NASA TECHNICAL NOTE

N73-14901

NASA TN D-7115

NASA TN D-7115

CASE FILE COPY

MINIMUM WEIGHT DESIGN OF STRUCTURES VIA OPTIMALITY CRITERIA

by J. Kiusalaas

George C. Marshall Space Flight Center
Marshall Space Flight Center, Ala. 35812

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1972

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO. NASA TN D-7115	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE Minimum Weight Design of Structures Via Optimality Criteria		5. REPORT DATE December 1972 6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) J. Kiusalaas		8. PERFORMING ORGANIZATION REPORT # M379
9. PERFORMING ORGANIZATION NAME AND ADDRESS George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812		10. WORK UNIT NO. 124-12-11-0000 11. CONTRACT OR GRANT NO.
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D. C. 20546		13. TYPE OF REPORT & PERIOD COVERED Technical Note 14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Astronautics Laboratory, Science and Engineering

16. ABSTRACT

This report reviews the state of the art of automated structural design through the use of optimality criteria, with emphasis on aerospace applications. The contents include constraints on stresses, displacements, and buckling strengths under static loading, as well as lower bound limits on natural frequencies and flutter speeds. It is presumed that the reader is experienced in finite element methods of analysis, but is not familiar with optimal design techniques.

17. KEY WORDS	18. DISTRIBUTION STATEMENT

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
Unclassified	Unclassified	72	$3.00

MSFC - Form 3292 (May 1969)

*For sale by the National Technical Information Service, Springfield, Virginia 22151

ACKNOWLEDGMENT

The research documented in this report was accomplished while the author held a National Research Council Postdoctoral Resident Research Associateship supported by the George C. Marshall Space Flight Center, Huntsville, Ala.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

DEFINITION OF SYMBOLS

Symbol	Definition
$[A]$	Air force matrix
A_i	Design variables (element sizes)
b	Semichord length in flutter
C_i	Redesign scale factors
c	Constant
E	Modulus of elasticity
$\{f\}$	Generalized force vector
$\{f_i\}$	Generalized force vector of the i^{th} element
G	Geometric potential energy
g_i	Direction cosines of redesign vector
$[H]$	Geometric stiffness matrix
I	Imaginary part of a variable; moment of inertia
$i = \sqrt{-1}$	
$[K]$	Stiffness matrix
$[K_i]$	Stiffness matrix of i^{th} element
$k = b\omega/V$	Reduced frequency (V is air speed)
$\left[k_i^{(m)}\right]$	Unit stiffness matrices of i^{th} element:

$$[K_i] = \sum_{m=0}^{3} \left[k_i^{(m)}\right] A_i^m$$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
L_i	Length or surface area of i^{th} element
ℓ	Length
M	Number of behavioral constraints
$[M]$	Mass matrix
$[M_i]$	Mass matrix of i^{th} element
$\left[m_i^{(m)}\right]$	Unit mass matrices of i^{th} element: $\left[M_i\right] = \sum_{m=0}^{1} \left[m_i^{(m)}\right] A_i^m$
P	Reference to point P; applied force
P_i	Axial force in i^{th} bar element
p_r	Eigenvalues
$Q_{ri} = \partial q_r / \partial A_i$	Gradients of behavioral variables
q_r	Behavioral variables
q_r^*	Maximum allowable value of q_r
R	Real part of a variable; ratio
r	Radius of tube
$\left[S_i^P\right]$	Stress-generator matrix: $\left\{\sigma_i^P\right\} = \left[S_i^P\right]\{f_i\}$
$\left[s_i^P\right]$	Unit stress-generator matrix: $\left[S_i^P\right] = \left[s_i^P\right]/A_i^\beta$

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
T	Kinetic energy
T_i	Kinetic energy of i^{th} element
t	Wall-thickness of element
U	Strain energy
U_i	Strain energy of i^{th} element
U_S	Strain energy of elastic supports
$\{u\}$	Generalized displacement vector
$\{u_i\}$	Generalized displacement vector of i^{th} element
V	Air speed; structural volume
V_r	Flutter speeds
$\{v\}$	Associated eigenvector
W	Structural weight
W_i	Structural weight of i^{th} element
α	Relaxation factor
β	Constant
δ_{ij}	Kronecker delta
ϵ	Constant
λ_r	Lagrangian multipliers
μ	Constant of proportionality

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
ν	Iteration number
ρ_i	Unit weight of i^{th} element
ω	Circular frequency: frequency $= \omega/(2\pi)$

$( )_{,i} = \partial( )/\partial A_i$

$( )' = \partial( )/\partial k$

$( )^*$	Denotes limiting allowable value of variable ( )
$( )^{(r)}$	Denotes variables associated with r^{th} load condition or r^{th} eigenmode
$( )^{(\nu)}$	Denotes variables evaluated after the ν^{th} design iteration

$\Delta( ) = ( )^{(\nu + 1)} - ( )^{(\nu)}$

MINIMUM WEIGHT DESIGN OF STRUCTURES VIA OPTIMALITY CRITERIA

INTRODUCTION

Every design task, whether applied to a structure or some other man-made object, is either directly or indirectly governed by certain optimality criteria. The designer is seldom required to create a product that will only serve its primary function — he is invariably expected to meet additional design objectives, such as the lowest possible weight or cost, minimal maintenance, maximum reliability, pleasing appearance, etc.

In structural mechanics, the term "optimal design" is commonly used in a much more restricted sense. It implies that the sole design objective is minimum structural weight.[1] Moreover, it is understood that the entire design process is carried out automatically on a digital computer.

The basic idea of computer-automated, minimum weight design can be best comprehended by viewing the optimizing algorithm as an evolutionary advance over the conventional design method. Consider a typical problem in which the designer is given the layout of the structure, the loading, and the constraints on the behavior of the structure (e.g., limits on stresses and displacements). He must then calculate the sizes of the members so that the total structural weight is minimized and no constraints are violated.

The design procedure is an iterative, trial-and-error process, each iteration consisting of two steps: an analysis of the current design, followed by a redistribution of structural material. In the conventional design method, the analysis, which we presume is carried out by a finite element computer program, is used primarily to check the behavioral constraints and provides little or no guidance for the material redistribution cycle. Consequently, the redesign is a creative task, based on empirical rules and the intuition and experience of the designer.

An obvious means of improving the technique is to extend the scope of the analysis cycle so that it will not only calculate the behavior of the current

---

1. Optimal weight design methods can also be used for minimum cost design by replacing the unit weight of each structural element by its unit cost.

design but also predict the changes in the behavior caused by material redistribution. As a specific example, consider a structure subjected to displacement constraints only. Let us assume that an analysis of the structure indicates that some of the displacements exceed the prescribed limits so that the designer must add to the structural stiffness by increasing the sizes of some of the members. The question is, which members and by how much?

To obtain a rational answer, the designer must have some idea of how sensitive the displacements are to changes in member sizes. This could be accomplished, at least partially, by computing the displacement gradients in addition to the displacements themselves. By displacement gradients we mean $\partial u_i / \partial W_j$, where u_i is the i^{th} generalized displacement and W_j represents the weight of the j^{th} member. The designer would then know which members are most effective in reducing the displacements that violate the constraints and by using the approximation $\Delta u_i = \sum_j (\partial u_i / \partial W_j) \Delta W_j$, he could also estimate the magnitude of the required redesign.

In practical problems such redesign data are too voluminous for the designer to handle. It is much more reasonable to process the data within the computer itself, i.e., to let the computer take care of the redesign in addition to the analysis. We have now arrived at a fully automated, optimal design algorithm.

It should be pointed out, however, that it seldom is feasible to obtain an efficient design without any human participation whatsoever. Structural layout, for example, is one branch of design that is unlikely to be automated in the near future, with the exception of some specific problems (e.g., trusses). Even optimal weight distribution with fixed layout, which is the topic of this report, cannot be entirely automated. In many problems it is neither practical to include all the constraints into the computer program (the subjective constraints defy formulation altogether) nor is it possible to arrive at an efficient choice of member types on the first attempt. All this requires continuous monitoring of the design process by an experienced engineer, who should have the means of stopping the computations and making appropriate changes in the problem formulation.

Structural optimization algorithms, or more specifically, the methods of material redistribution, can be divided into two broad categories, which we call "direct" and "indirect" techniques. In the first category, the redesign is viewed as a purely mathematical programming problem, where the merit function (weight) is to be minimized within certain constraints. Typically, the heart of the method is a systematic search procedure that works directly with the merit function and the constraints and converges to either the global or local minimum weight design.

The direct methods trace their origin to the operations research and optimal control theories and have dominated the literature on automated design since the birth of the subject, just over a decade ago. Variations of the technique are numerous — names like "feasible directions," "steepest descent," and "gradient search" are just a small sampling of the terminology used. A very readable review of direct methods has been presented by Schmit [1].

Despite the successful treatment of many problems, direct optimization does not appear to be a practical method for the design of large structures. It has become apparent that the number of design iterations required for convergence to optimal design increases very rapidly with the number of elements. Present estimates [2] place an upper limit of 150 design variables (structural elements) on the size of the structure that can be treated with acceptable computational economy. For this reason alone, direct methods have been increasingly overlooked in recent work on practical aerospace structures and will not be discussed further in this report.

A somewhat different approach to optimal structural design was developed in the late 1960's. The idea was to avoid the inconvenience and computational inefficiency of working directly with the merit function and the original constraints — hence the name, "indirect" method.

Indirect design methods are, loosely speaking, counterparts of the variational methods used in analysis. Each method is built around an <u>optimality criterion</u> that serves the same function as any well-known variational principle of mechanics; see Reference 3 for simple examples on the derivation and use of optimality criteria. The analogy is accentuated by the fact that the optimality criteria frequently specify the energy distribution in an optimal weight structure.

An optimality criterion is mathematically equivalent to the design objective and the constraints; consequently, its use also leads to a true (local) minimum weight structure. The advantage of the indirect method stems from the presence of the behavioral variables, rather than the total structural weight, in the optimality criterion. In contrast, the direct method displayed the behavioral variables in the constraint conditions only. Since the constraints are the source of most mathematical difficulties in optimization problems, the use of the optimality criterion usually leads to a more efficient design algorithm.

Despite the aformentioned superiority, a rigorous application of the optimality criteria to realistic design problems may still require unacceptably long computer runs. Consequently, it is a common practice to introduce convenient ad hoc modifications into the formulation of the problem, which have the effect of sacrificing some of the structural efficiency for computational economy. Although the resulting design will not have the true optimal weight, a modified method is still a substantial improvement over the traditional design technique.

Most of this report is devoted to the derivation of optimality criteria and their utilization in design algorithms. Although much of the material is derived from published descriptions of existing optimization programs, the report should not be considered merely as a literature survey. The publications listed either treat specific aspects of optimization, introduce special techniques, or are applicable to only a restricted class of elements. In contrast, the present work attempts to establish a general viewpoint to structural optimization. The optimality criteria and redesign techniques are first introduced in general terms, then specialized to different constraint conditions and compared with the methods proposed in the references.

It is important to keep in mind that computer-oriented structural optimization is still in the developmental stage. The only existing program with a sufficient capacity (3000 elements, 6000 degrees of freedom) to design a large aerospace structure [4] is largely limited to stress constraints[2] and restricted to elements with special properties. The remaining programs listed in the references are small demonstration algorithms involving no more than a few hundred elements with simple size-stiffness relations. Therefore, the design methods proposed here and in previous publications represent only the first steps toward a practical, fully automated structural optimization algorithm.

BASIC CONCEPTS

Restrictions

This report is restricted to the minimum weight design of linear, elastic structures. The layout of members and the loading (static, unless

2. Displacement constraints are treated in an indirect, rather inefficient manner — see the section, A Selected Survey of Optimization Programs.

specified otherwise) are assumed to be given, the only design variables being the sizes of structural members — the cross-sectional area and thickness for one- and two-dimensional elements, respectively. The size is presumed to be constant within each individual element. It is also assumed that finite element displacement methods are used in the analysis cycle.

Constraints

The behavioral constraints treated in the report are upper limits on stresses and displacements, and lower bounds on general buckling loads, natural frequencies, and flutter speeds. As these constraints play a major role in the derivation of optimality criteria, they can be classified as primary constraints.

In most design problems it is desirable to incorporate additional conditions, called secondary constraints, in the design algorithm. Minimum and maximum limits on member sizes, prohibition of local buckling, and equal size constraints (the requirement that the sizes of certain members be the same), fall into the last category. The secondary constraints mainly determine the details of programming, having little effect on the optimality criterion itself.

An insight into the relationship between the constraints and the optimal design can be obtained only through geometrical abstraction. To this end, we introduce the concept of design space — an N-dimensional Euclidean space, where N is the number of independent design variables. The coordinates are the design variables A_i , $i = 1, 2, \ldots N$, so that each point in the space represents a specific design of the structure.

The points that do not violate any constraints are known as feasible designs. The boundary between the feasible region and the remaining space is called the surface of active constraints, and the points on that surface are termed as critical designs. The optimal design problem involves finding that point of the feasible region that is associated with the lowest structural weight; it is invariably a critical design.

The three-bar truss in Figure 1a lends itself to a simple example of design space. All the members are assumed to be made of aluminum, with $E = 6.8948 \times 10^{10}$ N/m^2 (10^7 psi) (E is Young's modulus). The loading consists of two alternate forces of 88.96 N (20 kips) each, as shown. The structural volume (weight) is to be minimized subject to the following constraints:

$$|\sigma_i| \leq 137.88 \times 10^5 \text{ N/m}^2 \text{ (20 ksi)} \quad ,$$

$$u_v \leq 0.381 \text{ cm (0.15 in.)} \quad ,$$

$$u_h \leq 0.762 \text{ cm (0.3 in.)} \quad ,$$

and

$$A_i \geq 0.508 \text{ cm (0.2 sq. in.)} \quad ,$$

where σ_i is the stress in the i^{th} member, A_i is the cross-sectional area, and u_h and u_v represent the displacement components defined in Figure 1b.

