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# Linear and Nonlinear Stability Characteristics of Whistlers

by

Armando Larcher Brinca

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This work treats theoretically certain linear and nonlinear propagation characteristics of the right-hand polarized, slow electromagnetic, magnetoplasma waves known as "whistlers", and particularizes the results to conditions prevailing in the equatorial magnetosphere, i.e. to an effectively cold magnetoplasma permeated by a dilute nonthermal electron population with variable energy and pitch angle.

The linear studies treat whistlers with wavenormals parallel and oblique to the static magnetic field. For parallel propagation in a hot plasma with an arbitrary equilibrium velocity distribution, a stability criterion differentiating between absolute and convective instabilities is established by studying the asymptotic response of the system to impulse excitation. A discussion of true and pseudo-modes defined by the dispersion relation is given for the special case of resonance velocity distributions. The analysis of the linear growth rates of oblique whistlers with arbitrary frequency uses an energy conservation approach. The importance of the wave-particle interactions occurring for oblique propagation is assessed, and it is found that the growth rates do not maximize, in general, for whistlers propagating parallel to the static magnetic field.

The nonlinear studies consist of theories of artificially stimulated emissions, and of the whistler modulational instability in cold

and hot plasmas. By analyzing the distortion caused by a large amplitude triggering whistler on the nonthermal population of a magnetospheric plasma, it is found that test waves impressed on the perturbed plasma, and lying within two narrow bands centered on the frequency of the original wave, may experience strong growths at the early stages of the nonlinear wave-particle interaction. Identification of this creation of whistler sidebands with the commencement of the emission provides a mechanism for the stimulated emission onset. The main phase of this emission is then discussed by following the evolution in the geomagnetic mirror of the energetic electrons in Landau or cyclotron resonance with the triggering pulse. The characteristics of the whistler wavelets radiated by these resonant electrons may explain most of the spectral shapes observed in stimulated emissions. Finally, the modulational instability spectrum of whistlers in cold plasmas is determined, taking into consideration both ion motion and relativistic effects, and the modulational characteristics of whistlers in hot plasmas are derived.

# CONTENTS

|                                                       | <u>Page</u> |
|-------------------------------------------------------|-------------|
| ABSTRACT . . . . .                                    | iii         |
| TABLES . . . . .                                      | vii         |
| ILLUSTRATIONS . . . . .                               | viii        |
| NOTATION . . . . .                                    | xiii        |
| ACKNOWLEDGEMENTS . . . . .                            | xxiv        |
| 1. INTRODUCTION . . . . .                             | 1           |
| 2. LINEAR DISPERSION AND STABILITY . . . . .          | 2           |
| 2.1 Introduction . . . . .                            | 2           |
| 2.2 Dispersion Relation . . . . .                     | 2           |
| 2.2.1 Derivation . . . . .                            | 2           |
| 2.2.2 Discussion . . . . .                            | 7           |
| 2.2.3 True and Pseudo-Modes . . . . .                 | 10          |
| 2.3 Stability Criteria . . . . .                      | 15          |
| 2.3.1 Background . . . . .                            | 15          |
| 2.3.2 Instability in General . . . . .                | 18          |
| 2.3.3 Absolute and Convective Instabilities . . . . . | 22          |
| 3. OBLIQUELY PROPAGATING WHISTLERS . . . . .          | 27          |
| 3.1 Background . . . . .                              | 27          |
| 3.2 Growth Rate: Theory . . . . .                     | 29          |
| 3.3 Plasma Properties . . . . .                       | 37          |
| 3.4 Dissipation . . . . .                             | 42          |
| 3.5 Growth Rate: Results . . . . .                    | 49          |
| 3.6 Discussion . . . . .                              | 56          |
| 4. ARTIFICIALLY STIMULATED EMISSIONS . . . . .        | 57          |
| 4.1 Background . . . . .                              | 57          |
| 4.2 Characteristics . . . . .                         | 62          |
| 4.3 Onset: Whistler Sidebands . . . . .               | 64          |
| 4.3.1 Introduction . . . . .                          | 64          |
| 4.3.2 Distorted Distribution . . . . .                | 67          |
| 4.3.3 Sideband Stability . . . . .                    | 77          |
| 4.3.4 Discussion . . . . .                            | 87          |

## CONTENTS (Cont.)

|                                                                  | <u>Page</u> |
|------------------------------------------------------------------|-------------|
| 4.4 Main Phase . . . . .                                         | 95          |
| 4.4.1 Background . . . . .                                       | 95          |
| 4.4.2 Model of the Equatorial Magnetosphere . .                  | 97          |
| 4.4.3 Spectral Shapes . . . . .                                  | 99          |
| 4.4.4 Propagation . . . . .                                      | 108         |
| 4.5 Discussion . . . . .                                         | 112         |
| 5. MODULATIONAL INSTABILITY . . . . .                            | 114         |
| 5.1 Background . . . . .                                         | 114         |
| 5.2 Theory . . . . .                                             | 116         |
| 5.3 Cold Plasma Case . . . . .                                   | 119         |
| 5.4 Hot Plasma Case . . . . .                                    | 124         |
| 5.5 Discussion . . . . .                                         | 129         |
| 6. CONCLUSIONS . . . . .                                         | 134         |
| APPENDIX A. THE CAUCHY (HILBERT TRANSFORM)<br>INTEGRAL . . . . . | 136         |
| APPENDIX B. SOME PROPERTIES OF $D^{P,N}(\omega, k)$ . . . . .    | 142         |
| APPENDIX C. ASYMPTOTIC DISTRIBUTION AND<br>QUADRATURE . . . . .  | 143         |
| REFERENCES . . . . .                                             | 145         |

## TABLES

| <u>Number</u> |                                                               | <u>Page</u> |
|---------------|---------------------------------------------------------------|-------------|
| 3.1           | Signs of dissipation coefficients and<br>velocities . . . . . | 44          |
| 3.2           | Types of contributions for overall stability . .              | 46          |

# ILLUSTRATIONS

| <u>Figure</u> |                                                                                                                                                                                                                                                                                                                                                                                                             | <u>Page</u> |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| 2.1           | One possible set of branch cuts in the $\omega$ and $k$ planes. The fixed points of the branch cuts are denoted by $\times$ . . . . .                                                                                                                                                                                                                                                                       | 8           |
| 2.2           | Position of the branch cuts for $\omega_r < \Omega$ . . . . .                                                                                                                                                                                                                                                                                                                                               | 11          |
| 2.3           | Position of the branch cuts for $\omega_r > \Omega$ . . . . .                                                                                                                                                                                                                                                                                                                                               | 12          |
| 2.4           | Limiting positions of the modes $k(\omega = \omega_r - i\infty)$ . . . . .                                                                                                                                                                                                                                                                                                                                  | 14          |
| 2.5           | Evolution of the $K(W)$ modes as $W_i \rightarrow -\infty$ .<br>(a) $W_i = 0$ . (b) $W_i = -5 \times 10^{-3}$ .<br>(c) $W_i = -5 \times 10^{-2}$ . . . . .                                                                                                                                                                                                                                                  | 16          |
| 2.6           | Evolution of the $K(W)$ modes as $W_i \rightarrow -\infty$ .<br>(a) $W_i = -0.1$ . (b) $W_i = -1$ . . . . .                                                                                                                                                                                                                                                                                                 | 17          |
| 2.7           | Loci of $k_n(\omega')$ as $\omega' \rightarrow \omega'_r$ . . . . .                                                                                                                                                                                                                                                                                                                                         | 21          |
| 2.8           | Coalescence of colliding poles as $\omega' \rightarrow \bar{\omega}$ . . . . .                                                                                                                                                                                                                                                                                                                              | 25          |
| 3.1           | Oblique whistler propagation in a cold plasma<br>[ $W_p = \omega_p/\Omega = 10$ ]. (a) Polar diagrams of refractive index (LHR stands for 'lower hybrid resonance'). (b) Polarization ratios.<br>(c) Resonant parallel velocities for the Landau and fundamental gyro interactions<br>[ $V_L = \omega/k_{  }, V_G = (\omega - \Omega)/k_{  }$ ]. (d) Refractive index, phase and group velocities . . . . . | 38          |
| 3.2           | Dissipation coefficients.<br>[ $ m  \leq 3, W_p = 10, W = 0.5$ ] . . . . .                                                                                                                                                                                                                                                                                                                                  | 43          |
| 3.3           | Velocity gradients of the hot plasma distribution [ $\times(10c)^4$ ]. Note that $F(v_{\perp}, v_{  }) = F(v_{\perp}, -v_{  })$ , $F_{\perp}(v_{\perp}, v_{  }) = F_{\perp}(v_{\perp}, -v_{  })$ .<br>[ $q = 4.0, p = 0.5$ ] . . . . .                                                                                                                                                                      | 45          |

# ILLUSTRATIONS (Cont.)

| <u>Figure</u> |                                                                                                                                                                                                                                                                                                                                                                              | <u>Page</u> |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| 3.4           | Dependence of the dissipation coefficients for the Landau and fundamental gyro interactions on whistler frequency and cold plasma density. $[\theta = 45^\circ]$ . . . . .                                                                                                                                                                                                   | 47          |
| 3.5           | Contributions of the Landau ( $\chi^L$ ), fundamental gyro ( $\chi^G$ ), and other ( $\chi^0$ , $m \neq 0, -1$ ) interactions to the oblique whistler imaginary refractive index ( $= -\chi$ ). $[W_p = 10, q = 4.0, p = 0.5]$ . (a) $U = 300$ ev. (b) $U = 10$ kev . . .                                                                                                    | 50          |
| 3.6           | Dependence of the oblique whistler imaginary refractive index ( $= -\chi$ ) on the hot plasma parameters. $[W_p = 10, W = 0.5]$ . . . . .                                                                                                                                                                                                                                    | 53          |
| 3.7           | Dependence of the limiting condition $\chi''(0) = 0$ on the hot plasma parameters. A maximum (minimum) of the whistler growth occurs at parallel propagation when $\chi''(0) > 0$ ( $< 0$ ). $[W_p = 10, q = 4.0]$ . . . . .                                                                                                                                                 | 55          |
| 4.1           | Typical ASE spectrogram . . . . .                                                                                                                                                                                                                                                                                                                                            | 58          |
| 4.2           | Phase trajectories of untrapped ( $\kappa^2 < 1$ ) and trapped ( $\kappa^2 > 1$ ) electrons in the cyclotron resonance region. (Separatrix: $\kappa^2 = 1$ ) . . . . .                                                                                                                                                                                                       | 70          |
| 4.3           | Temporal evolution of $\langle w_0 \rangle$ . The distorted, averaged, electron velocity distribution in the resonance region is $\langle f \rangle \approx F_0(V_{G0}, v_\perp) + \langle w_0 \rangle F_\parallel(V_{G0}, v_\perp)$ . Note that both $\tau (= \omega_{NL} t)$ and $w^* (= 4\omega_{NL} / \pi k_0)$ depend on $v_\perp (\omega_{NL}^2 = ak_0 v_\perp)$ . . . | 76          |

# ILLUSTRATIONS (Cont.)

| <u>Figure</u> |                                                                                                                                                                                                                                                                                                                             | <u>Page</u> |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| 4.4           | Influence of the $v_{\perp}$ -monochromaticity on the linear hot plasma characteristics. (a) Root mean square perpendicular speed and $I_2/I_1$ . (b) Whistler growth rates . . . . .                                                                                                                                       | 86          |
| 4.5           | Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Lambda_M$ ), and linear attenuation ( $\Lambda_L$ ) in the absence of the large amplitude whistler ( $\beta$ is the fractional number density of the hot plasma). [ $\bar{v}_{\perp}/v_c = 8$ ] . . . . . | 88          |
| 4.6           | Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Lambda_M$ ), and linear attenuation ( $\Lambda_L$ ) in the absence of the large amplitude whistler ( $\beta$ is the fractional number density of the hot plasma). [ $\bar{v}_{\perp}/v_c = 4$ ] . . . . . | 89          |
| 4.7           | Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Lambda_M$ ), and linear attenuation ( $\Lambda_L$ ) in the absence of the large amplitude whistler ( $\beta$ is the fractional number density of the hot plasma). [ $\bar{v}_{\perp}/v_c = 2$ ] . . . . . | 90          |
| 4.8           | Geomagnetic dipolar field line coordinates . . .                                                                                                                                                                                                                                                                            | 98          |
| 4.9           | Electron motion in the geomagnetic mirror . . .                                                                                                                                                                                                                                                                             | 101         |
| 4.10          | Allowed frequency and obliquity emission bands for the gyro slab as a function of its position and parallel velocity. Numbers 1 and 2 identify possible emissions. [ $L = 3$ , $\Omega_0 = \Omega(z = 0)$ , $v_p(\theta = 0) = 0.05c$ at $z = 0$ for $\omega/\Omega_0 = 0.5$ ] . . . . .                                    | 102         |

# ILLUSTRATIONS (Cont.)

| <u>Figure</u> |                                                                                                                                                                                                                                                                                        | <u>Page</u> |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| 4.11          | Allowed frequency and obliquity emission bands for the Landau slab as a function of its position and parallel velocity. Numbers 1-6 identify possible emissions. [ $L = 3$ , $\Omega_0 = \Omega(z = 0)$ , $v_p(\theta = 0) = 0.05c$ at $z = 0$ for $\omega/\Omega_0 = 0.5$ ] . . . . . | 103         |
| 4.12          | Ducted hook radiated by gyro slab. [ $L = 3$ , $z_0 = 3,000$ km, $ v_G /\bar{v}_1 = 0.5$ at $z = 0$ ] . . . . .                                                                                                                                                                        | 105         |
| 4.13          | Path and obliquity of parallel whistlers launched in the equatorial plane at different altitudes . . . . .                                                                                                                                                                             | 109         |
| 4.14          | Whistler propagation time between $\lambda_i = 87^\circ$ , $L = 3$ and $\lambda_a = 116^\circ$ , $L_a$ as a function of initial obliquity $\theta_i$ . [Ducted propagation takes $t_{\min}$ seconds] . . . . .                                                                         | 111         |
| 5.1           | Possible evolution of a wave packet with group velocity $v_g$ in a modulationally unstable medium . . . . .                                                                                                                                                                            | 115         |
| 5.2           | Whistler modulational stability spectrum in cold plasmas (stable bands are hatched). (REL - Relativistic force equation used; ION - Ion motion considered.) [ $\omega \gg \Omega_i$ , $\omega_{pe}^2 \gg \Omega_e^2$ ] . . . . .                                                       | 125         |
| 5.3           | Whistler modulational stability spectrum in hot plasmas (stable bands are hatched). [ $\omega_{pe}/\Omega_e = 10$ ] . . . . .                                                                                                                                                          | 130         |

# ILLUSTRATIONS (Cont.)

| <u>Figure</u> |                                                                                                                                                                                                                    | <u>Page</u> |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| A.1           | Two-sheet Riemann surface of $G(\xi)$ . $G^{P,N}(\xi)$ are analytic on $P,N$ and $\tilde{G}^{P,N}(\xi)$ have singularities on $\tilde{P},\tilde{N}$ . . . . .                                                      | 138         |
| A.2           | Symmetry relations between the positive, $g^+(v)$ , and negative, $g^-(v)$ , frequency parts of $g(v)$ , a real valued function of $v_r$ . (a) Arbitrary $g(v_r)$ . (b) Even $g(v_r)$ . (c) Odd $g(v_r)$ . . . . . | 141         |

## NOTATION

### 1. General

#### 1.1 Vectors and Tensors

|                                                    |                                                                                                            |
|----------------------------------------------------|------------------------------------------------------------------------------------------------------------|
| $\underline{v}$                                    | vector                                                                                                     |
| $\hat{\underline{v}}$                              | unit vector                                                                                                |
| $v$                                                | modulus of $\underline{v}$                                                                                 |
| $\underline{v}_{\parallel}, \underline{v}_{\perp}$ | parallel and perpendicular components of $\underline{v}$ relative to the static magnetic field             |
| $v_x, v_y, v_z$                                    | components of $\underline{v}$ along the x-, y- and z-directions                                            |
| $\underline{a} \cdot \underline{b}$                | dot product of $\underline{a}$ and $\underline{b}$                                                         |
| $\nabla$                                           | $\hat{x} \partial/\partial x + \hat{y} \partial/\partial y + \hat{z} \partial/\partial z$ , nabla operator |
| $\partial/\partial \underline{v}$                  | $\hat{x} \partial/\partial v_x + \hat{y} \partial/\partial v_y + \hat{z} \partial/\partial v_z$            |
| $\underline{\underline{K}}$                        | tensor of second rank                                                                                      |
| $\underline{\underline{K}}^\dagger$                | hermitian conjugate of $\underline{\underline{K}}$ (complex conjugate of the transpose)                    |

#### 1.2 Complex Functions

|             |                                            |
|-------------|--------------------------------------------|
| $i$         | $(-1)^{1/2}$                               |
| $a_r$       | real part of $a$                           |
| $a_i$       | imaginary part of $a$                      |
| $a^*$       | complex conjugate of $a$                   |
| $\tilde{G}$ | analytic continuation of $G$               |
| $f^+$       | positive frequency part of $f$ [see (A.5)] |
| $f^-$       | negative frequency part of $f$ [see (A.5)] |
| $(\ )_R$    | $(\ )_x + i (\ )_y$                        |

#### 1.3 Derivatives and Quadratures

|                            |                                                                                               |
|----------------------------|-----------------------------------------------------------------------------------------------|
| $f$                        | Cauchy principal value                                                                        |
| $\int (\ ) d\underline{v}$ | $\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} (\ ) dv_z$ |

## NOTATION (Cont.)

$f'(a)$        $df/da$

$( )_k$        $\partial( )/\partial k$  [except in Chapter 5 and Appendix C];  
 $\int_{-\lambda/2}^{\lambda/2} ( ) \exp ikz d(z/\lambda)$  [in Chapter 5 and Appendix C]

$( )_\omega$        $\partial( )/\partial \omega$

### 1.4 Miscellaneous

$u_0$       zero order (equilibrium) component of  $u$  ;  
value of  $u$  for the triggering, or large amplitude wave;  
initial value of  $u$

$u_1$       first order component of  $u$

$( )_j$        $j = e$  value of  $( )$  for electrons;  
 $j = i$  value of  $( )$  for ions

$(^{\wedge})$       time-asymptotic value of  $( )$

## 2. Symbols

### 2.1 Latin Alphabet

| <u>Symbol</u>                                                                   | <u>Chapters where used;</u><br><u>page of definition</u> |
|---------------------------------------------------------------------------------|----------------------------------------------------------|
| $a$ wave amplitude . . . . .                                                    | 5; 116                                                   |
| $a$ $eB/m_e$ . . . . .                                                          | 4; 67                                                    |
| $c$ free space speed of light . . . . .                                         | 2-5; 4                                                   |
| $e$ magnitude of the electron charge . . . . .                                  | 2-5; 2                                                   |
| $f$ perturbed velocity distribution . . . . .                                   | 2,4; 4                                                   |
| $f$ equilibrium velocity distribution of the<br>nonthermal population . . . . . | 3; 41                                                    |
| $f$ $F - F_0$ . . . . .                                                         | 5,C; 126                                                 |
| $f_1$ fine structure of the velocity distribution .                             | 4; 77                                                    |
| $f^L$ linear solution of Vlasov's equation . . . . .                            | 5; 127                                                   |
| $\langle f \rangle$ averaged $f$ . . . . .                                      | 4; 75                                                    |

# NOTATION (Cont.)

| <u>Symbol</u>   | <u>Chapters where used;</u><br><u>page of definition</u>                             |
|-----------------|--------------------------------------------------------------------------------------|
| $g$             | perturbation in the velocity distribution<br>caused by the test wave . . . . . 4; 77 |
| $k$             | wavenumber . . . . . 2-5; 3                                                          |
| $\underline{k}$ | wavevector . . . . . 2-5; 3                                                          |
| $\bar{k}$       | saddle point in the $k$ -plane . . . . . 2; 22                                       |
| $\bar{k}$       | wavenumber of a modulational perturbation . . 5; 118                                 |
| $\bar{k}_L$     | wavenumber defined by (5.13) . . . . . 5; 118                                        |
| $\bar{k}_M$     | wavenumber defined by (5.15) . . . . . 5; 119                                        |
| $k_{n,m}$       | roots of $D_c^{P,N}(\omega', k_{n,m}) = 0$ . . . . . 2; 19                           |
| $m_e$           | electron rest mass . . . . . 2-5; 2                                                  |
| $n$             | refractive index . . . . . 3; 31                                                     |
| $n$             | number density . . . . . 2-5; 2                                                      |
| $n_h$           | number density of the nonthermal plasma . . . 3-5; 91                                |
| $p$             | real constant . . . . . 3,5; 41                                                      |
| $p_{u,t}$       | functions defined by (4.36) and (4.37) . . . 4; 80                                   |
| $q$             | real constant . . . . . 3,5; 41                                                      |
| $q$             | nome . . . . . 4; 73                                                                 |
| $q$             | electric charge . . . . . 5; 120                                                     |
| $r$             | geocentric distance . . . . . 4; 97                                                  |
| $\underline{r}$ | position vector . . . . . 2-5; 29                                                    |
| $r_e$           | earth's radius . . . . . 4; 97                                                       |
| $s$             | displacement along wavelet path . . . . . 4; 106                                     |
| $t$             | time variable . . . . . 2-5; 3                                                       |
| $\hat{t}$       | transit time in wave packet . . . . . 4; 92                                          |

