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LONGITUDINAL OSCILLATION OF LAUNCH VEHICLES

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16. Abstract

During powered flight a vehicle may develop longitudinal self-excited oscillations, so-called "POGO" oscillations, of its structure. The energy supplying the vibration is tapped from the thrust by the activity of the system itself; that is, oscillation of the structure causes oscillation of the propellant system, especially of the pumps. In this way an oscillating thrust can be created that, by a feedback loop, may sustain the structural oscillation under certain circumstances.

Two special features of the system prove to be essential for creation of instability. One is the effect of the inherent time interval that the thrust oscillation is lagging behind the structural oscillation. The other is the decrease of system mass caused by the exhausting of gas. The latter feature may cause an initially stable system to become unstable.

To examine the stability of the system, a single mass-spring model, which is the result of a one-term Galerkin approach to the equation of motion, has been considered. The Nyquist stability criterion leads to a stability graph that shows the stability conditions in terms of the system parameter and also demonstrates the significance of time lag, feedback magnitude, and loss of mass.

An important conclusion can be drawn from the analysis: large relative displacements of the pump-engine masses favor instability. This is also confirmed by flight measurements. During some of the Apollo flights, severe POGO oscillations of the S-II stage occurred. The relative soft support of the center engines by a pin-ended crossbeam led to this situation. Data obtained at the time the oscillation occurred show extremely large displacements of the crossbeam at the attachment point.

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DEFINITION OF SYMBOLS

Symbol	Definition
$a_j(t); j = 1, 2 \ldots (n - 1)$ $a(t) = a_1(t)$	coefficient of the jth mode defined by equation (8)
d	damping coefficient
d_{crit}	critical damping coefficient
$\mathbf{D} = [d_{ij}]$	damping matrix of order n
$\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$	vector having the dimension n, consisting of $(n-1)$ zeros and the unit at the station n to which the pumps belong
$f(\lambda)$	complex function defined by equations (19)
i	imaginary unit of the complex plane
$\mathbf{K} = [k_{ij}]$	stiffness matrix of order n
k^*	spring constant defined by equation (15)
K_j	jth modal spring defined by equation (7a)
$\mathbf{M} = [m_j]$	diagonal mass matrix of the order n
$m_j\ ; j = 1,2, \ldots n$	system masses
$M_o = \sum_{j=1}^{n} m_j$	total system mass
$M_j\ ; j = 0,1, \ldots n - 1$	modal masses defined by equation (6)
$m_1{}^* = m^*$	generalized masses defined by equation (15)
n	number of lumped system masses
n_0	number of encirclements of the point $(-1, 0)$ of the complex λ-plane by $f(\lambda)$
$r(\sigma) = \mid f(i\omega\sigma) \mid$	defined by equations (24)

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
s	factor of proportionality defined by equation (11)
t	time
T	thrust
ΔT	oscillating thrust component
$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$	displacement vector of the lumped n mass-spring system
$\mathbf{X}_j = \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \\ \vdots \\ x_n^{(j)} \end{bmatrix}$	jth mode of the lumped n mass-spring system
$\mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	rigid body mode of the lumped n mass-spring system
$\ddot{x}_c$	acceleration of the vehicle mass center, caused by ΔT
x_n	displacement of the station n to which the pumps belong
$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	Kronecker's symbol
ϵ_k, ϵ	dissipation coefficients
ζ	damping ratio

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
η	dimensionless constant defined by equations (16c)
$\lambda_j = \mu^{(j)} + i\gamma^{(j)}$	roots of the transcendental equation (17b)
γ	frequency of the steady state oscillation
σ	real variable
$\sigma_I ; \sigma_m ; \sigma_{II}$	defined by equations (32) and (33b)
$\sigma_j ; j = 1, 2, 3 \ldots$	infinite many roots of equation (28)
τ	time lag
φ	angle defined by equations (22) and (23)
$\left.\begin{array}{l}\omega_j ; j = 1, 2, \ldots n-1 \\ \omega_1 = \omega\end{array}\right\}$	natural (circular) frequencies
ω_s	defined by equations (16c) and (16d)

TECHNICAL NOTE D–

LONGITUDINAL OSCILLATION OF LAUNCH VEHICLES

INTRODUCTION

Any rocket (missile or launch vehicle) represents an elastic structure. During flight, any disturbance caused by wind gusts or irregularities of the thrust may cause longitudinal oscillations of the structure. In general oscillations of this kind are of a transient nature and are damped out after a certain time. The vibrating system is said to be stable.

It is well known, however, that under certain circumstances the system proves to be unstable; that is, vibration energy can be tapped from the thrust by the activity of the system itself and sustain the structural oscillations. The amplitudes increase until some nonlinear effects limit any further increase. In other words, self-excited oscillations, so called "POGO" oscillations named for the pogo-stick-like motion of the vehicle, may result.

The first time, POGO oscillations occurred during test flights of the Titan-II missile [1]. Later, the first and second stages of the Saturn V Launch vehicle were plagued by POGO oscillations [2].

The supply of energy is caused by an interaction of the structure and the propellant system. Oscillation of the structure, especially of the pumps, causes pressure oscillation of the liquid propellants. Consequently, an oscillating pump discharge and hence an oscillating thrust is generated that in return can excite the structure. In this way, the feedback loop is closed (Fig. 1). An inherent mechanism of the system may cause feedback of the thrust oscillation to sustain the structural oscillation at least for a certain time interval (Fig. 2).

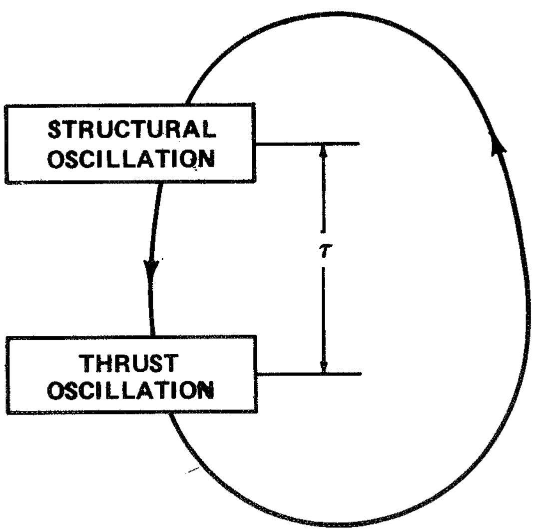


Figure 1. Feedback system with constant time lag τ.

To examine the stability of the vehicle, a proper model must be established. Then the characteristic equation must be investigated to determine whether there are roots having positive real parts that imply instability. Commonly this will be done by application of an appropriate stability criterion. In this way, the stability conditions in terms of the system parameter can be obtained.

