \tag{123}
$$

In this special case, for which Eq. (105) becomes $I\ddot{\gamma}+K\gamma=0$ with K symmetric, there is no accommodation of energy dissipation in the mathematical model. This represents a departure from physical reality that can have serious consequences. Yet, it is a very difficult task to determine the proper damping characteristics to be assigned to any structural joint or connection, and it is virtually impossible to assign the proper characteristics of an analytical joint, introduced by the analyst to represent discretely a structural deformation which is actually distributed over an elastic body. The difficulty of incorporating the damping characteristics of a nonspinning discretized flexible structure can be met with the introduction of *modal damping*, as that concept has long been used by structural dynamicists. When the vehicle is spinning, the application of this concept is not entirely straightforward, although with some reinterpretation the basic idea can still be applied.

For a nonspinning elastic body whose modal-coordinate free oscillations are described by $\ddot{\eta} + \sigma^2\eta = 0$ as in Eq. (103), the introduction of modal damping takes the form of the incorporation of a viscous damping term in each of the uncoupled modal vibration equations, to obtain $\ddot{\eta} + 2\zeta\sigma\dot{\eta} + \sigma^2\eta = 0$, as in Eq. (104). The analyst's task is to assign values to the diagonal elements $\zeta_1, \cdots, \zeta_\nu$ of the matrix ζ, on the basis of experiments or related experience. These values generally range from 0.005 to 0.05 for the class of elastic bodies used in space vehicles. In terms of the equations of motion for the nonspinning multiple-rigid-body systems considered in this report, the incorporation of modal damping would involve the introduction into Eq. (121) of the matrix $-2\bar{\zeta}^j\bar{\sigma}^j\dot{\bar{\eta}}^j$ as a part of $\bar{\phi}^{jT}R^j$, for $j = 2, \cdots, N+1$. This step is equivalent to the assumption that J^k in Eq. (105) is a polynomial in I^k and K^k (generally, J^k is treated as a linear combination of I^k and K^k).

For a spinning system, Eq. (121) is replaced by Eq. (118), and the incorporation of modal damping becomes a more ambiguous procedure. The basic objective is to replace the imaginary eigenvalues of the undamped structure by eigenvalues with appropriate negative real parts. As shown in Ref. 21 (pp. 45 and 46), this amounts to replacing the quantities typified by $\lambda_j = i\sigma_j$ and $\lambda_j^* = -i\sigma_j$ in the diagonal matrix λ in Eq. (90) by $-\zeta_j\sigma_j + i\sigma_j$ and $-\zeta_j\sigma_j - i\sigma_j$. The matrix λ in Eq. (90) came from $\Phi'^TQ\Phi$, and in Eq. (118), neither λ nor $\Phi'^TQ\Phi$ appears explicitly. There will be found, however, on the right side of Eq. (118) a product which has the nominal value $\Phi'^{kT}Q^k\Phi^k = \lambda^k$, so it is appropriate to augment Eq. (118) by the matrix product

$$-\begin{bmatrix} \zeta_1^k\sigma_1^k & & & & & 0 \\ & \ddots & & & & \\ & & \zeta_{\bar{\nu}}^k\sigma_{\bar{\nu}}^k & & & \\ & & & \zeta_1^k\sigma_1^k & & \\ & & & & \ddots & \\ 0 & & & & & \zeta_{\bar{\nu}}^k\sigma_{\bar{\nu}}^k \end{bmatrix} \begin{bmatrix} \\ \bar{\mathrm{Y}}^k \\ \\ \end{bmatrix}$$

as a means of incorporating modal damping. It should be noted that this step no longer implies the relationship between I^k, K^k, and the damping terms in J^k which were implied in the nonspinning case. Here $\bar{\nu}$ represents the number of modal coordinates retained after truncation.

F. Sample Problem Formulation

1. Partial linearization. In this section, we shall extend the sample problem considered in Section IIE (see Fig. 4) as an illustration of the unrestricted discrete-coordinate equations. Now we shall examine a set of partially linearized equations for this system, and explore the possibilities of coordinate transformation and truncation.

In what follows, we shall assume that in Fig. 4, the angles γ_5, γ_6, γ_7, γ_8, and γ_9 remain small, while γ_1 is a prescribed function of time and ω^0 has a nominal value in the direction of g^1. All other circumstances are as described in Section IIE. This is a rather complex example, possessing most of the difficulties of the general case.

In order to illustrate special cases, we shall occasionally consider restricted versions of this example.

Equations (25) provide the first-order equations of kinematics and dynamics of this system without restriction on the size of any variables. These equations are equivalent to the kinematical equations in Eq. (24) and the dynamical equations in Eq. (1).

For the partially linearized case, we replace Eq. (1) by Eq. (30c) as our starting point, retaining Eqs. (24) without change. Since γ_1 is prescribed, γ_1 does not appear on the left side of the version of Eq. (30c) to be written here. The variables in the differentiated column matrix on the left side consist of ω^0 and

$$\gamma^1 \triangleq [\gamma_2\, \gamma_3\, \gamma_4]^T; \qquad \gamma^2 \triangleq [\gamma_5\, \gamma_6\, \gamma_7\, \gamma_8\, \gamma_9]; \qquad \gamma^3 \triangleq [\gamma_{10}] \tag{124}$$

and the equations of motion are

$$\left[\begin{array}{c|c|c|c} \bar{a}^{00} + \hat{a}^{00} & \bar{a}^{01} + \hat{a}^{01} & \bar{a}^{02} & \bar{a}^{03} + \hat{a}^{03} \\ \hline \bar{a}^{10} + \hat{a}^{10} & \bar{a}^{11} + \hat{a}^{11} & \bar{a}^{12} & \bar{a}^{13} + \hat{a}^{13} \\ \hline \bar{a}^{20} + \hat{a}^{20} & \bar{a}^{21} + \hat{a}^{21} & \bar{a}^{22} & \bar{a}^{23} + \hat{a}^{23} \\ \hline \bar{a}^{30} + \hat{a}^{30} & \bar{a}^{31} + \hat{a}^{31} & \bar{a}^{32} & \bar{a}^{33} + \hat{a}^{33} \end{array}\right] \left[\begin{array}{c} \dot{\omega}^0 \\ \hline \ddot{\gamma}^1 \\ \hline \ddot{\gamma}^2 \\ \hline \ddot{\gamma}^3 \end{array}\right] =$$

$$\left[\begin{array}{c} \sum\limits_{k\in\mathcal{B}} [(\bar{C}^{0k} + \hat{C}^{0k})\bar{A}^k + \bar{C}^{0k}\hat{A}^k] - (\bar{a}_{01} + \hat{a}_{01})\ddot{\gamma}_1 \\ \hline R^1(\omega^0, \gamma^1, \gamma^2, \gamma^3, \dot{\gamma}^1, \dot{\gamma}^2, \dot{\gamma}^3, t) \\ \hline R^2(\omega^0, \gamma^1, \gamma^2, \gamma^3, \dot{\gamma}^1, \dot{\gamma}^2, \dot{\gamma}^3, t) \\ \hline R^3(\omega^0, \gamma^1, \gamma^2, \gamma^3, \dot{\gamma}^1, \dot{\gamma}^2, \dot{\gamma}^3, t) \end{array}\right] \tag{125}$$

as a special case of Eq. (30c). Explicit values of the submatrices in the partitions on the left side of Eq. (125) must be constructed from the comparison of Eqs. (30a, b, c) and the substitution of expressions found in Eqs. (36) and (37). The matrices $\bar{a}^{00}$ and $\hat{a}^{00}$ are available, respectively, from Eqs. (36a) and (37a), while all other matrix partitions must be expanded before the expressions in Eqs. (36) and (37) can be used. For example, the matrix $\bar{a}^{01} + \hat{a}^{01}$ can be expanded as

$$\bar{a}^{01} + \hat{a}^{01} = [\bar{a}_{02} + \hat{a}_{02} \,|\, \bar{a}_{03} + \hat{a}_{03} \,|\, \bar{a}_{04} + \hat{a}_{04}] \tag{126}$$

and $\bar{a}^{10} + \hat{a}^{10}$ is the transpose of this expression. Explicit values for the 3 by 1 matrices in the partitions of Eq. (126) are available from Eqs. (36b) and (37b) As a second example, the matrix $\bar{a}^{12}$ can be expanded as

$$\bar{a}^{12} = \begin{bmatrix} \bar{a}_{25} & \bar{a}_{26} & \bar{a}_{27} & \bar{a}_{28} & \bar{a}_{29} \\ \bar{a}_{35} & \bar{a}_{36} & \bar{a}_{37} & \bar{a}_{38} & \bar{a}_{39} \\ \bar{a}_{45} & \bar{a}_{46} & \bar{a}_{47} & \bar{a}_{48} & \bar{a}_{49} \end{bmatrix} \tag{127}$$

and the scalars in this matrix are available from Eq. (36d). A third example of particular interest is the 5 by 5 matrix

$$\bar{a}^{22} = \begin{bmatrix} \bar{a}_{55} & \bar{a}_{56} & \bar{a}_{57} & \bar{a}_{58} & \bar{a}_{59} \\ \bar{a}_{65} & \bar{a}_{66} & \bar{a}_{67} & \bar{a}_{68} & \bar{a}_{69} \\ \bar{a}_{75} & \bar{a}_{76} & \bar{a}_{77} & \bar{a}_{78} & \bar{a}_{79} \\ \bar{a}_{85} & \bar{a}_{86} & \bar{a}_{87} & \bar{a}_{88} & \bar{a}_{89} \\ \bar{a}_{95} & \bar{a}_{96} & \bar{a}_{97} & \bar{a}_{98} & \bar{a}_{99} \end{bmatrix} \tag{128}$$

the elements of which can also be obtained from Eq. (36d).

In order to obtain explicit expressions for the matrix partitions on the right side of Eq. (125), one must recognize the definitions in Eq. (124) and then examine Eqs. (30a, b). The top 3 by 1 partition on the right side of Eq. (127) is identical to the right side of Eq. (30a) except for the transfer of the term $(\bar{a}_{01} + \hat{a}_{01})\ddot{\gamma}_1$ from the left side of Eq. (30a)—this because $\gamma_1(t)$ is prescribed. The 3 by 1 matrix R^2 is from Eqs. (124) and (30b), given by

$$R^2 = \begin{bmatrix} g^{5T} \sum\limits_{k\epsilon\mathcal{P}} \varepsilon_{5k} [\bar{C}^{5k}\hat{A}^k + (\bar{C}^{5k} + \hat{C}^{5k})\bar{A}^k] + \bar{\tau}_5 + \hat{\tau}_5 - (\bar{a}_{51} + \hat{a}_{51})\ddot{\gamma}_1 \\ g^{6T} \sum\limits_{k\epsilon\mathcal{P}} \varepsilon_{6k} [\bar{C}^{6k}\hat{A}^k + (\bar{C}^{6k} + \hat{C}^{6k})\bar{A}^k] + \bar{\tau}_6 + \hat{\tau}_6 - (\bar{a}_{61} + \hat{a}_{61})\ddot{\gamma}_1 \\ \cdot \\ \cdot \\ \cdot \\ g^{9T} \sum\limits_{k\epsilon\mathcal{P}} \varepsilon_{9k} [\bar{C}^{9k}\hat{A}^k + (\bar{C}^{9k} + \hat{C}^{9k})\bar{A}^k] + \bar{\tau}_9 + \hat{\tau}_9 - (\bar{a}_{91} + \hat{a}_{91})\ddot{\gamma}_1 \end{bmatrix} \tag{129}$$

Expressions for R^1 and R^3 are exactly parallel, differing only in dimension and numerical indices, as required by the definitions in Eq. (124).

As in the sample problem formulation in Section IIE, it is convenient for subsequent analysis and computations to rewrite Eq. (125) as a first-order matrix equation, and to combine it with a set of kinematical equations, such as Eq. (24). This can be accomplished exactly as in Eq. (25), that is, by

$$V\dot{x} = W \tag{130a}$$

where

$$\begin{aligned} x &\triangleq [\theta_1\,\theta_2\,\theta_3\,\omega_1^0\,\omega_2^0\,\omega_3^0\,\gamma_2\,\dot{\gamma}_2 \;\cdots\; \gamma_{10}\,\dot{\gamma}_{10}]^T \\ &\triangleq [\theta^T \,\vdots\, \omega^{0T} \,\vdots\, \gamma^{1T} \,\vdots\, \gamma^{2T} \,\vdots\, \gamma^{3T}]^T \end{aligned} \tag{130b}$$

Because we intend in this section to restrict the results in Section IIE by partial linearization, we now express V and W in terms of partitioned matrices which reflect the linearization. These results (to be compared to the unrestricted counterparts in Eqs. 26 and 27) are given by Eqs. (130c).

$$
V\dot{x} = \left[\begin{array}{c|c|c|c|c|c|c|c}
U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \bar{a}^{00}+\hat{a}^{00} & 0 & \bar{a}^{01}+\hat{a}^{01} & 0 & \bar{a}^{02} & 0 & \bar{a}^{03}+\hat{a}^{03} \\ \hline
0 & 0 & U & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \bar{a}^{10}+\hat{a}^{10} & 0 & \bar{a}^{11}+\hat{a}^{11} & 0 & \bar{a}^{12} & 0 & \bar{a}^{13}+\hat{a}^{13} \\ \hline
0 & 0 & 0 & 0 & U & 0 & 0 & 0 \\ \hline
0 & \bar{a}^{20}+\hat{a}^{20} & 0 & \bar{a}^{21}+\hat{a}^{21} & 0 & \bar{a}^{22} & 0 & \bar{a}^{23}+\hat{a}^{23} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & U & 0 \\ \hline
0 & \bar{a}^{30}+\hat{a}^{30} & 0 & \bar{a}^{31}+\hat{a}^{31} & 0 & \bar{a}^{32} & 0 & \bar{a}^{33}+\hat{a}^{33}
\end{array}\right]
\left[\begin{array}{c}
\dot{\theta} \\ \hline \dot{\omega}^0 \\ \hline \dot{\gamma}^1 \\ \hline \ddot{\gamma}^1 \\ \hline \dot{\gamma}^2 \\ \hline \ddot{\gamma}^2 \\ \hline \dot{\gamma}^3 \\ \hline \ddot{\gamma}^3
\end{array}\right] =
$$

$$
\left[\begin{array}{c}
P_{\omega^0} \\ \hline
\sum_{k\in\mathcal{B}} [(\bar{C}^{ok}+\hat{C}^{ok})\bar{A}^k + \bar{C}^{ok}\hat{A}^k] - (\bar{a}_{01}+\hat{a}_{01})\ddot{\gamma}_1 \\ \hline
\dot{\gamma}^1 \\ \hline
R^1 \\ \hline
\dot{\gamma}^2 \\ \hline
R^2 \\ \hline
\dot{\gamma}^3 \\ \hline
R^3
\end{array}\right] = W \qquad (130c)
$$

If the partially linearized Eq. (130c) is to have any immediate computational advantage over its unrestricted counterpart in Eqs. (25)–(27), then this advantage is most apt to be found in the simplification of V, since it is the necessity of repeatedly inverting this matrix (actually or effectively) which makes the numerical integration of $V\dot{x} = W$ such a costly operation. In order to minimize this task, we can in either case partition the matrix into constant and variable submatrices in order to take advantage of the algorithm for matrix inversion in terms of matrix partitions (see Ref. 27, p. 640, or illustration in Ref. 21, p. 49).

As we examine the matrix V in Eq. (130), and compare with Eq. (26), we are reminded that the caret over a matrix is an indication that the matrix may depend upon the linearized variables, while the bar over a matrix indicates the absence of small-deformation variables; in both classes of matrices, however, we must expect variation with other kinematical variables. In order to detect the possibility of constancy of the coefficient matrices in Eq. (130c), we must examine the explicit expressions for their elements, as found in Eqs. (36) and (37). These elements must therefore be examined individually to establish conditions for their constancy.

From Eq. (36a), the matrix $\bar{a}^{00}$ (which is the same as $\bar{a}_{00}$) is seen to have a definitely constant part given by Φ^{00} (the inertia matrix of the augmented reference body) and a generally variable part involving the direction cosines $\bar{C}^{ok}$ for $k \in \mathcal{P}$. Only if *all* relative motions are assumed small (so $\bar{C}^{ok} = U$ for all k) do we have a constant matrix $\bar{a}^{00}$. The matrix $\hat{a}^{00}$ (or $\hat{a}_{00}$) from Eq. (37a) is even more disagreeable, since it experiences time variations even for small deformations. If $\dot{\omega}^0$ is also small (even if ω^0 is large), terms such as $\hat{a}^{00}\dot{\omega}^0$ can be ignored; only then does the matrix $\bar{a}^{00} + \hat{a}^{00}$ in V become constant. It is, in fact, clear that if all variables in $\dot{x}$ are assumed arbitrarily small (except perhaps $\dot{\theta}$, which is coupled only to ω^0, which may be large if nearly constant), and if there are no kinematically prescribed variables of large amplitude, then the matrix V must be a constant, to be inverted only once in advance of numerical integration. This is, however, not the case for the example under consideration.

Matrices in the second row-partition and second column-partition of V are available in terms of Eqs. (36b, c) and (37b, c). These matrices also will not be effectively constant except in the extreme case noted, in which all kinematical variables are small (except possibly θ, $\dot{\theta}$, and ω^0, which may be large providing that $\ddot{\theta}$ and $\dot{\omega}^0$ are small).

All other submatrices in V are composed of elements defined by Eqs. (36d) and (37d). The scalar $\bar{a}^{ik}$ in Eq. (36d) can be constant only if the path elements such as ε_{ir} adopt zero values when multiplied by time-varying direction-cosine matrices. For example, the submatrix $\bar{a}^{12}$ shown in Eq. (127) contains the scalar $\bar{a}_{39}$, which, from Eq. (36d), is

$$\bar{a}_{39} = g^{3T} \sum_{r\in\mathcal{P}} \varepsilon_{3r}\varepsilon_{9r}\bar{C}^{3r}\Phi^{rr}\bar{C}^{r9}g^9$$

$$- \mathcal{M} g^{3T} \sum_{r\in\mathcal{P}} \sum_{s\in\mathcal{P}-r} \varepsilon_{3r}\varepsilon_{9s} \left(\bar{C}^{3r}D^{srT}\bar{C}^{sr}D^{rs} - \bar{C}^{3s}D^{sr}D^{rsT}\right)\bar{C}^{r9}g^9 \qquad (131a)$$

Recall from Def. 27 that the path elements are defined by

$$\varepsilon_{sk} \triangleq \begin{cases} 1 \text{ if } h_s \text{ lies between } b_0 \text{ and } b_k \\ 0 \text{ otherwise} \end{cases}$$

Inspection of the configuration for this sample problem, shown in Fig. 4, now reveals that $\varepsilon_{3r} = 0$ unless $r = 3$ or 4 and $\varepsilon_{9s} = 0$ unless $s = 9$. (This information is also available from Table 2.) Thus, the first series of terms in Eq. (131a) vanishes, and we retain only

$$\bar{a}_{39} = -\mathcal{M} g^{3T} \left[\left(\bar{C}^{33}D^{93T}\bar{C}^{93}D^{39} - \bar{C}^{39}D^{93}D^{39T}\right)\bar{C}^{39}\right.$$

$$\left. + \left(\bar{C}^{34}D^{94T}\bar{C}^{94}D^{49} - \bar{C}^{39}D^{94}D^{49T}\right)\bar{C}^{49}\right] g^9 \qquad (131b)$$

Further simplification is afforded by the interpretation of Def. 35, which provides

$$D^{93} = D^{94} = D^{90}; \qquad D^{39} = D^{30}; \qquad D^{49} = D^{40}$$

Moreover, Eqs. (32), (33) may be interpreted to provide

$$\bar{C}^{33} = U; \qquad \bar{C}^{93} = \bar{C}^{03}; \qquad \bar{C}^{94} = \bar{C}^{04}; \qquad \bar{C}^{39} = \bar{C}^{30}; \qquad \bar{C}^{49} = \bar{C}^{40}$$

Thus, Eq. (131b) becomes

$$\bar{a}_{39} = -\mathcal{M} g^{3T} \left[\left(D^{90T}\bar{C}^{03}D^{30} - \bar{C}^{30}D^{90}D^{30T}\right)\bar{C}^{30}\right.$$

$$\left. + \left(\bar{C}^{34}D^{90T}\bar{C}^{04}D^{40} - \bar{C}^{30}D^{90}D^{40T}\right)\bar{C}^{40}\right] g^9 \qquad (131c)$$

Evidently, in this case, the matrix in square brackets varies with time, so that only for very special configurations could $\bar{a}_{39}$ be constant. (For example, if the matrix in square brackets has a principal axis remaining parallel to g^9 while g^3 remains perpendicular to g^9, then $\bar{a}_{39} = 0$; for the sample problem, we don't have such good fortune.)