# NOTATION (Cont.)

| <u>Symbol</u>        |                                                             | <u>Chapters where used;</u><br><u>page of definition</u> |
|----------------------|-------------------------------------------------------------|----------------------------------------------------------|
| $t_d$                | slab drifting time . . . . .                                | 4; 106                                                   |
| $t_g$                | growth time of test waves . . . . .                         | 4; 93                                                    |
| $t_p$                | wavelet propagation time . . . . .                          | 4; 106                                                   |
| $u$                  | invariant of motion . . . . .                               | 4; 68                                                    |
| $u$                  | $(v_x + iv_y)/v_A$ . . . . .                                | 5.3; 121                                                 |
| $u$                  | $v_x + iv_y$ . . . . .                                      | 5.4; 126                                                 |
| $\underline{v}$      | velocity . . . . .                                          | 2-5; 2                                                   |
| $\underline{v}_A$    | Alfvén velocity . . . . .                                   | 5; 120                                                   |
| $\underline{v}_g$    | group velocity . . . . .                                    | 3-5; 32                                                  |
| $v'_{g0}$            | $\partial v_g / \partial k$ at $k = k_0$ . . . . .          | 5; 117                                                   |
| $v_0$                | constant speed . . . . .                                    | 3,5; 41                                                  |
| $v_p$                | constant parallel speed . . . . .                           | 4; 83                                                    |
| $\underline{v}_p$    | phase velocity . . . . .                                    | 3,4; 39                                                  |
| $v_\theta$           | $(\kappa_B T/m_e)^{1/2}$ , electron thermal speed . . . . . | 2; 13                                                    |
| $\overline{v}_\perp$ | constant perpendicular speed . . . . .                      | 4; 83                                                    |
| $v_\perp^*$          | perpendicular speed defined by (4.20) . . . . .             | 4; 73                                                    |
| $v_{\perp c}$        | root of $F_\perp(v_{\perp c}, V_m) = 0$ . . . . .           | 3; 44                                                    |
| $w$                  | $v_\parallel - V_{G0}$ . . . . .                            | 4,C; 68                                                  |
| $w^*$                | parallel velocity defined by (4.28) . . . . .               | 4; 75                                                    |
| $\langle w \rangle$  | average of $w$ over $\psi$ . . . . .                        | 4; 72                                                    |
| $x$                  | displacement along x-axis . . . . .                         | 2-5; 29                                                  |
| $y$                  | displacement along y-axis . . . . .                         | 2-5; 29                                                  |
| $z$                  | displacement along z-axis . . . . .                         | 2-5; 2                                                   |
| $A$                  | Stix's cold plasma parameter . . . . .                      | 3; 30                                                    |

# NOTATION (Cont.)

| <u>Symbol</u>     |                                                                | <u>Chapters where used;</u><br><u>page of definition</u> |
|-------------------|----------------------------------------------------------------|----------------------------------------------------------|
| A                 | real constant . . . . .                                        | 5; 119                                                   |
| $A_n$             | normalizing constant . . . . .                                 | 3,4; 41,83                                               |
| B                 | Bromwich contour . . . . .                                     | 2; 4                                                     |
| B                 | Stix's cold plasma parameter . . . . .                         | 3; 30                                                    |
| $\underline{B}$   | wave magnetic flux density . . . . .                           | 2-5; 4                                                   |
| $B_N$             | normalizing constant . . . . .                                 | 4; 83                                                    |
| $B_0$             | static magnetic flux density . . . . .                         | 2-5; 2                                                   |
| $\underline{B}_t$ | test wave magnetic flux density . . . . .                      | 4; 77                                                    |
| C                 | Stix's cold plasma parameter . . . . .                         | 3; 30                                                    |
| C                 | enhancement of density in center of ducts . . . . .            | 4; 99                                                    |
| C                 | function defined by (A.11) . . . . .                           | A; 140                                                   |
| $C_{  }^L$        | $C_{  }^m$ for $m = 0$ . . . . .                               | 3,4; 46                                                  |
| $C_{  ,\perp}^G$  | $C_{  ,\perp}^m$ for $m = -1$ . . . . .                        | 3; 46                                                    |
| $C_{  ,\perp}^m$  | dissipation coefficients . . . . .                             | 3; 42                                                    |
| D                 | dispersion function . . . . .                                  | 2,3; 5                                                   |
| D                 | Stix's cold plasma parameter . . . . .                         | 3; 30                                                    |
| $\overline{D}$    | $D(\overline{\omega}, \overline{k})$ . . . . .                 | 2; 22                                                    |
| $D^{P,N}$         | dispersion functions defined by (2.15)<br>and (2.16) . . . . . | 2; 7                                                     |
| $D_c^{P,N}$       | $D^{P,N} \cup \tilde{D}^{P,N}$ . . . . .                       | 2; 7                                                     |
| $\underline{E}$   | wave electric field . . . . .                                  | 2-5; 4                                                   |
| $E_1$             | complete elliptic integral of the second<br>kind . . . . .     | 4,C; 72                                                  |
| $\underline{E}_t$ | test wave electric field . . . . .                             | 4; 77                                                    |

# NOTATION (Cont.)

| <u>Symbol</u>          | <u>Chapters where used;</u><br><u>page of definition</u>                                      |
|------------------------|-----------------------------------------------------------------------------------------------|
| $E^{P,N}$              | entire functions defined by (2.25) . . . . . 2; 19                                            |
| $F$                    | elliptic integral of the first kind . . . . . 4; 69                                           |
| $F$                    | perturbed exact velocity distribution . . . . . 5; 126                                        |
| $F_P$                  | time averaged kinetic power flux . . . . . 3; 34                                              |
| $F_{EM}$               | time averaged Poynting flux . . . . . 3; 34                                                   |
| $F_{\parallel, \perp}$ | $\partial f / \partial v_{\parallel, \perp}$ . . . . . 3, 4; 31                               |
| $G_{1,2}$              | functions defined by (3.4) . . . . . 3; 31                                                    |
| $G^{P,N}$              | functions defined by (A.2) and (A.3) . . . . . A; 136                                         |
| $H$                    | $(B_x + i B_y) / B_0$ . . . . . 5.3; 121                                                      |
| $H$                    | $B_x + i B_y$ . . . . . 5.4; 126                                                              |
| $\underline{H}$        | wave magnetic field . . . . . 3; 30                                                           |
| $I$                    | integral defined by (2.30) . . . . . 2; 20                                                    |
| $\vec{I}$              | unit tensor . . . . . 3; 34                                                                   |
| $I_{1,2}$              | integrals defined by the first and second<br>terms in the integrand of (4.31) . . . . . 4; 81 |
| $\underline{J}$        | current density . . . . . 2-5; 4                                                              |
| $J_n$                  | Bessel function of integer order $n$ . . . . . 3; 30                                          |
| $\underline{J}_s$      | source current density . . . . . 2; 4                                                         |
| $K$                    | $kc/\omega$ . . . . . 2; 15                                                                   |
| $K$                    | complete elliptic integral of the first kind . . . . . 4, C; 73                               |
| $\vec{K}$              | plasma permittivity tensor . . . . . 3; 29                                                    |
| $\vec{K}^a$            | $-i$ times the antihermitian part of $\vec{K}$ . . . . . 3; 33                                |
| $\vec{K}^h$            | hermitian part of $\vec{K}$ . . . . . 3; 33                                                   |
| $L$                    | Stix's cold plasma parameter . . . . . 3; 30                                                  |

# NOTATION (Cont.)

| <u>Symbol</u> | <u>Chapters where used;</u><br><u>page of definition</u>             |
|---------------|----------------------------------------------------------------------|
| L             | McIlwain geomagnetic coordinate . . . . . 3-5; 62                    |
| L             | length of triggering whistler packet . . . . . 4; 92                 |
| $\vec{M}$     | tensor defined by (3.1) . . . . . 3; 29                              |
| $M_{A,B,C,D}$ | wave modes . . . . . 2; 13                                           |
| P             | Stix's cold plasma parameter . . . . . 3; 30                         |
| $P_D$         | time averaged power dissipated by the wave . 3; 34                   |
| $P_D^m$       | component of $P_D$ defined by (3.33) . . . . . 3; 42                 |
| R             | Stix's cold plasma parameter . . . . . 3; 30                         |
| S             | function of $v_{\parallel}$ defined by (2.12) . . . . . 2; 5         |
| S             | Stix's cold plasma parameter . . . . . 3; 30                         |
| T             | function of $v_{\parallel}$ defined by (2.12) . . . . . 2; 5         |
| T             | absolute temperature . . . . . 2-5; 41                               |
| U             | mean electron kinetic energy . . . . . 3; 41                         |
| $U_M$         | time averaged magnetic energy density . . . . . 3; 34                |
| $U_T$         | $U_M + U_{E+P}$ . . . . . 3; 36                                      |
| $U_{E+P}$     | time averaged electric and kinetic energy<br>density . . . . . 3; 34 |
| $V_G$         | $(\omega - \Omega)/k_{\parallel}$ . . . . . 3-5; 36                  |
| $V_L$         | $\omega/k_{\parallel}$ . . . . . 3-5; 36                             |
| $V_m$         | $(\omega + m\Omega)/k_{\parallel}$ . . . . . 3; 31                   |
| W             | $\omega/\Omega$ . . . . . 2-5; 15                                    |
| $W_p$         | $\omega_p/\Omega$ . . . . . 2-5; 15                                  |

## NOTATION (Cont.)

### 2.2 Greek Alphabet

| <u>Symbol</u>  |                                                | <u>Chapters where used;</u><br><u>page of definition</u> |
|----------------|------------------------------------------------|----------------------------------------------------------|
| $\alpha$       | positive real constant defined in (B.1) . . .  | 2,B; 142                                                 |
| $\alpha$       | $n_h/n_o$ . . . . .                            | 5; 129                                                   |
| $\alpha$       | electron-ion mass ratio . . . . .              | 3; 31                                                    |
| $\alpha$       | function defined by (4.26) . . . . .           | 4; 75                                                    |
| $\alpha_{u,t}$ | functions defined by (4.36) and (4.37) . . .   | 4; 80                                                    |
| $\beta$        | positive real constant defined by (B.2) . . .  | 2,B; 142                                                 |
| $\beta$        | $n_h/n_o$ . . . . .                            | 4; 91                                                    |
| $\beta$        | $v_\theta/c$ . . . . .                         | 2; 15                                                    |
| $\beta$        | function defined by (4.26) . . . . .           | 4; 75                                                    |
| $\beta$        | positive real constant defined by (5.1) . . .  | 5; 116                                                   |
| $\gamma$       | 0.57721..., Euler's constant . . . . .         | 2; 23                                                    |
| $\gamma$       | growth rate . . . . .                          | 4,5; 78                                                  |
| $\gamma$       | $10^{-5}$ Gauss . . . . .                      | 4; 91                                                    |
| $\gamma$       | $(1 - v^2/c^2)^{-1/2}$ . . . . .               | 5; 120                                                   |
| $\gamma_L$     | linear growth rate . . . . .                   | 4; 84                                                    |
| $\delta$       | $a/\Omega$ . . . . .                           | 4; 68                                                    |
| $\delta$       | $n_h/n_o$ . . . . .                            | 3; 29                                                    |
| $\delta(x)$    | Dirac delta function . . . . .                 | 2-5; 6                                                   |
| $\delta\Omega$ | $\omega_r(t=0) - \omega_r(t=\infty)$ . . . . . | 5; 124                                                   |
| $\epsilon$     | whistler equivalent permittivity . . . . .     | 5; 128                                                   |
| $\epsilon_0$   | free space permittivity . . . . .              | 2-5; 4                                                   |
| $\eta$         | normalized variable defined by (2.22) . . . .  | 2; 15                                                    |
| $\eta$         | $l/\kappa$ , trapping parameter . . . . .      | 4; 74                                                    |

# NOTATION (Cont.)

| <u>Symbol</u>    |                                                       | <u>Chapters where used;</u><br><u>page of definition</u> |
|------------------|-------------------------------------------------------|----------------------------------------------------------|
| $\theta$         | angle between $\underline{k}$ and $\hat{z}$ . . . . . | 3-5; 29                                                  |
| $\theta$         | $\bar{\omega}_{NL} t$ . . . . .                       | 4; 83                                                    |
| $\theta$         | argument of $\varphi$ . . . . .                       | 5; 117                                                   |
| $\theta_c$       | semi-angle of the resonance cone . . . . .            | 3; 37                                                    |
| $\kappa$         | trapping parameter . . . . .                          | 4,C; 69                                                  |
| $\kappa_B$       | Boltzmann's constant . . . . .                        | 2-5; 41                                                  |
| $\lambda$        | magnetic colatitude . . . . .                         | 4; 97                                                    |
| $\lambda$        | wavelength . . . . .                                  | 2-5; 126                                                 |
| $\mu$            | real part of the refractive index . . . . .           | 3; 31                                                    |
| $\mu$            | frequency shift coefficient defined by (5.2). . . . . | 5; 116                                                   |
| $\mu_{Or}$       | coefficient defined by (5.24) . . . . .               | 5; 122                                                   |
| $\mu_{Oi}$       | coefficient defined by (5.25) . . . . .               | 5; 123                                                   |
| $\nu$            | collision frequency . . . . .                         | 2; 9                                                     |
| $\xi$            | $(\omega - \Omega)/k$ . . . . .                       | 2; 5                                                     |
| $\xi$            | complex variable . . . . .                            | A; 136                                                   |
| $\xi$            | angle defined by (4.8) . . . . .                      | 4; 69                                                    |
| $\xi$            | $z - v_{g0} t$ . . . . .                              | 5; 117                                                   |
| $\vec{\sigma}$   | conductivity tensor . . . . .                         | 3; 34                                                    |
| $\vec{\sigma}^h$ | hermitian part of $\vec{\sigma}$ . . . . .            | 3; 34                                                    |
| $\tau$           | $t$ , time . . . . .                                  | 5; 117                                                   |
| $\Phi$           | $\tan^{-1}(v_y/v_x)$ . . . . .                        | 2,5,C; 4                                                 |
| $\Phi$           | spatial Fourier transform defined by (5.4). . . . .   | 5; 116                                                   |
| $\varphi$        | $\tan^{-1}(v_y/v_x)$ . . . . .                        | 4; 67                                                    |

# NOTATION (Cont.)

| <u>Symbol</u>                 | <u>Chapters where used;</u><br><u>page of definition</u>                                                           |
|-------------------------------|--------------------------------------------------------------------------------------------------------------------|
| $\varphi$                     | complex amplitude of the envelope of<br>the packet . . . . . 5; 117                                                |
| $\varphi_{n,m}$               | $\arg k_{n,m}$ . . . . . 2; 20                                                                                     |
| $\chi$                        | invariant of motion defined by (4.4) . . . . 4; 68                                                                 |
| $\chi$                        | minus the imaginary part of the refractive<br>index . . . . . 3; 31                                                |
| $\chi^{L,G,0}$                | contributions to $\chi$ of the Landau (L),<br>fundamental gyro (G) and other (0) inter-<br>actions . . . . . 3; 49 |
| $\psi$                        | angle between $\underline{B}$ and $\underline{v}_\perp$ . . . . .                                                  |
| $\psi$                        | complex amplitude of the wave packet . . . . 5; 117                                                                |
| $\omega$                      | angular frequency . . . . . 2-5; 3                                                                                 |
| $\omega'$                     | frequency on the Bromwich contour . . . . . 2; 19                                                                  |
| $\underline{\omega}$          | frequency defined by (5.27) . . . . . 5; 124                                                                       |
| $\bar{\omega}$                | branch point in the $\omega$ -plane . . . . . 2; 22                                                                |
| $\bar{\omega}$                | frequency of the modulational perturbation . 5; 118                                                                |
| $\omega_L$                    | linear frequency . . . . . 5; 127                                                                                  |
| $\omega_p$                    | plasma frequency . . . . . 2-5; 5                                                                                  |
| $\bar{\omega}_p$              | frequency defined by (5.27) . . . . . 5; 124                                                                       |
| $\omega_{iM}$                 | growth rate defined by (5.14) . . . . . 5; 118                                                                     |
| $\omega_{NL}$                 | $(ak_0 v_\perp)^{1/2}$ . . . . . 4, 5, C; 68                                                                       |
| $\bar{\omega}_{NL}$           | $\omega_{NL}$ at $v_\perp = \bar{v}_\perp$ . . . . . 4; 83                                                         |
| $\langle \omega_{NL} \rangle$ | $\omega_{NL}$ at $v_\perp = \text{rms}(v_\perp)$ . . . . . 4; 67                                                   |
| $\Gamma$                      | operator defined by (3.4) . . . . . 3; 31                                                                          |
| $\Gamma$                      | gamma function . . . . . 5; 129                                                                                    |

# NOTATION (Cont.)

| <u>Symbol</u> |                                                                             | <u>Chapters where used;</u><br><u>page of definition</u> |
|---------------|-----------------------------------------------------------------------------|----------------------------------------------------------|
| $\Gamma_I$    | operator defined by (3.21) . . . . .                                        | 3; 35                                                    |
| $\Delta$      | angle defined by (2.29) . . . . .                                           | 2; 20                                                    |
| $\Lambda$     | operator defined by (3.4) . . . . .                                         | 3; 31                                                    |
| $\Lambda_I$   | operator defined by (3.21) . . . . .                                        | 3; 35                                                    |
| $\Lambda_L$   | total linear attenuation . . . . .                                          | 4; 85                                                    |
| $\Lambda_M$   | maximal consecutive growth . . . . .                                        | 4; 85                                                    |
| $\Psi$        | Tricomi's confluent hypergeometric function<br>of the second kind . . . . . | 2; 20                                                    |
| $\Omega$      | $eB_0/m_e$ . . . . .                                                        | 2-4; 2                                                   |
| $\Omega$      | frequency defined by (5.1) . . . . .                                        | 5; 116                                                   |
| $\Omega_0$    | $\Omega$ at the Equator . . . . .                                           | 4; 102                                                   |

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## 1. INTRODUCTION

The central theme of this dissertation is the whistler, one of the characteristic modes of propagation in a magnetoplasma. This circularly polarized electromagnetic wave is prominent in the literature of space and laboratory plasmas because static magnetic fields permeate the natural plasmas in the Universe, and commonly contribute to the containment of man-made plasmas.

The topics of linear and nonlinear whistler theory to be discussed here were selected for their relevance to whistlers propagating in the earth's magnetosphere. The only partial exception lies in the clarification of some fundamental aspects of the linear theory of stability and dispersion which, although stimulated by our interest in whistlers, have broader applicability. These clarifications on stability and dispersion treated in Chapter 2, and the analysis in Chapter 3 of the stability of whistlers propagating at an angle to the static magnetic field, comprise our linear whistler studies. The topics concerned with nonlinear whistlers are artificially stimulated emissions, discussed in Chapter 4, and modulational instability, discussed in Chapter 5. Critical descriptions of previous work, and detailed assessments of the results obtained, are given in the individual chapters. We have attempted to make these independent without being repetitious.

The contributions of our research on whistler theory may be summarized as follows: Chapter 2 generalizes the applicability of linear stability criteria to whistlers in arbitrary magnetoplasmas. Chapter 3 extends previous work on the stability of oblique low frequency whistlers to high frequencies, and establishes a criterion for the existence of maximal growth along the static magnetic field. Chapter 4 derives a new mechanism for the onset of stimulated emissions, and speculates on the potentiality of slab radiation to account for most of the observed emission spectral shapes. Chapter 5 corrects previous work on the whistler modulational instability in cold plasmas, and analyzes this whistler instability in hot plasmas for the first time.

## 2. LINEAR DISPERSION AND STABILITY

### 2.1 Introduction

A basic prerequisite for the understanding of the nonlinear characteristics of large amplitude whistlers is a thorough familiarity with their linear, small amplitude, characteristics. Because of its importance in areas ranging from magnetospheric physics to thermonuclear fusion, this first linear step has been the subject of considerable research both in cold and hot plasmas. Comprehensive reviews have been given by Helliwell [1965] and Lee [1969]. Rather than striving for completeness, we shall concentrate our linear studies here on a number of significant topics that have not so far received adequate attention in the literature.

In Section 2.2, the discussion of the whistler dispersion relation clarifies some subtleties in its interpretation that are relevant to the general question of distinguishing between true and pseudo-modes: not all the mathematical solutions of a dispersion relation represent physically realizable waves. Because the existing stability criteria for plasma waves [Briggs, 1964; Derfler, 1967; Derfler, 1970] are not applicable to arbitrary equilibrium velocity distributions, we shall extend Derfler's work in Section 2.3 to remove this restriction.

### 2.2 Dispersion Relation

#### 2.2.1 Derivation

For purposes of this section, the whistler is assumed to propagate in a uniform, time-invariant, collisionless magnetoplasma composed of a hot, nonrelativistic, electron population neutralized by an immobile ion background. The static magnetic field direction defines the positive z-axis,  $\underline{B}_0 = B_0 \hat{z}$ , and the whistler wavenumber lies along this axis,  $\underline{k} = k \hat{z}$ . The magnetoplasma is characterized by the electron gyrofrequency defined by the static magnetic field,  $\Omega = e B_0 / m_e$  [ $e$  and  $m_e$  represent the charge (magnitude) and mass of the electron]; the equilibrium electron number density,  $n_0$ , and the equilibrium velocity distribution,  $F_0(\underline{v})$ , normalized to unity. Some of the foregoing restrictive assumptions will be removed later when the problems

under consideration warrant the additional complication.