Since the POGO problem developed, numerous stability analyses have been performed. A review of the efforts made for the Saturn V/Apollo vehicle is given in Reference 3. In References 3 and 4 a description of one of the latest models and its stability analysis is outlined. According to these reports the POGO model is based on the idea that a model must be sufficiently complex to serve as a prediction tool. It should cover all experimental and flight data. In other words, the basic idea is that the more accurately the model copies the actual system, the better it will display all the important features,

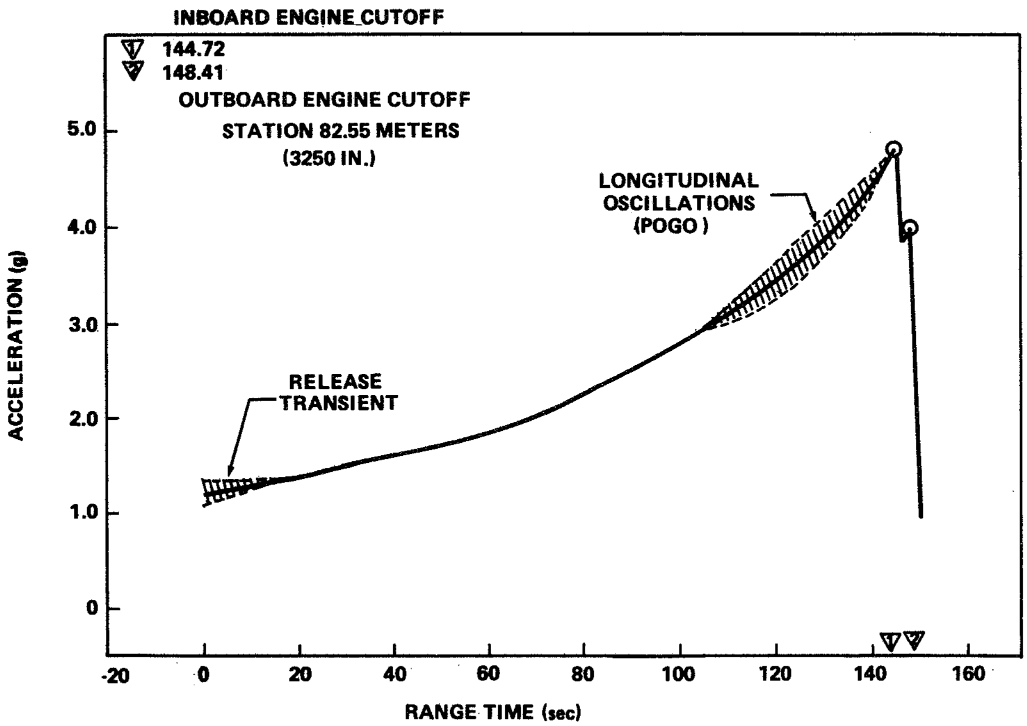


Figure 2. S-IC POGO oscillations; SA-502; Apollo 6.

especially the one of stability. Accordingly, the model is a coupled system of structural modes, propellant line modes, and inboard and outboard engine gain and phase transfer functions. The structural modes were obtained from a 280-degree-of-freedom structural model. The propellant line modes were derived also from many degree-of-freedom systems composed essentially of lumped liquid masses connected by springs. The stability analysis leads to a set of equations that forms a closed-loop system. The roots of the characteristic equations were obtained by the root locus technique.

Although opinions about the validity of such complex models may differ, one peculiarity is evident. Even if one assumes the model and analysis to be correct, one cannot consider both of them to be completely satisfying. The reason is that no understanding of the mechanism and the actual physical cause of self-excitation can be obtained in this manner. Any decision concerning the stability of the system is left to the computer.

In this note, the POGO problem shall be examined from another point of view. Two special features of the system prove to be crucial for creation of instability.

1. There is an inherent time lag of the thrust oscillations behind the structural oscillations (Fig. 1). The time lag arises from the (constant) time interval required to convert the pump inlet pressure oscillations into thrust oscillations.

2. The decrease of mass caused by the exhausting of gas.[1]

1. During the burn phase a vehicle loses approximately 70 percent of its mass.

Systems with time lag or systems possessing retarded action, as they are sometimes referred to, are known and have been examined previously [5-8]. The point is that systems with time lag are capable of self-excited oscillations or instability. In general, retarded action is present in all closed-loop control systems [5,8]. However, an example of a mechanical system was given by Minorsky in 1942 [5]. The equations of motion of problems possessing retarded action are amenable to so-called difference-differential equations [5,7]. This notation stems from the fact that these equations contain at least one term, called a retarded term, which depends not on the time t itself but on the difference $t-\tau$, where τ represents the time lag. In its linear form, having one degree-of-freedom only, the problem is already of transcendental character. It always leads to an infinite spectrum of frequencies with which the system can oscillate. In practice however, only a few of them (most frequently only one) are actually responsible for self-excitation [7].

To illustrate the importance of the time lag and difference-differential equation concepts for systems possessing retarded actions, Minorsky may be quoted. In essence he pointed out [5]:

> "In fact in all closed-loop systems there always exists a time lag in the cyclic process of operation. A mathematical description of such a process in terms of a difference-differential equation simplifies the commonly established procedure of 'couplings' between the different parts of the system which generally leads to an ordinary differential equation of a relatively high order; this usually involves considerable difficulties if one attempts to discuss the characteristic equation algebraically. <u>Furthermore the nature of coupling very frequently remains unknown. On the other hand, in establishing a difference-differential equation of</u> a retarded process one must know only the over-all time lag τ, <u>without the necessity of knowing the intermediate links in the system.</u>"

The conclusions to be drawn for the problem at hand are obvious. The combined work of the single parts of the propellant system must not be considered. Only the system's total effect on the vibrating structure is of significance. This effect is determined by two quantities, however: the time lag and the magnitude of the feedback. As far as the structure is concerned, a number of structural modes may serve for its representation.

The model derived in this note follows this line. However, the model is intended to serve first for understanding the basic mechanism of self-excitation; thus, further simplifications have been made. Accordingly, nonessential system details that may efface the actual cause of instability have been omitted. In doing so, the model has been reduced to one structural mode. This is in accordance with experience because all flight measurements show one POGO frequency only. Another simplification concerns the oscillating thrust component. Although the entire propellant system occurs in its creation, only the oscillation of the pumps is considered. The importance of the latter is shown in Reference 9. By linear approximation the respective amplitudes are assumed to be proportional. The thrust oscillation, however, is delayed by the time interval τ.

Accordingly, the simple model proposed is a single mass-spring model possessing retarded action. It has been obtained by a one-term Galerkin approach to the equation of motion. The stability is examined with the aid of Nyquist's criterion [6,8]. Special emphasis has been placed on showing the effect of the mass decrease. The possible importance of the latter was quoted in Reference 10. Furthermore, it is shown that a significant conclusion can be drawn from this simple model.