Generalizing from this experience with $\bar{a}_{39}$, we can see that, except for very special configurations, matrices $\bar{a}^{ij}$ (i odd, j even), as typified by $\bar{a}^{12}$, are not constant, because in every case there is at least one unrestricted angle lying between each body in the set $\mathscr{P} - \mathscr{A}$ and any body in the set $\mathscr{A}$. The same result emerges for $\bar{a}^{jk}$ (j, k odd), and the careted matrices all vary with time unless they are zero.

Thus, the only matrix partitions in V in Eq. (130c) which retain the possibility of constancy are $\bar{a}^{22}$ and (of course) the unit matrix U. As shown explicitly in Eq. (128), the elements of $\bar{a}^{22}$ are the scalars $\bar{a}_{ik}$ found in Eq. (36d) for $i, k \in \mathscr{A}$. Because every term in $\bar{a}_{ik}$ is multiplied by ε_{ir}, and i is confined to the set $\mathscr{A}$, we can discard all terms with superscripts r unless either $r \in \mathscr{A}$ or $\mathscr{b}_r$ is separated from $\mathscr{b}_0$ by a joint whose index is in $\mathscr{A}$. In application to the sample problem in Fig. 4, this rules out all r for which $\gamma_r \in \gamma^1$, leaving only $r \in \mathscr{A}$ and $r = 10$. Moreover, the path element ε_{ks} is also zero unless either $s \in \mathscr{A}$ or $\mathscr{b}_s$ is separated from $\mathscr{b}_0$ by a joint whose index is in $\mathscr{A}$. Since $s \neq r$, we have in this sample problem confined s to $\mathscr{A}$. Thus, the expression for $\bar{a}_{ik}$ in this case becomes

$$\begin{aligned}
\bar{a}_{ik} = {} & g^{i^T} \sum_{r \in \mathscr{A}} \varepsilon_{ir} \varepsilon_{kr} \bar{C}^{ir} \Phi^{rr} \bar{C}^{rk} g^k + g^{i^T} \varepsilon_{i,10} \varepsilon_{k,10} \bar{C}^{i,10} \Phi^{10,10} \bar{C}^{10,k} g^k \\
& - \mathcal{M} g^{i^T} \sum_{r \in \mathscr{A}} \sum_{s \in \mathscr{A} - r} \varepsilon_{ir} \varepsilon_{ks} \left(\bar{C}^{ir} D^{sr^T} \bar{C}^{sr} D^{rs} - \bar{C}^{is} D^{sr} D^{rs^T} \right) \bar{C}^{rk} g^k \\
& - \mathcal{M} g^{i^T} \sum_{r \in \mathscr{A}} \varepsilon_{ir} \varepsilon_{k,10} \left(\bar{C}^{ir} D^{10,r^T} \bar{C}^{10,r} D^{r,10} - \bar{C}^{i,10} D^{10,r} D^{r,10^T} \right) \bar{C}^{r,k} g^k \\
& - \mathcal{M} g^{i^T} \sum_{s \in \mathscr{A}} \varepsilon_{i,10} \varepsilon_{ks} \left(\bar{C}^{i,10} D^{s,10^T} \bar{C}^{s,10} D^{10,s} - \bar{C}^{is} D^{s,10} D^{10,s^T} \right) \bar{C}^{10,k} g^k \qquad (132)
\end{aligned}$$

All terms in Eq. (132) are constant except those involving $\bar{C}^{i,10}$ and $\bar{C}^{10,k}$, and these are given for all $i, k \in \mathscr{A}$ by $\bar{C}^{10,k} = C^{10,6}$ and $\bar{C}^{i,10} = C^{6,10} = (C^{10,6})^T$. From Eq. (4), we have

$$C^{10,6} = U \cos \gamma_{10} - \tilde{g}^{10} \sin \gamma_{10} + g^{10} g^{10^T} (1 - \cos \gamma_{10}) \qquad (133)$$

which obviously varies with time. Although many of these time-varying direction-cosine matrices are filtered out of Eq. (132) by the path elements, such terms survive when either $i \in \{5, 6\}$ or $k \in \{5, 6\}$. Thus, the lower right 3 by 3 matrix in Eq. (128c) is constant.

In generalizing from this example, we can conclude that the matrix $\bar{a}^{jj}$ is constant for even j whenever the angles in γ^j belong to a terminal appendage, and not otherwise except for very special configurations. For a flexible substructure which is not a terminal appendage, the corresponding matrix $\bar{a}^{jj}$ (j even) can be partitioned into a constant part and a time-varying part.

Partial linearization with the immediate objective of accelerating the numerical integration process would appear to be justified only if substantial and identifiable portions of the V matrix in $V\dot{x} = W$ are replaced by constants in the linearization process. For the sample problem, the V matrix is 24 by 24, and even without linearization, half of the terms are partitioned into constant (often zero) matrices. The suggested linearization in $\gamma_5, \cdots, \gamma_9$ renders constant only a 3 by 3 partition of $\bar{a}^{22}$, changing only nine of the 288 potentially variable elements to constants. This situation would improve significantly if γ_{10} and its derivatives were also assumed small, since then γ^2 would embrace γ^3, the substructure would become a terminal appendage, and a new 6 by 6 constant matrix $\bar{a}^{22}$ would emerge, changing 36 of the 288 potentially variable elements to constants.

Partial linearization begins to offer significant advantages in computation when the number of bodies in a flexible substructure is large, and particularly when the substructure is a terminal appendage. In this same combination of circumstances, more dramatic computational advantage can be realized by transforming to distributed coordinates, if one can justify truncation of high-frequency modes. Although these advantages are not very significant for the small sample problem chosen here, this will continue to be the vehicle for illustrating the general theory.

2. *Coordinate transformation.* From Eq. (130c), we can extract

$$\bar{a}^{22}\ddot{\gamma}^2 + \hat{a}^{20}\dot{\omega}^0 + \hat{a}^{21}\ddot{\gamma}^1 + \hat{a}^{23}\ddot{\gamma}^3 = R^2(\omega^0, \gamma^1, \gamma^2, \gamma^3, \dot{\gamma}^1, \dot{\gamma}^2, \dot{\gamma}^3, t) - \bar{a}^{20}\dot{\omega}^0 - \bar{a}^{21}\ddot{\gamma}^1 - \bar{a}^{23}\ddot{\gamma}^3 \tag{134}$$

All terms on the left side of Eq. (134) depend on γ^2 or its time derivatives, as does the term R^2 on the right. Since coordinate transformations are to be based on a homogeneous equation in γ^2, which is obtained from Eq. (134), we must examine Eq. (129), in which we have explicitly displayed R^2, and move those (careted) terms depending on γ^2 or its derivatives to the left side of Eq. (134). If we then record only the homogeneous counterpart to Eq. (134), we have

$$\bar{a}^{22}\ddot{\gamma}^2 + \hat{a}^{20}\dot{\omega}^0 + \hat{a}^{21}\ddot{\gamma}^1 + \hat{a}^{23}\ddot{\gamma}^3 - \begin{bmatrix} g^{5^T} \sum\limits_{k \in \mathcal{P}} \varepsilon_{5k} [\bar{C}^{5k}\hat{A}^k + \hat{C}^{5k}\bar{A}^k] + \hat{\tau}_5 - \hat{a}_{51}\ddot{\gamma}_1 \\ \vdots \\ g^{9^T} \sum\limits_{k \in \mathcal{P}} \varepsilon_{9k} [\bar{C}^{9k}\hat{A}^k + \hat{C}^{9k}\bar{A}^k] + \hat{\tau}_9 - \hat{a}_{91}\ddot{\gamma}_1 \end{bmatrix} = 0 \tag{135a}$$

In comparison of this particular example with the general theory in Section IIIC, we can identify Eq. (135a) as the specific counterpart of the general set of equations established by Eq. (39). As demonstrated in the general arguments following Eq. (39), it is next appropriate to replace the linear, variable-coefficient equations in Eq. (135a) by a corresponding set of linear, constant-coefficient equations, which will have the structure of Eq. (42), that is,

$$I\ddot{\gamma} + J\dot{\gamma} + K\gamma = 0 \tag{135b}$$

In this application,

$$\gamma \triangleq [\gamma_5\ \gamma_6\ \gamma_7\ \gamma_8\ \gamma_9]^T$$

$$I \overset{\Delta}{=} \bar{\bar{a}}^{22} \overset{\Delta}{=} \bar{a}^{22}\big|_{\text{nominal}}$$

and the elements of J and K are as defined in Eqs. (43) and (44).

More explicitly, the elements of the matrix I are established by the nominal (constant) values of the elements displayed in Eq. (128). Although any "nominal" might be selected, it may be most convenient to choose the values when all direction-cosine matrices are the unit matrix U, so that all angles γ_j for $j \in \mathcal{P} - \mathcal{A}$ are zero. The element $\bar{a}_{ij}$ in Eq. (128) is then replaced by $\bar{\bar{a}}_{ij}$, available from the nominal value in Eq. (36d) as

$$I_{ij} \overset{\Delta}{=} \bar{\bar{a}}_{ij} = g^{iT} \sum_{r \in \mathcal{P}} \varepsilon_{ir} \varepsilon_{jr} \Phi^{rr} g^j$$

$$- \mathcal{M} g^{iT} \sum_{r \in \mathcal{P}} \sum_{s \in \mathcal{P} - r} \varepsilon_{ir} \varepsilon_{js} \left(U D^{sr^T} D^{rs} - D^{sr} D^{rs^T} \right) g^j \qquad (i, j \in \mathcal{A}) \qquad \text{(136a)}$$

Similarly, the elements of J in Eq. (43) become

$$J_{ij} = -g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \frac{\partial \bar{\hat{A}}^k}{\partial \dot{\gamma}_j} - \frac{\partial \bar{\hat{\tau}}_i}{\partial \dot{\gamma}_j} \qquad (i, j \in \mathcal{A}) \qquad \text{(136b)}$$

for the indicated selection of the nominal case. With the substitution of Eqs. (49) and (46), and the recollection of items (7) and (8) from the input list at the beginning of Section IIE, Eq. (136b) becomes

$$J_{ij} = g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \left[\Phi^{kk} \varepsilon_{jk} \tilde{\bar{\omega}}^0 g^j + \varepsilon_{jk} \tilde{g}^j \Phi^{kk} \bar{\omega}^0 + \tilde{\bar{\omega}}^0 \Phi^{kk} \varepsilon_{jk} g^j \right.$$

$$\left. - \mathcal{M} \sum_{s \in \mathcal{B} - k} \varepsilon_{js} \left[\tilde{D}^{ks} \left(\tilde{g}^j \tilde{\bar{\omega}}^0 + \tilde{\bar{\omega}}^0 \tilde{g}^j \right) D^{sk} + \left(U D^{sk^T} D^{ks} - D^{sk} D^{ks^T} \right) \tilde{\bar{\omega}}^0 g^j \right] + d_i \delta_{ij} \qquad \text{(136c)}$$

where δ_{ij} is the Kronecker delta, and $\bar{\omega}^0$ is the nominal value of ω^0.

The elements of K in Eq. (135b) are obtained from Eq. (44) in similar fashion, incorporating Eqs. (46), (47), and (50) and replacing all direction-cosine matrices by U. The result is

$$K_{ij} = -g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \left\{ \left(\varepsilon_{jk} - \varepsilon_{ji} \right) \tilde{g}^j \left(-\tilde{\bar{\omega}}^0 \Phi^{kk} \bar{\omega}^0 + \mathcal{M} \sum_{s \in \mathcal{B} - k} \tilde{D}^{ks} \tilde{\bar{\omega}}^0 \tilde{\bar{\omega}}^0 D^{sk} \right) \right.$$

$$+ \varepsilon_{jk} \left[\left(\tilde{g}^j \tilde{\bar{\omega}}^0 - \tilde{\bar{\omega}}^0 \tilde{g}^j \right) \Phi^{kk} \bar{\omega}^0 + \tilde{\bar{\omega}}^0 \Phi^{kk} \tilde{g}^j \bar{\omega}^0 \right]$$

$$+ \mathcal{M} \sum_{s \in \mathcal{B} - k} \tilde{D}^{ks} \left[\left(\varepsilon_{is} - \varepsilon_{jk} \right) \tilde{g}^j \tilde{\bar{\omega}}^0 \tilde{\bar{\omega}}^0 - \varepsilon_{js} \left(\tilde{g}^j \tilde{\bar{\omega}}^0 - \tilde{\bar{\omega}}^0 \tilde{g}^j \right) \tilde{\bar{\omega}}^0 \right.$$

$$\left. \left. + \tilde{\bar{\omega}}^0 \left(-\varepsilon_{js} \tilde{g}^j \tilde{\bar{\omega}}^0 + \tilde{\bar{\omega}}^0 \varepsilon_{js} \tilde{g}^j \right) \right] D^{sk} \right\} + k_i \delta_{ij}$$

or

$$K_{ij} = -g^{iT} \sum_{k \in \mathcal{P}} \varepsilon_{ik} \left\{ \varepsilon_{ji} \tilde{g}^j \left(\tilde{\bar{\omega}}^0 \Phi^{kk} \bar{\omega}^0 - \mathcal{M} \sum_{s \in \mathcal{B} - k} \tilde{D}^{ks} \tilde{\bar{\omega}}^0 \tilde{\bar{\omega}}^0 D^{sk} \right) \right.$$

$$+ \varepsilon_{jk} \left[\tilde{\bar{\omega}}^0 \left(\Phi^{kk} \tilde{g}^j - \tilde{g}^j \Phi^{kk} \right) \bar{\omega}^0 + \mathcal{M} \sum_{s \in \mathcal{B} - k} \left(\tilde{g}^j \tilde{D}^{ks} - \tilde{D}^{ks} \tilde{g}^j \right) \tilde{\bar{\omega}}^0 \tilde{\bar{\omega}}^0 D^{sk} \right]$$

$$\left. + \mathcal{M} \sum_{s \in \mathcal{B} - k} \varepsilon_{js} \tilde{D}^{ks} \tilde{\bar{\omega}}^0 \tilde{\bar{\omega}}^0 \tilde{g}^j D^{sk} \right\} + k_i \delta_{ij} \qquad \text{(136d)}$$

The next step is to find a coordinate transformation of Eq. (135b) which will permit this equation to be replaced by a system of uncoupled equations; for guidance in this procedure, we consider Section IIID. In order to determine which of the coordinate transformations in Section IIID apply to this sample problem, we must establish the structure of matrices J and K in Eq. (135b); we already know that I is real, symmetric, and nonsingular, and that J and K are real.

In order to illustrate the two most useful coordinate transformations, this sample problem will be continued along two branches: (1) We shall assume the nominal value of ω^0 appearing in Eqs. (136c, d) to be zero, and the structural connections to be purely elastic, so that these equations become $J_{ij} = 0$ and $K_{ij} = k_i \delta_{ij}$; and (2) we shall examine the most general case represented by Eqs. (136c, d).

In the first case, Eq. (135b) becomes

$$I\ddot{\gamma} + K\gamma = 0 \tag{137}$$

as in Eq. (65). The transformation $\gamma = \phi\eta$, recorded as Eq. (97), applies directly, and Eq. (137) can be rewritten as

$$\ddot{\eta} + \sigma^2 \eta = 0 \tag{138}$$

as indicated by Eq. (103). Because σ^2 is a diagonal matrix, the five scalar equations implied by Eq. (138) can be solved independently. If we were to consider inhomogeneous counterparts to Eqs. (137) and (138), introducing forcing functions on the right side of both equations, then it would still be possible to obtain the five scalar solutions in Eq. (138) independently, and it might be feasible for certain purposes to approximate the total response in all five of the γ_i of Eq. (137) by

$$\gamma \cong \overline{\phi}\overline{\eta} \tag{139}$$

where $\overline{\eta}$ and $\overline{\phi}$ are truncated versions of η and ϕ. In the extreme case, $\overline{\eta}$ might be a single scalar, perhaps η_1, and $\overline{\phi}$ the corresponding column of ϕ, which, for η_1, would be the first column.

Coordinate truncation can be an enormously valuable device for improving the efficiency of dynamic analysis, but it can also be a dangerous oversimplification. In particular, for this sample problem, one must guard against eliminating a distributed coordinate which might be excited to large and potentially destructive values by base motions of corresponding frequency, and one must avoid truncation of a coordinate whose modest response might degrade the pointing accuracy of the spacecraft or a spacecraft component. This sample problem is further complicated by the body $\mathscr{b}_{10}$ (see Fig. 4), which can perform large rotations relative to the contiguous body $\mathscr{b}_6$. If $\mathscr{b}_{10}$ is an instrument with pointing requirements, it might necessitate the retention of appendage modal coordinates which affect $\mathscr{b}_0$ only negligibly; if $\mathscr{b}_{10}$ is an actively controlled device, it might destructively force the appendage at the natural frequency of an otherwise insignificant mode. If $\mathscr{b}_{10}$ is massive and subject to gross departures from the nominal value assumed for the modal analysis, this will seriously jeopardize the entire truncation process. In any case, the truncation operation must be undertaken cautiously, with systematic evaluation of consequences.

Once some level of truncation has been chosen (perhaps tentatively), we return to the partially linearized equations (Eq. 130c), and substitute the truncated transformation $\gamma^2 = \bar{\phi}\bar{\eta}$ (see Eq. 139 or Eq. 121), together with $\dot{\gamma}^2 = \bar{\phi}\dot{\bar{\eta}}$ and $\ddot{\gamma}^2 = \bar{\phi}\ddot{\bar{\eta}}$. The fifth and sixth row-partition equations must then be multiplied by $\bar{\phi}^T$ in order to obtain the reduced set of equations. We may observe that the fifth row-partition equation now says $\bar{\phi}^T\bar{\phi}\dot{\bar{\eta}} = \bar{\phi}^T\bar{\phi}\dot{\bar{\eta}}$, and we can gain computational advantage by replacing this with $U\dot{\bar{\eta}} = U\dot{\bar{\eta}}$. We might wish also to incorporate modal damping in our system with the addition of $-2\bar{\zeta}\bar{\sigma}\dot{\bar{\eta}}$ to the term $\bar{\phi}^T R^2$.

In the more general case, for which $\omega^0 \neq 0$ nominally, we have no simplification of Eq. (135b), so that coordinate transformation must be preceded by its representation in the first-order form

$$P\dot{\Gamma} = Q\Gamma \tag{140}$$

as in Eq. (70). The appropriate transformation is

$$\Gamma = \Phi Y \tag{141}$$

as in Eq. (74) or Eq. (109). With the same truncation rationale as previously, we can instead substitute the approximation

$$\Gamma = \bar{\Phi}\bar{Y} \tag{142}$$

into Eq. (130c), where Γ appears in the several forms

$$\begin{bmatrix} \gamma^2 \\ \hline \dot{\gamma}^2 \end{bmatrix} \triangleq \Gamma; \qquad \gamma^2 = [U \,\vdots\, 0]\,\Gamma; \qquad \dot{\gamma}^2 = [0 \,\vdots\, U]\,\Gamma$$

and rewrite Eq. (130c) in the form established by Eq. (118).

Although the additional algebraic complexity of the more general case with a nominally spinning base is easily accommodated, there remain two considerations which make this case much more difficult to deal with than the nominally nonspinning case; the problem areas involve *computation* and *modeling*.

The computational difficulty arises because the elements of the eigenvectors which comprise the columns of $\bar{\Phi}$ in Eq. (142) are in general complex numbers, whereas the eigenvectors $\bar{\phi}$ in Eq. (139) are all real. Computation of $\bar{\Phi}$ is thus a significantly more time-consuming task than is the computation of $\bar{\phi}$, although recent experience (Refs. 28, 29) suggests that even the general problem can be handled efficiently with the proper eigenvalue–eigenvector program.

More troublesome problems must be faced in the initial construction of an appropriate multiple-rigid-body mathematical model of a spinning, flexible spacecraft. The first major task is the discretization of the spacecraft. This involves not only subdividing the vehicle into rigid-body segments but also assigning the proper description to the connection characteristics. In the given sample problem, we must assign d_i and k_i $(i = 5, \cdots, 9)$ as well as system geometry and mass properties. This is necessary whether the vehicle is spinning or not, but the assignment may be complicated by spin.