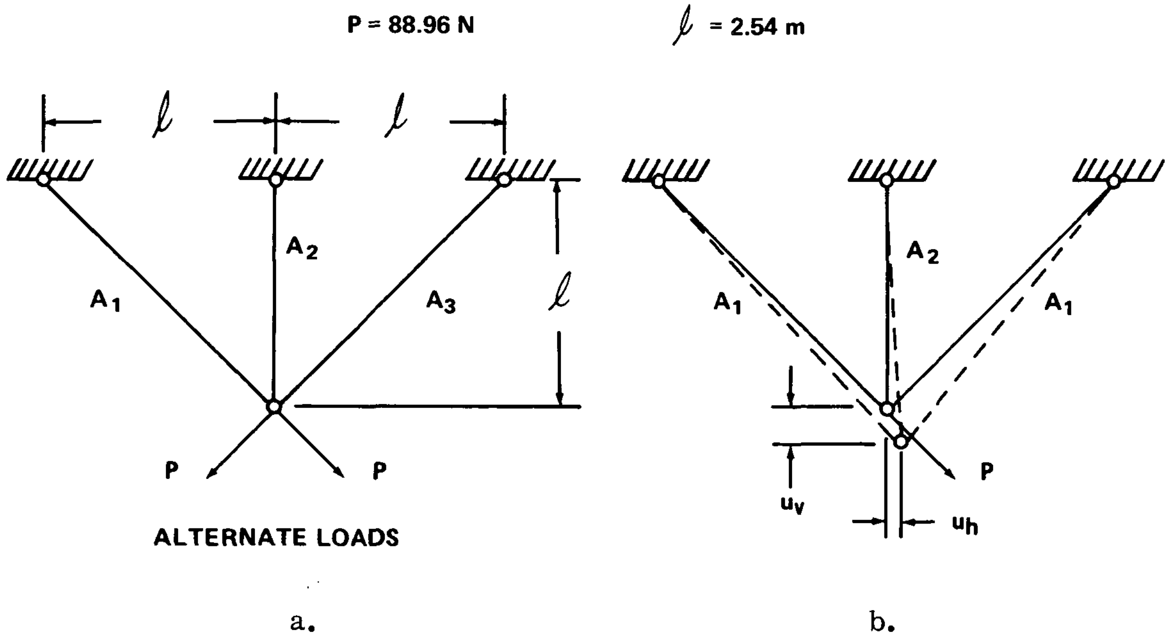


Figure 1. Three-bar truss.

The symmetry of loads and structural layout enable us to simplify the problem somewhat by considering only one of the loads and imposing the additional condition $A_3 = A_1$ (Fig. 1b). This reduces the dimensions of the design space from three to two.

In this simple example it is easy to analyze the structure with arbitrary values of the design variables. The results are

$$\sigma_1 = \frac{A_2 + \sqrt{2}A_1}{2A_2 + \sqrt{2}A_1} \frac{P}{A_1} ,$$

$$\sigma_2 = \frac{\sqrt{2}A_2}{2A_2 + \sqrt{2}A_1} \frac{P}{A_2} ,$$

$$\sigma_3 = \frac{A_2}{2A_2 + \sqrt{2}A_1} \frac{P}{A_1} ,$$

$$u_v = \frac{P\ell}{(\sqrt{2}A_2 + A_1)E} ,$$

and

$$u_h = \frac{P\ell}{A_1 E} .$$

All the constraints can now be shown in the design space ($A_1 - A_2$ plane) by plotting the lines $\sigma_1 = 137.88 \times 10^5\ \mathrm{N/m^2}$ (20 ksi), $u_v = 0.381$ cm (0.15 in.), etc., as has been done in Figure 2. The active constraints are determined by the envelope of all the constraint lines.

Figure 2 also shows constant volume contours of the structure, obtainable from $V = (2\sqrt{2}A_1 + A_2)\ell$ (V denotes the material volume). The optimal design can readily be found by inspection; it is represented by the point where a volume contour is tangent to the active constraint line.

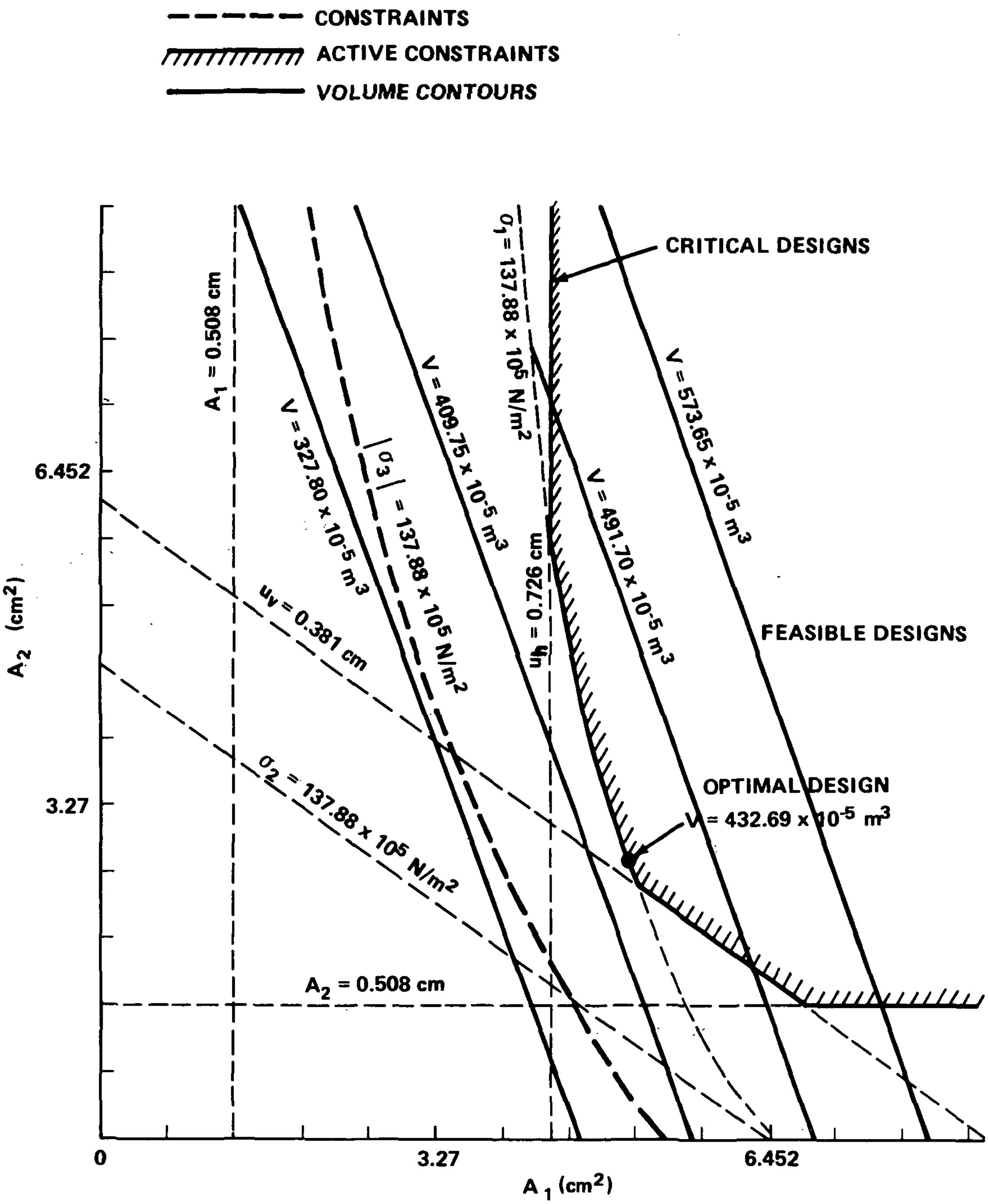
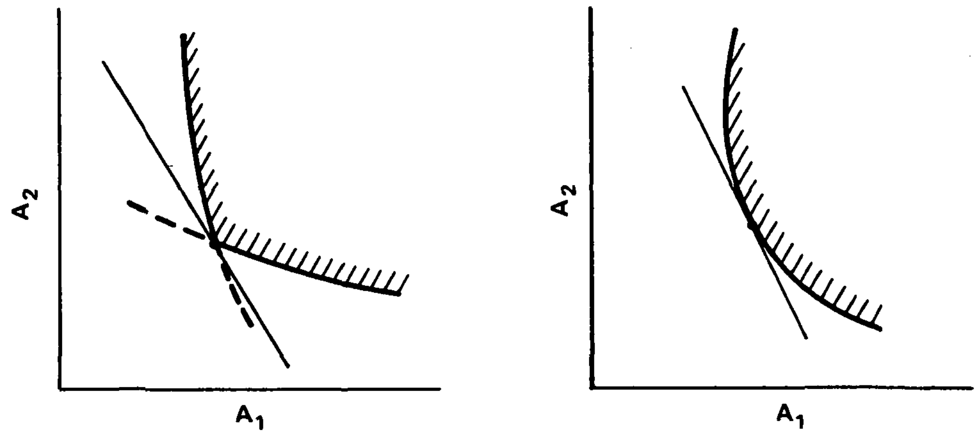


Figure 2. Design space for three-bar truss.

The location of the optimal point in Figure 2 is atypical. In the majority of problems it is found at the intersection of two or more constraint surfaces, as shown in Figure 3a; that is, the optimal design is commonly determined by several constraints simultaneously, rather than by a stationary point (Fig. 3b).



a. Intersection of constraint surfaces.

b. Stationary point.

Figure 3. Examples of optimal points.

An important feature of optimal design topology, which is apparent in Figure 2, is that only some of the constraints imposed on the problem are active. Moreover, only a portion of these constraints are active at the optimal design point itself, i.e., are directly involved in determining the minimum weight point.

It turns out that it is not too difficult to construct an optimal design algorithm if the constraints that are active at the optimal point are known beforehand. Unfortunately, this can be done only in a few, small-scale problems (see Reference 5 for an example). In the majority of structures a large portion of the computational effort must be expended, directly or indirectly, on finding the constraints associated with the optimal point.

Optimality Criteria

Consider at first one load condition and a single primary constraint,

$$q_r \leq q_r^* \quad , \tag{1}$$

where q_r is a behavioral variable and q_r^* its prescribed upper limit. In addition, we permit constraints on element sizes:

$$(A_i)_{min} \leq A_i \leq (A_i)_{max} \quad . \tag{2}$$

Let $\underset{\sim}{A}$ represent a design which is assumed to be critical, i.e.,

$$(q_r)_{\underset{\sim}{A}} = q_r^* \quad , \tag{3}$$

but not necessarily an optimal one. We presume that it is possible to compute the derivatives

$$Q_{ri} = \left(\frac{\partial q_r}{\partial A_i}\right)_{\underset{\sim}{A}} \quad . \tag{4}$$

The changes Δq_r of the constraint variable due to changes ΔA_i of the design can be estimated from

$$\Delta q_r = \sum_i Q_{ri} \, \Delta A_i \quad . \tag{5}$$

Equation (5) is an approximation for finite values of ΔA_i ; it is exact only when $|\Delta \underset{\sim}{A}|$ is infinitesimal.

The corresponding change of the structural weight is given by

$$\Delta W = \sum_i \rho_i L_i \Delta A_i \tag{6}$$

where ρ_i is the unit weigh of the i^{th} element and L_i represents the length or surface area for a one- or two-dimensional element, respectively.

The design $\underset{\sim}{A}$ can be improved if ΔA_i exist that do not violate the size constraints and for which $\Delta W < 0$, $\Delta q_r \leq 0$. We will now investigate the possibilities of weight reduction in detail.

1. If $Q_{ri} \geq 0$ for an element, it is evident by inspection of equations (5) and (6) that we can obtain a lighter-weight feasible design by simply decreasing the size of that element. It follows that at optimum weight design we must have

$$A_i = (A_i)_{min} \quad \text{if} \quad Q_{ri} \geq 0 \quad . \tag{7}$$

2. Consider any two elements, denoted by i and j , for which $Q_{ri} < 0$ and $Q_{rj} < 0$. It might be possible to save weight by removing some material from element i and adding a portion of that material to element j, or vice versa. To avoid violating the primary constraint, we consider only design changes which lead to $\Delta q_r = 0$. According to equation (5), the last constraint is equivalent to

$$\Delta A_j = -\frac{Q_{ri}}{Q_{rj}} \Delta A_i \quad . \tag{8}$$

The weight is reduced if $\Delta W < 0$, i.e.,

$$\rho_i L_i \Delta A_i + \rho_j L_j \Delta A_j < 0 \quad .$$

Substituting from equation (8), we get

$$\left(\rho_i L_i - \rho_j L_j \frac{Q_{ri}}{Q_{rj}}\right) \Delta A_i < 0 \quad . \tag{9}$$

If the size constraints are not active, we can satisfy equation (9) by choosing the appropriate sign for ΔA_i as long as the term in the parenthesis is not zero. Consequently, the design can be an optimal one only when the term vanishes, i.e.,

$$\frac{\rho_i L_i}{\rho_j L_j} = \frac{Q_{ri}}{Q_{rj}} \quad .$$

The last expression is equivalent to the equations,

$$\rho_i L_i + \lambda_r Q_{ri} = 0 \tag{10a}$$

and

$$\rho_j L_j + \lambda_r Q_{rj} = 0 \quad , \tag{10b}$$

where λ_r is a positive constant (Lagrangian multiplier); its value is determined from the condition $q_r = q_r^*$. The positive sign requirement follows the inequalities $\rho_i L_i > 0$ and $Q_{ri} < 0$.

Next consider the case $A_i = (A_i)_{min}$. Because of the minimum size constraint, the design change is restricted to $\Delta A_i > 0$. It follows from an inspection of equation (9) that the weight can be lowered only if the quantity in the parenthesis is negative. No weight reduction is possible if and only if

$$\rho_i L_i - \rho_j L_j \frac{Q_{ri}}{Q_{rj}} \geq 0 \quad .$$

Assuming A_j to be already at its optimal value, we can use equation (10b) to substitute $\rho_j L_j / Q_{rj} = -\lambda_r$, obtaining the optimality criterion

$$\rho_i L_i + \lambda_r Q_{ri} \geq 0 \quad \text{if} \quad A_i = (A_i)_{min} \ . \tag{11a}$$

Similarly, it can be shown that

$$\rho_i L_i + \lambda_r Q_{ri} \leq 0 \quad \text{if} \quad A_i = (A_i)_{max} \tag{11b}$$

at the optimal design.

The optimality criteria, equations (10) and (11), can be generalized for cases where the optimal point is determined by several primary constraints and load conditions [6]. The results can be stated as follows: The design $\underset{\sim}{A}$ is an optimal one if

$$\rho_i L_i + \sum_r \lambda_r Q_{ri} \begin{cases} = 0 & \text{if} \quad (A_i)_{min} < A_i < (A_i)_{max} \\ \geq 0 & \text{if} \quad A_i = (A_i)_{min} \\ \leq 0 & \text{if} \quad A_i = (A_i)_{max} \end{cases} \tag{12}$$

where

$$\lambda_r \begin{cases} = 0 & \text{if} \quad q_r < q_r^* \\ > 0 & \text{if} \quad q_r = q_r^* \end{cases} \ . \tag{13}$$

Equation (13) requires additional explanation. If there are M_L load conditions, each with M_c constraints, the total number of primary constraints in the problem is $M = M_L M_c$; consequently $r = 1, 2, \ldots M$. Only some of these constraints are active $(q_r = q_r^*)$ at optimal design and thus enter the optimality criterion. The inactive constraints $(q_r < q_r^*)$ are eliminated by setting $\lambda_r = 0$ for the appropriate values of r .