At this point, an important fact must be mentioned. The actual vibrating system is a nonlinear one whereby its nonlinearity represents an essential feature. The model derived must obviously be a linear one which covers a small vibration of the system only. As is well known, the linear theory is insufficient to describe the actual motion of the system. Especially, limit cycles cannot be determined in this way. By using linear analysis, the conditions only can be obtained under which the system is stable or unstable.

EQUATION OF MOTION

During powered flight the vehicle mass center is moving along the flight path. If an oscillating thrust component ΔT is acting on the vehicle, it causes vibrational motion of the vehicle mass center according to Newton's first law. Consequently, inertia forces of the vehicle masses are created that equilibrate the oscillating thrust component ΔT. This equilibrium system of forces, however, excites free longitudinal vibration of the elastic vehicle structure that superimposes the rigid body motion. If the vehicle structure is reduced to a lumped mass-spring system, the equation of motion of the vibrating masses relative to the mass center is

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \Delta T \mathbf{e}_n - \ddot{x}_c(t)\, \mathbf{M}\mathbf{X}_0 \tag{1}$$

where, according to Newton's first law,

$$\Delta T = M_0 \ddot{x}_0 \quad \text{and} \quad M_0 = \sum_{j=1}^{n} m_j \tag{2}$$

Substitution of equation (2) into equation (1) results in

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \Delta T \left(\mathbf{e}_n - \frac{1}{M_0} \mathbf{M}\mathbf{X}_0 \right) \tag{3}$$

Equation (3) shall be solved by a Galerkin approach [11]. Restriction to one term yields the desired one-degree-of-freedom model mentioned previously.

The eigensolutions of the above problem,

$$\omega_0 = 0 \; ; \; \mathbf{X}_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{(rigid body motion)}$$

and

$$\omega_j \neq 0 \; ; \; \mathbf{X}_j = \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \\ \vdots \\ x_n^{(j)} \end{bmatrix} \quad ; \quad j = 1,2,\ldots n\text{-}1 \tag{4}$$

are the solutions of the matrix eigenvalue problem

$$(\mathbf{K} - \omega^2 \mathbf{M})\, \mathbf{x} = \mathbf{0} \quad . \tag{5}$$

The eigenvectors satisfy the orthogonality conditions:

$$\mathbf{X}_i{}' \mathbf{M} \mathbf{X}_j = \sum_{k=1}^{n} m_k \, x_k^{(i)} \, x_k^{(j)} = \delta_{ij} M_j \; ; \quad i,j = 0,1,2,\ldots n\text{-}1 \quad , \tag{6}$$

where M_0 has been defined by equation (2). From equations (5) and (6) it follows that

$$\mathbf{X}_i{}' \mathbf{K} \mathbf{X}_j = \omega_i^2 \, \delta_{ij} M_j = \delta_{ij} K_j \quad , \qquad i,j = 0,1,2,\ldots n\text{-}1 \tag{7a}$$

hence,

$$\omega_j^2 = \frac{K_j}{M_j} \quad ; \qquad j = 0,1,2,\ldots n\text{-}1 \tag{7b}$$

To perform the Galerkin approach, it is assumed that

$$\mathbf{x}(t) = \sum_{j=1}^{n-1} a_j(t)\, \mathbf{X}_j \tag{8}$$

where a_j (j=1,...n-1) has the dimension of a length. Substitution of equation (8) into equation (3) and left multiplication by the eigenvector $\mathbf{X}_k{}'$ of the eigenvectors (4) yields

$$M_k \ddot{a}_k + \sum_{j=1}^{n-1} \dot{a}_j \, \mathbf{X}_k{}' \mathbf{D} \mathbf{X}_j + K_k a_k = \Delta T \, x_n^{(k)} \quad ; \qquad k = 1,2,\ldots n\text{-}1 \quad ; \tag{9}$$

thereby, equations (6) and (7a), are considered.

In Reference 12, it is explained that no considerable error results if one assumes

$$\mathbf{X}_k' \, \mathbf{D}\mathbf{X}_j = 2\epsilon_k M_k \delta_{kj} \quad ; \qquad k,j = 1,2,...n\text{-}1$$

Thus, it follows from equation (7b) and (9)

$$\ddot{a}_k(t) + 2\epsilon_k \dot{a}_k(t) + \omega_k^2 a_k(t) = \frac{x_n^{(k)}}{M_k} \Delta T \quad ; \qquad k = 1,2,...n\text{-}1 \tag{10}$$

Now, ΔT must be expressed. As mentioned previously, it is assumed that the oscillating thrust is caused primarily by the pump oscillations. In addition it is assumed that the magnitudes of both oscillations are linear-dependent. Because ΔT represents the retarded quantity that lags the time interval τ behind the structural oscillations, one may write

$$\Delta T = -s\ddot{x}_n(t-\tau) \tag{11}$$

where x_n is the coordinate of the mass at station n to which the pumps belong and the factor of proportionality s has the dimension of a mass.

The acceleration of the pumps can be composed of that of the mass center and relative acceleration. The latter is given by equation (8). Thus,

$$\ddot{x}_n = \ddot{x}_c + \sum_{j=1}^{n-1} \ddot{a}_j x_n^{(j)} \tag{12}$$

However, the magnitude of the oscillating thrust ΔT is not large enough to accelerate the vehicle mass center considerably. In other words, $\ddot{x}_c$ is a small quantity compared with the relative acceleration and can therefore be neglected.

Then, from equations (11) and (12), it follows that

$$\Delta T = -s \sum_{j=1}^{n-1} \ddot{a}_j(t-\tau) x_n^{(j)} \tag{13}$$

Substitution of equation (13) into equations (10) yields the following system of equations:

$$\ddot{a}_k(t) + 2\epsilon_k \dot{a}_k(t) + \omega_k^2 a_k(t) = -\frac{s}{M_k} x_n^{(k)} \sum_{j=1}^{n-1} \ddot{a}_j(t-\tau) x_n^{(j)} \quad ; \ k = 1,2 \ldots n\text{-}1$$

Because a one-term approach is considered, the system reduces to

$$\ddot{a}_1(t) + 2\epsilon_1 \dot{a}_1(t) + \omega_1{}^2 a_1(t) = -\frac{s}{M_1} x_n^{(1)^2} \ddot{a}_1(t-\tau) \quad . \tag{14}$$