The system is described self-consistently by Maxwell's equations and the Vlasov equation. The latter implies that the equilibrium velocity distribution must be a function of the electron speeds parallel and perpendicular to  $\underline{B}_0$ . Using cylindrical coordinates in velocity space, so that  $v_{\parallel} = v_z$ ,  $v_{\perp}^2 = v_x^2 + v_y^2$ , and  $\phi = \tan^{-1}(v_y/v_x)$ , the above restriction yields  $\partial F_0 / \partial \phi = 0$ , i.e.  $F_0(\underline{v}) = f_0(v_{\parallel}, v_{\perp})$ . The equilibrium state about which small perturbations will be allowed is thus characterized by  $\underline{B}_0$  and  $n_0 f_0(v_{\parallel}, v_{\perp})$ .

The dispersion relation can be readily derived by assuming all first order quantities to vary as  $\exp i(\omega t - kz)$ . However, this artifice does not correspond to a well-posed problem, and precludes the correct interpretation of the final result; a transient problem should always be studied, involving a superposition of modes. We shall adopt Laplace (time) and Fourier (space) transforms for the first order quantities, and study the response of the system to an impulsive current source applied at  $t=0$ . The existence of the spatial Fourier transform follows from the finite extent of the perturbation which ensures its square integrability. The existence of the Laplace transform is not so obvious: in principle, the system may support modes which grow too rapidly in time to have Laplace transforms. For the Landau problem, Backus [1960] has found an upper bound on the temporal growth rate of the solution. We shall assume that such a situation prevails in the whistler case. Thus the first order quantities will be determined by

$$e(z, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_B d\omega e(k, \omega) \exp i(\omega t - kz), \quad (2.1)$$

with the space-time transforms

$$e(k, \omega) = \int_{-\infty}^{\infty} dz \int_0^{\infty} dt e(z, t) \exp[-i(\omega t - kz)], \quad (2.2)$$

where  $\underline{B}$  denotes a straight line in the lower half of the complex  $\omega$ -plane, parallel to the real axis and, to ensure causality, lying below all singularities of the integrand (Bromwich contour).

Transforming Maxwell's equations, and recalling that the system is unperturbed at  $t=0$ , i.e. there are no initial-value terms, we obtain

$$\underline{k} \times \underline{E} = \omega \underline{B}, \quad \epsilon_0 c^2 \underline{k} \times \underline{B} = -\omega \epsilon_0 \underline{E} + i (\underline{J}_s + \underline{J}). \quad (2.3)$$

Here  $\underline{B}$  and  $\underline{E}$  are the wave field transforms; the spectral dependence  $(\underline{k}, \omega)$  is to be understood;  $\underline{J}_s$  is the source current density transform;  $\epsilon_0$  and  $c$  are the permittivity and speed of light in free space, and

$$\underline{J}(\underline{k}, \omega) = -e \int \underline{f}_1(\underline{k}, \underline{v}, \omega) \underline{v} d\underline{v}, \quad (2.4)$$

where the (total) perturbed velocity distribution is

$$f(\underline{z}, \underline{v}, t) = n_0 f_0 + f_1(\underline{z}, \underline{v}, t). \quad (2.5)$$

Eliminating the wave magnetic field in (2.3); noting that  $\underline{E} \cdot \underline{k} = 0$ , i.e.  $E_z = 0$ , for whistlers propagating parallel to  $\underline{B}_0$ ; using (2.4), and combining the vector components, yields

$$(k^2 c^2 - \omega^2)(E_x + iE_y) = \frac{i\omega e}{\epsilon_0} \int (\underline{v}_x + i\underline{v}_y) \underline{f}_1 d\underline{v} - \frac{i\omega}{\epsilon_0} (J_{sx} + iJ_{sy}). \quad (2.6)$$

The transformed, linearized, Vlasov equation determines the transformed perturbation in the velocity distribution through

$$i(\omega - k v_{\parallel}) \underline{f}_1 - \frac{e}{m_e} \underline{v} \times \underline{B}_0 \cdot \frac{\partial \underline{f}_1}{\partial \underline{v}} = \frac{en_0}{m_e} \left[ \underline{E} + \frac{1}{\omega} \underline{v} \times (\underline{k} \times \underline{E}) \right] \cdot \frac{\partial \underline{f}_0}{\partial \underline{v}}. \quad (2.7)$$

Going to cylindrical coordinates in velocity space

$(\underline{v}_x + i\underline{v}_y = v_{\perp} \exp i\phi)$ , and recalling that  $E_z = 0$ , we obtain

$$i(\omega - kv_{\parallel})f_1 + \Omega \frac{\partial f_1}{\partial \phi} = \frac{en_0}{m_e} (E_x \cos \phi + E_y \sin \phi) \left[ \frac{\partial f_0}{\partial v_{\perp}} \left( 1 - \frac{kv_{\parallel}}{\omega} \right) + \frac{\partial f_0}{\partial v_{\parallel}} \frac{kv_{\perp}}{\omega} \right]. \quad (2.8)$$

The transverse fields of the linear parallel-propagating whistler rotate with angular frequency  $\omega$  in the direction of the electron cyclotron gyration, so that  $E_x - i E_y = 0$ . The solution of (2.8) is then

$$f_1(k, \underline{v}, \omega) = \frac{en_0}{2m_e} \frac{E_x + i E_y}{i(\omega - kv_{\parallel} - \Omega)} \left[ \frac{\partial f_0}{\partial v_{\perp}} \left( 1 - \frac{kv_{\parallel}}{\omega} \right) + \frac{\partial f_0}{\partial v_{\parallel}} \frac{kv_{\perp}}{\omega} \right] \exp(-i\phi). \quad (2.9)$$

Using (2.9) in (2.6), we obtain

$$E_x + i E_y = \frac{i}{\omega \epsilon_0} \frac{1}{D(\omega, k)} (J_{sx} + i J_{sy}), \quad (2.10)$$

where  $D(\omega, k)$  is given by

$$\begin{aligned} D(\omega, k) &= 1 - \left( \frac{kc}{\omega} \right)^2 + \pi \left( \frac{\omega}{\omega_p} \right)^2 \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} dv_{\perp} \frac{(\omega - kv_{\parallel}) \frac{\partial f_0}{\partial v_{\perp}} + kv_{\perp} \frac{\partial f_0}{\partial v_{\parallel}}}{(\omega - kv_{\parallel} - \Omega)} v_{\perp}^2 \\ &= 1 - \left( \frac{kc}{\omega} \right)^2 - \left( \frac{\omega_p}{\omega} \right)^2 + \left( \frac{\omega_p}{\omega} \right)^2 \int_{-\infty}^{\infty} \frac{(\Omega/k) S(v_{\parallel}) - T(v_{\parallel})}{v_{\parallel} - \xi} dv_{\parallel}, \end{aligned} \quad (2.11)$$

with

$$\omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m_e}, \quad \xi = \frac{\omega - \Omega}{k},$$

$$S(v_{\parallel}) = 2\pi \int_0^{\infty} v_{\perp} f_0(v_{\parallel}, v_{\perp}) dv_{\perp}, \quad (2.12)$$

$$T(v_{\parallel}) = \pi \int_0^{\infty} v_{\perp}^3 \frac{\partial f_0}{\partial v_{\parallel}} dv_{\perp}.$$

The result obtained in (2.10) is to be expected in a linear, time- and space-invariant system. The transform of the response,  $E_x + i E_y$ , is related to the transform of the excitation,  $J_{sx} + i J_{sy}$ , through the transfer function (impulse response transform) of the system. The normal modes must be solutions of the parallel whistler dispersion relation,

$$D(\omega, k) = 0. \quad (2.13)$$

At this stage, the formal expression of  $D(\omega, k)$  in (2.11) is meaningless, because we did not indicate how to perform the improper integral over  $v_{\parallel}$ . Appendix A and Section 2.2.2 will discuss this matter. Before proceeding further, however, we may identify the physical mechanism causing the mathematical singularity, and retrieve the dispersion relation for parallel whistlers in cold plasmas, where the singularity does not occur.

The electrons of the unperturbed hot magnetoplasma gyrate about the static magnetic field,  $\underline{B}_0$ , with angular frequency  $\Omega$  and move along it with parallel velocity  $v_{\parallel}$ . The wave field may be in resonance with this unperturbed motion if electrons move in the opposite direction to the whistler with parallel velocity  $\xi$  ( $< 0$ ) such that  $\omega' = \omega - k\xi = \Omega$ . In this case, because the whistler fields rotate at frequency  $\omega$  ( $< \Omega$ ) in the same direction as the electrons, the Doppler-shifted wave frequency felt by the electrons,  $\omega'$ , coincides with their local cyclotron frequency.

The whistler dispersion relation for parallel propagation in cold plasmas is obtained from (2.11) with the use of the appropriate equilibrium distribution,  $f_0 = \delta(v_{\parallel}) \delta(v_{\perp}) / 2\pi v_{\perp}$ . We obtain, from  $D(\omega, k) = 0$ ,

$$\frac{k^2 c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega(\Omega - \omega)}. \quad (2.14)$$

In this case, cyclotron resonance can only occur when  $\omega = \Omega$ , since the unperturbed electrons are at rest ( $v_{\parallel} = v_{\perp} = 0$ ).

### 2.2.2 Discussion

Utilizing the properties of the Cauchy integral described in Appendix A, we find that (2.11) defines two functions given by

$$D^P(\omega, k) = 1 - \left(\frac{kc}{\omega}\right)^2 - \left(\frac{\omega_p}{\omega}\right)^2 + 2\pi i \left(\frac{\omega_p}{\omega}\right)^2 \left[\frac{\Omega}{k} S^+(\xi) - T^+(\xi)\right] \quad \xi_i > 0, \quad (2.15)$$

$$D^N(\omega, k) = 1 - \left(\frac{kc}{\omega}\right)^2 - \left(\frac{\omega_p}{\omega}\right)^2 - 2\pi i \left(\frac{\omega_p}{\omega}\right)^2 \left[\frac{\Omega}{k} S^-(\xi) - T^-(\xi)\right] \quad \xi_i < 0, \quad (2.16)$$

where the superscripts  $+$  and  $-$  denote the positive and negative frequency parts of  $S$  and  $T$ , as defined in (A.5) and (A.6). Some properties of  $D^P$  and  $D^N$  are analyzed in Appendix B.

The real axis of the complex  $\xi$ -plane [ $\xi_i = 0 = k_r \omega_i - k_i(\omega_r - \Omega)$ ] is the (branch cut) boundary between the two regions where the dispersion relations  $D^P = 0$  and  $D^N = 0$  hold. Mapping  $\xi_i = 0$  into the  $\omega$ -plane ( $k$ -plane) for a fixed value of  $k(\omega)$ , we obtain straight lines going through  $\omega = \Omega$  ( $k = 0$ ), as shown in Figure 2.1. They divide each plane into two halves where  $D^P$  and  $D^N$  are separately valid. Using analytic continuation, as indicated in Appendix A, we can derive  $D_c^P$  and  $D_c^N$  which are valid everywhere in the complex planes  $\xi$ ,  $\omega$  and  $k$ . Symbolically (see Figures 2.1 and A.1), we can write  $D_c^P = D^P \cup \tilde{D}^P$  and  $D_c^N = D^N \cup \tilde{D}^N$ , where  $U$  denotes the union of two sets.

The symmetry properties given in (A.12) - (A.14) show that the relation

$$[D_c^P(\omega, k)]^* = D_c^N(\omega^*, k^*) \quad (2.17)$$

always holds. When the equilibrium velocity distribution,  $f_0(v_{\parallel}, v_{\perp})$ , is even in  $v_{\parallel}$ , so that  $S(v_{\parallel}) = S(-v_{\parallel})$  and  $T(v_{\parallel}) = -T(-v_{\parallel})$ , the following relations also hold:

$$D_c^P(\omega, -k) = D_c^N(\omega, k) = [D_c^P(\omega^*, k^*)]^* . \quad (2.18)$$

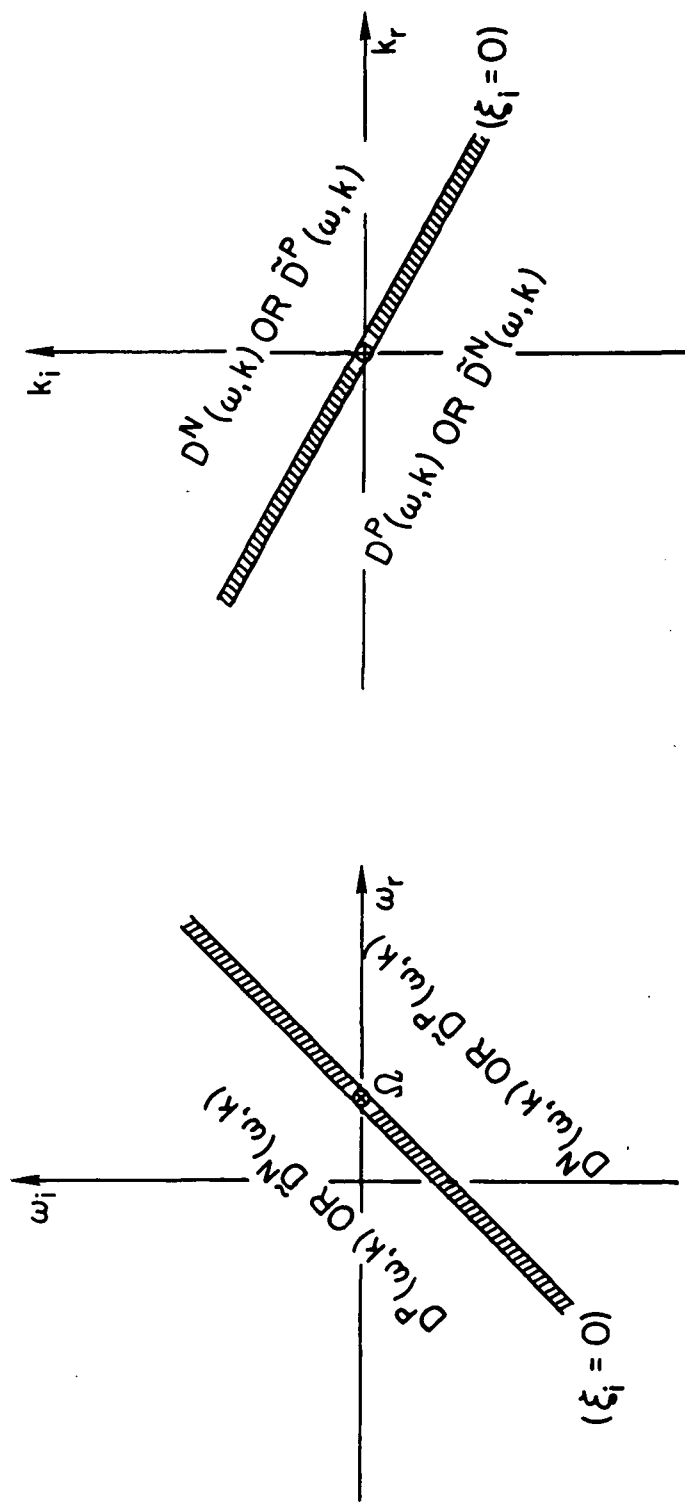


FIG. 2.1. One possible set of branch cuts in the  $\omega$  and  $k$  planes. The fixed points of the branch cuts are denoted by  $\times$ .

These symmetry considerations are important in Section 2.3 because the stability criteria for hot plasmas published so far [Derfler, 1967; Derfler, 1970] imply the satisfaction of (2.18) or  $D_c^P(\omega^*, k^*) = [D_c^N(-\omega, -k)]^*$ , whereas we have just seen that in the parallel whistler case (2.18) does not hold for a general equilibrium velocity distribution, and  $D_c^P(\omega^*, k^*) = [D_c^N(-\omega, -k)]^*$  is not satisfied.

For completeness, we note that if collisions are accounted for by the introduction of a phenomenological relaxation term [Lee, 1969], we would obtain

$$[D_c^P(\omega, k; \nu)]^* = D_c^N(\omega^*, k^*; -\nu) ,$$

with  $\xi = (\omega - \Omega - i\nu)/k$ , where  $\nu$  is a velocity independent collision frequency.

Inverting the Fourier-Laplace transforms in (2.10), and using  $(\ )_R = (\ )_x + i(\ )_y$ , we obtain

$$E_R(z, t) = \frac{1}{2\pi} \int_B d\omega E_R(z, \omega) \exp i\omega t , \quad (2.19)$$

where  $E_R(z, \omega)$  represents the inverse spatial Fourier transform of  $E_R(k, \omega)$ . Taking into consideration  $\text{sgn } \xi_i$ ; the notation introduced in Appendix B, and recalling that, in (2.20),  $\omega$  has a fixed value on the Bromwich contour with  $\omega_i < -\max(\alpha, \beta) < 0$ , we can write

$$E_R(z, \omega) = \frac{i}{2\pi\omega\epsilon_0} \left\{ \int_{-\infty}^0 dk \frac{J_{Rs}(\omega, k)}{D^P(\omega, k)} \exp(-ikz) + \int_0^{\infty} dk \frac{J_{Rs}(\omega, k)}{D^N(\omega, k)} \exp(-ikz) \right\} . \quad (2.20)$$

The role of the dispersion functions  $D^P$  and  $D^N$  is to define the response of the system to a given excitation. Specification of them and the excitation source,  $J_{Rs}$ , yields the parallel whistler mode fields for all  $z$  and  $t$  by direct computation of (2.19) and (2.20). Recalling (B.1) and (B.2) we note that the quadratures do not involve improper integrals.

The integral expressions in (2.19) and (2.20) will now be used to differentiate between true and pseudo-modes, and to establish stability criteria. The general interpretation of dispersion relations which are single-valued in  $\omega$  and  $k$ , i.e.  $D_c^P \equiv D_c^N \equiv D(\omega, k)$ , as is the case for the cold plasma description, or the warm plasma treatment based on replacement of the Vlasov equation by its velocity moments, is given by Rognlien and Self [1972]. They discuss boundary and initial value problems, and finite geometry effects.

### 2.2.3 True and Pseudo-Modes

APPROACH. If we attempt to compute (2.20) by contour integration, we find a branch cut separating the regions of validity of  $D^P$  and  $D^N$  (see Figure 2.1) which crosses the path of integration at  $k = 0$ . Derfler [1961] has shown that the problem can be solved by placing an upper bound on the speed  $|v_{||}|$  of the particles, and then let it tend to  $\infty$  (nonrelativistic dynamics) after integration. This artifice opens a "window" in the branch cut around  $k = 0$ , so that we have the situations sketched in Figures 2.2 and 2.3.

To proceed, we assume that  $J_{Rs}(\omega, k)$  either has no singularities in the complex  $\omega$ - and  $k$ -planes and hence is a constant (Liouville's theorem), or is analytic in the whole complex planes with the exception of the points at infinity (entire function). Then, noting that the contributions of the (infinite radius) circular arcs are zero, we can express  $E_R(z, \omega)$  as a sum of discrete normal modes of the system (residues of the poles) and continuous modes [Van Kampen, 1955] arising from the contributions from the branch cut. It is important to realize that the true modes contributing to the response of the system are the roots of  $D^P = 0$  in the regions where  $\xi_i > 0$ , and  $D^N = 0$  in the regions where  $\xi_i < 0$ . The roots of the analytic continuations,  $\tilde{D}^P = 0$  and  $\tilde{D}^N = 0$ , are "covered" by the Riemann sheets of the main functions  $D^N$  and  $D^P$  (see Figure A.1) and do not contribute to the response [Derfler, 1961]: they represent pseudo-modes of the system.

