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CANONICAL FLUID THERMODYNAMICS

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ABSTRACT

The space-time integral of the thermodynamic pressure plays
the role of the thermodynamic potential for compressible, adiabatic
flow in the sense that the pressure integral for stable flow is less
than for all slightly different flows. This stability criterion can
be converted into a variational minimum principle by requiring the
molar free-enthalpy and the temperature, which are the arguments
of the pressure function, to be generalized velocities, that is, the
proper-time derivatives of scalar spare-time functions which are
generalized coordinates in the canonical formalism. In a fluid
context, proper-time differentiation must be expressed in terms
of three independent quantities that specify the fluid velocity.
This can be done in several ways, all of which lead to different
variants (canonical transformations) of the same constraint-free
action integral whose Euler-Lagrange equations are just the
well-known equations of motion for adiabatic compressible flow.

The variational minimum principle provides a means for
comparing the relative stability of different flows, and indicates
that there is a natural tendency, caused by the action of dif-
fusion and turbulent mixing, for a fluid to settle into potential
flow, if this is consistent with the imposed boundary condi-
tions. When the boundary conditions have a very high degree
of symmetry, as in the case of a uniformly expanding spherical
box of gas, the most stable flow is rectilinear flow for which the
world-trajectory of each particle is a straight line, and its
three-velocity as seen by any observer embedded in the fluid is
proportional to the distance of the particle from the observer.
To the extent that the behavior of the interior regions of a freely-
expanding cosmic cloud can be expected to be similar to that of
a fluid in a uniformly expanding spherical box, this suggests
that the Cosmological Principle is a consequence of the laws of
thermodynamics rather than just an ad hoc postulate.

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CANONICAL FLUID THERMODYNAMICS

I. INTRODUCTION

1.1   Fluid Dynamics as a Derived Consequence of Thermodynamics

Nearly all the formulations of fluid dynamics that have been developed in the past have taken the postulates of particle mechanics-either in differential or variational form-as the starting point.   These postulates have then been modified or supplemented to accommodate the needs of a fluid possessing internal degrees of freedom.   Thermodynamics is often intimately involved in such a procedure, but always as a supplement or modification of something more fundamental, namely Newton's Second Law (conservation of momentum), or one of its generalizations (such as General Relativity), or one of its alternatives (sucn as D'Alembert's or Hamilton's Principles).

There exists an alternative possibility, however, namely that of taking the First and Second Laws of Thermodynamics to be the fundamental postualtes, and then finding a way of incorporating suitable kinematical variables into the thermodynamical formalism in such a way as to give a valid description of the state of motion of the fluid, as well as of its thermodynamic state.   This approach has been almost totally neglected.   There is a historical reason for this: The laws of mechanics were developed long before those of thermodynamics.   Moreover, mechanics was characterized by a conceptual and formal precision that thermodynamics began to approach only with the advent of the twentieth century.   The heuristic and phenomenological origins of thermodynamics were hardly conducive to its being regarded as fundamental in any profound sense.

Aside from these historical reasons for a bias in favor of mechanics over thermodynamics, there was the incontrovertible fact that classical thermodynamics not only had no prescription for determining the time-evolution of a system, but with its concepts of "equilibrium states" and "quasi-static processes" appeared to exclude time and evolution from its universe.

Helmholtz[1] was the first to realize that a marriage of thermodynamics with canonical formalism would provide each partner with exactly what the other lacked and most needed: By identifying a suitable thermodynamic potential as a Lagrangean or Hamiltonian, thermodynamics would immediately acquire the vast and elegant canonical formalism that had been developed for the express purpose of describing the time-evolution of a system.   For its part, the highly abstract canonical formalism could only gain from association with a methodology

that systematized empirical data and cast it into exactly the mathematical form—
a Pfaffian differential—that was so important to the canonical formalism.

For the most part, these ideas remained only implicit and undeveloped in
Helmholtz's work. He could not in any case have carried them much further,
because an indispensible ingredient was missing, namely a fundamental postulate
that would specify the time-dependence of a function given its spatial dependence.
Special Relativity provides just such a postulate by requiring invariance under a
group—the Lorentz group—that, unlike the Galilean group, mixes the time co-
ordinate with the spatial coordinates.

Figure 1 illustrates an intuitive argument that shows how the postulate of
Lorentz invariance can suffice to enable thermodynamics to specify the time-
evolution of a system as well as its static equilibrium state. Figure 1A shows
a closed thermally-isolated box containing movable, rigid, adiabatic diaphragms
that separate the fluid in the box into contiguous, leak-proof compartments of
variable volume. From the point of view of thermodynamics, the positions of
the diaphragms represent unconstrained "internal variables" that will assume
values that minimize the appropriate thermodynamic potential, which in this
case is the total internal energy $\mathbb{U}$. (In sec. 2.4, however, we shall see that
an equally good potential would be $\mathbb{J} = \sum_\lambda P_\lambda V_\lambda$, where P_λ and V_λ are the
pressure and volume respectively of the λ-th compartment, and the summation
extends over all the compartments in the box. The use of $\mathbb{J}$ as a thermodynamic
potential is intimately related to the Principle of Virtual Work, whereas the use
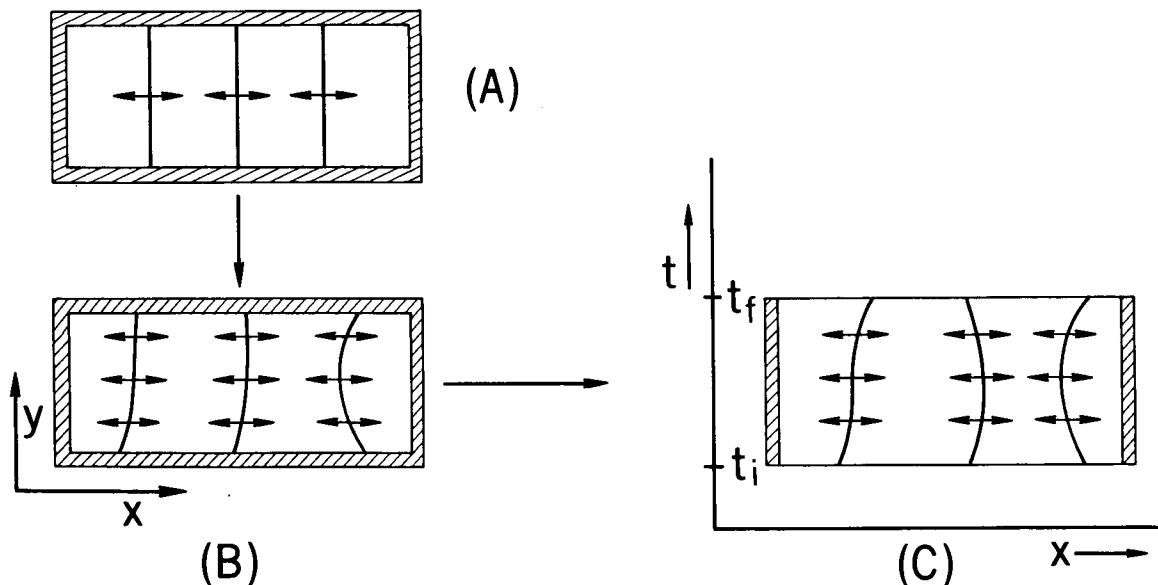of $\mathbb{U}$ is obviously just the Principle of Minimum Energy.

Figure 1. Fluid Motion Determined by Time-Like "Diaphragms"

The same argument holds if the diaphragms are flexible instead of rigid, the only difference being that we now have a non-denumerable infinity of "internal variables" instead of a finite number. This case is illustrated in Figure 1B.

If now we invoke the requirement that, because of Lorentz invariance, the time coordinate must be on an equal footing with any of the spatial coordinates, it must follow that the same argument applies if the diaphragms extend into the t-direction as shown in Figure 1C. But if we were to fill the box with a great number of such flexible diaphragms, the specification of their space-time loci would be equivalent to specifying the fluid flow within the box, and this would have been done by a purely thermodynamical argument without in any way postulating the conservation of momentum.

It is easy to understand how a combination of thermodynamics with relativity could produce a statement of the conservation of momentum as a derived result. Thermodynamics is built on two laws: The First Law is a statement of conservation of energy, and the Second Law provides a stability criterion. In relativity, energy and momentum are simply different components of the same four-vector. Therefore, in a relativistic context, a conservation condition imposed on energy automatically implies momentum conservation. Minkowski[2] had already made this point in 1908 without, however, mentioning thermodynamics. This argument also indicates that the same thermodynamical formalism that yields the equations of motion (which are really just statements of the various necessary conservation laws) can be expected to provide a stability criterion because the Second Law will have been built into the formalism as well as the First Law. The lack of such a built-in stability criterion should be regarded as a deficiency of the standard approach to mechanics via Newton's Laws.

1.2   Kinematic Parameters as Thermodynamic Variables

As the flexible-diaphragm example given in Figure 1 illustrates, the analytical problem of incorporating motion into thermodynamics can be pictured as the problem of suitably parameterizing three imaginary families of surfaces that are embedded in the fluid and swept along with it. This is illustrated in Figure 2 for the case of two spatial coordinates and the corresponding two families of surfaces that are labeled with the parameters λ^1 and λ^2. These surfaces have been defined as having coincided with the spatial Cartesian coordinate mesh at the initial time t_i, but this definition is just an arbitrary choice. Because (in three spatial dimensions) the same particle is always found at the intersection of the three surfaces specified by certain given and constant values of the parameters (λ^1, λ^2, λ^3), the values of these parameters effectively

CONVECTED CELL

λ_C^1 λ_C^2

λ_B^2

R'

λ_A^2

λ_B^1

S'

λ_A^1

Q

P'

λ_C^2

λ_B^2

λ_A^2

R

Q

S

P

y_C

y_B

y_A

x_A

x_B

x_C

$x \longrightarrow$

$y \longrightarrow$

t_i

t_f

$t \uparrow$

t

$\lambda_A^1 = x_A$ AT t_i
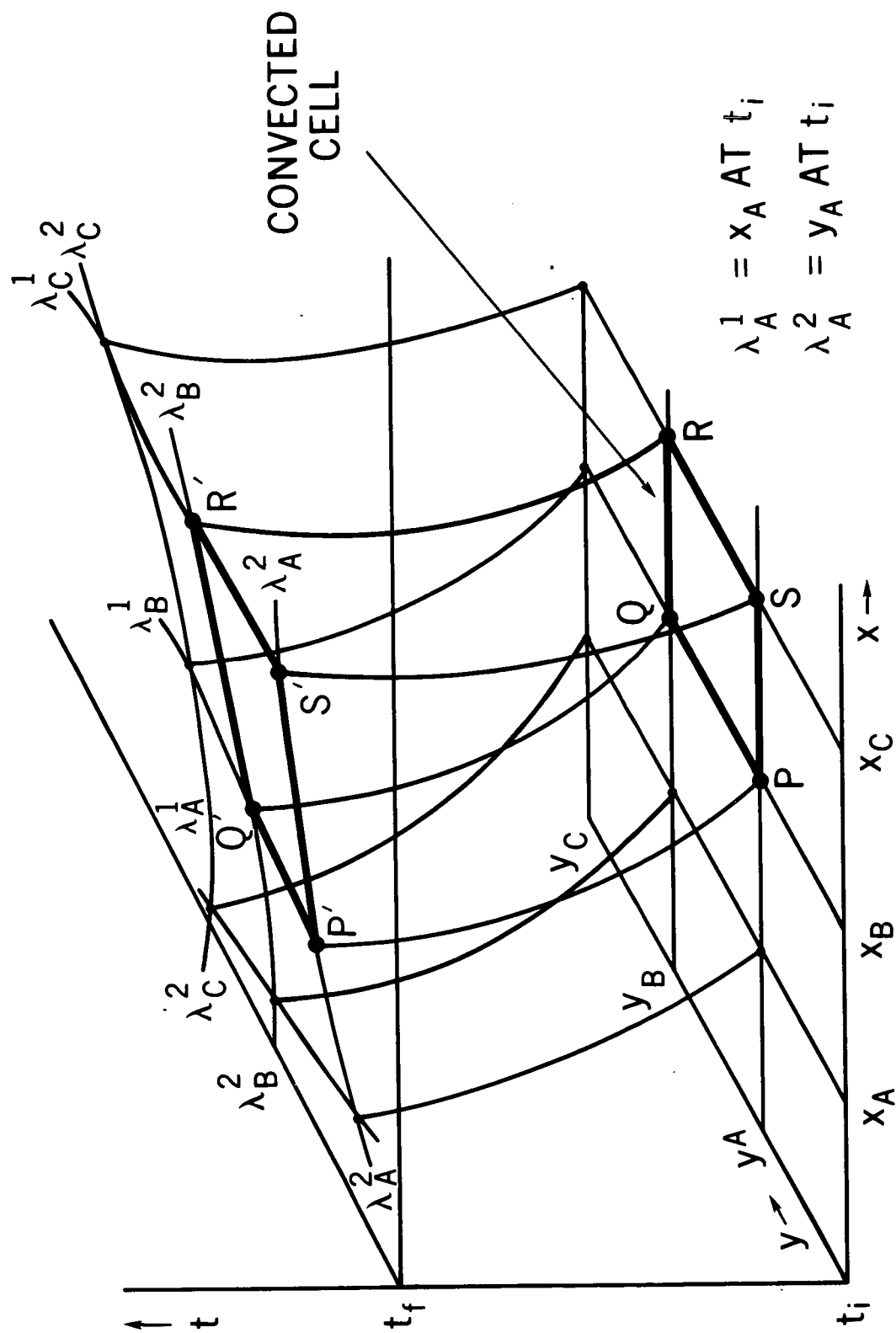
$\lambda_A^2 = y_A$ AT t_i

Figure 2. Converted Particle–Identity Parameter of the Flow

identify the particle, and will be called the identification parameters. The inter-
sections of the convected surfaces define convected parallelepipeds that will be
called the "convected cells". For compressible flow, the volume as well as the
shape of each of these cells changes with time, but the number and identity of the
particles contained within it remain unchanged.

There is a simple two-step procedure for incorporating the identity param-
eters (or, more exactly, their space-time derivatives) into the thermodynamic
formalism: First, each intensive quantity (in particular, the temperature, the
molar free-enthalpy, and the molar enthalpy) must be identified as a generalized
velocity, i.e., the proper-time derivative of a new variable which is to be re-
garded as a generalized coordinate in the sense of the general canonical forma-
lism. Second, proper-time differentiation which has thus been introduced into
the formalism must be defined in terms of the fluid four-velocity, which in turn
is defined in terms of the four-gradients of the identity parameters. This pro-
gram is carried out in sec. 3.6. An alternative procedure is to work within the
curvilinear convected coordinate system defined by the identity parameters, and
regard the position function $\vec{r}$ (t) for each particle as the kinematic functions
to be incorporated into the thermodynamic formalism. This is done in sec. 3.5.

The recognition that intensive quantities should be identified as generalized
velocities can be traced back to Helmholtz[3] who made such an identification for
temperature, but the idea has had a curious history that is briefly sketched in
sec. 2.2. The generalized coordinate corresponding to temperature was called
the "thermasy" by Van Dantzig[4], and extending this nomenclature the generalized
coordinates corresponding to the molar free-enthalpy, and the enthalpy will be
called the "ergasy" and the "enthalpasy" respectively.

1.3   Plan of Presentation

At the beginning of each of the following sections is a résumé of the section.
Besides serving to introduce and summarize each section, these résumés can
be read in sequence   as a detailed abstract of the entire article.

The broad plan of presentation is as follows: In sec. 2 the fundamental
thermodynamic inequality and the Additivity Postulate for thermodynamic po-
tentials are discussed, and two competing minimum principles (Dirichlet's and
Castigliano's), which have been extensively applied to the solution of static con-
tinuum problems, are derived. It is then shown that when the requirement of
relativistic covariance is imposed, only the generalization of Castigliano's
principle (for which the corresponding potential function is the space-time
integral of the pressure), remains acceptable.

In sec. 3 this stability criterion is converted into a variational minimum principle by the device of identifying intensive quantities as generalized velocities. It is shown that a solution of a problem using this variational minimum principle is also a solution of the well-known system of partial differential equations for compressible adiabatic flow.

Section 4 explores insights provided by the variational minimum principle into the question of the relative stability of different categories of adiabatic compressible flow.

A brief literature survey and historical sketch is given in the appendix.

1.4 Notation

The signature used for the Minbowski metric $g_{jk} = g^{jk}$ is

$$(g_{00}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1) . \tag{1.1}$$

The summation convention is used with lower-case Latin indices ranging over (0, 1, 2, 3) for four-vector and four-tensor components and lower-case Greek indices ranging over (1, 2, 3) for the three scalar identity parameters. Lower-case Gothic indices ranging over all the thermodynamic and kinematic degrees of freedom may be regarded as vector indices in configuration space.

Bold-face symbols designate four-vectors, and three vectors are indicated by overhead arrows.

The time-like and space-like components of a four-vector (usually in contra-variant form) are displayed in the following manner as illustrated for the position vector $\underset{\sim}{x}$, the velocity vector $\underset{\sim}{V}$ and the gradient operator $\underset{\sim}{\partial}$:

$$\underset{\sim}{x} = (x^{j}) = (ct, \vec{x}) \tag{1.2}$$

$$\underset{\sim}{V} = (V^{j}) = \Gamma(c, \vec{v}) ; \quad \Gamma \equiv (1 - \beta^{2})^{-1/2} , \quad \beta \equiv v/c \tag{1.3}$$

$$\underset{\sim}{\partial} = (\partial^{j}) = (\partial_{0}, -\vec{\nabla}) ; \quad (\partial_{j}) = (\partial_{0}, \vec{\nabla}) ; \quad \partial_{0} \equiv \frac{\partial}{\partial ct} \tag{1.4}$$

The inner product of two four-vectors or two three-vectors is indicated by a dot. Thus the derivative with respect to proper time τ is defined as

$$\frac{d}{d\tau} \equiv \underset{\sim}{\hat{V}} \cdot \underset{\sim}{\partial} \equiv g_{jk} V^{j} \partial^{k} = \Gamma \frac{d}{dt} \tag{1.5}$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$.

An overhead caret indicates a normalized quantity. Thus $\hat{\underset{\sim}{V}} = \underset{\sim}{V}/c$ is the four-velocity normalized to unity, and $\hat{\wedge}$ is a volume normalized to one role, i. e., $\hat{\wedge} = 1/n$ is the molar volume, where n is the molar density.

Extensive thermodynamic quantities of an entire system are indicated by enhanced (but light-faced) capitals. Thus $\mathbb{U}$, $\mathbb{N}$, and $\mathbb{S}$ are respectively the total internal energy, total mole number, and total entropy respectively. Intensive thermodynamic quantities, including molar amounts of extensive quantities, are indicated by simple light-faced capitals. Thus, for example, T, G, H, and S are respectively the temperature, molar free-enthaply (or molar Gibbs function), molar enthalpy, and molar entropy. Densities of extensive quantities are designated by lower-case light-face Latin letters. Thus u, n, and s are respectively internal energy density, mole-number density, and entropy density.

The scalar functions that can be used to describe a mapping in four-space are designated by lower-case, light-face Greek letters. Thus λ^{α} (α = 1, 2, 3) are the three identity parameters, and θ, γ and η are respectively the thermasy, the ergasy, and the enthalpasy.

II. COVARIANT STABILITY CRITERION FOR A FLUID

Résumé

In sec. 2.1 it is shown that the internal energy density and the pressure are dual thermodynamic potentials in the sense that the arguments of one are equal to the respective partial derivatives of the other, and that the two potential functions are actually just different representations of the same body of information. In sec. 2.3 the convexity property of both potential functions is explained in terms of stability under fluctuations.

In sec. 2.2 the idea that intensive quantities should be regarded as generalized velocities is explained, and a brief historical sketch of the development of this idea is given.

In sec. 2.4 the usual postulate of the additivity of extensive thermodynamic potentials is invoked in a three-dimensional context to arrive at the two competing, but actually equivalent, minimum principles that historically have been the most important analytical tools for the treatment of static continuum problems such as that presented by an elastic medium.

In sec. 2.5 it is shown that, when the requirement of relativistic covariance is imposed, these two stability criteria for a continuum are no longer equivalent, in the sense that while one is in complete harmony with the requirements of relativity, the other would require that as non-invariant quantity serve as a stability criterion, which is obviously unacceptable.

2.1 Energy Density and Pressure as Dual Thermodynamic Potentials

The essence of the thermodynamic-potential approach developed mainly by Gibbs, Helmholtz, and Planck[5] is that one is not condemned to work with the non-integrable Pfaffian forms that had characterized the older approach that had concentrated on thermodynamic processes (i.e., infinitesimal changes in a system). Rather it was shown that, if all of the relevant degrees of freedom are included in the analysis, there exists a potential function capable of describing all possible processes. This is the total energy $\mathbb{U}$ of the system. (The total negative entropy ("negentropy") $-\mathbb{S}$ can also be used as the fundamental potential function, giving rise to the so-called entropy representation of thermodynamics. The formalism developed in this article, however, will be based on the energy representation which takes $\mathbb{U}$ to be the fundamental potential.)

It is vital that $\mathbb{U}$ be regarded as a function it its "natural variables":
$\mathbb{N} \equiv$ total mole number, $\mathbb{S} \equiv$ total entropy, and $\mathbb{V} \equiv$ total volume. Other sets of arguments for $\mathbb{U}$ are in fact possible, and the change from one set of variables to another is the analog of a canonical transformation in mechanics.[6] The range of admissible sets of variables is, however, far more restricted than would appear from the process-approach to thermodynamics.

If an equilibrium state of the system corresponds to the numerical values $\bar{\mathbb{N}}, \bar{\mathbb{S}}, \bar{\mathbb{V}}$, and $\bar{\mathbb{U}} \equiv \mathbb{U}(\bar{\mathbb{N}}, \bar{\mathbb{S}}, \bar{\mathbb{V}})$ then the $\mathbb{U}$ corresponding to neighboring values of $\mathbb{N}$, $\mathbb{S}$, and $\mathbb{V}$ is given by

$$\mathbb{U} = \bar{\mathbb{U}} + d^{(1)}\mathbb{U} + d^{(2)}\mathbb{U} \tag{2-1a}$$

where

$$d^{(1)}\mathbb{U} = \bar{G}\, d\mathbb{N} + \bar{T}\, d\mathbb{S} - \bar{P}\, d\mathbb{V} \qquad \text{(def. of } \bar{G},\, \bar{T},\, \bar{P}) \tag{2-1b}$$

and

$$d^{(2)}\mathbb{U} \equiv \mathbb{U} - \bar{\mathbb{U}} - d^{(1)}\mathbb{U} \geqslant 0\ . \qquad \text{(stability condition)} \tag{2-1c}$$

As implied by (2-1c), the symbol $d^{(2)}\mathbb{U}$ represents the second and <u>higher-order</u> terms in the expansion for $\mathbb{U} - \bar{\mathbb{U}}$.

In the thermodynamic-potential approach, the first partial derivatives with respect to $\mathbb{N}$, $\mathbb{S}$, and $\mathbb{V}$ define (as functions of these variables) the equilibrium values of the respective intensive quantitives $\tilde{G} \equiv$ molar Gibbs function, or molar free-enthalpy, $\bar{T} \equiv$ temperature, and $\bar{P} \equiv$ pressure. Eq. (2-1b) is, in fact, the statement of the First Law of Thermodynamics (conservation of energy) as it applies to infinitesimal displacements from the equilibrium state.

The inequality (2-1c) is the condition that characterizes a stable state, and is equivalent to the statement

$$d\mathbb{U} \geqslant d\mathbb{Q} + d\mathbb{W} \tag{2-2a}$$

where

$$d\mathbb{Q} \equiv \bar{T}\, d\mathbb{S} \qquad \text{and} \qquad d\mathbb{W} \equiv \bar{G}\, d\mathbb{N} - \bar{P}\, d\mathbb{V} \tag{2-2b}$$

which simply says that any displacement from a stable state is characterized by an increase in energy $d\mathbb{U}$ that exceeds the increase that would equal the sum of the heat energy $d\mathbb{Q}$ and ordered energy $\bar{G}d\mathbb{N}$ introduced into the system, and the mechanical work $(-\bar{P}d\mathbb{V})$ done on the system. That is, a further source of

$\mathcal{B}$

energy is necessary to produce the displacement. This is one of the many ways of stating the Second Law of Thermodynamics. Thus all of the necessary thermodynamic information is summarized in (2-1).

The form of the thermodynamic inequality that has greatest relevance to the variational formalism developed below results from substituting (2-1b) into (2-1c) and making the replacement $d\mathbb{N} = \mathbb{N} - \bar{\mathbb{N}}$, etc:

$$d^{(2)}\mathbb{U} \equiv \mathbb{U} - \bar{\mathbb{U}} - \bar{G}(\mathbb{N} - \bar{\mathbb{N}}) - \bar{T}(\mathbb{S} - \bar{\mathbb{S}}) + \bar{P}(\mathbb{V} - \bar{\mathbb{V}}) \geqslant 0. \tag{2-3}$$

There is a redundancy implicit in the use of both $\mathbb{N}$ and $\mathbb{V}$ as independent variables. These are scale factors,[7] either one of which would suffice to specify the system. This redundancy is reflected in the fact that the relation

$$\mathbb{U} = G\mathbb{N} + T\mathbb{S} - P\mathbb{V} \equiv (\partial\mathbb{U}/\partial\mathbb{N})\mathbb{N} + (\partial\mathbb{U}/\partial\mathbb{S})\mathbb{S} + (\partial\mathbb{U}/\partial\mathbb{V})\mathbb{V} \tag{2-4}$$

is identically satisfied. This is the Euler identity for a function that is homogeneous of degree one.

We shall be especially interested in the formalism that results from the specification $\mathbb{V} = 1$, because then the extensive quantities $\mathbb{U}$, $\mathbb{N}$, and $\mathbb{S}$ go over into the corresponding densities u, n and s, and (2-1b) becomes

$$du = G\,dn + T\,ds \tag{2-5}$$

where the overhead bars have been dropped to emphasize that this relation holds for all n and s if we regard G and T to be functions G (n, s) and T (n, s) rather than fixed values. Dividing (2-4) by $\mathbb{V}$ we obtain

$$u = Gn + Ts - P \quad, \tag{2-6}$$

and making use of this identity (which can now be regarded as the definition of P), (2-5) can be converted into its dual form

$$dP = n\,dG + s\,dT \tag{2-7}$$

(This equation is more commonly written in the form dG = (1/n)dP - SdT, where S = s/n is the molar entropy, and is known as the Gibbs-Duhem equation.)

The dual nature of the functions u(n, s) and P(G, T) results from the interchange of the intensive and extensive quantities. In fact, they are simply two different ways of representing the fundamental potential function u, which are illustrated in Figure 3.
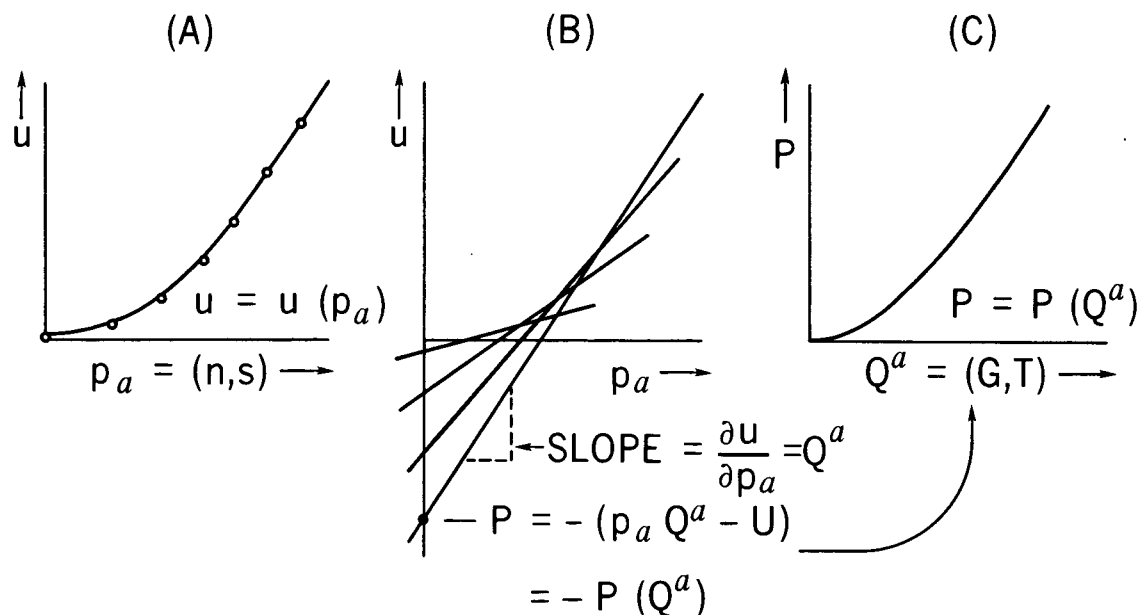
Figure 3. Extensive and Intensive Representations of Thermodynamic Potential Function

In case A of Figure 3, u is represented as a connected "string of points", whereas in case B it is represented as the envelope of tangent planes. Each such tangent plane is characterized by an intercept with the u-axis (at the point $-P$ on this axis) which is a definite function of the components (G, T) of the gradient of u in the (n, s) space. In Figure 3c the (negative) intercept is plotted as a function of the gradient components (G, T). (In Figure 3 the extensive densities (n, s) and the intensive quantities (G, T) have been represented collectively by p_a and Q^α respectively, where a is a two-value index. This notation can be generalized to any number of degrees of freedom.)