According to equations (6) and (7a)

$$\frac{M_1}{x_n^{(1)^2}} = \sum_{j=1}^{n} m_j \left(\frac{x_j^{(1)}}{x_n^{(1)}}\right)^2 = m_1{}^* \quad ; \quad \frac{K_1}{x_n^{(1)^2}} = k_1{}^* \tag{15}$$

where $m_1{}^*$, $k_1{}^*$ represent generalized mass and spring constant of the first mode normalized to the unit at station n. One realizes that the smaller $x_n^{(1)}$ is relative to the other coordinates of the first mode, the larger is the generalizes mass m_1^* :

If the subscript 1 is omitted, one obtains from equations (14) and (15)

$$\ddot{a}(t) + 2\zeta\omega\dot{a}(t) + \omega^2 a(t) = -\eta\ddot{a}(t-\tau) \tag{16a}$$

thereby are

$$\zeta = \frac{d}{d_{crit}} = \frac{\epsilon_1}{\omega} \tag{16b}$$

$$\eta = \frac{s}{m^*} = \frac{\omega^2}{\omega_s{}^2} \tag{16c}$$

where

$$\omega_s{}^2 = \frac{k_1{}^*}{s} \tag{16d}$$

In the following, the stability of the simple system represented by equations (16) will be examined.

APPLICATION OF NYQUISTS STABILITY CRITERION

By the assumption

$$a(t) = ae^{\lambda t} \tag{17a}$$

it follows from equation (16a) that

$$\lambda^2 + 2\zeta\omega\lambda + \omega^2 + \lambda^2 \eta e^{-\lambda\tau} = 0 \tag{17b}$$

This is a transcendental equation having an infinite number of roots that, in general, are complex:

$$\lambda_j = \mu^{(j)} \pm i\gamma^{(j)} \quad ; \qquad j = 1,2... \tag{17c}$$

If interpreted in the complex λ-plane, one may state: The vibration is stable [6, 8] if and only if no λ_j lies in the right half-plane. In other words, the vibration is stable if

$$\mu^{(j)} < 0 \quad ; \qquad j = 1,2...$$

Because the spring-mass system which follows from the system (16a) by omitting the right side

$$\ddot{a} + 2\zeta\omega\dot{a} + \omega^2 a = 0$$

can be assumed to be a vibrating system with positive (not overcritical) damping, its characteristic equation

$$\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0 \tag{18}$$

does not have roots in the right half-plane. Consequently, one may divide equation (17b) by the left side of the equation (18) without gaining any poles in the right half-plane. Thus, instead of equation (17b), the following equation may be examined:

$$1 + \eta \frac{\lambda^2 e^{-\lambda\tau}}{\lambda^2 + 2\zeta\omega\lambda + \omega^2} = 0$$

or

$$1 + f(\lambda) = 0$$

where

$$f(\lambda) = \eta \frac{\lambda^2 e^{-\lambda\tau}}{\lambda^2 + 2\zeta\omega\lambda + \omega^2} . \tag{19}$$

To study the stability of the system at hand, the Nyquist analysis must be applied [8]. The Nyquist criterion is:

> The system is stable if and only if $f(\lambda)$ does not encircle the point (-1,0) of the $f(\lambda)$-plane if λ travels along the so-called Nyquist path.

The Nyquist path is a closed contour in the λ-plane that completely encloses the entire right half of the λ-plane. In Reference 6 it has been shown that steady-state solutions of equation (16a) exist under very special conditions only. Thus, the general case,

$$\mu^{(j)} \neq 0 \quad ; \qquad j = 1,2,... \tag{20}$$

shall first be studied. In doing so, one may consider the Nyquist path to be going along the imaginary axis [6, 8]. To enclose the right half-plane, a large semicircular path is drawn in the right half-plane having the radius R that is interpreted as being infinite in the limit (Fig. 3).

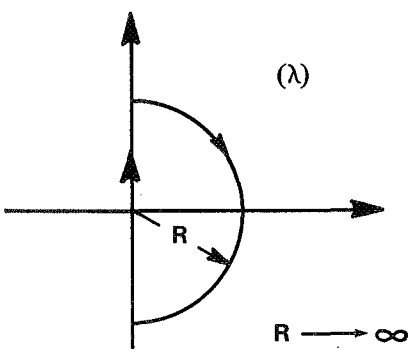


Figure 3. Nyquist path in the complex λ-plane.

Because

$$\lim_{R \to \infty} \lambda^2 e^{-\lambda\tau} = 0$$

and, hence,

$$\lim_{R \to \infty} f(\lambda) = 0$$

the traveling along the semicircle does not contribute (if $R \to \infty$) and therefore can be omitted [6].

Thus, if one substitutes

$$\lambda = i\omega\sigma \tag{21a}$$

the Nyquist path is represented by the following:

$$-\infty \leqslant \sigma \leqslant \infty$$

From equations (19) and (21a) one realizes that

$$f(i\omega\sigma) = \eta \frac{\sigma^2 e^{-i\omega\tau\sigma}}{\sigma^2 - 1 - 2i\zeta\sigma} \tag{21b}$$

Now $f(i\omega\sigma)$ shall be expressed by its radius vector $r(\sigma)$ and its argument. First, the denominator:

$$\sigma^2 - 1 - 2i\zeta\sigma = \sqrt{(\sigma^2 - 1)^2 + 4\zeta^2\sigma^2} \quad e^{-i\phi}$$

where

$$e^{-i\phi} = \cos\phi - i\sin\phi \quad ,$$

$$\sin\phi = 2\zeta \frac{\sigma}{\sqrt{(\sigma^2 - 1)^2 + 4\zeta^2\sigma^2}} \tag{22}$$

and

$$\cot\text{an}\,\phi = \frac{1}{2\zeta} \frac{\sigma^2 - 1}{\sigma} \tag{23}$$

Thus, equation (21b) can be written

$$f(i\omega\sigma) = r(\sigma)\, e^{i(\phi - \omega\tau\sigma)} \tag{24a}$$

where

$$r(\sigma) = \eta \frac{\sigma^2}{\sqrt{(\sigma^2 - 1)^2 + 4\zeta^2\sigma^2}} \tag{24b}$$

and ϕ is given by equations (22) and (23).

Because

$$f(-i\omega\sigma) = \bar{f}(i\omega\sigma)$$

whereby the conjugate complex $\bar{f}$ is symmetrical to f, the real axis being the symmetry axis, one may restrict the travel path to positive σ. Thus, the Nyquist criterion can be expressed as follows:

The system is stable if and only if $f(\lambda)$ does not encircle the point (-1,0) of the $f(\lambda)$-plane if σ travels along

$$0 < \sigma \leqslant \infty \tag{25}$$

where λ and σ are linked by equation (21a).

Now, one has to determine under what conditions $f(\lambda)$ encircles (-1,0) if σ travels along the interval (25).

