NASA TM X- 66151

EQUATORIAL RADIUS OF THE EARTH: A DYNAMICAL DETERMINATION

MOHAMMAD ASADULLAH KHAN

December 1972

GSFC —— GODDARD SPACE FLIGHT CENTER ——
GREENBELT, MARYLAND

EQUATORIAL RADIUS OF THE EARTH:

A DYNAMICAL DETERMINATION

Mohammad Asadullah Khan*

Geodynamics Branch

NASA

Goddard Space Flight Center

Greenbelt, Maryland 20771

---

*On leave of absence from Hawaii Institute of Geophysics, University of Hawaii, Honolulu, Hawaii, 96822.

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

EQUATORIAL RADIUS OF THE EARTH:

A DYNAMICAL DETERMINATION

Mohammad Asadullah Khan
Geodynamics Branch

ABSTRACT

An interesting variation on the familiar method of determining the earth's equatorial radius a_e, from a knowledge of the earth's equatorial gravity is suggested. The value of equatorial radius thus found is 6378,142 ± 5 meters. The associated parameters are GM = $3.986008 \pm .000004 \times 10^{20}$ cm^3 sec^{-2} which includes the relative mass of atmosphere $\sim 10^{-6} \times$ GM, the equatorial gravity γ_e = 978,030.9 milligals (constrained in this solution by the Potsdam Correction of 13.67 milligals as the Potsdam Correction is more directly, or less indirectly, measurable than the equatorial gravity) and an ellipsoidal flattening of f = 1/298.255.

CONTENTS

ILLUSTRATIONS

TABLES

Preceding page blank

EQUATORIAL RADIUS OF THE EARTH:

A DYNAMICAL DETERMINATION

INTRODUCTION

The problem of determining the earth's equatorial radius a_e, consists of scaling

a mean ellipsoid which best fits an accurately determined detailed geoid. The

importance of the problem is well known in geodesy and many of the geodetic

and geophysical applications of artificial earth satellites. Several theoretical

approaches are available to solve the problem, namely, gravimetric methods

(see for example, Heiskenan and Moritz, 1967), astrogeodetic methods (see for

example, Fricke et al., 1965), dynamic and geometric satellite methods (see for

example, Kaula, 1967; Anderle, 1967 or Veis, 1967) but their applications is

limited by the density and precision of the available data required for the purpose.

In presatellite era a value of a_e = 6378,388 meters, computed by Hayford (1909)

from the isostatically reduced astrogeodetic data in the United States, was

adopted to define the equatorial radius of the "international reference ellipsoid."

The artificial earth satellite data analyses and space probe studies have indicated

much smaller values (see, for example, Veis, 1967) ranging approximately be-

tween a_e = 6378,124 meters and a_e = 6378,160 meters, with the latest determi-

nations concentrated around a value of a_e = 6378,140 meters approximately.

This can be seen from Table 1 which, by the way, is by no means anywhere near

a complete listing of all the important work on the determination of a_e.

Table 1

Comparisons of the Recent Estimates

of the Earth's Equatorial Radius a$_e$

Reference	Equatorial Radius a$_e$ (meters)
Vies (1965)	6378,169
Geodetic Reference System, 1967	6378,160
Kaula, 1967	6378,153
Present paper (atmospheric mass included)	6378,144
Present paper (atmospheric mass excluded)	6378,142
Veis (1965)	6378,142
Lambeck (1971)	6378,140
Rapp (1970)	6378,138
Strange et al. (1971)	6378,124