Taking into consideration the different domains of applicability of the main functions  $D^P$  and  $D^N$  for the two cases ( $\omega_r \gtrless \Omega$ ) considered in Figures 2.2 and 2.3, we conclude that the ambiguities in the character

$$\omega_r < \Omega$$

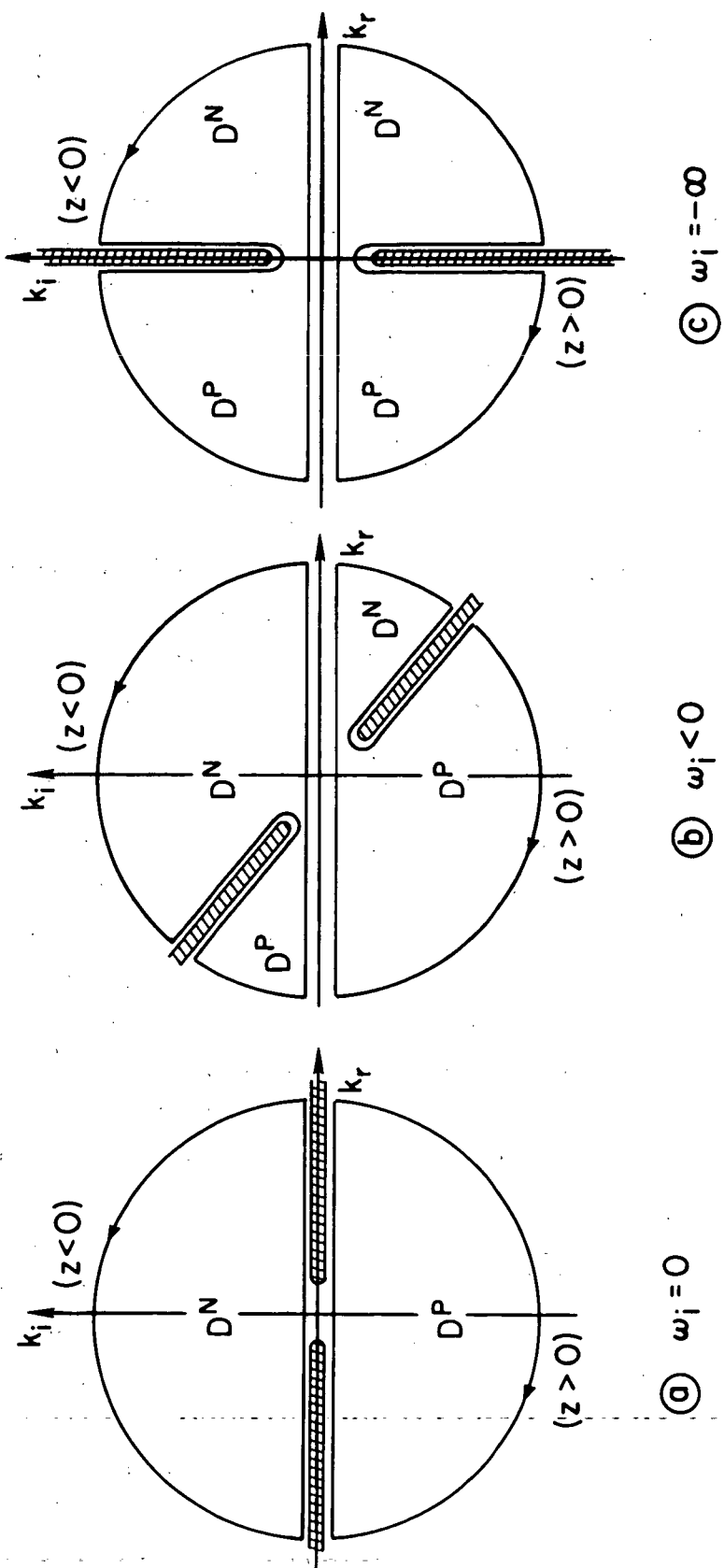


FIG. 2.2. Position of the branch cuts for  $\omega_r < \Omega$ .

$$\boxed{\omega_r > \Omega}$$

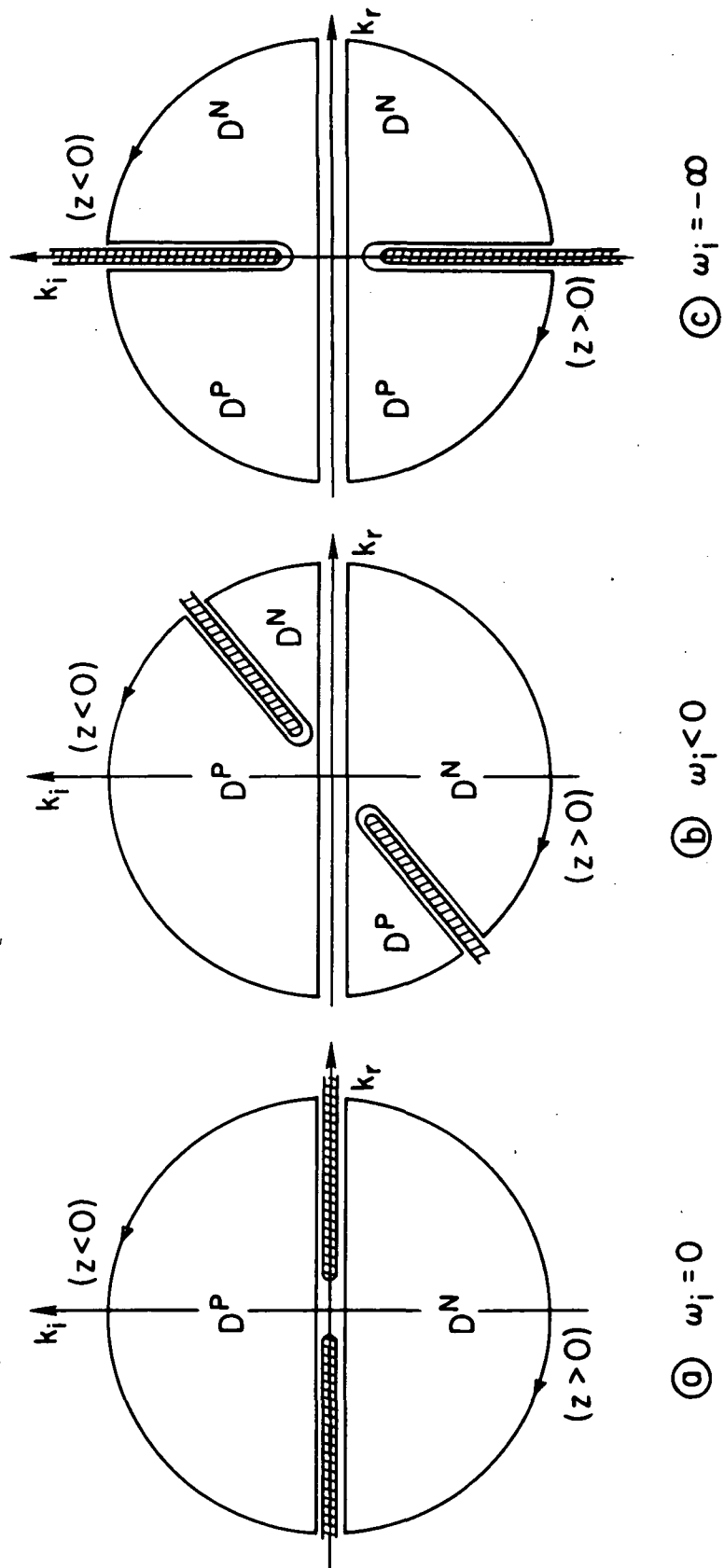


FIG. 2.3. Position of the branch cuts for  $\omega_r > \Omega$ .

of the modes, i.e. whether they are true or pseudo-modes, and whether they correspond to propagation along  $+z$  or  $-z$ , are only avoided for the case  $\omega_i \rightarrow -\infty$ . When this limiting value is reached (note that it is a valid Bromwich contour), each mode  $k(\omega = \omega_r - i\infty)$ , which is a root of  $D_c^P = 0$  or  $D_c^N = 0$ , will occupy one of the four limiting positions represented schematically in Figure 2.4. These modes depend on the dispersion relation used to determine them,  $D_c^P = 0$  or  $D_c^N = 0$ . When  $D_c^P = 0$  is used, Figure 2.4 and Figures 2.2 or 2.3 (for  $\omega_i = -\infty$ ) show that the limiting mode positions correspond to the following situations.  $M_A$  and  $M_D$  are pseudo-modes;  $M_B$  is a true mode propagating in the  $-z$  direction, and  $M_C$  is a true mode propagating in the  $+z$  direction. Similarly, if  $D_c^N = 0$  determines the limiting mode positions as  $\omega_i \rightarrow -\infty$ , we find that  $M_B$  and  $M_C$  are pseudo-modes;  $M_A$  is a true mode propagating in the  $-z$  direction, and  $M_D$  is also a true mode propagating along  $+z$ .

EXAMPLE. As an example of the procedure just described, we shall consider a separable velocity distribution,  $f_0 = f_{\perp}(v_{\perp})f_{\parallel}(v_{\parallel})$ , with  $f_{\parallel}$  given by a second-order resonance (Cauchy) distribution:

$$f_0(v_{\parallel}, v_{\perp}) = \frac{2 v_{\theta}^3}{\pi} \frac{f_{\perp}(v_{\perp})}{(v_{\parallel}^2 + v_{\theta}^2)^2}, \quad (2.21)$$

$$v_{\theta}^2 = \langle v_{\perp}^2 \rangle / 2 = \langle v^2 \rangle / 3.$$

The adoption of this distribution has the advantage of yielding algebraic dispersion relations with a finite (five) number of modes when  $D_c^{P,N} = 0$  is solved for  $k(\omega)$ ; a Maxwellian distribution would result in transcendental dispersion relations with an infinite number of modes.

Using (2.21) in (2.15), we obtain an algebraic expression for  $D_c^P$ , so that its analytic continuation is trivial,  $D_c^P \equiv D^P$ :

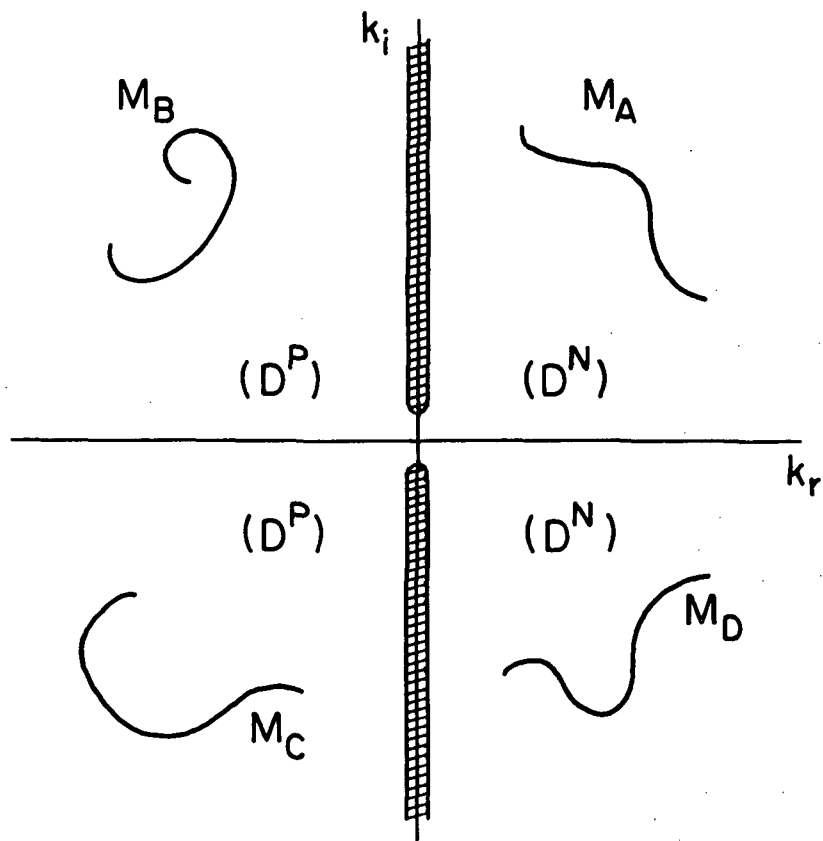


FIG. 2.4. Limiting positions of the modes  
 $k(\omega = \omega_r - i \infty)$ .

$$D_c^P = 1 - \left(\frac{K}{W}\right)^2 + \left(\frac{W_p}{W}\right)^2 \left[ -1 + \frac{i}{2\beta K} \sum_{m=0}^1 \frac{(m+1)! 2^{-m/2}}{(2^{-1/2} - i\eta)^{2-m}} + \frac{1}{4} \sum_{m=0}^1 \frac{(m+1)(2-m)! 2^{m/2}}{(2^{-1/2} - i\eta)^{m+2}} \right], \quad (2.22)$$

where the following normalized variables have been introduced  $K = kc/\Omega$ ,  $W = \omega/\Omega$ ,  $W_p = \omega_p/\Omega$ ,  $\beta = v_\theta/c$ , and  $\eta = (W-1)/2^{1/2}\beta K$ . Figures 2.5 and 2.6 show the evolution of the limiting procedure described above for  $\beta = 4.4 \times 10^{-3}$  (10 ev) and  $W_p^2 = 40$ . The five modes  $K(W)$  defined by  $D_c^P = 0$  are followed, as  $W_i \rightarrow \infty$ , until their final positions are clearly defined. As a result we find that, of the original five modes, only mode A corresponds to a true mode (propagating in the  $-z$  direction); the others are pseudo-modes. A similar situation is obtained for third and fourth order resonance velocity distributions, with only one true mode among the original  $K(W_r)$  modes. Since a Maxwellian distribution can be obtained as the limit of the  $n$ -th order resonance distribution when  $n \rightarrow \infty$  and the thermal speed is kept constant, these results stimulate the interesting speculation that, among the infinity of modes defined by a Maxwellian dispersion relation  $D_c^P = 0$ , perhaps only one is a true mode. We have not, however, investigated this point.

## 2.3 Stability Criteria

### 2.3.1 Background

A general approach to the study of linear wave stability consists of applying an impulsive excitation in time and space to the system and following its time-asymptotic response. The evolution of the system may occur in three different ways. If it is stable, the maximal amplitude of the asymptotic response at any location decreases with time and approaches zero. When this evolution does not obtain, the system is unstable and its asymptotic response may be of two distinct kinds: if the response grows in time at every point in space, precluding the attainment of a steady state, the instability is absolute or

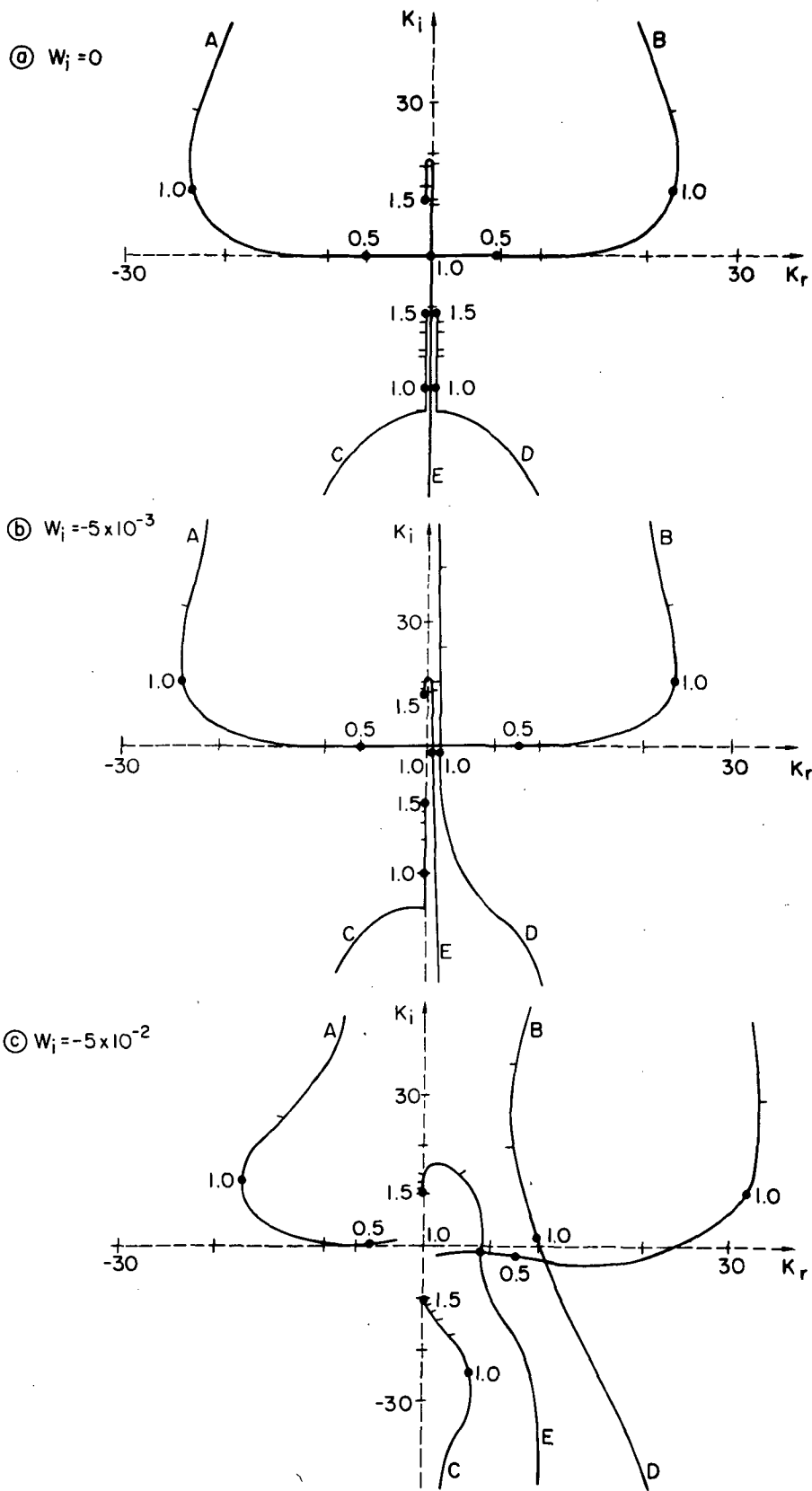


FIG. 2.5. Evolution of the  $K(W)$  modes as  $W_i \rightarrow -\infty$ . (a)  $W_i = 0$ .  
 (b)  $W_i = -5 \times 10^{-3}$ . (c)  $W_i = -5 \times 10^{-2}$ .

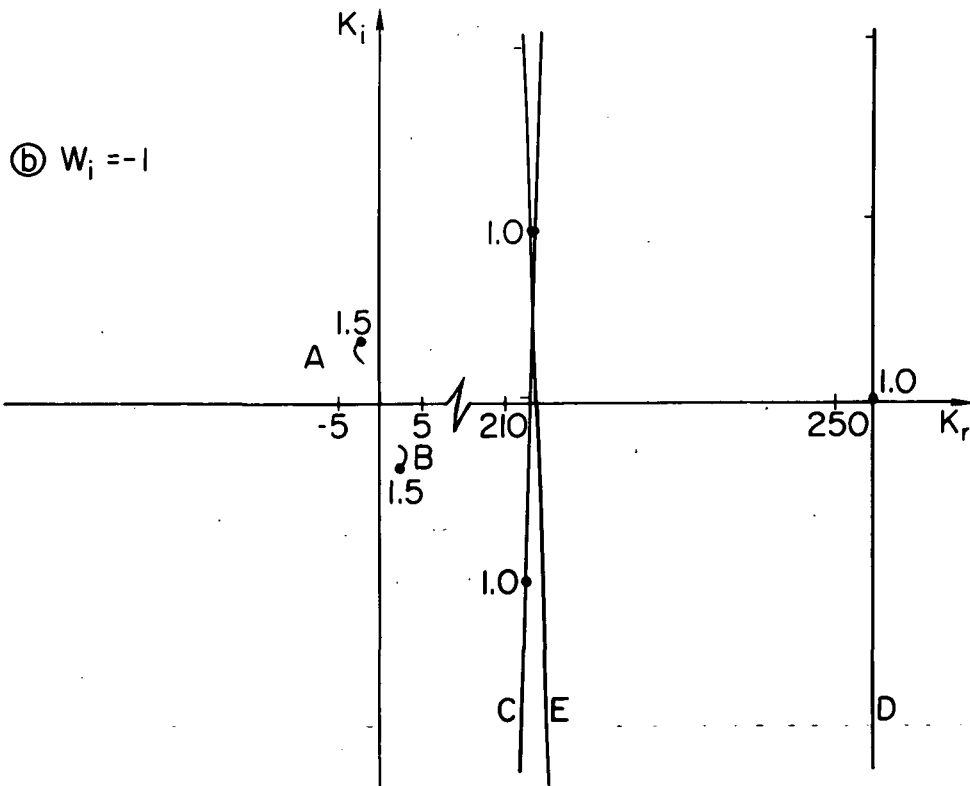
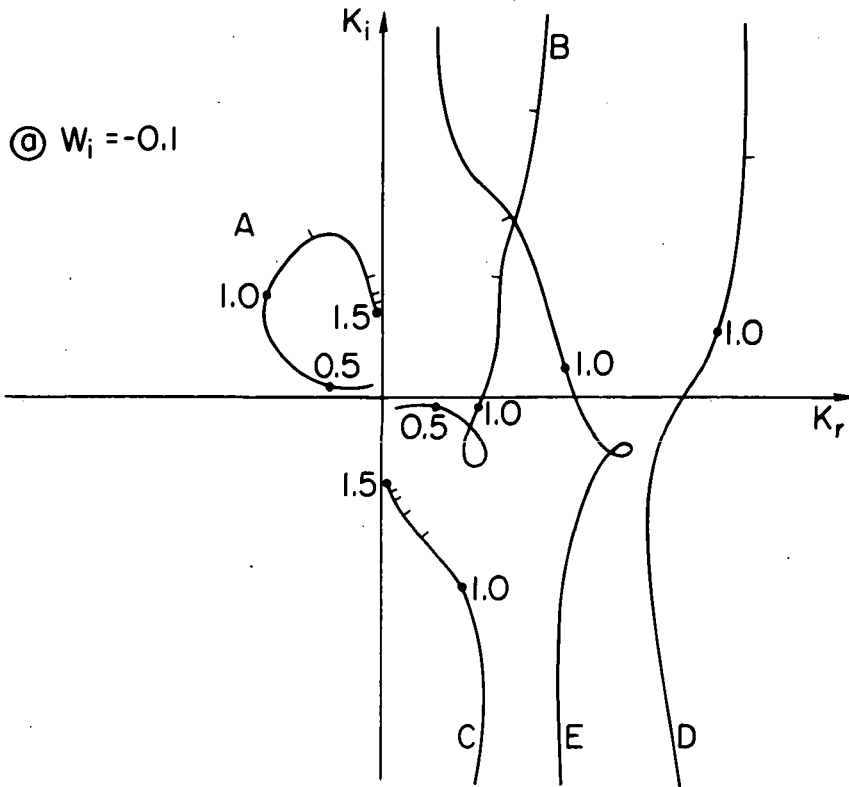


FIG. 2.6. Evolution of the  $K(W)$  modes as  $W_i \rightarrow -\infty$ . (a)  $W_i = -0.1$ .  
(b)  $W_i = -1$ .

non-convective; if the pulse disturbance propagates away from the excitation source while growing in time, eventually leaving the system quiescent at any point, the instability is convective. In contrast to the situation prevailing in absolutely unstable systems, it is possible to approach a steady state in convective instabilities in which a sinusoidal oscillation excited at a fixed location is spatially amplified.