Unless there is actually a physical damping device at a physical hinge, it is probably best to ignore damping torques at hinge connections until transformation to modal coordinates has been accomplished; incorporation of a modal damping matrix can then be accomplished both for spinning and for nonspinning structures, as described at the end of Section IIIE.

The stiffness characteristics of each connection are defined by a single scalar, so that $k_5, \cdots, k_9$ must be specified for this sample problem. The determination of these spring constants is a routine chore when the structure is not spinning, although the calculation can be greatly complicated if variations in spacecraft thermal distortions result in variations in the undeformed nominal state of the vehicle. One must wonder, however, whether the structural properties of a nonspinning structure are changed by the steady-state forces borne by structural members when the vehicle is spinning.

As noted in Ref. 25, it is necessary when dealing with a spinning elastic structure, described conventionally in terms of finite elements, to augment the elastic stiffness matrix with a new matrix called the *geometric stiffness matrix* or the *preload stiffness matrix.* The purpose of the new matrix is to reflect the influence on structural response of steady-state loads in the members—loads induced, in this case, by spin. Without such correction terms, there would, for example, be no stiffness in the direction of the spin axis for a finite element model of a spinning cable, and we know from experience that a cable, even without bending moment capacity when at rest, gains effective stiffness from spin. If the spinning cable (or a spinning beam) is modeled as a continuum, the partial differential equations include terms which reflect the foreshortening of the structure in the direction normal to the spin axis, and in this way the steady-state load in the structure changes the effective stiffness. We should examine the possibility of the necessity of such correction terms in the stiffness characteristics of a multiple-rigid-body model of a spinning flexible structure.

Let us examine this question once again in the context of the cable disposed perpendicularly to the spin axis. Now we idealize the cable as a chain of rigid-body segments, connected by hinge joints, as depicted in Fig. 11. We imagine that, when the vehicle is not spinning, the cable has no bending stiffness, so that in the absence of spin there are no springs at the joints. Should we artificially introduce "effective spring constants" when the vehicle spins? In this case, the answer is clearly negative, since the foreshortening is automatically accommodated by the selection of variables of rotation rather than translation.

Limited investigation suggests that the lesson of the segmented cable can be generalized. It is not necessary to augment the elastic stiffnesses of the hinge connections of a multiple-rigid-body model of an elastic structure to accommodate steady-state loads due to spin. Since the determination of the geometric stiffness matrix for a finite element model is a task of major proportions, this would seem to offer an extra advantage to the multiple-rigid-body model.

IV. Summary and Conclusions

A. Summary

This report is confined in scope to the dynamic analysis of mathematical idealizations consisting of hinge-connected rigid bodies in a topological tree. The equations of unrestricted motion of the system are recorded as Eq. (1), in a matrix form that follows naturally from the previous developments of Hooker and Margulies (Refs. 9, 14). Several partially linearized forms of these equations appear as Eqs. (30a) through (30d), with the objective of providing approximations of Eq. (1), for which numerical integration is facilitated. Equations (1) and (30) are formulated in terms designed to minimize the demands on the user of a digital computer program for the numerical integration of these equations. This report includes the conceptual outlines of such a program and illustrative examples for its use; however, we reserve for later documentation the detailed listing of the computer program and instructions for the user.

The bulk of this report is devoted to the exposition of what seems to be an entirely new idea, which is based upon the application of a coordinate transformation to a subset of the deformation variables in the partially linearized Eqs. (30), and subsequent coordinate truncation. The resulting hybrid-coordinate equations appear as Eq. (118). Because the distributed (modal) coordinates in this formulation are linear combinations of "small" variables which represent relative rotations of contiguous pairs of rigid bodies, their use does not imply that the deformations of the elastic substructure modeled by a chain or branch of elastically connected rigid bodies are "small" in any global sense, but only that the strains or local deformations are "small." Thus, we have accomplished a discretized version of the equations of elasticity for large deformations with small strains. The distributed coordinates introduced here are called *large-deformation modal coordinates.* We expect these results to have applicability to a class of spacecraft for which deformations are so large as to preclude the use of modal coordinates in the usual sense of structural dynamics (as in Ref. 21), and yet we can gain the advantages of computational efficiency normally associated with hybrid-coordinate analysis (as opposed to discrete-coordinate analysis).

While Eq. (118) provides an alternative to Eqs. (30) which has advantages in computational efficiency, the former equations demand a much more complex specification of input information, including mode shapes and frequencies generated in a separate computer program for eigenvalue–eigenvector analysis. For this reason, the decision has been made at JPL to develop a distinct computer program for the numerical integration of Eq. (118), as opposed to its successively more comprehensive counterparts in Eqs. (30) and (1). This program will be documented separately when it becomes available.

B. Projection

As noted in the Introduction, the determination of the influence of spacecraft nonrigidity on mission performance is not a single problem but a family of problems, and a family of solution procedures is required for the efficient resolution of these problems. In this report, we have tried to deal with those vehicles for which a multiple-rigid-body tree model is appropriate. In a report now in preparation, we treat those vehicles for which it seems preferable to adopt an idealization consisting again of hinge-connected rigid bodies in a topological tree but with a small-deformation elastic appendage attached to each of the rigid bodies.* These equations are still quite tractable, and are amenable to digital computer numerical integration. More general formulations will be developed if it becomes clear that they meet the criteria of need and reasonable efficiency of simulation.

*The equations of motion appear in vector-dyadic form in Ref. 30.

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Appendix A

Derivation of Discrete-Coordinate Dynamical Equations*

In addition to the symbol definitions adopted in Section IIA, consider the following:

Def. A1. Let $\boldsymbol{\rho}^k$ be the position vector of c_k with respect to CM.

Def. A2. Let the system of forces applied to $\mathcal{b}_k$ by $\mathcal{b}_j$ for $k \in \mathcal{B}$ and $j \in \mathcal{B}_k$ be equivalent to a resultant force $\mathbf{f}^{kj}$ applied at the labeled point ($\mathcal{h}_j$ or $\mathcal{h}_k$) common to $\mathcal{b}_k$ and $\mathcal{b}_j$, plus a torque $\mathbf{T}^{kj}$.

Def. A3. Let $\mathbf{t}^{kj}$ be the *kinematical constraint torque* applied to $\mathcal{b}_k$ by $\mathcal{b}_j$, for $k \in \mathcal{B}$ and $j \in \mathcal{B}_k$; that is, let the *total* torque $\mathbf{T}^{kj}$ applied to $\mathcal{b}_k$ by its neighbor $\mathcal{b}_j$ be given by

$$\mathbf{T}^{kj} = \mathbf{t}^{kj} + \delta_{jN_k}\tau_k \mathbf{g}^k - \delta_{kN_j}\tau_j \mathbf{g}^j$$

so that if $k > j$, then $\mathbf{g}^k \cdot \mathbf{T}^{kj} = \tau_k$, and if $k < j$, then $\mathbf{g}^j \cdot \mathbf{T}^{kj} = -\tau_j$.

Def. A4. Let $\boldsymbol{\omega}^k$ be the inertial angular velocity of $\mathcal{b}_k$.

All dynamical information regarding the motion of the $n + 1$ rigid bodies whose indices comprise the set $\mathcal{B}$ must be contained in the translational equations

$$\mathbf{F}^r + \sum_{s \in \mathcal{B}_r} \mathbf{f}^{rs} - m_r(\ddot{\mathbf{X}} + \ddot{\boldsymbol{\rho}}^r) = 0 \qquad r \in \mathcal{B} \tag{A-1}$$

and the rotational equations

$$\mathbf{T}^k + \sum_{j \in \mathcal{B}_k} \mathbf{T}^{kj} + \sum_{j \in \mathcal{B}_k} \mathbf{p}^{kj} \times \mathbf{f}^{kj} - \mathbf{I}^k \cdot \dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k \times \mathbf{I}^k \cdot \boldsymbol{\omega}^k = 0 \qquad k \in \mathcal{B} \tag{A-2}$$

The objective is to recast these $2(n + 1)$ vector equations into $n + 6$ scalar independent equations which do not involve unknown scalar components of the kinematical constraint torques $\mathbf{t}^{kj}$ or the constraint forces $\mathbf{f}^{kj}$. To this end, sum all equations from Eq. (A-1) to find

$$\sum_{r \in \mathcal{B}} [\mathbf{F}^r + \sum_{s \in \mathcal{B}_r} \mathbf{f}^{rs}] = \sum_{r \in \mathcal{B}} [m_r \ddot{\mathbf{X}} + m_r \ddot{\boldsymbol{\rho}}^r]$$

The action–reaction principle (Newton's third law) provides

$$\sum_{r \in \mathcal{B}} \sum_{s \in \mathcal{B}_r} \mathbf{f}^{rs} = 0 \tag{A-3}$$

and the mass center definition furnishes

$$\sum_{r \in \mathcal{B}} m_r \boldsymbol{\rho}^r = 0 \tag{A-4}$$

so that the sum of Eqs. (A-1) becomes simply

$$\sum_{r \in \mathcal{B}} \mathbf{F}^r = \sum_{r \in \mathcal{B}} m_r \ddot{\mathbf{X}} \tag{A-5}$$

*The derivation following is the natural progeny of the derivations by Hooker and Margulies (Refs. 9 and 14), and represents a special case of that presented in Ref. 30.

or, with Defs. 25 and 30 from Section IIA,

$$\mathbf{F} = \mathcal{M}\ddot{\mathbf{X}} \tag{A-6}$$

This result is the vector counterpart of the matrix dynamical equation recorded in Section IIB as Eq. (2), but in addition to verifying that equation, it permits Eq. (A-1) to be rewritten as

$$\sum_{s \in \mathcal{B}_r} \mathbf{f}^{rs} = -\mathbf{F}^r + m_r \left[\sum_{u \in \mathcal{B}} \frac{\mathbf{F}^u}{\mathcal{M}} + \ddot{\boldsymbol{\rho}}^r \right] \qquad r \in \mathcal{B} \tag{A-7}$$

Summing selected equations from the set implied by Eq. (A-7) yields

$$\mathbf{f}^{jk} = \sum_{r \in \mathcal{B}_{kj}} \sum_{s \in \mathcal{B}_r} \mathbf{f}^{rs} = -\sum_{r \in \mathcal{B}_{kj}} \left[\left(\mathbf{F}^r - \frac{m_r}{\mathcal{M}} \sum_{u \in \mathcal{B}} \mathbf{F}^u \right) - m_r \ddot{\boldsymbol{\rho}}^r \right] \tag{A-8}$$

by virtue of the action–reaction principle. (Note that the indicated summation produces the translatory equation of motion for the nested set of bodies consisting of all bodies on the branch attached to $\mathscr{b}_j$ and commencing with $\mathscr{b}_k$.)

The action–reaction principle combines with Eq. (A-8) to give

$$\mathbf{f}^{kj} = -\mathbf{f}^{jk} = \sum_{r \in \mathcal{B}_{kj}} \left[\left(\mathbf{F}^r - \frac{m_r}{\mathcal{M}} \sum_{u \in \mathcal{B}} \mathbf{F}^u \right) - m_r \ddot{\boldsymbol{\rho}}^r \right] \qquad k \in \mathcal{B} \tag{A-9}$$

and substitution of this result into Eq. (A-2) eliminates all interbody forces from the rotational equations.

The term of concern in Eq. (A-2) can, with Eq. (A-9), be written (using Defs. 32, 34, and 35 from Section IIA) as

$$\begin{aligned}
\sum_{j \in \mathcal{B}_k} \mathbf{p}^{kj} \times \mathbf{f}^{kj} &= \sum_{j \in \mathcal{B}_k} \mathbf{p}^{kj} \times \sum_{r \in \mathcal{B}_{kj}} \left[\left(\mathbf{F}^r - \frac{m_r}{\mathcal{M}} \sum_{u \in \mathcal{B}} \mathbf{F}^u \right) - m_r \ddot{\boldsymbol{\rho}}^r \right] \\
&= \sum_{r \in \mathcal{B}} \left[\mathbf{L}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times \frac{m_r}{\mathcal{M}} \sum_{u \in \mathcal{B}} \mathbf{F}^u - \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r \right] \\
&= \sum_{r \in \mathcal{B}} \mathbf{L}^{kr} \times \mathbf{F}^r + \mathbf{D}^{kk} \times \sum_{u \in \mathcal{B}} \mathbf{F}^u - \sum_{r \in \mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r \\
&= \sum_{r \in \mathcal{B}} \mathbf{D}^{kr} \times \mathbf{F}^r - \sum_{r \in \mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r
\end{aligned} \tag{A-10}$$

It should be noted that, by Def. 32, $\mathbf{L}^{kk} = 0$; thus, certain of the terms in the preceding summations are zero.

Combining Eqs. (A-10) and (A-2) furnishes the vector equations

$$\mathbf{T}^k + \sum_{j \in \mathcal{B}_k} \mathbf{T}^{kj} + \sum_{r \in \mathcal{B}} \mathbf{D}^{kr} \times \mathbf{F}^r - \sum_{r \in \mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r - \mathbf{I}^k \cdot \dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k \times \mathbf{I}^k \cdot \boldsymbol{\omega}^k = 0 \qquad k \in \mathcal{B} \tag{A-11}$$

Next, sum individual equations from Eq. (A-11) over those values of k corresponding to the indices of a nested set of bodies consisting of all bodies on a branch attached to b_{N_s} and commencing with b_s; the resulting equation, which by virtue of the action–reaction principle involves no interbody torques except $\mathbf{T}^{sN_s}$, is given by

$$\mathbf{T}^{sN_s} + \sum_{k\in\mathcal{B}_{N_s s}} [\mathbf{T}^k + \sum_{r\in\mathcal{B}} (\mathbf{D}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r) - \mathbf{I}^k \cdot \dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k \times \mathbf{I}^k \cdot \boldsymbol{\omega}^k] = 0 \qquad s \in \mathcal{P} \tag{A-12}$$

In what follows, $\sum\limits_{k\in\mathcal{B}_{N_s s}}$ will be replaced by $\sum\limits_{k\in\mathcal{P}} \varepsilon_{sk}$, which, by Def. 27, is equivalent.

Definition A3 permits the substitution

$$\mathbf{T}^{sN_s} = \mathbf{t}^{sN_s} + \tau_s \mathbf{g}^s \tag{A-13}$$

so that dot-multiplication of Eq. (A-12) by $\mathbf{g}^s$ yields the scalar equations

$$\tau_s + \mathbf{g}^s \cdot \sum_{k\in\mathcal{P}} \varepsilon_{sk} [\mathbf{T}^k + \sum_{r\in\mathcal{B}} (\mathbf{D}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r) - \mathbf{I}^k \cdot \dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k \times \mathbf{I}^k \cdot \boldsymbol{\omega}^k] = 0 \qquad s \in \mathcal{P} \tag{A-14}$$

Equation (A-14) provides n scalar equations of the desired character, being free of any involvement with constraint forces or torques. The required three additional scalar equations are readily obtained by summing the individual vector equations represented by Eq. (A-11) over all values of k in $\mathcal{B}$. Since the action–reaction principle in this case eliminates all interbody torques, the result is

$$\sum_{k\in\mathcal{B}} [\mathbf{T}^k + \sum_{r\in\mathcal{B}} (\mathbf{D}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r) - \mathbf{I}^k \cdot \dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k \times \mathbf{I}^k \cdot \boldsymbol{\omega}^k] = 0 \tag{A-15}$$

In order to prove that Eqs. (A-14) and (A-15) establish the validity of Eq. (1) of Section IIB, one must first use available kinematical relationships to express $\ddot{\boldsymbol{\rho}}^r$, $\boldsymbol{\omega}^k$, and $\dot{\boldsymbol{\omega}}^k$ (for $r, k \in \mathcal{B}$) in terms of the kinematical variables appearing in Eq. (1).

To eliminate $\boldsymbol{\rho}^r$, let $\mathcal{C}_{rj}$ be the set of indices of bodies lying on the path between b_r and b_j, and note, from Defs. A1 and 32, that

$$\boldsymbol{\rho}^r - \boldsymbol{\rho}^j = \mathbf{L}^{jr} + \sum_{s\in\mathcal{C}_{rj}} (\mathbf{L}^{sr} - \mathbf{L}^{sj}) - \mathbf{L}^{rj} \tag{A-16}$$

See Fig. A-1 for an illustration of this relationship. Having recorded Eq. (A-16), one can recognize more easily that the right side is unnecessarily complex; since, from Def. 32, $\mathbf{L}^{jj} = \mathbf{L}^{rr} = 0$, and for any index s in the set $\mathcal{B} - \mathcal{C}_{rj}$ the sum $\mathbf{L}^{sr} - \mathbf{L}^{sj}$ is zero (as in Fig. A-1 for $s = x, y$, or z), Eq. (A-16) can be replaced by

$$\boldsymbol{\rho}^r - \boldsymbol{\rho}^j = \sum_{s\in\mathcal{B}} (\mathbf{L}^{sr} - \mathbf{L}^{sj}) \qquad r, j \in \mathcal{B} \tag{A-17}$$

with the obvious consequence,

$$\frac{m_j}{\mathcal{M}} \boldsymbol{\rho}^r - \frac{m_j}{\mathcal{M}} \boldsymbol{\rho}^j = \sum_{s\in\mathcal{B}} \frac{m_j}{\mathcal{M}} (\mathbf{L}^{sr} - \mathbf{L}^{sj}) \qquad r, j \in \mathcal{B} \tag{A-18}$$

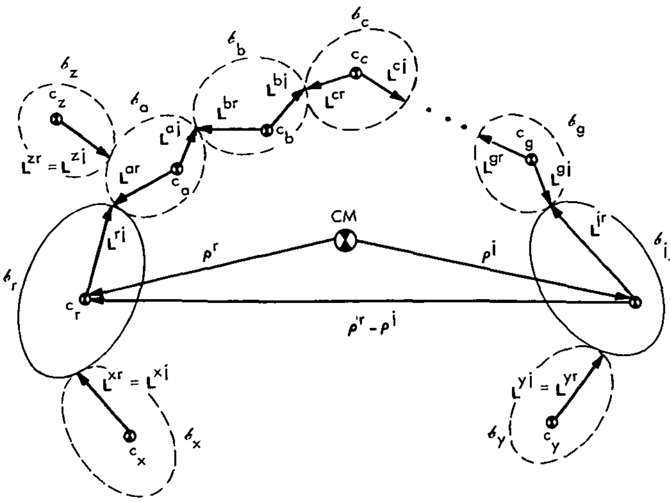

Fig. A-1. Vector chain for $\mathcal{C}_{rj} = \{a, b, \cdots, g\}$

For any given value of r, there are $n + 1$ equations implied by Eq. (A-18); summing these provides

$$\sum_{j\in\mathcal{B}} \frac{m_j}{\mathcal{M}} \boldsymbol{\rho}^r - \sum_{j\in\mathcal{B}} \frac{m_j}{\mathcal{M}} \boldsymbol{\rho}^j = \sum_{j\in\mathcal{B}} \sum_{s\in\mathcal{B}} \frac{m_j}{\mathcal{M}} (\mathbf{L}^{sr} - \mathbf{L}^{sj}) \qquad \text{(A-19)}$$

Equation (A-4) removes the second term on the left, and the definition of $\mathcal{M}$ (Def. 30) simplifies the first, yielding

$$\boldsymbol{\rho}^r = \sum_{j\in\mathcal{B}} \sum_{s\in\mathcal{B}} \frac{m_j}{\mathcal{M}} (\mathbf{L}^{sr} - \mathbf{L}^{sj}) \qquad \text{(A-20)}$$

Reversing the summation sequence and expanding produces

$$\begin{aligned}\boldsymbol{\rho}^r &= \sum_{s\in\mathcal{B}} \left[\sum_{j\in\mathcal{B}} \frac{m_j}{\mathcal{M}} \mathbf{L}^{sr} - \sum_{j\in\mathcal{B}} \frac{m_j}{\mathcal{M}} \mathbf{L}^{sj} \right] \\ &= \sum_{s\in\mathcal{B}} \left[\mathbf{L}^{sr} - \sum_{j\in\mathcal{B}} \frac{m_j}{\mathcal{M}} \mathbf{L}^{sj} \right] \qquad \text{(A-21)}\end{aligned}$$

Definitions 33 and 35 now permit the representation

$$\boldsymbol{\rho}^r = \sum_{s\in\mathcal{B}} [\mathbf{L}^{sr} + \mathbf{D}^{ss}] = \sum_{s\in\mathcal{B}} \mathbf{D}^{sr} \qquad \text{(A-22)}$$