THE METHOD

The method suggested here stems from Khan's (1972) work showing that the

Potsdam Correction d, can be expressed as

$$d = (\gamma_P^I - \gamma^S) + (\triangle g^I - \triangle g^S) \tag{1}$$

The quantities γ and $\triangle g$ are the theoretical gravity and gravity anomaly on the

appropriate reference surface respectively. The superscript I denotes the

international reference ellipsoid and the superscript S the appropriate "satellite reference ellipsoid." The subscript P refers to the "Potsdam Gravity System." The theoretical gravity γ on the surface of an arbitrary reference ellipsoid, is given by the well-known formula

$$\gamma = \gamma_e (1 + \alpha_2 \sin^2 \phi + \alpha_4 \sin^2 2\phi) \qquad (2)$$

where the equatorial gravity γ_e is related to GM, a_e and the flattening f as

$$GM = \gamma_e a^2 (1 - f) \left(1 + \frac{3}{2} m + \frac{3}{7} m f + \frac{9}{4} m^2\right) \qquad (3)$$

and the coefficients α_2 and α_4 are given as

$$\alpha_2 = -f + \frac{5}{2} m + \frac{1}{2} f^2 - \frac{26}{7} m f + \frac{15}{4} m^2$$

and $\qquad (4)$

$$\alpha_4 = \frac{1}{8} f^2 - \frac{5}{8} m f$$

with

$$m = \frac{\omega^2 a^3 (1 - f)}{GM}$$

Note that the above formulas are accurate only to the $O(f^2)$ as they neglect quantities $O(f^3)$.

The appropriate choice of γ_e, α_2 and α_4 in the above equations will yield γ^I or γ^S as the case may be. In practice, the quantity γ_P^I is computed from the

3

international gravity formula in which the value of equatorial gravity is tied to the Potsdam gravity system. Hence any error in the system is reflected in this quantity. The quantity γ^S is computed from the socalled "satellite gravity formula" (not the "Gravity Formula 1967") in which the ellipsoid flattening is held fixed at $f = 1/298.255$ but GM and a_e are treated as variables.

The gravity anomaly Δg is given by

$$\Delta g = \frac{GM}{a_e^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^{n+2} (n-1)(\delta C_{nm} \cos m\lambda + \delta S_{rm} \sin m\lambda) P_{nm}(\sin \phi) \quad (5)$$

where

$$\begin{matrix} \delta C_{nm} \\ \delta S_{nm} \end{matrix} = \text{observed} \begin{matrix} C_{nm} \\ S_{nm} \end{matrix} - \text{reference} \begin{matrix} C_{nm} \\ S_{nm} \end{matrix}$$

and other symbols have the usual meaning (see for example, Khan, 1969). In practice, however, only C_{20} and C_{40} are considered for the reference ellipsoid. Consequently all other δC_{nm} and δS_{nm} (except n = 2 and 4) equal the observed geopotential coefficients. The largest of the coefficient differences, involved in the computation of Δg^I and Δg^S or the difference $\Delta g^I - \Delta g^S$, does not exceed 10^{-5}. Hence, neither of these quantities is sensitive to small changes in GM and a_e. Thus the sensitivity of γ^S to small variations in GM or a_e or more exactly the ratio GM/a_e^2, on which the equatorial gravity γ_e^S, used in computing γ^S, is based makes the relation stated in Equation (1) highly useful as it can be utilized eventually to determine a_e.

Note from Equation (3) and the expression for m that the equatorial gravity γ_e is automatically fixed once the earth's rotational speed ω, GM, a_e and the flattening

4

f are defined. Of these quantities ω and f are well-determined. Of the remaining three i.e., GM, a_e and γ_e, knowledge of any two will determine the third uniquely. The quantity GM is independently determined from lunar and planetary probes and although these determinations still have a considerable uncertainty associated with them, GM is far more strongly determined than γ_e or a_e. The quantity γ_e is not directly measurable and must be either derived theoretically from expressions involving a_e (see Equation 3) which we wish to compute in this paper or from surface gravity data which does not form an adequate network. (This was true until recently but the situation is likely to change with the increased use of the portable free fall absolute gravity measuring equipment developed by Hammond and Faller, 1971).

This difficulty can be overcome, however, by controlling the value of equatorial gravity γ_e^S (entering Equation (1) via γ^S) through the Potsdam gravity correction d as demonstrated in Equation (1). Because of the recent advances in electronic technology, the absolute gravity value at a particular site can be measured to an accuracy of a few parts in 10^8 of absolute gravity g (Hammond and Faller, 1971). If such precise measurements could be made at the Potsdam site itself, the Potsdam gravity correction could be determined with the same precision. However, since no such measurements are available (in case they are, they have not been released to the scientific community), we can use indirect estimates of d made through highly accurate measurements of absolute gravity at sites previously tied to the Potsdam gravity system. Such measurements yield (Cook and Hammond, 1969) a value of approximately d = 13.67 milligals which is probably correct to better than a milligal.