If, instead of choosing $\mathbb{V} = 1$, we were to choose $\mathbb{V} = \mathbb{V}_c$ where $\mathbb{V}_c$ is a constant volume not subject to change, then (2-6) — (2-7) would remain unchanged except that (u, n, s) would go over into $(\mathbb{U}, \mathbb{N}, \mathbb{S})$, and P would go over into $\mathbb{J} = P\mathbb{V}_c$. The value of the extensive quantity $\mathbb{J}$ that corresponds to $\mathbb{V}_c = 1$ is the density $j \equiv P$. That is, P is an intensive quantity, but the same function multiplied by unit volume, which is "invisible" in the formalism, becomes an energy density. To avoid conceptual ambiguity, either the symbol j or the symbol P will be used, according to the meaning intended.

The energy density j is just the work that would be required to blow up a balloon to unit volume in the presence of the surrounding fluid, assuming that the final volume of such a balloon is negligibly small compared to the total volume occupied by the fluid. This guarantees that the ambient pressure remains constant during the expansion of the balloon. Thus j is just the "interaction

potential energy" of a sample of fluid occupying unit volume, in the sense that if we could contrive to surround the sample with a "rigid balloon", we could first extract the intrinsic energy u from the sample. At the end of this process, the sample would be at zero temperature and would exert zero pressure (if we ignore quantum-mechanical zero-point vibrations). We could then contrive with the help of appropriate mechanisms to allow the balloon to diminish in volume in a reversible manner, whereby we would obtain (from the surrounding fluid) an additional amount of work equal to j. Thus

$$\text{total energy density} = u + j = n\,(U+P/n) = nH \qquad (2\text{-}8)$$

where U is the molar internal energy, $1/n$ is the molar volume, and H is the molar enthalpy.

It will be understood throughout that U, G, and H all include the cold molar rest-energy Mc^2. Thus, in a fluid context, H is to be regarded as the total energy (intrinsic plus interaction) of one mole of fluid, and by the relativistic mass-energy equivalence, (H/c^2) is the corresponding total molar mass.

2.2 Intensive Quantities as Generalized Velocities

The idea of identifying intensive quantities in thermodynamics as <u>generalized velocities</u> in analogy with the fact that velocities in mechanics are intensive quantities independent of the quantity of matter goes back to Helmholtz[3]. This simple identification immediately brings thermodynamics within the framework of the standard canonical formalism. Thus if we make the identifications

$$(G,\,T) \equiv Q^\alpha \equiv \dot{q}^\alpha \equiv dq^\alpha/d\tau \qquad \text{(generalized velocities)} \quad (2\text{-}9a)$$

and

$$(n,\,s) \equiv p_\alpha\,, \qquad \text{(generalized momentum densities)} \quad (2\text{-}9b)$$

then the thermodynamic relations (2-5) — (2-7) assume the form

$$du = \dot{q}^\alpha d\,p_\alpha \qquad\qquad u(p_\alpha) \equiv \text{Hamiltonian density} \quad (2\text{-}10a)$$

$$dP = p_\alpha\, d\,\dot{q}^\alpha \qquad\qquad P(\dot{q}^\alpha) \equiv \text{Lagrangean density} \quad (2\text{-}10b)$$

$$u = \dot{q}_\alpha\, p^\alpha - P\,. \qquad\qquad (2\text{-}10c)$$

It is easy to see that the canonical equations of motion arising from this identification are

$$\dot{n} = \dot{s} = 0 .$$

(2-10c)

That is, the quiescent equilibrium state of a homogeneous fluid is characterized as the solution of the canonical equations of motion.

As a matter of historical interest, it should be noted that the identifications made by Helmholtz were somewhat different from those of (2-10). He was concerned with a homogeneous body of fixed volume, so he identified his Lagrangean with $\mathbb{L} = \frac{1}{2} M v^2 - \breve{\mathbb{F}}$ where $\breve{\mathbb{F}}(T, V) = \breve{U} - T \breve{\mathbb{S}}$ is the free-energy (now known as the Helmholtz function). The overhead hooks indicate non-relativistic quantites. (Actually Helmholtz worked with $-\mathbb{L}$, which he called the "kinetic potential". He then introduced a compensating change of sign in the right-hand side of (2-10c.) With the identifications $T \equiv \dot{\Theta}$ and $\vec{v} \equiv \dot{\vec{\lambda}}$, Helmholtz's Hamiltonian turns out to be $\breve{U}$. Planck[8] generalized this prodecure to the relativistic case by effectively making the identification $\mathbb{L} \equiv -\mathbb{F}(1 - v^2/c^2)^{-1/2}$ where $\mathbb{F}$ now includes the rest-energy Mc^2.

The curious thing is that, although both Helmholtz and Planck treated the temperature T as a generalized velocity $\dot{\Theta}$, neither one ever gave the generalized coordinate a symbol or a name. Laue gave it a symbol, thereby making the assertion that temperature should be regarded as a generalized velocity open and visible. (In the papers of Helmholtz and Planck, this assertion is so diffidently made that it is very easily overlooked. Although Laue made the identification explicit, he did not go beyond the treatment of Helmholtz and Planck. Van Dantzig[9] did go well behond this, and he gave Θ a name ("thermasy") and a vital role in his formalism. Curiously enough, however, Van Dantzig regarded the differential relation connecting T and Θ to be non-holonomic (i.e., path-dependent) and hence in general non-integrable. This point of view prevented him from arriving at many of the most important results that follow from regarding the thermasy as a full-fledged generalized coordinate to be manipulated in the canonical and variational formalisms just like any other generalized coordinate.

Meanwhile, the thermasy was independently rediscovered on several occasions in the context of fluid dynamics[10], in the sense that its introduction was found to allow a formal integration of Euler's equation of motion for a fluid. The corresponding generalized coordinate for the free-enthalpy, the ergasy γ ($G \equiv \dot{\gamma}$) had long since found its way into fluid dynamics in the guise of the potential function of the flow.

2.3 Convexity Property of Energy Density and Pressure

Applying (2-3) to the case of a constant volume $\mathbb{V} = \overline{\mathbb{V}}$ and dividing by the volume, we find

$$d^{(2)}u \equiv u - \bar{u} - \overline{G}(n - \bar{n}) - \overline{T}(s - \bar{s}) \geqslant 0$$

$$= u - \not\mu \geqslant 0 \tag{2-11}$$

where

$$\not\mu \equiv \bar{u} + \overline{(\partial u/\partial n)}(n - \bar{n}) + \overline{(\partial u/\partial s)}(s - \bar{s})$$

$$= \bar{u} + \overline{G}(n - \bar{n}) + \overline{T}(s - \bar{s}) \tag{2-12}$$

$$= -\overline{P} + \overline{G}n + \overline{T}s \quad . \qquad \text{(using (2-6))}$$

The linear function $\not\mu(n, s ; \overline{G}, \overline{T})$ in which (n, s) are the variables and $(\overline{G}, \overline{T})$ are constant parameters, is the plane that is tangent to the surface $u(n, s)$ at the point $(\bar{n}, \bar{s})$. This is illustrated in Figure 4. The geometric significance of the inequality (2-11) is that in a stable region the surface u must lie _above_ all of its tangent planes. That is, viewed in the direction of the positive u-axis, the surface $u(p_a)$ must appear _convex_.

This convexity property can be simply explained in terms of stability with respect to fluctuations. From the point of view of the small sample of fluid occupying unit volume, the surrounding fluid behaves like a combination particle (or mole number) reservoir, and a heat (i.e., entropy) reservoir. The sample and the surrounding fluid constantly exchange matter and entropy back and forth. These exchanges cause the point representing the thermodynamic state of the sample to oscillate in a random fashion about the equilibrium point, always remaining on the surface $u(p_a)$. The linear function represents (to within an additive constant) the potential function of the "reservoir" (i.e., the surrounding fluid). Because the total mole number and total entropy are conserved during the random exchange, the point representing the thermodynamic state of the reservoir makes complementory oscillations within the tangent plane. It is easy to show that the energy $d^{(2)}u$, which is called the "excess energy density" in Figure 4, is the energy that would be lost from the combined sample-reservoir system if the necessary amounts of n and s were transferred from the sample in its instantaneous non-equilibrium state to the reservoir in order to
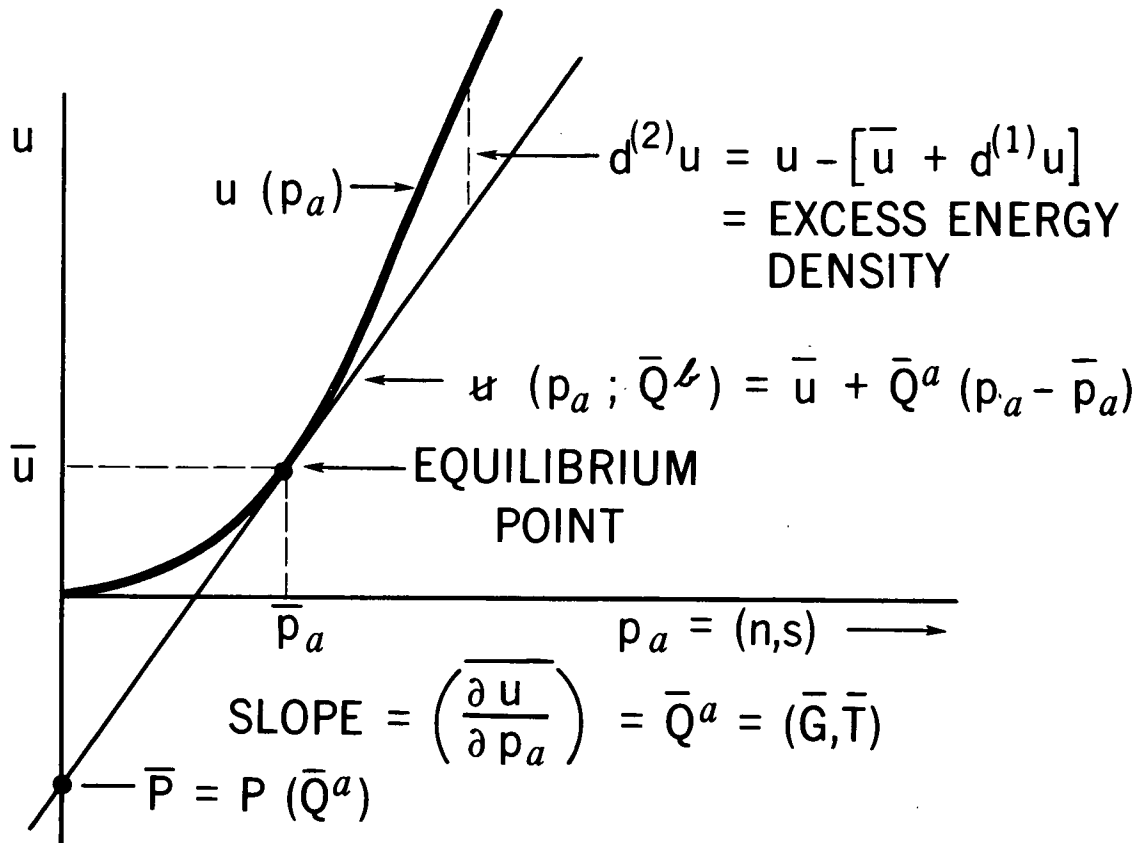
Figure 4. Convexity Property of Energy Density

bring the sample back to the equilibrium state. In principle, if suitable mechanisms were available, this transfer could be accompanied by conversion of the excess energy into useful work. In actual fact, such mechanisms are nearly always not present, and the excess energy represents the still-ordered fluctuation energy (e.g., the kinetic energy of turbulent eddies) that is destined to be degraded into disordered heat energy as the fluctuations die out.

The convexity property simply says that this fluctuation energy is always positive. In other words, stability is characterized by the fact that energy must be provided from outside in order to make the system fluctuate.

Inasmuch as the function $P(Q^a)$ is just a different representation of $u(p_a)$, there must exist an analogous convexity property for P with a corresponding interpretation in terms of stability against fluctuations.

The convexity property of P can be demonstrated as follows: Substituting the third line of (2-12) into (2-11) we have

$$d^{(2)} u = u + \bar{P} - \bar{G} n - \bar{T} s \geqslant 0$$

$$= \bar{P} - P - n(\bar{G} - G) - s(\bar{T} - T) \geqslant 0 \quad \text{(using (2-6))}$$

$$= \bar{P} - \not{P} = d\bar{P} - d^{(1)}\bar{P} \geqslant 0 \quad (2\text{-}13)$$

$$= d^{(2)} \bar{P} \equiv d^{(2)} P(\bar{Q}^{\alpha}) \geqslant 0$$

In this case the barred quantities are regarded as variable and the unbarred quantities are fixed parameters that specify the equilibrium point. In the terms of Figure 4, letting the barred quantities vary corresponds to specifying the equilibrium state as a fixed point on the u-surface and letting the tangent plane oscillate about this point. Obviously the excess energy density must still be a positive quantity.

We should think of an oscillating tangent plane as representing the thermodynamic potential of a zero-capacity probe, such as an ideal thermometer, instead of an infinite-capacity reservoir. Such a probe would be sensitive to the fluctuations in T and G that occur at a point, but are so localized that they have no effect on the total values of n and s for the whole sample. In this case the sample effectively serves as a reservoir for the probe. Small though n and s may be, they are still infinite compared with the extensive quantities that can be absorbed by the zero-capacity probe.

We could draw a convexity figure for $P(\bar{Q}^{\alpha})$ and its tangent plane that would have the same form as Figure 4. In such a figure the tangent plane would be specified by the fixed values (n, s) and would be fixed. We would then have the same picture of a point on the convex surface fluctuating about the point of tangency. In this case we would give the following physical interpretation to the convexity condition: A sample of fluid in a stable condition is characterized by the fact that the average pressure for a fluctuating sample is greater than the pressure of a quiescent sample at equilibrium. In other words, fluctuations in a stable fluid have the effect of "stiffening" the fluid. This is obviously a necessary condition for stability, because if the reverse were true fluctuations would induce a collapse of the sample.

The analytical statement of the above stability requirement is

$$\langle P \rangle \equiv \frac{1}{(\tau_f - \tau_i)} \int_{\tau_i}^{\tau_f} P \, d\tau \;\geqslant\; \bar{P} \equiv P(\langle \dot{q}^\alpha \rangle) \qquad \text{(2-14a)}$$

where

$$\langle \dot{q}^\alpha \rangle \equiv \frac{1}{(\tau_f - \tau_i)} \int_{\tau_i}^{\tau_f} \dot{q}^\alpha \, d\tau = \frac{(q_f - q_i)}{(\tau_f - \tau_i)} \; . \qquad \text{(2-14b)}$$

This can be restated as follows: The equilibrium values of the intensive quantities $\bar{Q}^\alpha$ are given by the functions $\dot{q}^\alpha(\tau)$ where the functional dependence of $q^\alpha(\tau)$ is such that

$$\int_{\tau_i}^{\tau_f} P(\dot{q}^\alpha) \, d\tau = \text{minimum for } (q_i^\alpha, \, q_f^\alpha) \text{ held constant } . \qquad \text{(2-15)}$$

The Euler-Lagrange equations that result from this variational minimum principle are just those given in (2-10c), namely $\dot{n} = \dot{s} = o$. The significant thing about characterizing the equilibrium state by means of (2-15) is that the same formalism describes all the non-equilibrium states (i.e., fluctuating states) as well as the equilibrium state. This is something that classical thermodynamics cannot do: The thermodynamic potential function describes the equilibrium state and nothing else. The key step in expanding the descriptive powers of the thermodynamic potential function was to replace this function by the corresponding functional that results automatically when we identify the intensive quantities as generalized velocities.

Note that a corresponding variational principle in terms of u instead of P is not possible. This is the first indication that the potentials u and P, which heretofore have been treated as equivalent, are in fact not equivalent. The pressure P (or j) is more fundamental.

Finally, we note that there is an alternative to the "stiffening-under-fluctuations" interpretation of stability that was given above. In terms of the interaction potential energy j that was introduced in (2-8), and which is identically equal to P except for its conceptual content, we would write (2-13) as

$$d^{(2)} u = d^{(2)} j \;\geqslant\; 0 \; . \qquad \text{(2-16)}$$

This means that we could invoke exactly the same kind of fluctuation-energy argument to explain stability in terms of j that was already given in terms of u.

The important difference is that the energy being referred to would be the inter-action energy j that the sample possesses by virtue of its interaction with the rest of the fluid, rather than the intrinsic energy of the sample itself.

It seems rather surprising that equilibrium statements in terms of these two very different energies should lead to the same result. The explanation of this puzzle was given in 1929 by K.O. Friedrichs when he showed that minimum energy principles applied to dual energy functions such as u and j do in fact approach the same extremal (i.e., equilibrium) state. In fact, this duality pro-vides a very useful technique for estimating the error of an approximate solution for the equilibrium state. This will be discussed more fully in Sec. 3.7.

2.4 Thermodynamic Additivity in Three-Space

If there is no action-at-a-distance within the system, i.e., if u and j are functions only of local quantities, then the usual thermodynamic assumption of the addivity of extensive quantities is justified[11] and we can ascribe the usual properties to the functions $\mathbb{U}$ and $\mathbb{J}$ where

$$\mathbb{U} \equiv \int_{\mathbb{V}} u \, d^3(\vec{n}) \quad \text{and} \quad \mathbb{J} \equiv \int_{\mathbb{V}} j \, d^3(\vec{n}) = \int_{\mathbb{V}} P \, d^3(\vec{n}) \,. \quad (2\text{-}17a,b)$$

In particular, if $d^{(2)} u \geqslant 0$ (which implies $d^{(2)} j \geqslant 0$ and vice versa) everywhere throughout the volume $\mathbb{V}$, it follows that

$$d^{(2)} \mathbb{U} \geqslant 0 \quad \text{and} \quad d^{(2)} \mathbb{J} \geqslant 0 \quad (\text{if } d^{(1)} u = d^{(1)} j \geqslant 0). \quad (2\text{-}18a,b)$$

If u and j (or P) are non-uniform throughout $\mathbb{V}$, then we can take the point of view that $\mathbb{U} = \mathbb{U}(\mathbb{N}, \mathbb{S}, \mathbb{Z})$ where $\mathbb{Z}$ represents an infinity of internal degrees of freedom with respect to which $\mathbb{U}$ must be convex (because of (2-18a)). These will adjust themselves, to the extent that constraints and boundary condi-tions allow, to minimize $\mathbb{U}$. But, a minimum for $\mathbb{U}$ with respect to the internal degrees of freedom means that

$$\left[d^{(1)} \mathbb{U} \right]_{\mathbb{N}, \mathbb{S}} = 0 \,. \qquad \text{(equilibrium condition)} \qquad (2\text{-}19)$$

It follows from (2-18b) that J is also convex with respect to all the internal degrees of freedom. Thus the condition

$$\left[d^{(1)}J\right]_{N,S} = 0 \quad , \tag{2-20}$$

together with (2-18b), identifies a configuration characterized by minimum interaction energy.

It is a consequence of the Friedrichs Transformation referred to above that the two very different-looking conditions (2-19) and (2-20) do in fact specify the same configuration, because the integrands of U and J are dual functions.

In the context of static elasticity theory, the two different minimum principles based on (2-19) and (2-20) had been in use long before Friedrichs provided the formal proof of their equivalence. They had grown out of the analysis of trusses and were known respectively as the Principles of Dirichlet[12] and of Castigliano (or more accurately, of Castigliano and Ménabréa)[13]. It was merely assumed that they led to the same solution, and application to special cases always bore out this assumption.

Dirichlet's Principle is simply the application of the thermodynamic principle of minimum internal energy to the case in which the internal energy involved is the elastic energy. Castigliano's Principle is just the Principle of Virtual Work applied to the case of trusses or of an elastic medium. It is easy to show that minimizing J is equivalent to the Principle of Virtual Work applied to the case of a fluid.

These considerations show that with respect to an extended, static medium, such as a fluid at rest, the same equivalence exists between U and J as exists locally between u and j. We shall now see that this equivalence does not survive the introduction of relative motion within the system.

2.5 Thermodynamic Additivity in Four-Space

In the case of quiescent systems, nothing in sec. 2.4 would be changed by integrating U and J with respect to time, and then using these time integrals to determine the equilibrium configuration by means of the same minimum principles. That is, we could use the space-time integrals

$$\mathcal{U} \equiv \int_{t_i}^{t_f} dt \int_V u \, d^3(\vec{r}) \quad \text{and} \quad \mathcal{J} \equiv \int_{t_i}^{t_f} dt \int P \, d^3(\vec{r}) \tag{2-21a,b}$$

as thermodynamic potentials to determine the equilibrium configuration.

If, however, the various parts of the system have relative motion, then the equivalence of $\mathscr{U}$ and $\mathscr{P}$ breaks down, and only one of these can survive as a valid thermodynamic potential. We shall see that the survivor is $\mathscr{P}$.

The inadequacy of $\mathscr{U}$ is immediately evident with respect to the requirement of relativistic covariance. Because stability is an invariant property that is completely independent of the observer's frame of reference or coordinate system, any quantitity that is to be used as a stability criterion must be invariant, not only under Lorentz Transformations, but also under general curvilinear coordinate transformations. Because P is a scalar quantity, $\mathscr{P}$ is also a scalar if the four-volume of integration is defined in a way that is independent of the observer's reference frame and coordinate system, that is, if it has an objective physical definition independent of the observer.

The energy density u, however, is not a scalar. It is a scalar <u>density</u>. Thus it depends on the coordinate system used, and consequently so does $\mathscr{U}$.

Even without invoking covariance under general curvilinear coordinate transformations, $\mathscr{U}$ is not satisfactory. Because u is the energy density in the local rest-frame rather than the laboratory frame (inertial frame) over which the integration is carried out, it should actually be written as the norm of the energy flux-density four-vector $\underset{\sim}{u}$, that is u must be defined as

$$u \equiv (\underset{\sim}{u} \cdot \underset{\sim}{u})^{1/2} = \left[(u^0)^2 - \vec{u} \cdot \vec{u} \right]^{1/2}. \tag{2-22}$$

But this is an nonlinear expression, and if we were to use it in (2-21a), we would no longer have thermodynamic additivity with respect to coexisting but non-interacting fluids, which physically we know must be valid.

This problem of nonlinearity does not arise with $\mathscr{P}$. Because P is a true scalar (not a scalar density) it is independent of the local state of motion of the fluid with respect to the laboratory frame. In the case of coexisting non-interacting fluids, the pressures are additive, and hence so are the corresponding $\mathscr{P}$'s.

One way of stating the relation between P and u is to say that they are dual in the sense of tensor analysis, as well as in the thermodynamic sense. Just as in the case of dual antisymmetric tensors, one of which is in fact a tensor <u>density</u>, so in the dual pair P and u, one is a scalar and the other a scalar <u>density</u>, which means that their fundamental natures are very different even though in many respects they might appear to be interchangeable. We shall see in sec. 3.7 that the Friedrichs Transformation is still valid in the non-static

case, and in that sense $\mathcal{U}$ still retains its equivalence to P. But it will also be clear that the role of $\mathcal{U}$ is definitely subordinate to that of P.

Covariant Stability Criterion

In summary, the covariant stability criterion for a fluid with internal motion is

$$\oint \equiv \int_{(\underset{\sim}{x})} P d^+(\underset{\sim}{x}) \quad = \text{minimum,} \tag{2-23}$$

in which it is understood that the four-space region of integration is defined in a way that is independent of the observer.

The boundary conditions to be imposed on the integral $\oint$, i.e., the space-time behavior of the containing walls, and the conditions to be imposed on the variables at these walls, are familiar and require no special elaboration. The terminal conditions to be imposed on the variables throughout three-space at the initial and final times will be discussed in the next section.

III. VARIATIONAL MINIMUM PRINCIPLE FOR A FLUID

Résumé

In sec. 3.1 the stability criterion that in sec. 2 was shown to satisfy the requirement of relistivistic covariance is converted to a variational principle by implementing the idea that intensive quantitives are generalized velocities. In a fluid-dynamical context, the proper-time derivative that is involved in defining a velocity must itself be defined in terms of unconstrained kinematical variables that describe the flow. This can be done in various ways: In sec. 3.5 the analysis is carried out in terms of a coordinate system that is embedded in the fluid and swept along with it, and in sec. 3.6 the analysis is repeated in terms of a system of Cartesian coordinates fixed in the laboratory frame of reference. In both cases it is shown that the Euler-Lagrange equations that result from the variational principle are equivalent to the well-known system of partial differential equations for adiabatic compressible flow. The terminal conditions that must be satisified during the variation are discussed in Sections 3.3, 3.4, and 3.5.

Although from the formal point of view the replacement of intensive quantities by their corresponding generalized coordinates amounts to nothing more than a "change of variable", the physical implications are far-reaching. A brief intuitive explanation of these implications is presented in sec. 3.2.

The replacement of the temperature by its corresponding generalized coordinate, the thermasy, permits two different formalisms for relativistic thermodynamics, developed respectively by Planck and Van Dantzig, that heretofore have appeared to be irreconcilable, to be presented from a unified point of view that harmonizes the apparent differences. This unified approach is presented in sec. 3.4. It is closely related to the idea of describing fluid flow in terms of a four-dimensional mapping, which is discussed at greater length in sec. 4.3 where still a third alternative formalism, differing from both those of Planck and Van Dantzig, is presented. Although the non-covariant appearance of the Planck formalism renders it less attractive in a relativistic context than any of its manifestly covarient competitors, it does permit the most familiar-seeming introduction to the variational formalism, and for that reason it is employed in sec. 3.5.

In sec. 3.7 the variational principle is cast into a modified form that permits an estimate of the accuracy of an approximate solution. This modification in fact amounts to a simplified derivation of the Friedrichs Transformation, at the same time generalizing it from static to dynamic problems. Both the simplification and the generalization result from the identification of intensive quantities

with generalized velocities, a feature that is not included in the standard form of the Friedrichs Transformation.