The implementation of this approach is beset with mathematical difficulties arising from the inversion of the Fourier-Laplace transforms of the impulse response of the system. Thus it is not surprising that, although both the method and the instability classification were clearly established by Sturrock as long ago as 1958 [Sturrock, 1958], only much later did stability criteria applicable to a wide range of systems become available [Briggs, 1964; Derfler, 1967; Derfler, 1970].

Because the work of Briggs is essentially restricted to cold plasmas, we shall be concerned mainly with Derfler's approach, which is suited to hot plasma systems with dispersion relations that are multivalued functions of  $\omega$  and  $k$ . The criteria derived by Derfler were established in the context of electrostatic waves in hot plasmas and, as pointed out in Section 2.2.2, rely on the existence of certain symmetry relations that do not hold in general for parallel whistlers in hot magnetoplasmas. Our aim will be to extend Derfler's analysis to cover the general parallel whistler case. (Previous applications of Derfler's criteria to parallel whistlers [Lee and Crawford, 1970] were made for situations where the symmetry relations used by Derfler did hold, i.e. when  $f_0(v_{\parallel}, v_{\perp}) = f_0(-v_{\parallel}, v_{\perp})$ , or  $f_0$  is monochromatic in  $v_{\parallel}$ .)

### 2.3.2 Instability in General

PLASMA WAVE FUNCTION. The impulsive excitation of the system is assumed to be of the form

$$J_{xs}(z, t) = \epsilon_0 \delta'(t) \delta(z) \quad , \quad J_{xs}(k, \omega) = -i \omega \epsilon_0 \quad , \quad (2.23)$$

$$J_{ys}(z, t) = J_{ys}(k, \omega) = 0 \quad ,$$

so that the temporal Fourier transform of the response is obtained from (2.20) as

$$2\pi E_R(z, \omega') = \int_{-\infty}^0 dk \frac{\exp(-ikz)}{D^P(\omega', k)} + \int_0^{\infty} dk \frac{\exp(-ikz)}{D^N(\omega', k)}, \quad (2.24)$$

where  $\omega'$  is on the Bromwich contour. We recall that  $D^P$  and  $D^N$  are analytic in the half-planes  $\xi_i > 0$  and  $\xi_i < 0$ . They have unique analytic continuations into  $\xi_i \leq 0$  and  $\xi_i \geq 0$  that permit us to obtain  $D_c^P$  and  $D_c^N$  valid everywhere. Thus the only singularities of the functions  $1/D_c^P(\omega', k)$  and  $1/D_c^N(\omega', k)$  in the  $k$ -plane are poles (meromorphic functions). Assuming these poles to be simple (we shall remove this restriction below), we obtain new entire functions  $E^P$  and  $E^N$  by removing the terms corresponding to poles in the Laurent's series of  $1/D_c^P$  and  $1/D_c^N$ . That is

$$E^{P,N}(k) = \frac{1}{D^{P,N}(\omega', k)} - \sum_{n,m} \frac{1}{(k - k_{n,m}) D_{ck}^{P,N}(\omega', k_{n,m})}, \quad (2.25)$$

where  $D_c^{P,N}(\omega', k_{n,m}) = 0$  and  $( )_k = \partial( ) / \partial k$ . Because the entire functions  $E^{P,N}(k)$  do not have singularities at  $k = \infty$  [note that  $1/D_c^{P,N}(\omega', \infty) = 0$ ], it follows from Liouville's theorem that they are constants whose value, by inspection of (2.25), is zero. Thus we find the partial fraction expansions

$$\frac{1}{D_c^{P,N}(\omega', k)} = \sum_{n,m} \frac{1}{(k - k_{n,m}) D_{ck}^{P,N}(\omega', k_{n,m})} \quad (2.26)$$

which, when used in (2.24), yield

$$2\pi E_R(z, \omega') = \sum_n \frac{1}{D_{ck}^P(\omega', k_n)} \int_{-\infty}^0 dk \frac{\exp(-ikz)}{k - k_n} + \sum_m \frac{1}{D_{ck}^N(\omega', k_m)} \int_0^{\infty} dk \frac{\exp(-ikz)}{k - k_m}. \quad (2.27)$$

[The results obtained in Appendix B impose restrictions on the locations of  $k_n$  and  $k_m$ , i.e.  $k_n(k_m)$  cannot be on the negative (positive) real  $k$  axis,

$$\pi > \arg k_n = \varphi_n > -\pi \quad , \quad 2\pi > \arg k_m = \varphi_m > 0 \quad . \quad (2.28)]$$

Using the definition of Tricomi's confluent hypergeometric function of the second kind [Erdélyi, 1953],

$$\Psi(1,1;x) = \int_0^\infty dt \frac{\exp(-xt)}{1+t} \quad \begin{aligned} |\Delta| &< \pi \quad , \\ |\Delta + \arg x| &< \pi/2 \quad , \end{aligned} \quad (2.29)$$

we readily see that

$$I = \int_{-\infty}^0 dk \frac{\exp(-ikz)}{k - k_n} = -\Psi(1,1; -i k_n z) \quad . \quad (2.30)$$

This plasma wave function  $\Psi$  was studied extensively by Derfler [1964] and Simonen [1966] in connection with plasma wave propagation. Using  $x = R \exp iQ$ , the functions  $\Psi_r(1,1;x)$  and  $\Psi_i(1,1;x)$ , plotted as a function of  $R$ , yield (oscillatory) evanescent waves for the range  $|Q| < 3\pi/2$ , and increasing (unstable) waves otherwise. Since  $Q = \arg(-i k_n z) = \varphi_n - (\pi/2) \operatorname{sgn} z$ , the oscillatory evanescent waves occur when  $|\varphi_n| < \pi$ , the range specified in (2.28) for  $\arg k_n$ . This means that while  $\omega'$  lies on the Bromwich contour, each one of the  $I$  integrals entering into (2.27) will give rise to a stable wave. This component of the response is not the asymptotic behavior we wish to determine, however.

**TIME-ASYMPTOTIC RESPONSE.** To obtain the behavior as  $t \rightarrow \infty$  we must push  $\omega'$  towards the real axis [Briggs, 1964], and study the implications of this process of analytic continuation. As  $\omega' \rightarrow \omega'_r$ , the roots  $k_n(\omega')$  move in the  $k$ -plane. Two possible loci are sketched in Figure 2.7. The locus traced by  $k_n(\omega')$  in Case (a) always has  $|\varphi_n| < \pi$  and thus

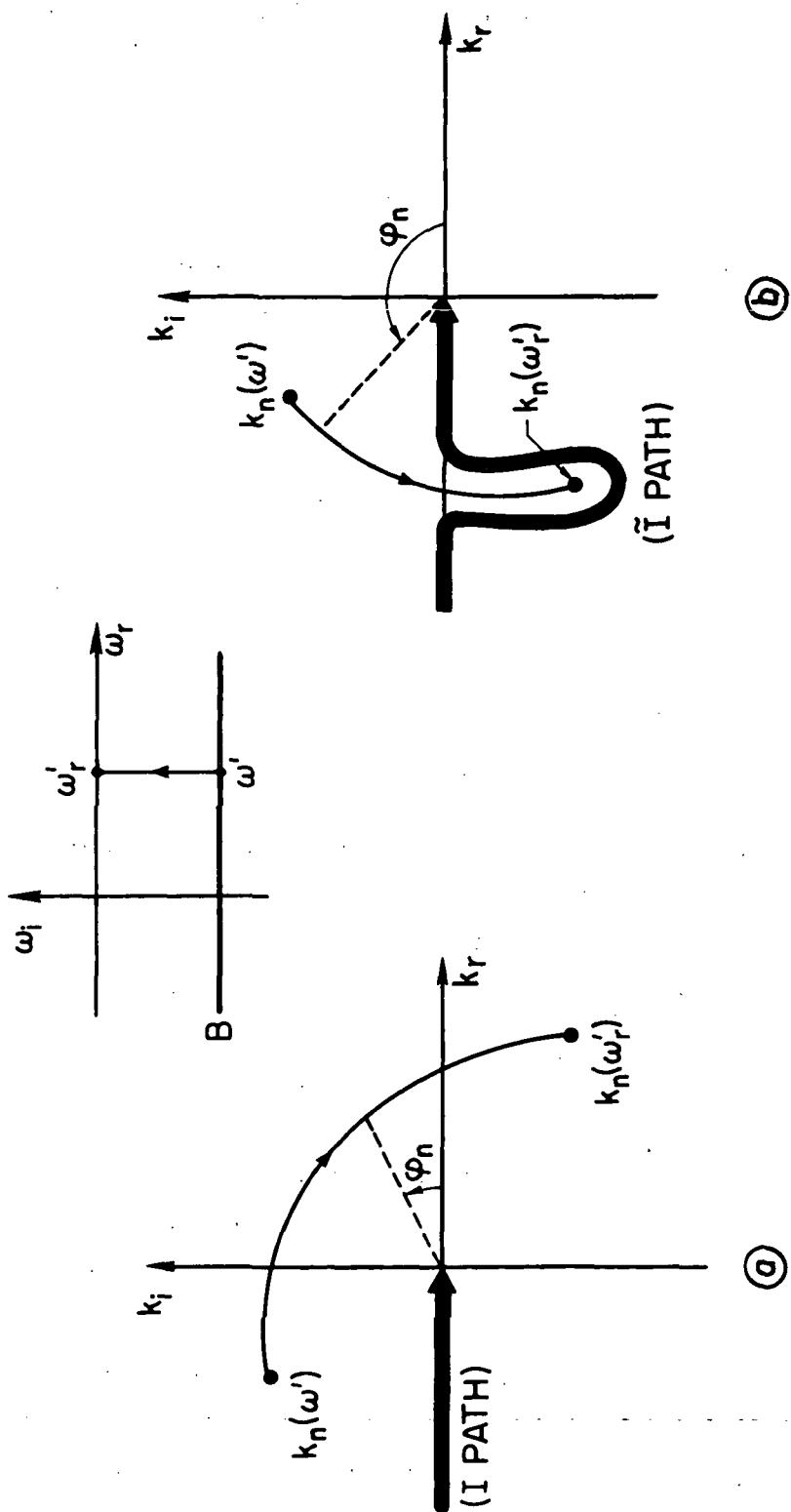


FIG. 2.7. Loci of  $k_n(\omega')$  as  $\omega' \rightarrow \omega'_r$ .

yields a stable plasma wave function. In Case (b), the locus crosses the original path of integration ( $k = k_r < 0$ ) of the quadrature defining I. We thus need to deform this original path, as indicated in Figure 2.7(b), to obtain the proper analytic continuation of I (denoted by  $\tilde{I}$ ). Because this locus has regions with  $|\varphi_n| > \pi$ , the plasma wave function increases as  $z \rightarrow -\infty$ : the system is unstable in some sense. (For a pole originally in the lower k-plane, the conclusions are identical, except that now the plasma wave function,  $\Psi$ , increases as  $z \rightarrow \infty$ .) In other words, it is sufficient for instability to occur that the poles  $k_n(\omega')$ , solutions of  $D_c^P(\omega', k_n) = 0$ , cross the negative real k-axis as  $\omega' \rightarrow \omega'_r$ . Had we worked with  $I' = \int_0^\infty dk \exp(-ikz)/(k-k_m)$ , we would have found the instability arising if  $k_m(\omega')$ , the solution of  $D_c^N(\omega', k_m) = 0$ , had crossed the positive real k-axis as  $\omega' \rightarrow \omega'_r$ . Naturally, a stable system displays the proper type of loci in the k-plane when pushed towards the real axis ( $\omega \rightarrow \omega_r$ ) for all  $\omega$  on the Bromwich contour. These results will be used at the end of Section 2.3.3 to formulate a general stability criterion.

### 2.3.3 Absolute and Convective Instabilities

The approach adopted above must be reviewed when  $D_{ck}^P(\omega', k_n)$  or  $D_{ck}^N(\omega', k_m)$  are equal to zero. The expansions used in (2.25) cease to be valid. This corresponds to double roots of the dispersion relations ( $D = D_k = 0$ ), so we must analyze further this situation. Concentrating on  $D_c^P$ , these singular points  $(\bar{\omega}, \bar{k})$  satisfy

$$\bar{D}_c^P = D_c^P(\bar{\omega}, \bar{k}) = 0, \quad \bar{D}_{ck}^P = D_{ck}^P(\bar{\omega}, \bar{k}) = 0. \quad (2.31)$$

In the vicinity of such a point, i.e. a saddle point of  $\omega(k)$  and a branch point of  $k(\omega)$ , the following Taylor expansions hold:

$$\begin{aligned} D_c^P(\omega, k) = 0 &= (\omega - \bar{\omega}) \bar{D}_{c\omega}^P + \frac{1}{2} (\omega - \bar{\omega})^2 \bar{D}_{c\omega\omega}^P + (\omega - \bar{\omega})(k - \bar{k}) \bar{D}_{c\omega k}^P \\ &+ \frac{1}{2} (k - \bar{k})^2 \bar{D}_{ckk}^P + \dots, \end{aligned} \quad (2.32)$$

$$D_{ck}^P(\omega, k) = (\omega - \bar{\omega}) \bar{D}_{c\omega k}^P + (k - \bar{k}) \bar{D}_{ckk}^P + \dots$$

Hence, neglecting higher order terms, the poles that collide to form the double root are given by

$$k_{n_{1,2}} = \bar{k} \pm i \left[ 2(\omega - \bar{\omega}) \frac{\bar{D}_{c\omega}^P}{\bar{D}_{ckk}^P} \right]^{1/2}, \quad (2.33)$$

yielding

$$D_{ck}^P(\omega, k_{n_{1,2}}) = \pm i \left[ 2(\omega - \bar{\omega}) \bar{D}_{c\omega}^P \bar{D}_{ckk}^P \right]^{1/2}. \quad (2.34)$$

From the expansion [Erdélyi, 1953] of the plasma wave function

$$\Psi(1,1;x) = -(\ln x + \gamma) \exp(-x) + \sum_{n=1}^{\infty} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \frac{x^n}{n!} \quad (2.35)$$

where  $\gamma = 0.57721\dots$  is Euler's constant, we obtain

$$\lim_{k_{n_{1,2}} \rightarrow \bar{k}} \left[ \Psi(1,1; -i k_{n_1} z) - \Psi(1,1; -i k_{n_2} z) \right] = -i(\phi_{n_1}^L - \phi_{n_2}^L) \exp(-i\bar{k}z), \quad (2.36)$$

with

$$\phi_{n_{1,2}}^L = \lim_{k_{n_{1,2}} \rightarrow \bar{k}} (\arg k_{n_{1,2}}). \quad (2.37)$$

The contribution of the two colliding poles to  $2\pi E_R(z, \omega)$  is then [see (2.27) - (2.29)]

$$\begin{aligned}
& - \frac{\Psi(1,1;-k_{n_1} z)}{D_{ck}^P(\omega, k_{n_1})} - \frac{\Psi(1,1;-ik_{n_2} z)}{D_{ck}^P(\omega, k_{n_2})} \\
& = i \left[ \Psi(1,1;-ik_{n_1} z) - \Psi(1,1;-ik_{n_2} z) \right] \left[ 2(\omega - \bar{\omega}) \bar{D}_{c\omega}^P \bar{D}_{ckk}^P \right]^{-1/2} \\
& = \left( \phi_{n_1}^L - \phi_{n_2}^L \right) \left[ 2(\omega - \bar{\omega}) \bar{D}_{c\omega}^P \bar{D}_{ckk}^P \right]^{-1/2} \exp(-ikz) , \tag{2.38}
\end{aligned}$$

and, through  $\Delta\phi^L = \phi_{n_1}^L - \phi_{n_2}^L$ , depends on how the collision of the poles took place. Figure 2.8 shows the two possibilities. In Case (a) the colliding poles coalesce without pinching the negative real  $k$ -axis, so that  $\Delta\phi^L = 0$ ; the contributions of the two poles to  $E_R(z, \omega)$  cancel each other. In Case (b) the poles merge across  $k = k_r < 0$ , precluding the analytic continuation of  $I$  by deformation of the integration path [as in Figure 2.7(b)]. This yields  $\Delta\phi^L = 2\pi$ . The asymptotic response due to the double root, i.e. the inverse Fourier transform of  $E_R(z, \omega)$ , will satisfy

$$E_R(z, t) \propto \exp i(\bar{\omega}t - \bar{k}z) , \tag{2.39}$$

and hence will give rise to an absolute instability if  $\bar{\omega}_i < 0$ . Working with  $D_c^N$  and the  $k_m$  poles yields identical conclusions when the collision pinches the positive real  $k$ -axis.

SUMMARY. Given the dispersion functions  $D_c^P(\omega, k)$  and  $D_c^N(\omega, k)$ , the formulation of the generalized (colliding pole) stability criterion for parallel whistlers can thus be made as follows:

- (i) Using  $D_c^P(\omega, k) = 0$ , map the negative real  $k$ -axis into the  $\omega$ -plane;
- (ii) using  $D_c^N(\omega, k) = 0$ , map the positive real  $k$ -axis into the  $\omega$ -plane.

If in neither (i) nor (ii) does the mapping enter the half plane  $\omega_i < 0$ , the system is stable; otherwise it is unstable. If unstable,

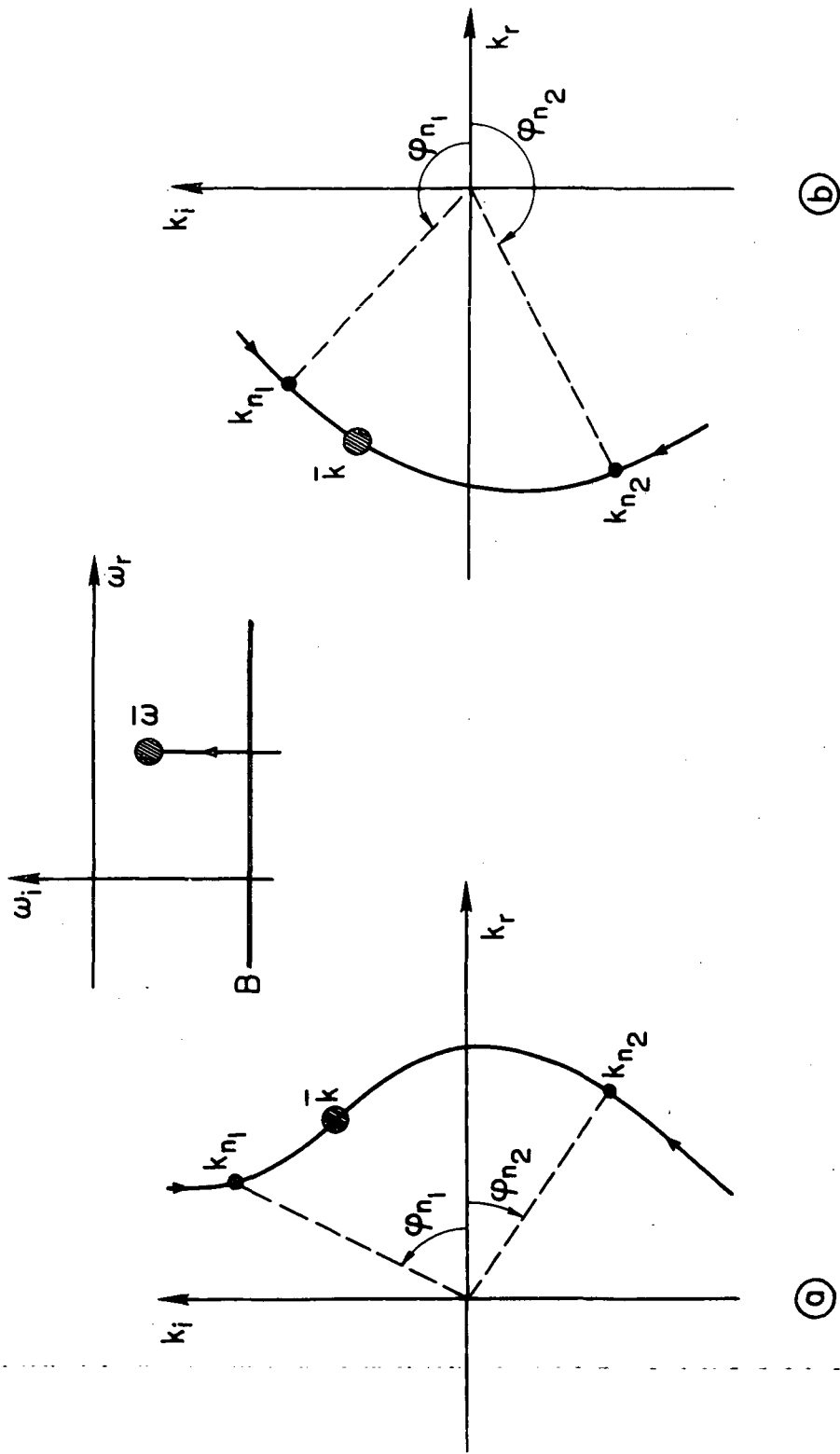


FIG. 2.8. Coalescence of colliding poles as  $\omega' \rightarrow \bar{\omega}$ .

select a Bromwich contour  $\omega_i < -\max(\alpha, \beta) < 0$  so that  $D_c^P(\omega, k) \neq 0$  for all negative real  $k$ , and  $D_c^N(\omega, k) \neq 0$  for all positive real  $k$ . Start a process of analytic continuation towards real  $\omega$ , and observe

- (iii) the coalescence of poles  $D_c^P(\omega, k_{n_1}) = 0$  and  $D_c^P(\omega, k_{n_2}) = 0$  at saddle points  $k = \bar{k}_P(\bar{\omega}_P)$ ;
- (iv) the coalescence of poles  $D_c^N(\omega, k_{m_1}) = 0$  and  $D_c^N(\omega, k_{m_2}) = 0$  at saddle points  $k = \bar{k}_N(\bar{\omega}_N)$ .