The advantage of the method suggested in Equation (1) lies in the fact that the less cooperative equatorial gravity value is controlled through the Potsdam Correction d which is relatively more directly, <u>or in fact less indirectly,</u> measured. This is clearly shown by Equations (1) through(4). These equations also demonstrate that the method is equivalent to applying the Potsdam Correction to the equatorial gravity value γ_e^I in the international gravity formula, projecting the corrected value onto an ellipsoid with a flattening of f = 1/298.255 and using the resulting value as a known parameter in Equation (3) to determine a_e. Thus Equation (1) is really an interesting variation on the well-known relation stated in Equation (3).

RESULTS

The solution of Equation (1) is given in Figure 1 which shows plots of the earth's equatorial radius a_e as functions of GM and Potsdam gravity correction d. The quantities γ_P^I and $\triangle g^I$ are based on the well-known parameters of the international gravity formula and the international reference ellipsoid. The quantities γ^S and $\triangle g^S$ are based on an ellipsoidal flattening of $f^S = 1/298.255$. The value of $\triangle g^I - \triangle g^S$ is kept compatible with the indicated values of GM and a_e, though small discrepancies in the value of GM and a_e will not affect $\triangle g^I - \triangle g^S$ to the degree of accuracy (a few parts in 10^8 of absolute gravity g) considered here. All the parameters used in computing γ^S, which makes chief contribution to the changes in d (see Equation (1) and Figure 1), are computed exactly from the appropriate values of GM and a_e.

As pointed out earlier, a well-accepted value of Potsdam correction (Cook and Hammond, 1969; Khan, 1972), obtained from the free fall determinations of
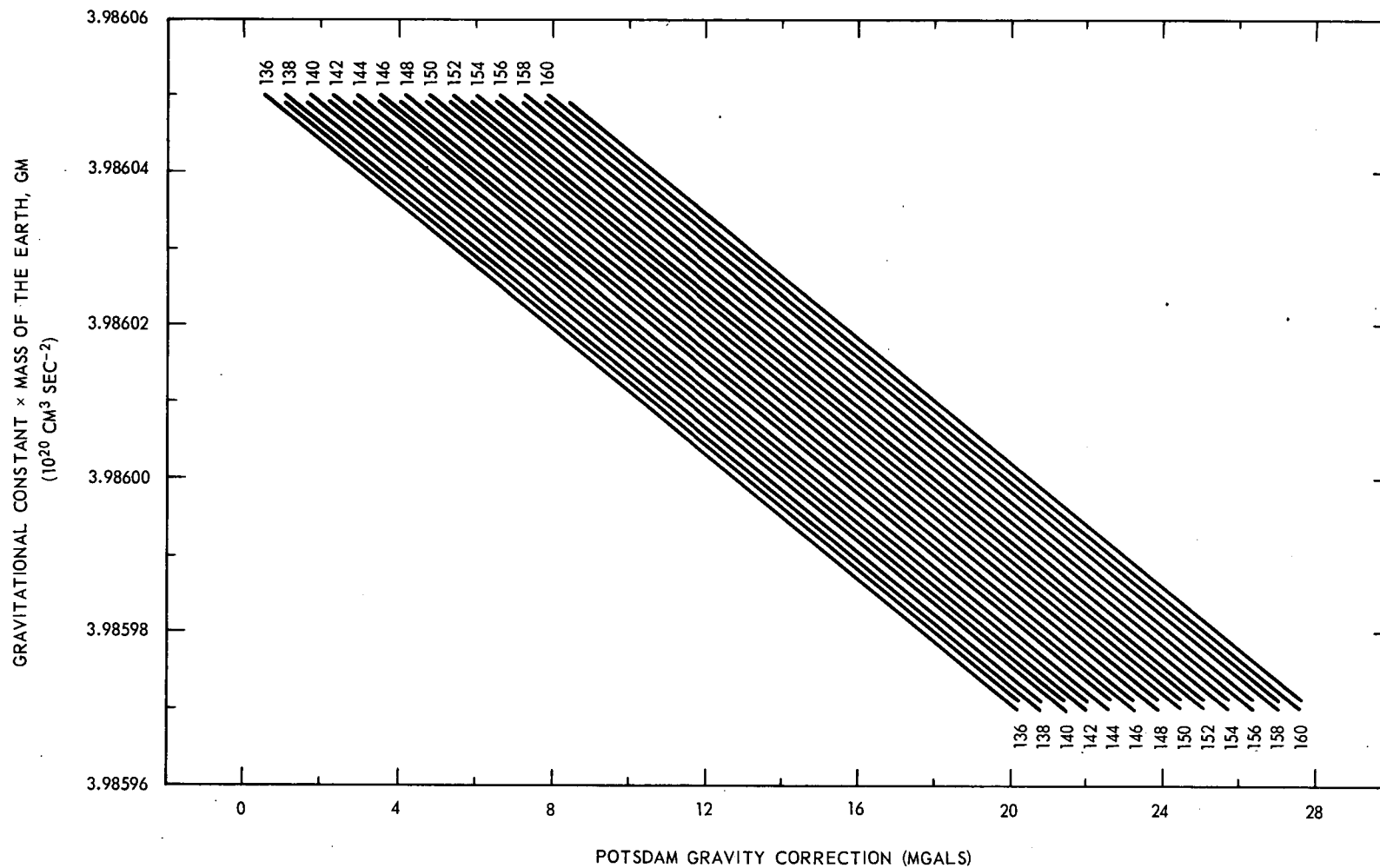
Figure 1. Earth's equatorial radius a_e as function of GM and d. The labels on a_e curves should be read with 6378 as prefix. For example, the rightmost slant curve has a value of 6378,161 meters and the leftmost 6378,136 meters.