Equation (5), from which the optimality criteria were derived, is a linear approximation of the constraint surface in the neighborhood of the design $\underset{\sim}{A}$. It follows that the optimality criteria are valid only in a small region around $\underset{\sim}{A}$; that is, they are conditions for local optimality and do not guarantee that $\underset{\sim}{A}$ is a global minimum weight design. Proof of global optimality is a difficult problem, which has been resolved only for some special cases.

Weight Reduction Cycle

We assume again, for the sake of simplicity, that the optimal design is governed by only one primary constraint: $q_r \leq q_r^*$. Let $\underset{\sim}{A}^{(\nu)}$ be a point in the design space, which we call the current design. It does not have to be a critical design, nor does it have to lie in the feasible region. Our task is to modify the current design in such a way that the new design $\underset{\sim}{A}^{(\nu+1)}$ is closer to the optimal point than $\underset{\sim}{A}^{(\nu)}$. In particular, we want a repeated application of the weight reduction cycle to produce a sequence of designs that converges uniformly to the minimum weight configuration, as shown in Figure 4.

A computationally efficient redesign equation can be obtained from the optimality criteria (12) and (13). For a single constraint, the optimality criterion for active members, that is, members governed by primary constraints rather than size limits, is

$$\rho_i L_i + \lambda_r Q_{ri} = 0 \ .$$

Multiplying both sides of the equation by $(1 - \alpha)A_i$, where α is a constant to be determined later, and rearranging the terms, we can write

$$A_i = C_i A_i \qquad (14)$$

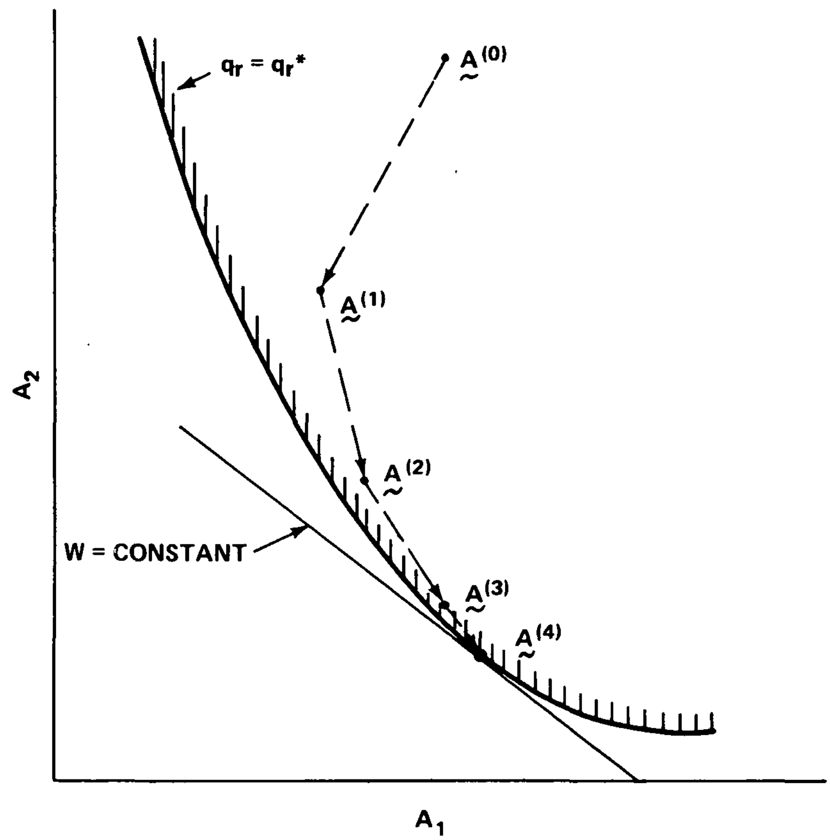


Figure 4. Example of a convergent sequence of designs produced by repeated application of weight reduction operation.

where

$$C_i = \alpha - (1 - \alpha)\lambda_r \frac{Q_{ri}}{\rho_i L_i} \quad . \tag{15}$$

Equation (14), being the optimality criterion, is satisfied identically for each active member when $\underset{\sim}{A}$ is the optimal design, regardless of the value of α . On the other hand, if $\underset{\sim}{A}$ is a nonoptimal point, equation (14) can be used as the redesign formula for active members: $A_i^{(\nu + 1)} = C_i A_i^{(\nu)}$. Repeated applications of the formula are equivalent to the solution of the optimality criteria by the method of relaxation. The parameter α , $-1 < \alpha < 1$, is known as the relaxation factor; its value determines the rate of convergence. It has been found that the optimal rate of convergence usually requires some underrelaxation $(\alpha > 0)$.

In order to include passive members — members governed by size constraints — into the redesign equation, the latter is rewritten as

$$A_i^{(\nu+1)} = \begin{cases} C_i A_i^{(\nu)} & \text{if} \quad (A_i)_{min} \leq C_i A_i^{(\nu)} \leq (A_i)_{max} \\ (A_i)_{min} & \text{if} \quad C_i A_i^{(\nu)} < (A_i)_{min} \\ (A_i)_{max} & \text{if} \quad C_i A_i^{(\nu)} > (A_i)_{max} \end{cases} \tag{16}$$

where C_i is given by equation (15).

Before C_i can be evaluated, however, the value of the Lagrangian multiplier λ_r must be found. As pointed out before, λ_r is determined in such a way that the design $\underset{\sim}{A}^{(\nu+1)}$ is critical, i.e., from the condition $q_r^{(\nu+1)} = q_r^*$.

We assume for the time being that the identity of the active and passive members is known a priori. The change in q_r due to the design changes in the passive members is, according to equation (5),

$$(\Delta q_r)_{pass} = \sum_{i \text{ pass } 1} Q_{ri} \left[(A_i)_{min} - A_i^{(\nu)} \right] + \sum_{i \text{ pass } 2} Q_{ri} \left[(A_i)_{max} - A_i^{(\nu)} \right] \quad . \tag{17}$$

The two sums represent the effects of members governed by minimum and maximum size constraints, respectively.

The contribution of active members to change in the constraint variable is

$$(\Delta q_r)_{act} = \sum_{i\ act} Q_{ri}\, \Delta A_i \qquad . \tag{18}$$

From equation (16) we obtain for active members

$$\Delta A_i = A_i^{(\nu+1)} - A_i^{(\nu)} = (C_i - 1)\, A_i^{(\nu)} \quad .$$

Substituting for C_i from equation (15) yields

$$\Delta A_i = -(1-\alpha)\left(1 + \lambda_r \frac{Q_{ri}}{\rho_i L_i}\right) A_i^{(\nu)} \qquad .$$

Therefore, equation (18) becomes

$$(\Delta q_r)_{act} = -(1-\alpha) \sum_{i\ act}\left(Q_{ri} + \lambda_r \frac{Q_{ri}^2}{\rho_i L_i}\right) A_i^{(\nu)} \qquad . \tag{19}$$

The total change in q_r is

$$\Delta q_r = (\Delta q_r)_{pass} + (\Delta q_r)_{act} \qquad . \tag{20}$$

We set $\Delta q_r = q_r^* - q_r^{(\nu)}$, which is equivalent to the requirement

$q_r^{(\nu + 1)} = q_r^*$, and substitute for $(\Delta q_r)_{act}$ from equation (19). The resulting equation can be solved for the Lagrangian multiplier:

$$\lambda_r = - \frac{q_r - q_r^{(\nu)} - (\Delta q_r)_{pass} + (1 - \alpha) \sum_{i\ act} Q_{ri} A_i^{(\nu)}}{(1 - \alpha) \sum_{i\ act} \frac{Q_{ri}^2}{\rho_i L_i} A_i^{(\nu)}} . \quad (21)$$

Equation (21) can be evaluated only if the identities of active and passive members are known beforehand. Since this is generally not the case, the weight minimization cycle itself must be carried out by a trial-and-error procedure outlined as follows.

1. Analyze the current design $\underset{\sim}{A}^{(\nu)}$ and compute the gradients Q_{ri} of the constraint variable.

2. Set $A_i^{(\nu + 1)} = (A_i)_{min}$ if $Q_{ri} > 0$ and consider these members to be passive.

3. For the remaining members, use the same division into passive and active groups that occurred at the end of the previous redesign cycle. If the present redesign operation is the first one, assume that all these members are active.

4. Compute λ_r from equation (21).

5. Use equations (15) and (16) to calculate $A_i^{(\nu + 1)}$ and use the results to reclassify the members into passive and active groups.

6. If the classification of the members has remained unchanged, the redesign cycle is completed; otherwise, use the new classification to repeat steps 4, 5, and 6.

Experience reported in References 2 and 7 and in a manuscript now being prepared for publication[3] has shown the method to be efficient. In most

3. J. Kiusalaas, Optimal Design of Structures with Buckling Constraints.

cases only a few iterations are needed for the active-passive classification of members. Moreover, the optimization technique itself — repeated application of the redesign equation (16) — appears to be the most economical method of minimum weight design in use at the present time. Three or four cycles usually yield a structural weight that differs only by a few percent from the true optimal weight, regardless of the size of the structure.

In the preceding discussion it was assumed that the design is determined by a single primary constraint. It is not difficult to revise the redesign equations for multiple constraints (or several load conditions). Equation (16) will remain valid but now

$$C_i = \alpha - \frac{1-\alpha}{\rho_i L_i} \sum_{p\ act} \lambda_p Q_{pi} \quad , \tag{22}$$

where the sum is taken over all the constraints that are active at the optimal design. The requirement that $q_r^{(\nu+1)} = q_r^*$, where r applies to active constraints only, yields a set of simultaneous equations for λ_r :

$$- (1-\alpha) \sum_{i\ act} \left(\frac{Q_{ri} A_i^{(\nu)}}{\rho_i L_i} \sum_{p\ act} \lambda_p Q_{pi} \right)$$

$$= (1-\alpha) \sum_{i\ act} Q_{ri} A_i^{(\nu)} + q_r^* - q_r^{(\nu)} - (\Delta q_r)_{pass} \qquad r = 1, 2, \ldots M \, . \tag{23}$$

The contribution of passive members $(\Delta q_r)_{pass}$ is obtained again from equation (17).

As can be seen, optimization under multiple constraints is considerably more complex than design with respect to a single constraint. The main difficulty is that the active constraints can be identified only by a trial-and-error routine, similar to the one used in separating active and passive elements.

The general idea is to initially assume that all constraints are active and to solve equation (23) for the Lagrangian multipliers. The constraints associated with negative values of λ_r are then considered passive [the multipliers must not be negative; see equation (13)] and the corresponding rows and columns are eliminated from equation (23). The equations are solved again and the process is repeated until all the remaining values of λ_r are positive. Since the identity of passive and active members is also unknown at the beginning, the calculation for the Lagrangian multipliers must be combined with the procedure for classifying the elements.

If only a few constraints are imposed, the method is practical and efficient. It has been successfully applied to design with respect to buckling,[4] where two primary constraints (lower bounds on the first two buckling loads) were employed. Most practical problems, unfortunately, involve numerous constraints and load conditions, so that the multiconstraint optimization method just outlined would be exceedingly expensive. Within the confines of the present state of the art, economically acceptable computer times can be obtained only by resorting to approximate methods. Two of these methods are described in this report in the section, A Selected Survey of Optimization Programs.

Behavior Modification Cycle

It is sometimes necessary to modify the behavior of the structure by means other than the weight minimization operation. The sole function of the behavior modification cycle is to bring the design as close to the active constraint surface as possible within the limitations of the linear approximation (5). Therefore, the criterion in choosing the appropriate redesign formula is the accuracy with which the behavioral changes can be predicted; the resulting change in structural weight is of secondary importance.

The need for a behavior modification cycle stems from the approximate nature of the expression for changes in the behavioral variables (5). Experience has shown that equation (5) is sometimes a very poor approximation if the design change $\Delta \underset{\sim}{A}$ is obtained from the weight minimization operation described in the last section. In certain cases, such as structures on elastic supports,[5] a repeated application of the weight minimization equations may even lead to divergence of th design from the active constraint surface.

---

4. Ibid.

5. Ibid.

In a behavior modification operation the member sizes are changed more or less uniformly, in which case equation (5) has been observed to be a more satisfactory approximation to behavioral changes.

Figure 5 illustrates the application of behavior modification. A band of acceptable, near-critical designs is defined by

$$q_r^* - \epsilon_1 < q_r < q_r^* + \epsilon_2$$

where ϵ_1 and ϵ_2 are predetermined constraints. Whenever the current design is outside the acceptable band, a behavior modification cycle is used to bring the design closer to the active constraint surface. The weight minimization formula is used only on designs that lie inside the acceptable band.

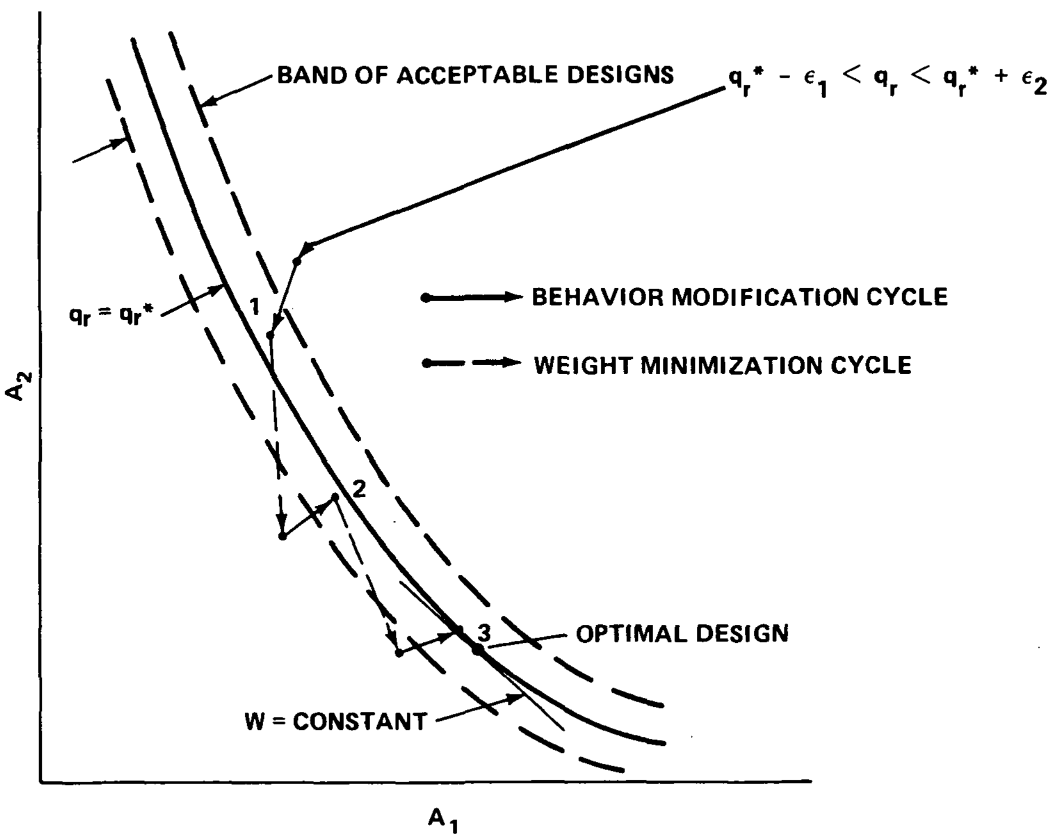


Figure 5. Example of a sequence of acceptable designs produced by behavior modification and weight reduction operations.