Absolute instability arises if, and only if, two poles collide in (iii) across the negative real  $k$ -axis, with the corresponding branch point having  $\bar{\omega}_{Pi} < 0$ , and/or two poles collide in (iv) across the positive real  $k$ -axis, with the corresponding branch point having  $\bar{\omega}_{Ni} < 0$ . Otherwise, the (unstable) system supports convective instabilities.

It is important to note that in general (i) and (ii), or (iii) and (iv), are not equivalent. However, since the symmetry relation (2.17) holds, it is always possible to check the stability of the system using only one of the two dispersion relations. When the symmetry relations are stronger, as given in (2.18), the mappings (i) and (ii), or (iii) and (iv), are equivalent. In this case, the stability criteria formulated by Derfler [Derfler, 1967; Derfler, 1970] are valid for the parallel whistler.

### 3. OBLIQUELY PROPAGATING WHISTLERS

#### 3.1 Background

This chapter analyzes the linear stability of whistlers with arbitrary frequency and direction of propagation in a cold plasma permeated by a dilute energetic electron population. This approaches conditions typical of the magnetosphere. The derived dependences on whistler obliquity of the growth rate and strength of the two most important wave-particle interactions (Landau and fundamental cyclotron), are relevant to the nonlinear problem to be studied in Chapter 4: VLF emissions stimulated by whistlers. The results obtained are applicable to the triggering and triggered waves, and suggest a possible mechanism for the emission onset.

Among the considerable literature on whistler stability analysis [Sagdeev and Shafranov, 1961; Scarf, 1962; Tidman and Jaggi, 1962; Noerdlinger, 1963; Sudan, 1963; Bell and Buneman, 1964; Kennel, 1966; Kennel and Thorne, 1967; Liemohn, 1967; Scharer and Trivelpiece, 1967; Liemohn, 1969; Lee and Crawford, 1970], only a few papers [Kennel, 1966; Kennel and Thorne, 1967] consider propagation at an angle to the static magnetic field (oblique propagation). This imbalance may be explained by the fact that whistlers recorded on the ground indicate the occurrence of trapping in ducts of enhanced ionization, i.e. wavenormal orientations within a relatively narrow cone aligned with the geomagnetic field line path, and by the substantial simplification brought about in the analysis by the assumption of parallel propagation.

Satellite observations of whistlers have shown, however, that ducted whistlers are the exception, rather than the rule [Walter and Angerami, 1969], and indicate the necessity of studying oblique whistler stability: the limitations on wavenormal orientation do not occur for nonducted propagation.

The studies of Kennel [1966], and Kennel and Thorne [1967], were restricted to low-frequency ( $\omega \ll \Omega$ , where  $\Omega$  is the electron cyclotron frequency) and some of the conclusions derived in this context cannot be extended to high-frequency whistlers. In particular, the existence of a maximum in the growth rate for parallel propagation is not general.

The work to be described here is of interest to the problem of artificially stimulated emissions because recent satellite observations [Angerami and Bell, 1971] have shown that nonducted, oblique, whistlers are able to stimulate such emissions. The choice of parameters for calculations in this chapter will reflect clearly this interest. The region that we wish to model lies in the vicinity of the magnetospheric equatorial plane, at a geocentric distance between three and four earth radii ( $3 < L < 4$ ). The cold plasma consists of electrons and protons with densities such that  $\omega_p/\Omega \sim 10$ , where  $\omega_p$  is the electron plasma frequency. The characteristics of the assumed energetic population take into consideration that the region of interest is contained within the Van Allen zone of stably trapped fluxes of relatively high-energy electrons [Kennel and Thorne, 1967], and the choice of the whistler frequency,  $\omega/\Omega \sim 0.5$ , also reflects our motivation [Angerami and Bell, 1971; Carpenter et al., 1969]. Nevertheless, in the presentation of the analytic results, we have avoided using simplifications justified by this particular situation, so that the derived expressions, with appropriate parametrization, may be used in other domains of oblique whistler stability.

In obtaining the whistler growth rates we have used an energy conservation approach involving the resonant particles and the wave. This method is not valid in general because it neglects the wave-particle momentum conservation and the energy exchange with the nonresonant particles [Hollweg and Völk, 1971]. However, for the situation analyzed in this chapter (cold plasma permeated by a dilute energetic electron population), the energy conservation approach is both appropriate, reproducing the results derived from the dispersion relation, and desirable, using quantities with familiar physical meanings to study the oblique whistler stability.

Section 3.2 casts the discussion of the exact expressions for the growth rates in terms of the power dissipated by the oblique whistler in the propagating medium. Section 3.3 analyzes the characteristics of the two (cold and nonthermal) plasma populations which influence the wave stability. Section 3.4 assesses the importance of the wave-particle

resonances, and discusses in detail the Landau and fundamental gyro interactions. Section 3.5 presents numerical results for the spatial growth rates and derives a criterion for the existence of maximal growth at parallel propagation. Section 3.6 discusses the possible relevance of the derived results to artificially stimulated emissions, and so provides an introduction to Chapter 4.

### 3.2 Growth Rate: Theory

**GROWTH RATE VIA DISPERSION RELATION.** The whistler mode of angular frequency  $\omega$  is assumed to propagate in a uniform, time-invariant, collisionless, magnetoplasma composed of a dominant, two-species (electrons and ions), cold plasma permeated by a tenuous energetic electron population; the static magnetic field direction defines the positive z-axis, and the whistler wavenumber,  $\underline{k}$ , lies in the x-z plane at an angle  $\theta$  to the unit vector  $\hat{z}$ . The electron and ion number densities,  $(1-\delta)n_0$  and  $n_0$ , together with the electron gyrofrequency defined by the static magnetic field,  $\Omega = eB_0/m_e > 0$ , characterize the cold magnetoplasma. The hot electron population has a number density  $\delta n_0$  and an equilibrium velocity distribution  $f(v_{\parallel}, v_{\perp})$ , normalized to unity. Plane wave propagation as  $\exp i(\omega t - \underline{k} \cdot \underline{r})$  is assumed, and the theoretical possibility of having a non-convective instability [Lee and Crawford, 1970] can be excluded for the parameters of interest to us.

Application of Fourier (space) and Laplace (time) transforms to the linearized Maxwell-Vlasov system of equations leads to the following wave equation for the electric field in cartesian coordinates, neglecting the contribution of the initial-value terms:

$$\vec{M} \cdot \underline{E} = 0, \quad \vec{M} = \vec{K} + \frac{k^2 c^2}{\omega^2} \begin{bmatrix} -\cos^2 \theta & 0 & \sin \theta \cos \theta \\ 0 & -1 & 0 \\ \sin \theta \cos \theta & 0 & \sin^2 \theta \end{bmatrix}, \quad (3.1)$$

where  $c$  is the free space speed of light and  $\vec{K}$  is the plasma permittivity tensor defined by the transformed Maxwell's equation

$\underline{k} \times \underline{H} = -\epsilon_0 \omega \vec{\underline{K}} \cdot \underline{E}$ . The components of this tensor are derivable from the results of Kennel [1966]. We obtain,

$$\begin{aligned}
K_{xx} &= S + \frac{\delta}{4} \Gamma (J_{m-1} + J_{m+1})^2 G_1, \\
K_{xy} &= -K_{yx} = i D + i \frac{\delta}{4} \Gamma (J_{m-1}^2 - J_{m+1}^2) G_1, \\
K_{xz} &= -\frac{\delta}{2} \Gamma (J_{m-1} + J_{m+1}) G_2, \\
K_{yy} &= S + \frac{\delta}{4} \Gamma (J_{m-1} - J_{m+1})^2 G_1, \\
K_{yz} &= i \frac{\delta}{2} \Gamma (J_{m-1} - J_{m+1}) G_2, \\
K_{zx} &= -\frac{\delta}{2} \Lambda J_m (J_{m-1} + J_{m+1}) G_1, \\
K_{zy} &= -i \frac{\delta}{2} \Lambda J_m (J_{m-1} - J_{m+1}) G_1, \\
K_{zz} &= P + \delta \Lambda J_m G_2,
\end{aligned} \tag{3.2}$$

where the Bessel functions  $J_n$  of integer order  $n$  have argument  $(k_{\perp} v_{\perp} / \Omega)$ ; we have used the cold plasma parameters introduced by Stix [1962],

$$\begin{aligned}
R &= 1 - \frac{\alpha \omega_p^2}{\omega(\omega + \Omega)} - \frac{(1-\delta)\omega_p^2}{\omega(\omega - \Omega)}, \quad L = 1 - \frac{\alpha \omega_p^2}{\omega(\omega - \Omega)} - \frac{(1-\delta)\omega_p^2}{\omega(\omega + \Omega)}, \\
S &= (R+L)/2, \quad D = (R-L)/2, \\
P &= 1 - (1+\alpha-\delta) \omega_p^2 / \omega^2, \quad C = PRL, \\
A &= S \sin^2 \theta + P \cos^2 \theta, \quad B = RL \sin^2 \theta + PS (1 + \cos^2 \theta),
\end{aligned} \tag{3.3}$$

and introduced the following quantities,

$$\Gamma = -\frac{2\pi\omega^2}{\omega_k} \sum_{\parallel} \int_0^{\infty} dv_{\perp} v_{\perp}^2 \int_{-\infty}^{\infty} dv_{\parallel} \frac{1}{v_{\parallel} - V_m} ,$$

$$\Lambda = -\frac{2\pi\omega^2}{\omega_k} \sum_{\parallel} \int_0^{\infty} dv_{\perp} v_{\perp} \int_{-\infty}^{\infty} dv_{\parallel} \frac{v_{\parallel}}{v_{\parallel} - V_m} ,$$

$$G_1 = \left(1 - \frac{k_{\parallel} v_{\parallel}}{\omega}\right) F_{\perp} + \frac{k_{\parallel} v_{\perp}}{\omega} F_{\parallel} ,$$

$$G_2 = J_m \left[ \left(1 + \frac{m\Omega}{\omega}\right) F_{\parallel} - m \frac{\Omega v_{\parallel}}{\omega v_{\perp}} F_{\perp} \right] , \quad (3.4)$$

$$n = \mu - i\chi = \frac{ck}{\omega} , \quad \omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m_e} , \quad V_m = \frac{\omega + m\Omega}{k_{\parallel}} ,$$

$$F_{\parallel} = \frac{\partial f}{\partial v_{\parallel}} , \quad F_{\perp} = \frac{\partial f}{\partial v_{\perp}} .$$

$\alpha$  stands for the electron-ion mass ratio, and the symbols  $(\parallel)$  and  $(\perp)$  refer to the direction of the static magnetic field.

The components of this permittivity tensor, (3.2), do not satisfy, in general, the Onsager reciprocal relations [De Groot, 1951]. Manipulation of (3.2) and (3.4) shows that the symmetry properties ( $K_{xy} = -K_{yx}$ ,  $K_{xz} = K_{zx}$ ,  $K_{yz} = -K_{zy}$ ) are recovered when the hot electron distribution satisfies  $I = \int_{-\infty}^{\infty} v_{\parallel} f(v_{\parallel}, v_{\perp}) dv_{\parallel} = 0$ . Recalling that the Onsager relations are based on the symmetry of the mechanical equations of motion of individual particles with respect to time, the condition  $I = 0$  has a clear physical meaning: the static magnetic field caused by a net parallel drift of the hot electron population,  $I \neq 0$ , hinders the microscopic reversibility of the particle's motion. (The role of the main static magnetic field is taken into account in the usual formulation of the symmetry properties [Allis et al., 1963].)

The electron distributions considered in the following are even in  $v_{\parallel}$ , so that  $I = 0$ . This assumption simplifies the expressions for the components of the anti-hermitian part of  $\vec{K}$ , (3.20), but does not alter the expressions derived for the power dissipation, (3.33), and growth rate, (3.31), of the whistler.

The dispersion relation for electromagnetic waves is obtained from (3.1),

$$D(\omega, \underline{k}) = |\underline{\tilde{M}}| = 0, \quad (3.5)$$

and can be expanded in powers of the small parameter  $\delta$ , yielding

$$D(\omega, \underline{k}) \approx D_0(\omega, \underline{k}) + \delta D_1(\omega, \underline{k}) + \delta^2 D_2(\omega, \underline{k}) + \dots, \quad (3.6)$$

where  $D_0(\omega, \underline{k}) = A n^4 - B n^2 + C$ . For a dilute energetic population ( $\delta \ll 1$ ), the cold plasma dispersion relation will suffice to give the real parts of the frequency and wavenumber. Since we are only interested in this situation, from here on the real part of the refractive index,  $\mu$ , will be taken as the whistler root of  $D_0 = 0$ , given by

$$\mu^2 = [-B - (B^2 - 4AC)^{1/2}] / 2A. \quad (3.7)$$

The evaluation of the growth rates can be performed as in Kennel [1966] by expanding (3.6) about the cold plasma solution, and neglecting terms of order higher than  $\delta$ . Because the energetic population is considered to be small, the dispersion relation is satisfied approximately by real frequencies and wavenumbers ( $|\omega_r| \gg |\omega_i|, |\underline{k}_r| \gg |\underline{k}_i|$ ). The imaginary parts are then related through

$$\omega_i = \left[ \frac{\partial D}{\partial \omega} \right]^{-1} \underline{k}_i \cdot \frac{\partial D}{\partial \underline{k}} \approx \left[ \frac{\partial D_0}{\partial \omega} \right]^{-1} \underline{k}_i \cdot \frac{\partial D_0}{\partial \underline{k}} = - \underline{k}_i \cdot \underline{v}_g, \quad (3.8)$$

where  $\underline{v}_g$  is the cold plasma group velocity of the oblique whistler. The spatial growth rate,  $\underline{k}_i$ , is obtained by Taylor expanding  $D_0(\omega, \underline{k})$  in (3.6),

$$D(\omega_r, \underline{k}_r + i \underline{k}_i) \approx D_0 + i \underline{k}_i \cdot \frac{\partial D_0}{\partial \underline{k}} + \delta D_{1r} + i \delta D_{1i} \approx 0, \quad (3.9)$$

with the second member evaluated at  $(\omega_r, \underline{k}_r)$ , and taking the imaginary part,

$$\underline{k}_i \cdot \frac{\partial D_0}{\partial \underline{k}} = - \delta D_{1i}. \quad (3.10)$$

This result may be written as

$$k_i = - \frac{\delta D_{li}}{\sin \theta \frac{\partial D_0}{\partial k_i} + \cos \theta \frac{\partial D_0}{\partial k_{\parallel}}} , \quad (3.11)$$

so that,

$$\chi = - \frac{ck_i}{\omega} = \frac{\delta D_{li}}{2 \mu (2A \mu^2 - B)} . \quad (3.12)$$

**GROWTH RATE VIA DISSIPATION.** It will be demonstrated that the explicit form of  $D_{li}$  is rather complicated. Hence, it seems desirable to cast the stability discussion in terms of quantities with a more transparent physical meaning. Following Allis et al. [1963], and Bekefi [1966], we shall use an energy conservation relation for first order quantities derived from the transformed Maxwell's equations. Allowing for small imaginary parts in  $\omega$  and  $\underline{k}$ , and using the expansion

$$\omega \underline{\underline{K}}(\omega, \underline{k}) \approx \omega \underline{\underline{K}}_r + i k_i \cdot \frac{\partial(\omega \underline{\underline{K}})}{\partial \underline{k}} + i \omega_i \frac{\partial(\omega \underline{\underline{K}})}{\partial \omega} , \quad (3.13)$$

with the right-hand side evaluated at  $(\omega_r, \underline{k}_r)$ , we obtain

$$\underline{k}_i \cdot (\underline{F}_{EM} + \underline{F}_P) = \omega_i (U_M + U_{E+P}) - \frac{1}{2} P_D , \quad (3.14)$$

where we have

$$\begin{aligned} \underline{F}_{EM} &= \frac{1}{2} (\underline{E} \times \underline{H}^*)_r , \quad \underline{F}_P = - \frac{1}{4} \epsilon_0 \underline{E}^* \cdot \frac{\partial(\omega \underline{\underline{K}}^h)}{\partial \underline{k}} \cdot \underline{E} , \\ U_M &= \frac{1}{4} \mu_0 |\underline{H}|^2 , \quad U_{E+P} = \frac{1}{4} \epsilon_0 \underline{E}^* \cdot \frac{\partial(\omega \underline{\underline{K}}^h)}{\partial \omega} \cdot \underline{E} , \\ P_D &= - \frac{1}{2} \epsilon_0 \underline{E}^* \cdot (\omega \underline{\underline{K}}^a) \cdot \underline{E} , \end{aligned} \quad (3.15)$$

$$2 \underline{\underline{K}}^h = \underline{\underline{K}} + \underline{\underline{K}}^\dagger , \quad 2i \underline{\underline{K}}^a = \underline{\underline{K}} - \underline{\underline{K}}^\dagger ,$$

with  $\dagger$  denoting the complex conjugate of the transpose (hermitian conjugate).

The terms involved in the energy relation (3.14) are interpreted as follows [Bekefi, 1966]:  $\underline{F}_{EM}$ , is the time averaged Poynting flux, representing the flow of electromagnetic energy;  $\underline{F}_p$ , stands for non-electromagnetic flux, due to the energy of the particles;  $U_M$ , is the time averaged magnetic energy density;  $U_{E+P}$  represents the sum of the electrical energy density and that part of the kinetic energy of the charged particles which is excited by the wave;  $P_D$ , as shown below, is the time averaged power dissipated by the wave in the medium. Indeed, denoting the (transformed) current density by  $\underline{J}$ , and using the conductivity tensor  $\underline{\sigma} = i\omega\epsilon_0 (\underline{\underline{K}} - \underline{\underline{I}})$ , so that  $\underline{J} = \underline{\sigma} \cdot \underline{E}$ , we have

$$P_D = \langle \underline{J} \cdot \underline{E} \rangle = \frac{1}{4} (\underline{J} \cdot \underline{E}^* + \underline{J}^* \cdot \underline{E}) = \frac{1}{4} [\underline{E}^* \cdot \underline{\sigma} \cdot \underline{E} + (\underline{\sigma} \cdot \underline{E})^* \cdot \underline{E}] , \quad (3.16)$$

or, since  $(\underline{\sigma} \cdot \underline{E})^* \cdot \underline{E} = \underline{E}^* \cdot \underline{\sigma}^\dagger \cdot \underline{E}$ , we may write

$$P_D = \frac{1}{2} \underline{E}^* \cdot \underline{\sigma}^h \cdot \underline{E} , \quad (3.17)$$

which agrees with the former expression for  $P_D$ , noting that  $\underline{\sigma}^\dagger = -i\omega\epsilon_0 \underline{\underline{K}}^\dagger + i\omega\epsilon_0 \underline{\underline{I}}$ , i.e.,  $\underline{\sigma}^h = -\omega\epsilon_0 \underline{\underline{K}}^a$ .

To proceed, we need explicit expressions for the hermitian ( $\underline{\underline{K}}^h$ ) and antihermitian ( $i\underline{\underline{K}}^a$ ) parts of the permittivity tensor. They can be obtained by noting that the integrals over  $v_{\parallel}$  which occur in the components of  $\underline{\underline{K}}$  may be evaluated through the use of Dirac's relation (3.18) [Kennel, 1966]; that our assumption of  $\delta \ll 1$  implies  $|\omega_i| \ll |\omega_r|$ , and that causality in the Laplace transform for our phase convention requires  $\omega_i < 0$ , i.e., in the initial value problem,  $V_{mi} \leq 0$ , and we have approximately

$$\int_{-\infty}^{\infty} \frac{u(v_{\parallel})}{v_{\parallel} - V_m} dv_{\parallel} = \oint_{-\infty}^{\infty} \frac{u(v_{\parallel})}{v_{\parallel} - V_m} dv_{\parallel} - i\pi u(V_m) , \quad V_{mi} < 0 , \quad (3.18)$$

where  $\oint$  denotes the principal part in the Cauchy sense, and the half plane  $V_{mi} > 0$  can be explored by analytic continuation using Landau's method [Stix, 1962].