The term in the dynamical equations (A-14) and (A-15) which requires the substitution of $\boldsymbol{\rho}^r$ from Eq. (A-22) is

$$-\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r = -\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \sum_{s\in\mathcal{B}} \ddot{\mathbf{D}}^{sr} = -\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \Big(\ddot{\mathbf{D}}^{kr} + \sum_{s\in\mathcal{B}-k} \ddot{\mathbf{D}}^{sr}\Big) \qquad \text{(A-23)}$$

The first term in parentheses in Eq. (A-23) contributes (with the use of Def. 35)

$$-\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\mathbf{D}}^{kr} = \sum_{r\in\mathcal{B}} (\mathbf{D}^{kk} - \mathbf{D}^{kr}) \times m_r \ddot{\mathbf{D}}^{kr}$$

$$= \mathbf{D}^{kk} \times \sum_{r\in\mathcal{B}} m_r \ddot{\mathbf{D}}^{kr} - \frac{d}{dt} \left(\sum_{r\in\mathcal{B}} \mathbf{D}^{kr} \times m_r \dot{\mathbf{D}}^{kr}\right) \tag{A-24}$$

The interpretation of b_k as the mass center of the augmented body $\mathcal{b}'_k$ permitted the recognition in Section IIA of the relationship

$$\sum_{r\in\mathcal{B}} m_r \mathbf{D}^{kr} = 0 \tag{A-25}$$

which eliminates the first term on the right of Eq. (A-24). The second term involves the inertial time derivatives of vectors $\mathbf{D}^{kr}$, which are fixed in $\mathcal{b}_k$, so that, with Def. A4, one can write

$$\dot{\mathbf{D}}^{kr} = \boldsymbol{\omega}^k \times \mathbf{D}^{kr} \tag{A-26}$$

Equation (A-24) thus becomes

$$-\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\mathbf{D}}^{kr} = -\frac{d}{dt} \left[\sum_{r\in\mathcal{B}} \mathbf{D}^{kr} \times (m_r \boldsymbol{\omega}^k \times \mathbf{D}^{kr})\right]$$

$$= -\frac{d}{dt} \left\{\left[\sum_{r\in\mathcal{B}} m_r (\mathbf{D}^{kr} \cdot \mathbf{D}^{kr} \mathbf{U} - \mathbf{D}^{kr}\mathbf{D}^{kr})\right] \cdot \boldsymbol{\omega}^k\right\}$$

$$= -\frac{d}{dt} (\mathbf{K}^k \cdot \boldsymbol{\omega}^k) \tag{A-27}$$

where $\mathbf{U}$ is the unity dyadic and the dyadic $\mathbf{K}^k$ is defined by

$$\mathbf{K}^k \triangleq \sum_{r\in\mathcal{B}} m_r (\mathbf{D}^{kr} \cdot \mathbf{D}^{kr} \mathbf{U} - \mathbf{D}^{kr}\mathbf{D}^{kr}) \tag{A-28}$$

Thus, Eq. (A-23) becomes

$$-\sum_{r\in\mathcal{B}} \mathbf{L}^{kr} \times m_r \ddot{\boldsymbol{\rho}}^r = -\frac{d}{dt} (\mathbf{K}^k \cdot \boldsymbol{\omega}^k) - \sum_{r\in\mathcal{B}} \sum_{s\in\mathcal{B}-k} m_r \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{sr} \tag{A-29}$$

The second term on the right in Eq. (A-29) can be expanded to find (using Eq. A-25 and $\mathbf{L}^{kk} \triangleq 0$)

$$-\sum_{r\in\mathcal{B}-k} \sum_{s\in\mathcal{B}-k-r} m_r \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{sr} - \sum_{r\in\mathcal{B}-k} m_r \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{rr}$$

$$= -\sum_{r\in\mathcal{B}-k} \sum_{s\in\mathcal{B}-k-r} m_r \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{sr} + \sum_{r\in\mathcal{B}-k} \sum_{s\in\mathcal{B}-r} m_s \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{rs}$$

$$= -\sum_{r\in\mathcal{B}-k} \sum_{s\in\mathcal{B}-k-r} m_s \mathbf{L}^{ks} \times \ddot{\mathbf{D}}^{rs} + \sum_{r\in\mathcal{B}-k} \sum_{s\in\mathcal{B}-r} m_s \mathbf{L}^{kr} \times \ddot{\mathbf{D}}^{rs} \tag{A-30}$$

where in the final step, the indices of the summed quantities in the first term have been exchanged, taking advantage of the fact that the double summation ranges over a set of indices r, s which is symmetric in the sense that a, b belongs to the set if and only if b, a belongs to it as well. This observation (first noted in Ref. 9) permits the factorization of the terms in Eq. (A-30), and the recognition that $\mathbf{L}^{kk} \triangleq 0$ allows the further simplification

$$\sum_{r\in\mathcal{B}-k}\left[\sum_{s\in\mathcal{B}-r} m_s\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{rs} - \sum_{s\in\mathcal{B}-k-r} m_s\mathbf{L}^{ks}\times\ddot{\mathbf{D}}^{rs}\right] = \sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-r} m_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\times\ddot{\mathbf{D}}^{rs} \tag{A-31}$$

The quantity $\mathbf{L}^{kr} - \mathbf{L}^{ks}$ is zero for any index s corresponding to a body which lies anywhere on the branch that begins with $b_{N_{kr}}$ and includes b_r; for any other index s, the quantity $\mathbf{D}^{rs}$ is also $\mathbf{D}^{rk}$. Thus, $\ddot{\mathbf{D}}^{rs}$ can be changed to $\ddot{\mathbf{D}}^{rk}$. At the same time, $s\in\mathcal{B}-r$ is replaced by $s\in\mathcal{B}$, since there is no contribution when $s = r$. With Defs. (33) and (35), Eq. (A-31) then becomes

$$\begin{aligned}\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-r} m_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\times\ddot{\mathbf{D}}^{rs} &= \sum_{r\in\mathcal{B}-k}\left[\sum_{s\in\mathcal{B}} m_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\right]\times\ddot{\mathbf{D}}^{rk} \\ &= \sum_{r\in\mathcal{B}-k}\left[\mathcal{M}\mathbf{L}^{kr} - \sum_{s\in\mathcal{B}} m_s\mathbf{L}^{ks}\right]\times\ddot{\mathbf{D}}^{rk} \\ &= \sum_{r\in\mathcal{B}-k}\left[\mathcal{M}\mathbf{L}^{kr} + \mathcal{M}\mathbf{D}^{kk}\right]\times\ddot{\mathbf{D}}^{rk} = \mathcal{M}\sum_{r\in\mathcal{B}-k}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}\end{aligned} \tag{A-32}$$

Finally, Eqs. (A-29)–(A-32) combine to provide

$$-\sum_{r\in\mathcal{B}}\mathbf{L}^{kr}\times m_r\ddot{\boldsymbol{\rho}}^r = -\frac{d}{dt}(\mathbf{K}^k\cdot\boldsymbol{\omega}^k) + \mathcal{M}\sum_{r\in\mathcal{B}-k}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk} \tag{A-33}$$

thereby permitting Eqs. (A-14) and (A-15) to be written in a more useful vector-dyadic form as

$$\begin{aligned}\tau_s + \mathbf{g}^s\cdot\sum_{k\in\mathcal{P}}\varepsilon_{sk}\Big[\mathbf{T}^k + \sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\times\mathbf{F}^r - \boldsymbol{\Phi}^{kk}\cdot\dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k\times\boldsymbol{\Phi}^{kk}\cdot\boldsymbol{\omega}^k \\ + \mathcal{M}\sum_{r\in\mathcal{B}-k}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}\Big] = 0 \qquad s\in\mathcal{P}\end{aligned} \tag{A-34}$$

and

$$\sum_{k\in\mathcal{B}}\Big[\mathbf{T}^k + \sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\times\mathbf{F}^r - \boldsymbol{\Phi}^{kk}\cdot\dot{\boldsymbol{\omega}}^k - \boldsymbol{\omega}^k\times\boldsymbol{\Phi}^{kk}\cdot\boldsymbol{\omega}^k + \mathcal{M}\sum_{r\in\mathcal{B}-k}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}\Big] = 0 \tag{A-35}$$

where the relationship between $\mathbf{K}^k$ in Eq. (A-28) and $\boldsymbol{\Phi}^{kk}$ in Def. 36 has been noted to provide $\mathbf{K}^k + \mathbf{I}^k = \boldsymbol{\Phi}^{kk}$.

Observing that $\mathbf{D}^{rk}$ is fixed in b_r, we can substitute

$$\ddot{\mathbf{D}}^{rk} = \dot{\boldsymbol{\omega}}^r\times\mathbf{D}^{rk} + \boldsymbol{\omega}^r\times(\boldsymbol{\omega}^r\times\mathbf{D}^{rk}) \tag{A-36}$$

The final sums in Eqs. (A-34) and (A-35) thus become

$$\begin{aligned}\mathcal{M}\sum_{r\in\mathcal{B}-k}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk} &= \mathcal{M}\sum_{r\in\mathcal{B}-k}\{-\mathbf{D}^{kr}\times(\mathbf{D}^{rk}\times\dot{\boldsymbol{\omega}}^r) + \mathbf{D}^{kr}\times[\boldsymbol{\omega}^r\times(\boldsymbol{\omega}^r\times\mathbf{D}^{rk})]\} \\ &= \mathcal{M}\sum_{r\in\mathcal{B}-k}\{-\mathbf{D}^{rk}\mathbf{D}^{kr}\cdot\dot{\boldsymbol{\omega}}^r + \mathbf{D}^{rk}\cdot\mathbf{D}^{kr}\dot{\boldsymbol{\omega}}^r \\ &\quad + \mathbf{D}^{kr}\times[\boldsymbol{\omega}^r\times(\boldsymbol{\omega}^r\times\mathbf{D}^{rk})]\}\end{aligned} \tag{A-37}$$

The final form of the vector-dyadic equations of motion is available upon substitution into Eqs. (A-34) and (A-35) of Eq. (A-37) and the kinematic expansions

$$\boldsymbol{\omega}^k = \boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}} \varepsilon_{rk}\dot{\gamma}_r\mathbf{g}^r \tag{A-38a}$$

and

$$\dot{\boldsymbol{\omega}}^k = \dot{\boldsymbol{\omega}}^0 + \sum_{r\epsilon\mathscr{P}} \varepsilon_{rk}(\mathbf{g}^r\ddot{\gamma}_r + \boldsymbol{\omega}^r \times \mathbf{g}^r\dot{\gamma}_r) \tag{A-38b}$$

The result of these substitutions into Eq. (A-35) may be written in the form (with judicious switching of index labels to permit later comparisons)

$$\begin{aligned}
&\Big\{\sum_{k\epsilon\mathscr{B}} \boldsymbol{\Phi}^{kk} - \mathcal{M}\sum_{k\epsilon\mathscr{B}}\sum_{j\epsilon\mathscr{B}-k}(\mathbf{D}^{jk}\cdot\mathbf{D}^{kj}\mathbf{U} - \mathbf{D}^{jk}\mathbf{D}^{kj})\Big\}\cdot\dot{\boldsymbol{\omega}}^0 \\
&\quad + \sum_{r\epsilon\mathscr{B}}\Big[\boldsymbol{\Phi}^{rr} - \mathcal{M}\sum_{j\epsilon\mathscr{B}-r}(\mathbf{D}^{jr}\cdot\mathbf{D}^{rj}\mathbf{U} - \mathbf{D}^{jr}\mathbf{D}^{rj})\Big]\cdot\sum_{k\epsilon\mathscr{P}}\varepsilon_{kr}\mathbf{g}^k\ddot{\gamma}_k = \\
&\qquad \sum_{k\epsilon\mathscr{B}}\Big\{\mathbf{T}^k + \sum_{j\epsilon\mathscr{B}}\mathbf{D}^{kj}\times\mathbf{F}^j - \boldsymbol{\Phi}^{kk}\cdot\sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\dot{\gamma}_r\Big(\boldsymbol{\omega}^0 + \sum_{s\epsilon\mathscr{P}}\varepsilon_{sr}\dot{\gamma}_s\mathbf{g}^s\Big)\times\mathbf{g}^r \\
&\qquad - \Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\mathbf{g}^r\dot{\gamma}_r\Big)\times\boldsymbol{\Phi}^{kk}\cdot\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\mathbf{g}^r\dot{\gamma}_r\Big) \\
&\qquad + \mathcal{M}\sum_{j\epsilon\mathscr{B}-k}\mathbf{D}^{kj}\times\Big[\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\mathbf{g}^r\dot{\gamma}_r\Big)\times\Big\{\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\mathbf{g}^r\dot{\gamma}_r\Big)\times\mathbf{D}^{jk}\Big\}\Big] \\
&\qquad + \mathcal{M}\sum_{j\epsilon\mathscr{B}-k}(\mathbf{D}^{jk}\cdot\mathbf{D}^{kj}\mathbf{U} - \mathbf{D}^{jk}\mathbf{D}^{kj})\cdot\sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\dot{\gamma}_r\Big(\boldsymbol{\omega}^0 + \sum_{s\epsilon\mathscr{P}}\varepsilon_{sr}\dot{\gamma}_s\mathbf{g}^s\Big)\times\mathbf{g}^r\Big\}
\end{aligned} \tag{A-39a}$$

If this equation is written in the symbolic form

$$\boldsymbol{\alpha}^{00}\cdot\dot{\boldsymbol{\omega}}^0 + \sum_{k\epsilon\mathscr{P}}\mathbf{a}^{0k}\ddot{\gamma}_k = \sum_{k\epsilon\mathscr{B}}\mathbf{A}^k \tag{A-39b}$$

then, by comparison with Defs. 38, 39, and 41, it becomes apparent that Eq. (A-39) is equivalent to the first three scalar equations recorded in Section IIB as Eq. (1). This recognition requires the identification of the matrix a_{00} in Def. (38) as the representation of the dyadic $\boldsymbol{\alpha}^{00}$ in vector basis $\{\mathbf{b}^0\}$, i.e., $\boldsymbol{\alpha}^{00} = \{\mathbf{b}^0\}^T a_{00}\{\mathbf{b}^0\}$. Similarly, it must be recognized that $\mathbf{a}^{0k} = \{\mathbf{b}^0\}^T a_{0k}$ and $\mathbf{A}^k = \{\mathbf{b}^k\}^T A^k$.

The remaining scalar equations in Eq. (1) must be confirmed by Eq. (A-34) after substitution of Eqs. (A-37) and (A-38). With some switching of indices to facilitate comparison with Eq. (1), Eq. (34) may be written in the form

$$\begin{aligned}
&\mathbf{g}^s\cdot\sum_{r\epsilon\mathscr{P}}\varepsilon_{sr}\Big[\boldsymbol{\Phi}^{rr} - \mathcal{M}\sum_{j\epsilon\mathscr{B}-r}(\mathbf{D}^{jr}\cdot\mathbf{D}^{rj}\mathbf{U} - \mathbf{D}^{jr}\mathbf{D}^{rj})\Big]\cdot\dot{\boldsymbol{\omega}}^0 \\
&\quad + \mathbf{g}^s\cdot\Big[\sum_{r\epsilon\mathscr{P}}\varepsilon_{sr}\Big[\boldsymbol{\Phi}^{rr} - \mathcal{M}\sum_{i\epsilon\mathscr{B}-r}(\mathbf{D}^{ir}\cdot\mathbf{D}^{ri}\mathbf{U} - \mathbf{D}^{ir}\mathbf{D}^{ri})\Big]\cdot\sum_{j\epsilon\mathscr{P}}\varepsilon_{jr}\mathbf{g}^j\ddot{\gamma}_j = \\
&\qquad \mathbf{g}^s\cdot\sum_{k\epsilon\mathscr{P}}\varepsilon_{sk}\Big\{\mathbf{T}^k + \sum_{j\epsilon\mathscr{B}}\mathbf{D}^{kj}\times\mathbf{F}^j - \boldsymbol{\Phi}^{kk}\cdot\sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\dot{\gamma}_r\Big(\boldsymbol{\omega}^0 + \sum_{s\epsilon\mathscr{P}}\varepsilon_{sr}\dot{\gamma}_s\mathbf{g}^s\Big)\times\mathbf{g}^r \\
&\qquad - \Big[\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\mathbf{g}^r\dot{\gamma}_r\Big)\times\boldsymbol{\Phi}^{kk}\cdot\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rk}\mathbf{g}^r\dot{\gamma}_r\Big) \\
&\qquad + \mathcal{M}\sum_{j\epsilon\mathscr{B}-k}\mathbf{D}^{kj}\times\Big[\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\mathbf{g}^r\dot{\gamma}_r\Big)\times\Big\{\Big(\boldsymbol{\omega}^0 + \sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\mathbf{g}^r\dot{\gamma}_r\Big)\times\mathbf{D}^{jk}\Big\}\Big] \\
&\qquad + \mathcal{M}\sum_{j\epsilon\mathscr{B}-k}(\mathbf{D}^{jk}\cdot\mathbf{D}^{kj}\mathbf{U} - \mathbf{D}^{jk}\mathbf{D}^{kj})\cdot\sum_{r\epsilon\mathscr{P}}\varepsilon_{rj}\dot{\gamma}_r\Big(\boldsymbol{\omega}^0 + \sum_{s\epsilon\mathscr{P}}\varepsilon_{sr}\mathbf{g}^s\dot{\gamma}_s\Big) + \mathbf{g}^r\Big\} \\
&\qquad + \tau_s \qquad (s\,\epsilon\,\mathscr{P})
\end{aligned} \tag{A-40a}$$

with the symbolic equivalent (for $s \epsilon \mathscr{P}$)

$$\mathbf{a}_{s0} \cdot \dot{\boldsymbol{\omega}}^0 + \sum_{j \epsilon \mathscr{P}} a_{sj} \ddot{\gamma}_j = \mathbf{g}^s \cdot \sum_{k \epsilon \mathscr{P}} \varepsilon_{sk} \mathbf{A}^k + \tau_s \tag{A-40b}$$

Once the vectors in this scalar equation are recast as matrices and comparison is made with Eq. (1) and Defs. 39 and 40, it becomes clear that with Eq. A-40 we have established the validity of the last n scalar equations in Eq. (1). Thus, Eqs. (39) and (40) together constitute a proof of Eq. (1).

Appendix B

Exact Scalar Equations for a Three-Body Example

In order to reveal in explicit detail the structure of the scalar equations of motion derived in generic matrix terms in the body of this report, these matrix equations are expanded here for application to the special three-body system illustrated in Fig. B-1, in the absence of external forces and torques.

Body $\mathscr{b}_0$ has a planar slit through its center c in the $\mathbf{b}_2^0, \mathbf{b}_3^0$ plane, and a dumbbell body $\mathscr{b}_1$ has its center attached to $\mathscr{b}_0$ at c, so that $\mathscr{b}_1$ can rotate relative to $\mathscr{b}_0$ only about a hinge axis parallel to $\mathbf{b}_1^0$ (which is also labeled $\mathbf{b}_1^1$).

Body $\mathscr{b}_2$ is a solid cylinder embedded in a cavity in $\mathscr{b}_0$, with the cylinder axis (and the single degree of rotational freedom) established by $\mathbf{b}_1^0$ (which is also labeled $\mathbf{b}_1^2$). Point c is the mass center of the combination $\mathscr{b}_0 + \mathscr{b}_2$, and principal axes of inertia of $\mathscr{b}_0 + \mathscr{b}_2$ are defined by $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$. The point c is also the mass center of $\mathscr{b}_1$, and hence of the system $\mathscr{b}_0 + \mathscr{b}_1 + \mathscr{b}_2$.

The equations of motion are given by Eq. (1) as

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}^0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \end{bmatrix} = \begin{bmatrix} A^0 + C^{01}A^1 + C^{02}A^2 \\ u^{1T}A^1 + \tau_1 \\ u^{1T}A^2 + \tau_2 \end{bmatrix} \tag{B-1}$$

where the symbol $u^1 \triangleq [1 \;\; 0 \;\; 0]^T$ represents g^1 and g^2, since $\mathbf{g}^1 = \mathbf{g}^2 = \mathbf{b}_1^0 = \mathbf{b}_1^1 = \mathbf{b}_1^2$.

From Def. 41 in Section IIA, we have simply

$$A^0 = -\tilde{\omega}^0 \Phi^{00} \omega^0 \tag{B-2}$$

because both $\mathscr{b}_1$ and $\mathscr{b}_2$ are supported at their mass centers, which are also their barycenters, so that $D^{20} = D^{10} = 0$.