absolute gravity acceleration at various sites, is 13.67 milligals, with an accuracy probably better than a milligal. The most recent value of GM is 3.986008×10^{20} cm^3 sec^{-2} (Esposito and Wong, 1972). Corresponding to these quantities, the value of equatorial radius is found to be $a_e = 6378,144$ meters (to the nearest meter). The correction for the mass of the atmosphere reduces GM to 3.986005×10^{20} cm^3 sec^{-2} and the corresponding value of the equatorial radius is $a_e = 6378,142$ meters (to the nearest meter). Compare it with $a_e = 6378,160$ meters adopted by IUGG (Moscow, August 1971) on one extreme and with $a_e = 6378,124$ meters (Strange et al., 1971) on the other extreme. Some of the important recent values of a_e are listed in Table 1.

The maximum error introduced by the approximations made in the second order theory used here is 5×10^{-8} milligals. The error contributed by this factor in the determination of a_e, therefore, cannot exceed a few tenths of a meter. The chief sources of error are the uncertainties in GM, Potsdam Correction and the value of equatorial gravity in the International Reference System. Of these the uncertainty in GM is $\pm 4 \times 10^{14}$ cm^3 sec^{-2}. This corresponds to an uncertainty of ± 3.2 meters in the value of a_e. The maximum uncertainty in Potsdam Correction is not believed to exceed 1 milligal. This corresponds to a maximum uncertainty of 3 meters. Additional uncertainties could stem from some yet undetected systematic shift in the Potsdam gravity correction or the value of absolute gravity acceleration at the Potsdam site but it seems unlikely as the Potsdam correction obtained from various observation sites remains consistent. The tacit assumption made here is that the gravity value obtained from the international gravity formula at the projection of the Potsdam site onto the