Because

$$-1 = e^{\pm i(2j-1)\pi} \quad ; \quad j = 1,2 \ldots$$

it follows from equations (24) that encirclement will occur if and only if a positive σ exists so that

$$\phi - \omega\tau\sigma = \pm(2j-1)\pi \quad ; \quad j = 1,2 \ldots \tag{26}$$

where ϕ is determined by equations (22) and (23), and

$$r(\sigma) > 1 \tag{27}$$

By combining equations (22), (23), (25), and (26), it follows:

$$\operatorname{cotan} \omega\tau\sigma = \frac{1}{2\zeta} \; \frac{\sigma^2 - 1}{\sigma} \tag{28}$$

and

$$(2j-1)\pi < \omega\tau\sigma < 2j\pi \quad ; \quad j = 1,2,\ldots \tag{29}$$

Equation (28) is a transcendental equation having infinite many roots. Let

$$\sigma_j \qquad j = 1,2,3,\ldots \tag{30}$$

be those positive roots which belong to the intervals (29). Then, considering equation (24b) and the condition (27), one may state: Encirclement will occur if for at least one root σ_k of the roots (30)

$$\eta^2 > \left(1 - \frac{1}{\sigma_k^2}\right)^2 + \left(\frac{2\zeta}{\sigma_k}\right)^2 \tag{31a}$$

No encirclement can occur if for all roots (30)

$$\eta^2 < \left(1 - \frac{1}{\sigma_j^2}\right)^2 + \left(\frac{2\zeta}{\sigma_j}\right)^2 \qquad j = 1,2,3... \tag{31b}$$

Figure 4 is a graph of

$$g(\sigma) = \left(1 - \frac{1}{\sigma^2}\right)^2 + \left(\frac{2\zeta}{\sigma}\right)^2 \tag{32}$$

The function $g(\sigma)$ represents the right side of the conditions (31). It shows two stationary points at

$$\sigma_m = \sqrt{\frac{1}{1 - 2\zeta^2}} \qquad \text{and} \qquad \sigma = \infty$$

The respective values of $g(\sigma)$ are

$$g(\sigma_m) = 4\zeta^2 (1 - \zeta^2)$$

$$g(\infty) = 1$$

The first represents a minimum, and the second represents a maximum. The horizontal line through $\sigma = 1$ is the asymtote of $g(\infty)$.

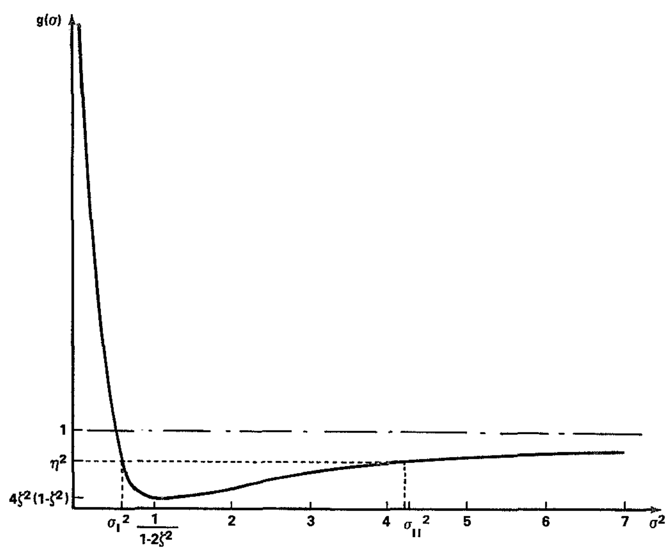


Figure 4. Graph of $g(\sigma)$.

The abscissa σ_I, σ_{II} are the positive solutions of

$$g(\sigma) = \eta^2 \tag{33a}$$

One obtains

$$\sigma_{I,II}^2 = \frac{1 - 2\zeta^2}{1 - \eta^2}\left[1 \pm \sqrt{1 - \frac{1 - \eta^2}{(1 - 2\zeta^2)^2}}\right] \tag{33b}$$

For any σ_k satisfying

$$\sigma_I < \sigma_k < \sigma_{II} \tag{34}$$

condition (31a) is valid as can be seen from Figure 3. Thus, encirclement can occur if and only if the interval (34) contains at least one of the roots (30).

Let n_o be the number of encirclements of $(-1, 0)$; then, from Figure 4 or equation (33b), one concludes

$$n_o = 0 \qquad \text{if} \qquad \eta < 2\zeta\sqrt{1 - \zeta^2} \tag{35a}$$

$$n_o = \infty \qquad \text{if} \qquad \eta \geqslant 1 \tag{35b}$$

In the latter case, namely, the length of the interval (34) becomes infinite.

The conclusions which can be drawn from Figure 4 and equation (33b) are gathered in Table 1.

Now, the special case of the steady-state oscillation shall be examined. It occurs if at least one of the real parts (20) equals zero. Then, according to equation (17c), one may assume

$$\lambda = \pm i\gamma \quad ; \quad \gamma > 0 \tag{36}$$

Now, from equations (16), (17), and (36), it follows that

$$\ddot{a} + (2\zeta\omega + \eta\gamma\sin\gamma\tau)\dot{a} + (\omega^2 - \eta\gamma^2\cos\gamma\tau)a = 0.$$

TABLE 1. σ_I AND σ_{II} AS FUNCTIONS OF η.

η	σ_I	σ_{II}
$< 2\zeta\sqrt{1-\zeta^2}$	Congugate Complex	
$2\zeta\sqrt{1-\zeta^2}$	$\sigma_I = \sigma_{II} = \sqrt{\frac{1}{1-2\zeta^2}}$	
2ζ	1	$\sqrt{\frac{1}{1-4\zeta^2}}$
$2\zeta < \eta < 1$	< 1	> 1
$\to 1$	$\lim \sigma_I = \frac{1}{\sqrt{2}}\sqrt{\frac{1}{1-2\zeta^2}}$	$\lim \sigma_{II} = \infty$

For this equation to represent the steady-state oscillation, the coefficients must reduce to

$$2\zeta\omega + \eta\gamma\sin\gamma\tau = 0$$

$$\omega^2 - \eta\gamma^2\cos\gamma\tau = \gamma^2$$

or if notation (21a) is used $(\sigma > 0)$

$$\eta\sigma\sin\omega\tau\sigma = -2\zeta \tag{37a}$$

$$\eta\sigma\cos\omega\tau\sigma = -\frac{\sigma^2 - 1}{\sigma} \tag{37b}$$

Equations (37) are equivalent to the following equations:

$$\text{cotan}\,\omega\tau\sigma = \frac{1}{2\zeta}\,\frac{\sigma^2 - 1}{\sigma}$$

$$(2j - 1)\pi < \omega\tau\sigma < 2j\pi$$

$$\eta^2 = \left(\frac{\sigma^2 - 1}{\sigma^2}\right)^2 + \left(\frac{2\zeta}{\sigma}\right)^2 \tag{38}$$

These equations, in turn, are identical to equations (28), (29), (32), and (33a). Thus, the special conditions under which steady-state oscillations may occur are characterized by coincidence of one of the roots (30) with either σ_I or σ_{II}. In that case,

$$\gamma_1 = \omega\sigma_I \quad \text{or} \quad \gamma_2 = \omega\sigma_{II}$$

are the frequencies of steady-state oscillations.