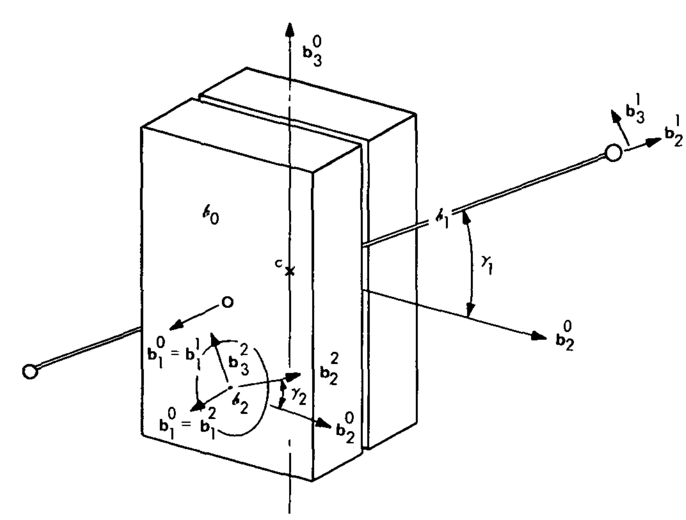

Fig. B-1. Three-body example with mass-center connection points

The 3 by 1 matrices A^1 and A^2 are, from Def. 41,

$$A^1 = -\Phi^{11}\dot{\gamma}_1 C^{10}\tilde{\omega}^0 C^{01}u^1 - (C^{10}\tilde{\omega}^0 C^{01} + \dot{\gamma}_1\tilde{u}^1)\,\Phi^{11}(C^{10}\omega^0 + \dot{\gamma}_1 u^1) \tag{B-3}$$

and

$$A^2 = -\Phi^{22}\dot{\gamma}_2 C^{20}\tilde{\omega}^0 C^{02}u^1 - (C^{20}\tilde{\omega}^0 C^{02} + \dot{\gamma}_2\tilde{u}^1)\,\Phi^{22}(C^{20}\omega^0 + \dot{\gamma}_2 u^1) \tag{B-4}$$

where advantage has been taken of the identities $\tilde{u}^1 u^1 = 0$ and $D^{21} = D^{12} = 0$.

Further simplification of A^1 and A^2 is afforded by the substitution from Eqs. (4) and (5) of

$$C^{01} = U\cos\gamma_1 + \tilde{u}^1 \sin\gamma_1 + u^1 u^{1T}(1 - \cos\gamma_1) \tag{B-5}$$

$$C^{02} = U\cos\gamma_2 + \tilde{u}^1 \sin\gamma_2 + u^1 u^{1T}(1 - \cos\gamma_2) \tag{B-6}$$

$$C^{10} = C^{01T} \qquad C^{20} = C^{02T} \tag{B-7}$$

which provide

$$C^{01}u^1 = u^1\cos\gamma_1 + 0 + u^1(1 - \cos\gamma_1) = u^1 \tag{B-8}$$

$$C^{02}u^1 = u^1\cos\gamma_2 + 0 + u^1(1 - \cos\gamma_2) = u^1 \tag{B-9}$$

With Eqs. (B-8) and (B-9), A^1 and A^2 become

$$A^1 = -\dot{\gamma}_1\Phi^{11}C^{10}\tilde{\omega}^0 u^1 - (C^{10}\tilde{\omega}^0 C^{01} + \dot{\gamma}_1\tilde{u}^1)\,\Phi^{11}(C^{10}\omega^0 + \dot{\gamma}_1 u^1) \tag{B-10}$$

$$A^2 = -\dot{\gamma}_2\Phi^{22}C^{20}\tilde{\omega}^0 u^1 - (C^{20}\tilde{\omega}^0 C^{02} + \dot{\gamma}_2\tilde{u}^1)\,\Phi^{22}(C^{20}\omega^0 + \dot{\gamma}_2 u^1) \tag{B-11}$$

Before expanding Eq. (B-1) in scalar terms, we must examine the definitions of the elements of the coefficient matrix on the left side. From Defs. 38–40, we have

$$a_{00} = \Phi^{00} + C^{01}\Phi^{11}C^{10} + C^{02}\Phi^{22}C^{20} \tag{B-12}$$

$$a_{01} = C^{01}\Phi^{11}u^1 \qquad a_{10} = u^{1T}\Phi^{11}C^{10} \tag{B-13}$$

$$a_{02} = C^{02}\Phi^{22}u^1 \qquad a_{20} = u^{1T}\Phi^{22}C^{20} \tag{B-14}$$

$$a_{11} = u^{1T}\Phi^{11}u^1; \qquad a_{12} = 0; \qquad a_{21} = 0; \qquad a_{22} = u^{1T}\Phi^{22}u^1 \tag{B-15}$$

If $\mathscr{b}_1$ is a dumbbell of length $2L$, with tip particle masses equal to m and a massless connecting rod, then

$$\Phi^{11} = \begin{bmatrix} 2mL^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2mL^2 \end{bmatrix} \tag{B-16}$$

since (from Def. 36) $\Phi^{11} = I^{11}$ plus terms which in this case are zero.

Similarly, for the cylinder $\mathscr{b}_2$, we can write

$$\Phi^{22} = \begin{bmatrix} J & 0 & 0 \\ 0 & J' & 0 \\ 0 & 0 & J' \end{bmatrix} \tag{B-17}$$

where J and J' are moments of inertia of the cylinder for the symmetry axis and a transverse axis, respectively.

Use of Eqs. (B-6), (B-7), and (B-17) provides, upon expansion,

$$C^{02}\Phi^{22}C^{20} = \Phi^{22} \tag{B-18}$$

so that Eq. (B-12) becomes

$$a_{00} = \Phi^{00} + \Phi^{22} + C^{01}\Phi^{11}C^{10} \tag{B-19}$$

Because c is the mass center of $\mathscr{b}_0 + \mathscr{b}_2$, c is also the barycenter b_0. Since Φ^{00} is the inertia matrix of the augmented body $\mathscr{b}'_0$, $\Phi^{00} + \Phi^{22}$ is the inertia matrix of the system $\mathscr{b}_0 + \mathscr{b}_2$ about the system mass center at c. Since by hypothesis, $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ are principal axes of this system, we can write

$$\Phi^{00} + \Phi^{22} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{B-20}$$

where I_1, I_2, I_3 are principal axis moments of inertia.

Combination of Eqs. (B-16), (B-5), and (B-7) produces, upon expansion,

$$C^{01}\Phi^{11}C^{10} = 2mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\gamma_1 & -\sin\gamma_1\cos\gamma_1 \\ 0 & -\sin\gamma_1\cos\gamma_1 & \cos^2\gamma_1 \end{bmatrix} \tag{B-21}$$

With this information, we can expand Eqs. (B-12)–(B-15) to obtain

$$a_{00} = \begin{bmatrix} I_1 + 2mL^2 & 0 & 0 \\ 0 & I_2 + 2mL^2\sin^2\gamma_1 & -2mL^2\sin\gamma_1\cos\gamma_1 \\ 0 & -2mL^2\sin\gamma_1\cos\gamma_1 & I_3 + 2mL^2\cos^2\gamma_1 \end{bmatrix} \tag{B-22}$$

$$a_{01} = \begin{bmatrix} 2mL^2 \\ 0 \\ 0 \end{bmatrix} = 2mL^2u^1 \tag{B-23}$$

$$a_{10} = [2mL^2 \quad 0 \quad 0] = 2mL^2u^{1T} \tag{B-24}$$

$$a_{02} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix} = Ju^1 \tag{B-25}$$

$$a_{20} = [J \quad 0 \quad 0] = J u^{1T} \tag{B-26}$$

$$a_{11} = 2mL^2; \qquad a_{12} = a_{21} = 0; \qquad a_{22} = J \tag{B-27}$$

Having expressed the left side of Eq. (B-1) wholly in scalar terms, we can return to Eqs. (B-2) and (B-10)–(B-11) and focus again on the right side. Now we can utilize Eqs. (B-16)–(B-21) to simplify the upper partition to

$$\begin{aligned}
A^0 + C^{01}A^1 + C^{02}A^2 = & -\tilde{\omega}^0\Phi^{00}\omega^0 - \dot{\gamma}_1 (C^{01}\Phi^{11}C^{10})\tilde{\omega}^0 u^1 - \tilde{\omega}^0 (C^{01}\Phi^{11}C^{10})\,\omega^0 \\
& - \tilde{\omega}^0 C^{01}\Phi^{11}\dot{\gamma}_1 u^1 - C^{01}\dot{\gamma}_1\tilde{u}^1\Phi^{11}C^{10}\omega^0 - \dot{\gamma}_1^2 C^{01}\tilde{u}^1\Phi^{11}u^1 \\
& - \dot{\gamma}_2 (C^{02}\Phi^{22}C^{20})\tilde{\omega}^0 u^1 - \tilde{\omega}^0 (C^{02}\Phi^{22}C^{20})\,\omega^0 - \tilde{\omega}^0 C^{02}\Phi^{22}\dot{\gamma}_2 u^1 \\
& - \dot{\gamma}_2 C^{02}\tilde{u}^1\Phi^{22}C^{20}\omega^0 - \dot{\gamma}_2^2 C^{20}\tilde{u}^1\Phi^{22}u^1 \\
= & -\tilde{\omega}^0 (\Phi^{00} + \Phi^{22} + C^{01}\Phi^{11}C^{10})\,\omega^0 \\
& - \dot{\gamma}_1 (C^{01}\Phi^{11}C^{10}\tilde{\omega}^0 u^1 + \tilde{\omega}^0 C^{01}\Phi^{11}u^1 + C^{01}\tilde{u}^1\Phi^{11}C^{10}\omega^0 \\
& + \dot{\gamma}_1 C^{01}\tilde{u}^1\Phi^{11}u^1) - \dot{\gamma}_2 (\Phi^{22}\tilde{\omega}^0 u^1 + \tilde{\omega}^0 C^{02}\Phi^{22}u^1 \\
& + C^{02}\tilde{u}^1\Phi^{22}C^{20}\omega^0 + \dot{\gamma}_2 C^{20}\tilde{u}^1\Phi^{22}u^1) \\
= & -\tilde{\omega}^0 a_{00}\omega^0 + \dot{\gamma}_1 (C^{01}\Phi^{11}C^{10}\tilde{u}^1 + 2mL^2\tilde{u}^1 - C^{01}\tilde{u}^1\Phi^{11}C^{10})\,\omega^0 \\
& + \dot{\gamma}_2 (\Phi^{22}\tilde{u}^1 + J\tilde{u}^1 - C^{02}\tilde{u}^1\Phi^{22}C^{20})\,\omega^0
\end{aligned}$$

But

$$\begin{aligned}
C^{02}\tilde{u}^1\Phi^{22}C^{20} &= J' C^{02}\tilde{u}^1 C^{20} \\
&= J' (\tilde{u}^1 \cos\gamma_2 + \tilde{u}^1\tilde{u}^1 \sin\gamma_2)(U \cos\gamma_2 - \tilde{u}^1 \sin\gamma_2) \\
&= J' (\tilde{u}^1 \cos^2\gamma_2 - \tilde{u}^1\tilde{u}^1\tilde{u}^1 \sin^2\gamma_2) = J'\tilde{u}^1 (\cos^2\gamma_2 + \sin^2\gamma_2) \\
&= J'\tilde{u}^1
\end{aligned}$$

so that

$$\Phi^{22}\tilde{u}^1 - C^{02}\tilde{u}^1\Phi^{22}C^{20} = J'\tilde{u}^1 - J'\tilde{u}^1 = 0$$

Moreover, the identity

$$-(C^{01}\tilde{u}^1\Phi^{11}C^{10})^T = C^{01}\Phi^{11}\tilde{u}^1 C^{10} = C^{01}\Phi^{11}C^{10}\tilde{u}^1$$

permits further simplification. The final matrix expression is

$$A^0 + C^{01}A^1 + C^{02}A^2 = -\tilde{\omega}^0 a_{00}\omega^0 + (2C^{01}\Phi^{11}C^{10} + 2mL^2 U)\,\dot{\gamma}_1\tilde{u}^1\omega^0 + J\dot{\gamma}_2\tilde{u}^1\omega^0$$

Substitution of the scalar expansions of the coefficient matrices (see Eqs. B-21 and B-22) yields

$$A^0 + C^{01}A^1 + C^{02}A^2 = \begin{bmatrix} 0 & \omega_3 I_2 + 2mL^2 s(\omega_3 s + \omega_2 c) & -\omega_2 I_3 - 2mL^2(\omega_2 c^2 + \omega_3 sc) \\ -\omega_3(I_1 + 2mL^2) & -(\omega_1 + 2\dot{\gamma}_1)\, 2mL^2 sc & \omega_1 I_3 - J\dot{\gamma}_2 + 2mL^2(\omega_1 c^2 - 2\dot{\gamma}_1 s^2) \\ \omega_2(I_1 + 2mL^2) & -\omega_1 I_2 + J\dot{\gamma}_2 + 2mL^2(2\dot{\gamma}_1 c^2 - \omega_1 s^2) & 2mL^2 sc(\omega_1 + 2\dot{\gamma}_1) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= \begin{bmatrix} (I_2 - I_3)\,\omega_2\omega_3 + 2mL^2[(\omega_2^2 - \omega_3^2)\, sc + \omega_2\omega_3(s^2 - c^2)] \\ (I_3 - I_1)\,\omega_3\omega_1 - J\dot{\gamma}_2\omega_3 - 2mL^2(\omega_1 + 2\dot{\gamma}_1)\, s(\omega_3 s + \omega_2 c) \\ (I_1 - I_2)\,\omega_1\omega_2 + J\dot{\gamma}_2\omega_2 + 2mL^2(\omega_1 + 2\dot{\gamma}_1)\, c(\omega_3 s + \omega_2 c) \end{bmatrix} \qquad \text{(B-28)}$$

where s denotes $\sin\gamma_1$ and c denotes $\cos\gamma_1$.

Returning to the original equation of motion, Eq. (B-1), we see that all that remains of the task of obtaining scalar equations is the determination, from Eqs. (B-10) and (B-11), of the scalars $u^{1T}A^1$ and $u^{1T}A^2$. This operation is simplified by the relationships

$$u^{1T}\tilde{u}^1 = 0; \qquad u^{1T}\Phi^{11} = 2mL^2 u^{1T}; \qquad u^{1T}\Phi^{22} = Ju^{1T}$$

$$u^{1T}C^{10} = u^{1T}\cos\gamma_1 + u^{1T}(1 - \cos\gamma_1) = u^{1T}; \qquad u^{1T}C^{20} = u^{1T}$$

$$u^{1T}\tilde{\omega}^0 C^{01}u^1 = u^{1T}\tilde{\omega}^0 u^1 = 0; \qquad \Phi^{11}u^1 = 2mL^2 u^1$$

$$u^{1T}\tilde{\omega}^0 C^{02}u^1 = u^{1T}\tilde{\omega}^0 u^1 = 0; \qquad \Phi^{22}u^1 = Ju^1$$

With these substitutions, Eq. (B-10) provides

$$u^{1T}A^1 = -\dot{\gamma}_1(u^{1T}\Phi^{11}C^{10}\tilde{\omega}^0 u^1) - u^{1T}(C^{10}\tilde{\omega}^0 C^{01} + \dot{\gamma}_1\tilde{u}^1)\,\Phi^{11}(C^{10}\omega^0 + \dot{\gamma}_1 u^1)$$

$$= -u^{1T}\tilde{\omega}^0(C^{01}\Phi^{11}C^{10})\,\omega^0 \qquad \text{(B-29)}$$

and Eq. (B-11) provides (noting Eq. B-18)

$$u^{1T}A^2 = -\dot{\gamma}_2(u^{1T}\Phi^{22}C^{20}\tilde{\omega}^0 u^1) - u^{1T}(C^{20}\tilde{\omega}^0 C^{02} + \dot{\gamma}_2\tilde{u}^1)\,\Phi^{22}(C^{20}\omega^0 + \dot{\gamma}_2 u^1)$$

$$= -u^{1T}\tilde{\omega}^0\Phi^{22}\omega^0 \qquad \text{(B-30)}$$

Substitution into Eqs. (B-29) and (B-30) of the scalar expansions given by Eqs. (B-17) and (B-21) permits the representation

$$u^{1T}A^1 = [0 \quad \omega_3 \quad -\omega_2]\,(C^{01}\Phi^{11}C^{10})\,\omega^0$$

$$= 2mL^2\,[0 \quad s(\omega_3 s + \omega_2 c) \quad -c(\omega_3 s + \omega_2 c)]\,\omega^0$$

$$= 2mL^2\,[(\omega_2^2 - \omega_3^2)\, sc + \omega_2\omega_3(s^2 - c^2)] \qquad \text{(B-31)}$$

and

$$u^{1T}A^2 = [0 \quad \omega_3 \quad -\omega_2]\,\Phi^{22}\omega^0 = J'\,[0 \quad \omega_3 \quad -\omega_2]\,\omega^0 = 0 \qquad \text{(B-32)}$$

Finally, we can combine Eqs. (B-22)–(B-28), (B-31), and (B-32) in Eq. (B-1), to obtain the required set of five scalar equations of motion in the form

$$(I_1 + 2mL^2)\,\dot{\omega}_1 + 2mL^2\ddot{\gamma}_1 + J\ddot{\gamma}_2 = (I_2 - I_3)\,\omega_2\omega_3 + 2mL^2\,[\omega_2\omega_3\,(\sin^2\gamma_1 - \cos^2\gamma_1) + (\omega_2^2 - \omega_3^2)\sin\gamma_1\cos\gamma_1] \tag{B-33}$$

$$(I_2 + 2mL^2\sin^2\gamma_1)\,\dot{\omega}_2 - 2mL^2\,\dot{\omega}_3\sin\gamma_1\cos\gamma_1 = (I_3 - I_1)\,\omega_3\omega_1 - J\dot{\gamma}_2\omega_3 - 2mL^2\,(\omega_1 + 2\dot{\gamma}_1)\sin\gamma_1\,(\omega_3\sin\gamma_1 + \omega_2\cos\gamma_1) \tag{B-34}$$

$$(I_3 + 2mL^2\cos^2\gamma_1)\,\dot{\omega}_3 + 2mL^2\dot{\omega}_2\sin\gamma_1\cos\gamma_1 = (I_1 - I_2)\,\omega_1\omega_2 + J\dot{\gamma}_2\omega_2 + 2mL^2\,(\omega_1 + 2\dot{\gamma}_1)\cos\gamma_1\,(\omega_3\sin\gamma_1 + \omega_2\cos\gamma_1) \tag{B-35}$$

$$2mL^2\,(\dot{\omega}_1 + \ddot{\gamma}_1) = 2mL^2\,[(\omega_2^2 - \omega_3^2)\sin\gamma_1\cos\gamma_1 + \omega_2\omega_3\,(\sin^2\gamma_1 - \cos^2\gamma_1)] + \tau_1 \tag{B-36}$$

$$J\,(\dot{\omega}_1 + \ddot{\gamma}_2) = \tau_2 \tag{B-37}$$

Needless to say, it is *not* recommended that Eq. (1) be used (as in this appendix) for the purpose of deriving equations of motion for simple three-body systems such as that illustrated in Fig. B-1; the final scalar equations of motion obtained here can be derived from first principles by an experienced analyst in much less than half the time it will take him to execute the steps outlined in this appendix, obtaining the special set of equations from the generic matrix equations. Even for a much more complicated system, scalar equations of motion can probably be obtained in literal scalar form more efficiently by starting from first principles, although as the complexity of the system grows, the question becomes moot.

The purpose of the generic matrix equations of motion in this report is not to give the analyst a better starting point for the derivation of equations of motion but to give the digital computer a set of equations in a format it can readily process. At the present, the computer is programmed to obtain the scalar equations from the given matrix equations numerically, performing the necessary multiplications and additions after each integration interval. In the future, as digital computer capacity for the manipulation of literal algebraic symbols is improved, it may become desirable for these manipulations to be performed symbolically only once, in advance of all numerical integrations. At no time is it intended that the analyst perform the manipulations, unless (as in this appendix) it is appropriate for demonstration and checking. Equations (B-33)–(B-37) were derived also from direct time differentiations of angular momentum vectors, in order to establish some measure of confidence in the general equations, and to develop some subjective appreciation of the analytical labor involved in each of the alternative derivations.

Appendix C

Linearized Scalar Equations for a Three-Body Example

In Appendix B, we considered a relatively simple example of a three-body system, and obtained from Eq. (1) a complete set of five scalar equations of motion for the system. Now we consider another three-body system with five degrees of freedom, and in this case (see Fig. C-1), we no longer make the support points of $\mathscr{b}_1$ and $\mathscr{b}_2$ on $\mathscr{b}_0$ coincident with the corresponding mass centers, as they were for the preceding appendix (see Fig. B-1).