This method of design has two advantages over the repeated application of the weight minimization cycle alone. The first and most important of these is the "stabilizing" effect that the behavior modification has on convergence of the design procedure. The use of behavior modification can not only turn a divergent process into a convergent one but will also improve convergence in general. Second, the sequence of acceptable designs (points 1, 2, and 3 in Fig. 5) is useful in monitoring the design process. A weight comparison of successive designs can be used to terminate the design process whenever the weight reduction becomes small or ceases altogether. This method of termination is particularly valuable when approximate, multiconstraint optimization techniques are used which converge to a point other than the true minimum weight design.

Uniform scaling is perhaps the most efficient means of obtaining a near-critical design since it yields a good prediction of behavioral changes. The sizes of all active members are scaled by the same factor μ, i.e., by the operation

$$A_i^{(\nu+1)} = \mu A_i^{(\nu)} \quad . \tag{24}$$

The passive members can be accounted for by using equation (16) as the scaling equations, with

$$C_i = \mu \quad . \tag{25a}$$

The value of μ is computed in the same manner as λ_r, namely from the requirement $q_r^{(\nu+1)} = q_r^*$. The result is

$$\mu = \frac{q_r^* - q_r^{(\nu)} - (\Delta q_r)_{pass} + \sum_{i\ act} Q_{ri} A_i^{(\nu)}}{\sum_{i\ act} Q_{ri} A_i^{(\nu)}} \quad . \tag{25b}$$

In all other respects the scaling operation is identical to the iterative weight reduction cycle described previously in the subsection, Weight Reduction Cycle.

In multiconstraint design problems a value of μ should be evaluated for each constraint and the largest value used as the scale factor. In practice it is sufficient to confine the calculations to a few constraints, the choice being based on the magnitude of the ratio

$$R_r = \frac{q_r^{(\nu)} - q_r^*}{q_r^*} \quad . \tag{26}$$

The constraints yielding the largest values of R_r are more likely to determine the critical design.

Another behavior modification operation that has been used is the gradient travel mode [8, 9, 10]. The idea is to bring the design to the active constraint surface with a smaller change in weight than in uniform scaling. In gradient travel the weight change of each active member is proportional to its effectiveness in changing the constraint variable:

$$\Delta W_i = \mu \frac{\partial q_r}{\partial W_i} \quad , \tag{27}$$

where μ is the constant of proportionality. Substituting

$$\Delta W_i = \rho_i L_i \left(A_i^{(\nu+1)} - A_i^{(\nu)} \right)$$

and

$$\partial q_r / \partial W_i = Q_{ri} / (\rho_i L_i) \quad ,$$

we can rewrite equation (27) in the form

$$A_i^{(\nu+1)} = C_i A_i^{(\nu)} \quad ,$$

where

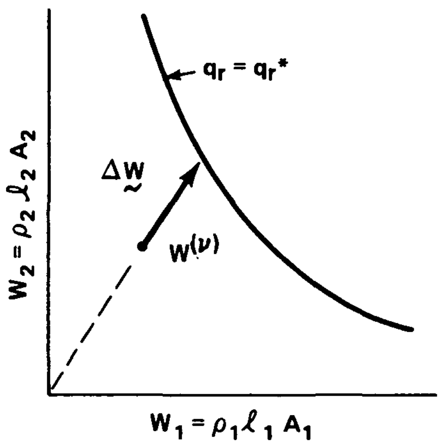
$$C_i = 1 + \mu \frac{Q_{ri}}{(\rho_i L_i)^2 A_i^{(\nu)}} \quad . \tag{28a}$$

As in the case of uniform scaling, we can include passive members in the gradient travel mode by adopting equations (16) as the redesign formulas. The scale factors C_i would, of course, be computed from equation (28a). The requirement $q_r^{(\nu+1)} = q_r^*$ yields the value of μ :

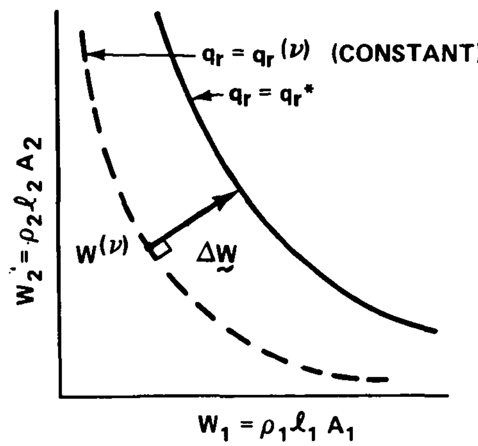
$$\mu = \frac{q_r^* - q_r^{(\nu)} - (\Delta q_r)_{pass}}{\sum\limits_{i\ act} \left(\frac{Q_{ri}}{\rho_i L_i} \right)^2} \quad . \tag{28b}$$

The gradient travel mode represents, within the linear approximation, the shortest distance in the design space between the point $\underset{\sim}{W}^{(\nu)}$ and the active portion of the $q_r = q_r^*$ surface. In the absence of passive members the redesign vector is normal to the q_r = constant surface at $\underset{\sim}{W}^{(\nu)}$, as shown in Figure 6b. In comparison, the redesign vector used in uniform scaling passes through the origin of the design space (Fig. 6a).

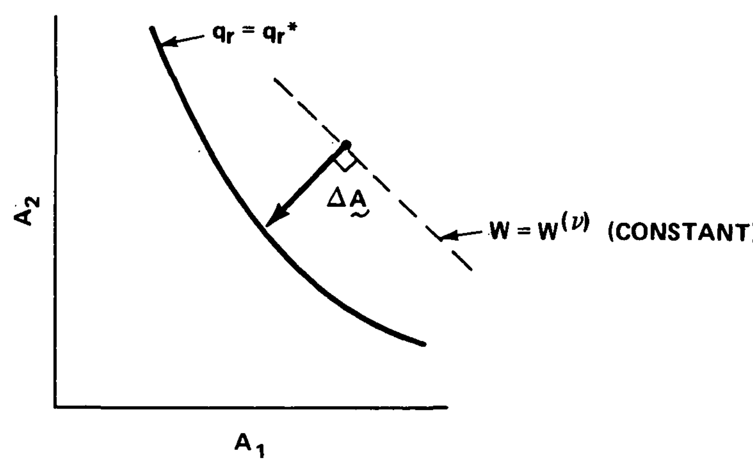
The gradient travel operation does make sense only if the behavior modification cycle requires an increase in stiffness (and in structural weight). If a decrease in stiffness is required, the gradient travel mode conflicts with the design objective because it reduces the stiffness with a minimal saving of structural weight.



a. Uniform scaling.



b. Gradient travel.



c. Weight gradient travel.

Figure 6. Examples of behavior modification operations.

A behavior modification operation that is better suited for reducing structural stiffness is the weight gradient travel operation [10],

$$\Delta A_i = \mu \frac{\partial W}{\partial A_i} = \mu \rho_i L_i \;, \tag{29}$$

where i applies to the active members only. Using $\Delta A_i = A_i^{(\nu + 1)} - A_i^{(\nu)}$, equation (29) becomes

$$A_i^{(\nu + 1)} = C_i A_i^{(\nu)}$$

where

$$C_i = \mu \frac{\rho_i L_i}{A_i^{(\nu)}} - 1 \quad . \tag{30a}$$

The expression for μ, obtained again from $q_r^{(\nu + 1)} = q_r^*$, is

$$\mu = \frac{q_r^* - q_r^{(\nu)} - (\Delta q_r)_{pass}}{\sum\limits_{i\ act} \rho_i L_i Q_{ri}} \quad . \tag{30b}$$

The weight gradient travel mode in the design space is shown in Figure 6c. The design change vector is normal to the constant weight contour, the direction of steepest descent on the weight surface.

GRADIENTS OF CONSTRAINT VARIABLES

General Considerations

The key to indirect optimization (as well as direct optimization) lies in the computation of the constraint variable gradients. The calculations are, with the exception of stress constraints, straightforward and economically feasible, provided that the relationships between the member sizes and the element stiffness matrices are specified in advance.

We denote the stiffness matrix of the i^{th} element by $[K_i]$ and the total stiffness matrix of the structure by $[K]$. The generalized displacement vectors of the i^{th} element and the structure are written as $\{u_i\}$ and $\{u\}$, respectively. The rules for compiling $[K]$ from the element stiffness are based, as usual, on the invariance of the total strain energy U:

$$U = \frac{1}{2}\{u\}^T[K]\{u\} = \frac{1}{2}\sum_i \{u_i\}^T[K_i]\{u_i\} + U_S \quad ,$$

where the sum applies to all the elements in the structure and U_S denotes the strain energy of elastic supports.

To account for equal area constraints, we introduce the group stiffness matrix $[K_g]$ and the group displacement vector $\{u_g\}$, defined by

$$U_g = \{u_g\}^T[K_g]\{u_g\} = \sum_{i \text{ group } g} \{u_i\}^T[K_i]\{u_i\} \quad ,$$

where the sum is taken over the elements belonging to the g^{th} equal size group; that is, elements that must have the same size A_g. It is assumed that all the elements in any one group have an identical size-stiffness relation.

Henceforth, no distinction will be made between problems with equal size constraints and those without the constraints. All the formulas that follow are valid for both types of problems, provided that A_i, $\{u_i\}$, $[K_i]$, etc., are interpreted as belonging to the i^{th} group if equal size constraints are applied. In other words, all members belonging to an equal size group are simply lumped together into an equivalent element for redesign purposes.

Development of the general theory requires no a priori knowledge of the exact form of the element size-stiffness relations. It is sufficient to assume that the derivatives of the stiffness matrices, namely $\partial [K_i] / \partial A_i$, are calculable. On the other hand, the size-stiffness relations play an important role in some programming details, particularly in the storage of the element stiffness matrices.

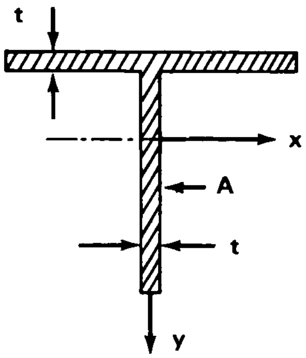
The great majority of design problems can be handled by using element stiffness matrices of the form

$$[K_i] = \sum_{m=0}^{3} \left[k_i^{(m)} \right] A_i^{m} , \qquad (31)$$

where the unit stiffness matrices $\left[k_i^{(m)} \right]$ are independent of the element size A_i. The stiffness due to the nonvariable portion of the element is represented by $\left[k_i^{(0)} \right]$ and the direct stresses contribute to $\left[k_i^{(1)} \right] A_i$; the bending and twisting stiffnesses are included in $\left[k_i^{(1)} \right] A_i$, $\left[k_i^{(2)} \right] A_i^2$, or $\left[k_i^{(3)} \right] A_i^3$, depending on the type of the element. Three samples of size-stiffness relations are shown in Figure 7.

The derivatives of element stiffness matrices are

$$[K_{i,i}] = \frac{\partial [K_i]}{\partial A_i} = \sum_{m=1}^{3} m \left[k_i^{(m)} \right] A_i^{m-1} . \qquad (32)$$



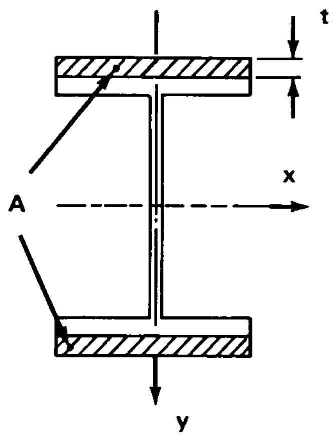
CHANGE IN GAGE THICKNESS t ONLY:

$I_x = c_1 A$ $\quad\quad I_y = c_2 A$ $\quad\quad J_2 = c_3 A^3$

PROPORTIONAL CHANGE IN ALL DIMENSIONS:

$I_x = c_4 A^2$ $\quad\quad I_y = c_5 A^2$ $\quad\quad J = c_6 A^2$

a.

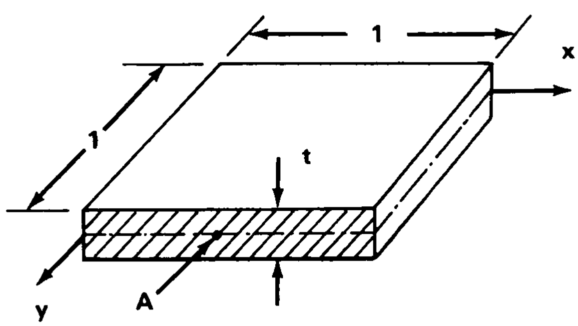


CHANGE IN THICKNESS OF REINFORCEMENT t ONLY:

$I_x = c_0 + c_1 A$ $\quad\quad I_y = c_2 + c_3 A$

$$J = c_4 + c_5 A + c_6 A^2 + c_7 A^3$$

b.



CHANGE IN PLATE THICKNESS t:

$I_x = I_y = c_1 A^3$ $\quad\quad J = c_2 A^3$

c.

Figure 7. Examples of size-stiffness relations.

Since both $[K_i]$ and $[K_{i,i}]$ are used once in each design cycle (the stiffness matrix is required in analysis and its derivative in the redesign cycle), it is necessary to store $\left[k_i^{(m)}\right]$, $m = 0, 1, 2, 3$ separately for each element. This would require a modification of conventional analysis programs if they are to be used in an optimization algorithm.