Now, combining this and previous statements, especially equations (35), Nyquist's criterion for the case at hand is:

1. The system is stable if

$$\eta < 2\zeta\sqrt{1 - \zeta^2}$$

2. If

$$\eta \geqslant 2\zeta\sqrt{1 - \zeta^2}$$

the system is stable if none of the roots (30) lie within the interval

$$\sigma_I \leqslant \sigma \leqslant \sigma_{II} \tag{39}$$

which is determined by equations (33). If at least one of the roots (30) belongs to this interval, the system is unstable.

3. If $\eta \geqslant 1$, the system is unstable.

For some practical applications, another formulation of Nyquist's criterion is appropriate. It follows from equation (22), (26) conditions of encirclement (31), and equation (38) that the system is stable or unstable accordingly as

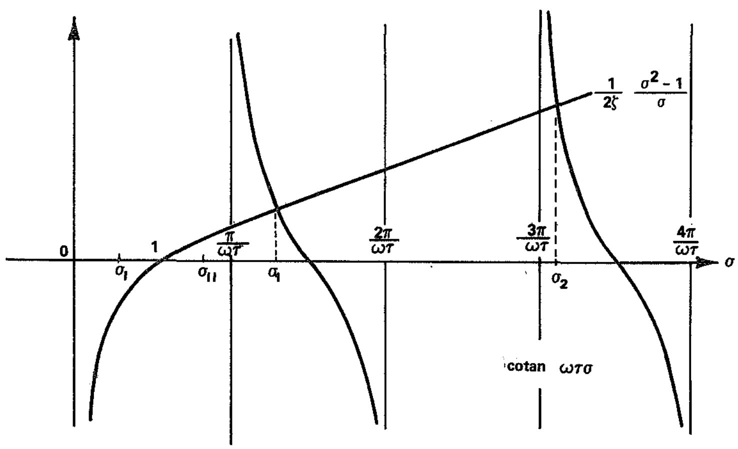
$$\eta < -\frac{2\zeta}{\sigma \sin \omega\tau\sigma}$$

for all of the roots (30) or

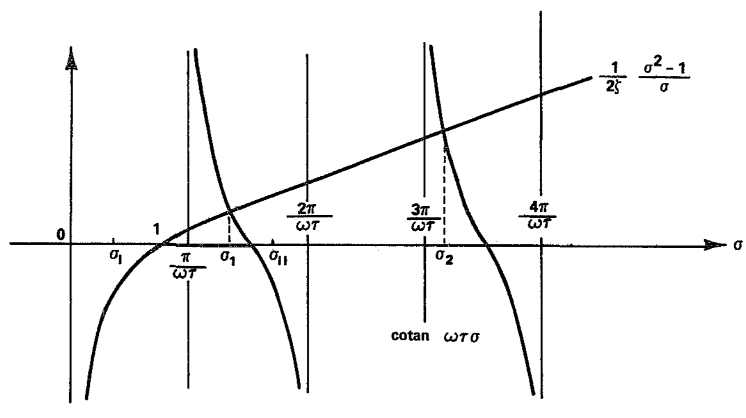
$$\eta \geqslant -\frac{2\zeta}{\sigma \sin \omega\tau\sigma}$$

for at least one of these roots.

Figure 4 is a graphic interpretation of the stability problem. For determination of the roots (30), the curves (28) are drawn in. Here, cotan $\omega\tau\sigma$ is represented by the first and the second branch of the intervals (29), j = 1, 2. Figure 4a shows stability because σ_1 lies to the right of σ_{II}. Consequently all roots (30) lie outside the interval (30). However, Figure 4b is an unstable case because σ_1 lies between σ_I and σ_{II}.



(a) STABLE CASE



(b) UNSTABLE CASE

Figure 5. Stability graphs.

STABILITY OWING TO THE DECREASE IN MASS

One realizes that the stability of the system depends, apart from the damping ratio ζ, on two dimensionless parameters, namely

$$\omega \tau \text{ and } \omega_s \tau$$

While the first one determines the roots (30), the ratio

$$\eta = \left(\frac{\omega \tau}{\omega_s \tau}\right)^2 = \frac{s}{m^*}$$

determines the boundaries σ_I and σ_{II} of the critical interval (39).

The quantities s and τ represent specific quantities of the vibrating system and can therefore be considered to be constant. Now, the decrease in mass may cause a change over from the state of stability (Fig. 4a) to one of instability (Fig. 4b) and vice versa. The reason is that a decrease in mass affects both of the parameters η and $\omega\tau$. The decrease in mass causes, in general, an increase of the natural period ω [11] [2] and a decrease of the generalized mass m* as it can be seen by a closer examination.

At the beginning of the powered flight, the liquid propellant masses are large; consequently, η is a small quantity. The increase of η may first cause σ_I and σ_{II} to become real and later on may increase the length $\sigma_{II} - \sigma_I$ of the interval (39) as shown in Figure 3 and Table 1. As result of the increase of ω , however, the roots (30) (Fig. 4) travel in the negative direction of the σ-axis. Consequently one of the roots, in general the first one, approaches the interval (39) and finally may enter it. However, this implies instability. Further decrease of mass may cause this root to leave the interval. If at that time the next one is still outside the interval, the system regains stability; otherwise, the instability persists. In this way, the state of stability may change over to one of instability and vice versa.

To demonstrate this behavior a numerical case shall be considered. Assumed is a single mass-spring system ($m^* = m$). The damping ratio of the system shall be

$$\zeta = 0.06$$

The state of decreasing mass shall be characterized by an interval of the natural frequency ω

$$4.6 \leqslant \omega\tau \leqslant 5.9 \tag{40}$$

---

2. In exceptional cases, ω remains constant as the mass decreases.

The Nyquist criterion is applied to some successive (discrete) values of $\omega\tau$ from this interval. Five cases determined by different values of $\omega_s\tau$ are investigated. All values of $\omega\tau$ and $\omega_s\tau$ are gathered in Table 2.

TABLE 2. STABILITY STATE OF A ONE-DOF SYSTEM DEPENDENT ON NATURAL FREQUENCY AND FEEDBACK MAGNITUDE.