Utilization of this result in the components of  $\vec{\underline{K}}$ , and application of the definitions of  $\vec{\underline{K}}^h$  and  $\vec{\underline{K}}^a$ , yields

$$\vec{\underline{K}}^h = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix}, \quad (3.19)$$

and, for hot electron distributions even in  $v_{\parallel}$ :

$$\begin{aligned} K_{xx}^a &= \frac{\delta}{4} \Gamma_I (J_{m-1} + J_{m+1})^2 G_1, \\ K_{xy}^a &= -K_{yx}^a = i \frac{\delta}{4} \Gamma_I (J_{m-1}^2 - J_{m+1}^2) G_1, \\ K_{xz}^a &= K_{zx}^a = -\frac{\delta}{2} \Gamma_I (J_{m-1} + J_{m+1}) G_2, \\ K_{yy}^a &= \frac{\delta}{4} \Gamma_I (J_{m-1} - J_{m+1})^2 G_1, \\ K_{yz}^a &= -K_{zy}^a = i \frac{\delta}{2} \Gamma_I (J_{m-1} - J_{m+1}) G_2, \\ K_{zz}^a &= \delta \Lambda_I J_m G_2, \end{aligned} \quad (3.20)$$

where we have

$$\begin{pmatrix} \Gamma_I \\ \Lambda_I \end{pmatrix} = \frac{2\pi^2 \omega_p^2}{\omega k_{\parallel}} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dv_{\perp} \begin{pmatrix} v_{\perp}^2 \\ 1 \\ v_{\perp} \end{pmatrix} \int_{-\infty}^{\infty} dv_{\parallel} \begin{pmatrix} 1 \\ v_{\parallel} \end{pmatrix} \delta(v_{\parallel} - v_m), \quad (3.21)$$

and we have neglected the contribution of the dilute energetic population to  $\vec{\underline{K}}^h$ , thus retrieving the cold plasma permittivity tensor.

The power dissipated by the wave in the plasma,  $P_D$ , is now determined from (3.15) and (3.20). We find

$$\begin{aligned} P_D &= -\frac{\delta \omega \epsilon_0}{4} |E_y|^2 \frac{(S - \mu^2)}{D^2 (\mu^2 \sin^2 \theta - P)} \left\{ \frac{\mu^2 \sin^2 \theta - P}{2(S - \mu^2)} \Gamma_I [(R - \mu^2) J_{m-1} + (L - \mu^2) J_{m+1}]^2 G_1 \right. \\ &\quad \left. - 2[(S - \mu^2 \cos^2 \theta)(S - \mu^2) - D^2] \Lambda_I J_m G_2 - \mu^2 \sin \theta \cos \theta \Gamma_I [(R - \mu^2) J_{m-1} + (L - \mu^2) J_{m+1}] G_2 \right\}. \end{aligned} \quad (3.22)$$

An equivalent expression for  $P_D$  is

$$P_D = - \frac{\delta\omega\epsilon_0}{2} |E_y|^2 \frac{S - \mu^2}{D^2(\mu^2 \sin^2\theta - P)} D_{li} , \quad (3.23)$$

where  $D_{li}$  coincides with the expression given by Kennel [1966] when the contributions of the hot ion population to his equation (3.9) are neglected.

It is clear from (3.22) that the dissipation depends upon the interactions of the oblique whistler with energetic electrons whose parallel velocities are equal to  $V_m$ . The resonances for  $m = 0$ ,  $V_L = \omega/k_{\parallel}$ , and  $m = -1$ ,  $V_G = (\omega - \Omega)/k_{\parallel}$ , are the familiar Landau and fundamental gyro interactions. For energetic electron distributions decreasing monotonically with parallel energy, they are expected to yield the most important contributions, since they involve the less energetic, and hence more numerous, particles. Note that the Landau resonance ( $m=0$ ) is non-existent for parallel whistlers ( $\theta=0$ ) because the wave is then purely transverse, and has no component of the electric field along  $\underline{B}_0$ . The other interactions ( $m \neq 0, -1$ ) come about due to the spatial variations of the wave fields in planes perpendicular to  $\underline{B}_0$ , and also disappear for parallel propagation.

The temporal growth rate for the initial value problem is obtained from (3.14) by putting  $k_i = 0$ . This yields

$$\omega_i = \frac{P_D}{2(U_M + U_{E+P})} . \quad (3.24)$$

In the plasma being considered, the energy density,  $U_T = U_M + U_{E+P}$ , is essentially determined by the cold plasma propagation characteristics [see (3.28) and (3.29)], and is a positive definite quantity, in contrast to other active media that might display negative energy densities [Bekefi, 1966]. It follows that whistlers will experience amplification ( $\omega_i < 0$ ), or attenuation ( $\omega_i > 0$ ), according to the sign of the dissipation  $P_D$ .

Although the existence of the hot electron population is a requisite for the growth process, its properties are by no means solely determined by the energetic distribution. Inspection of the expressions

entering into the growth rate shows that its behavior will depend on both the cold and hot plasma characteristics. Before embarking on a detailed discussion of whistler stability, we shall consequently review some relevant aspects of these two components of the plasma.

### 3.3 Plasma Properties

**COLD PLASMA.** The cold plasma will determine the gross features of the whistler mode. Of particular interest to oblique propagation is the existence of a resonance cone of semi-aperture  $\theta_c$ , aligned with  $\underline{B}_0$  (see Figure 3.1); oblique whistlers whose wave normals lie outside this cone ( $|\theta| > \theta_c$ ) are evanescent waves and are not considered in our discussion [Helliwell, 1965]. The critical angle  $\theta_c$  is defined by  $\tan^2 \theta_c = -P/S$  and, for whistler frequencies below the lower hybrid resonance, is imaginary, meaning that the whistler refractive index surface becomes closed and all wave normal orientations yield propagating waves. Our interest lies in frequencies where  $\theta_c$  is real and can be approximated by  $\theta_c = \cos^{-1}(\omega/\Omega)$ ; unless specified otherwise, the results presented later will refer to wave and plasma frequencies of  $\omega/\Omega = 0.5$  and  $\omega_p/\Omega = 10$ .

The growth rates will be directly affected by the cold plasma characteristics through several factors that can be readily associated with physical properties of the propagation, namely, the whistler polarization, the resonant and group velocities, and the energy density. The weighting coefficients determining the partial contribution of each interaction to the total dissipation, though related to the previous factors, are better displayed graphically and will be analyzed later.

The wave polarization is deduced from (3.1), using the cold plasma value for  $\vec{M}$ , i.e., with  $\vec{K} = \vec{K}^h$ . The whistler fields available for interaction with the energetic electrons then satisfy

$$\frac{E_x}{E_y} = -i \frac{S-\mu^2}{D}, \quad \frac{E_z}{E_y} = \frac{E_x}{E_y} \frac{\mu^2 \sin\theta \cos\theta}{\mu^2 \sin^2\theta - P}, \quad (3.25)$$

and the magnitudes of these ratios are shown in Figure 3.1 as a function of  $\theta$ . Both ratios increase steadily as the resonance cone is

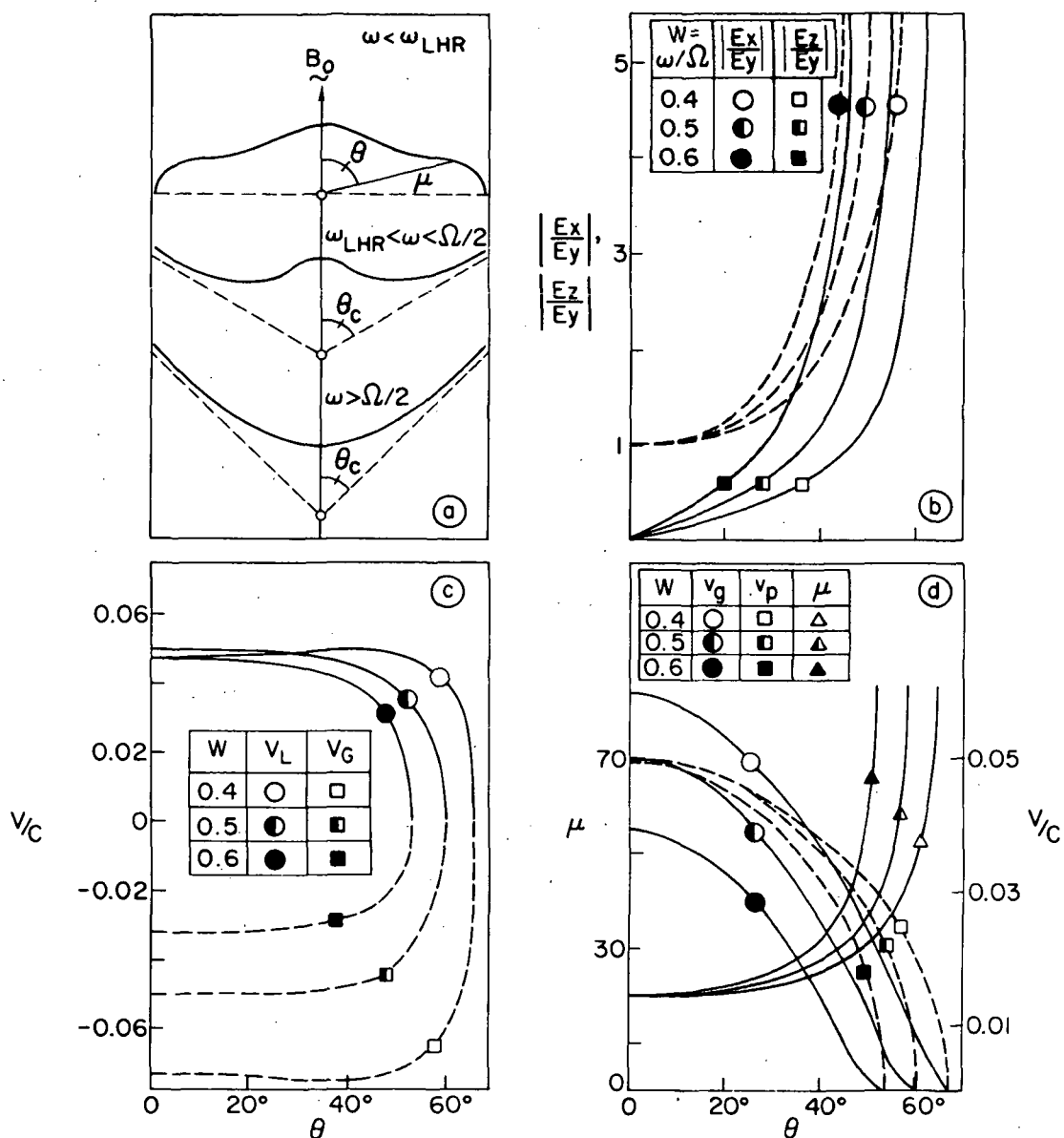


FIG. 3.1. Oblique whistler propagation in a cold plasma [ $W_p = \omega_p/\Omega = 10$ ]. (a) Polar diagrams of refractive index (LHR stands for 'lower hybrid resonance'). (b) Polarization ratios. (c) Resonant parallel velocities for the Landau and fundamental gyro interactions [ $V_L = \omega/k_{\parallel}$ ,  $V_G = (\omega - \Omega)/k_{\parallel}$ ]. (d) Refractive index, phase and group velocities.

approached. The absence of Landau interactions for parallel propagation becomes clear:  $E_{\parallel} = E_z = 0$  at  $\theta = 0$ . The polarization ratios, and the expression for the whistler refractive index, can be simplified considerably when  $|\text{PD} \cos \theta| \gg |(\text{RL-PS}) \sin^2 \theta|$ . This inequality is well satisfied for the range of parameters that we use, and is related to the well known 'quasi-longitudinal' [Stix, 1962], or 'quasi-circular' [Allis et al., 1963] approximations; consistent with our nomenclature, we will term it 'quasi-parallel'. It allows us to write

$$\frac{E_x}{E_y} = -i \left( \frac{D-S \cos \theta}{\text{RL}} \right) \mu^2, \quad \frac{E_z}{E_y} = -i \frac{\sin \theta}{P} \mu^2, \quad (3.26)$$

$$\mu^2 = \text{RL} / (S - D \cos \theta). \quad (3.27)$$

Resonances with the nonthermal electrons occur for particles whose parallel velocity is  $V_m = (\omega + m\Omega)/k_{\parallel}$ . The strength of each interaction will depend strongly, although not exclusively, on the availability of electrons with these velocities. In Figure 3.1 we plot the Landau, and fundamental gyro, resonant velocities,  $\omega/k_{\parallel}$  and  $(\omega - \Omega)/k_{\parallel}$ , so that the other ( $m \neq 0, -1$ ) velocities are readily pictured. They all approach zero as  $\theta$  tends to  $\theta_c$ , and thus diminish the importance of the availability of particles in determining the interaction strength.

The behavior of the refractive index  $\mu$ , phase velocity  $v_p = c/\mu$ , and group velocity  $v_g = c[\partial(\omega\mu)/\partial\omega]^{-1}$  is shown in Figure 3.1. The reduction of the group velocity as the resonance cone is approached is important when casting the stability analysis in terms of temporal growth,  $\omega_i$ , since the whistler packet logarithmic growth is proportional to  $-\int (\omega_i/v_g) dz$ . A wave propagation experiment relies on spatial growth, and we will give the stability results in terms of  $\chi = -c k_i/\omega$ .

Finally, we note from (3.24) that, for a fixed level of dissipation, the energy density,  $U_T = U_M + U_{E+P}$ , will determine the temporal growth rate. From the definition of  $U_M$  and  $U_{E+P}$ , Maxwell's equations and (3.15), (3.19), and (3.25), we find

$$\begin{aligned}
U_M &= \frac{\epsilon_0}{4} \mu^2 |E_y|^2 \left\{ 1 + \cos^2 \theta \left[ \frac{P(S-\mu^2)}{D(P-\mu^2 \sin^2 \theta)} \right]^2 \right\} , \\
U_{E+P} &= \frac{\epsilon_0}{8} |E_y|^2 \left[ \left( \frac{L-\mu^2}{D} \right)^2 \frac{\partial}{\partial \omega} (\omega R) + \left( \frac{R-\mu^2}{D} \right)^2 \frac{\partial}{\partial \omega} (\omega L) \right. \\
&\quad \left. + 2 \left( \frac{S-\mu^2}{D} \right)^2 \left( \frac{\mu^2 \sin \theta \cos \theta}{\mu^2 \sin^2 \theta - P} \right)^2 \frac{\partial}{\partial \omega} (\omega P) \right] , \quad (3.28)
\end{aligned}$$

or, using the quasi-parallel approximation for  $(\omega_p/\Omega)^2 \gg 1$  and  $(\omega/\Omega) \sim 0.5$ ,

$$\begin{aligned}
U_M &= \frac{\epsilon_0}{2} \mu^2 |E_y|^2 , \\
U_{E+P} &= \frac{\epsilon_0}{8} |E_y|^2 \mu^4 \left[ \left( \frac{1+\cos \theta}{R} \right)^2 \frac{\partial}{\partial \omega} (\omega R) + \left( \frac{1-\cos \theta}{L} \right)^2 \frac{\partial}{\partial \omega} (\omega L) \right. \\
&\quad \left. + 2 \frac{\sin^2 \theta}{P^2} \frac{\partial}{\partial \omega} (\omega P) \right] , \quad (3.29)
\end{aligned}$$

where we have  $\mu^2 \approx \omega_p^2 / \omega(\Omega \cos \theta - \omega)$ .

The increase in energy density as  $\theta$  tends to  $\theta_c \approx \cos^{-1}(\omega/\Omega)$  is strong ( $U_M \sim \mu^2$ ,  $U_{E+P} \sim \mu^4$ ). Equation (3.24) suggests that  $U_T$  might compete effectively with the variations in the dissipation,  $P_D$ . Note that similar, but milder, competition will occur for spatial growth, since we have for the energy fluxes  $F \sim v_g U_T$  with  $v_g \sim \mu^{-3}$ , so that  $k_i \sim P_D/F \sim P_D/\mu$ .

**HOT PLASMA.** We wish to use a velocity distribution for the energetic electrons that is both mathematically tractable and flexible enough to simulate, approximately, nonthermal electron distributions that might exist near the magnetospheric equatorial plane at  $3 < L < 4$ , and within the plasmasphere. In particular, we wish to be able to vary the pitch

angle anisotropy, and the kinetic energy of the hot electrons. The Vlasov equation determines the evolution of the hot plasma and implies that its distribution function is constant along trajectories in phase space, i.e., an equilibrium distribution should be a function of the constants of motion. Although significant interactions might occur with other types of energetic populations, we will adopt a distribution which is at equilibrium in the geomagnetic mirror. The distribution should then be a function of the kinetic energy,  $m_e(v_\perp^2 + v_\parallel^2)/2$ , and the first adiabatic invariant,  $m_e v_\perp^2/(2B_0)$ . In particular, for the whistler path near the equatorial plane, where  $B_0$  is almost constant, we can use [Dysthe, 1971]

$$f(v_\parallel, v_\perp) = A_N \frac{v_\perp^{2p}}{(v_\perp^2 + v_\parallel^2 + v_0^2)^q}, \quad (3.30)$$

$$2\pi \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty f dv_\parallel = 1,$$

with the normalization defining  $A_N$ , and the mean square velocity,  $\langle v^2 \rangle$ , determining  $v_0^2$ . Later in the discussion, we will assess the effects that might be expected on the whistler stability from other distributions.

The parameters of the assumed distribution ( $p$ ,  $q$ ,  $v_0^2$ ) can be related to overall properties of the energetic population by taking appropriate moments of (3.30) in velocity space. Denoting the mean electron kinetic energy by  $U$ , we obtain,

$$\frac{2U}{m_e} = \langle v^2 \rangle = \langle v_\perp^2 \rangle + \langle v_\parallel^2 \rangle = \left( \frac{p+3/2}{q-p-5/2} \right) v_0^2, \quad (3.31)$$

$$\frac{T_\perp}{T_\parallel} = \frac{\langle v_\perp^2 \rangle}{2\langle v_\parallel^2 \rangle} = p + 1, \quad (3.32)$$

where the last relation defines a 'temperature' anisotropy.

### 3.4 Dissipation

OBLIQUE PROPAGATION. Using (3.4) and (3.21) in (3.22), the expression derived for the whistler power dissipation, we find

$$P_D = \frac{\delta\omega\epsilon_0}{4} |E_y|^2 \frac{\mu^2 - S}{D^2(\mu^2 \sin^2\theta - P)} \sum_{m=-\infty}^{\infty} P_D^m ,$$

$$P_D^m = \int_0^{\infty} \left( \frac{\mu^2 \sin^2\theta - P}{2(S - \mu^2)} \left[ (R - \mu^2) J_{m-1}^2 + (L - \mu^2) J_{m+1}^2 \right] v_{\perp}^2 - 2V_m^2 \left[ (S - \mu^2 \cos^2\theta)(S - \mu^2) - D^2 \right] J_m^2 \right. \\ \left. - \mu^2 \sin\theta \cos\theta v_{\perp} V_m \left[ (R - \mu^2) J_{m-1} + (L - \mu^2) J_{m+1} \right] J_m \right) \\ \times \left[ \frac{k v_{\perp}}{\omega} F_{\parallel}(v_{\perp}, V_m) - m \frac{\Omega}{\omega} F_{\perp}(v_{\perp}, V_m) \right] dv_{\perp} . \quad (3.33)$$

The expression for  $P_D^m$  can be written conveniently as

$$P_D^m = - \int_0^{\infty} \left[ C_{\parallel}^m F_{\parallel}(v_{\perp}, V_m) + C_{\perp}^m F_{\perp}(v_{\perp}, V_m) \right] dv_{\perp} . \quad (3.34)$$

The partial contribution of the interaction of order  $m$  to whistler growth will lead to amplification (attenuation) when  $P_D^m$  is negative (positive). More specifically, the integration defining  $P_D^m$  will contribute to amplification (attenuation) in the interval  $(v_{\perp}, v_{\perp} + dv_{\perp})$  when the integrand  $(C_{\parallel}^m F_{\parallel} + C_{\perp}^m F_{\perp})$  is positive (negative). These conclusions are useful to establish the stability type of the interaction because, as will be seen below, the dissipation coefficients  $C_{\parallel}^m$  and  $C_{\perp}^m$  do not change sign for a given  $m$ , and, for reasonably smooth distributions, the behavior of the derivatives  $F_{\parallel}$  and  $F_{\perp}$  evaluated at  $V_m$  is easily predictable.