A special case is obtained when

$$[K_i] = \left[k_i^{(m)}\right] A_i^m \quad . \tag{33}$$

The size-stiffness relation in equation (33) makes the uniform scaling operation particularly simple if all the members are active (no size constraints). If μ is the uniform scale factor, such that $A_i^{(\nu+1)} = \mu A_i^{(\nu)}$, then

$$\{f\}^{(\nu+1)} = \{f\}^{(\nu)} \quad ,$$

$$\{u\}^{(\nu+1)} = \{u\}^{(\nu)}/\mu^m \quad ,$$

and (34)

$$p_r^{(\nu+1)} = \mu^m p_r^{(\nu)} \quad ,$$

where $\{f\}$, $\{u\}$, and p represent generalized forces, displacements, and buckling loads, respectively (it is assumed that applied loads do not depend on member sizes). Consequently, no reanalysis is required after a uniform scaling cycle.

The linear size-stiffness relationship $(m=1)$ is of particular significance in aerospace structures. Shear panels, membrane elements, and thin-walled beams with variable gage thickness (Fig. 7a) fall into this category.

Uniform scaling, together with equations (34), has been extensively used in aerospace-oriented optimization programs [2, 4, 7, 11, 12, 13].

Although the simple uniform scaling procedure (scaling of all members) appears to be very attractive, its value is somewhat dubious because the operation is inconsistent with constraints on member sizes. The selective scaling procedure in equation (16), based on active-passive member categories, seems to be preferable.

Stress Constraints

Rigorous design with stress constraints is still an unresolved problem in the optimization of large structures. The approximate methods that have been used, and are described in this report, fall somewhat outside the general optimization theory developed in the preceding sections.

Because of notational complexities, it is difficult to present the ideas in any generality. They are best displayed for the simplest class of problems — the design of trusses.

We consider first the constraints on the tensile stresses,

$$\sigma_r \leq \sigma^*_{r\,\text{tens}} \quad .$$

The last inequality is identical to equation (1) if we set $\sigma_r = q_r$ and $\sigma^*_{r\,\text{tens}} = q^*_r$. For members of a truss,

$$\sigma_r = \frac{P_r}{A_r} \quad ,$$

where P_r is the axial force in member r . Consequently,

$$Q_{ri} = \left(\frac{\partial \sigma_r}{\partial A_i}\right)^{(\nu)} = - \frac{P_r^{(\nu)}}{\left(A_r^{(\nu)}\right)^2} \delta_{ri} + \frac{1}{A_r^{(\nu)}} \left(\frac{\partial P_r}{\partial A_i}\right)^{(\nu)} \qquad (35)$$

with δ_{ri} denoting the Kronecker delta.

It is a common practice to neglect the last term in equation (35), i.e., to assume that the change in member forces during redesign is negligible. This assumption leads to an enormous simplification of the design algorithm, apart from avoiding the difficulties of computing $\partial P_r / \partial A_i$.

The advantages of the simplified formula, which can be written

$$Q_{ri} = - \frac{\sigma_r^{(\nu)}}{A_r^{(\nu)}} \delta_{ri} \quad , \tag{36}$$

are most apparent in multiconstraint design problems. It can be verified by an inspection of equation (23) that the equations for the Lagrangian multipliers λ_p become decoupled upon substitution of equation (36). As a result each stress constraint and load condition combination can be handled as a single-constraint problem. The largest member size predicted by all the load conditions is chosen as the new design $A_i^{(\nu + 1)}$.

Substituting equation (36) in the equations of the weight reduction cycle, we obtain from equation (21),

$$\lambda_r = \frac{(2 - \alpha)\, \sigma_r^{(\nu)} - \sigma^*_{r\,\mathrm{tens}}}{(1 - \alpha)\left(\sigma_r^{(\nu)}\right)^2} \rho_r L_r A_r^{(\nu)} \quad .$$

Equation (15) then gives the redesign scale factor for the r^{th} element:

$$C_r = \alpha + \frac{(2 - \alpha)\, \sigma_r^{(\nu)} - \sigma^*_{r\,\mathrm{tens}}}{\sigma_r^{(\nu)}} \qquad \text{if} \quad \sigma_r^{(\nu)} > 0 \quad . \tag{37a}$$

Constraints on compressive stresses

$$\sigma_r \geq - \sigma^*_{r\,comp}$$

can be handled by setting $q_r = - \sigma_r$ in equations (21) and (15). The result is

$$C_r = \alpha + \frac{(2 - \alpha)\,\sigma_r^{(\nu)} + \sigma^*_{r\,comp}}{\sigma_r^{(\nu)}} \quad \text{if} \quad \sigma_r^{(\nu)} < 0 \quad . \tag{37b}$$

As mentioned before, if there are several load conditions, a value of C_r is computed for each condition and the largest value is chosen as the scale factor in the redesign equations:

$$A_i^{(\nu + 1)} = \begin{cases} C_i A_i^{(\nu)} & \text{if} \quad C_i A_i^{(\nu)} \geq (A_i)_{min} \\ (A_i)_{min} & \text{if} \quad C_i A_i^{(\nu)} < (A_i)_{min} \end{cases} \quad . \tag{38}$$

The statement $A_i^{(\nu + 1)} = (A_i)_{max}$ if $C_i A_i^{(\nu)} > (A_i)_{max}$ has been omitted from equation (38) since the simplified design equations are obviously incapable of handling upper limits on the design variables. This is one of the major drawbacks of the approximate method.

It can be seen from equations (37a) and (37b) that no change in the member size occurs ($C_r = 1$), when $\sigma_r^{(\nu)} = \sigma^*_{r\,tens}$ or $\sigma_r^{(\nu)} = - \sigma^*_{r\,comp}$. Therefore, the final design is known as the fully stressed design, where each element is stressed to the allowable value under at least one load condition or is governed by the minimum size constraint.

The fully stressed design always coincides with the minimum weight design for statically determinate structures. In the case of static indeterminacy, it is generally just an approximation to the optimal weight configuration. The error (weight penalty) is claimed to be small in most practical design problems but this contention has not yet been proved conclusively. The fact is that special problems can easily be conceived where the discrepancy between the minimum weight and the fully stressed designs is significant.

Equations (37a) and (37b) have actually not been used in the existing optimization programs. The fully stressed design is traditionally obtained more directly by the stress ratio redesign method [2, 4, 7]. The assumption that member forces do not change with redesign is equivalent to the equations,

$$\sigma_r^{(\nu)} A_r^{(\nu)} = \sigma^*_{r\,\mathrm{tens}} A_r^{(\nu+1)} \qquad \text{if} \qquad \sigma_r^{(\nu)} > 0$$

and

$$-\sigma_r^{(\nu)} A_r^{(\nu)} = \sigma^*_{r\,\mathrm{comp}} A_r^{(\nu+1)} \qquad \text{if} \qquad \sigma_r^{(\nu)} < 0 \quad .$$

Solving the equations for $A_r^{(\nu+1)}$ yields the scale factors to be used in the redesign formulas (38):

$$C_r = \begin{cases} \sigma_r^{(\nu)} / \sigma^*_{r\,\mathrm{tens}} & \text{if} \quad \sigma_r^{(\nu)} > 0 \\ -\sigma_r^{(\nu)} / \sigma^*_{r\,\mathrm{comp}} & \text{if} \quad \sigma_r^{(\nu)} < 0 \quad . \end{cases} \tag{39}$$

No benefit has been obtained by introducing a relaxation factor into the stress ratio redesign equations [14].

For a general finite element the stress vector at point P can be obtained from the nodal forces $\{f_i\}$ by the operation

$$\left\{ \sigma_i^P \right\} = \left[S_i^P \right] \{ f_i \} \quad , \tag{40}$$

where the matrix $\left[S_i^P \right]$ depends on A_i and the location of P . The stress constraint

$$\sigma_{i\ \mathrm{eff}} \leq \sigma_i^* \tag{41}$$

is imposed on the effective stress in the member, the latter being given by

$$\sigma_{i\ \mathrm{eff}} = \max_P \ \mathfrak{F}\left(\left\{ \sigma_i^P \right\} \right) \quad . \tag{42}$$

The operation $\mathfrak{F}$ is determined by the failure criterion and is independent of A_i and the location of P .

For example, if the Von Mises yield criterion in two dimensions is used,

$$\sigma_{i\ \mathrm{eff}} = \max_P \sqrt{\left(\sigma_i^P\right)_x^2 + \left(\sigma_i^P\right)_y^2 - \left(\sigma_i^P\right)_x \left(\sigma_i^P\right)_y + 3\left(\tau_i^P\right)_{xy}^2} \quad .$$

The stress ratio redesign is usable only for the special case

$$\left[S_i^P \right] = \left[s_i^P \right] / A_i^{\beta} \quad , \tag{43}$$

where $\left[s_i^P \right]$ does not depend on the member size. It is readily verified that to obtain $\sigma_{i\ \mathrm{eff}}^{(\nu + 1)} = \sigma_i^*$, we must use

$$C_i = \left(\frac{\sigma_{i\,\mathrm{eff}}^{(\nu)}}{\sigma_i^*} \right)^{\frac{1}{\beta}} \tag{44}$$

in the redesign equation (38). All of the elements used in References 2, 4, 7, 11, and 12, apart from having linear size-stiffness relations, are of the type described by equation (43), with $\beta = 1$. The only exception is the plate element used in Reference 4.

Stress-constrained design with more general elements appears to be a very difficult problem and thus far no entirely satisfactory methods have been proposed. Certain elements with a relatively simple relationship between the effective stress and the nodal forces, such as plates, and circular thin-walled tubes with a fixed t/r ratio seem to be manageable (we assume combined action of direct forces, bending, and torsion). However, these elements must be designed by equations similar to (37a), (37b), and (38) since the stress ratio method would not be applicable. The alternative is to use specially tailored failure criteria, as has been done in Reference 4.

Apart from the failure criterion $\sigma_{i\,\mathrm{eff}} \leq \sigma_i^*$, an optimization algorithm should also provide for local buckling constraints. Again, one can make use of the assumption that nodal forces are unchanged during redesign. With the nodal forces known, the minimum element size required to prevent it from buckling can be computed provided, of course, that the buckling design data for the elements are available [4]. This size is then compared with $(A_i)_{min}$ and the largest value used as the lower bound in the design with respect to stress. The major difficulty of the method lies in providing buckling design data for each element.

Before leaving the subject of fully stressed design, another flaw of the method should be mentioned. The finite element, stiffness method generally does not yield stress fields that are continuous between elements, i.e., they do not satisfy equilibrium conditions exactly. As a result the member sizes of the final design reflect these discontinuities and may produce a weight distribution that has an intuitively "wrong" appearance.

The problem is solved in Reference 4 by a stress smoothing procedure, called the nodal stress method. Roughly speaking, the method redistributes the nodal forces predicted by the finite element analysis in such a manner that the net force acting on each node vanishes. A bonus of the nodal stress method is a considerably faster convergence of successive designs to the fully stressed state.

The difficulties with abnormal weight distribution have been tackled in References 11 and 12 by an energy approach. The scale factor in the redesign equations (38) is chosen as

$$C_i = \lambda \sqrt{U_i^{(\nu)} / U_i^*} \quad , \tag{45}$$

where U_i is the strain energy of the i^{th} member (the maximum energy produced by any one load system is used if multiple load conditions exist) and U_i^* is called the strain energy capacity of the element. The latter is defined as the energy stored in the member at failure, assuming a uniform strain field (constant tension in bars, constant bending in beams, etc.). The constant λ is the same for all elements and is adjusted so as to bring the design $A_i^{(\nu + 1)}$ to the active constraint surface.

If the structure is composed of constant strain members, such as bars and membrane elements, the energy approach is identical to the stress ratio method ($\lambda = 1$ would be used). A comparison of the results produced by the two methods is not available for more complex elements.

Displacement Constraints

A displacement constraint can be expressed as

$$u_r \leq u_r^*$$

where u_r^* is the prescribed upper limit on the generalized displacement u_r. The above inequality has the same form as the standard behavioral constraint (1). Since the displacement gradients can be calculated in a straightforward manner, the design algorithm can be constructed directly upon the general theory developed in this report.

The most economical way of evaluating displacement gradients is based on the dummy load method. To obtain the derivatives of the generalized displacement component u_r, we first place a unit dummy load on the structure in the direction of the r-coordinate. Denoting the dummy load vector by $\{f^{(r)}\}$, where $f_j^{(r)} = \delta_{jr}$, and the resulting displacement vector by $\{u^{(r)}\}$, equilibrium equations of the structure under the dummy load are

$$[K]\{u^{(r)}\} = \{f^{(r)}\} \quad . \tag{46}$$

Multiplying both sides of the equation by $\{u\}^T$, where $\{u\}$ is the displacement vector due to real loads, we obtain

$$\{u\}^T [K]\{u^{(r)}\} = u_r \quad . \tag{47}$$

The right-hand side of equation (47) was obtained by

$$\{u\}^T\{f^{(r)}\} = \sum_i u_i \delta_{ir} = u_r \quad .$$

Differentiation of equation (47) yields

$$\frac{\partial u_r}{\partial A_i} = \{u\}^T \left[K_{,i}\right]\{u^{(r)}\} + \{u_{,i}\}^T [K]\{u^{(r)}\} + \{u\}^T [K]\{u_{,i}^{(r)}\} \quad , \tag{48}$$

where we used the notation $\{u_{,i}\} = \partial\{u\}/\partial A_i$. Equation (48) can be simplified considerably. Differentiating both sides of equation (46), we get

$$[K]\{u_{,i}^{(r)}\} = -\left[K_{,i}\right]\{u^{(r)}\} \quad .$$

Similarly, differentiation of the equilibrium equations due to real loads, $[K]\{u\} = \{f\}$ or $\{u\}^T[K] = \{f\}^T$, yields, assuming $\{f\}$ to be independent of A_i,

$$\{u_{,i}\}^T[K] = -\{u\}^T[K_{,i}] \quad .$$

Consequently, equation (48) becomes

$$\frac{\partial u_r}{\partial A_i} = -\{u\}^T[K_{,i}]\{u^{(r)}\} \quad .$$

Only the stiffness matrix of the i^{th} element contributes to $[K_{,i}]$, i.e. $[K_{,i}] = [K_{i,i}]$.

Therefore,

$$Q_{ri} = \frac{\partial u_r}{\partial A_i} = -\{u_i\}^T[K_{i,i}]\left\{u_i^{(r)}\right\} \quad . \qquad (49)$$

References 2 and 7, in which the dummy load approach was first used, imply that equation (49) is an approximation, valid only when the changes in internal forces are negligible during redesign. The equation is, in fact, an exact expression for the displacement gradients within the framework of the finite element theory.