The objective of this appendix is to display the structure of Eq. (1) in this particular application, noting the complexity introduced by the off-center mounting of $\mathscr{b}_1$ and $\mathscr{b}_2$ on $\mathscr{b}_0$, and then to extract from these general equations of motion a new set linearized in the variables γ_1 and γ_2 and their derivatives. These results are then to be compared to those obtained from the generic linearized equations developed as Eq. (30) in this report.

In this example (see Fig. C-1), $\mathscr{b}_0$ is a cylinder of radius r, and $\mathscr{b}_1$ and $\mathscr{b}_2$ are identical uniform, thin rods of mass m and length $2L$, attached to $\mathscr{b}_0$ by hinges through points p_1 and p_2. Unit vectors $\mathbf{b}_1^0, \mathbf{b}_2^0, \mathbf{b}_3^0$ are fixed in $\mathscr{b}_0$, with $\mathbf{b}_3^0$ along the symmetry axis. Position vectors from the mass center c_0 of $\mathscr{b}_0$ to $\mathscr{p}_1$ and $\mathscr{p}_2$ are given, respectively, by

$$\mathbf{p}^{01} = r\mathbf{b}_2^0 \qquad \mathbf{p}^{02} = -r\mathbf{b}_2^0 \tag{C-1}$$

Thus, the barycenter b_0 coincides with c_0, and we have

$$\mathbf{D}^{00} = 0; \qquad \mathbf{D}^{01} = \mathbf{L}^{01} = \mathbf{p}^{01}; \qquad \mathbf{D}^{02} = \mathbf{L}^{02} = \mathbf{p}^{02} \tag{C-2}$$

In $\mathscr{b}_1$ and $\mathscr{b}_2$, the barycenters are displaced from the mass centers, such that

$$\mathbf{D}^{11} = (L - R)\,\mathbf{b}_2^1; \qquad \mathbf{D}^{10} = -R\mathbf{b}_2^1; \qquad \mathbf{D}^{22} = -(L - R)\,\mathbf{b}_2^2; \qquad \mathbf{D}^{20} = R\mathbf{b}_2^2 \tag{C-3}$$

where $R = mL/\mathcal{M}$.

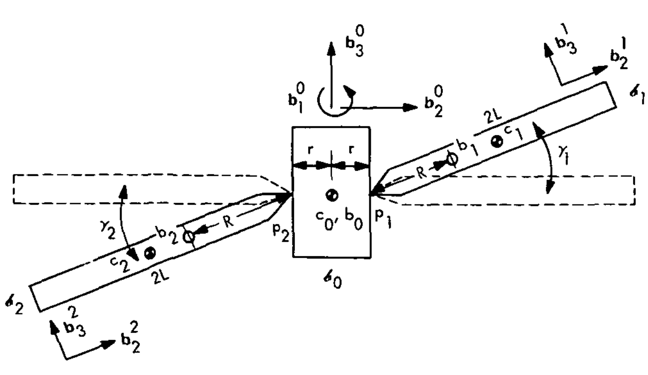

Fig. C-1. Three-body example

The hinge axes are defined by

$$\mathbf{g}^1 = \mathbf{g}^2 = \mathbf{b}_1^0 \tag{C-4}$$

and the corresponding elastic hinge torques by

$$\tau_1 \mathbf{b}_1^1 = -k\gamma_1 \mathbf{b}_1^1; \qquad \tau_2 \mathbf{b}_1^2 = -k\gamma_2 \mathbf{b}_1^2 \tag{C-5}$$

The vectors in Eqs. (C-1)–(C-5) appear in the equations of motion in this report in the form of matrices or scalars. In this example, it is convenient to express many of these quantities in terms of the symbols

$$u^1 \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad u^2 \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad u^3 \triangleq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{C-6}$$

Then, some of the quantities of interest become

$$D^{00} = 0 \qquad D^{01} = ru^2 \qquad D^{02} = -ru^2$$

$$D^{11} = u^2 (\mathcal{M} - m) \frac{R}{m}$$

$$D^{10} = -Ru^2 = D^{12}$$

$$D^{22} = -u^2 (\mathcal{M} - m) \frac{R}{m}$$

$$D^{20} = Ru^2 = D^{21}$$

$$g^1 = g^2 = u^1$$

$$\tau_1 = -k\gamma_1 \qquad \tau_2 = -k\gamma_2 \tag{C-7}$$

The equations of motion are given by Eq. (1) as

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}^0 \\ \ddot{\gamma}_1 \\ \ddot{\gamma}_2 \end{bmatrix} = \begin{bmatrix} A^0 + C^{01}A^1 + C^{02}A^2 \\ u^{1T}A^1 - k\gamma_1 \\ u^{1T}A^2 - k\gamma_2 \end{bmatrix} \tag{C-8}$$

corresponding exactly to Eq. (B-1) for the simpler system in Appendix B.

From Def. 41 and Eq. (B-7),

$$\begin{aligned} A^0 = & -\tilde{\omega}^0 \Phi^{00} \omega^0 + \mathcal{M} [r\tilde{u}^2 C^{01} (C^{10} \tilde{\omega}^0 C^{01} + \dot{\gamma}_1 \tilde{u}^1)^2 (-Ru^2) \\ & - r\tilde{u}^2 C^{02} (C^{20} \tilde{\omega}^0 C^{02} + \dot{\gamma}_2 \tilde{u}^1)^2 (Ru^2) \\ & + (-Uu^{2T} C^{10} u^2 + C^{01} u^2 u^{2T}) rR\dot{\gamma}_1 (\tilde{\omega}^0 C^{01} + \dot{\gamma}_1 C^{01} \tilde{u}^1) u^1 \\ & + (-Uu^{2T} C^{20} u^2 + C^{02} u^2 u^{2T}) rR\dot{\gamma}_2 (\tilde{\omega}^0 C^{02} + \dot{\gamma}_2 C^{02} \tilde{u}^1) u^1 \end{aligned} \tag{C-9}$$

This equation simplifies immediately upon noting that

$$\tilde{u}^1 u^1 = 0; \qquad \tilde{u}^1 u^2 = u^3; \qquad u^2 u^{2T} = [0 \,\vdots\, u^2 \,\vdots\, 0] \stackrel{\Delta}{=} U^2$$

Further simplification is afforded by the substitution from Eqs. (4) and (5) of

$$C^{01} = U \cos\gamma_1 + \tilde{u}^1 \sin\gamma_1 + u^1 u^{1T}(1 - \cos\gamma_1) \tag{C-10}$$

$$C^{02} = U \cos\gamma_2 + \tilde{u}^1 \sin\gamma_2 + u^1 u^{1T}(1 - \cos\gamma_2) \tag{C-11}$$

$$C^{10} = C^{01T}; \qquad C^{20} = C^{02T} \tag{C-12}$$

where $u^1 u^{1T} = [u^1 \,\vdots\, 0 \,\vdots\, 0] \stackrel{\Delta}{=} U^1$. With the substitutions

$$u^{2T} C^{20} u^2 = u^{2T} u^2 \cos\gamma_2 - u^{2T} \tilde{u}^1 u^2 \sin\gamma_2 + u^{2T} u^1 u^{1T} u^2 (1 - \cos\gamma_2) = \cos\gamma_2 \tag{C-13}$$

$$u^{2T} C^{10} u^2 = u^{2T} u^2 \cos\gamma_1 - u^{2T} \tilde{u}^1 u^2 \sin\gamma_1 + u^{2T} u^1 u^{1T} u^2 (1 - \cos\gamma_1) = \cos\gamma_1 \tag{C-14}$$

$$C^{01} u^2 u^{2T} = C^{01} U^2 = \cos\gamma_1 \, U^2 + \sin\gamma_1 \, \tilde{u}^1 U^2 \tag{C-15}$$

$$C^{02} u^2 u^{2T} = C^{02} U^2 = \cos\gamma_2 U^2 + \sin\gamma_2 \, \tilde{u}^1 U^2 \tag{C-16}$$

$$C^{01} u^1 = u^1 \cos\gamma_1 + u^1 (1 - \cos\gamma_1) = u^1 \tag{C-17}$$

$$C^{02} u^1 = u^1 \cos\gamma_2 + u^1 (1 - \cos\gamma_2) = u^1 \tag{C-18}$$

$$C^{02} u^2 = u^2 \cos\gamma_2 + u^3 \sin\gamma_2 \tag{C-19}$$

$$C^{01} u^2 = u^2 \cos\gamma_1 + u^3 \sin\gamma_1 \tag{C-20}$$

$$\tilde{u}^2 C^{01} = \tilde{u}^2 \cos\gamma_1 + \tilde{u}^2 \tilde{u}^1 \sin\gamma_1 \tag{C-21}$$

$$\tilde{u}^2 C^{02} = \tilde{u}^2 \cos\gamma_2 + \tilde{u}^2 \tilde{u}^1 \sin\gamma_2 \tag{C-22}$$

and such operations as

$$\begin{aligned} \tilde{u}^2 C^{01} \tilde{u}^1 \tilde{u}^1 u^2 &= (\tilde{u}^2 \cos\gamma_1 + \tilde{u}^2 \tilde{u}^1 \sin\gamma_1) \tilde{u}^1 u^3 \\ &= -\tilde{u}^2 \tilde{u}^1 u^2 \sin\gamma_1 = -u^1 \sin\gamma_1 \end{aligned} \tag{C-23}$$

we can rewrite A^0 in Eq. (C-9) as

$$\begin{aligned} A^0 = {} & -\tilde{\omega}^0 \Phi^{00} \omega^0 + \mathcal{M} r R \, [u^1 \sin\gamma_1 \dot{\gamma}_1^2 - \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 (u^2 \cos\gamma_1 + u^3 \sin\gamma_1) \\ & - \tilde{u}^2 \tilde{\omega}^0 C^{01} u^3 \dot{\gamma}_1 - \tilde{u}^2 C^{01} \tilde{u}^1 C^{10} \tilde{\omega}^0 (u^2 \cos\gamma_1 + u^3 \sin\gamma_1) \dot{\gamma}_1 \\ & - \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 (u^2 \cos\gamma_2 + u^3 \sin\gamma_2) - (\tilde{u}^2 \cos\gamma_2 + \tilde{u}^2 \tilde{u}^1 \sin\gamma_2) \tilde{u}^1 u^3 \dot{\gamma}_2^2 \\ & - \tilde{u}^2 \tilde{\omega}^0 C^{02} u^3 \dot{\gamma}_2 - \tilde{u}^2 C^{02} \tilde{u}^1 C^{20} \tilde{\omega}^0 (u^2 \cos\gamma_2 + u^3 \sin\gamma_2) \dot{\gamma}_2 \\ & + (-U \cos\gamma_1 + U^2 \cos\gamma_1 + \tilde{u}^1 U^2 \sin\gamma_1) \dot{\gamma}_1 \tilde{\omega}^0 u^1 \\ & + (-U \cos\gamma_2 + U^2 \cos\gamma_2 + \tilde{u}^1 U^2 \sin\gamma_2) \dot{\gamma}_2 \tilde{\omega}^0 u^1] \end{aligned}$$

With the further substitutions

$$C^{02}u^3 = u^3 \cos\gamma_2 - u^2 \sin\gamma_2 \tag{C-24}$$

$$C^{01}u^3 = u^3 \cos\gamma_1 - u^2 \sin\gamma_1 \tag{C-25}$$

$$\begin{aligned}\tilde{u}^2 C^{01} \tilde{u}^1 C^{10} &= (\tilde{u}^2 \cos\gamma_1 + \tilde{u}^2\tilde{u}^1 \sin\gamma_1)(\tilde{u}^1 \cos\gamma_1 - \tilde{u}^1\tilde{u}^1 \sin\gamma_1) \\ &= \tilde{u}^2\tilde{u}^1 \cos^2\gamma_1 - \tilde{u}^2\tilde{u}^1\tilde{u}^1\tilde{u}^1 \sin^2\gamma_1\end{aligned} \tag{C-26}$$

$$\begin{aligned}\tilde{u}^2 C^{02} \tilde{u}^1 C^{20} &= (\tilde{u}^2 \cos\gamma_2 + \tilde{u}^2\tilde{u}^1 \sin\gamma_2)(\tilde{u}^1 \cos\gamma_2 - \tilde{u}^1\tilde{u}^1 \sin\gamma_2) \\ &= \tilde{u}^2\tilde{u}^1 \cos^2\gamma_2 - \tilde{u}^2\tilde{u}^1\tilde{u}^1\tilde{u}^1 \sin^2\gamma_2\end{aligned} \tag{C-27}$$

A^0 becomes

$$\begin{aligned}A^0 = & -\tilde{\omega}^0\Phi^{00}\omega^0 + \mathcal{M}rR\,\{u^1(\dot{\gamma}_1^2 \sin\gamma_1 + \dot{\gamma}_2^2 \sin\gamma_2) \\ & - \tilde{u}^2\tilde{\omega}^0\tilde{\omega}^0\,[u^2(\cos\gamma_1 + \cos\gamma_2) + u^3(\sin\gamma_1 + \sin\gamma_2)] \\ & - \tilde{u}^2\tilde{\omega}^0\,[u^3(\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2) - u^2(\dot{\gamma}_1 \sin\gamma_1 + \dot{\gamma}_2 \sin\gamma_2)] \\ & - (\tilde{u}^2\tilde{u}^1 \cos^2\gamma_1 - \tilde{u}^2\tilde{u}^1\tilde{u}^1\tilde{u}^1 \sin^2\gamma_1)\tilde{\omega}^0(u^2\dot{\gamma}_1 \cos\gamma_1 + u^3\dot{\gamma}_1 \sin\gamma_1) \\ & - (\tilde{u}^2\tilde{u}^1 \cos^2\gamma_2 - \tilde{u}^2\tilde{u}^1\tilde{u}^1\tilde{u}^1 \sin^2\gamma_2)\tilde{\omega}^0(u^2\dot{\gamma}_2 \cos\gamma_2 + u^3\dot{\gamma}_2 \sin\dot{\gamma}_2) \\ & - (U - U^2)\tilde{\omega}^0u^1(\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2) + \tilde{u}^1U^2\tilde{\omega}^0u^1(\dot{\gamma}_1 \sin\gamma_1 + \dot{\gamma}_2 \sin\gamma_2)\}\end{aligned} \tag{C-28}$$

Expansion and multiplication establish the identities

$$\tilde{u}^2\tilde{u}^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C-29}$$

$$\tilde{u}^2\tilde{u}^1\tilde{u}^1\tilde{u}^1 = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\tilde{u}^2\tilde{u}^1 \tag{C-30}$$

$$U - U^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U^1 + U^3 \tag{C-31}$$

$$\tilde{u}^1U^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{C-32}$$

Although these substitutions would all simplify the computation of A^0, the second and third alone provide

$$
\begin{aligned}
A^0 = &-\tilde{\omega}^0\Phi^{00}\omega^0 + \mathcal{M}rR\,\{u^1(\dot{\gamma}_1^2\sin\gamma_1 + \dot{\gamma}_2^2\sin\gamma_2) \\
&- \tilde{u}^2\tilde{\omega}^0\tilde{\omega}^0\,[u^2(\cos\gamma_1 + \cos\gamma_2) + u^3(\sin\gamma_1 + \sin\gamma_2)] \\
&- \tilde{u}^2\tilde{\omega}^0\,[u^3(\dot{\gamma}_1\cos\gamma_1 + \dot{\gamma}_2\cos\gamma_2) - u^2(\dot{\gamma}_1\sin\gamma_1 + \dot{\gamma}_2\sin\gamma_2)] \\
&- \tilde{u}^2\tilde{u}^1\tilde{\omega}^0\,[u^2(\dot{\gamma}_1\cos\gamma_1 + \dot{\gamma}_2\cos\gamma_2) + u^3(\dot{\gamma}_1\sin\gamma_1 + \dot{\gamma}_2\sin\gamma_2)] \\
&- (U^1 + U^3)\,\tilde{\omega}^0u^1(\dot{\gamma}_1\cos\gamma_1 + \dot{\gamma}_2\cos\gamma_2) + \tilde{u}^1U^2\tilde{\omega}^0u^1(\dot{\gamma}_1\sin\gamma_1 + \dot{\gamma}_2\sin\gamma_2)\}
\end{aligned}
\tag{C-33}
$$

Further reduction of A^0 requires expansion of ω^0 as $\omega^0 = [\omega_1 \quad \omega_2 \quad \omega_3]^T$. This explicit representation provides

$$
\tilde{u}^2\tilde{\omega}^0\tilde{\omega}^0 = \begin{bmatrix} \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \\ 0 & 0 & 0 \\ (\omega_2^2 + \omega_3^2) & -\omega_1\omega_2 & -\omega_1\omega_3 \end{bmatrix} \tag{C-34}
$$

$$
\tilde{u}^2\tilde{\omega}^0 = \begin{bmatrix} -\omega_2 & \omega_1 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_3 & -\omega_2 \end{bmatrix} \tag{C-35}
$$

$$
\tilde{u}^2\tilde{u}^1\tilde{\omega}^0 = \begin{bmatrix} \omega_3 & 0 & -\omega_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C-36}
$$

$$
(U^1 + U^3)\,\tilde{\omega}^0u^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\omega_2 \end{bmatrix} = -\omega_2u^3 \tag{C-37}
$$

$$
\tilde{u}^1U^2\tilde{\omega}^0u^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix} = \omega_3u^3 \tag{C-38}
$$

so that A^0 becomes

$$
\begin{aligned}
A^0 = &-\tilde{\omega}^0\Phi^{00}\omega^0 + \mathcal{M}rR\,\Bigg\{u^1(\dot{\gamma}_1^2\sin\gamma_1 + \dot{\gamma}_2^2\sin\gamma_2) \\
&- \begin{bmatrix} \omega_2\omega_3 \\ 0 \\ -\omega_1\omega_2 \end{bmatrix}(\cos\gamma_1 + \cos\gamma_2) + \begin{bmatrix} \omega_1^2 + \omega_2^2 \\ 0 \\ \omega_1\omega_3 \end{bmatrix}(\sin\gamma_1 + \sin\gamma_2) \\
&+ \begin{bmatrix} 0 \\ 0 \\ \omega_2 \end{bmatrix}(\dot{\gamma}_1\cos\gamma_1 + \dot{\gamma}_2\cos\gamma_2) - \begin{bmatrix} \omega_1 \\ 0 \\ \omega_3 \end{bmatrix}(\dot{\gamma}_1\sin\gamma_1 + \dot{\gamma}_2\sin\gamma_2)
\end{aligned}
$$

$$-\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2) + \begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix} (\dot{\gamma}_1 \sin\gamma_1 + \dot{\gamma}_2 \sin\gamma_2)$$

$$\left. + \begin{bmatrix} 0 \\ -\omega_3 \\ \omega_2 \end{bmatrix} (\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2) + \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix} (\dot{\gamma}_1 \sin\gamma_1 + \dot{\gamma}_2 \sin\gamma_2) \right\}$$

or

$$A^0 = -\tilde{\omega}^0 \Phi^{00} \omega^0 + \mathcal{M} r R \{ u^1 (\dot{\gamma}_1^2 \sin\gamma_1 + \dot{\gamma}_2^2 \sin\gamma_2)$$

$$+ (\omega_1 \omega_2 u^3 - \omega_2 \omega_3 u^1)(\cos\gamma_1 + \cos\gamma_2) + [(\omega_1^2 + \omega_2^2) u^1 + \omega_1 \omega_3 u^3](\sin\gamma_1 + \sin\gamma_2)$$

$$+ 2\omega_2 u^3 (\dot{\gamma}_1 \cos\gamma_1 + \dot{\gamma}_2 \cos\gamma_2)\} \tag{C-39}$$

From Def. 41, we also find

$$A^1 = -\Phi^{11} \dot{\gamma}_1 (C^{10} \tilde{\omega}^0 C^{01} + \dot{\gamma}_1 \tilde{u}^1) u^1 - [C^{10} \tilde{\omega}^0 C^{01} + \dot{\gamma}_1 \tilde{u}^1] \Phi^{11} [C^{10} \omega^0 + \dot{\gamma}_1 u^1]$$

$$+ \mathcal{M} [-R \tilde{u}^2 C^{12} (C^{20} \tilde{\omega}^0 C^{02} + \dot{\gamma}_2 \tilde{u}^1)^2 R u^2 - R \tilde{u}^2 C^{10} (\tilde{\omega}^0 \tilde{\omega}^0) u^2 r$$

$$+ (-U u^{2T} C^{21} u^2 R^2 + C^{12} u^2 u^{2T} R^2) \dot{\gamma}_2 (C^{10} \tilde{\omega}^0 C^{01} + \dot{\gamma}_2 \tilde{u}^1) u^1] \tag{C-40}$$

and

$$A^2 = -\Phi^{22} \dot{\gamma}_2 (C^{20} \tilde{\omega}^0 C^{02} + \dot{\gamma}_2 \tilde{u}^1) u^1 - (C^{20} \tilde{\omega}^0 C^{02} + \gamma_2 \tilde{u}^1) \Phi^{22} (C^{20} \omega^0 + \dot{\gamma}_2 u^1)$$

$$+ \mathcal{M} [-\tilde{u}^2 C^{21} (C^{10} \tilde{\omega}^0 C^{01} + \dot{\gamma}_1 \tilde{u}^1)^2 R^2 u^2 - R \tilde{u}^2 C^{20} (\tilde{\omega}^0 \tilde{\omega}^0) u^2 r$$

$$+ (-U u^{2T} C^{12} u^2 R^2 + C^{21} u^2 u^{2T} R^2) \dot{\gamma}_1 (C^{20} \tilde{\omega}^0 C^{02} + \dot{\gamma}_1 \tilde{u}^1) u^1] \tag{C-41}$$

These expressions are certainly no simpler than that found previously for A^0, and we must expect that their expansion and reduction to expressions of minimum complexity would again be a substantial chore (although one that a computer can execute numerically in very little time, since only multiplications and additions are involved).