3.1 Construction of a Variational Minimum Principle from the Stability Criterion

In the previous section we derived the following covariant stability criterion for a continuum:

$$d^{(1)} \int P \, d^4(\underset{\sim}{x}) = 0 \tag{3-1a}$$

(equilibrium criterion)

$$d^{(2)} \int P \, d^4(\underset{\sim}{x}) \geqslant 0 \quad \longleftrightarrow \quad d^{(2)} P \geqslant 0 \quad \text{everywhere.} \tag{3-1b}$$

(stability criterion)

The choice of arguments of P is not unique, but in everything that follows we shall regard P = P (G, T) to be a function of the molar Gibbs function or free-enthalpy G (which is understood to include the relativistic rest-mass energy Mc^2) and the absolute temperature T, which will be assumed to be expressed in energy units. (This could be expressed more explicitly by writing RT instead of T, where R is the molar gas constant, but for simplicity the R will be omitted.) The choice of G and T is not unique. For example a very useful variant of the formalism can be derived on the basis of P(H, S). Such changes in the choice of arguments amount to canonical transformations.[6]

The essential step in converting the stability criterion of (3-1) into a variational minimum principle that includes the motion, as well as the thermodynamic properties of the fluid, in a way that allows a determination of the flow by means of the standard formalism of the calculus of variations is to identify the intensive quantities G and T as generalized velocities. Thus G and T are replaced as primitive variables by scalars γ and Θ, whose proper-time derivatives are defined to be G and T. Extending the nomenclature introduced by Van Dantzig,[4] these are called ergasy and thermasy respectively.

$$G \equiv d\gamma/d\tau \, , \quad \text{thus} \quad \gamma = \int G \, d\tau \quad \text{(ergasy)} \, . \tag{3-2a}$$

$$T \equiv d\Theta/d\tau \, , \quad \text{thus} \quad \Theta = \int T \, d\tau \quad \text{(thermasy)}. \tag{3-2b}$$

In a fluid context, proper-time differentation is defined as

$$\frac{d}{d\tau} \equiv c \, \underset{\sim}{\hat{V}} \cdot \underset{\sim}{\partial} \equiv c \, \hat{V}^j \, \partial_j \tag{3-3a}$$

where

$$\hat{\underset{\sim}{V}} = \underset{\sim}{V}/c = (\hat{V}^j) = (\Gamma, \Gamma \vec{v}/c) \quad \text{where } \Gamma \equiv (1-\beta^2)^{-1/2}, \quad \beta \equiv v/c \; ; \quad \text{(3-1b)}$$

$$\underset{\sim}{\partial} \equiv (\partial^j) = (\partial_o, -\vec{\nabla}) \; ; \quad (\partial_j) = (\partial_o, \vec{\nabla}) \; ; \quad \partial_o \equiv \frac{\partial}{\partial ct} \; . \quad \text{(3-3c)}$$

(Occasionally the four-velocity normalized to c rather than to 1 will be used and will be designated by $\underset{\sim}{V}$.)

In this way the fluid velocity enters the formalism. The velocity $\hat{\underset{\sim}{V}}$ has four components, but because of the normalization condition that it must satisfy it has only three degrees of freedom. Thus this four-vector cannot be regarded as a primitive variable. Rather it must be expressed in terms of three truly independent and unconstrained variables. This can be done in various ways, two of which are discussed in detail later in this section.

The three variables that determine the fluid velocity, together with the ergasy and thermasy, constitute the five basic undetermined space-time functions, that is, the dependent variables in a variational problem in which P is a functional of these five undetermined dependent variables. These are determined by the requirement that the action integral $\oint$, which is the space-time integral of the pressure functional, be stationary. It will be shown later in this section that the Euler-Lagrange equations resulting from variation of γ and Θ give the equations for conservation of mole number (i.e., particles) and entropy respectively. Thus the requirement that, to first order in the displacements from the equilibrium condition, the mole number and entropy contained in each convective cell of the fluid must remain constant is indeed satisfied. The variation of the three functions that determine the fluid velocity will be shown to yield Euler-Lagrange equations which, together with the conservation equations resulting from variation of γ and Θ, suffice to guarantee local conservation of energy and momentum.

The necessary and sufficient condition that the flow be stable is that the pressure be convex at every space-time point of the four-volume over which the action integral extends. When the convexity inequality is expressed in terms of the five dependent variables, it becomes the well-known Weierstrass condition[14] for a stationary point to be a minimum. In this connection it should be noted that in the two variants of the formalism discussed below, only derivatives of the dependent functions enter into the Lagrangean density P. (This special case is referred to in the variational literature as being cyclic or kinosthenic.) For this special case the Weierstrass condition is sufficient as well as necessary to guarantee a minimum. Other variants (canonical transformations) of the formalism exist in which some of the dependent functions, as well as their

derivatives, enter into the pressure functional. In this case the Weierstrass condition (which involves only the derivative-dependence of the pressure functional) is a necessary but not sufficient condition for a minimum. It guarantees stability under fluctuations having very high frequency and negligible amplitude. It is easy to show that in order to guarantee stability under fluctuations of finite amplitude a necessary condition is an extension of the Weierstrass condition that includes the generalised work associated with the finite-amplitude displacement. Together these two conditions are sufficient to guarantee a minimum.

In an actual direct numerical solution of the problem, it would not be necessary to bother with these conditions. Since the necessary and sufficient condition is that the Lagrangean density P be a convex function of G and T, and since for a given substance the stable ranges of G and T are known in advance, it would only be necessary to calculate for each set of trial functions the values of G and T at each point of space-time in order to check whether they fall within the known stable range. It may turn out that it is impossible to find a set of trial functions that yield a stationary value of the action integral and lie everywhere in the stable range of variables. In this case it would be necessary to introduce an interior boundary surface which would correspond to the locus of a physical discontinuity such as a shock front or a condensation front. The locus of this front would be parameterized, and the parameters involved would be treated as additional variables.

3.2 Fluid Reservoir Concept

Although from the mathematical point of view, the replacement of G and T by γ and Θ is a formal device—a change of variable—that converts the thermodynamic stability criterion of (3-1) into a variational principle, from the physical point of view it has far-reaching implications. From (3-2) and (3-3) it is evident that

$$G = \hat{\underset{\sim}{V}} \cdot \underset{\sim}{\mathscr{J}} \quad \text{where} \quad \underset{\sim}{\mathscr{J}} = c \underset{\sim}{\partial} \gamma \quad ; \qquad (3\text{-}4a)$$

$$T = \hat{\underset{\sim}{V}} \cdot \underset{\sim}{\mathscr{S}} \quad \text{where} \quad \underset{\sim}{\mathscr{S}} = c \underset{\sim}{\partial} \Theta \quad . \qquad (3\text{-}4b)$$

The physical interpretation to be given these relations is that $\underset{\sim}{\mathscr{J}}$ and $\underset{\sim}{\mathscr{S}}$ represent the local energy four-vectors of the particle and entropy (heat) reservoirs respectively. The eqs. (3-4) require that in the rest-frame of the fluid, $\mathscr{J}^\circ = G$ and $\mathscr{S}^\circ = T$. This is just the condition that must be fulfilled in order to insure that the fluid is in <u>reversible</u> contact with the reservoirs.[15]

The fact that the energy four-vector of the heat reservoir is not in general collinear with the fluid velocity vector $\hat{\underset{\sim}{V}}$ is a consequence of the fact that there can exist a heat flux with respect to the fluid matter. In fact, we could visualize the heat reservoir as a photon gas in reversible interaction with the material fluid. Although for a uniform fluid at rest, one would expect that in its final equilibrium state the photon gas would be at rest with respect to the matter, this need not be the case for transient conditions produced by acceleration of the matter. When relative motion between the matter and heat reservoir is present, heat exchange between the two will necessarily be accompanied by a momentum exchange.

Similar considerations hold in the case of interaction with the particle reservoir. The term "particle reservoir" is actually not quite correct, since G is a measure of the total _ordered_ energy per mole of fluid (in the same sense that T is a measure of the disordered thermal energy). Thus the reservoir described by $\underset{\sim}{\hat{g}}$ should be regarded as supplying not only energy that is bound to the molecules (such as mass-energy, nuclear energy, and chemical energy), but also ordered energy, such as that contained in waves propagating through the matter, whose energy four-vector need not be collinear with $\hat{\underset{\sim}{V}}$. Because such "unbound" energy is very small compared with the rest-mass energy, it is a good approximation to regard the ordered energy as being bound to the particles, and this is the justification for speaking simply of a "particle reservoir" rather than of an "ordered-energy reservoir".

Even when all of the ordered energy is bound to the particles, however, particle exchange between the fluid and the particle reservoir can involve a momentum exchange. The explanation for this is given by the tendency of blobs of fluid that are exchanged between neighboring sites because of turbulent mixing to maintain their angular momentum during the exchange. This is illustrated in Figure 5.

In case (A), a blob of fluid that jumps from point P to point Q in a rigidly rotating vortex, conserving its angular momentum during the jump, finds itself at Q with a much larger velocity than the surrounding fluid, and so in the process of being absorbed by the surrounding fluid at Q, it will transfer momentum to the fluid as well as the energy that it brings with it. In the case of a potential vortex, however, after making the jump from P to Q, the blob finds itself with a velocity (determined by conservation of its angular momentum) exactly equal to the velocity of the ambient fluid. Thus its absorption by the fluid will involve no momentum transfer.

It is clear that turbulent mixing tends to produce the kind of flow illustrated in Case B. We shall see in sec. 4 that this is a special case of a general tendency for a fluid to go over to potential flow (to the extent that the imposed boundary
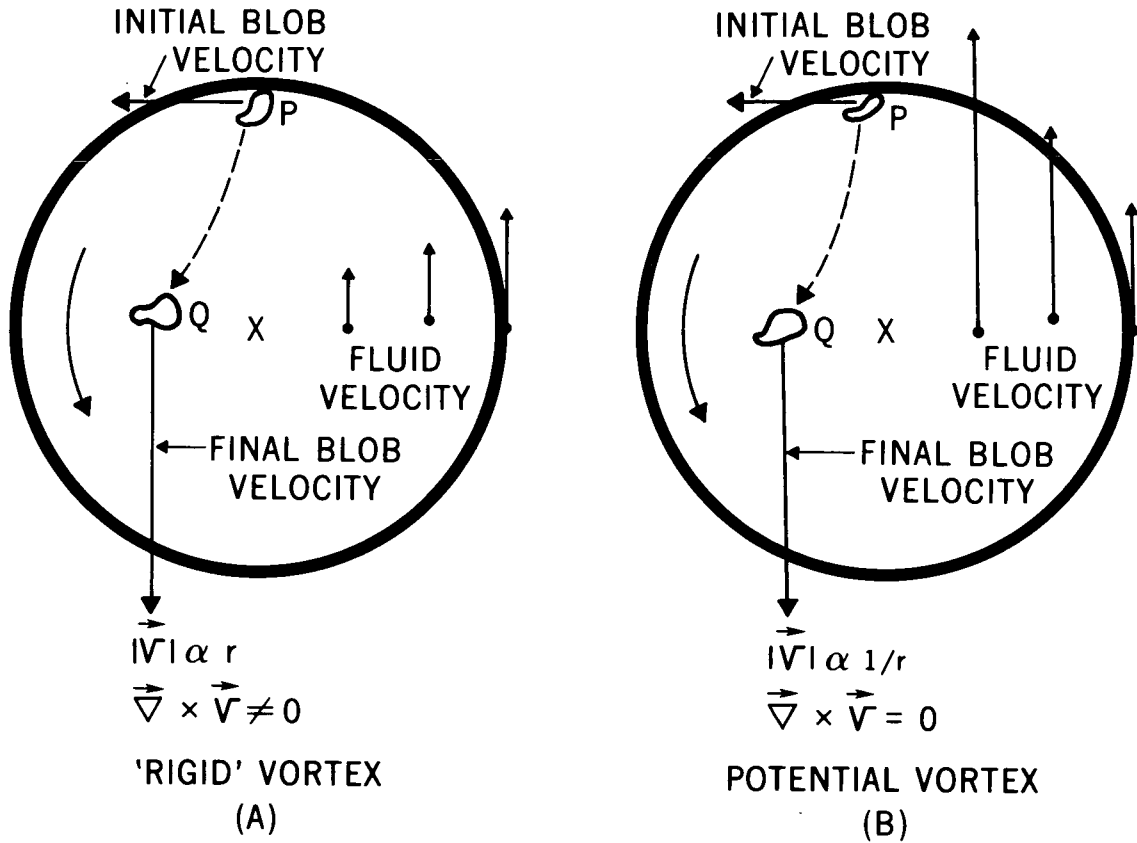
Figure 5. Momentum Exchange Resulting Mass-Transport in Rotating Fluid

conditions permit it). In fact, we shall see that potential flow results when the fluid has completely "forgotten" its initial conditions. All that remains are the requirements imposed by the boundaries and the physical properties of the fluid.

Note that, because $\underset{\sim}{g} = c \underset{\sim}{\partial} \gamma$ and $\underset{\sim}{\mathcal{S}} = c \underset{\sim}{\partial} \theta$ the two reservoirs are themselves examples of potential flow. Thus they may be pictured as fluid flows that have existed long enough to have forgotten their origins. It is possible also to give an argument based on the concept of the Carnot cycle to justify the assertion that a fluid heat reservoir must be in potentital flow.[16]

The vector $\underset{\sim}{H} \equiv H \overset{\wedge}{\underset{\sim}{V}}$ is the four-vector generalization of the molar enthalpy, and $S \underset{\sim}{\mathcal{S}}$ is the generalization of the unavailable energy ST that cannot be converted into useful work. Thus $\underset{\sim}{H} - S \underset{\sim}{\mathcal{S}}$ is the energy four-vector that represents the energy and momentum that could be converted into ordered form and will be called the molar free-enthalpy four-vector of the fluid. In general this is not equal to the reservoir molar free-enthalpy $\underset{\sim}{g}$, and it is necessary to introduce a vector $\underset{\sim}{K}$ to relate the two:

$$\underset{\sim}{H} - S \underset{\sim}{\mathcal{S}} = \underset{\sim}{g} + c \underset{\sim}{K} \, . \tag{3-5a}$$

From the formal point of view, this equation (which is just the definition of $\underset{\sim}{K}$) simply gives the decomposition of the molar free-enthalpy four-vector on the left into an irrotational part $\underset{\sim}{\mathcal{A}} = c \underset{\sim}{\partial} \gamma$, and a rotational part $c \underset{\sim}{K}$. We shall see that the vector $\underset{\sim}{K}$ plays an important role in the formalism.

A similar decomposition can be carried out for the molar momentum four-vector $\underset{\sim}{H}/c = (H/c^2) \underset{\sim}{V}$. (Since H is the total energy of a mole of fluid, H/c^2 is the total rest-mass).

This decomposition is

$$\underset{\sim}{H}/c = (H/c^2) \underset{\sim}{V} = \underset{\sim}{\partial} \eta + \underset{\sim}{D} \tag{3-5b}$$

where

$$\eta = \gamma + S \theta \qquad \text{and} \qquad \underset{\sim}{D} = - \theta \underset{\sim}{\partial} S + \underset{\sim}{K} \tag{3-5c}$$

The scalar η , called the enthalpasy, is both the potential function of the flow and the quantity that generates the enthalpy reservoir for the fluid whose energy four-vector is $\underset{\sim}{\mathcal{H}} \equiv c \underset{\sim}{\partial} \eta$ corresponding to the momentum four-vector $\underset{\sim}{\mathcal{H}}/c = \underset{\sim}{\partial} \gamma$. The vector $\underset{\sim}{D}$, called the "drift vector" may be regarded as giving the drift momentum of the fluid with respect to its enthalpy reservoir.

3.3   Terminal Conditions for the Action Integral

In a closed system such as a box, or in the case of flow through a tube or nozzle, the conditions imposed by spatial boundaries (i.e., walls) are the same for the action integral as for a differential equation approach to the problem. The novel feature of the action integral is that, instead of specifying the initial value q_i of each dependent variable and its time derivative $\partial q_i / \partial t$ at the initial time, we specify the initial and final values q_i and q_f at the initial and final times t_i and t_f . As illustrated in Figure 6, this equivalent to specifying the time-average of the time-derivative of the variable. Thus, as illustrated in Figure 6, we effectively specify the average generalized velocity throughout the time interval. Whether this is the average velocity experienced by a given sample of fluid, or the average observed at a fixed point in the laboratory coordinates depends on the independent coordinates we use. This, and other subtleties of a relativistic nature, will be further discussed in connection with the two cases that are worked out in detail below. Without going into these subtleties here, we can say simply that by specifying q_i and q_f for the ergasy, thermasy, and the three identity parameters, we are effectively specifying the

AVERAGE GENERALIZED VELOCITY

$$\left\langle \frac{dq}{dt} \right\rangle = \frac{q_f - q_i}{t_f - t_i}$$

EXTREMAL PATH
q_f (COHERENT FLOW)

NON-EXTREMAL PATHS
(INCOHERENT FLOW)

q_i

$q \longrightarrow$

(A)
CONSTRAINED INITIAL
AND FINAL VALUES

MOST STABLE EVOLUTION
(ACTION INTEGRAL IS MINIMUM)

COHERENT
EVOLUTIONS

$q \longrightarrow$

(B)
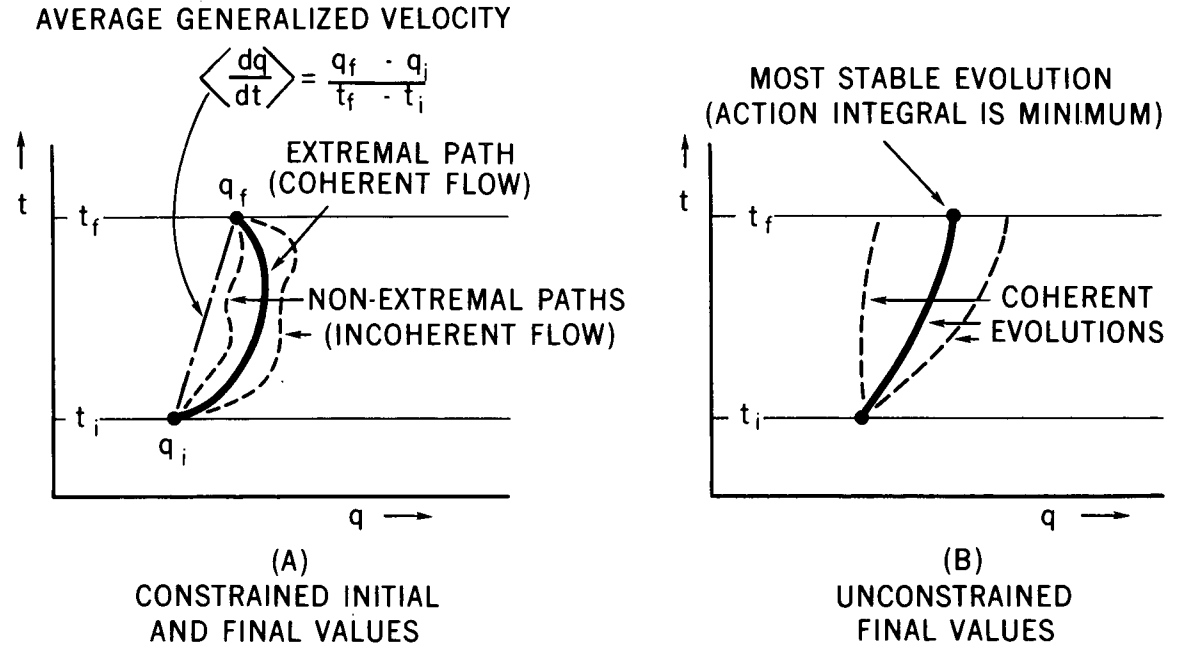UNCONSTRAINED
FINAL VALUES

Figure 6. Basic Types of Terminal Conditions

time-averages of the molar free-enthalpy, the temperature, and the fluid velocity.

Because the space-time dependencies for the q's which yield a minimum for the action integral are influenced by the behavior of the fluid throughout the entire space-time region, and are not dependent on the history of just the single particle in question, this choice of functional dependences will be referred to as the "coherent evolution" of the system, as opposed to the case of any non-extremal set of trial functions, which will be called an "incoherent evolution".

There exists a different coherent evolution for each different choice of initial and final values of the q's, i.e., for each choice of average generalized velocities. The action integral may be regarded as the thermodynamic potential function whose arguments are these average values. As such, there is a correlation between the value of the action integral $\oint$ for a given coherent evolution, and the relative stability of this evolution compared with other coherent evolutions corresponding to different average-values for the generalized velocities. The lower the value of $\oint$, the more stable the evolution.

The reason that $\oint$ serves as a stability criterion for variation of the values of q_f (or of q_i) on the final hyperplane t_f, as well as for variation in the values of the q's on interior hyperplanes between t_i and t_f, is that, from the point of view of the stability criterion given in (3-1), there is no distinction between the q's on the bounding hyperplanes t_i and t_f and interior values of the q's. The stability

criterion treats them all as internal degrees of freedom that seek those values that minimize $\mathcal{J}$.

Thus if we allow the q_f's to vary freely during the variation procedure, or if we allow them to vary in such a way that certain imposed constraints are satisfied, then the coherent evolution for which $\mathcal{J}$ is stationary with respect to variation of the q_f's is more stable than all the other coherent evolutions of the class under study that correspond to the various allowed choices of the q_f's. That is, the same condition ($d^{(1)} P \geqslant 0$ everywhere throughout the four-volume of integration) that guarantees the stability of each coherent evolution of the class under study, also guarantees that the evolution that gives a <u>stationary</u> value for $\mathcal{J}$ with respect to variation of the q_f's in fact gives a <u>minimum</u> value for $\mathcal{J}$.

A well-known physical example of the case of a thermodynamic minimum principle corresponding to fixed terminal values of the dependent function is provided by a soap film stretched between the rims of two funnels. This is shown in cross-section in Figure 7 (A). It corresponds to Figure 6 (A). If the upper funnel is large enough so that the film terminates on the interior surface of the upper funnel instead of on its rim, then we have a free boundary problem in which the film seeks out the terminal point that minimizes its internal energy. This case is shown in Figure 7 (B). It corresponds to Figure 6 (B). The fact that the soap film approaches its upper terminus perpendicularly to the surface of the funnel corresponds to the fact that the action integral is stationary (minimum) with respect to small displacements of its terminal point. That is, a slight displacement of the terminal point does not change the total length (in cross-section) of the soap film, and hence does not increase its total surface-tension energy,which is the thermodynamic function being minimized.

3.4   Unified Approach to the Planck and Van Dantzig Formalisms

In the convected-frame analysis that is closest to non-relativistic fluid dynamics, the independent variables are the time t and the three scalar identity parameters $\lambda^\alpha = (\lambda^1, \lambda^2, \lambda^3)$. As mentioned earlier, the time t should also be regarded as a <u>scalar</u> quantity, namely the projection of the position four-vector onto a characteristic four-velocity vector that is defined in terms of the boundaries of the problem and is independent of the observer. For example, in the case of a fluid contained within a rigid box, t would be the projection of the position four-vector onto the four-velocity vector of the box. That is, t would be the proper time of the box. The present analysis will be carried out in the preferred frame in which the characteristic four-velocity vector has only a time-like component, i.e., the rest-frame of the enclosing box in the example cited.

FUNNEL

RIGHT · ANGLE

SOAP FILM

FUNNEL

(A)
CONSTRAINED
TERMINAL POINTS

(B)
UNCONSTRAINED
UPPER
TERMINUS
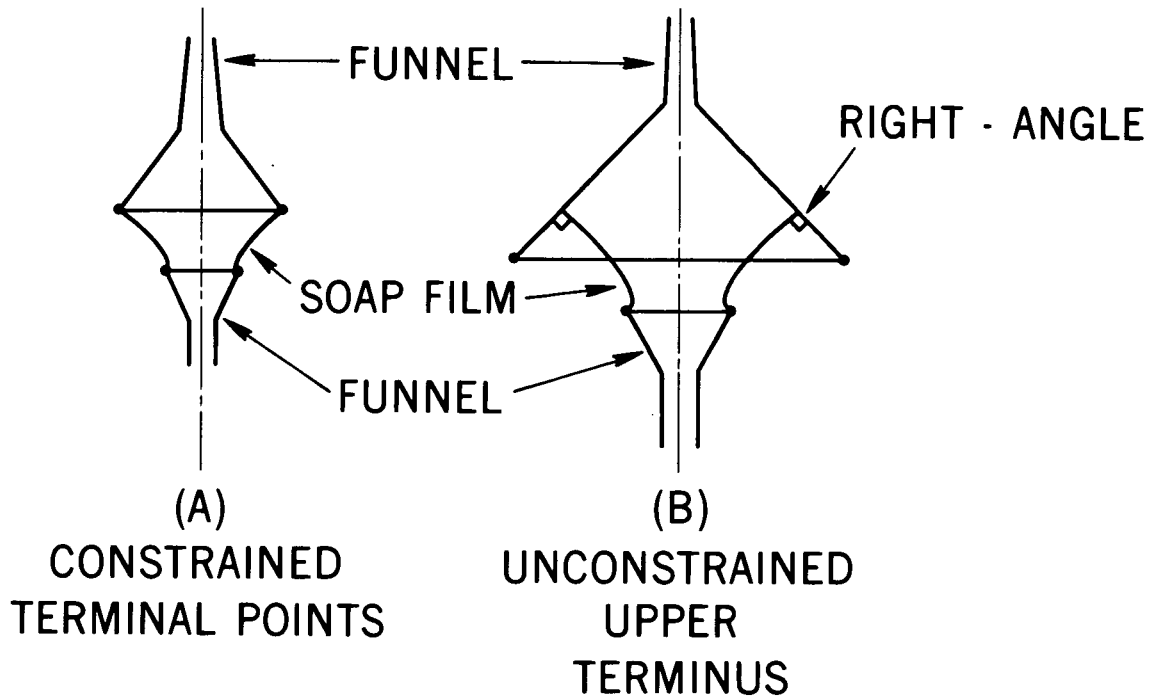
Figure 7. Soap Film Illustration of Basic Types of Terminal Conditions

In the convected frame, the substantial time derivative d/dt is defined as the partial derivative with respect to t holding the identity parameters constant. Holding these three parameters constant amounts to moving along with the particle that is characterized by the numerical values of these parameters. Thus

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_\lambda \qquad (3\text{-}6)$$

where the λ - subscript indicates that all three parameters are held constant.

The fluid three-velocity $\vec{v}$ is defined as

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \equiv \left(\frac{\partial \vec{x}}{\partial t}\right)_\lambda \qquad (3\text{-}7)$$

where $\vec{x}(t, \lambda^\alpha)$ is the position vector at time t of the particle whose identity parameters are λ^α .

The definitions of the functionals G and T follow from (3-2), (3-6) and (3-7):

$$G \equiv d\gamma/d\tau = G^*(1-\beta^2)^{-1/2} \quad \text{where} \quad G^* \equiv d\gamma/dt = (1-\beta^2)^{1/2} G \qquad (3\text{-}8a)$$

$$T \equiv d\theta/d\tau = T^*(1-\beta^2)^{-1/2} \quad \text{where} \quad T^* \equiv d\theta/dt = (1-\beta^2)^{1/2} T \qquad (3\text{-}8b)$$

The velocity dependence of G* and T* is the same as that of the contracted three-volume of a moving body. (The asterisk will always be used to designate a scalar quantity multiplied by $(1 - \beta^2)^{1/2}$.) This velocity dependence of intensive thermodynamic quantities is characteristic of the Planck formalism for relativistic thermodynamics. This formalism is by no means unique, and the reasons for its original introduction by Planck (1907) have more to do with the timing of Planck's work relative to the evolution of relativity theory than with any intrinsic merit of the formalism itself.[17] In particular, Planck did his work before the importance of the concepts of covariance and irreducibility had evolved. The strikingly non-covariant appearance of the Planck formalism can be explained in the present context in terms of the hybrid nature of the independent coordinates — three scalar quantities λ^α, and the time parameter t, which on the face of it appears to be one of the components of a four-vector. As mentioned above, however, t should properly be regarded as a scalar quantity, in which case the four families of surfaces corresponding to $(t, \lambda^1, \lambda^2, \lambda^3)$ do indeed constitute an unambiguous four-space mapping that describes the aggregate of particle trajectories and the hypersurfaces of simultaneity defined by the "natural time coordinate" t. From this point of view $\beta = V/c$, like t, is a scalar quantity, and thus so are G* and T*.

The Planck formalism can be seen in a larger perspective if it is recognized that the choice of time coordinate is not unique. The description of fluid dynamics is essentially the problem of describing mappings in four-space. The three families of hypersurfaces corresponding to the functions $\lambda^\alpha(\underset{\sim}{x})$ can be supplemented by a family of space-like hypersurfaces corresponding to the space-time function $\epsilon(\underset{\sim}{x})$. If ϵ has the dimensions of action, then

$$d\epsilon/d\tau = (\partial\epsilon/\partial\tau)_\lambda \equiv \hat{E} \qquad (3-9)$$

has the dimensions of energy, and may be regarded as a "natural unit of energy."