The displacements $\{u\}$ and $\{u^{(r)}\}$ can be calculated simultaneously during the analysis of the structure by adding $\{f^{(r)}\}$ to the matrix of real loads. The extra cost of computation would be relatively small.

The special size-stiffness relationship (33) produces an interesting result. Because

$$[K_i] = \left[k_i^{(m)}\right] A_i^{m} \qquad ,$$

we have

$$[K_{i,i}] = m\,[K_i]/A_i \qquad .$$

Therefore, equation (49) takes the form

$$Q_{ri} = -\,m\,U_i^{(r)}/A_i \qquad ,$$

where

$$U_i^{(r)} = \{u_i\}^T [K_i] \left\{u_i^{(r)}\right\}$$

may be called the "dummy energy" of the i^{th} element — the work done by the real internal forces as they undergo the displacements of the dummy load. The optimality criterion (12) now becomes for active members

$$m \sum_r \lambda_r U_i^{(r)} / W_i = 1 \qquad ,$$

where

$$W_i = \rho_i L_i A_i$$

is the weight of the i^{th} element. For a single constraint, the criterion is

$$U_i^{(r)}/W_i = c ,$$

i.e., the dummy energy density must be constant throughout the structure. The value of c is determined by the allowable displacement u_r^* .

Buckling Constraints

We assume that all the loads acting on the structure can be considered to be proportional to a single parameter p . The values of p at buckling are denoted by p_r and are presumed to be arranged in an ascending order:

$$p_1 \leq p_2 \leq p_3 \ldots \quad .$$

If the structure is to be safe against buckling, the constraint conditions are

$$p_r \geq p^* ,$$

where p^* is the desired value of p at buckling. The inequality can be brought to the standard form of equation (1) by multiplying both sides by minus one and setting $q_r = -p_r$, $q_r^* = -p^*$.

At casual glance it may appear that the design could be based only on p_1 , the fundamental buckling load. This approach is indeed adequate if $p_1 < p_2$ at the optimal design. It can be shown,[6] however, that the minimum weight structure may possess two fundamental buckling modes $(p_1 = p_2)$, which requires the use of a multiconstraint design approach. (There is a slight possibility of having more than two fundamental modes at the optimal design.)

---

6. Ibid.

The buckling problem is governed by the incremental equilibrium equation

$$[K]\{u\} = p\,[H]\{u\} \quad , \tag{50}$$

where $\{u\}$ is the vector of generalized buckling displacements, and $[H]$ represents the geometric stiffness matrix of the structure. The latter is symmetric and is assumed to be independent of the member sizes.[7]

Differentiating both sides of equation (50) with respect to A_i , we obtain

$$[K_{,i}]\{u\} + [K]\{u_{,i}\} = \frac{\partial p}{\partial A_i}[H]\{u\} + p[H]\{u_{,i}\} \quad . \tag{51}$$

Multiplying equation (50) from the left by $\{u_{,i}\}^T$ and equation (51) by $\{u\}^T$ and then subtracting (50) from (51) yields

$$\{u\}^T[K_{,i}]\{u\} = \frac{\partial p}{\partial A_i}\{u\}^T[H]\{u\} \quad .$$

Finally, upon substituting $[K_{,i}] = [K_{i,i}]$, we obtain

$$Q_{ri} = -\frac{\partial p_r}{\partial A_i} = -\frac{\left\{u_i^{(r)}\right\}^T[K_{i,i}]\left\{u_i^{(r)}\right\}}{\left\{u^{(r)}\right\}^T[H]\left\{u^{(r)}\right\}} \quad . \tag{52}$$

---

7. The assumption is strictly valid only when the prebuckling state is statically determinate. For the case of static indeterminacy, the assumption is a convenient approximation; optimal design can be obtained by recomputing the forces in the prebuckling state after each redesign cycle.

The superscript (r) signifies that the displacements of the r^{th} buckling mode are to be used in the equation. There is a striking similarity between equations (52) and (49), which is accentuated when the special size-stiffness relations (33) are used. With the substitutions

$$U_i^{(r)} = \left\{u_i^{(r)}\right\}^T [K_i] \left\{u_i^{(r)}\right\} \quad ,$$

$$G^{(r)} = \{u^{(r)}\}^T [H] \{u^{(r)}\} \quad ,$$

the optimality criterion becomes

$$m \sum_r \left(\lambda_r / G^{(r)}\right)\left(U_i^{(r)} / W_i\right) = 1$$

for the multiconstraint case and $U_i^{(r)}/W_i = c$ for a single constraint. As in the design for displacement constraints, the last equation also requires a uniform energy density — in the present case the strain energy of buckling — throughout the structure.

Natural Frequency Constraints

Constraints on natural frequencies are handled in essentially the same manner as buckling constraints. We introduce $p_r = \omega_r^2$, $p_1 \leq p_2 \leq p_3 \ldots$, where $\omega_r/(2\pi)$ are the natural frequencies of the structure. It is assumed that the design objective is to eliminate all frequencies below a certain value ω^* . Consequently, the behavioral constraints are, as they were for buckling,

$$p_r \geq p^* \quad .$$

Again, the standard form of the behavioral inequality is obtained with $q_r = -p_r$ and $q_r^* = -p^*$.

The importance of optimizing the design with respect to at least two modes, which was discussed in the preceding subsection, is also applicable to frequency-constrained design.

The free vibration equation is an eigenvalue problem of the same form as equation (50):

$$[K]\{u\} = p[M]\{u\} \quad , \tag{53}$$

where $\{u\}$ represents the buckling mode and $[M]$ is the mass matrix. If the rotary inertia is neglected, the element mass matrixes can be written as

$$[M_i] = \left[m_i^{(0)}\right] + \left[m_i^{(1)}\right] A_i \quad , \tag{54}$$

where $\left[m_i^{(k)}\right]$ are independent of A_i. In addition to the contribution of the individual elements (54), $[M]$ is also allowed to contain nonstructural inertia terms (due to masses attached to the structure).

The gradients of p_r are obtained in a manner identical to the method used for buckling constraints, and the derivations are not repeated here (one must not forget, however, that the derivatives of $[M]$ are nonzero). The result is

$$Q_{ri} = -\frac{\partial p_r}{\partial A_i}$$

$$= -\frac{\left\{u_i^{(r)}\right\}^T [K_{i,i}]\left\{u_i^{(r)}\right\} - p_r\left\{u_i^{(r)}\right\}^T [M_{i,i}]\left\{u_i^{(r)}\right\}}{\{u^{(r)}\}^T [M]\{u^{(r)}\}} \quad . \tag{55}$$

For the special case $[K_i] = \left[k_i^{(m)}\right] A_i^m$ and $[M_i] = \left[m_i^{(1)}\right] A_i$ we introduce the potential and kinetic energies of the i^{th} element,

$$U_i^{(r)} = \left\{u_i^{(r)}\right\}^T [K_i] \left\{u_i^{(r)}\right\}$$

and

$$T_i^{(r)} = p_r \left\{u_i^{(r)}\right\}^T [M_i] \left\{u_i^{(r)}\right\} \quad ,$$

respectively, and the kinetic energy of the entire structure

$$T^{(r)} = p_r \{u^{(r)}\}^T [M] \{u^{(r)}\} \quad .$$

Equation (55) then is

$$Q_{ri} = -\left(p_r / T^{(r)}\right) \left(m\, U_i^{(r)} - T_i^{(r)}\right) \Big/ A_i^{(\nu)}$$

and the optimality criterion (12) becomes for the active members,

$$\sum_r \left(\lambda_r p_r / T^{(r)}\right) \left(m U_i^{(r)} - T_i^{(r)}\right) \Big/ W_i = 1 \quad .$$

If a single constraint is used, this reduces to

$$\left(m U_i^{(r)} - T_i^{(r)}\right) \Big/ W_i = c \quad .$$

Before leaving the topic, we should point out a peculiarity of the uniform scaling operation. Assuming the special size-stiffness relation just described and an absence of nonstructural inertia terms in $[M]$, the uniform scaling operation, when applied to all members, would result in

$$[K]^{(\nu+1)} = \mu^{m}[K]^{(\nu)} \quad ,$$

$$[M]^{(\nu+1)} = \mu[M]^{(\nu)} \quad ,$$

where μ is the scaling factor. The vibration equation (53) for the new design would be

$$[K]^{(\nu)}\{u\}^{(\nu+1)} = \left(p^{(\nu+1)}\Big/\mu^{m-1}\right)[M]^{(\nu)}\{u\}^{(\nu+1)} \quad .$$

A comparison with the equation of the previous design,

$$[K]^{(\nu)}\{u\}^{(\nu)} = p^{(\nu)}[M]^{(\nu)}\{u\}^{(\nu)} \quad ,$$

leads us to the conclusion that

$$u^{(\nu+1)} = u^{(\nu)}$$

and

$$p^{(\nu+1)} = \mu^{m-1}p^{(\nu)} \quad .$$

We note that when $m = 1$, no change in the natural frequencies will occur. Consequently, the uniform scaling operation will be ineffective in modifying the behavior of the structure if the size-stiffness relations are linear unless the terms that are independent of A_i dominate the mass matrix.

Flutter Velocity Constraints

Design with respect to flutter velocity completes the trio of eigenvalue-constrained problems, the other two being designs with respect to buckling and natural frequency constraints. Denoting the flutter speeds of the structure by V_r, $V_1 \leq V_2 \leq V_3 \ldots$, and the desired lowest flutter velocity by V^*, the constraint conditions are

$$V_r \geq V^* \quad .$$

Therefore, $q_r = -V_r$ and $q_r^* = -V^*$.

The equations governing steady-state motions of the structure can be written in the form [10],

$$[K]\{u\} = p\,([M] + [A])\{u\} \tag{56}$$

where $[A]$ is the air force matrix and $p = \omega^2$, $\omega_r/2\pi$ being the frequency of flutter oscillations. The air force matrix is complex, asymmetric, and a function of the reduced frequency

$$k = \frac{b\omega}{V} \quad , \tag{57}$$

b being the semichord length. The exact form of $[A]$ depends on the aerodynamic theory used.

Multiplying both sides of equation (57) by V and differentiating, we obtain

$$k\frac{\partial V}{\partial A_i} = b\frac{\partial \omega}{\partial A_i} - V\frac{\partial k}{\partial A_i} = \frac{b}{2\omega}\frac{\partial p}{\partial A_i} - \frac{b\omega}{k}\frac{\partial k}{\partial A_i} \quad ,$$

from which

$$Q_{ri} = -\frac{\partial V_r}{\partial A_i} = -\frac{b}{2\omega_r k_r}\frac{\partial p_r}{\partial A_i} + \frac{b\omega_r}{k_r^2}\frac{\partial k_r}{\partial A_i} \quad . \tag{58}$$

To obtain $\partial p / \partial A_i$, we differentiate both sides of (56):

$$\left[K,_i\right]\{u\} + [K]\left\{u,_i\right\} = \frac{\partial p}{\partial A_i}\Big([M] + [A]\Big)\{u\} + p\Big([M,_i] + [A,_i]\Big)\{u\}$$
$$+ p\Big([M] + [A]\Big)\left\{u,_i\right\} \quad . \tag{59}$$

The next step is to eliminate the derivative of the eigenvector $\{u_{,i}\}$. Since $[A]$ is asymmetric, we need the help of the associated eigenvector $\{v\}$, given by the solution of

$$\{v\}^T[K] = p\{v\}^T\Big([M] + [A]\Big) \quad . \tag{60}$$

It can be shown that the eigenvalues p_r of equations (56) and (60) are identical.

Multiplying equation (59) from the left by $\{v\}^T$ and equation (60) from the right by $\{u_{,i}\}$ and subtracting, we have

$$\{v\}^T[K_{,i}]\{u\} = \frac{\partial p}{\partial A_i}\{v\}^T\Big([M] + [A]\Big)\{u\}$$

$$+ p\{v\}^T\Big([M_{,i}] + [A_{,i}]\Big)\{u\} \quad .$$

Substituting

$$[A_{,i}] = (\partial[A] / \partial k)\ (\partial k / \partial A_i)$$

and using the notation

$$\partial[A] / \partial k = [A'] \quad ,$$

we get

$$\frac{\partial p}{\partial A_i} = \frac{\{v\}^T[K_{,i}]\{u\} - p\{v\}^T\left([M_{,i}] + \dfrac{\partial k}{\partial A_i}[A']\right)\{u\}}{\{v\}^T\Big([M] + [A]\Big)\{u\}} \quad . \tag{61}$$

At this point we recall that if an arbitrary value of V is used in equation (56), the resulting eigenvalues are generally complex. A real eigenvalue p_r, signifying a steady-state motion, can be obtained only when $V = V_r$, $r = 1, 2, 3 \ldots$. Since we are designing with respect to flutter, i.e., steady-state oscillations, we must use $p = p_r$ and $k = k_r$ (both real) in equation (61) and also restrict $\partial p_r / \partial A_i$ and $\partial k_r / \partial A_i$ to real values. The last requirement essentially establishes an interdependence between

$$dp_r = \sum_i (\partial p_r / \partial A_i)\ dA_i$$

and

$$dk_r = \sum_i (\partial k_r / \partial A_i)\, dA_i \quad ,$$

which assures us that $p_r + dp_r$ and $k_r + dk_r$ also correspond to the flutter conditions (are real) for the design $A_i + dA_i$.

Following the technique developed in Reference 10, we separate the terms appearing in equation (61) into real and imaginary parts:

$$\{v^{(r)}\}^T\Big([K_{,i}] - p_r[M_{,i}]\Big)\{u^{(r)}\} = R_1^{(r)} + i\, I_1^{(r)} \quad ,$$

$$p_r\{v^{(r)}\}^T [A']\{u^{(r)}\} = R_2^{(r)} + i\, I_2^{(r)} \quad , \tag{62}$$

and

$$\{v^{(r)}\}^T\Big([M] + [A]\Big)\{u\} = R_3^{(r)} + i\, I_3^{(r)} \quad .$$

With the terms introduced, $\partial p_r / \partial A_i$ can also be divided into real and imaginary portions:

$$\frac{\partial p_r}{\partial A_i} = \frac{\left(R_1^{(r)} - R_2^{(r)} \dfrac{\partial k_r}{\partial A_i}\right) R_3^{(r)} - \left(I_1^{(r)} - I_2^{(r)} \dfrac{\partial k_r}{\partial A_i}\right) I_3^{(r)}}{\left(R_3^{(r)}\right)^2 + \left(I_3^{(r)}\right)^2}$$

$$+\, i\, \frac{\left(I_1^{(r)} - I_2^{(r)} \dfrac{\partial k_r}{\partial A_i}\right) R_3^{(r)} - \left(R_1^{(r)} - R_2^{(r)} \dfrac{\partial k_r}{\partial A_i}\right) I_3^{(r)}}{\left(R_3^{(r)}\right)^2 + \left(I_3^{(r)}\right)^2} \tag{63}$$

The last term in equation (63) must vanish if $\partial p_r / \partial A_i$ is to be real, which yields

$$\frac{\partial k_r}{\partial A_i} = \frac{R_3^{(r)} I_1^{(r)} - R_1^{(r)} I_3^{(r)}}{R_3^{(r)} I_2^{(r)} - R_2^{(r)} I_3^{(r)}} \quad . \tag{64}$$

From the real part of equation (63) we now obtain

$$\frac{\partial p_r}{\partial A_i} = \frac{R_1^{(r)} R_3^{(r)} - I_1^{(r)} I_3^{(r)}}{\left(R_3^{(r)}\right)^2 + \left(I_3^{(r)}\right)^2}$$

$$- \frac{\left(R_2^{(r)} R_3^{(r)} - I_2^{(r)} I_3^{(r)}\right)\left(R_3^{(r)} I_1^{(r)} - R_1^{(r)} I_3^{(r)}\right)}{\left(R_3^{(r)} I_2^{(r)} - R_2^{(r)} I_3^{(r)}\right)\left[\left(R_3^{(r)}\right)^2 + \left(I_3^{(r)}\right)^2\right]} \quad . \tag{65}$$

Substitution of equations (64) and (65) in equation (58) completes the expression for Q_{ri} .