	$\omega_s\tau$				
	1	2	3	4	5
$\omega\tau$	14.40	14.33	14.25	14.00	13.40
4.60	S	S	S	S	S
5.00	S	S	S	U	U
5.17	S	S	U	U	U
5.30	S	U	U	U	U
5.39	S	S	U	U	U
5.60	S	S	S	U	U
5.90	S	S	S	S	U

Note: S = stable; U = unstable.

According to the stability criterion, if

$$\omega_s\tau > \frac{\omega\tau}{\sqrt{2\zeta\sqrt{1-\zeta^2}}}$$

the system is stable. Thus, for the above ζ and the upper limit of the interval (40) ($\omega\tau = 5.9$), it follows that for

$$\omega_s\tau > 17.34$$

the system is stable for the entire interval (40). Because the $\omega_s\tau$ values of Table 2 are smaller than the above value, special consideration must be given to the position of the roots (30) relative to the respective intervals (39) according to the stability criterion. The result of this investigation is shown by Table 2.

Cases 2 through 5 of Table 2 show transition from stability to instability and, with exception of the last one, return to stability.

The Nyquist criterion is not an algebraic criterion. It is based on fundamentals of the theory of the complex variables. For that reason, it lacks some perspicuity. It therefore seems to be convenient to perform also a direct integration of equation (16a). This has been done for the cases 3 and 4 of Table 2.

The integration is performed by the method of Runge-Kutta (Table 3 and 4) and also by analog computer (Figs. 6 and 7). It is assumed that

$$\tau = 1.$$

The initial conditions are

$$a(o) = .2 \quad ; \quad \dot{a}(o) = 0.$$

As far as the Runge-Kutta analysis is concerned, the amplitude maxima only are printed (Table 3 and 4).

TABLE 3. CASE 3 (ω_s = 14.25) MAXIMA OF a(t).

ω															
5.0	0.200	0.151	0.152	0.153	0.151	0.151	0.150	0.150	0.149	0.149	0.148	0.148	0.147	0.147	0.145
5.3	0.200	0.148	0.153	0.153	0.154	0.155	0.155	0.156	0.157	0.157	0.157	0.158	0.158	0.159	0.159
5.6	0.200	0.144	0.150	0.149	0.148	0.147	0.146	0.146	0.145	0.145	0.143	0.143	0.142	0.142	0.141

Owing to the transient condition at the beginning, the values of a(t) behave nonuniform, but, later, the tendency to stability or instability, respectively, can easily be observed.

TABLE 4. CASE 4 (ω_s = 14.00) MAXIMA OF a(t).

ω															
4.6	0.200	0.152	0.150	0.145	0.141	0.135	0.131	0.128	0.125	0.121	0.117	0.113	0.110	0.107	0.103
5.3	0.200	0.149	0.156	0.157	0.160	0.162	0.165	0.167	0.168	0.171	0.174	0.176	0.180	0.182	0.183
5.9	0.200	0.140	0.146	0.141	0.140	0.137	0.134	0.132	0.129	0.128	0.125	0.124	0.121	0.118	0.117

Once again, one realizes the transition from stability to instability and the return to stability, caused by the decrease of mass. However, these results are obtained by linear analysis which, as mentioned previously, is insufficient to describe the actual motion. As in the case of Nyquist's criterion, the conditions only are displayed under which the system is stable or unstable.

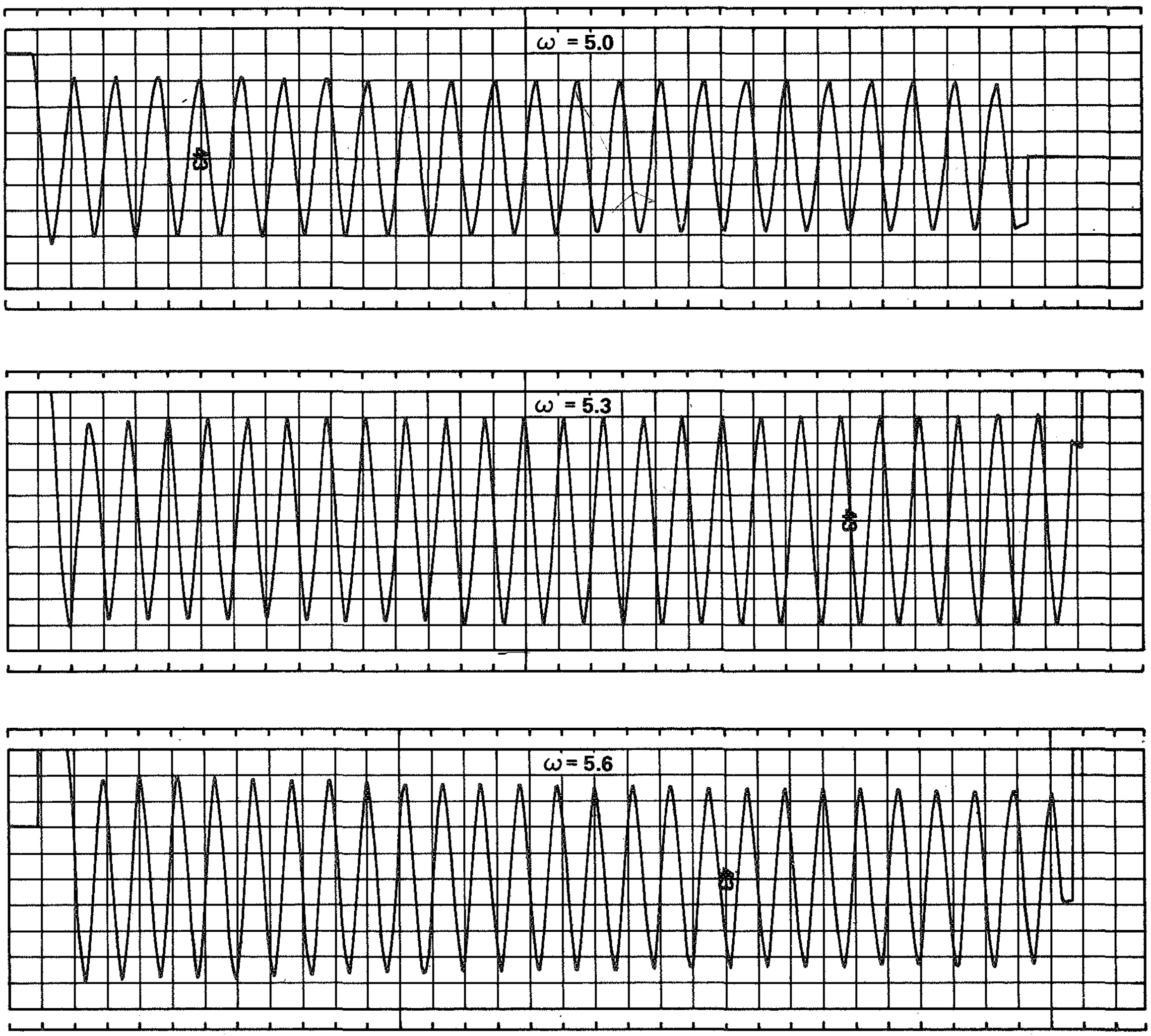


Figure 6. Analog computer run of case 3 (ω_s = 14.25).