Note that the use of the proper time in (3-9) assumes that the particle trajectories are known, since $c\tau$ is defined as the arclength of the trajectory. This fact makes τ unsuitable for use as an independent coordinate in a variational context. The generalized time coordinate $\epsilon(\underset{\sim}{x})$ is, however, a space-time function whose definition at any point is independent of the history of the particle passing through that point.

Note also that if we were to use ϵ as the time coordinate in a convected-frame analysis, the basic functions would become

$$(\partial\gamma/\partial\epsilon)_\lambda = G/\hat{E} \ , \quad (\partial\theta/\partial\epsilon)_\lambda = T/\hat{E} \ , \quad (\partial\underset{\sim}{x}/\partial\epsilon)_\lambda = \underset{\sim}{V}/\hat{E} \ . \qquad (3-10)$$

Thus, by using ϵ as the time coordinate, we would effectively be measuring G, T, and the rest-energy vector $Mc\underset{\sim}{V}$ in terms of the "natural energy unit" $\hat{E}$.

The Planck formalism (with an inconsequential change of normalization) results from the choice $\epsilon \equiv Mc^2 t$. In this case

$$\hat{E} = Mc^2(dt/d\tau) = M^\circ c^2$$

where
$$M^\circ = M(1 - \beta^2)^{-1/2}. \qquad (Planck)$$

Thus, if we choose to regard t as the time coordinate of an arbitrarily chosen observer, rather than as a scalar in the manner outlined above, what the Planck formalism does is to use the molar mass-energy of the fluid as seen by the observer as the "natural energy unit". When we express G and T in terms of this unit, we obtain $G/M^\circ c^2 = G^*/Mc^2$ and $T/M^\circ c^2 = T^*/Mc^2$. Thus the fact that, in the Planck formalism the temperature appears to decrease as the speed of the observer relative to the fluid increases, can be explained simply by saying that the basic energy unit that the observer is using appears to increase.

If we make the choice $\epsilon \equiv \Theta$, then we obtain the Van Dantzig formalism. In this case $\hat{E} = (\partial\Theta/\partial\tau)_\lambda = T$ and (3-10) becomes

$$(\partial\gamma/\partial\Theta)_\lambda = G/T \, , \, (\partial\underset{\sim}{\mathcal{N}}/\partial\Theta)_\lambda = \underset{\sim}{V}/T \, , \, \left[(\partial\underset{\sim}{\mathcal{N}}/\partial\Theta)_\lambda \cdot (\partial\underset{\sim}{\mathcal{N}}/\partial\Theta)_\lambda\right]^{1/2} = c^2/T. \quad (3\text{-}11)$$

Comparing this with (3-7) and (3-8), we see that what has been done is to exchange the roles of t and Θ. The thermasy Θ replaces the time t as an independent variable and drops out of the list of dependent variables, while ct joins the three components of $\overrightarrow{\mathcal{N}}$ to complete the four-vector $\underset{\sim}{\mathcal{N}}$. The dependent variables thus become γ and $\underset{\sim}{\mathcal{N}}$, and we have a manifestly covariant formalism. If this formalism is further developed, it can be shown that P is effectively replaced by P/T, and the energy density u is replaced by the negentropy density $-s$. This corresponds to the fact that $-s$ and P/T are dual potentials in the sense that

$$d(-s) = (G/T)dn - (1/T)du \qquad (3\text{-}12a)$$

and

$$d(P/T) = n\,d(G/T) - u\,d(1/T) \, . \qquad (3\text{-}12b)$$

Thus the use of the thermasy as the time coordinate amounts to using the entropy representation of thermodynamics instead of the energy representation.

It can be shown that fluids in reversible thermal interaction with one another must all share the same thermasy. This thermasy can be used as a common scalar time coordinate in such a problem, and this fact can be shown to have important conceptual consequences, as well as obvious formal advantages. In particular, it means that when the formalism is extended to General Relativity, there exists a physically preferred coordinate system in the case of gravitationally interacting fluids in thermal equilibrium with each other - namely the coordinate system corresponding to the normal congruence defined by the family of surfaces of constant thermasy. The physical significance of using this preferred coordinate system is that we are viewing the interacting fluids in the local rest-frame of their common heat reservoir.

It is of historical interest to note that, although Van Dantzig discovered and named the thermasy, its most important advantages escaped him because he defined it to be a path-dependent quantity like proper time (i.e., the line integral $\int T d\gamma$ along the trajectory), rather than a space-time function that is independent of the particle trajectories. Thus he could not use it as a dependent variable in a variational formalism, or as part of a mapping. His primary objective was to construct a covariant alternative to Planck's non-covariant formalism, and in this he succeeded. Unfortunately, his work was largely ignored and then nearly totally forgotten until the essential features had been rediscovered by others. [17]

3.5   Convected-Frame Analysis in Planck Formalism

Despite the formal awkwardness of the Planck formalism that results from using (t, λ^α) as the independent coordinates, this choice does have the pedagogical advantage of providing the most familiar-looking approach to the variational formalism, and for this reason it will now be presented, as an introduction to the manifestly covariant formalism that is developed in sec. 3.6.

Once the functionals G and T have been defined (cf. (3-8)), it is only necessary to substitute them into the function $P(G, T)$ to arrive at the functional $P(\frac{d\lambda}{dt}, \frac{d\theta}{dt}, \frac{d\lambda}{dt})$ which is all that is needed in order to apply the minimum principle. The construction of this functional will now be illustrated for the case of a perfect gas.

The raw empirical information is contained in the function $P(\breve{G}, T)$ where $\breve{G}$ is the non-relativistic free-enthalpy, which is related to the relativistic free-enthalpy G by

$$G = \breve{G} + M c^2.$$

(3-13)

Substituting $\breve{G} = G - Mc^2$ into $P(\breve{G}, T)$, we obtain the desired function $P(G, T)$. For a perfect gas this is

$$P = A T^{-C_p/R} e^{\breve{G}/RT} = A T^{-C_p/R} \exp\left[(G - Mc^2)/RT\right] \qquad (3\text{-}14)$$

where A is a constant, R is the molar gas constant, and C_p is the molar constant-pressure specific heat. (For a monatomic gas, $C_p/R = 5/2$.)

Because $\partial P/\partial \breve{G} = (\partial P/\partial G)(\partial G/\partial \breve{G}) = \partial P/\partial G$, we have

$$dP = n\,dG + s\,dT = n\,d\breve{G} + s\,dT \qquad \text{(in general)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3\text{-}15)$$

$$= (P/RT)\,dG + (P/RT)(C_p - \breve{G}/T)\,dT \qquad \text{(for a perfect gas)}.$$

Since, for a perfect gas, we have

$$P/RT = n$$

$$C_p = \breve{H}/T = (\breve{G} + ST)/T = (\breve{G}/T) + S \qquad (3\text{-}16)$$

it is evident that the second line of (3-15) does in fact agree with the first.

Substituting (3-8) into the first line of (3-15) we find

$$dP = n^\circ\,dG^* + n^\circ\,S\,dT^* + n^\circ (H^\circ/c^2)\,\vec{v}\cdot d\vec{v} \qquad (3\text{-}17)$$

$$= (1-\beta^2)^{-\frac{1}{2}} n\left[d(d\breve{G}/dt) + S\,d(d\Theta/dt) + (1-\beta^2)^{-\frac{1}{2}}(H/c^2)\vec{v}\cdot d\vec{v}\right].$$

The second line of (3-17) is all we need for the purposes of the minimum principle. It is, however, of historical interest to note that the first line of (3-17) can be transformed into

$$d(H^\circ - P/n^\circ) = T^*\,dS - P\,d(1/n^\circ) + \vec{v}\cdot d\left[(H^\circ/c^2)\vec{v}\right] \qquad (3\text{-}18)$$

where $(1/n^\circ)$ is the molar volume of the moving fluid as seen in the laboratory frame. This is the equation that was regarded as the natural generalization of the rest-equation

$$dU = T\,dS - P\,d(1/n) \qquad (3\text{-}19)$$

for a mole of gas, and was taken as the starting point of the Planck formalism.

The action integral in the convected-frame Planck formalism is

$$\mathcal{J} = \int_{(t, \lambda^\alpha)} P\left(\frac{d\breve{G}}{dt}, \frac{d\Theta}{dt}, \frac{d\vec{\lambda}}{dt}\right) \mathcal{N}^* dt\,d^3(\lambda^\alpha) \qquad (3\text{-}20)$$

where

$$\mathcal{N}^* \equiv \det(\partial\vec{\varkappa}/\partial\lambda^\alpha) = (\partial\vec{\varkappa}/\partial\lambda^1) \cdot \left[(\partial\vec{\varkappa}/\partial\lambda^2) \times (\partial\vec{\varkappa}/\partial\lambda^3)\right] \quad \text{(3-21)}$$

is the cell volume corresponding to $d^3(\lambda^\alpha) = 1$, as seen in laboratory frame. The Euler-Lagrange equations are determined by the stationary requirement

$$\delta \mathcal{J} = \delta \int_{(t,\lambda^\alpha)} \mathcal{L}(\partial q^\alpha/\partial t, \partial q^\alpha/\partial\lambda^\alpha) \, dt \, d^3(\lambda^\alpha) = 0 \quad \text{(3-22)}$$

where

$$\mathcal{L} \equiv P \mathcal{N}^* \quad \text{and} \quad q^\alpha = (\gamma, \Theta, \vec{\varkappa}) . \quad \text{(3-23)}$$

The variation is to be carried out for constant terminal values q_i^α and q_f^α. (As previously noted, the terminal values may also be varied if we wish to compare the stability of different coherent evolutions.) Because

$$\left(\frac{q_f^\alpha - q_i^\alpha}{t_f - t_i}\right)_\lambda = \frac{1}{(t_f - t_i)} \int_{t_i}^{t_f} \left(\frac{\partial q^\alpha}{\partial t}\right)_\lambda dt \equiv \left\langle \frac{dq^\alpha}{dt} \right\rangle , \quad \text{(3-24)}$$

it is evident that the variation process is carried out while holding constant the values of

$$\langle d\gamma/dt \rangle = \langle G^* \rangle, \quad \langle d\Theta/dt \rangle = \langle T^* \rangle, \quad \langle d\vec{\varkappa}/dt \rangle = \langle \vec{v} \rangle. \quad \text{(3-25)}$$

for each individual particle of the fluid.

As is well known, the Euler-Lagrange equation that results from the variation of q^α is

$$\delta q^\alpha : \quad \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial q^\alpha/\partial t)}\right] + \frac{\partial}{\partial\lambda^\alpha}\left[\frac{\partial \mathcal{L}}{\partial(\partial q^\alpha/\partial\lambda^\alpha)}\right] = \frac{\partial \mathcal{L}}{\partial q^\alpha}. \quad \text{(3-26)}$$

In order to apply this, we need in addition to (3-17) an identity involving $\mathcal{N}^*$. This is derived as follows: From (3-21) we find

$$\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha\vec{\varkappa})} = (\partial_\beta\vec{\varkappa}) \times (\partial_\gamma\vec{\varkappa}), \quad \left[\text{where } \partial_\alpha \equiv \frac{\partial}{\partial\lambda^\alpha} \text{ and the values of } (\alpha,\beta,\gamma) \text{ are in cyclic order}\right] \quad \text{(3-27)}$$

and from this, it is easily confirmed that

$$\partial_\alpha\left[\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha\vec{\varkappa})}\right] = 0 . \quad \text{(3-28)}$$

From (3-27) and (3-21) we also note that

$$(\partial_\epsilon\vec{\varkappa}) \cdot \left[\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha\vec{\varkappa})}\right] = \mathcal{N}^* \delta_\epsilon^\alpha . \quad (\delta_\epsilon^\alpha \equiv \text{Kronecker delta}) \quad \text{(3-29)}$$

Thus the vector triplets

$$\vec{\Lambda}_\epsilon \equiv \partial_\epsilon \vec{\pi}$$
$$\equiv (\partial \vec{\pi}/\partial \lambda^\epsilon)_t \quad \text{and}$$

$$\vec{\Lambda}^\alpha \equiv \frac{1}{\mathcal{N}^*}\left[\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha \vec{\pi})}\right]$$
$$= \left[(\partial_\beta \vec{\pi}) \times (\partial_\gamma \vec{\pi})\right]/\mathcal{N}^* \quad (\alpha, \beta, \gamma \text{ cyclic})$$

(3-30)

constitute two reciprocal tripods for which

$$\vec{\Lambda}_\epsilon \cdot \vec{\Lambda}^\alpha = \delta_\epsilon^\alpha \; . \tag{3-31}$$

But it is also true that

$$\vec{\Lambda}_\epsilon \cdot (\vec{\nabla}\lambda^\alpha)_t = \left(\frac{\partial \vec{\pi}}{\partial \lambda^\epsilon}\right)_t \cdot (\vec{\nabla}\lambda^\alpha)_t = \delta_\epsilon^\alpha \; . \tag{3-32}$$

Because the tripod that is reciprocal to the tripod $\vec{\Lambda}_\epsilon$ is unique, it follows from (3-29) and (3-32) that

$$(\vec{\nabla}\lambda^\alpha)_t = \frac{1}{\mathcal{N}^*}\left[\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha \vec{\pi})}\right] = \vec{\Lambda}^\alpha \; . \tag{3-33}$$

Using (3-28) and (3-33), it follows that

$$\partial_\alpha\left[P\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha \vec{\pi})}\right] = \left[\frac{\partial \mathcal{N}^*}{\partial(\partial_\alpha \vec{\pi})}\right](\partial_\alpha P)_t$$
$$= (\mathcal{N}^* \vec{\nabla}\lambda^\alpha)(\partial P/\partial \lambda^\alpha)_t = \mathcal{N}^*(\vec{\nabla}P)_t$$

(3-34)

which is the identity we need. Using this and the fact that

$$\mathcal{N}^* n^\circ = \mathcal{N} n = \mathcal{M} \tag{3-35}$$

where $\mathcal{N}$ is the cell volume in the local fluid rest-frame and $\mathcal{M}$ is the number of moles in the cell, we find the following Euler-Lagrange equations:

$$\delta \gamma: \quad \left[\partial(n^\circ \mathcal{N}^*)/\partial t\right]_\lambda = d\mathcal{M}/dt = 0 \tag{3-36}$$

$$\delta \theta: \quad \left[\partial(n^\circ S \mathcal{N}^*)/\partial t\right]_\lambda = d(\mathcal{M}S)/dt = \mathcal{M}\,dS/dt = 0 \tag{3-37}$$

$$\delta \vec{\pi}: \quad d\left[\mathcal{M}(H^\circ/c^2)\vec{v}\right]/dt + \mathcal{N}^*\vec{\nabla}P = 0 \tag{3-38a}$$

$$\text{or}$$
$$d\left[(H^\circ/c^2)\vec{v}\right]/dt = -(\vec{\nabla}P)/n^\circ \; . \tag{3-38b}$$

These are just the statements of conservation of mole number (particles), entropy, and momentum respectively.

It is interesting to note that for the case in which $\mathcal{N}^* = \mathcal{N}_c^*$ is a constant, and the two reservoirs are constant and at rest in the laboratory frame so that

$$\gamma = G_c^* t, \quad \Theta = T_c^* t \quad \text{and} \quad G = \Gamma G^*, \quad T = \Gamma T^* \qquad (3\text{-}39)$$

we have

$$\frac{G - Mc^2}{RT} = \frac{\Gamma G_c^* - Mc^2}{R\Gamma T_c^*} = \left(\frac{G_c^*}{RT_c^*}\right) - \left(\frac{Mc^2}{RT_c^*}\right)\left[1 - \frac{d\vec{\pi}}{dt}\cdot\frac{d\vec{\pi}}{dt}/c^2\right]^{1/2} \quad (3\text{-}40)$$

and

$$\mathscr{L} = P\mathscr{N}^* = \mathscr{N}_c^* A' \exp\left[-\left(\frac{Mc}{RT_c^*}\right)\left(c^2 - \frac{d\vec{\pi}}{dt}\cdot\frac{d\vec{\pi}}{dt}\right)^{1/2}\right] \qquad (3\text{-}41)$$

where $\mathscr{N}_c^*$, A' and (Mc/RT_c^*) are all constants. The argument of the exponential is just the relativistic free-particle Lagrangean. Thus, except for the weighting provided by the exponential, the variational principle for a compressible gas reduces to the relativistic/free-particle case when the compressibility and thermodynamics no longer play a role.

The transition to the non-relativistic case is made by making the identification

$$T \approx T^* = d\gamma/dt \qquad (3\text{-}42)$$

and replacing γ with $\breve{\gamma}$ by means of the substitution

$$\gamma \equiv Mc^2 t + \breve{\gamma}(t, \lambda^\alpha) . \qquad (3\text{-}43)$$

Neglecting $\breve{G}/Mc^2$ and terms of order higher than β^2, we find

$$d\breve{\gamma}/dt \approx \breve{G} - \tfrac{1}{2} Mv^2 = -\breve{L} \qquad (3\text{-}44)$$

where $\breve{L}$ is the Lagrangean of a particle of mass M with potential energy $\breve{G}$.

The non-relativistic pressure functional is

$$P\{\breve{G}, T\} = P\left\{\left[(d\breve{\gamma}/dt) + \tfrac{1}{2} M(d\vec{\pi}/dt)\cdot(d\vec{\pi}/dt)\right], (d\Theta/dt)\right\} \qquad (3\text{-}45)$$

for which

$$dP = n\left[d(d\breve{\gamma}/dt) + M(d\vec{\pi}/dt)\cdot d(d\vec{\pi}/dt) + S d(d\Theta/dt)\right]. \qquad (3\text{-}46)$$

Using this, the non-relativistic Euler-Lagrange equations are found exactly as in the relativistic case.

Although the convected-frame treatment presented above provides the most familiar-appearing introduction to the variational formalism, an analysis in

inertial coordinates (the laboratory frame) that would be common to any number of interacting fluids is not only more appropriate for treating a problem involving several fluids, but also provides much greater insight into the relative stability of different categories of flow. Such a laboratory frame analysis will now be carried out.

3.6 Laboratory Frame Analysis

The formalism in the laboratory frame is manifestly covariant because the independent variables are the four components $(x^j) = (ct, \vec{x})$ of the position four-vector, and the five dependent variables γ, Θ and λ^α are all scalars.

The main problem in the laboratory frame analysis is to find a way of expressing the local fluid velocity $\hat{\underset{\sim}{V}}$ directly in terms of the identity parameters λ^α. Obviously the functional form of $\hat{\underset{\sim}{V}}$ ($\underset{\sim}{x}$) is implicitly contained in the functions $\lambda^\alpha(\underset{\sim}{x})$ because the identity parameters may be thought of as three families of imaginary surfaces embedded in the fluid and swept along with it. The motion of the mutual intersection point of three such surfaces, one from each family, defines the motion of the particle that is identified by the three numbers characterizing the interesecting surfaces. The array of parallelepipeds formed by these surfaces are the unit cells of volume ν^* (as observed in the laboratory frame), and the reciprocal of ν^* is the cell density ρ°, i.e., $\rho^\circ \equiv 1/\nu^*$. Since these cells are swept along with the fluid, we can define a cell flux-density vector $\underset{\sim}{\rho}$:

$$\underset{\sim}{\rho} \equiv \rho \hat{\underset{\sim}{V}} \quad \text{or} \quad \underset{\sim}{\rho} = (\rho^j) = \rho^\circ(1, \vec{V}/c) . \tag{3-47}$$

This vector $\underset{\sim}{\rho}$ is the key to the kinematic description of the fluid in the laboratory frame because it can be expressed very simply in terms of the functions $\lambda^\alpha(\underset{\sim}{x})$:

$$\rho^j \equiv \varepsilon^{jk\ell n} \partial_k \lambda^1 \partial_\ell \lambda^2 \partial_n \lambda^3 \tag{3-48}$$

where $\varepsilon^{jk\ell n}$ is the Levi-Cevita symbol. $(\varepsilon^{jk\ell n} = +1$ if $j \neq k \neq \ell \neq n$ and the order of the numerical values of j k l n is an even permutation of the order 0 1 2 3 and $\varepsilon^{jk\ell n} = -1$ if the order is an odd permutation.$)$

It is obvious from (3-48) that the contraction of $\underset{\sim}{\rho}$ with any of the three gradients $\underset{\sim}{\partial}\lambda^\alpha$ is identically zero:

$$\underset{\sim}{\rho} \cdot \underset{\sim}{\partial}\lambda^\alpha = (\rho/c) d\lambda^\alpha/d\tau = 0 . \quad (\alpha = 1, 2, 3) \tag{3-49}$$

These three equations are the proof that $\underset{\sim}{\rho}$ is proportional to $\overset{\wedge}{\underset{\sim}{V}}$, because the definition of $\underset{\sim}{\rho}$ given in (3-48) indeed guarantees that the λ^{α}'s are constants of motion which, from their definition, they obviously must be.

An indication that the magnitude ρ of $\underset{\sim}{\rho}$ may be interpreted as the cell density in the local fluid rest-frame is provided by the fact that (3-48) guarantees the identity

$$\underset{\sim}{\partial} \cdot \underset{\sim}{\rho} = 0 \qquad \text{(conservation of the convected cells) (3-50)}$$

which must be satisfied if the mapping represented by the functions $\lambda^{\alpha}(\underset{\sim}{x})$(together with a suitable scalar time coordinate) is to be non-singular.

The verification that the ρ^{0} defined by (3-48) is indeed the reciprocal of the ν^{*} defined in (3-21) follows from direct calculation of ρ^{0} from (3-48):

$$\rho^{0} = (\vec{\nabla}\lambda^{1}) \cdot [(\vec{\nabla}\lambda^{2}) \times (\vec{\nabla}\lambda^{3})] = \det(\partial\lambda^{\alpha}/\partial\vec{x}) = 1/\nu^{*}. \quad (3-51)$$

From (3-47) and (3-48), we find the following expression for the local three-velocity of the fluid.

$$\vec{v} = -(\partial\lambda^{\alpha}/\partial t)_{\hat{x}} \vec{\Lambda}_{\alpha} = v^{\alpha}\vec{\Lambda}_{\alpha} \qquad (3-52a)$$

where

$$\vec{\Lambda}_{\alpha} \equiv (\partial\vec{x}/\partial\lambda^{\alpha})_{t} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}[(\vec{\nabla}\lambda^{\beta})_{t} \times (\vec{\nabla}\lambda^{\gamma})_{t}]/\rho^{0} \qquad (3-52b)$$

$$(\alpha, \beta, \gamma \text{ in cyclic order})$$

is the same tripod introduced in (3-30), but now expressed in terms of $\lambda^{\alpha}(\underset{\sim}{x})$ instead of $\vec{x}(t, \lambda^{\alpha})$ as in (3-30). Taking the dot product of (3-52a) with $\vec{\Lambda}^{\beta}$ and using (3-31), we find

$$\vec{v} \cdot \vec{\Lambda}^{\beta} = v^{\beta} = -(\partial\lambda^{\beta}/\partial t)_{\vec{x}}. \qquad (3-52c)$$

The physical meaning of (3-52) is illustrated in Figure 8. The three numbers v^{β} are just the projections of $\vec{v}$ onto the vectors of the tripod, which is illustrated in the figure by the solid arrows. These numbers represent the velocity components that an observer fixed in the laboratory frame at the point P would measure by observing the time-rate of change of the identity labels passing the point P. The reason for the minus sign that appears in the definition of v^{β}
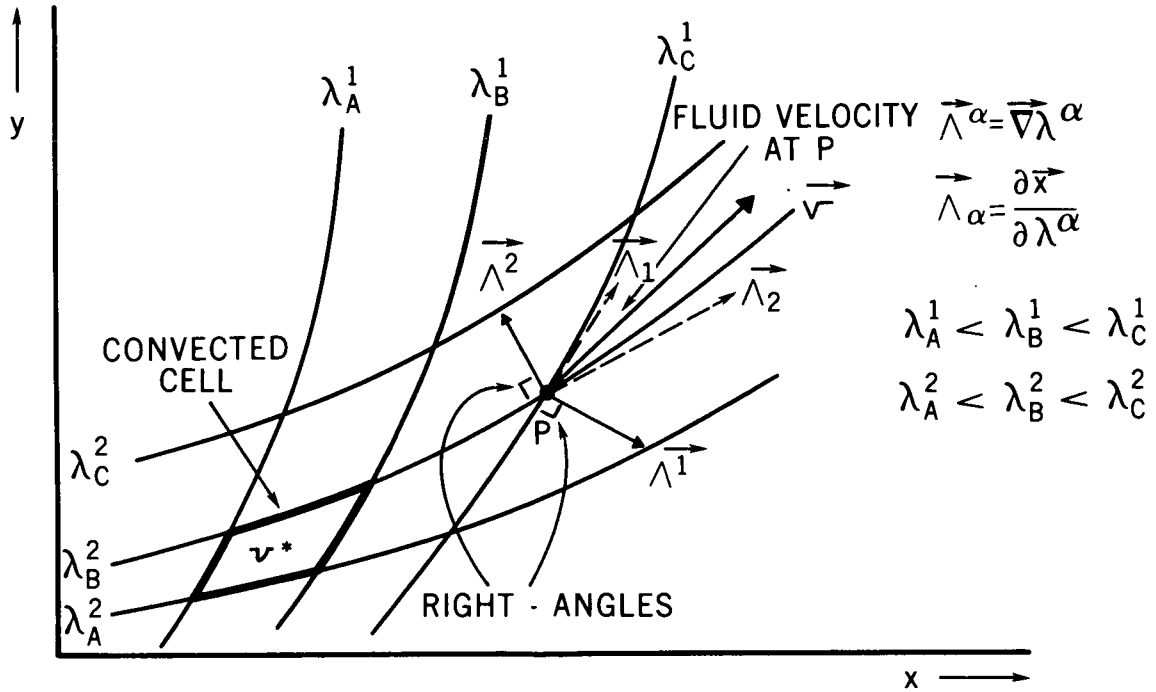
Figure 8. The Convected Coordinate System

in (3-52c) is evident from the figure. If the subscripts A, B, and C indicate parameters of increasing magnitude, then if the flow is positive, i.e., in the direction of increasing parameters which is the direction of the $\vec{\Lambda}^{\beta}$ vectors as indicated in the figure, the observer at P will observe parameters of decreasing magnitude as time passes.

The vectors $\vec{\Lambda}_{\alpha}$ and $\vec{\Lambda}^{\alpha}$ introduced in sec. 3.5 and illustrated in Figure 8 constitute the appropriate pair of tripods to express any vector that lies in the hyperplane t = constant, i.e. an ordinary three-vector as seen in the laboratory frame. In actual fact, we are dealing with a pair of tetrapods rather than tripods, the fourth vector of each tetrapod being the velocity four-vector of the laboratory itself.

In order to refer vectors to the local rest-frame of the fluid, we need a different pair of tetrapods. The time-like vector of each tetrapod is now chosen to be the local normalized fluid velocity $\hat{\underset{\sim}{V}}$. . From (3-49) we see that

$$\hat{\underset{\sim}{V}} \cdot \underset{\sim}{\Lambda}^{\alpha} = 0 \quad \text{where} \quad \underset{\sim}{\Lambda}^{\alpha} \equiv \underset{\sim}{\partial} \lambda^{\alpha} \qquad (3\text{-}53a,b)$$

Thus the three independent four vectors $\underset{\sim}{\Lambda}^{\alpha}$ span the hyperplane that is orthogonal to $\hat{\underset{\sim}{V}}$ at the point in question, and the tetrapod $(\hat{\underset{\sim}{V}}, \Lambda^{\alpha})$ constitutes a complete set of base vectors in terms of which any arbitrary vector can be expressed.