It is becoming clear that the task of obtaining exact equations of motion for the case illustrated in Fig. C-1 is much more arduous than for the simpler system analyzed in Appendix B (Fig. B-1), although both systems involve three bodies and five degrees of freedom. Having made this point, we now attack our second objective, and seek an approximation of Eq. (C-8) in which γ_1, γ_2, $\dot{\gamma}_1$, $\dot{\gamma}_2$, $\ddot{\gamma}_1$, and $\ddot{\gamma}_2$ are small enough to warrant linearization in these variables.

The matrix A^0 in Eq. (C-39) then simplifies to

$$A^0 \cong -\tilde{\omega}^0 \Phi^{00} \omega^0 + \mathcal{M} r R \{2(\omega_1 \omega_2 u^3 - \omega_2 \omega_3 u^1)$$

$$+ (\gamma_1 + \gamma_2)[(\omega_1^2 + \omega_2^2) u^1 + \omega_1 \omega_3 u^3] + 2(\dot{\gamma}_1 + \dot{\gamma}_2)\omega_2 u^3\} \tag{C-43}$$

For A^1 and A^2, we can substitute the direction cosine approximations (see Eqs. C-10–C-12)

$$C^{01} \cong U + \gamma_1 \tilde{u}^1 \qquad\qquad C^{10} \cong U - \gamma_1 \tilde{u}^1$$

$$C^{02} \cong U + \gamma_2 \tilde{u}^1 \qquad\qquad C^{20} \cong U - \gamma_2 \tilde{u}^1$$

$$C^{12} = C^{10} C^{02} \cong U - \gamma_1 \tilde{u}^1 + \gamma_2 \tilde{u}^1$$

$$C^{21} = U + \gamma_1 \tilde{u}^1 - \gamma_2 \tilde{u}^1$$

into Eqs. (C-40) and (C-41). The desired approximations are

$$\begin{aligned}
A^1 \cong\ & -\Phi^{11} \dot{\gamma}_1 \tilde{\omega}^0 u^1 - \tilde{\omega}^0 \Phi^{11} \omega^0 + \gamma_1 \tilde{u}^1 \tilde{\omega}^0 \Phi^{11} \omega^0 \\
& - \gamma_1 \tilde{\omega}^0 \tilde{u}^1 \Phi^{11} \omega^0 - \dot{\gamma}_1 \tilde{u}^1 \Phi^{11} \omega^0 + \tilde{\omega}^0 \Phi^{11} \gamma_1 \tilde{u}^1 \omega^0 - \tilde{\omega}^0 \Phi^{11} \dot{\gamma}_1 u^1 \\
& + \mathcal{M} \left[-\tilde{u}^2 (U - \gamma_1 \tilde{u}^1 + \gamma_2 \tilde{u}^1) \tilde{\omega}^0 \tilde{\omega}^0 R^2 u^2 \right. \\
& - \tilde{u}^2 (\tilde{\omega}^0 \dot{\gamma}_2 \tilde{u}^1 + \dot{\gamma}_2 \tilde{u}^1 \tilde{\omega}^0 - \gamma_2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 + \tilde{\omega}^0 \tilde{\omega}^0 \gamma_2 \tilde{u}^1) R^2 u^2 \\
& \left. - R \tilde{u}^2 (U - \gamma_1 \tilde{u}^1) \tilde{\omega}^0 \tilde{\omega}^0 u^2 r - R^2 (U - u^2 u^{2T}) \dot{\gamma}_2 \tilde{\omega}^0 u^1 \right] \\
=\ & -\tilde{\omega}^0 \Phi^{11} \omega^0 - \mathcal{M} R^2 \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 u^2 - \mathcal{M} R r \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 u^2 \\
& - \dot{\gamma}_1 \left[\Phi^{11} \tilde{\omega}^0 u^1 + \tilde{u}^1 \Phi^{11} \omega^0 + \tilde{\omega}^0 \Phi^{11} u^1 \right] \\
& - \dot{\gamma}_2 \left[\mathcal{M} R^2 \tilde{u}^2 (\tilde{\omega}^0 \tilde{u}^1 + \tilde{u}^1 \tilde{\omega}^0) u^2 + \mathcal{M} R^2 (U - u^2 u^{2T}) \tilde{\omega}^0 u^1 \right] \\
& + \gamma_1 \left[\tilde{u}^1 \tilde{\omega}^0 \Phi^{11} \omega^0 - \tilde{\omega}^0 \tilde{u}^1 \Phi^{11} \omega^0 + \tilde{\omega}^0 \Phi^{11} \tilde{u}^1 \omega^0 \right. \\
& \left. + \mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 + \mathcal{M} R r \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 \right] \\
& + \gamma_2 \left[-\mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 + \mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 - \mathcal{M} R^2 \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 \tilde{u}^1 u^2 \right] \\
=\ & -\tilde{\omega}^0 \Phi^{11} \omega^0 + \mathcal{M} (R^2 + Rr) \tilde{u}^2 \tilde{\omega}^0 \tilde{u}^2 \omega^0 \\
& + \dot{\gamma}_1 \left[\Phi^{11} \tilde{u}^1 - \tilde{u}^1 \Phi^{11} + (\Phi^{11} u^1)^{\sim} \right] \omega^0 \\
& + \dot{\gamma}_2 \left[\mathcal{M} R^2 \tilde{u}^2 \tilde{u}^3 + \mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{u}^2 + \mathcal{M} R^2 (U - u^2 u^{2T}) \tilde{u}^1 \right] \omega^0 \\
& + \gamma_1 \left[\tilde{u}^1 \tilde{\omega}^0 \Phi^{11} - \tilde{\omega}^0 \tilde{u}^1 \Phi^{11} + \tilde{\omega}^0 \Phi^{11} \tilde{u}^1 - \mathcal{M} R (R + r) \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{u}^2 \right] \omega^0 \\
& + \gamma_2 \left[\mathcal{M} R^2 \tilde{u}^2 \tilde{\omega}^0 \tilde{u}^3 \omega^0 \right]
\end{aligned} \tag{C-44}$$

and

$$\begin{aligned}
A^2 =\ & -\Phi^{22} \dot{\gamma}_2 \tilde{\omega}^0 u^1 - \tilde{\omega}^0 \Phi^{22} \omega^0 + \gamma_2 \tilde{u}^1 \tilde{\omega}^0 \Phi^{22} \omega^0 \\
& - \gamma_2 \tilde{\omega}^0 \tilde{u}^1 \Phi^{22} \omega^0 - \dot{\gamma}_2 \tilde{u}^1 \Phi^{22} \omega^0 + \tilde{\omega}^0 \Phi^{22} \gamma_2 \tilde{u}^1 \omega^0 \\
& - \tilde{\omega}^0 \Phi^{22} \dot{\gamma}_2 u^1 + \mathcal{M} \left[-\tilde{u}^2 (U + \gamma_1 \tilde{u}^1 - \gamma_2 \tilde{u}^1) \tilde{\omega}^0 \tilde{\omega}^0 R^2 u^2 \right. \\
& - \tilde{u}^2 (-\gamma_1 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 + \gamma_1 \tilde{\omega}^0 \tilde{\omega}^0 \tilde{u}^1 + \dot{\gamma}_1 \tilde{u}^1 \tilde{\omega}^0 + \tilde{\omega}^0 \dot{\gamma}_1 \tilde{u}^1) R^2 u^2
\end{aligned}$$

$$
\begin{aligned}
&- Rr\tilde{u}^2 (U - \tilde{u}^1 \gamma_2) \tilde{\omega}^0 \tilde{\omega}^0 u^2 - R^2 (U - u^2 u^{2T}) \dot{\gamma}_1 \tilde{\omega}^0 u^1] \\
= &-\tilde{\omega}^0 \Phi^{22} \omega^0 - \mathcal{M} R^2 \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 u^2 - \mathcal{M} Rr \tilde{u}^2 \tilde{\omega}^0 \tilde{\omega}^0 u^2 \\
&- \dot{\gamma}_1 [\mathcal{M} R^2 \tilde{u}^2 (\tilde{\omega}^0 \tilde{u}^1 + \tilde{u}^1 \tilde{\omega}^0) u^2 + \mathcal{M} R^2 (U - u^2 u^{2T}) \tilde{\omega}^0 u^1] \\
&- \dot{\gamma}_2 [\Phi^{22} \tilde{\omega}^0 u^1 + \tilde{u}^1 \Phi^{22} \omega^0 + \tilde{\omega}^0 \Phi^{22} u^1] \\
&+ \gamma_1 [-\mathcal{M} R^2 \tilde{u}^2 (\tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 - \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 + \tilde{\omega}^0 \tilde{\omega}^0 \tilde{u}^1 u^2)] \\
&+ \gamma_2 [\tilde{u}^1 \tilde{\omega}^0 \Phi^{22} \omega^0 - \tilde{\omega}^0 \tilde{u}^1 \Phi^{22} \omega^0 + \tilde{\omega}^0 \Phi^{22} \tilde{u}^1 \omega^0 + \mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2 \\
&+ \mathcal{M} Rr \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{\omega}^0 u^2] \\
= &-\tilde{\omega}^0 \Phi^{22} \omega^0 + \mathcal{M} (R^2 + Rr) \tilde{u}^2 \tilde{\omega}^0 \tilde{u}^2 \omega^0 \\
&+ \dot{\gamma}_1 [\mathcal{M} R^2 \tilde{u}^2 \tilde{u}^3 + \mathcal{M} R^2 \tilde{u}^2 \tilde{u}^1 \tilde{u}^2 + \mathcal{M} R^2 (U - u^2 u^{2T}) \tilde{u}^1] \omega^0 \\
&+ \dot{\gamma}_2 [\Phi^{22} \tilde{u}^1 - \tilde{u}^1 \Phi^{22} - (\Phi^{22} u^1)^{\sim}] \omega^0 + \gamma_1 [\mathcal{M} R^2 \tilde{u}^2 \tilde{\omega}^0 \tilde{u}^3 \omega^0] \\
&+ \gamma_2 [\tilde{u}^1 \tilde{\omega}^0 \Phi^{22} - \tilde{\omega}^0 \tilde{u}^1 \Phi^{22} + \tilde{\omega}^0 \Phi^{22} \tilde{u}^1 \\
&- \mathcal{M} R (R + r) \tilde{u}^2 \tilde{u}^1 \tilde{\omega}^0 \tilde{u}^2] \omega^0
\end{aligned}
\tag{C-45}
$$

The augmented body inertia matrices are available from Def. 36 as

$$\Phi^{00} = I^0 + m (D^{01T} D^{01} U - D^{01} D^{01T}) + m (D^{02T} D^{02} U - D^{02} D^{02T})$$

$$\Phi^{11} = I^{11} + m (D^{11T} D^{11} U - D^{11} D^{11T}) + (\mathcal{M} - m) (D^{10T} D^{10} U - D^{10} D^{10T})$$

$$\Phi^{22} = I^{22} + m (D^{22T} D^{22} U - D^{22} D^{22T}) + (\mathcal{M} - m) (D^{20T} D^{20} U - D^{20} D^{20T})$$

With Eqs. (C-7), these matrices become

$$\Phi^{00} = I^0 + mr^2 (U - U^2) + mr^2 (U - U^2) = I^0 + 2mr^2 (U^1 + U^3) \tag{C-46}$$

$$
\begin{aligned}
\Phi^{11} &= I^{11} + (\mathcal{M} - m)^2 \left(\frac{R^2}{m}\right) (U - U^2) + (\mathcal{M} - m) R^2 (U - U^2) \\
&= I^{11} + (\mathcal{M} - m) \left(\frac{\mathcal{M} R^2}{m}\right) (U - U^2) = I^{11} + (\mathcal{M} - m) \left(\frac{mL^2}{\mathcal{M}}\right) (U^1 + U^3)
\end{aligned}
$$

and, since

$$I^{22} = I^{11} = \left(\frac{mL^2}{3}\right) (U^1 + U^3)$$

$$\Phi^{22} = \Phi^{11} = \left(\frac{mL^2}{3\mathcal{M}}\right) (4\mathcal{M} - 3m) (U^1 + U^3) \triangleq (J - \mathcal{M} R^2) (U^1 + U^3) \tag{C-47}$$

defining

$$J \triangleq \frac{4}{3} mL^2 \tag{C-48}$$

as the moment of inertia of a rod of mass m and length $2L$ about an axis normal to the rod and passing through one end of the rod.

The combination of Eqs. (C-44)–(C-48) is simplified by the following identities:

$$U^1\tilde{u}^1 = 0; \qquad -\tilde{u}^1U^1 = (U^1\tilde{u}^1)^T = 0$$

$$U^1u^1 = u^1; \qquad U^3u^1 = 0$$

$$\tilde{u}^2\tilde{u}^3 = U^3\tilde{u}^1; \qquad \tilde{u}^2\tilde{u}^1 = -U^1\tilde{u}^3$$

$$U - u^2u^{2T} = U - U^2 = U^1 + U^3$$

$$\tilde{u}^2\tilde{u}^1\tilde{u}^2 = 0$$

$$\tilde{u}^1(U - U^3) = \tilde{u}^1(U^1 + U^2) = \tilde{u}^1U^2 = U^3\tilde{u}^1$$

Now Eqs. (C-44) and (C-45) can be written as

$$\begin{aligned}
A^1 \cong & -(J - \mathcal{M}R^2)\tilde{\omega}^0(U^1 + U^3)\omega^0 + \mathcal{M}R(R + r)\tilde{u}^2\tilde{\omega}^0\tilde{u}^2\omega^0 \\
& + (J - \mathcal{M}R^2)\dot{\gamma}_1(U^3\tilde{u}^1 - \tilde{u}^1U^3 + \tilde{u}^1)\omega^0 + 2\mathcal{M}R^2\dot{\gamma}_2U^3\tilde{u}^1\omega^0 \\
& + \gamma_1\{(J - \mathcal{M}R^2)[\tilde{u}^1\tilde{\omega}^0(U^1 + U^3) - \tilde{\omega}^0\tilde{u}^1U^3 + \tilde{\omega}^0U^3\tilde{u}^1] \\
& - \mathcal{M}R(R + r)\tilde{u}^2\tilde{u}^1\tilde{\omega}^0\tilde{u}^2\}\omega^0 + \gamma_2\mathcal{M}R^2\tilde{u}^2\tilde{\omega}^0\tilde{u}^3\omega^0 \\
= & -(J - \mathcal{M}R^2)\tilde{\omega}^0(U^1 + U^3)\omega^0 + \mathcal{M}R(R + r)\tilde{u}^2\tilde{\omega}^0\tilde{u}^2\omega^0 \\
& + 2J\dot{\gamma}_1U^3\tilde{u}^1\omega^0 - 2\mathcal{M}R^2(\dot{\gamma}_1 - \dot{\gamma}_2)U^3\tilde{u}^1\omega^0 \\
& + \gamma_1\{J[\tilde{u}^1\tilde{\omega}^0(U^1 + U^3) + \tilde{\omega}^0(U^3\tilde{u}^1 - \tilde{u}^1U^3)]\omega^0 \\
& - \mathcal{M}R^2[\tilde{u}^1\tilde{\omega}^0(U^1 + U^3) + \tilde{\omega}^0(U^3\tilde{u}^1 - \tilde{u}^1U^3) - U^1\tilde{u}^3\tilde{\omega}_0\tilde{u}^2]\omega^0 \\
& + \mathcal{M}RrU^1\tilde{u}^3\tilde{\omega}^0\tilde{u}^2\omega^0\} + \gamma_2\mathcal{M}R^2\tilde{u}^2\tilde{\omega}^0\tilde{u}^3\omega^0
\end{aligned}$$

or, after expansion,

$$\begin{aligned}
A^1 \cong & -(J - \mathcal{M}R^2)(\omega_2\omega_3u^1 - \omega_1\omega_2u^3) - \mathcal{M}R(R + r)(\omega_2\omega_3u^1 - \omega_1\omega_2u^3) \\
& + 2J\dot{\gamma}_1\omega_2u^3 - 2\mathcal{M}R^2(\dot{\gamma}_1 - \dot{\gamma}_2)\omega_2u^3 \\
& + \gamma_1\{J[(\omega_2^2 - \omega_3^2)u^1 + \omega_1\omega_3u^3] - \mathcal{M}R^2[(\omega_2^2 - \omega_3^2 + \omega_3^2 + \omega_1^2)u^1 + \omega_1\omega_3u^3] \\
& - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)u^1\} + \gamma_2\mathcal{M}R^2[(\omega_1^2 + \omega_2^2)u^1 + \omega_1\omega_3u^3] \\
= & \; u^1\{-(J + \mathcal{M}Rr)\omega_2\omega_3 + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) \\
& - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\gamma_1 + \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\gamma_2\} \\
& + u^3\{(J + \mathcal{M}Rr)\omega_1\omega_2 + 2J\omega_2\dot{\gamma}_1 - 2\mathcal{M}R^2\omega_2(\dot{\gamma}_1 - \dot{\gamma}_2) \\
& + (J - \mathcal{M}R^2)\omega_1\omega_3\gamma_1 + \mathcal{M}R^2\omega_1\omega_3\gamma_2\}
\end{aligned} \tag{C-49}$$

and, by parallel construction,

$$A^2 \cong u^1\{-(J + \mathcal{M}Rr)\,\omega_2\omega_3 + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\,\gamma_2$$

$$+ \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\,\gamma_1\} + u^3\{(J + \mathcal{M}Rr)\,\omega_1\omega_2 + 2J\omega_2\dot{\gamma}_2 + 2\mathcal{M}R^2\omega_2(\dot{\gamma}_1 - \dot{\gamma}_2)$$

$$+ (J - \mathcal{M}R^2)\,\omega_1\omega_3\gamma_2 + \mathcal{M}R^2\omega_1\omega_3\gamma_1\} \qquad \text{(C-50)}$$

Equations (C-43), (C-49), and (C-50) are in final form, ready to be substituted into Eq. (C-8) to obtain linearized equations of motion. These equations also require the appropriate approximations for the coefficients on the left side of Eq. (C-8), as obtained from Defs. 38–40.