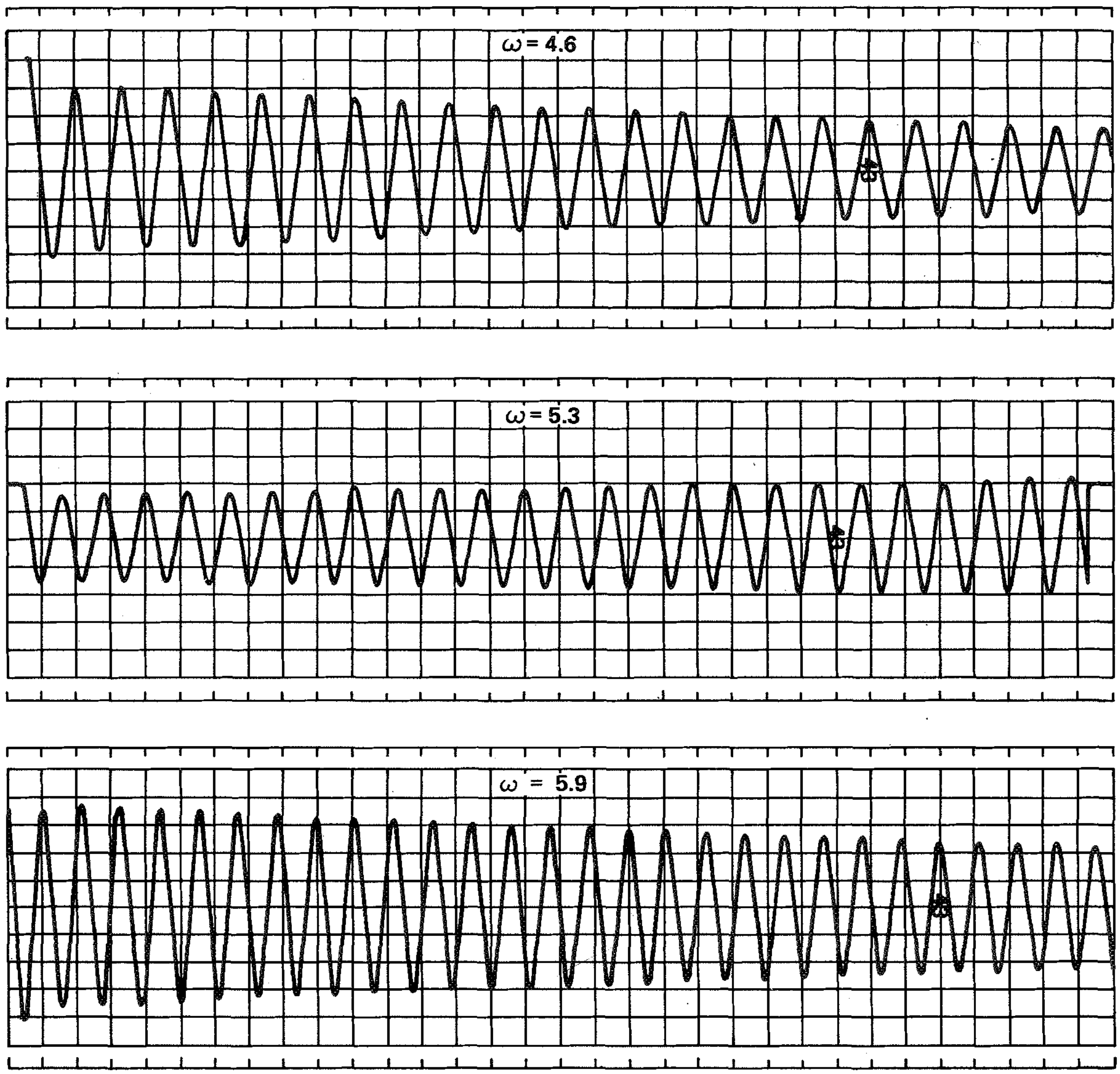


Figure 7. Analog computer run of case 4 (ω_s = 14.00).

SUMMARY AND CONCLUSION

Because of its elastic structure, a launch vehicle is capable of longitudinal oscillation. In this regard, three basic features prove to be essential:

1. The elastic vehicle structure and the propellant system form a feedback system.

2. A time delay exists in the feedback loop.

3. The loss of mass because of the depletion of liquid propellant.

The intent of this report is to show that the presence of these features impart the tendency to self-excitation.

It is well known that feedback systems with time lag are sensitive to self-excitation or instability. However, the loss of mass changes the systems mass and mass distribution continuously. Consequently, the entire system changes continuously and also may change its stability behavior. At one time point, the system may be stable, and at another time point, it may be unstable.

From this point of view, the changing of the stability state of a vehicle during powered flight must be considered a certainly peculiar but absolutely understandable feature of the vehicle. It may be difficult or impossible to design a vehicle which proves to be stable during the entire powered flight time. Above all, it seems to be very difficult to predict the stability behavior of a vehicle. The only remaining alternative is to keep the effect of self-excitation unsignificant. Preventive measures exist.

First, the quantity η should remain small; consequently, the effect of feedback and also the interval of instability (39) remain small. This can be achieved by large m^*. According to the remark following equation (15), the coordinate x_n must be kept as small as possible relative to the other coordinates of the first mode. This implies to support the engines as stiff as possible. The connection between soft engine support and self-excitation was demonstrated in the past. First, the connection was demonstrated during test flights of the Titan-II missile, whereby POGO oscillations first occurred. The spring support of the engine-pump masses of this missile is striking [1]. Later, POGO oscillation of the S-II stage during some of the Apollo flights were observed. Especially, severe oscillations occurred during the flight of Apollo 13 [2]. Flight data obtained at that time show extremely large relative displacements of the center engine. The soft support of the center engine by a pin-ended cross beam led to this situation.

The use of gas-filled accumulators [1,3] in some of the suction lines represents another preventive measure. An accumulator is a device which is able to absorb pressure fluctuations in the suction system before they could propagate through the pumps and produce significant thrust oscillations. In the case of the Titan-II missile and also of the S-IC stage of the Saturn V launch vehicle, these "POGO fixes" proved to be successful [1, 3].

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