From (3-47) and (3-48) it is evident that the vectors

$$\tilde{\underset{\sim}{\Lambda}}{}_{\alpha}^{j} \equiv -(1/\rho)\, \varepsilon^{jk\ell n}\, \hat{\underset{\sim}{V}}_{k}\, \Lambda^{\beta}_{\ell}\, \Lambda^{\gamma}_{n} \qquad (\alpha,\beta,\gamma \text{ in cyclic order}) \quad (3\text{-}54)$$

have the properties

$$\tilde{\underset{\sim}{\Lambda}}{}_{\alpha}\cdot\hat{\underset{\sim}{V}} = 0 \;,\quad \tilde{\underset{\sim}{\Lambda}}{}_{\alpha}\cdot\Lambda^{\alpha} = 1 \;,\quad \tilde{\underset{\sim}{\Lambda}}{}_{\alpha}\cdot\Lambda^{\beta} = 0 \quad (\alpha \neq \beta) \qquad (3\text{-}55)$$

Thus the tetrapod $(\hat{\underset{\sim}{V}}, \tilde{\underset{\sim}{\Lambda}}{}_{\alpha})$ is reciprocal to the tetrapod $(\hat{\underset{\sim}{V}}, \Lambda^{\alpha})$.
Note that, although the space-like part of Λ^{α} (in covariant form) is equal
to the tripod three vector $\vec{\Lambda}^{\alpha} = (\vec{\nabla}\lambda^{\alpha})_{t}$ that was used in sec. 3.5, the
space-like part of $\tilde{\underset{\sim}{\Lambda}}{}_{\alpha}$ is not simply related to the vector $\vec{\Lambda}_{\alpha} \equiv (\partial\vec{\pi}/\partial\lambda^{\alpha})_{t}$.
The tilde has been placed over Λ^{α} to emphasize this fact. Note also that
whereas in sec. 3.5 normalization was defined by $\vec{\Lambda}_{\alpha}\cdot\vec{\Lambda}^{\alpha} = 1$, it is now
defined by

$$1 = \tilde{\underset{\sim}{\Lambda}}{}_{\alpha}\cdot\Lambda^{\alpha} = (\tilde{\Lambda}{}_{\alpha}^{0}\Lambda^{\alpha}_{0} - \vec{\tilde{\Lambda}}_{\alpha}\cdot\vec{\Lambda}^{\alpha}) \qquad (3\text{-}56)$$

where $\vec{\Lambda}^{\alpha}$ and $\vec{\tilde{\Lambda}}_{\alpha}$ are the respective space-like parts of Λ^{α} and $\underset{\sim}{\Lambda}_{\alpha}$ (both
in either contravariant or covariant form). The minus sign in (3-56) is a result
of the signature used for the Minkowski metric, and constitutes another difference
between the tetrapod formalism being developed here and the tripod formalism
developed in sec. 3.5.

It is convenient to write (3-54) in the form

$$\tilde{\underset{\sim}{\Lambda}}{}_{\alpha} = \hat{\underset{\sim}{V}}\cdot(\underset{\sim}{\Lambda}^{\beta}\wedge\underset{\sim}{\Lambda}^{\gamma})/\rho \qquad (3\text{-}57a)$$

where

$$(\underset{\sim}{\Lambda}^{\beta}\wedge\underset{\sim}{\Lambda}^{\gamma})^{jk} \equiv \varepsilon^{jk\ell n}\, \Lambda^{\beta}_{\ell}\, \Lambda^{\gamma}_{n} \;.$$

From (3-48) we find

$$\partial\rho^{j}/\partial\Lambda^{\alpha}_{k} = \varepsilon^{jk\ell n}\,\Lambda^{\beta}_{\ell}\,\Lambda^{\gamma}_{n} \quad \text{or} \quad \partial\underset{\sim}{\rho}/\partial\underset{\sim}{\Lambda}^{\alpha} = \underset{\sim}{\Lambda}^{\beta}\wedge\underset{\sim}{\Lambda}^{\gamma} \;. \qquad (3\text{-}58)$$

(Note that in the bold-face form of this relation it is important to use the solidus
notation for the derivative, because this indicates the order of the indices of the
tensor equation.)

Using (3-57) and (3-58) it is a simple matter to derive the relation

$$\partial \rho / \partial \underset{\sim}{\Lambda}{}^{\alpha} \equiv \partial (\underset{\sim}{\rho} \cdot \underset{\sim}{\rho})^{1/2} / \partial \underset{\sim}{\Lambda}{}^{\alpha} = \rho \underset{\sim}{\widetilde{\Lambda}}_{\alpha} \qquad (3\text{-}59)$$

which provides us with an alternative expression for the $\underset{\sim}{\widetilde{\Lambda}}_{\alpha}$ vectors in terms of the $\underset{\sim}{\Lambda}{}^{\alpha}$ tetrapod. Summarizing the various expressions, we have

$$\underset{\sim}{\widetilde{\Lambda}}_{\alpha} = \underset{\sim}{\hat{v}} \cdot (\underset{\sim}{\Lambda}{}^{\beta} \wedge \underset{\sim}{\Lambda}{}^{\gamma}) / \rho = (\partial \rho / \partial \underset{\sim}{\Lambda}{}^{\alpha}) / \rho = -(\partial \nu / \partial \underset{\sim}{\Lambda}{}^{\alpha}) / \nu \qquad (3\text{-}60)$$

where $\nu = 1/\rho$ is the cell volume in the local rest-frame of the fluid. The last form for $\underset{\sim}{\widetilde{\Lambda}}_{\alpha}$ that involves ν is the analog of the expression (3-33) for $\underset{\sim}{\overset{\rightharpoonup}{\Lambda}}{}^{\alpha}$. The difference in sign is a consequence of the sign difference noted following (3-56).

It should be noted that the tetrapod formalism that has just been developed is a truncated version of the formalism that results in the case of a complete mapping in four-space in which the time-like vector, as well as the three space-like ones, is expressible as the gradient of some scalar function $\in (\underset{\sim}{\alpha})$. In such a case the vectors $\underset{\sim}{\widetilde{\Lambda}}_{\alpha}$ are expressible as functions of $(\in, \lambda^{\alpha})$ regarded as independent variables: $\underset{\sim}{\widetilde{\Lambda}}_{\alpha} = (\partial \underset{\sim}{\alpha} / \partial \lambda^{\alpha})_{\in}$ where $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}(\in, \lambda^{\gamma})$. The time-like vectors of the two tetrapods are $\underset{\sim}{\Lambda}{}^{\in} \equiv \underset{\sim}{\partial} \in$ and $\underset{\sim}{\Lambda}_{\in} \equiv (\partial \underset{\sim}{\alpha} / \partial \in)_{\lambda}$, and in general $\underset{\sim}{\Lambda}{}^{\in} \neq \underset{\sim}{\Lambda}_{\in}$. When they are equal, the mapping becomes what in mathematics is called a <u>normal congruence</u>, and in fluid dynamics is called <u>potential flow</u>. We shall discuss such a case in greater detail in section 4.

The laboratory frame analysis is based on the identification

$$G \equiv \underset{\sim}{\hat{v}} \cdot \underset{\sim}{\beta} = -\underset{\sim}{\rho} \cdot \underset{\sim}{\partial} \gamma / (\underset{\sim}{\rho} \cdot \underset{\sim}{\rho})^{1/2} \qquad (3\text{-}61a)$$

$$T \equiv \underset{\sim}{\hat{v}} \cdot \underset{\sim}{\mathcal{F}} = c \underset{\sim}{\rho} \cdot \underset{\sim}{\partial} \theta / (\underset{\sim}{\rho} \cdot \underset{\sim}{\rho})^{1/2} \qquad (3\text{-}61b)$$

where $\underset{\sim}{\rho}$ in turn is expressed in terms of the gradients of the three functions $\lambda^{\alpha}(\underset{\sim}{\alpha})$ as given in (3-48). As an introduction, however, it is more convenient at first to regard the four components of $\underset{\sim}{\rho}$ to be independent functions. Using (3-61) in P(G, T), the relation $dP = n d G + s d T$ becomes

$$dP = \underset{\sim}{n} \cdot d \underset{\sim}{\beta} + S \underset{\sim}{n} \cdot d \underset{\sim}{\mathcal{F}} - (\in n / \rho) \underset{\sim}{K} \cdot d \underset{\sim}{\rho} \qquad (3\text{-}62a)$$

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where

$$\underset{\sim}{n} \equiv n \overset{\wedge}{\underset{\sim}{V}} \quad \text{and} \quad c \underset{\sim}{K} \equiv \underset{\sim}{H} - \underset{\sim}{\cancel{\mu}} - S \underset{\sim}{\mathcal{J}} = H \overset{\wedge}{\underset{\sim}{V}} - c \underset{\sim}{\partial} \gamma - c S \underset{\sim}{\partial} \Theta. \quad \text{(3-62b)}$$

Because

$$c \underset{\sim}{K} \cdot \overset{\wedge}{\underset{\sim}{V}} = H - \overset{\wedge}{\underset{\sim}{V}} \cdot \underset{\sim}{\cancel{\mu}} - S \overset{\wedge}{\underset{\sim}{V}} \cdot \underset{\sim}{\mathcal{J}} = H - G - ST = 0 \quad \text{(3-63)}$$

the expansion of $\underset{\sim}{K}$ in terms of the $(\overset{\wedge}{\underset{\sim}{V}}, \underset{\sim}{\Lambda}^{\alpha})$ tetrapod will contain no time-like component.

$$\underset{\sim}{K} = K_{\alpha} \underset{\sim}{\Lambda}^{\alpha} = K_{\alpha} \underset{\sim}{\partial} \lambda^{\alpha} \quad \text{where} \quad K_{\alpha} = \underset{\sim}{K} \cdot \underset{\sim}{\tilde{\Lambda}}_{\alpha}. \quad \text{(3-64)}$$

Substituting this into (3-62a) and using the fact that $\underset{\sim}{\Lambda}^{\alpha} \cdot \underset{\sim}{\rho} = 0$ we find

$$dP = \underset{\sim}{n} \cdot \left[d \underset{\sim}{\cancel{\mu}} + S d \underset{\sim}{\mathcal{J}} + c K_{\alpha} d \underset{\sim}{\Lambda}^{\alpha} \right] ; \quad \text{(3-65a)}$$

$$dP = c \underset{\sim}{n} \cdot \left[d(\underset{\sim}{\partial} \gamma) + S d(\underset{\sim}{\partial} \Theta) + K_{\alpha} d(\underset{\sim}{\partial} \lambda^{\alpha}) \right]. \quad \text{(3-65b)}$$

In the local fluid rest-frame this becomes

$$dP = n d \cancel{\mu}_{rf}^{\circ} + s d \mathcal{J}_{rf}^{\circ} + (\vec{K} \cdot \vec{\tilde{\Lambda}}_{\alpha})_{rf} (dv^{\alpha})_{rf} \quad \text{(3-66)}$$

where by (3-52c) $c \Lambda^{\alpha}_{o} = \partial \lambda^{\alpha} / \partial t = -v^{\alpha}$ and $\vec{K}$ and $\vec{\tilde{\Lambda}}_{\alpha}$ are the space-like parts of $\underset{\sim}{K}$ and $\underset{\sim}{\tilde{\Lambda}}_{\alpha}$ respectively. Note that although $v^{\alpha} = 0$ in the local rest-frame, in general $(dv^{\alpha})_{rf} \neq 0$. The vector $\vec{K}$ is just that part of the total molar momentum $\vec{H}/c$ that is independent of the particle and heat reservoirs, and $(\vec{K} \cdot \vec{\Lambda})_{rf}$ is just the rest-frame value of $\vec{K}$ referred to the "natural coordinate axes" of the fluid.

The relation (3-65b) is all we need to derive the Euler-Lagrange equations of the minimum principle, which in the laboratory frame is

$$\cancel{f} = (1/c) \int_{(\underset{\sim}{x})} P(\underset{\sim}{\partial} \gamma, \underset{\sim}{\partial} \Theta, \underset{\sim}{\partial} \lambda^{\alpha}) d^{4}(\underset{\sim}{x}) \quad = \text{minimum.} \quad \text{(3-67)}$$

The variation is to be carried out holding the terminal values and constant. Since

$$\left(\frac{q_f^\alpha - q_i^\alpha}{t_f - t_i}\right)_{\vec{\mathcal{R}}} = \left\langle \frac{\partial q}{\partial t} \right\rangle_{\vec{\mathcal{R}}} \qquad (3\text{-}68)$$

the physical significance of holding the terminal conditions fixed during the variation is that, at a given <u>laboratory position</u> $\vec{\mathcal{R}}$ the time averages

$$\left\langle \partial \gamma / \partial t \right\rangle_{\mathcal{R}} = \left\langle b^o \right\rangle_{\vec{\mathcal{R}}} \, , \quad \left\langle \partial \Theta / \partial t \right\rangle_{\vec{\mathcal{R}}} = \left\langle \mathcal{S}^o \right\rangle_{\vec{\mathcal{R}}} \, , \quad \left\langle -\partial \lambda^\alpha / \partial t \right\rangle_{\vec{\mathcal{R}}} = \left\langle r^\alpha \right\rangle_{\vec{\mathcal{R}}} \quad (3\text{-}69)$$

are to be the same for all incoherent evolutions (trial solutions) as for the coherent evolution (extremal solution that minimizes $\not\!f$). Comparing this with (3-25), we see that in the convected frame analysis the averages that were held constant referred to a given particle, whereas in the laboratory frame analysis we are dealing with averages over time at a given position in the laboratory frame.

Using the fact that in the laboratory frame the Euler-Lagrange equations for an arbitrary Lagrangean density are

$$\underset{\sim}{\partial} \cdot \left[\frac{\partial \mathcal{L}}{\partial (\underset{\sim}{\partial} q^\alpha)} \right] = \frac{\partial \mathcal{L}}{\partial q^\alpha} \qquad (3\text{-}70)$$

we find the following equations:

$$\delta \gamma : \qquad c \underset{\sim}{\partial} \cdot \underset{\sim}{n} = 0 \qquad (3\text{-}71)$$

$$\delta \Theta : \qquad c \underset{\sim}{\partial} \cdot (\underset{\sim}{n} S) = n \, dS / d\tau = 0 \qquad (3\text{-}72)$$

$$\delta \lambda^\alpha : \qquad c \underset{\sim}{\partial} \cdot (\underset{\sim}{n} K_\alpha) = n \, dK_\alpha / d\tau = 0 \, . \qquad (3\text{-}73)$$

This means that the flow is characterized by the four constants of motion (S, K_α). These four scalars (under Lorentz transformations) can, in fact be regarded as the components of a single four-vector if general curvilinear coordinate transformations are admitted. This can be seen as follows: From the relation

$$(\underset{\sim}{H} - \underset{\sim}{\mathcal{H}})/c = S \underset{\sim}{\mathcal{S}}/c + K_\alpha \underset{\sim}{\Lambda}^\alpha = S \underset{\sim}{\partial} \Theta + K_\alpha \underset{\sim}{\partial} \lambda^\alpha \qquad (3\text{-}74)$$

we see that $(S\tilde{\mathcal{S}}/c + K_\alpha \Lambda^\alpha)$ is equal to the molar four-momentum $\underset{\sim}{H}/c$ of the fluid referred to that of the particle reservoir $\underset{\sim}{\mu}/c$. That is, $(S\tilde{\mathcal{S}}/c + K_\alpha \Lambda^\alpha)$ is the amount of the excess four-momentum by which the fluid is out of equilibrium with its own particle reservoir. If we use $(\tilde{\mathcal{S}}/c, \underset{\sim}{\Lambda}^\alpha)$ as our reference tetrapod instead of the four Cartesian unit vectors of the laboratory frame, we see that the four numbers (S, K_α) are the projections of the excess four-momentum $(\underset{\sim}{H} - \underset{\sim}{\mu})/c$ onto the vectors of the tetrapod $(\tilde{\mathcal{S}}/c, \underset{\sim}{\Lambda}^\alpha) = (\underset{\sim}{\partial}\Theta, \underset{\sim}{\partial}\lambda^\alpha)$ which is obviously reciprocal to the tetrapod $(\partial\underset{\sim}{x}/\partial\Theta, \partial\underset{\sim}{x}/\partial\lambda^\alpha)$ which characterizes the Van Dantzig formalism in which the independent coordinates are (Θ, λ^α). In this sense, (S, K_α) is the generalized momentum four-vector that is conjugate to the curvilinear coordinates (Θ, λ^α) that determine the mapping that describes the fluid motion.

To summarize these remarks, we would say that a complete description of the fluid requires first a knowledge of the function $\gamma(\underset{\sim}{x})$ whose gradient
$$c\,\underset{\sim}{\partial}\gamma = \underset{\sim}{\mu}$$
provides the reference energy four-vector. Second, we need the four functions $[\Theta(\underset{\sim}{x}), \lambda^\alpha(\underset{\sim}{x})]$ which determine a "natural coordinate system". Finally we need the components of the four-vector (in the general curvilinear sense) (S, K_α) which specifies, in the natural coordinate system determined by (Θ, λ^α), the value of the fluid energy four-vector $\underset{\sim}{H}$ with respect to the reference $\underset{\sim}{\mu}$. Since the components (S, K_α) are constants of motion, they are determined by the initial conditions, i.e. they are the retained memory of the fluid's past. This memory can be regarded as having been erased if the right-hand side of (3-74) becomes the gradient of a scalar which could be absorbed into $\underset{\sim}{\mu}$ by a redefinition of γ. In particular, we will be interested in the case for which $K_\alpha = 0$ and $\underset{\sim}{\partial}S = 0$. For this case (3-74) can be written

$$\underset{\sim}{H} = c\,\underset{\sim}{\partial}(\gamma + S\Theta) \equiv c\,\underset{\sim}{\partial}\eta \equiv \underset{\sim}{\mu} . \qquad (3\text{-}75)$$

This is the case of <u>potential flow</u> that will be discussed at greater length in sec. 4. Suffice it simply to note at this point that such a flow is self-perpetuating in the sense that, as eqs. (3-72) and (3-73) indicate, once either of the conditions $\underset{\sim}{\partial}K_\alpha = 0$ or $\underset{\sim}{\partial}S = 0$ becomes established throughout a sample of the fluid through turbulence, diffusion, or any other mechanism it will be maintained in that sample throughout all later time.

The equations (3-71) — (3-73), together with (3-74) (which is really just the definition of K_α) imply the relativistic Euler equation, which is just the statement of local conservation of energy-momentum. It is easy to show that, if we

contract the curl of (3-74) with $\underset{\sim}{V} = c\hat{\underset{\sim}{V}}$ and use $dS/d\tau = d K_\alpha/d\tau = d\lambda^\alpha/d\tau = 0$ and $\underset{\sim}{V} \cdot \underset{\sim}{S}/c = \dot{T}$, we obtain

$$d[(H/c^2)\underset{\sim}{V}]/d\tau = \underset{\sim}{\partial} H - T\underset{\sim}{\partial} S = (\underset{\sim}{\partial} P)/n \qquad \text{(3-76a)}$$

or

$$d H^\circ/d\tau = (\partial P/\partial t)/n \qquad \text{(3-76b)}$$

$$d[(H^\circ/c^2)\vec{r}]/d\tau = -(\vec{\nabla} P)/n \qquad \text{(3-76c)}$$

where (3-76c) corresponds to the contravariant form of (3-76a) in which $\underset{\sim}{V} = (V^j) = (c\Gamma, \Gamma\vec{v})$ and $\underset{\sim}{\partial} = (\partial^j) = (\partial/c\partial t, -\vec{\nabla})$. The conversion of the right-hand side of (3-76a) into $(\underset{\sim}{\partial} P)/n$ is simply on application of the relation

$$dP = ndG + nSdT = n(dH - TdS) \qquad \text{(3-77)}$$

in which the differentials are successively taken to correspond to unit displacements in each of the orthogonal directions of four-space.

3.7 Accuracy test of a Direct Solution

A direct solution of a problem using the variational minimum principle would usually involve expressing each of the dependent functions $\Upsilon(\underset{\sim}{x})$, $\Theta(\underset{\sim}{x})$, and $\lambda^\alpha(\underset{\sim}{x})$ in terms of a chosen class of functions containing undetermined parameters. This class would be selected to satisfy the imposed boundary and terminal conditions and to imitate as closely as possible the anticipated physical behavior of the dependent function. The values of all the undetermined parameters would then be chosen to minimize the action integral. Let the dependent functions corresponding to this minimal choice of parameters be $\overline{\Upsilon}(\underset{\sim}{x})$, $\overline{\Theta}(\underset{\sim}{x})$ and $\overline{\lambda}^\alpha(\underset{\sim}{x})$. Using (3-61) we could now calculate $\overline{G}(\underset{\sim}{x})$ and $\overline{T}(\underset{\sim}{x})$ for our solution.

In general this solution will be only approximate because the class of functions that we used as trial solutions will have been too restrictive, and it would be good to have a simple test for the accuracy of the approximate solution obtained. Ideally the error should be expressed in terms of a single number that

is positive-definite and approaches zero as the error approaches zero. Such a measure of the error does in fact exist, and will now be derived.

From the first line of (2-13) we have

$$d^{(2)}u = P(\bar{G}, \bar{T}) + u(n,s) - \bar{G}n - \bar{T}s \geqslant 0. \quad \text{(3-78)}$$

The equality holds when

$$\left[(\partial\bar{P}/\partial\bar{G}) = n \,;\, (\partial\bar{P}/\partial\bar{T}) = s\right] \,;\, \left[(\partial u/\partial n) = \bar{G} \,;\, (\partial u/\partial s) = \bar{T}\right]$$

$$(3\text{-}79a,b)$$

which are just the equilibrium conditions. If we identify $\bar{G}$ and $\bar{T}$ with the approximate space-time functions obtained by minimizing the action integral, and <u>define</u> $n(\bar{G}, \bar{T})$ and $s(\bar{G}, \bar{T})$ to be the functions that we obtain from (3-79a), then if we substitute these functions into the known function $u(n, s)$ and calculate

$$G \equiv \frac{\partial u[n(\bar{G}, \bar{T}), s(\bar{G}, \bar{T})]}{\partial n} \,;\, T \equiv \frac{\partial u[n(\bar{G}, \bar{T}), s(\bar{G}, \bar{T})]}{\partial s} \quad \text{(3-80)}$$

we will arrive at functions $G(\bar{G}, \bar{T})$ and $T(\bar{G}, \bar{T})$ which in general are different from $\bar{G}$ and $\bar{T}$. Thus, although the conditions of (3-79a) will have been satisfied by definition, the second set in general will not be satisfied, and the number $[d^{(2)}u]$ in (3-79) will be positive. Thus

$$\int_{(\underset{\sim}{x})}[d^{(2)}u] \, d^4(\underset{\sim}{x}) = \int_{(\underset{\sim}{x})}[P(\bar{G}, \bar{T}) + u(n,s) - \bar{G}n - \bar{T}s] d^4(\underset{\sim}{x}) \geqslant 0. \quad \text{(3.81)}$$

This number will be zero only when the solution to the problem is exact. The ratio of this number to the calculated numerical value of the action integral may be regarded as a crude "error percentage" of the solution.

Friedrichs Transformation

There exists an alternative procedure for testing the accuracy of the solution that is closely related to the so-called canonical form of the variational principle (which as we shall see is not nearly so powerful or convenient for problem-solving as the standard form given in (3-67)).

We can rewrite (3-81) in the form

$$\int_{(x)} P(G,T)\, d^+(x) \geqslant \int_{(x)} \mathcal{P}\, d^+(x) \tag{3-82a}$$

where

$$\mathcal{P} \equiv n\,G + s\,T - u(n,s)\,. \tag{3-82b}$$

(The overhead bars have been dropped because it is understood that n and s are to be entirely independent of G and T). Referring to (2-13) we see that $d\mathcal{P} = d^{(1)}P$. In terms of the figure we could draw that would be the analog of Figure 4 with $P(Q^a)$ replacing $u(p_a)$, it would be evident that $\mathcal{P}$ is just the plane tangent to the function $P(G,T)$ at the point determined by n and s.

Thus we write (3-82) in the form

$$\int_{(x)} P(G,T)\, d^+(x) \geqslant \int_{(x)} \left[n\,G + s\,T - u(n,s) \right] d^+(x) \tag{3-83a}$$

where on both sides of the equation

$$\left. \begin{aligned} G &= G(\partial\gamma,\partial\lambda^\alpha) \equiv \underset{\sim}{v}\cdot\partial\gamma \\ T &= T(\partial\Theta,\partial\lambda^\alpha) \equiv \underset{\sim}{v}\cdot\partial\Theta \end{aligned} \right\} \quad \begin{aligned} \underset{\sim}{v} &\equiv c\underset{\sim}{\rho}/(\underset{\sim}{\rho}\cdot\underset{\sim}{\rho})^{1/2} \\ \rho^j &\equiv \varepsilon^{jk\ell n}\,\partial_k\lambda^1\,\partial_\ell\lambda^2\,\partial_n\lambda^3 \end{aligned} \tag{3-83b}$$

and n and s on the right-hand side are regarded as <u>completely independent</u> functions of $\underset{\sim}{x}$.

We now use the following procedure to approach the solution of the variational problem: For <u>given</u> trial functions for γ, Θ and λ^α (hence for given G and T), which we hold fixed, we vary n and s so as to make the right-hand side a <u>maximum</u>. It is apparent from (3-78) that

$$\overline{G}\,n + \overline{T}\,s - u(n,s) \leq P(\overline{G},\overline{T}) \tag{3-84}$$

where the bars now indicate the fixed values of G and T. This means that as we vary n <u>and</u> s, the left-hand side of (3-84) is bounded from above by the fixed value $P(\overline{G},\overline{T})$. We then choose new functional dependencies for $\gamma(x)$, $\Theta(x)$ and $\lambda^\alpha(x)$ so as to <u>decrease</u> the left-hand side of (3-83a). We then repeat the first process on the right-hand side, using the new values of G and T just determined as fixed parameters as we vary n and s.

In this way the left-hand side of (3-83a) is made to approach a __minimum__, and the right-hand side a __maximum__. For the exact solution of the problem the two sides would be equal, but for a given class of trial functions there will in general remain an irreducible gap between the two sides which is a measure of the inadequacy of the chosen class of trial functions to represent the exact solution.

This process, in particular the construction of an integral that approaches a maximum from one that approaches a minimum, is known in the mathematical literature as the __Friedrichs Transformation__.[18]

Friedrichs was stimulated to find this transformation by the desire to show the equivalance of the Dirichlet and Castigliano Principles in static elasticity theory which, as previously remarked, correspond respectively to the right-hand side of (3-83a) (with sign reversed) and to the left-hand side.

Canonical Form of the Variational Principle

Using the definitions (3-61) for G and T as functionals of $\gamma(\underset{\sim}{x})$, $\Theta(\underset{\sim}{x})$ and $\lambda^{\alpha}(\underset{\sim}{x})$, the right-hand side of (3-82a) can be used by itself as a variational principle:

$$\delta \int_{(\underset{\sim}{x})} \left[n \underset{\sim}{V} \cdot \underset{\sim}{\partial} \gamma + s \underset{\sim}{V} \cdot \underset{\sim}{\partial} \Theta - u(n,s) \right] d^+(\underset{\sim}{x}) = 0 \qquad (3\text{-}85)$$

where $\underset{\sim}{V}$ is the same function of $\underset{\sim}{\partial}\lambda^{\alpha}$ as previously used. This is the so-called _canonical form of the variational principle_. It is easy to show that the variations with respect to γ, Θ, and λ^{α} yield the same Euler-Lagrange equations as previously displayed in (3-71) — (3-73). The equations, resulting from variation of n and s are simply

$$\delta n : \quad \underset{\sim}{V} \cdot \underset{\sim}{\partial} \gamma = \partial u/\partial n \qquad (3\text{-}86a)$$

$$\delta s : \quad \underset{\sim}{V} \cdot \underset{\sim}{\partial} \Theta = \partial u/\partial s \qquad (3\text{-}86b)$$

which are just the conditions (3-79b) characterizing equilibrium.

The variational principle given in (3-85) is no longer a minimum (or a maximum) principle because the variations with respect to γ, Θ and λ^{α} for fixed n and s are of indeterminate curvature. This type of indeterminacy, which essentially destroys the utility of the principle for direct numerical solutions, is

characteristic of a variational principle that involves constraints and Lagrange multipliers. [19] In (3-85) we are effectively requiring that $(-u)$ be maximum subject to the two constraints represented by (3-86).