From Def. 38, the linear approximation of a_{00} is given by

$$a_{00} \cong \Phi^{00} + \Phi^{11} + \gamma_1\tilde{u}^1\Phi^{11} - \gamma_1\Phi^{11}\tilde{u}^1 + \Phi^{22} + \gamma_2\tilde{u}^1\Phi^{22} - \gamma_2\Phi^{22}\tilde{u}^1$$

$$- \mathcal{M}\{(D^{10^T}D^{01}U - D^{10}D^{01^T}) + (D^{20^T}D^{02}U - D^{20}D^{02^T})$$

$$- \gamma_1(D^{10^T}\tilde{u}^1D^{01}U + \tilde{u}^1D^{10}D^{01^T}) - \gamma_2(D^{20^T}\tilde{u}^1D^{02}U + \tilde{u}^1D^{20}D^{02^T})$$

$$+ (D^{01^T}D^{10}U - D^{01}D^{10^T})(U - \gamma_1\tilde{u}^1) + \gamma_1\tilde{u}^1D^{01^T}D^{10} + \gamma_1D^{01^T}\tilde{u}^1D^{10}U$$

$$+ (D^{21^T}D^{12}U - D^{21}D^{12^T})(U - \gamma_1\tilde{u}^1) + \gamma_1\tilde{u}^1D^{21^T}D^{12} + (\gamma_1 - \gamma_2)\,D^{21^T}\tilde{u}^1D^{12}U$$

$$- \gamma_2\tilde{u}^1D^{21}D^{12^T} + (D^{02^T}D^{20}U - D^{02}D^{20^T})(U - \gamma_2\tilde{u}^1)$$

$$+ \gamma_2\tilde{u}^1D^{02^T}D^{20} + \gamma_2D^{02^T}\tilde{u}^1D^{20}U + (D^{12^T}D^{21}U - D^{12}D^{21^T})(U - \gamma_2\tilde{u}^1)$$

$$+ \gamma_2\tilde{u}^1D^{12^T}D^{21} - (\gamma_1 - \gamma_2)\,D^{12^T}\tilde{u}^1D^{21}U - \gamma_1\tilde{u}^1D^{12}D^{21^T}\}$$

Substituting Eqs. (C-47) and (C-7) simplifies this lengthy expression somewhat, providing

$$a_{00} \cong \Phi^{00} + \Phi^{11} + \Phi^{22} + (J - \mathcal{M}R^2)(\tilde{u}^1U^3 - U^3\tilde{u}^1)(\gamma_1 + \gamma_2)$$

$$- \mathcal{M}R\{-2r(U - U^2) + r(\gamma_1 + \gamma_2)\,\tilde{u}^1U^2$$

$$- r(U - U^2)(U - \gamma_1\tilde{u}^1) - r\gamma_1\tilde{u}^1$$

$$- R(U - U^2)(U - \gamma_1\tilde{u}^1) - R\gamma_1\tilde{u}^1 + R\gamma_2\tilde{u}^1U^2$$

$$- r(U - U^2)(U - \gamma_2\tilde{u}^1) - r\gamma_2\tilde{u}^1$$

$$- R(U - U^2)(U - \gamma_2\tilde{u}^1) - R\gamma_2\tilde{u}^1 + R\gamma_1\tilde{u}^1U^2\}$$

$$= \Phi^{00} + \Phi^{11} + \Phi^{22} + J(\tilde{u}^1U^3 - U^3\tilde{u}^1)(\gamma_1 + \gamma_2)$$

$$+ \mathcal{M}R^2\{(U^3\tilde{u}^1 - \tilde{u}^1U^3)(\gamma_1 + \gamma_2) + 2(U^1 + U^3)$$

$$- U^3\tilde{u}^1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\,\tilde{u}^1U^2 + (\gamma_1 + \gamma_2)\,\tilde{u}^1\}$$

$$+ \mathcal{M}Rr\{2(U^1 + U^3) - (\gamma_1 + \gamma_2)\,\tilde{u}^1U^2 + 2(U^1 + U^3)$$

$$- U^3\tilde{\mathcal{U}}^1(\gamma_1+\gamma_2) + (\gamma_1+\gamma_2)\tilde{\mathcal{U}}^1\}$$

$$= \Phi^{00} + \Phi^{11} + \Phi^{22} + J(\tilde{\mathcal{U}}^1 U^3 - U^3\tilde{\mathcal{U}}^1)(\gamma_1+\gamma_2)$$

$$+ \mathcal{M}R^2[2U^1 + 2U^3 + (\gamma_1+\gamma_2)(\tilde{\mathcal{U}}^1 - \tilde{\mathcal{U}}^1 U^2 - \tilde{\mathcal{U}}^1 U^3)]$$

$$+ \mathcal{M}Rr[4U^1 + 4U^3 + (\gamma_1+\gamma_2)(\tilde{\mathcal{U}}^1 - \tilde{\mathcal{U}}^1 U^2 - U^3\tilde{\mathcal{U}}^1)] \qquad \text{(C-51)}$$

By expanding the constituents of this expression, one can verify that

$$\Phi^{00} + \Phi^{11} + \Phi^{22} + (2\mathcal{M}R^2 + 4\mathcal{M}rR)(U^1 + U^3)$$

$$= I^0 + 2(J + 2\mathcal{M}Rr + mr^2)(U^1 + U^3)$$

$$= I^0 + \left(2mr^2 + 4mrL + \frac{8}{3}mL^2\right)(U^1 + U^3) \triangleq I \qquad \text{(C-52)}$$

where the new symbol I is the inertia matrix of the total system with respect to the system mass center when $\gamma_1 = \gamma_2 = 0$. Thus, we have

$$a_{00} \cong I + (\gamma_1+\gamma_2)[J(\tilde{\mathcal{U}}^1 U^3 - U^3\tilde{\mathcal{U}}^1) + \mathcal{M}R^2(\tilde{\mathcal{U}}^1 - \tilde{\mathcal{U}}^1 U^2 - \tilde{\mathcal{U}}^1 U^3)$$

$$+ \mathcal{M}Rr(\tilde{\mathcal{U}}^1 - \tilde{\mathcal{U}}^1 U^2 - U^3\tilde{\mathcal{U}}^1)]$$

But

$$\tilde{\mathcal{U}}^1 - \tilde{\mathcal{U}}^1 U^2 = \tilde{\mathcal{U}}^1 U^3$$

so we have

$$a_{00} \cong I + (\gamma_1+\gamma_2)(J + \mathcal{M}Rr)(\tilde{\mathcal{U}}^1 U^3 - U^3\tilde{\mathcal{U}}^1) \qquad \text{(C-53)}$$

Expanding I, J, and R from Eqs. (C-52), (C-48), and (C-3) provides

$$a_{00} \cong I^0 + 2m\left(r^2 + 2rL + \frac{4}{3}L^2\right)(U^1 + U^3) + m\left(\frac{4}{3}L^2 + rL\right)(\tilde{\mathcal{U}}^1 U^3 - U^3\tilde{\mathcal{U}}^1)(\gamma_1+\gamma_2)$$

$$= \begin{bmatrix} I_1^0 + 2m\left(r^2 + 2rL + \frac{4}{3}L^2\right) & 0 & 0 \\ 0 & I_2^0 & -m(\gamma_1+\gamma_2)\left(\frac{4}{3}L^2 + rL\right) \\ 0 & -m(\gamma_1+\gamma_2)\left(\frac{4}{3}L^2 + rL\right) & I_3^0 + 2m\left(r^2 + 2rL + \frac{4}{3}L^2\right) \end{bmatrix} \qquad \text{(C-54)}$$

It may be noted that a_{00} is the inertia matrix in basis $\{\mathbf{b}^0\}$ of the total system relative to the system mass center. Detailed construction of this matrix from physical definitions confirms that Eq. (C-54) provides the correct linearized approximation of a_{00}.

Construction of the equations of motion from Eq. (C-8) requires expansion of the linearized approximations of a_{10} and a_{20} from Def. 39. The expansion provides

$$a_{10} \cong u^{1T}\Phi^{11}(U - \gamma_1\tilde{\mathcal{U}}^1) - \mathcal{M}u^{1T}[D^{10T}D^{01}U - D^{10}D^{01T}$$

$$- \gamma_1 D^{10T}\tilde{\mathcal{U}}^1 D^{01}U - \gamma_1 D^{10T}D^{01}\tilde{\mathcal{U}}^1 + D^{12T}D^{21}U - D^{12}D^{21T}$$

$$- (\gamma_1 - \gamma_2) D^{12T}\tilde{\mathcal{U}}^1 D^{21}U - \gamma_1 D^{12T}D^{21}\tilde{\mathcal{U}}^1 + \gamma_2 D^{21T}D^{12}\tilde{\mathcal{U}}^1]$$

Equation (C-47) yields

$$u^{1T}\Phi^{11}(U - \gamma_1\tilde{\mathcal{U}}^1) = (J - \mathcal{M}R^2)\,u^{1T}(U^1 + U^3)(U - \gamma_1\tilde{\mathcal{U}}^1) = (J - \mathcal{M}R^2)\,u^{1T}$$

Equation (C-7) yields

$$u^{1T}D^{10}D^{01} = u^{1T}D^{12}D^{21} = u^{1T}\tilde{\mathcal{U}}^1 = 0 \qquad D^{10T}u^1D^{01} = D^{12T}\tilde{\mathcal{U}}^1D^{21} = 0$$

Thus, a_{10} simplifies* to

$$a_{10} \cong (J - \mathcal{M}R^2)\,u^{1T} - \mathcal{M}(D^{10T}D^{01} + D^{12T}D^{21})\,u^{1T}$$

$$= (J - \mathcal{M}R^2 + \mathcal{M}R^2 + \mathcal{M}Rr)\,u^{1T} = (J + \mathcal{M}Rr)\,u^{1T}$$

$$= m\left(\frac{4}{3}L^2 + rL\right)u^{1T} \tag{C-55}$$

Similar calculations produce

$$a_{20} = a_{10} \cong m\left(\frac{4}{3}L^2 + rL\right)u^{1T} \tag{C-56}$$

For the remainder of the terms in Eq. (C-8), only constant-term approximations are required. Defs. 39, 40 provide

$$a_{02} = a_{01} \cong u^1 m\left(\frac{4}{3}L^2 + rL\right) \tag{C-57}$$

$$a_{12} = a_{21} \cong -\mathcal{M}u^{1T}(D^{21T}D^{12}U - D^{21}D^{12T})\,u^1$$

$$= \mathcal{M}R^2 = \frac{m}{\mathcal{M}}\,mL^2 \tag{C-58}$$

$$a_{11} \cong u^{1T}\Phi^{11}u^1 = (J - \mathcal{M}R^2)\,u^{1T}(U^1 + U^3)\,u^1$$

$$= J - \mathcal{M}R^2 = mL^2\left(\frac{1}{3} - \frac{m}{\mathcal{M}}\right) \tag{C-59}$$

$$a_{22} \cong u^{1T}\Phi^{22}u^1 = J - \mathcal{M}R^2 = mL^2\left(\frac{1}{3} - \frac{m}{\mathcal{M}}\right) \tag{C-60}$$

*Note that in this example, the quantities $\hat{a}_{10}$ and $\hat{a}_{20}$ (defined by Eq. 37b and available as the variable parts of the linear approximations shown here for a_{10} and a_{20}) are zero. This convenient result stems from the diagonality of Φ^{11} and the parallelism of $\mathbf{D}^{10}$, $\mathbf{D}^{01}$, $\mathbf{D}^{12}$, and $\mathbf{D}^{21}$; it is not a general result.

Equations (C-54)–(C-60) provide all of the coefficients on the left side of the equations of motion (Eq. C-8), and all of the terms on the right are available from Eqs. (C-49), (C-50), and (C-43), utilizing also the expressions for C^{01} and C^{02}. The resulting linearized scalar equations are obtained, after simplification of terms, as follows:

$$I_1\dot{\omega}_1 + (J + \mathcal{M}Rr)(\ddot{\gamma}_1 + \ddot{\gamma}_2) = (I_2 - I_3)\,\omega_2\omega_3 + (J + \mathcal{M}Rr)(\omega_2^2 - \omega_3^2)(\gamma_1 + \gamma_2) \tag{C-61}$$

$$I_2\dot{\omega}_2 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\dot{\omega}_3 = (I_3 - I_1)\,\omega_3\omega_1 - (J + \mathcal{M}Rr)\,\omega_1\omega_2(\gamma_1 + \gamma_2) \tag{C-62}$$

$$\begin{aligned} I_3\dot{\omega}_3 - (J + \mathcal{M}Rr)(\gamma_1 + \gamma_2)\,\dot{\omega}_2 &= (I_1 - I_2)\,\omega_1\omega_2 \\ &\quad + 2(J + \mathcal{M}Rr)\,\omega_2(\dot{\gamma}_1 + \dot{\gamma}_2) + (\mathcal{M}Rr + J)\,\omega_1\omega_3(\gamma_1 + \gamma_2) \end{aligned} \tag{C-63}$$

$$\begin{aligned} (J + \mathcal{M}Rr)\,\dot{\omega}_1 + (J - \mathcal{M}R^2)\,\ddot{\gamma}_1 + \mathcal{M}R^2\ddot{\gamma}_2 &= -(J + \mathcal{M}Rr)\,\omega_2\omega_3 \\ + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\,\gamma_1 &+ \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\,\gamma_2 - k\gamma_1 \end{aligned} \tag{C-64}$$

$$\begin{aligned} (J + \mathcal{M}Rr)\,\dot{\omega}_1 + \mathcal{M}R^2\ddot{\gamma}_1 + (J - \mathcal{M}R^2)\,\ddot{\gamma}_2 &= -(J + \mathcal{M}Rr)\,\omega_2\omega_3 \\ + [J(\omega_2^2 - \omega_3^2) - \mathcal{M}R^2(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr(\omega_1^2 + \omega_3^2)]\,\gamma_2 &+ \mathcal{M}R^2(\omega_1^2 + \omega_2^2)\,\gamma_1 - k\gamma_2 \end{aligned} \tag{C-65}$$

where I_1, I_2, and I_3 are the diagonal elements of I as defined by Eq. (C-52), so that these are the principal axis moments of inertia of the total system when $\gamma_1 = \gamma_2 = 0$.

Note that in Eqs. (C-64) and (C-65) the coefficients of γ_1 and γ_2 are interchanged, so that their coefficient matrix is symmetric. This result was anticipated in Section IIIC, since the system in Fig. C-1 is an example of a "meridional deformation case" (see Fig. 11).

Equations (C-61)–(C-65) have been derived by the laborious process of linearizing Eq. (1) from Section IIA of this report. In work not shown here, the same equations have also been derived from first principles. Equations (C-61)–(C-63) may be obtained simply by taking the inertial frame time derivative of the system angular momentum for the system mass center. Equation (C-64) is available from the dot-product of $\mathbf{b}_1^0$ and the equation

$$\mathbf{M}^{p_1} = \dot{\mathbf{H}}^{p_1} + m\mathbf{c}_1 \times \ddot{\mathbf{P}}_1$$

where $\mathbf{M}^{p_1}$ is the external moment about p_1 applied to b_1, $\mathbf{H}^{p_1}$ is the inertial angular momentum of b_1 referred to p_1, $\mathbf{c}_1 \triangleq L\mathbf{b}_2^1$ and $\mathbf{P}_1$ is the vector from the system mass center to p_1. Equation (C-65) is available from a similar calculation. The derivation of Eqs. (C-61)–(C-65) from first principles is a much more rapid and more error-free process for an experienced analyst than is the application of Eq. (1) for this purpose, although at present a digital computer can accomplish only the latter task.

Although this appendix has served to provide a second explicit check on the linear terms in Eq. (1), its primary objective is to check these results against the generic linearized equations recorded as Eqs. (30) of Section IIIB.

For a three-body system with only small relative motions, Eq. (30a) becomes

$$(\bar{a}_{00} + \hat{a}_{00})\,\dot{\omega}^0 + \bar{a}_{01}\ddot{\gamma}_1 + \bar{a}_{02}\ddot{\gamma}_2 = \bar{A}^0 + \hat{A}^0 + (\bar{C}^{01} + \hat{C}^{01})\,\bar{A}^1 + \bar{C}^{01}\hat{A}^1 + (\bar{C}^{02} + \hat{C}^{02})\,\bar{A}^2 + \bar{C}^{02}\hat{A}^2 \tag{C-66}$$

and Eq. (30b) provides

$$(\bar{a}_{10} + \hat{a}_{10})\,\dot{\omega}^0 + \bar{a}_{11}\ddot{\gamma}_1 + \bar{a}_{12}\ddot{\gamma}_2 = g^{1T}(\bar{A}^1 + \hat{A}^1) + \bar{\tau}_1 + \hat{\tau}_1 \tag{C-67}$$

and

$$(\bar{a}_{20} + \hat{a}_{20})\,\dot{\omega}^0 + \bar{a}_{21}\ddot{\gamma}_1 + \bar{a}_{22}\ddot{\gamma}_2 = g^{2T}(\bar{A}^2 + \hat{A}^2) + \bar{\tau}_2 + \hat{\tau}_2 \tag{C-68}$$

Having struggled to derive Eqs. (C-61)–(C-65) from Eq. (1), we can now recognize that we have already executed the steps implied by Eqs. (C-66)–(C-68). Equations (C-53) and (C-54) are, in fact, expressions for $\bar{a}_{00} + \hat{a}_{00}$, and Eqs. (C-57) and (C-56) provide $\bar{a}_{01}$, $\bar{a}_{02}$, $\bar{a}_{10}$, and $\bar{a}_{20}$, with the further indication that $\hat{a}_{10} = \hat{a}_{20} = 0$. (This simplification is computationally significant, and its validity in every new case should be investigated.) The right-hand sides of Eqs. (C-66)–(C-68) apparently also coincide with results already obtained, with $\bar{A}^0 + \hat{A}^0$ given by Eq. (C-43), $\bar{A}^1 + \hat{A}^1$ given by Eq. (C-49), and $\bar{A}^2 + \hat{A}^2$ given by Eq. (C-50), and with $\bar{C}^{10} = \bar{C}^{20} = U$, $\hat{C}^{10} = -\gamma_1\tilde{u}^1$, and $\hat{C}^{20} = -\gamma_2\tilde{u}^1$.

These results also match those available from Eqs. (33a) and (33b) for $\bar{C}^{rj}$ and $\hat{C}^{rj}$ ($j = 0$ and $r = 1, 2$), as well as Eqs. (36) and (37) for the quantities $\bar{a}_{00}$, $\hat{a}_{00}$, $\bar{a}_{01}$, $\hat{a}_{01}$, $\bar{a}_{02}$, $\hat{a}_{02}$, $\bar{a}_{12} = \bar{a}_{21}$, $\bar{a}_{11}$, $\bar{a}_{22}$, $\bar{A}^0$, $\hat{A}^0$, $\bar{A}^1$, $\hat{A}^1$, $\bar{A}^2$, and $\hat{A}^2$. Thus, the generic partially linearized equations in Section IIIB have also been confirmed.

The final step in the development of hybrid-coordinate equations is the transformation of the small variables (γ_1 and γ_2) into a new set of variables η_1 and η_2, and (when justified) the truncation of the new set of variables to obtain for the total system a set of equations of reduced dimension. As noted in Sections IIIC and D, the quest for an appropriate transformation is, in general, a complex computational task involving eigenvector calculations. For the simple system under consideration in this appendix, however, the necessary transformation can be determined by inspection of Eqs. (C-61)–(C-65), or by physical argument.

We require the transformation

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{C-69a}$$

with the inverse transformation

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{C-69b}$$

We can replace $\gamma_1 + \gamma_2$ in Eqs. (C-61)–(C-63) by η_1, and replace Eqs. (C-64) and (C-65) by their sums and differences, to obtain

$$I_1\dot{\omega}_1 + (J + \mathcal{M}Rr)\,\ddot{\eta}_1 = (I_2 - I_3)\,\omega_2\omega_3 + (J + \mathcal{M}Rr)\,(\omega_2^2 - \omega_3^2)\,\eta_1 \tag{C-70}$$

$$I_2\dot{\omega}_2 - (J + \mathcal{M}Rr)\,\eta_1\dot{\omega}_3 = (I_3 - I_1)\,\omega_3\omega_1 - (J + \mathcal{M}Rr)\,\omega_1\omega_2\eta_1 \tag{C-71}$$

$$I_3\dot{\omega}_3 - (J + \mathcal{M}Rr)\,\eta_1\dot{\omega}_2 = (I_1 - I_2)\,\omega_1\omega_2 + 2\,(J + \mathcal{M}Rr)\,\omega_2\dot{\eta}_1 + (J + \mathcal{M}Rr)\,\omega_1\omega_3\eta_1 \tag{C-72}$$

$$J\ddot{\eta}_1 + 2\,(J + \mathcal{M}Rr)\,\dot{\omega}_1 = -2\,(J + \mathcal{M}Rr)\,\omega_2\omega_3 + [J\,(\omega_2^2 - \omega_3^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2) - k]\,\eta_1 \tag{C-73}$$

$$(J - 2\mathcal{M}R^2)\,\ddot{\eta}_2 = [J\,(\omega_2^2 - \omega_3^2) - 2\mathcal{M}R^2\,(\omega_1^2 + \omega_2^2) - \mathcal{M}Rr\,(\omega_1^2 + \omega_3^2) - k]\,\eta_2 \tag{C-74}$$

Because only the final equation involves η_2, and this equation is satisfied by $\eta_2 \equiv 0$, it is quite satisfactory for the purposes of dynamic analysis of rotational motions of the central body (defined by $\omega_1, \omega_2, \omega_3$) to truncate this system of equations, abandoning Eq. (C-74) entirely.

NASA — JPL — Coml., L.A., Calif.