The main difficulty with flutter optimization appears not to be in the redesign but in the analysis — the solution of the flutter equation for a realistic air force matrix $[A]$.

A SELECTED SURVEY OF OPTIMIZATION PROGRAMS

Stress and Displacement Constraints

One of the most troublesome aspects of optimal design is the treatment of multiple constraints other than stress constraints. The difficulties are most acute in displacement-constrained designs since it is not unusual to have a very large number of displacement limits placed on a single problem. As noted before, a rigorous use of optimality criteria is out of the question and approximate techniques must be found.

An effective method for handling stress and displacement constraints has been developed by Gellatly and Berke [2, 7]. In essence, they consider the stress constraints and each displacement constraint-load combination as a separate, autonomous optimization problem. If there are M displacement constraint-load combinations, then M + 1 different values are calculated for each element size during a redesign cycle (one value from stress constraints and M values from displacement constraints). The largest value is selected as the size for each element.

The separation of members into active and passive categories during each redesign cycle is accomplished by successive iterations, as shown in Figure 8. At first all the members are assumed to be active. During subsequent iterations the active members in each of the autonomous problems are limited to elements that were controlled by the same problem in the previous iteration. The procedure is repeated until no change takes place in the active-passive member categories.

Following each redesign cycle, the design is analyzed and scaled uniformly to the active constraint surface. Since only elements with linear size-stiffness relations are used in the program, the scaling operation predicts the corresponding behavioral changes exactly, eliminating the need for further analysis. The design procedure is terminated when the structural weight ceases to decrease between two successive critical designs.

The algorithm of Gellatly and Berke, like all approximate methods, does not converge to the true minimum weight design. The weight penalty has not yet been evaluated.

The redesign equation used during the weight reduction cycle in References 2 and 7 differs somewhat from the formula (15) proposed in this report. For elements with linear size-stiffness relations, equation (15) becomes

$$C_i = \alpha + (1 - \alpha)\lambda_r U_i^{(r)} / W_i \quad .$$

Gellatly and Berke, on the other hand, use

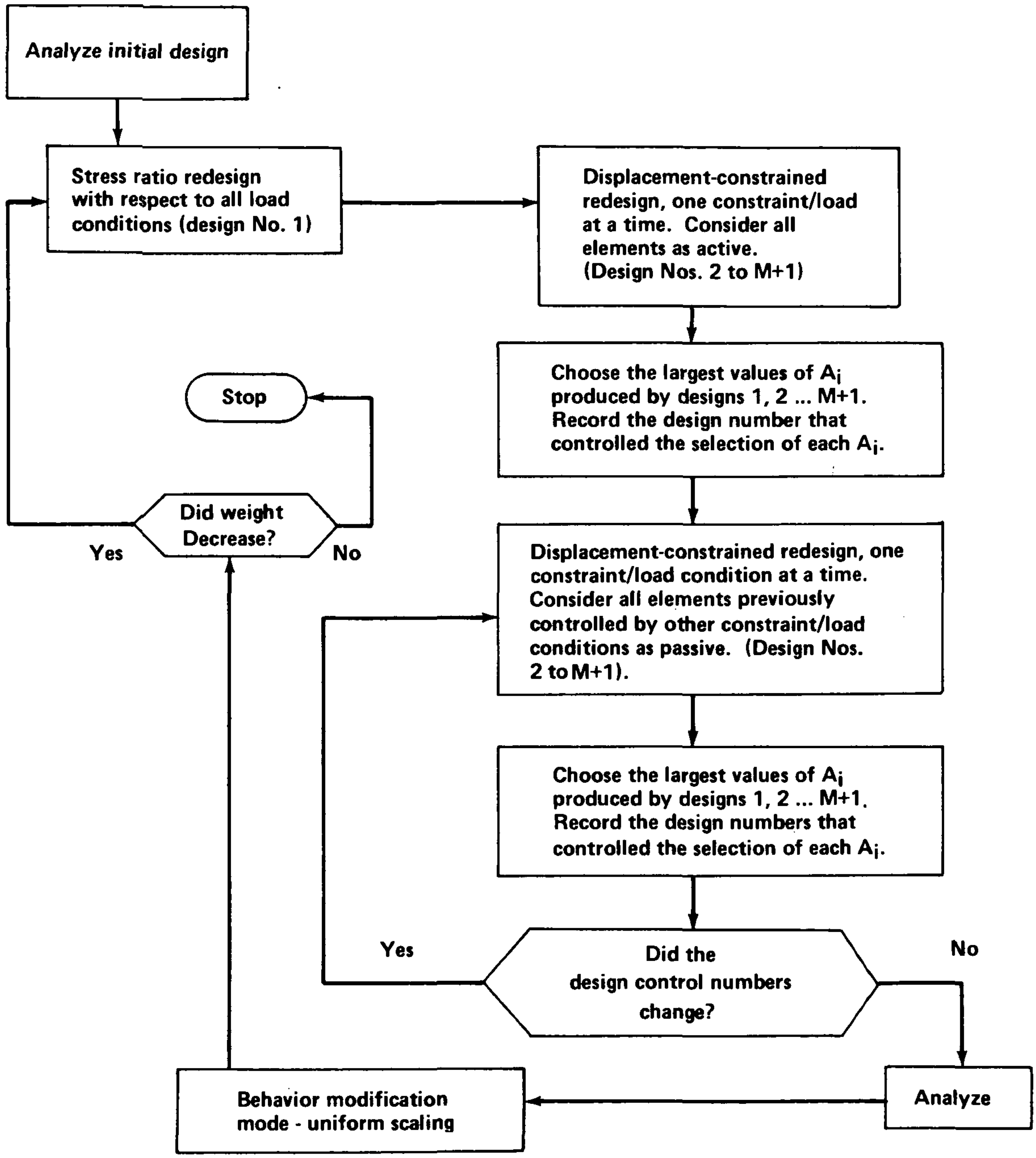


Figure 8. Flow diagram for program in References 2 and 7.

$$C_i = \sqrt{\lambda_r U_i^{(r)} / W_i} \quad , \tag{66}$$

which can be derived from the optimality criterion $\lambda_r U_i^{(r)} / W_i = 1$ upon the assumption that the internal forces do not change during redesign.

It is easily shown that the discrepancy between the two expressions for C_i is negligible for small design changes ΔA_i (i.e., if $C_i \approx 1$). With $C_i \approx 1$, we have

$$\sqrt{\lambda_r U_i^{(r)}/W_i} \approx \frac{1}{2}\left(1 + \lambda_r U_i^{(r)}/W_i\right) \quad ,$$

which coincides with equation (15) if we take $\alpha = \frac{1}{2}$. Since the redesign formulas are strictly applicable for small design changes anyway, neither formula can be considered to be more "accurate" than the other. Equation (15), however, has the advantage of greater flexibility: It is applicable to size-stiffness relations other than linear, it allows control of the convergence rate through the relaxation parameter α, and it can be used in rigorous multiconstraint design (the simultaneous equations to be solved for λ_r would be linear, as opposed to nonlinear equations if the Gellatly-Berke redesign formula were used).

A different approach to multiconstraint design is used by Dwyer et al. [4] (Fig. 9). The first phase of the design algorithm consists of repeated applications of the stress ratio redesign formula and the uniform scaling operation. The displacement constraints are accounted for by computing the uniform scale factor from all the behavioral constraints. Consequently, each design cycle produces a critical (usable) design. This procedure is repeated until the structural weight ceases to decrease.

The second phase is a displacement-constrained design algorithm that is used only when the last critical design was governed by a displacement constraint. The method of redesign used in this phase may be classified as a gradient search procedure, consisting of alternate steps of uniform scaling and gradient travel mode. It is not based on the optimality criterion and is not always effective in reducing weight.

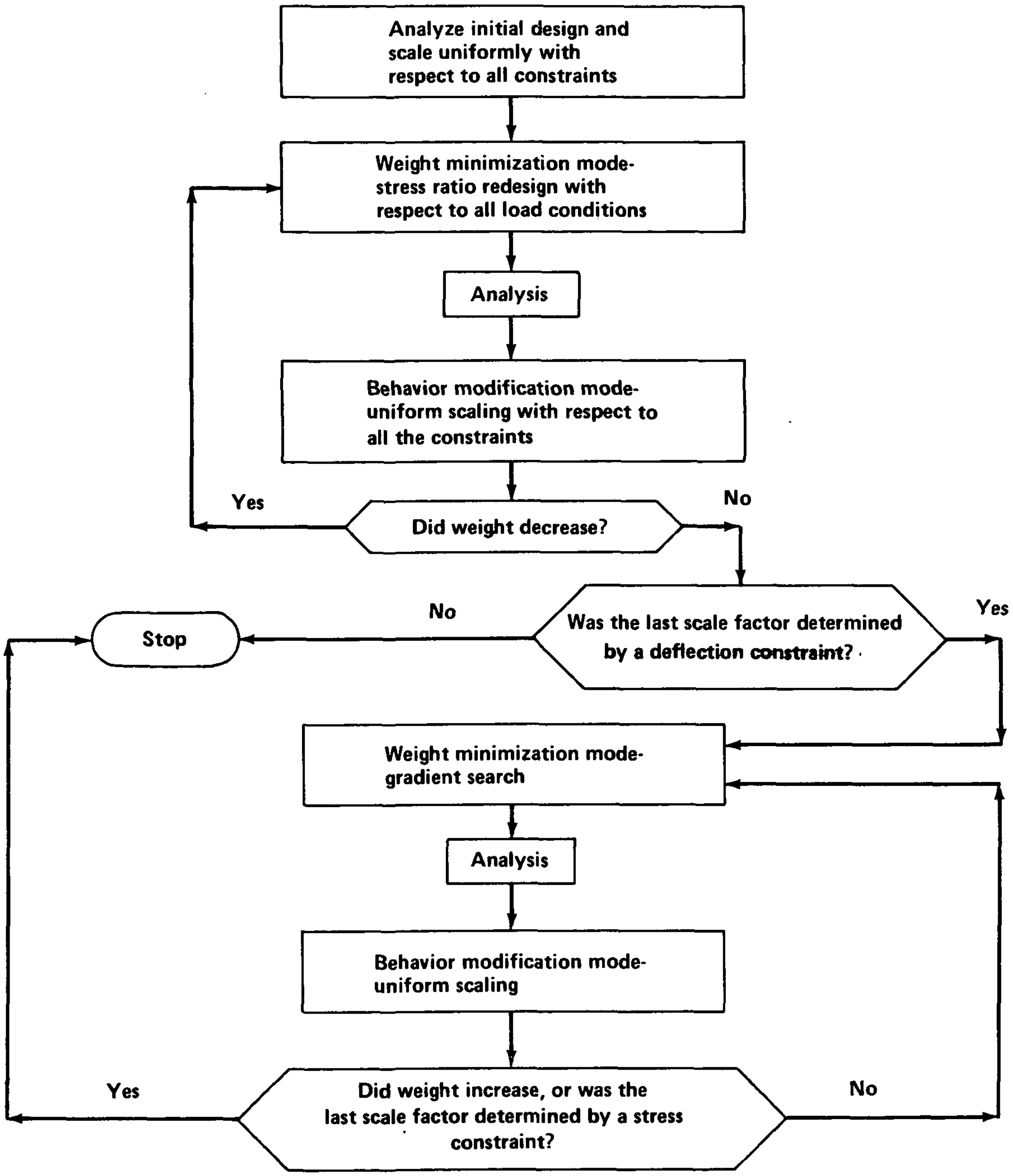


Figure 9. Flow diagram for program in Reference 4.

The optimization program of Venkayya [11, 12] uses essentially the same approach as Reference 4. The differences are minor: Venkayya replaces the stress ratio redesign formula with the energy ratio formula (45) and uses a slightly modified gradient travel mode in the displacement-constrained design phase. The method for computing the displacement gradients is inefficient in Reference 4 as well as in References 11 and 12.

Buckling Constraints

There appear to be no published accounts of buckling-constrained optimization programs. The ideas developed in this report were used by the author to assemble a frame optimization program,[8] which works well on a variety of problems where the prebuckling state is statically determinate. The flow diagram of the program is shown in Figure 10.

The program accepts elements with size-stiffness relations

$$[K_i] = \left[k_i^{(m)}\right] A_i^m \qquad ,$$

where $m = 1, 2$ or 3. In addition, elastic supports (discrete nodal supports or uniformly distributed element supports) are permitted. Minimum permissible element sizes and equal size constraints are also included in the program.

It was found essential to consider the problem as one of multiconstraint design, where the weight reduction cycle takes into account the first two buckling modes simultaneously [see equations (22) and (23)]. Without this feature, the design did not converge in cases where the first two buckling loads were equal at the optimal design, i.e., if the optimal design was located at the intersection of $p_1 = p^*$ and $p_2 = p^*$ constraint surfaces, as indicated in Figure 3a.

The redesign is carried out by either applying the weight reduction equations (22) and (23) or the uniform scaling operation, depending on whether the critical load of the current design lies in the acceptable band or not. In other words, the design process shown in Figure 5 is used. The program is stopped if the critical load is within the acceptable range and the optimality criterion (12) is satisfied within a prescribed latitude.