The form of the variational principle given in (3-85) is called "canonical" for the following reason: If we write the integral in the form

$$\int_{(t,\vec{x})} \left[n\, d\gamma/d\tau + s\, d\Theta/d\tau - u(n,s) \right] dt\, d^3(\vec{x}) \qquad (3\text{-}87)$$

it is evident that at the stationary point the integrand can be written in the form

$$P(Q^\alpha) = p_\alpha Q^\alpha - u(p_\alpha) \qquad (3\text{-}88a)$$

where

$$Q^\alpha \equiv (d\gamma/d\tau,\ d\Theta/d\tau) = (G, T)\ ;\ p_\alpha \equiv (n, s)\ . \qquad (3\text{-}88b)$$

This has the standard form of the canonical transformation between a Lagrangean P and a Hamiltonian u.

IV. FLOW-STABILITY HIERARCHY

Résumé

It follows from the equations of motion of the fluid, in the form that they take in the analysis given in sec. 3.6, that there exists a hierarchy of flow categories that are characterized by the amount of the information contained in the memory of the fluid's initial conditions that has been "erased" through the action of such mechanisms as diffusion and turbulent mixing. This hierarchy, which is discussed in sec. 4.1, amounts to an information-loss criterion for the relative stability of different fluid flows.

In sec. 4.2, it is shown that this criterion is correlated with another criterion of a local nature that is very much in the spirit of thermodynamics, namely the tendancy of the local value of the molar enthalpy to decrease as the rotational part of the flow vanishes. From the thermodynamic point of view, the degrees of freedom needed to describe the rotational component amount to "unconstrained internal variables" of a system that is adiabatically immersed in a constant-pressure reservoir. Classical thermodynamics says that the unconstrained internal variables of such a system will tend to assume those values that minimize the enthalpy. It is shown that this does indeed happen in the fluid. Alternatively, the inequality that represents this tendancy can be regarded as the generalization to the case of compressible flow of a theorem due to Kelvin for incompressible flow that indicates the tendency of an arbitrary flow to go over to potential flow, to the extent that this is consistent with the imposed boundary conditions.

The information-loss and local energy (i.e., enthalpy) criteria of stability that are developed in sec. 4.1 and 4.2 respectively, are shown in sec. 4.4 to be correlated to the role played by the action-integral as the thermodynamic potential for the entire system in the sense that the smaller the value of the action integral, the more stable the flow. It is also shown that increasing stability is correlated with an increase in the size of the group of infinitesimal changes in the terminal conditions that leave the value of the action-integral unchanged.

In sec. 4.3 compressible fluid-flow is described in terms of a mapping in four-space in the same sense that the complex-variable treatment of incompressible flow in two dimensions amounts to a two-dimensional mapping. The tendency of an arbitrary flow to go over to potential flow is interpreted as a tendency for the fluid to establish local equilibrium with its fluid enthalpy-reservoir.

Because of the unique importance of potential flow as the most stable flow that is consistent with a very wide class of imposed boundary conditions of practicle significance, the corresponding form of the variational minimum principle, which involves only one unknown function, is presented in sec. 4.5, and illustrated for the case of a perfect gas. It is shown that the case of an incompressible fluid can be represented as the limit in which the ratio of specific heats becomes infinite.

When the imposed boundary conditions possess a sufficiently high degree of symmetry, it is shown in sec. 4.6 that potential flow passes into the even more stable rectilinear flow characterized by straight world-trajectories of all the particles. In sec. 4.7 this is applied to the case of a fluid contained by a uniformly-expanding spherical box. In view of this example, it is pointed out that a result of characterizing fluid thermodynamics in terms of the variational minimum principle has been to replace the self-contradictory concept of a quasistatic change with the straight-forward concept of a time-evolution that minimizes the action-integral, which plays the role of a generalized thermodynamic potential of the system.

In sec. 4.8 the point is made that, to the extent that the fluid inside a spherical, uniformly-expanding box may be regarded as a good approximation for the interior regions of a freely-expanding cosmic cloud, the case of rectilinear flow — which, of all possible flows, is the most stable by the above criteria — accounts for the Cosmological Principle (or Hubble's Uniformity Principle), and indicates that it need not be regarded as an arbitrary ad hoc postulate, but rather may be regarded as a consequence of the Second Law of Thermodynamics.

4.1 Self-Perpetuating Flows

In sec. 3 we saw that general compressible adiabatic flow is characterized by four constants of motion, S and the three scalars K_α. Because they are constants of motion, if any of them for any reason becomes spatially constant throughout a sample of the fluid, the condition will perpetuate itself for all later time. In fact, when one of these four quantities becomes spatially uniform throughout a region, it means that information has been lost because the system will have "forgotten" part of the information that characterized its past history.

If we accept the postulate that loss of information is irreversible, then we can use the information loss of a system as a measure of stability - the greater the loss, the greater the stability. Thus complete uniformity of S and all the K_α means that the corresponding flow is more stable than another flow that

satisfies the same boundary conditions, but has a lesser degree of uniformity of (S, K_α). When one of the K_α is constant, it can be eliminated from the formalism altogether by absorption into the ergasy, and for this reason we shall speak simply of $K_\alpha = 0$ rather than a non-zero constant. It would not be physically proper to set entropy equal to zero, so complete loss of the information involved in the spatial dependence of S will be characterized by $\partial S = 0$.

The stability hierarchy resulting from memory loss in the flow in shown in Figure 9.

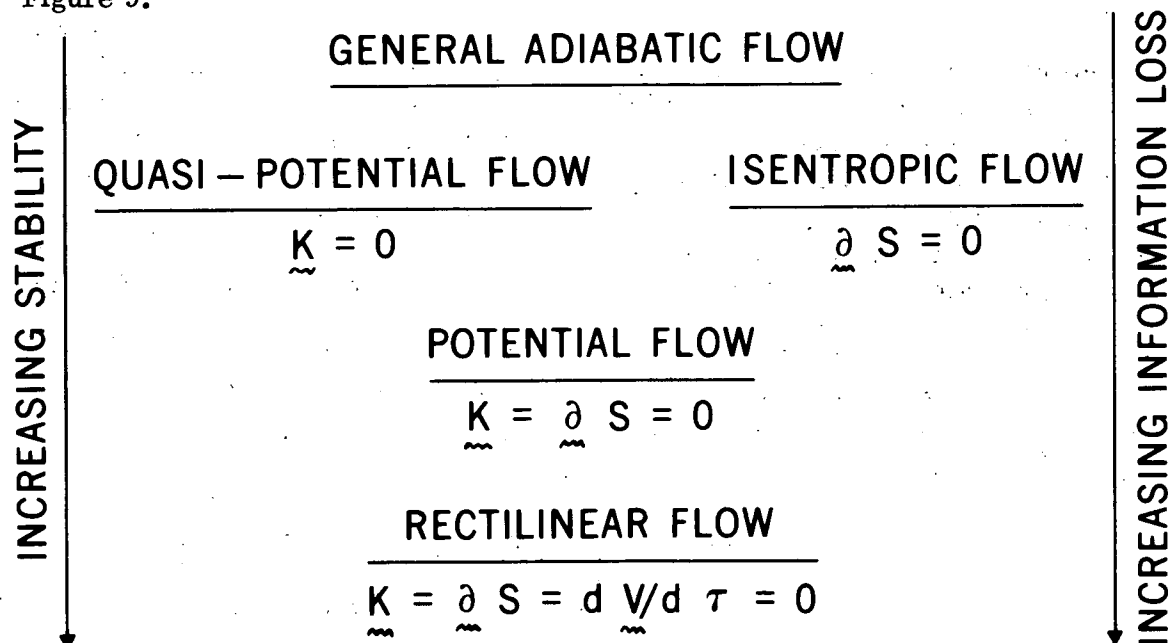Figure 9. Stability Hierarchy

The characteristics of the various cases are evident from the relation

$$ \underset{\sim}{H} \equiv H\hat{\underset{\sim}{V}} = \underset{\sim}{\mathcal{S}} + S\underset{\sim}{\mathcal{T}} + c\underset{\sim}{K} = c(\partial \gamma + S\partial\Theta + K_\alpha\partial\lambda^\alpha). \tag{4-1}$$

The case $K_\alpha = \underset{\sim}{K} = 0$ is characterized by the fact that the free-enthalpy momentum four-vector is the gradient of a scalar:

$$ (\underset{\sim}{H} - S\underset{\sim}{\mathcal{T}})/c = (H/c^2)\underset{\sim}{V} - S\partial\Theta = \partial\gamma. $$

(Quasi-Potential Flow) (4-2)

In the case of isentropic flow

$$ \underset{\sim}{H}/c = (H/c^2)\underset{\sim}{V} = \partial\eta + K_\alpha\partial\lambda^\alpha $$

(Isentropic Flow) (4-3a)

where

$$\eta = \gamma + S\theta . \qquad \text{(enthalpasy)} \qquad (4\text{-}3b)$$

Thus the molar momentum four-vector can be decomposed into irrotational and rotational parts, the latter of which is characterized by the three constants of motion K_α .

In the case of potential flow, we have simply

$$\underset{\sim}{H}/c = (H/c^2)\underset{\sim}{V} = \underset{\sim}{\partial}\eta . \qquad \text{(Potential Flow)} \qquad (4\text{-}4)$$

The case of rectilinear flow will be discussed later.

The fact that quasi-potential flow and isentropic flow are shown on the same level in Figure 9 does not indicate that the two cases are equally stable. To determine the relative stability of the two cases requires a quantitative measure which in fact the action integral provides. This will be discussed in greater detail below. The diagram in the figure is purely qualitative, and is characterized by the fact that each level is a special case of all levels above it,

4.2 Tendency of Molar Enthalpy to Decrease

Because S is a constant of motion, it can be expressed as some function of the λ^α 's. This means that, if we take (4-3b) to be the definition of the enthalpasy η even when $\underset{\sim}{\partial}S \neq 0$, we can write (4-1) in the form

$$\underset{\sim}{H}/c = (H/c^2)\underset{\sim}{V} = \underset{\sim}{\partial}\eta + \underset{\sim}{D} \equiv \underset{\sim}{H}/c + \underset{\sim}{D} . \qquad (4\text{-}5a)$$

whereas $\underset{\sim}{D}$ is the "drift momentum" defined by

$$\underset{\sim}{D} \equiv -\theta\underset{\sim}{\partial}S + K_\alpha \underset{\sim}{\partial}\lambda^\alpha = (K_\alpha - \theta\partial_\alpha S)\underset{\sim}{\partial}\lambda^\alpha . \qquad (4\text{-}5b)$$

The significant properties of $\underset{\sim}{D}$ are

$$\underset{\sim}{D} \cdot \underset{\sim}{V} = 0 \qquad (4\text{-}6a)$$

and

$$\underset{\sim}{D} \cdot \underset{\sim}{D} \leq 0 \qquad (4\text{-}6b)$$

because $\underset{\sim}{D}$ is a space-like vector. From (4-5) and (4-6) we find

$$H = \overset{o}{H} + D \geqslant \overset{o}{H} \tag{4-7a}$$

where

$$H \equiv (\underset{\sim}{H} \cdot \underset{\sim}{H})^{1/2} \; ; \quad \overset{o}{H} \equiv (\underset{\sim}{\overset{o}{H}} \cdot \underset{\sim}{\overset{o}{H}})^{1/2} \; ; \quad D \equiv (- \underset{\sim}{D} \cdot \underset{\sim}{D})^{1/2} \; . \tag{4-7b}$$

The physical significance of the inequality in (4-7a) is that, if the irrotational part of the decomposition for $\underset{\sim}{H}/c$ given in (4-5a) is held constant, the molar enthalpy of a sample of fluid attains its minimum value when $\underset{\sim}{D}$, the rotational part of $\underset{\sim}{H}/c$, vanishes. This may be regarded as a special case of the general thermodynamic theorem that says that the internal degrees of freedom of a thermally isolated system maintained at constant pressure will tend to relax in such a way as to minimize the enthalpy. A small sample of fluid embedded in the surrounding fluid satisfies these conditions and, to the extent that it is possible to maintain the boundary constraints without changing $\partial \eta$, the components of $\underset{\sim}{D}$ behave like internal degrees of freedom that assume the values that minimize H. The inequality (4-7a) just says that in such a case minimum H corresponds to $\underset{\sim}{D}$ = O.

This is just a generalization to the case of adiabatic compressible flow of the minimum energy theorem[20] that Kelvin (1849) proved for the case of incompressible flow. He showed that potential flow has less kinetic energy than any other flow with the same normal motion at the boundary. The simplest case of this theorem is the statement that flow straight through a pipe without rotation (potential flow) has less kinetic energy than the same flow with a rotation about the pipe axis superimposed on it. Kelvin effectively used the kinetic energy as the appropriate thermodynamic potential to determine stability. In the present context, we are using the enthalpy as the stability measure. Alternatively, we could say that potential flow corresponds to a minimum (namely vanishing) value of the kinetic energy associated with the drift momentum which represents the momentum of the fluid relative to its enthalpy reservoir.

It should be emphasized that it has not been proved that potential flow will always establish itself. There will be situations in which the boundaries will maintain the rotational part of the flow. Referring to Figure 10 (A), such a situation would occur if it were impossible to find a function $\eta(\underset{\sim}{x})$ whose gradient, which is proportional to the fluid velocity, possessed no normal component at any point on the bounding walls. In such a case, in order to maintain the boundary condition that no fluid pass through the walls, it would be necessary to maintain a non-potential flow component in the neighborhood of the walls.

Assuming, however, that the form of the walls is such that a potential func-
tion can be found that would satisfy the boundary conditions, it may still happen
that potential flow does not become established. This could happen, for example,
if it were necessary to surmount an energy barrier in order to make the transi-
tion from the existing rotational flow to the kinematically possible potential flow,
and sufficient energy were not available. Such barriers are characteristic of
changes that must be brought about by collective action throughout a finite region
of the fluid. If, however, sufficient energy is available in the form of turbulent
motion, for example, then a transition to potential flow can be expected. For
this reason, such transitions are more likely when the Reynolds number of the
flow is very large.

Cases (A) and (B) in Figure 10 illustrate (in two dimensions) the transition
from rotational to potential flow. In mathematical terms, this would be described
as a transition from a non-normal to a normal congruence. Case (C) in Figure
10 represents the further transition to rectilinear flow which is characterized
not only by the fact that the trajectories are straight lines, but also by the fact
that any two η = constant surfaces are everywhere equidistant. This case will
be discussed later.

(A)
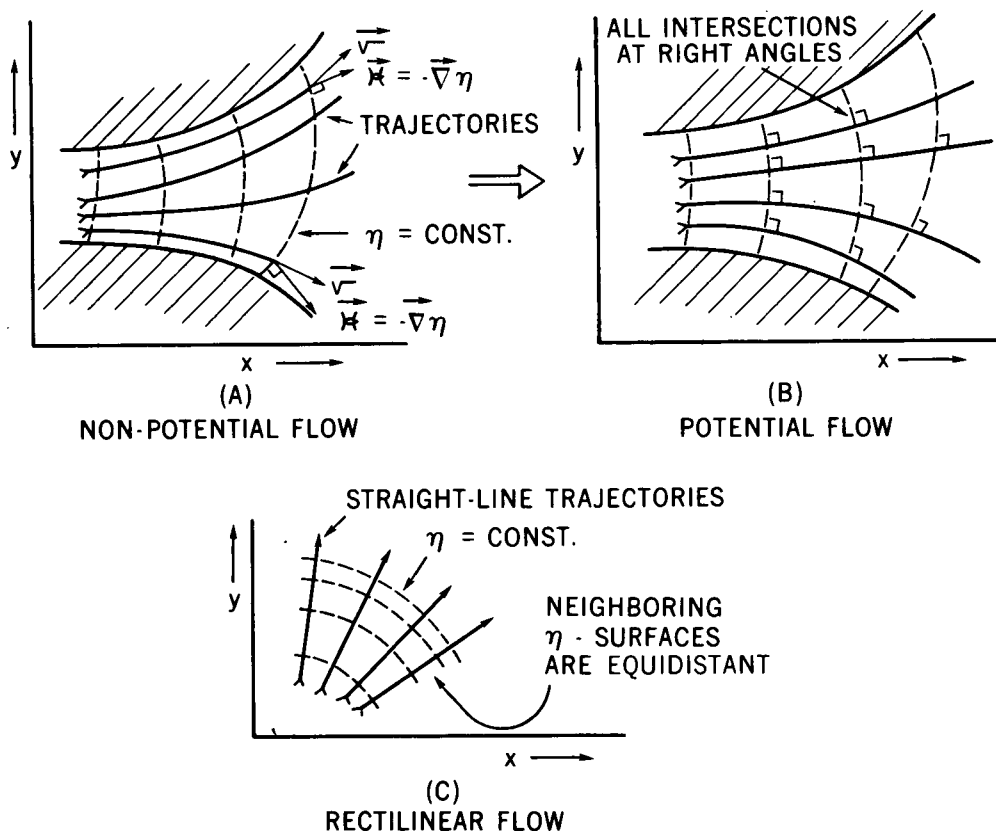NON-POTENTIAL FLOW

(B)
POTENTIAL FLOW

(C)
RECTILINEAR FLOW

Figure 10. Various Flow Categories

4.3 Mapping and Fluid-Reservoir Descriptions of the Flow

The time-like and space-like components of (4-5a) (in contravariant form) are

$$H^\circ = (1-\beta^2)^{-1/2}(Mc^2 + \breve{H}) = \not{H}^\circ + cD^\circ = (\partial\eta/\partial t) + (1-\beta^2)^{-1/2}\vec{v}\cdot\vec{D}_{rf} \quad (4\text{-}8a)$$

$$\vec{H}/c = (H^\circ/c^2)\vec{v} = \not{H}/c + \vec{D} = -\vec{\nabla}\eta + \vec{D} \quad (4\text{-}8b)$$

where the space-like parts of the sundry four-vectors (in contravariant form) are indicated by overhead arrows, and $\vec{D}_{rf}$ is the space-like part of the drift momentum four-vector in the fluid rest-frame. ($D^\circ_{rf} = 0$.)

These relations can be given a thermodynamic interpretation in which the drift vector $\underset{\sim}{D}$ plays the role of the <u>excess energy-momentum</u> in the sense of Figure 4. In this case the abscissa would be the drift momentum $\underset{\sim}{D}$. Its fluctuations produce excursions back and forth along the function $U(\underset{\sim}{D})$ where $U(\underset{\sim}{D})$ is the molar internal energy for fixed entropy. The basic property of the internal energy function is its <u>convexity</u> for variations in <u>all variables.</u> Under the conditions of constant S and P,

$$\left(\partial^2 U/\partial\underset{\sim}{D}^2\right)_{S,P} = \left(\partial^2 H/\partial\underset{\sim}{D}\right)_{S,P} \geqslant 0 \quad (4\text{-}9)$$

so $H(\underset{\sim}{D})$ must also be a convex function. If we identify $\not{H}$ in (4-7a) as the equilibrium enthalpy $\bar{H}$ about which the sample is fluctuating, then (4-7a) can be written in the following form:

$$\text{excess energy} \equiv D \equiv (-\underset{\sim}{D}\cdot\underset{\sim}{D})^{1/2} = H - \bar{H} \geqslant 0 \,. \quad (4\text{-}10)$$

Thus the changing value of D that we observe as we travel along any one of the trajectories in Figure 10 (A) could be regarded as fluctuations in time of the excess energy, and the different values that we would observe over a single $\eta =$ constant surface could be interpreted as spatial fluctuations in the excess energy. By contrast, the excess energy is everywhere zero in the potential flow shown in Figure 10 B, which means that the fluid is everywhere in equilibrium with its reservoir.

In a relativistic context, energy and momentum are parts of a single entity, the energy-momentum four-vector. Thus it is to be expected that the fluctuations about the equilibrium energy that were described above have their counterpart in fluctuations about a locally defined equilibrium momentum. These momentum fluctuations are illustrated in Figure 11.

RESERVOIR MOMENTUM

$$\vec{H}/c = -\vec{\nabla}\eta$$

DRIFT MOMENTUM

$$\vec{D}$$

FLUID MOMENTUM
$$\vec{H}/c = (H°/c^2)\,\vec{v}$$
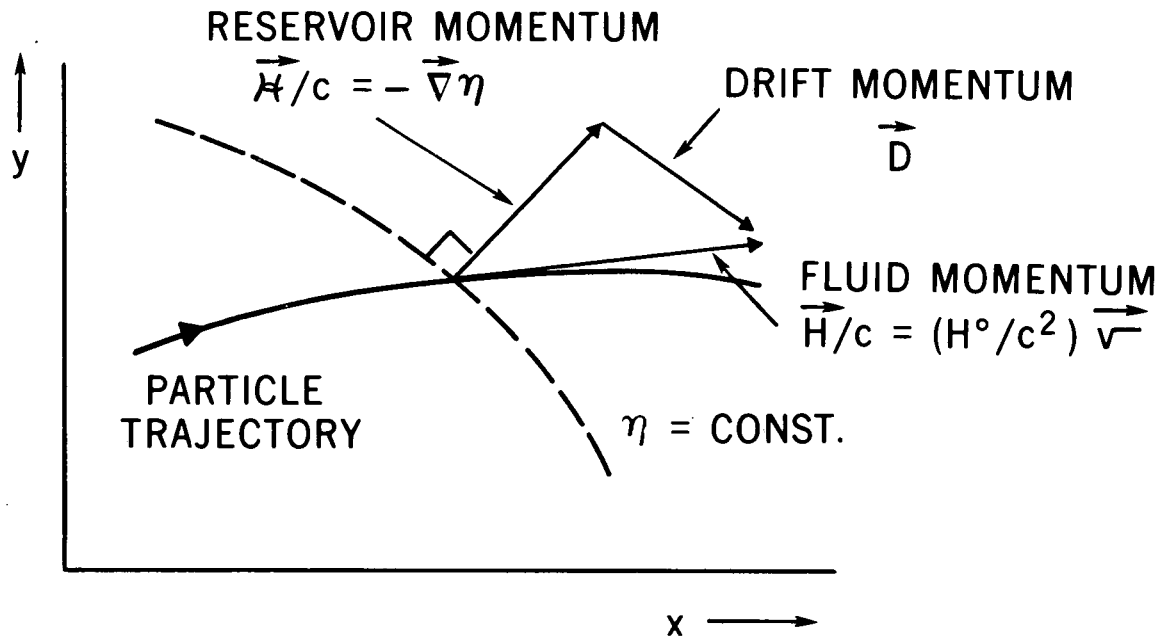
PARTICLE
TRAJECTORY

$$\eta = \text{CONST.}$$

Figure 11. Fluid "Drift" with Respect to its Local Reservoir

Potential flow, illustrated in Figure 10 (B), is characterized not only by a vanishing of the excess energy, but also by a vanishing of the "drift" of the fluid with respect to the local reservoir defined by $\underset{\sim}{H} = c \underset{\sim}{\partial} \eta$, and consequently a vanishing of the drift momentum $\vec{D}$. Referring to Figure 5 (A), $\vec{D}$ is just the discrepancy between the momentum per mole of the displaced blob of fluid in its new position and the momentum per mole of the ambient fluid. The vanishing of this drift is associated with the transition to the potential vortex shown in Figure 5 (B). This illustrates a special case of the fact that

$$\underset{\sim}{\partial} \times \underset{\sim}{H}/c = \underset{\sim}{\partial} \times \underset{\sim}{D} \quad \text{and so} \quad \underset{\sim}{\partial} \times \underset{\sim}{H}/c = 0 \quad \text{when} \quad \underset{\sim}{D} = 0.$$

Thus $\underset{\sim}{D}$ plays a double role: the drift momentum from the thermodynamical point of view, and the vector potential of the vorticity from the fluid-dynamical point of view. The gradient $\underset{\sim}{\partial}\eta$ also plays a double role. It is the reservoir momentum $\underset{\sim}{H}/c$ in the thermodynamical interpretation, and the irrotational component of the flow in the fluid-dynamical interpretation. We shall now see that η has still a third role to play.

If the figures 10 and 11 were drawn in four-space rather than three-space, it would be evident that the function $\eta\,(\underset{\sim}{x})$ defines a "natural time coordinate". Integrating along a world-trajectory of a particle we have

$$\eta = \int H\, d\tau = \int (Mc^2 + \breve{H})\, d\tau \, . \tag{4-11}$$

(Note that this is a property of η , not the definition of it. That is, η is not a path-dependent function like the proper time τ .)

This relation says that η/Mc^2 could be pictured as the reading of a clock whose instantaneous rate is $H/Mc^2 = 1+\check{H}/Mc^2$, which obviously would differ very little (except for long-term cumulative error) from a clock reading ordinary proper time.

As previously mentioned, the introduction of a "natural scalar time coordinate" $\in(x)$ implies the existence of a "natural unit of energy" $\hat{E} \equiv d\in/d\tau$. In the present case, $\in = \eta$ and $\hat{E} = H$ which means, not surprisingly, that the molar enthalpy (which is the total energy of a mole of the fluid) serves as a natural unit of energy. Normalizing other energies, such as the molar Gibbs function G or the temperature T (actually RT where R is the molar gas constant) to the "natural energy unit" H corresponds to differentiating the ergasy and the thermasy with respect to η rather than τ :

$$ G/H = (d\gamma/d\tau)/(d\eta/d\tau) = d\gamma/d\eta \qquad (4\text{-}12a) $$

$$ T/H = (d\Theta/d\tau)/(d\eta/d\tau) = d\Theta/d\eta \qquad (4\text{-}12b) $$

Thus, by using H as the reference energy unit, we have eliminated τ from the formalism, which is a desirable thing to do because τ is not a point function, but rather is dependent on the entire past trajectory of the particle. Moreover, η has to be carried along in the formalism in any case because it has a vital dynamical role to play (the potential function of the fluid). Thus, by eliminating τ in favor of η , we are reducing the number of variables of the problem.

The elimination of τ in favor of η is equivalent to eliminating the normalized velocity $\hat{\underset{\sim}{V}}$ in favor of the "unnormalized velocity four-vector $\underset{\sim}{H} = H\hat{\underset{\sim}{V}}$. This has several advantages: it eliminates the annoying constraint implied by the normalization of $\hat{\underset{\sim}{V}}$. It packs useful information into all four of the unconstrained components of $\underset{\sim}{H}$. Finally, because $\underset{\sim}{H} = c\partial \eta$ is the gradient of a scalar, we are dealing with a complete mapping in four-space, and the tetrapod formalism that results is far more convenient to work with than the truncated formalism in which $\hat{\underset{\sim}{V}}$ was the time-like vector of the tetrapod.

Because we are dealing with a complete mapping, it means that we can use $(c\eta, \lambda^\alpha)$ as the independent coordinates rather than the choice (ct, λ^α) which characterized the Planck formalism. With this, we arrive at a formalism

characterized by

$$G/H = (\partial Y/\partial \eta)_\lambda \; ; \; T/H = (\partial \Theta/\partial \eta)_\lambda \; ; \; H = c\left[(d\underset{\sim}{x}/d\eta)\cdot(d\underset{\sim}{x}/d\eta)\right]^{-1/2} \quad (4\text{-}13)$$

in which $Y(\eta, \lambda^\alpha)$, $\Theta(\eta, \lambda^\alpha)$, and $\underset{\sim}{x}(\eta, \lambda^\alpha)$ are the dependent variables and the action integral is

$$\mathcal{J} = (1/c)\int_{(\eta, \lambda^\alpha)} P\Delta \, d(c\eta) \, d^3(\lambda^\alpha) \quad (4\text{-}14)$$

and Δ is the Jacobian connecting the convected frame with the laboratory frame:

$$\Delta = \frac{\partial(ct, \vec{x})}{\partial(c\eta, \lambda^\alpha)} = N/H \quad (4\text{-}15)$$

where N is the cell volume in the local rest-frame. With the relation

$$\Delta \, dP = m\left[d(\partial Y/\partial \eta)_\lambda + S\, d(\partial\Theta/\partial \eta)_\lambda - c^{-1}\underset{\sim}{H}\cdot d(\partial\underset{\sim}{x}/\partial\eta)_\lambda\right] (4\text{-}16)$$

it is easy to confirm that the Euler-Lagrange equations resulting from the variation of (4-14) are in fact the correct equations of motion.