reference ellipsoid is compatible with the old gravity value at the Potsdam site,
i.e., no error is assumed in the value of equatorial gravity γ_e^I derived by
Heiskenan (1928), except for the Potsdam Correction, and in the reduction of
the absolute gravity value at the Potsdam site to the geoid. With this granted,
the value of the equatorial radius of the best-fit earth ellipsoid is found to be
a_e = 6378,144 ± 5 meters with the mass of the atmosphere <u>included in GM</u> and
a_e = 6378,142 ± 5 meters <u>with GM corrected</u> for the mass of the atmosphere.
The corresponding value of equatorial gravity γ_e compatible with a_e = 6378,142
meters is γ_e = 978,030.9 milligals (rounded to the nearest tenth of a milligal).
Compare it to the equatorial gravity value of 978,031.1 milligals (see for ex-
ample, Veis, 1967) obtained from a much more extensive gravity coverage than
was available to Heiskenan (1928). It is remarkable that γ_e should display such
a stability. It is also remarkable that if Heiskenan had the present values of
GM, f and d available to him in 1928, he would have obtained a value of a_e very
close to the modern values of the earth's equatorial radius.

RECOMMENDATIONS

The Recommended Geodetic Parameters are:

GM (corrected for mass
    of the atmosphere)   = 3.986005 ± .000004 × 10^{20} cm^3 sec^{-2}

a_e = 6378,142 ± 5 meters

f^{-1} = 298.255 ± .005

d = 13.67 ± 1 milligals

γ_e = 978,030.9 ± 2.5 milligals

9

REFERENCES

1. R. Anderle, Geodetic parameter set NWL-5E-6 based on doppler satellite observations. The Use of Artificial Satellites for Geodesy, II (National Technical University, Athens, Greece (1967).

2. A. H. Cook and J. A. Hammond, The acceleration due to gravity at the National Physical Laboratory, Metrologia, 5, 141-142 (1969).

3. W. Fricke and several others, Report to the executive committee of the working group on the system of astronomical constants, Bull. Geod., No. 75, pp. 59-67 (1965).

4. J. F. Hayford, The figure of the earth and isostasy, from measurements in United States, U.S. Coast and Geodetic Survey (1909).

5. J. A. Hammond and J. E. Faller, Results of absolute gravity determinations at a number of sites, J. Geophys. Res. 76, 7850-7855 (1971).

6. W. A. Heiskenan, Ist die Erde ein dreiachsiges Ellipsoid? Gerlands Beitr. Geophys., v. 19, pp. 356-377.

7. W. A. Heiskenan and H. Moritz, Physical Geodesy, W. H. Freeman and Company (1967).

8. Kaula, W. M., Comparison and combination of satellites with other results for geodetic parameters, The Use of Artificial Satellites for Geodesy, II (National Technical University, Athens, Greece, 1967).

9. M. A. Khan, Potsdam Correction from the satellite determined geopotential, Nature Physical Sciences, vol. 239, 43-45 (1972).

10. M. A. Khan, Figure of the Earth and mass anomalies defined by satellite orbit perturbations, Am. Geophys. Union Monograph No. 13, 293-304 (1969).

11. K. Lambeck, The relation of some major geodetic datums to a geocentric reference system, Bull. Geodes. 99 (1971).

12. W. E. Strange, S. F. Vincent, R. H. Berry and J. G. Marsh, A detailed gravimetric geoid for the United States, NASA Preprint No. X-552-71-219 (1971).

13. R. H. Rapp, Methods for the computation of geoidal undulations from potential coefficients, Ohio State University, Department of Geodetic Science Report No. 132 (1970).

14. G. Veis, the deflection of the vertical of major geodetic datums and the semimajor axis of the earth's ellipsoid as obtained from satellite observations, Space Research V (North Holland Pub. Co., Amsterdam, 1965).

15. G. Veis, The determination of the radius of the Earth and other geodetic parameters as derived from optical satellite data, Smith. Astrophys. Obsy. Rep. 264 (1967).

Additional and more extensive bibliographical references are given in the papers cited above.

LIST OF ILLUSTRATIONS

Figure 1. Earth's equatorial radius a_e as function of GM and d. The labels on a_e curves should be read with 6378 as prefix. For example, the rightmost slant curve has a value of 6378,161 meters and the leftmost 6378,136 meters.