---

8. Kiusalaas, op. cit.

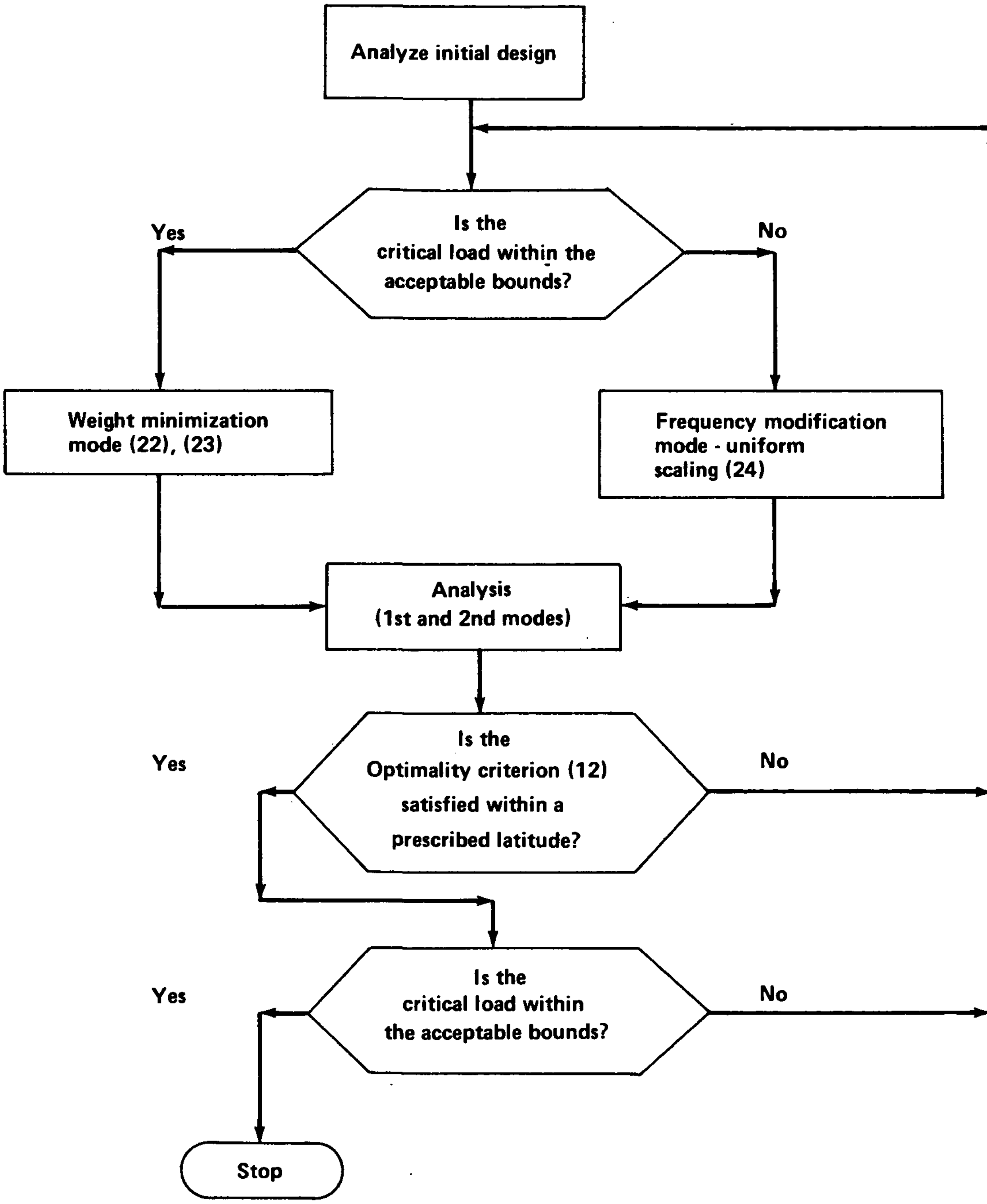


Figure 10. Flow diagram for program referenced in footnote 3.

Experience with the program indicates that the initial rate of convergence is very fast. Three or four redesigns were usually sufficient to produce a structural weight within a percent or two of the true optimal weight. It was noted that in some problems the structural behavior (buckling modes) became very sensitive to small design changes near the optimal point. This caused successive designs to oscillate between two near-optimal points and made further weight reduction impossible unless the relaxation factor was increased close to unity. Fortunately, in all cases tested, the oscillatory behavior occurred only when the design weight was within a percent of the optimal weight.

Natural Frequency Constraints

The first procedure for optimizing a complex-structure with respect to frequency constraints was published by Rubin [8, 9]. The program treats frames and restricts elements to linear size-stiffness reductions but accepts nonstructural masses.

The layout of the program, shown in Figure 11, is somewhat similar to the algorithm just described for buckling-constrained design. Two frequency modification modes are used: uniform scaling if the frequency is to be reduced and gradient travel [equation (28)] if an increase in frequency is required.

The weight minimization mode adopted by Rubin is a numerical search procedure known as gradient projection search. The redesign formula is $\Delta A_i = \mu g_i$, where g_i is chosen so as to maximize the weight loss $-\Delta W$, subject to constraint $\Delta p_1 = 0$ (no change in fundamental frequency). The magnitude of the redesign vector μ is obtained by trial and error.

As was already stated in the introduction, numerical search techniques are generally inferior to indirect design methods that are based on the optimality criteria. This is particularly true for the design of large structures. Another flaw of the program is the single-constraint design approach, which restricts its application to structures where the optimal design is a stationary point on the $p_1 = p^*$ constraint surface (Fig. 3b).

A more recent frequency-constrained optimization program that does make use of the optimality criterion has been published by Venkayya et al. [13]. The elements are again restricted to linear size-stiffness relations.

The weight minimization mode consists of the redesign equation

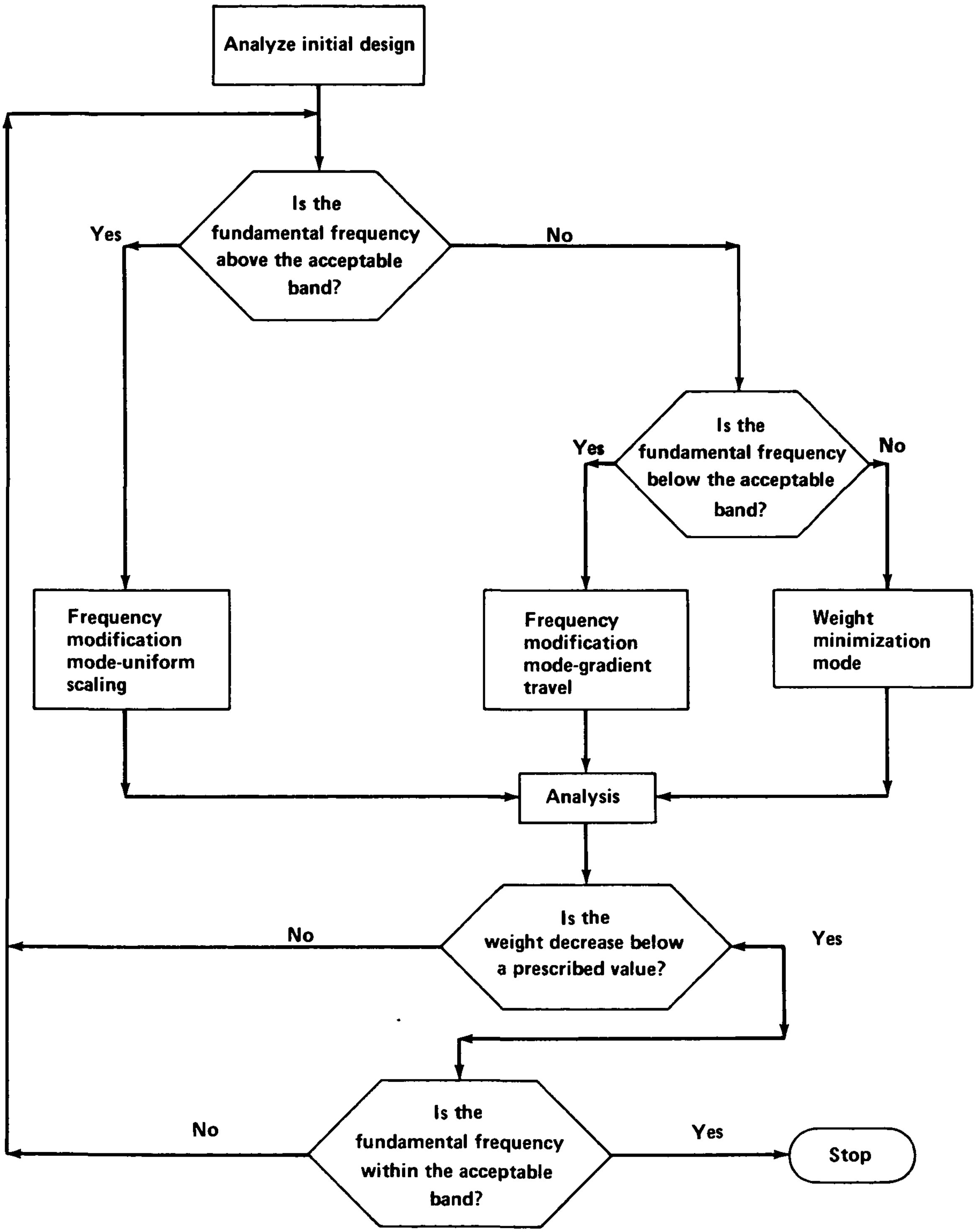


Figure 11. Flow diagram for program in References 8 and 9.

$$A_i^{(\nu+1)} = C_i A_i^{(\nu)} ,$$

where

$$C_i = \sqrt{c \frac{U_i^{(r)} - T_i^{(r)}}{W_i}} . \tag{67}$$

The origin of the formula is the optimality criterion

$$(c/W_i)(U_i^{(r)} - T_i^{(r)}) = 1 .$$

Similarly to equation (66), equation (67) can be shown to coincide with the redesign formula (15) proposed in this paper,

$$C_i = \alpha + (1 - \alpha) c \frac{U_i^{(r)} - T_i^{(r)}}{W_i} ,$$

provided that we take $\alpha = \frac{1}{2}$ and consider small design changes $(C_i \approx 1)$.

The algorithm in Reference 13 does not use a frequency modification mode — the design changes are obtained entirely by repeated application of equation (67). Experience with buckling-constrained design[9] has shown that the use of the weight minimization mode along can lead to a divergent design sequence in certain problems. It appears, therefore, that the absence of a frequency modification mode and the single-constraint approach make the program described in Reference 13 applicable to a limited class of problems only.

Flutter Velocity Constraints

Optimization with respect to flutter has been confined to very simple structural configurations. The most advanced paper published to date, written

9. Ibid.

by Rudisill and Bhatia [10], is rather similar to the frequency-constrained design algorithm used by Rubin [8, 9] and consequently suffers from the same drawbacks. The main contribution of the paper lies in the derivation of the expressions for the flutter velocity gradients.

The weight minimization mode used by Rudisill and Bhatia is also a gradient projection search procedure, where the incremental increase in the flutter speed is maximized while the total weight is held constant. For the behavior modification cycle they use flutter velocity gradient travel [equation (28)] to increase the flutter velocity and weight gradient travel [equation (30)] if a decrease in the velocity is desired.

CONCLUSIONS

There is little doubt that computer-automated, minimum weight design is an eminently practical means of structural design, even in its present stage of development. It is safe to predict that by the end of the next decade most structures, in aerospace as well as civil engineering applications, will be computer designed.

The principles and methods found in the present state of the art appear to be further advanced than their application; that is, none of the published design algorithms make full use of the existing knowledge and experience. Part of the blame must be placed on the high cost of program development — an optimization program requires about twice the programming effort of a corresponding analysis algorithm. In addition, structural optimization is still a peripheral area of structural design, known only to a small group of engineers. Consequently, funding agencies have been reluctant to underwrite the cost of practical (large) structural optimization programs, preferring more traditional areas of structural mechanics.

In view of the present situation, the next few years should be dominated by increased applications of optimal design, rather than new theoretical developments.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama 35812, Aug. 4, 1972
124-12-11-0000

REFERENCES

1. Schmit, L. A. Jr.: Structural Synthesis 1956-1969: A Decade of Progress. Recent Advances in Matrix Methods of Structural Analysis and Design, Gallagher, Yamada, and Oden, eds., University of Alabama Press, 1971.

2. Gellatly, R. A.; Berke, L.; and Gibson, W.: The Use of Optimality Criteria in Automated Structural Design. Third Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, 1971.

3. Prager, W.; and Taylor, J. E.: Problems of Optimal Structural Design. Journal of Applied Mechanics, vol. 35, 1968, pp. 102-106.

4. Dwyer, W. J.; Emerton, R. K.; and Ojalvo, I. U.: An Automated Procedure for the Optimization of Practical Aerospace Structures. Volume I: Theoretical Development and User's Information. Technical Report AFFDL-TR-70-118, April 1971.

5. Chern, J. M.; and Prager, W.: Minimum-Weight Design of Statically Determinate Trusses Subject to Multiple Constraints. International Journal of Solids and Structures, vol. 7, August 1971, pp. 931-940.

6. Prager, W.: Optimal Design of Multi-Purpose Structures. International Journal of Solids and Structures, vol. 4, 1968, pp. 469-475.

7. Gellatly, R. A.; and Berke, L.: Optimal Structural Design. Technical Report AFFDL-TR-70-165, April 1971.

8. Rubin, C. P.: Minimum Weight Design of Complex Structures Subject to a Frequency Constraint. AIAA Journal, vol. 8, May 1970, pp. 923-927.

9. Rubin, C. P.: Dynamic Optimization of Complex Structures. ASME/AIAA Tenth Structures, Structural Dynamics, and Materials Conference, New Orleans, La., 1969.

10. Rudisill, C. S.; and Bhatia, K. G.: Optimization of Complex Structures to Satisfy Flutter Requirements. AIAA Journal, vol. 9, August 1971, pp. 1487-1491.

11. Venkayya, V. B.: Design of Optimum Structures. International Journal of Computers and Structures, vol. 1, 1971, pp. 265-309.

12. Venkayya, V. B.; Khot, N. S.; and Reddy, V. S.: Energy Distribution in an Optimum Structural Design. Technical Report AFFDL-TR-68-156, March 1969.

13. Venkayya, V. B.; Khot, N. S.; Tischler, V. A.; and Taylor, R. F.: Design of Optimum Structures for Dynamic Loads. Third Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio, 1971.

14. Melosh, R. J.: Convergence in Fully-Stressed Design. Symposium on Structural Optimization, Advisory Group for Aerospace Research and Development, (AD 715 483), Paris, France, 1970.

NASA-Langley, 1972 —— 32 M379

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546