This represents a covariant alternative to the Van Dantzig formalism in which Θ, rather than η, was used as the time coordinate. One could also construct a formalism based on the use of Y as the time coordinate. These various formalisms could be regarded as canonical transformations of one another, although these transformations are produced by changing the scalar time-coordinate that is used rather than by adding an arbitrary divergence to the Lagrangean density. That is, the Lagrangean density (the pressure, in an inertial frame) remains unchanged, and has an absolute physical significance that cannot be ascribed to a Lagrangean density that can be changed at will by the addition of an arbitrary divergence. This fact is of capital importance when the formalism is extended to General Relativity. It means that, because the Lagrangean density has an absolute significance, the canonical transformations being just coordinate changes (in the sense of general curvilinear coordinates), then so also does the stress-energy tensor have a physical significance that is absolute to within a general curvilinear coordinate transformation. It has, of course, long been recognized that this must be the case, because in the Einstein field equations the stress-energy tensor is the source that generates the metric tensor.

It should be noted that, although the Lagrangean density has an absolute sense that is invariant under the canonical transformations generated by changing the

choice of scalar-time coordinate, this is not true of the stress-energy tensor derived from the Lagrangean density or of any Hamiltonian function that might be defined in terms of the stress-energy tensor. That is, there is a different Hamiltonian density for each different time coordinate, and these can be very different indeed, as for example, the internal energy density u and the negentropy - s which, as pointed out above, correspond to the time coordinates τ and Θ respectively.

Although the discussion of this section has given a central role to the enthalpasy η , in the case of the analysis of a single fluid, it should be repeated that the thermasy Θ has a stronger claim to universality because it is shared in common by several coexisting fluids in reversible thermal contact.[21]

4.4 Decrease of Action Integral with Increasing Stability

In the discussion of Figures 6 and 7, the point was made that the terminal values q_i^{α} and q_f^{α} can be treated like thermodynamic degrees of freedom just like the values of the q 's in the interior of the four-volume over which the action integral $\mathcal{J}$ is integrated. Thus it can be stated that when the action integral is stationary with respect to variation of any of the terminal values of the q's , as well as with respect to the functional forms of all the q's in the interior of the four-volume, the stationary value of the action integral is in fact a minimum. Because the action integral is based on the stability criterion stated in (3-1) (which applies to variation of terminal as well as interior values of the q's), a local minimum of $\mathcal{J}$ with respect to variation of the terminal values means that the corresponding coherent evolution (which will be called the minimum evolution) is more stable than any of the neighboring coherent evolutions.

It will now be shown that this stability criterion is consistent with the criterion illustrated in Figure 9 that was based on loss of information. Because the formalism is sensitive only to the difference $q_f^{\alpha} - q_i^{\alpha}$ rather than to the individual values q_f^{α} and q_i^{α}, we can, without any loss of generality, consider the initial values q_i^{α} to be held fixed, and study the effects of varying only the final values q_f^{α} .

For an arbitrary action integral $\mathcal{J} = \int \mathcal{L} \, d^4(\underset{\sim}{x})$ referred to the laboratory frame, if the Euler-Lagrange equations are satisfied, the change in $\mathcal{J}$ produced by changes in the terminal values of the dependent variables is

$$d\mathcal{J} = \int_{(\vec{x})} \left[\frac{\partial \mathcal{L}}{\partial (\partial_0 q^{\alpha})} \right] d q^{\alpha} d^3(\vec{x}) \qquad \text{at} \quad t = t_f \qquad (4\text{-}17)$$

which, in the case of the action integral given in (3-93), becomes

$$d\mathcal{J} = \int_{(\vec{\kappa})} n^0 [d\gamma + S d\Theta + K_\alpha d\lambda^\alpha] d^3(\vec{\kappa}) \qquad \text{at} \quad t = t_f \quad . \qquad (4\text{-}18)$$

The case of quasi-potential flow is characterized by $K_\alpha = 0$. Thus the action integral is insensitive to first-order changes in λ_f^α. Isentropic flow is characterized by an $S = S_c$ that is everywhere the same. Thus the action integral is insensitive to changes in γ_f and Θ_f such that $d\gamma_f = -S_f d\Theta_f$. That is, $\mathcal{J}$ is insensitive to one of the two degrees of freedom represented by γ_f and Θ_f. This corresponds to the fact that in the case of isentropic flow, as indicated in (4-3), the two functions γ and Θ can be replaced by the single function η.

The case of potential flow combines the above two cases, and we have

$$d\mathcal{J} = \int_{(\vec{\kappa})} (n^0 d\eta) d^3(\vec{\kappa}) \qquad \text{at} \quad t = t_f . \qquad (4\text{-}19)$$

If we regard $\mathcal{J} = \mathcal{J}(\gamma_f, \Theta_f, \lambda_f^\alpha ; \vec{\kappa})$ as a function of the five variables $\gamma_f, \Theta_f,$ and λ_f^α at each point $\vec{\kappa}$, then for fixed $\vec{\kappa}$ we could picture the surface $\mathcal{J} = $ constant in a space spanned by the five thermodynamical variables. Increasing stability then corresponds to increasing symmetry in the shape of this surface until finally, for the case of potential flow, it becomes the hyperplane corresponding to constant η.

In this way increasing stability can be related not only to decreasing $\mathcal{J}$, but also to an increasingly large symmetry group, namely the group of variations in the terminal conditions that do not change the value of $\mathcal{J}$.

The magnitude of $\mathcal{J}$ is a much more powerful stability criterion than the information-loss criterion depicted in Figure 9. First of all, it is a quantitative measure. Thus it would provide a means of comparing the relative stability of quasi-potential flow and isentropic flow, which was not possible in terms of information-loss.

Secondly, unlike the information-loss criteria which was based on the scalar constants of motion of the flow, the $\mathcal{J}$-criterion is valid for any set of variables. Thus, for example, if we were interested in wave motion or turbulence, we might find it useful to fourier-analyse some or all of the thermodynamical variables used heretofore. This would amount to a canonical transformation that would introduce an entirely new set of variables. Because we would be dealing with collective modes of the system, the concept of "constant of motion" would no longer apply, which invalidates (or at least greatly complicates) the information-loss approach to stability. The action-integral approach, however, is still

valid. It is only necessary to find a configuration such that $\mathcal{J}$ is insensitive to changes in the terminal values of the dependent variables in order to find a configuration that is stable with respect to small perturbations. The relative stability of widely differing configurations can be determined by comparing the corresponding values of the action integral. There is no guarantee, of course, that a state of lower stability (higher $\mathcal{J}$) will pass over into one of greater stability, because there might be an energy barrier between the two states and insufficient turbulent energy in the system or its environs to surmount the barrier.

Very often in specifying a fluid–dynamical problem, one does not have a complete specification of the terminal conditions (the equivalent of the initial conditions in a Cauchy problem). Rather, we know only certain averages over the whole system such as total mass transport or total kinetic energy. In such a problem, one could set up the trial functions to satisfy the known terminal conditions, and leave all other degrees of freedom open and subject to variation. The set of trial functions that minimize $\mathcal{J}$ in such a case represent the most stable solution that satisfies the known terminal conditions that must be satisfied by the flow.

4.5 Variational Principle for Potential Flow

The above stability considerations make it obvious that potential flow plays a uniquely important role in fluid dynamics. For this reason, it is important to specialize the minimum principle to the case for potential flow in such a way that η is the only surviving dependent variable (in a laboratory-frame analysis). This is in fact possible. The starting point is the potential function $P(H, S)$ for which $dP = n(dH - TdS)$.

For example, for a perfect gas we have

$$P = B \overset{\smile}{H}{}^{C_P/R} e^{-S/R} \tag{4-20}$$

where B is a constant. In the case of interest, S is everywhere constant, and can be absorbed into the constant B. Introducing the ratio of specific heats

$$r \equiv C_P/C_V = 1 + 2/f \tag{4-21}$$

where f is the number of degrees of freedom of the molecule that share in the energy equipartition (f = 3 for a monatomic gas), and using the relation $C_P = C_V + R$,

we find $C_P/R = r/(r-1)$ so that (4-20) can be rewritten for the isentropic case in the form

$$P = B \, \zeta^{r/(r-1)} \quad \text{where} \quad \zeta = \breve{H}/Mc^2 = (H/Mc^2) - 1 \qquad \text{(4-22a,b)}$$

where of course the constant B here is different from the one in (4-20). Using the fact that $n = (\partial P/\partial H)_S = d\,P/d\,H$ we find

$$n = \frac{r/(r-1)}{Mc^2 B^{(r-1)/r}} \, P^{1/r} \qquad \text{for perfect gas with} \quad r = C_P/C_V . \qquad \text{(4-23)}$$

This shows that an incompressible fluid can be regarded as the limiting case for which $r \to \infty$. This limit corresponds to $C_P/R \to 1$ or $C_V \to 0$. From (4-21), it follows that $C_V \to 0$ while C_P remains finite implies that $f \to 0$. That is, we are dealing with the limit in which there are no internal degrees of freedom to absorb work of compression. Hence the fluid is incompressible. From (4-23) it is clear that in this limit the rest-density n becomes

$$n = 1/(Mc^2 B) \qquad \text{(incompressible "perfect gas" — } r = \infty) \qquad \text{(4-24)}$$

To arrive at the minimum principle, we make the identification

$$H = \mathcal{H} = (\underset{\sim}{\mathcal{H}} \cdot \underset{\sim}{\mathcal{H}})^{1/2} = c(\underset{\sim}{\partial}\eta \cdot \underset{\sim}{\partial}\eta)^{1/2} . \qquad \text{(4-25)}$$

The Lagrangean density is thus $P(H) = P[c(\underset{\sim}{\partial}\eta \cdot \underset{\sim}{\partial}\eta)^{1/2}]$ and from $P = n d\,H$ we find

$$d P = c \underset{\sim}{n} \cdot d(\underset{\sim}{\partial}\eta) \quad \text{where} \quad \underset{\sim}{n} \equiv n \underset{\sim}{\hat{v}} = n \underset{\sim}{\partial}\eta/(\underset{\sim}{\partial}\eta \cdot \underset{\sim}{\partial}\eta)^{1/2} . \qquad \text{(4-26)}$$

The action integral for a perfect gas, for example, is

$$\oint = (B/c) \int_{(\underset{\sim}{x})} \left\{ [(\underset{\sim}{\partial}\eta \cdot \underset{\sim}{\partial}\eta)^{1/2}/Mc] - 1 \right\}^{r/(r-1)} d^4(\underset{\sim}{x}) . \qquad \text{(perfect gas)} \qquad \text{(4-27)}$$

(For a monatomic gas, $r/(r-1) = 5/3$.)

The only Euler-Lagrange equation is the one that results from varying $\mathcal{H}$. From (26), it is evident that it is

$$\delta \mathcal{H}: \quad c \underset{\sim}{\partial} \cdot \underset{\sim}{n} = 0 \qquad \text{(4-28)}$$

f

which is the conservation equation. There is no need for a dynamical equation corresponding to Euler's equation, because this is already implied in the identification

$$\underset{\sim}{H} = H \overset{\wedge}{\underset{\sim}{V}} = c\,\underset{\sim}{\partial}\,\eta \tag{4-29}$$

This implies $\underset{\sim}{\partial} \times \underset{\sim}{H} = 0$, and contracting this with $\overset{\wedge}{\underset{\sim}{V}}$ and using $d\,P = n\,d\,H$ we find

$$d(\underset{\sim}{H}/c)/d\tau \;=\; \underset{\sim}{\partial}\,H \;=\; \left(\underset{\sim}{\partial}\,P\right)/n \tag{4-30}$$

which is Euler's equation.

4.6 Rectilinear Flow

The tendency toward increasing stability that has been discussed thus far has been characterized as the tendency of a system to forget its initial conditions.

This tendency can also be described in the following manner: If we use $(c\,\eta\,,\,\lambda^{\alpha})$ as the independent coordinates in terms of which we describe the flow, then general adiabatic flow is characterized by the fact that $S = S(\lambda^{\beta})$ and $K_{\alpha} = K_{\alpha}(\lambda^{\beta})$ are functions of the space-like coordinates λ^{β} (the identity parameters) but not of the time-like parameter η (because they are constants of motion). The tendency toward greater stability can be described as a diffusion of the non-uniformity in S and K_{α} so that on any given η = constant hypersurface S and K_{α} become more and more uniform. This is the analog of the tendency of molar entropy to become uniform throughout the entire volume of a closed box of gas at rest for which the η=constant hypersurfaces are just the space-like hyperplanes that define the rest-frame of the box.

The space-like η = constant hypersurfaces define what is meant by "spatial uniformity" for any given flow. It is evident from the example of the rigid box of gas, as well as from cases (A) and (B) depicted in Figure 10, that the η =constant hypersurfaces in turn are largely specified by the boundaries (i.e. the walls) that confine the flow. More precisely, as the minimum principle for the case of potential flow makes clear, the functional form of $\eta(x)$ is determined by the functional $P(\underset{\sim}{\partial}\eta)$ (which in turn is determined by the physical properties of the fluid), the terminal conditions (which are equivalent to specifying the time-average $\langle \overset{\circ}{\mathcal{H}} \rangle_{\hat{x}}$ at every spatial point, and the boundary conditions (i.e. the form and motion of the containing walls). Through mechanisms such as diffusion and turbulence, the fluid itself could effectively seek out the terminal conditions

that minimize the action integral, but the boundary conditions are externally applied and cannot be influenced by an internal process within the fluid. This means that the boundary conditions set an upper limit on the degree of uniformity that can be achieved by the system. For example, from cases (A) and (B) in Figure 10 it is evident that the requirement that the η = constant hypersurfaces meet the bounding walls normally means that the perpendicular distance from one surface to its neighbor will not be constant over the entire surface. This is equivalent to saying that the gradient $\partial \eta = H/c$ will be a function of λ^β as well as of η . In other words, after S and K_α have become uniform with respect to the convected coordinates λ^β (which is the case of potential flow), there still remains the thermodynamic variable $H(\eta, \lambda^\beta)$. If this too were to lose its functional dependence on λ^β , i.e. became uniform on the η = constant hypersurfaces, the system would achieve a higher degree of uniformity, and hence stability. However, the upper boundary for such uniformity is established by the degree of symmetry possessed by the bounding walls themselves.

For example, case (C) of Figure 10 shows a flow for which H is no longer a function of λ^β . The η = constant surfaces are equidistant, which means that (in two dimensions) they are arcs of circles. This in turn implies that the particle trajectories are straight lines. This condition obviously requires that the bounding walls also be either straight lines or expanding areas of circles.

The fact that equidistant neighboring η-hypersurfaces automatically imply rectilinear flow follows directly from

$$\underset{\sim}{H}/c = (H/c^2)\underset{\sim}{V} = \underset{\sim}{\partial} \eta \qquad \text{(general potential flow)} \qquad (4\text{-}37a)$$

and

$$H = c(\underset{\sim}{\partial}\eta \cdot \underset{\sim}{\partial}\eta)^{1/2} = H(\eta) \qquad \text{(equidistant } \eta\text{-surfaces)} \qquad (4\text{-}37b)$$

When both these conditions are satisfied there exists a function $f(\eta)$ such that

$$\underset{\sim}{V} = \underset{\sim}{\partial} f(\eta) \quad \text{which implies that} \quad \underset{\sim}{\partial} \times \underset{\sim}{V} = 0 \qquad (4\text{-}38a)$$

Contracting the second of these relations with $\underset{\sim}{V}$, we find

$$\underset{\sim}{V} \cdot \underset{\sim}{\partial} \underset{\sim}{V} = d\underset{\sim}{V}/d\tau = 0 . \qquad \text{(rectilinear flow)} \qquad (4\text{-}38b)$$

IV-16

Thus the uniformity of H, as well as of S and K_α, on every η-hypersurface automatically implies rectilinear flow.

The simplest example of such flow arises in the case of potential flow within a uniformly expanding spherical box. Because of the three-dimensional symmetry of the situation, η must be a function of $(\vec{x} - \vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$ where $\vec{x}_0$ is the center of the spherical box. The requirement of relativistic covariance, however, necessitates that in the absence of a velocity four-vector that would serve to distinguish one space-like hyperplane from all others, the above three-dimensional scalar must be generalized to the four-dimensional scalar $(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)$. It is obvious then that (4-38a) can be written

$$\underset{\sim}{V} = c\, \underset{\sim}{\partial}(c\tau) \qquad \text{where} \quad c\tau \equiv \left[(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) \right]^{1/2}$$

$$\hspace{4cm} (4\text{-}39)$$

$$= (\underset{\sim}{x} - \underset{\sim}{x}_0)/\tau \quad ; \quad \underset{\sim}{V} \cdot \underset{\sim}{V} = (\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)/\tau^2 = c^2.$$

Expressed in terms of its components as seen in the frame of an arbitrarily chosen observer who is embedded in the fluid and moving on a straight world-line passing through $\underset{\sim}{x}_0$, this four-vector equation becomes

$$\Gamma c = c(t - t_0)/\tau \qquad \text{or} \quad \Gamma \tau = t - t_0 \qquad (4\text{-}40a)$$

$$\Gamma \vec{v} = (\vec{x} - \vec{x}_0)/\tau \qquad \text{or} \quad \vec{v} = (\vec{x} - \vec{x}_0)/(t - t_0). \qquad (4\text{-}40b)$$

Thus at a certain instant of time t, the three-velocity $\vec{v}$/of any particle is proportional to the distance $(\vec{x} - \vec{x}_0)$ from the observer to the particle.

This situation is illustrated in the observer's three space (assuming that the box as a whole appears to him to have a velocity $\vec{V}$) in Figure 12 (A), and in four-space in Figure 12 (B).

Note that the fact that η, and hence all the thermodynamical quantities, must be constant on the η=constant hyperboloids rather than on t=constant hyperplanes, is a purely relativistic effect that could not be obtained within the framework of the Galilean group. Formally this resulted from the observation that if three-dimensional symmetry requires η to be a function of $|\vec{x} - \vec{x}_0|$ at any instant of time t, then relativistic covariance requires that the time-dependence must be that involved in $|\underset{\sim}{x} - \underset{\sim}{x}_0|$. Within the framework of the Galilean group one could not make this assertion, because there are no symmetry operations connecting time with the spatial coordinates, which implies that the time dependence is entirely independent of the spatial dependence.
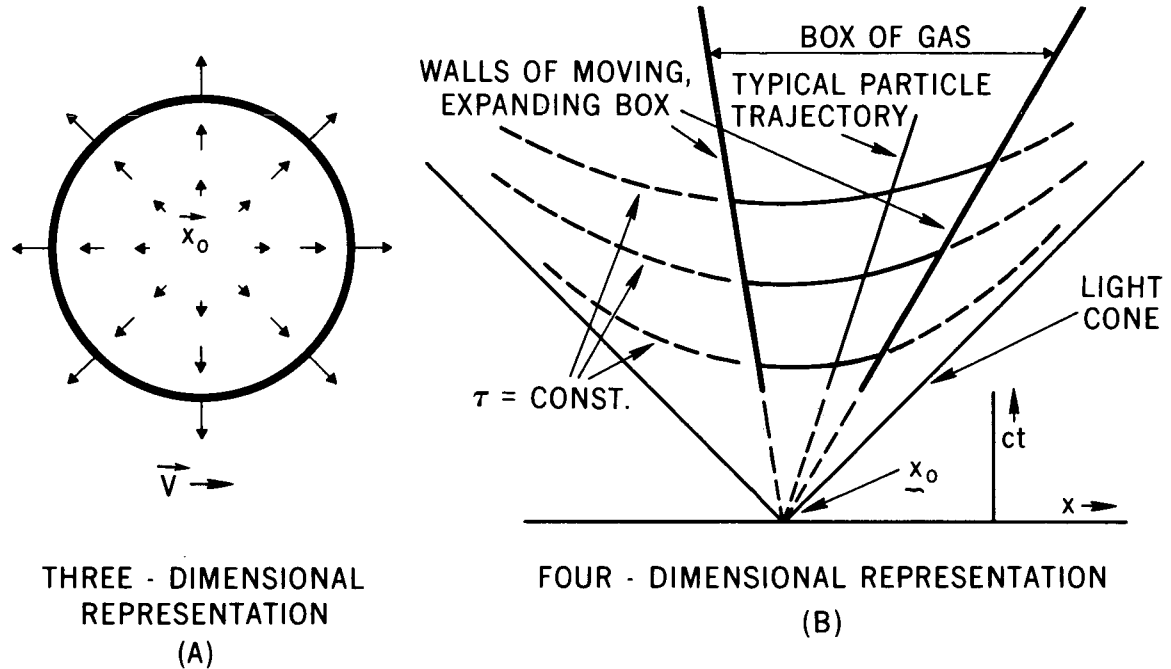
THREE - DIMENSIONAL
REPRESENTATION
(A)

FOUR - DIMENSIONAL REPRESENTATION
(B)

Figure 12. Uniformly Expanding Spherical Box of Gas (Rectilinear Flow)

Another way to say this is to note that spatial uniformity of the thermodyna-
mical quantities is connected with stability, which is a invariant property. Thus
in the case of boundary conditions which appear the same to all observers (except
for the observer-dependent apparent velocity of the box as a whole), "spatial
uniformity" must be defined in a way that is independent of the observer. This
means that the hypersurfaces on which the thermodynamical quantities are uni-
form must also be independent of the observer. The constant hyperboloids are
the only hypersurfaces that are invariant under the Lorentz group.

4.7 Implications for the Foundations of Thermodynamics

The introduction of the requirement of relativistic covariance into thermo-
dynamics has thus made it possible to define the equilibrium (i.e. most stable)
state of a uniformly changing (but unaccelerated) system without resort to the
Quasi-Static Hypothesis, which is really the assertion that a uniformly changing
system behaves in the same way as a static system, i.e. both are characterized
by uniformity on the $t =$ constant hyperplanes. Within the framework of the
Galilean group, some such hypothesis is needed to compensate for the fact that
there exists no fundamental relation connecting time dependence with spatial
symmetry. In the absence of such a fundamental relation, the Quasi-Static
Hypothesis has appeared to be the only way of papering over the implicit

contradiction in the fact that classical thermodynamics could deal only with equilibrium (i.e. unchanging) situations, but to be of any practical significance it had to be applicable to changing situations. The hypothesis is in fact an example of Zeno's Flying-Arrow Paradox.

The present approach is built on the concept of non-uniformity and evolution with time right from the beginning, and makes no use of the concept of "equilibrium". Rather, the concept of relative stability as measured by the magnitude of the action integral becomes the central idea. Thus, for fixed boundary and terminal conditions, the coherent evolution is more stable than all of the neighboring incoherent evolutions, and this is the reason we are more likely to observe it than the incoherent ones. This point of view does not, however, banish the incoherent evolutions from all consideration. The same formalism that describes a coherent evolution also describes the incoherent ones, and the possibility is admitted that we can expect to observe an incoherent evolution in nature. In fact, this possibility is required by the fact that near its minimum the action integral is very insensitive to slight changes in the evolution. Thus we can expect that we essentially never observe the strictly coherent evolution that corresponds to the exact minimum of the action integral, but rather the class of incoherent solutions that are clustered about the coherent one.

Because non-uniformity, motion, and evolution have been built into this formulation of thermodynamics right at the outset, the possibility exists for an easy and natural generalization that would include dissipative and transport properties. In the formalism described above only first derivatives have been incorporated into the Lagrangean density, and the resulting Euler-Lagrange equations are exact statements of the sundry conservation laws. If second derivatives are included, exact conservation will be broken. The Euler-Lagrange equations will include extra terms corresponding to the generation of entropy, and related force terms in the Euler equation. It would be a matter of heuristically incorporating new terms into the pressure functional (or the internal energy density if this seemed intuitively easier) such that the empirically known entropy generation and viscosity force terms appear in the Euler-Lagrange equations. This would simply be an extension of the customary thermodynamical procedure of constructing appropriate model state functions to account for known empirical data.

4.8 Implications for Astrophysics and Cosmology

For systems of laboratory dimensions there is so little difference between uniformity on a τ-hyperboloid on the one hand and uniformity on the t-hyperplane tangent to the hyperboloid on the other that the significance of the conclusions

represented in Figure 12 is conceptual rather than practical. On an astro-physical scale, however, the difference can be of very great observational importance.

In particular, we can expect in the case of a freely expanding gas cloud to have a situation that, except for the edge effects in the outer regions of the cloud, closely resembles the interior of the expanding spherical box of gas. That is, if the expansion is slow enough, and the dimensions of the cloud at some time in its past were small enough, so that most of the cloud (except for its outer edges) could reach its most stable configuration then this configuration would be char-acterized by uniformity on hyperboloids corresponding to some space-time point $\mathcal{N}_0$. Note, however, that this point could be - most likely would be - virtual in the sense that the cloud need never have been concentrated at the point in question. The backward extrapolation of the world-line of the particles in the cloud after it had fallen into the rectilinear flow state would all intersect at the point $\mathcal{N}_0$, and hence give the appearance of having originated in a con-centrated explosive event at the space-time point $\mathcal{N}_0$. But in actual fact, the early history of the cloud could have been very different. Once the characteristic rectilinear flow configuration has established itself because of the mutual inter-actions of the various parts of the cloud, purely kinematic considerations show that it will perpetuate itself for all later time. The result is a cloud that, by virtue of its great size and speed of expansion, could never hope to experience the necessary interaction between its various widely dispersed and fast-receding parts in order to establish a uniform state, yet nevertheless, in apparent defi-ance of causality, is uniform.

The evolution of such a cloud is illustrated in Figure 13. This figure also makes the point that, to the extent that they cannot extend their backward light-cones far enough into the past to discern edge effects, observers at different points on a given Υ - hyperboloid will all see the same thing as they look outward in space and backward in time. That is, in the frame of reference in which the observer Q appeared to be at rest (instead of the observer P as shown in the figure), and the observer P at the new position P', the backward light-cone from Q would reveal exactly the same picture (except for the possibility of edge ef-fects) as P's light-cone had revealed.

In a cosmological context this is known as the "Cosmological Principle", and is regarded as a necessary, but nevertheless artibrary, postulate. The above argument indicates that far from being an arbitrary postulate, it is the natural condition of systems that at some time in their past have been compact enough to have interacted sufficiently to reach the most stable state, that of rectilinear expansion.
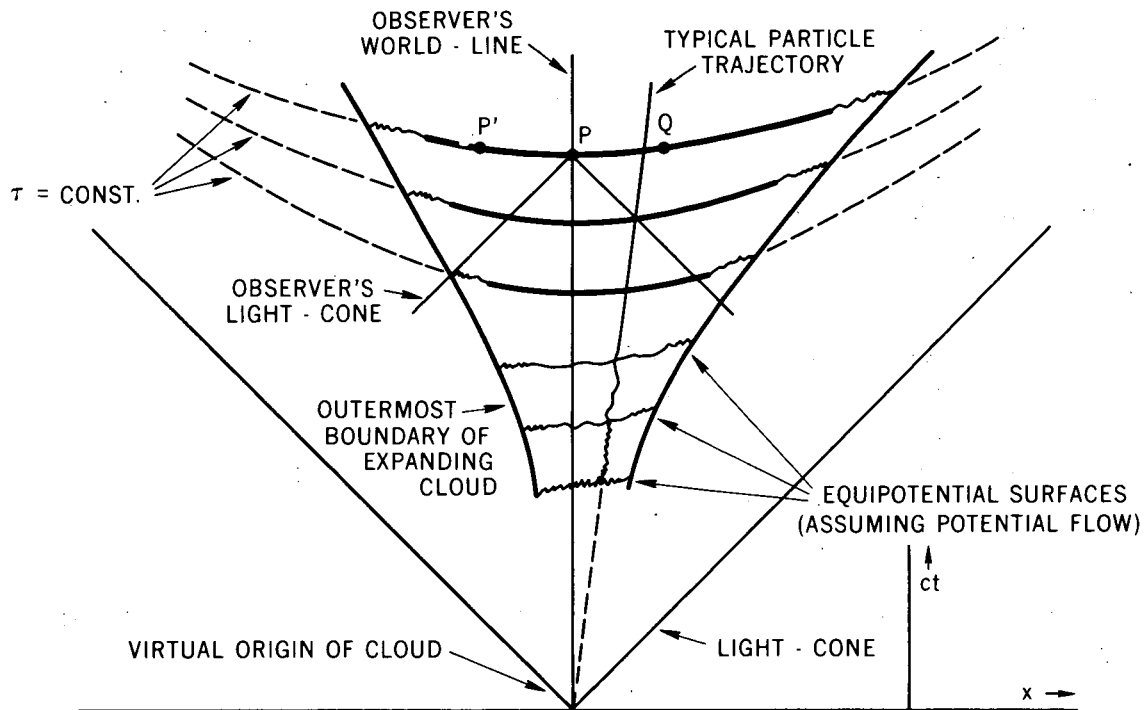
Figure 13. Free-Expansion Transition to Rectilinear Flow

This argument has omitted the effects of gravitation. It can be shown, however, that the whole formalism can be carried out within the framework of General Relativity. Rectilinear flow then becomes geodesic flow, and the metric tensor (gravitational potential) behaves like the other thermodynamical variables in its tendency to become uniform on hypersurfaces of constant thermasy, which is the most convenient scalar-time parameter to use. The tendency of the metric components to become uniform on the constant-ϑ hypersurfaces can be given the usual geometric interpretation in terms of a constant spatial curvature.

APPENDIX: BRIEF LITERATURE SURVEY

Short, readable historical surveys of the development of variational principles, especially the Least Action Principle, are to be found in the appendix of Funk's book, in Chapter 10 of Lanczos' book, and throughout the text of the book by Yourgrau and Mandelstam. Much more detailed surveys are given in the long review articles by Voss and Hellinger. Helmholtz's review (1887) devotes most of its space to a discussion of Leibniz's priority rights and to an examination of the different variation processes used by Hamilton and Jacobi. Extensive bibliographies are to be found in the articles by Cayley (1857) and Bateman (1945). Finally, all of the classical papers themselves are reprinted in volumes 46 and 47 of Ostwald's "Klassiker der exakten Wissenschaften". In the bibliography at the end of this article, only nineteenth-century papers are cited individually. Earlier papers are referred to one of the above-listed reviews.

The reader should be forewarned that the survey that follows is polemic in its intent, and that the selection and emphasis have been made accordingly. The position that is defended is the one outlined in the Introduction, namely that the relative significance of the roles played by mechanics and thermodynamics has been biased by the respective histories of the two fields. A brief review of the development of the Least Action Principle will attempt to show that, despite the fact that the development of the principle was based mainly on the Principle of Virtual Displacements, which is really an energy argument, and hence thermodynamic in nature, the Least Action Principle came to be regarded as simply an alternative form of Newton's Laws. A brief survey of the development of the energy concept will then be given in order to establish the relative chronology, in particular to emphasize how surprisingly late in the development of physics a clear-cut and universally accepted formulation of energy conservation came into being. Only after this development could variational minimum principles in the spirit of thermodynamics be established.

A.1 Early Development of the Least Action Principle

The Least Action Principle, in a rather muddled form, was first published in 1744 by Maupertuis, but the first clear statement of it was published too in 1744 by Euler who, however, made no claim to priority, deferring rather to Maupertuis. Helmholtz credits Leibniz with having set forth the Least Action idea in 1689 in a manuscript that was never published until 1860, except for a brief abstract published in 1695. In 1750 this claim on behalf of Leibniz was made by Samuel König, a Dutch mathematician of Swiss origin, and led to a comic-opera priority dispute with Maupertuis which, together with the far-reaching and far-fetched metaphysical significance that Maupertuis attributed to the Least Action Principle, did much to undermine the esteem in which it was

held by serious scientists. These metaphysical claims were based on the apparent teleological aspect of the principle which was represented as proving the existence of a Supreme Being. This argument was greatly weakened when counter-examples were invented to show that the action was not <u>always</u> a minimum.

In sharp contrast to Maupertuis' metaphysical approach, d'Alembert expressed the opinion that the Least Action Principle could be derived from Newton's Laws, and Lagrange showed that this was indeed the case by effectively casting d'Alembert's Principle into integral form. Having done this, Lagrange invented his multiplier technique for imposing constraints, and applied it to derive a variational principle for an incompressible fluid in which the pressure was the multiplier corresponding to the constant-density constraint.[22]

In order to understand the hidden thermodynamical nature of the Least Action Principle, it is necessary to appreciate the important role of the Principle of Virtual Displacements. Voss[23] attributes the first clear statement of the Principle of Virtual Displacements (Virtual "Velocities" in the older literature) to Johann Bernoulli (1717). D'Alembert (1743)[24] extended this principle from static to dynamic situations by means of the trick of treating the negative of a particle's acceleration like any of the other forces acting on it, and, as mentioned above, Lagrange (1788)[25] cast d'Alembert's Principle into integral form to derive the Least Action Principle.

From the thermodynamic point of view, the important feature of Lagrange's application of the Principle of Virtual Displacements is that he regarded it as an <u>equality</u>. As such, it contains no stability criterion, and when generalized to dynamics as in d'Alembert's Principle, it is indeed equivalent to Newton's Laws. Fourier (1798)[26] was the first to point out that the Principle of Virtual Displacements should actually be an <u>inequality</u>, which automatically implies a stability criterion. By this time, however, the notion that the Least Action Principle was just an alternative form of Newton's Laws had become well-established and this in turn caused interest in it to slacken. It was not until the work of Hamilton (1834, 1835) and Jacobi (1837) had shown that what we now call Hamilton's Principle was indeed much more than just an alternative form of Newton's Laws that interest in variational principles was revived, and with the development of the energy concept speculations about minimum variational principles were resumed, this time without any teleological overtones.

A.2 <u>Development of the Energy Concept</u>

Aside from some rather qualitative observations on the conservation of kinetic energy and its ability to perform various kinds of work, the first

analytical statement of the conservation and change of kinetic energy was made in the context of celestial mechanics by Daniel Bernoulli (1748)[27]. From Lagrange (1777)[28] came the concept of a potential function for discrete masses, and from Laplace (1782)[29] the concept of a potential function for continuous masses. In connection with the analysis of simple machines, the conversion of kinetic energy into work by mechanical means had by this time become a commonplace, but the conversion of kinetic energy into work via thermodynamic processes was first applied by S. Carnot (1824)[30].

To the physician Robert Mayer (1942) is due to the first general statement of the energy concept in the sense that he introduced the idea of expressing all arbitrary processes in terms of "equivalent mechanical work", but it is to another physician, Hermann von Helmholtz (1847), that we owe the mathematical formulation of Conservation of Energy as we know and use it today. Thus 1847 can be taken as the date when the energy concept, which had been developing for a century-and-a-half, finally crystallized. However, we have sobering evidence that even at this late date these ideas were still not widely recognized in the fact that the papers of both Mayer and Helmholtz were rejected by Poggendorff's Annalen.[31]

Simultaneously with the development of the energy concept, the concept of stability developed out of Fourier's observation that the proper statement of the Principle of Virtual Displacements involved an inequality, rather than an equality. (Gauss and Ostrogradsky later independently made the same discovery.[32]) This was extended to the continuum in the case of an elastic medium by Dirichlet (1846), and the combination of the Principle of Virtual Displacements with the energy concept led to the Principle of Virtual Work as embodied in the Principle of Castigliano (1879) and Ménabréa (1871, 1875), first applied to the study of trusses, and later extended to an elastic medium.

With the incorporation of thermodynamics into Hamilton's Principle that was carried out by Helmholtz in 1886[33], he came close to constructing a variational minimum principle by means of thermodynamics, but in actual fact he regarded Hamilton's Principle as fundamental, and thermodynamics as the means for specifying the potential function that must be inserted into the Lagrangian. He considered the case of a body of constant volume that is held at constant temperature by immersion in a constant-temperature heat reservoir. For such a case he showed that the appropriate potential function is what he called the free-energy (and what today is often called the Helmholtz Function).

By the end of the nineteenth centure, thermodynamics had recovered sufficiently from its late start vis-a-vis mechanics to arouse some enthusiasm for a many-faceted program known as Energetics[34], part of which consisted of the attempt to derive the equations of motion from the expression for the total energy

of the system. These attempts did not meet with much success, and the reason for this has already been indicated in the Introduction of this article, namely, an extra postulate is necessary in order to specify the time-evolution. In the development of this article, that postulate has been the requirement of Lorentz-invariance.

A.3 Recent Literature of Variational Principles for Fluid Flow

The purpose of the survey of this section is to trace the evolution from Helmholtz's variational principle, which applied to a rigid body, toward a variational minimum principle for a compressible fluid. Because the term "variational principle" has many different meanings in the literature, a rudimentary classification is necessary. A variational principle will be called "non-holonomic" if it can be stated only in differential form. Such principles are characteristic of systems, such as those involving dissipation, that are acted on by non-potential forces. Such principles are of no use whatever for direct numerical solutions because there exists no action integral to minimize. When an action integral does exist, the principle will be called "holonomic". Holonomic principles in turn can be divided into "non-minimum" and "minimum principles" according to whether the nature of the stationary point is indeterminate or is a minimum (for stable flow). A non-minimum principle has very limited utility for direct numerical solutions. It was noted in footnote 19 that the presence of Lagrange multipliers in the Lagrangean density renders the nature of the stationary point indeterminate. The vast majority of variational principles for fluid flow do involve constraints, and so fall into the non-minimum category.

This survey will concentrate on relativistic variational principles. Good surveys of non-relativistic variational principles for fluid dynamics are available in the review articles by Serrin[35] and by Truesdell and Toupin[36].

The relativistic generalization of Helmholtz's work for a rigid body of fixed volume was carried out by Planck (1907), and the generalization of Planck's work to relativistic fluid flow was carried out by his student Lamla (1912), and independently by Herglotz (1911). Both of these variational principles were non-holonomic. Much later, Taub (1948, 1954) used a somewhat different procedure to arrive at a holonomic principle, first non-relativistically, and then within the framework of General Relativity. His relativistic action integral contains a Lagrange multiplier, and so the variational principle that results is non-minimum.[37] The non-relativistic action integral is unconstrained, but the variational principle is nevertheless non-minimum for a subtler reason. Both Helmholtz's and Planck's principles were minimum for small variations from the stationary value so long as the volume was held constant. It is easy to show

that in their principles if the volume is allowed to vary, the Lagrangean is concave with respect to the volume-variation instead of convex. Thus the action integral is a maximum with respect to variation of volume and a minimum with respect to variation of the other variables, so the stationary point is a saddle-point. Although Helmholtz and Planck held the volume constant, in the fluid generalization this is not possible, so a saddle-point results. This is not as bad as a completely indeterminate stationary point, and has some potential for direct numerical solutions.

All of the principles described above were lineal descendents of Planck's variational principle. Two approaches that were very different from Planck's were taken by Minkowski (1908)[38] and by Born (1909). Both pieces of work are notable for the fact that, unlike Planck's, they are manifestly covariant. This results from the fact that both make use of the proper time, which is introduced in Minkowski's paper for the first time. Although Minkowski's variational principle is non-holonomic, it is interesting in that it amounts to a covariant application of the Principle of Virtual Displacements to derive a variational principle for a cold, relativistic gas. (No thermodynamics is involved in Minkowski's derivation.) No thermodynamics is involved in Born's work either. His Lagrangian density is the interaction term for a cold electon gas in an electromagnetic field. Whereas Minkowski worked in the laboratory frame, Born works in the convected frame, using the three identity parameters and the proper time as his independent variables. The use of proper time as an independent variable requires the imposition of a constraint, and this makes his variational principle, which unlike Minkowski's is holonomic, non-minimum. Born's action integral is in the spirit of the Castigliano Principle in that it involves only the interaction energy of the electron gas and the electromagnetic field, and leaves out of consideration the intrinsic energies of both. This tradition of manifestly covariant extensions to the fluid case of the simple minimum principles of classical physics was continued by Synge (1937) and then Lichnerowicz (1941) who developed within the framework of General Relativity a variational principle for barotropic flow that is the analog of Fermat's Least-Time Principle in optics.

The most frequently applied technique for constructing a variational principle for a fluid has been to add the necessary constraints (each one involving a Lagrange multiplier) to the Lagrangean density for a cold fluid. This is the method used by Khalatnikov (1954), Wei (1959), Sharma and Sharma (1963), Tam (1966, 1968), and Schmid (1969). All of these variational principles are holonomic, but non-minimum.

Of greater relevance to the variational minimum principle developed in the present article are principles whose Lagrangean densities are the unconstrained pressure functional in one form or another. Van Dantzig (1940a) presented such a principle, but it seems to have been limited to potential flow. The variational

principle derived by Schmid (1970b) is similar to the one derived in the present article, except that the identity parameters are represented as the four components of a four-vector instead of as three scalar functions, and the cell density flux vector is not parameterized in terms of the identity parameters. This makes for unnecessary redundancy, and does not allow the flow to be described in terms of a mapping in four-space. Schutz (1970) has given the relativistic generalization of a variational principle derived by Seliger and Whitham (1968). This theory starts with the thermodynamic pressure in the form $P(H, S)$. Referring to eq. (3-5b) of this article, it is clear that we could contract each side of the equation with itself and arrive at an expression for H in terms of the quantities on the right which we could use to eliminate the H in the pressure functional. This approach has the same disadvantages as those encountered in the case of the theory of Schmid (1970), namely redundancy and the inability to describe the flow in terms of a mapping. The fact that, in spite of their redundancy, these variational principles are minimum principles arises from the fact that they may be regarded as simply different variants of the same variational minimum principle, and the vital properties of this principle result from the independence of the variables (lack of constraints) and the fact that the Lagrangean density (the pressure functional) by its very nature must be convex with respect to fluctuations in all its degrees of freedom, regardless of how these might be parameterized.

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REFERENCES

H. Bateman (1945): Hamilton's Work in Dynamics and its Influence on Modern Thought; 51-63 of the Hamilton festschrift "A Collection of Papers in memory of Sir William Rowan Hamilton" (Scripta Mathematica-Yeshiva College — New York, 1945).

M. Born (1909); Die träge Masse und das Relativitätsprinzip, Ann. d. Physik (4) 28, 571-584.

F. Cajori (1929): A History of Physics (Dover, New York, 1962. This is a reissue of the 2nd edition published by Macmillan in 1929).

H. B. Callen (1960); Thermodynamics (Wiley, New York, 1960).

A. Castigliano (1879): Théorie de l'équilibre des systèmes élastiques et ses applications (Turin, 1879).

A. Cayley (1857): Report on the Recent Progress of Theoretical Dynamics (Report of the British Association for the Advancement of Science, 1857) — Collected Papers, Vol. 3, pp. 156-204 (Cambridge University Press, 1890).

H. C. Corben (1949): The Transformation Theory of Thermodynamics, Phys. Rev. 76, 166 (1949).

R. Courant and D. Hilbert (1953): Methods of Mathematical Physics, Vol. 1 (Interscience Publishers, New York, 1953).

D. Van Dantzig (1939a): On the Phenomenological Thermodynamics of Moving Matter, Physica 6, 673-704.
(1939b): On Relativistic Thermodynamics, Proc, Kon. Ned. Akad. Wet. 42, 601-607.
(1939c): On Relativistic Gas Theory, Proc. Kon. Ned. Akad. Wet. 42, 608-625.
(1940a): On the Thermo-Hydrodynamics of Perfectly Perfect Fluids, I, Proc. Kon. Ned. Akad. Wet. 43, 387-402.
(1940b): On the Thermo-Hydrodynamics of Perfectly Perfect Fluids, II, Proc. Kon. Ned. Akad. Wet. 43, 609-618.

P. G. L. Dirichlet (1846): Ueber die Stabilität des Gleichgewichts, J. reine angew. Math. (Crelle) 32, 85-88 — Werke, L. Kronecker and L. Fuchs, Eds. (G. Reimer, Berlin 1897), Vol. 2, pp. 3-8.

C. Eckart (1960): Variation Principles of Hydrodynamics, Phys. of Fluids 3, 421-427.

J. B. Fourier (1798): Mémoire sur la statique contenant la démonstration du principe des vitesses virtuelles et la théorie des moments, Journ. de l'École Polytechnique, Ve Cahier, p. 20, (1798) — Oeuvres de Fourier, G. Darboux, Ed. (Gauthier-Villars, Paris, 1890), Vol. 2, pp. 477-521.

K. Friedrichs (1929): Ein Verfahren der Variationsrechnung das Minimum eines Integrals als das Maximum eines anderen Ausdruck darzustellen, Nachr. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl. (1929), pp. 13-20.

P. Funk (1970): Variationrechnung und ihre Anwendung in Physik and Technik (Springer, Berlin, 1970), 2nd ed.

W. R. Hamilton (1834): On a General Method in Dynamics; by which the Study of Motions of all free Systems of attracting or repelling Points is reduced to the Search and Differentiation of one central Relation, or Characteristic Function, Phil. Trans. Roy. Soc. (Part 2 for 1834), 247-308 — The Mathematical Papers of Sir William Rowan Hamilton, A. W. Conway and A. J. McConnell, Eds., (Cambridge University Press, 1940), Vol. 2: Dynamics, pp. 103-161.
(1835): Second Essay on a General Method in Dynamics, Phil. Trans. Roy. Soc. (Part I for 1835), 95-144 — Mathematical Papers, Vol. 2, pp. 162-211.

E. Hellinger (1913): Die allgemeinen Ansätze der Mechanik der Kontinua, Enc. der Math. Wiss. IV^4 (Mechanik) (Teubner, Leipzig, 1907-1914) Art. 30, pp. 601-694.

H. von Helmholtz (1847): Die Erhaltung der Kraft. Read before the Physical Society of Berlin July 23, 1847. Published privately as a phamphlet, but now available in his Wissenschaftliche Abhandlungen (Barth, Leipzig, 1882), Vol. 1, pp. 12-75.
(1884): Studien zur Statik monocyklischer Systeme, Sitzber. Akad. Wiss. zu Berlin (1884), 159-177 — Abhandlungen (Barth, Leipzig, 1895) Vol. 3, pp. 119-141.
(1886): Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung, J. reine angew. Math. (Crelle) 100, 137-166, 213-222 — Abhandlungen (Barth, Leipzig, 1895), Vol. 3, pp. 203-248.
(1887): Zur Geschichte des Princips der kleinsten Action, Sitzber. Akad. Wiss. zu Berlin (1887), pp. 225-236.

G. Herglotz (1911): Über die Mechanik des deformierbaren Körpers vom Standpunkte der Relativitätstheorie, Ann. d. Physik (4) 36, 493-533.

C. G. J. Jacobi (1837): Zur Theorie der Variationsrechnung und der Differential-gleichungen, J. reine angew. Math. (Crelle) 17, 68-82 — Gesammelte Werke, K. Weierstrass, Ed. (Berlin 1886 — available in Chelsea edition, 1969), Vol. 4, pp. 39-55.
(1837): Über die Reduction der Integration der partiellen Differential-gleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewöhnlicher Differential-gleichungen, J. reine angew. Math. (Crelle) 17, 97-162 — Gesammelte Werke, Vol. 4, pp. 57-127.

Kelvin (William Thomson) (1849): On the Vis-Viva of a Liquid in Motion, Cambr. and Dubl. Math. Journ. 4, 90-94 — Papers, Vol. 1, pp. 107-112.

I. M. Khalatnikov (1954): Contributions to Relativistic Hydrodynamics (in Russian) Zhurn. Eks. i. Teor. Fiz. 27, 529-541.

H. Lamb (1932): Hydrodynamics (Cambridge University Press, 1932), 6th ed.

E. Lamla (1912): Über die Hydrodynamik des Relativitätsprinzips, Ann. d. Physik (4) 37, 772-796.

C. Lanczos (1970): The Variational Principles of Mechanics, (University of Toronto Press, 1970), 4th ed.

M. von Laue (1911- First ed.) Die Relativitätstheorie (Friedr. Vieweg & Sohn, Braunschweig, 1961), 7th ed. (See pp. 180-182 for Laue's review of Helmholtz's canonical formalism for thermodynamics.)

A. Lichnerowicz (1941): Sur l'invariant intégral de l'hydrodynamique relativiste, Ann. Sci Ecole Normale Superieure 58, 285-304.

J. R. Mayer (1842): Bemerkungen über die Kräfte der unbelebten Natur, Annalen der Chemie and Pharmacie 42, 233-240 (1842). Reprinted in translation by G. C. Foster as "Remarks on the Forces of Inorganic Nature", Phil. Mag. (Ser. 4) 24, 371-377 (1862).

L. F. Ménabréa (1871): Principe general pour déterminer les pressions et les tensions dans un système élastique, Accad. delle Science Torino (2) 25, 141-180.
(1875): Sulla determinazione delle tensioni e delle pressioni ne sistemi elastici, Rend. Acc. dei Lincei (2) 2, 201-220.

H. Minkowski (1908): Die Grundgleichungen für die electromagnetischen Vorgänge in bewegten Körpern, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. (1908), 53-111 — Gesammelte Abhandlungen von Hermann Minkowski (Leipzig, 1911 — available from Chelsea Publ. Co.), Vol. 2, pp. 352-404.

W. Ostwald (1894): Klassiker der exakten Wissenschaften, Nr. 46 & Nr. 47: Abhandlungen zur Variationsrechnung (W. Engelmann, Leipzig, 1894).

M. Planck (1907): Zur Dynamik bewegter Systeme, Sitzber. preuss. Akad. Wiss. (Berlin), (1907), 542-570. Reprinted verbatim in Ann. d. Physik (4) 26, 1-34 (1908).

O. Redlich (1968): Fundamental Thermodynamics Since Carathéodory, Rev. Mod. Phys. 40, 556-563.

H. Rund (1966): The Hamilton-Jacobi Theory in the Calculus of Variations (Van Nostrand, London, 1966).

L. A. Schmid (1967a): Heat Transfer in Relativistic Charged Fluid Flow, Nuovo Cimento 47B, 1-28.
(1967b): Larmor and Halmholtz Theorems for Charged Fluid Flow, Nuovo Cimento 52B, 288-312.
(1967c): Hamilton-Jacobi Equation for Relativistic Charge Fluid Flow, Nuovo Cimento 52B, 313-337.
(1970a): Effects of Heat Exchange on Relativistic Fluid Flow, pp. 161-201 of "A Critical Review of Thermodynamics", E. B. Stuart, B. Gal-Or, and A. J. Brainard, Eds. (Mono Book Corp., Baltimore, 1970).
(1970b): Variational Formulation of Relativistic Fluid Thermodynamics, Pure and Appl. Chem. 22, 493-500.

B. F. Schutz (1970): Perfect Fluids in General Relativity: Velocity Potentials and a Variational Principle, Phys. Rev. D2, 2762-2773.

R. L. Seliger & G. B. Whitham (1968): Variational Principles in Continuum Mechanics, Proc. Roy. Soc. A305, 1-25.

J. Serrin (1959): Mathematical Principles of Classical Fluid Mechanics, Handbuch der Physik VIII/1 (Springer, Berlin, 1959).

S. R. Sharma & N. K. Sharma (1963): Relativistic variational principle for a charged hydromagnetic compressible fluid, Indian J. Theor. Phys. 11, 39-46.

A. Sommerfeld (1950): Mechanics of Deformable Bodies, (Academic Press, New York, 1950).

J. L. Synge (1937): Relativistic Hydrodynamics, Proc. London Math. Soc. (2) 43, 376-416.

K.-K. Tam (1966): A Variational Principle in Relativistic Magnetohydrodynamics, Canad. J. Phys. 44, 2403-2409.
(1967): Relativistic Scalar Hydrodynamics, Canad. J. Phys. 45, 1415-1417.

A. H. Taub (1948): On Hamilton's Principle for Perfect Compressible Fluids, Proc. Symposia in Applied. Math., Vol. 1, 148-157.
(1954): General Relativistic Variational Principle for Perfect Fluids, Phys. Rev. 94, 1468-1470.
(1969): Stability of General Relativistic Gaseous Masses and Variational Principles, Commun. Math. Phys. 15, 235-254.

L. Tisza (1966): Generalized Thermodynamics (The M.I.T. Press, 1966)
(1970): The Thermodynamics of Phase Equilibrium: From the Phase Rule to the Scaling Laws, Pure and Appl. Chem. 22, 261-272.

C. Truesdell & R. Toupin (1960): The Classical Field Theories, Handbuch der Physik (Springer, Berlin, 1960), Vol. III/1.

A. Voss (1901): Prinzipien der rationellen Mechanik, Enc. d. Math. Wiss. IV[1] (Mechanik) (Teubner, Leibzig, 1901-1908), Art. 1 (pp. 3-121).

W. Yourgrau & S. Mandelstam (1960): Variational Principles in Dynamics and Quantum Theory, (Pitman Publ. Co., New York, 1960), 2nd ed.

FOOTNOTES

1. Helmholtz (1886). See esp. pp. 225-226 of Vol. 3 of his Abhandlungen.

2. Minkowski (1908). The remark appears shortly before eq. 23 of the Appendix.

3. Helmholtz (1884). He identifies the temperature as a velocity shortly after eq. 5 at the beginning of Sec. 3. (This is on p. 132 of Vol. 3 of his Abhandlungen.)

4. Van Dantzig (1939a). Van Dantzig introduced the term "thermasy" without citing any source or giving any etymology. Presumably it derives from the Greek $\theta\epsilon\rho\mu\alpha\nu\sigma\iota\varsigma$ (heat-generator) with elision of the ν and the final ς from the suffix $\alpha\nu\sigma\iota\varsigma$ denoting generator. By analogy, "ergasy" and "enthalpasy" would signify energy-generator and enthalpy-generator respectively. My thanks to my colleague E. G. Stassinopoulos for this etymological analysis.

5. For an excellent presentation of the thermodynamic-potential approach to thermodynamics, see the textbook by H. B. Callen (1960). For a review of the conceptual development of thermodynamics, see Chaps. 1 and 2 of Tisza (1966) and also Redlich (1968).

6. The analogy between thermodynamical transformations and canonical transformations has been pointed out by Corben (1949). See also Corben and Stehle (1950), Sec. 95 (pp. 329-337).

7. See Tisza (1966), p. 45, 116, and Tisza (1970).

8. Planck (1907).

9. Van Dantzig (1939a, b, c), (1940a, b).

10. Eckart (1960), Schmid (1967a, b, c).

11. See Tisza (1966), p. 109, 122-123.

12. Dirichlet (1846). See Funk (1970), pp. 32-33, Chap. 8.

13. Castigliano (1879), Ménabréa (1871, 1875).

14. See Funk (1970) pp. 65-67; Rund (1966), p. 38, 139.

15. Schmid (1967a), Sec. 4; Schmid (1970), Sec. IV-3.

16. Schmid (1970), Sec. V-3.

17. See the historical sketch of the early development of relativistic thermo-dynamics given in Schmid (1970), Appendix.

18. Friedrichs (1929). For a discussion of the Friedrichs Transformation, see Funk (1970), Chap. 8; Courant and Hilbert, Vol. 1 (1953), Chap. 4, Sec. 9.

19. See Funk (1970), footnote on p. 500.

20. Kelvin (1849). For discussions of Kelvin's Minimum Energy Theorem, see Lamb (1932), p. 47, and Truesdell and Toupin (1960), p. 368.

21. Schmid (1970), Sec. III-2.

22. See, for example, Sommerfeld (1950), p. 90.

23. Voss, p. 66, esp. footnote 181.

24. Voss, p. 77, footnote 209.

25. Hellinger, p. 615, footnote 23.

26. Voss, p. 73, footnote 201. The important remark occurs at the end of Sec. 13 of Fourier's paper.

27. Voss, p. 101, footnote 283.

28. Voss, p. 101, footnote 284.

29. Voss, p. 101, footnote 285

30. Voss, p. 103, footnote 294.

31. Cajori (1929), p. 222.

32. Voss, p. 74, footnotes 202 and 203.

33. Helmholtz (1886), pp. 225-226 of his Abhandlungen, Vol. 3.

34. See Voss, Sec. 50.

35. Sec. IV (pp. 144-150).

37. In a more recent paper, "Stability of Fluids Motions", Proc. of the 1967 Colloque sur fluides et champ gravitational (CNRS, Paris), that was cited by Taub (1969), he has developed a new variational principle for relativistic barotropic flow. I have not yet seen this paper, but it seems evident that this must be a minimum principle because in Taub (1969) he applies it with success to the problem of gravitational stability.

38. Minkowski derives his variational principle in the appendix of his 1908 paper.