In Figure 3.2 we have plotted the coefficients of these derivatives as a function of  $v_{\perp}/c$ , and up to orders  $m = \pm 3$ , for several angles of propagation,  $\theta = 30^\circ, 45^\circ, 57^\circ$ , with  $\theta_c \approx 60^\circ$ . Their signs are recorded in Table 3.1 where, for future reference, we have included  $V_m = (\omega + m\Omega)/k_{\parallel}$ . At  $\theta = 0$ , for parallel propagation, all coefficients

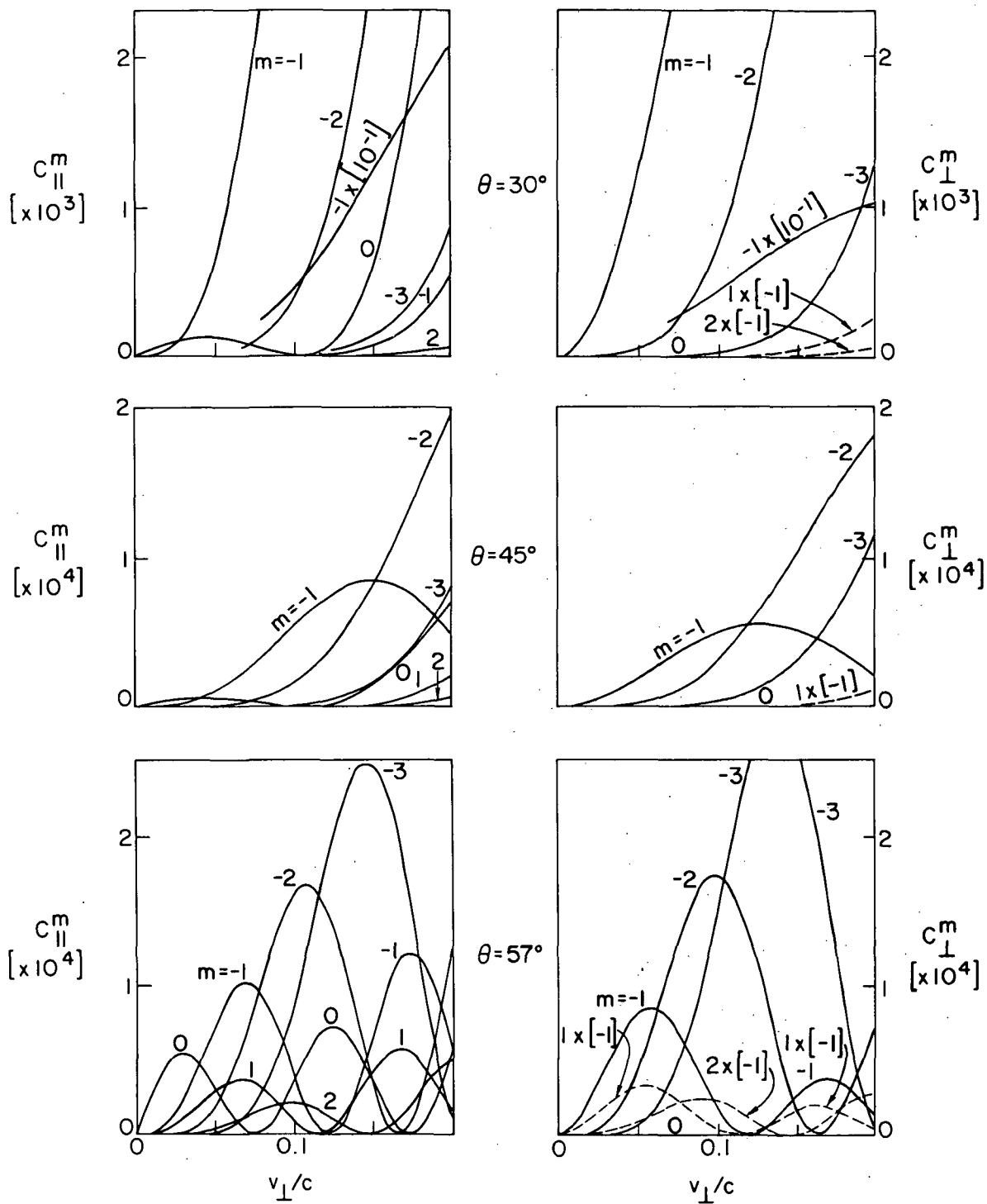


FIG. 3.2. Dissipation coefficients. [ $|m| \geq 3$ ,  $w_p = 10$ ,  $w = 0.5$ ].

are identically zero, except at  $m = -1$ , the fundamental gyro resonance.

TABLE 3.1. SIGNS OF DISSIPATION COEFFICIENTS AND VELOCITIES

|                   | $m < 0$ | $m = 0$ | $m > 0$ |
|-------------------|---------|---------|---------|
| $C_{\parallel}^m$ | +       | +       | +       |
| $C_{\perp}^m$     | +       | 0       | -       |
| $V_m$             | -       | +       | +       |

The behavior of the derivatives depends naturally on the velocity distribution adopted for the hot population. In Figure 3.3 we plot  $F_{\perp}$  and  $F_{\parallel}$  for energetic populations with a 'temperature' anisotropy of 1.5 ( $p = 0.5$ ), and mean energies of 300 ev and 10 kev; the plots are made for  $|v_{\parallel}/c| = 0.05$  and 0.023, where the former velocity corresponds to the phase velocity of the parallel whistler in the cold plasma. It is clear that  $\text{sgn } F_{\parallel} = -\text{sgn } V_m$ , and  $\text{sgn } F_{\perp} = (v_{\perp c} - v_{\perp})$ , where  $v_{\perp c}$  satisfies  $F_{\perp}(v_{\perp c}, V_m) = 0$  and goes to zero when the 'temperature' anisotropy disappears ( $p \rightarrow 0$ ).

For all distributions having these general properties, we can assess the type of contribution made to the overall stability of the whistler by the two components of the integrand,  $C_{\parallel}^m F_{\parallel}$  and  $C_{\perp}^m F_{\perp}$ , for any interaction (arbitrary  $m$ ). Recalling that amplification (attenuation) requires a positive (negative) integrand, we find the situation summarized in Table 3.2.

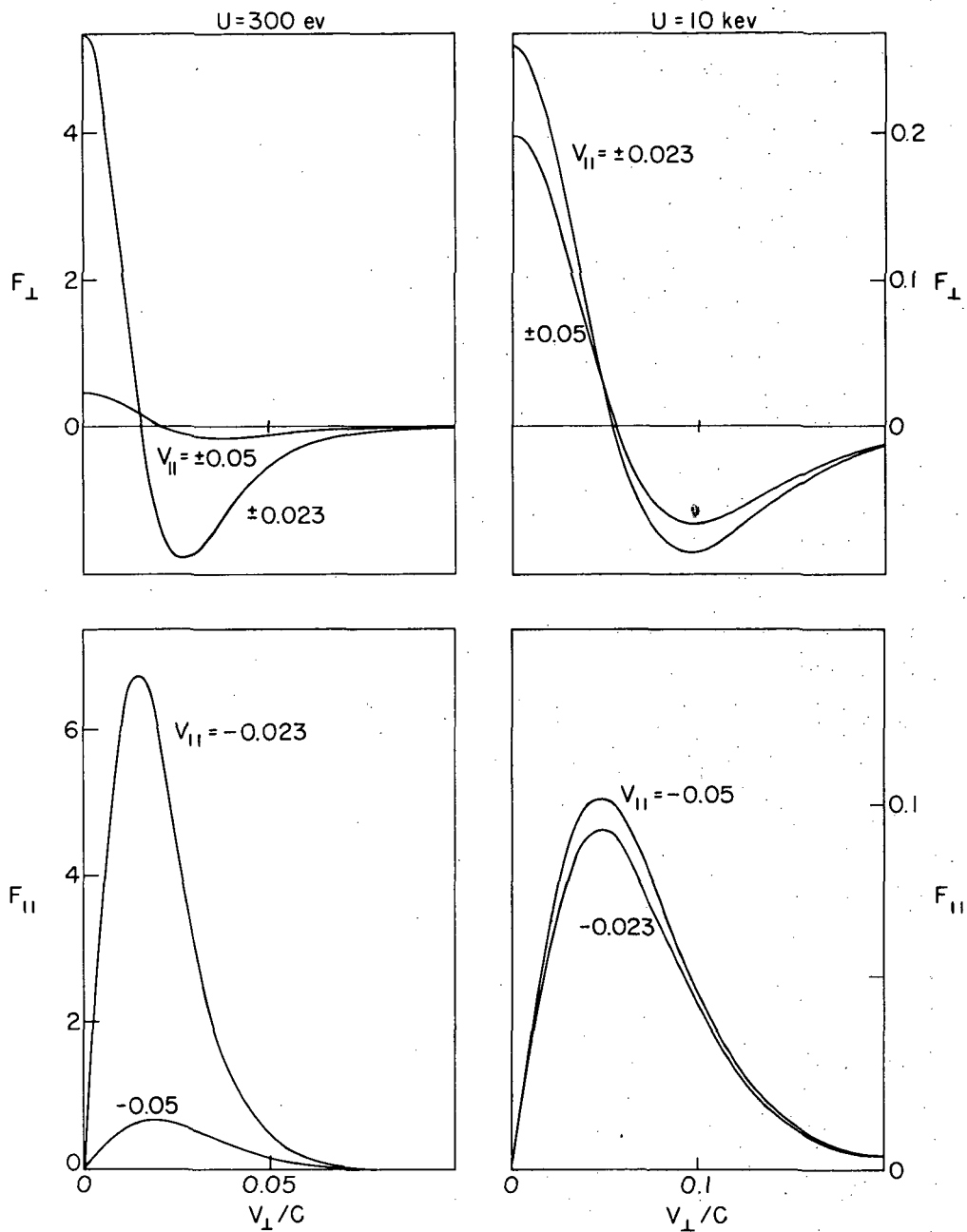


FIG. 3.3. Velocity gradients of the hot plasma distribution  $[X(10c)^4]$ . Note that  $F_{\parallel}(v_{\perp}, v_{\parallel}) = -F_{\parallel}(v_{\perp}, -v_{\parallel})$ ,  $F_{\perp}(v_{\perp}, v_{\parallel}) = F_{\perp}(v_{\perp}, -v_{\parallel})$ .  $[q = 4.0, p = 0.5]$ .

TABLE 3.2. TYPES OF CONTRIBUTIONS FOR OVERALL STABILITY

|                                 | m < 0                     |                           | m = 0                     |                           | m > 0                     |                           |
|---------------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
|                                 | $v_{\perp} < v_{\perp c}$ | $v_{\perp} > v_{\perp c}$ | $v_{\perp} < v_{\perp c}$ | $v_{\perp} > v_{\perp c}$ | $v_{\perp} < v_{\perp c}$ | $v_{\perp} > v_{\perp c}$ |
| $C_{\parallel}^m F_{\parallel}$ | +                         | +                         | -                         | -                         | -                         | -                         |
| $C_{\perp}^m F_{\perp}$         | +                         | -                         | 0                         | 0                         | -                         | +                         |

These results are useful in the analysis of the behavior of each interaction, but do not lead readily to an estimate of the overall whistler dissipation. A significant simplification is brought about if we consider only the interactions with  $m = -1$ , fundamental gyro resonance (G), and  $m = 0$ , Landau resonance (L). We note that this procedure is justified when we avoid both propagation near the resonance cone and highly energetic (relativistic) distributions for the hot electrons. The contribution of the higher order interactions to the dissipation is then reduced by two factors; first, the coefficients  $C_{\parallel}^m$  and  $C_{\perp}^m$  are negligible for nonrelativistic values of  $v_{\perp}$ , and, second, the availability of particles with the required resonant velocities, i.e. the values of  $F_{\parallel}$  and  $F_{\perp}$  at the pertinent  $V_m$ , are very small when compared to the velocity gradients achieved at  $V_G$  and  $V_L$ . Figure 3.5 in Section 3.5 confirms the validity of this approximation.

LANDAU AND GYRO RESONANCES. Focusing our interest on the common situations where the Landau and fundamental gyro interactions are the most significant, we find

$$\sum_m P_D^m \approx - \int_0^{\infty} \left[ C_{\parallel}^L F_{\parallel}(v_{\perp}, V_L) + C_{\parallel}^G F_{\parallel}(v_{\perp}, V_G) + C_{\perp}^G F_{\perp}(v_{\perp}, V_G) \right] dv_{\perp}. \quad (3.35)$$

The dependences on  $v_{\perp}$  and  $\theta$  of the three coefficients  $C_{\parallel}^L$ ,  $C_{\parallel}^G$ ,  $C_{\perp}^G$  were shown above in Figure 3.2. Now, in Figure 3.4, we display, for fixed

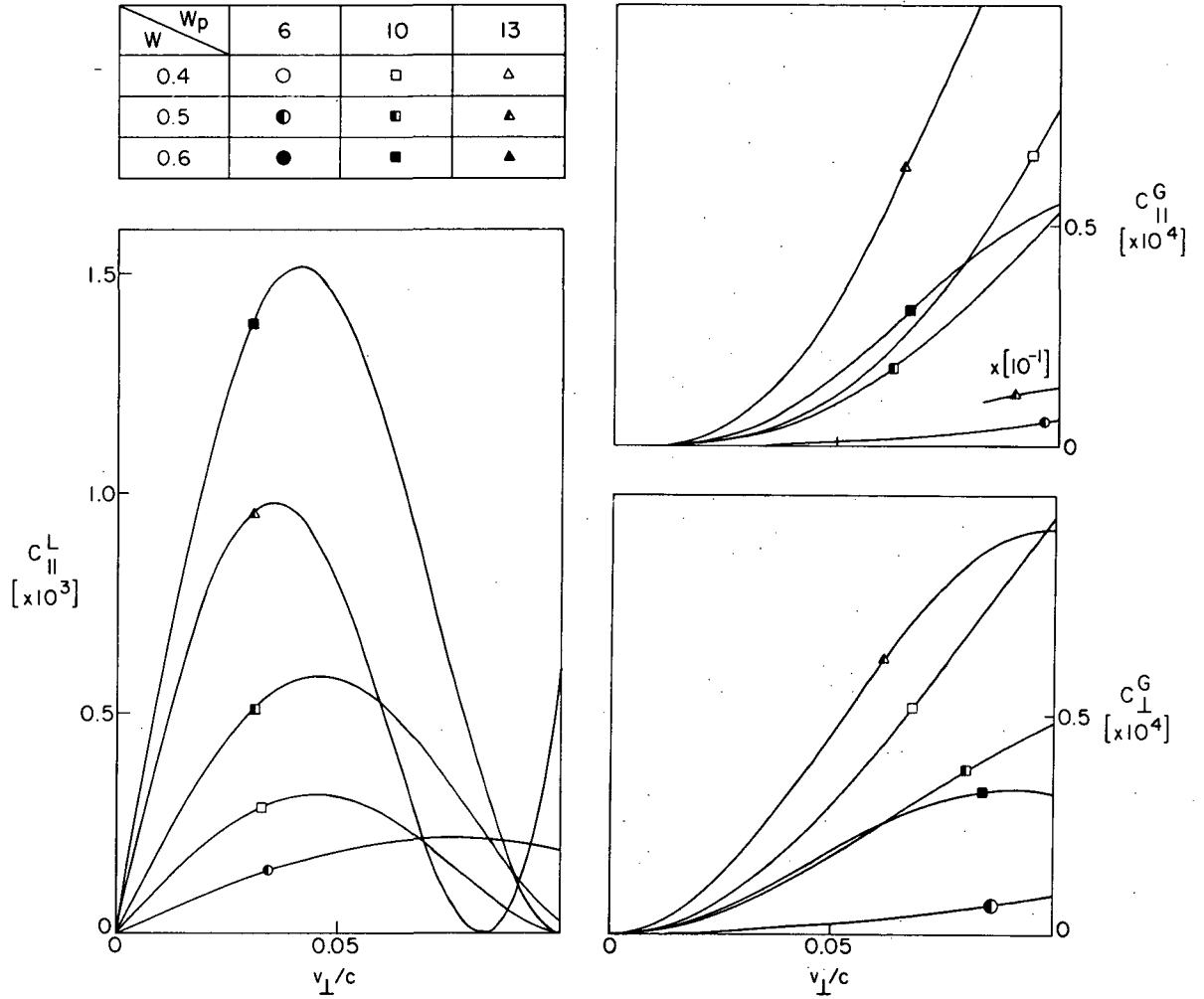


FIG. 3.4. Dependence of the dissipation coefficients for the Landau and fundamental gyro interactions on whistler frequency and cold plasma density.  
 $[\theta = 45^\circ, W = \omega/\Omega, w_p = \omega_p/\Omega].$

$\theta = 45^\circ$ , the dependence upon frequency and plasma density. It is interesting to recognize that the three terms making up the dissipation have quite distinct stability properties. Taking, for the sake of illustration, a velocity distribution of the type defined by (3.30), we realize that the Landau term,  $-\int C_{\parallel}^L F_{\parallel} dv_{\perp}$ , yields positive dissipation (whistler attenuation); similar to the situation met with electrostatic plasma waves, a positive parallel gradient is necessary to bring about negative Landau dissipation. In contrast, the first gyro resonance term,  $-\int C_{\parallel}^G F_{\parallel} dv_{\perp}$ , gives rise to negative dissipation (whistler amplification) when  $\text{sgn } F_{\parallel} = -\text{sgn } V_G$ , the usual situation when the distribution does not have a hump along  $v_{\parallel} < 0$ . Finally, the last gyro resonance term,  $-\int C_{\perp}^G F_{\perp} dv_{\perp}$ , depends on the anisotropy of the distribution. Integration by parts yields  $\int f(\partial C_{\perp}^G / \partial v_{\perp}) dv_{\perp}$ , and since, far from resonance, the plots show  $(\partial C_{\perp}^G / \partial v_{\perp})$  becoming negative only at large values of  $v_{\perp}$ , we might expect that it would be necessary to have very large anisotropies for the maximum of  $f$  to lie within the region of negative  $\partial C_{\perp}^G / \partial v_{\perp}$ , i.e. to obtain negative dissipation (whistler amplification) from this term.

**PARALLEL PROPAGATION.** As mentioned in Section 3.1, the situation that most of the whistler stability literature has analyzed is parallel propagation ( $\theta = 0$ ). In this case, using  $\mu^2 = R$  in (3.33), yields

$$P_D = -\frac{\delta\epsilon_0}{8} |E_y|^2 \int_0^\infty \left[ kv_{\perp}^3 F_{\parallel}(v_{\perp}, V_G) + \Omega v_{\perp}^2 F_{\perp}(v_{\perp}, V_G) \right] dv_{\perp}, \quad (3.36)$$

or, integrating the second term by parts,

$$P_D = \frac{\delta\epsilon_0}{8} k |E_y|^2 \left[ 2 \frac{\Omega}{k} \int_0^\infty v_{\perp} f(v_{\perp}, V_G) dv_{\perp} - \int_0^\infty v_{\perp}^3 F_{\parallel}(v_{\perp}, V_G) dv_{\perp} \right]. \quad (3.37)$$

The first term always has a stabilizing effect, contributing to the whistler dissipation regardless of the velocity distribution shape. The second term, for distributions satisfying  $\text{sgn } F_{\parallel} = -\text{sgn } V_G$ , i.e.,  $F_{\parallel}(v_{\perp}, V_G) > 0$ , or no humps along  $v_{\parallel} < 0$ , leads to whistler amplification. The outcome of the competition depends upon the actual distribution of the hot electrons. The tendency towards instability manifested by

anisotropic distributions favoring the perpendicular energy is clearly illustrated by assuming a separable distribution,  $f = E(v_{\parallel})G(v_{\perp})$ , so that  $P_D$  becomes

$$P_D = \frac{\delta \epsilon_0}{8\pi} k |E_y|^2 \left[ \frac{\Omega}{k} E(v_G) - \frac{\langle v_{\perp}^2 \rangle}{2} \frac{dE(v_G)}{dv_{\parallel}} \right]. \quad (3.38)$$

It is readily seen that, for distributions with  $dE(v_G)/dv_{\parallel} > 0$ , an increase in the perpendicular energy has a destabilizing effect that might lead to positive growth ( $P_D < 0$ ).

### 3.5 Growth Rate: Results

**INFLUENCE OF THE RESONANCES.** So far, we have discussed the separate effects which combine to yield the growth rate experienced by oblique whistlers. Now we shall compute this spatial growth rate. Combining (3.12), (3.22) and (3.23), we obtain

$$\chi = - \frac{ck_i}{\omega} = \frac{\delta}{4\mu(2A\mu^2 - B)} \left\{ \frac{\mu^2 \sin^2 \theta - P}{2(S\mu^2)} \Gamma_I \left[ (R\mu^2)_{J_{m-1}} + (L\mu^2)_{J_{m+1}} \right]^2 G_1 - \right. \\ \left. - 2 \left[ (S\mu^2 \cos^2 \theta)(S\mu^2) - D^2 \right] \Lambda_I J_m G_2 - \mu^2 \sin \theta \cos \theta \Gamma_I \left[ (R\mu^2)_{J_{m-1}} + (L\mu^2)_{J_{m+1}} \right] G_2 \right\} \quad (3.39)$$

where positive  $\chi$  corresponds to attenuation of the whistler wave.

We wish to obtain the behavior of  $\chi$  for arbitrary directions of whistler propagation in varying mixtures of cold and hot plasma components. Our first concern will be to assess quantitatively the importance of the interactions other than the Landau and (fundamental) gyro resonances. For this purpose, we subdivide the total  $\chi$  into three components,  $\chi = \chi^G + \chi^L + \chi^O$ , where  $\chi^G$  refers to the fundamental gyro resonance ( $m = -1$ );  $\chi^L$  to the Landau resonance ( $m = 0$ ), and  $\chi^O$  to the higher order ( $m \neq 0, -1$ ) interactions. In Figure 3.5 we plot  $\chi$  and the ratios  $\chi^G/\chi^L$  and  $\chi^O/\chi^L$ , for several energetic electron distributions and whistler frequencies, recollecting that, from Section 3.4,  $\chi^L$  is positive (or zero at  $\theta = 0$ ) for the assumed velocity distribution. In evaluating  $\chi^O$  we have neglected interactions of order  $m$  such that  $|m| > n$  when  $|\chi^{\pm n}| < 10^{-2} \sum_m \chi^m$  ( $|m| < n$ ). This procedure fails in the immediate

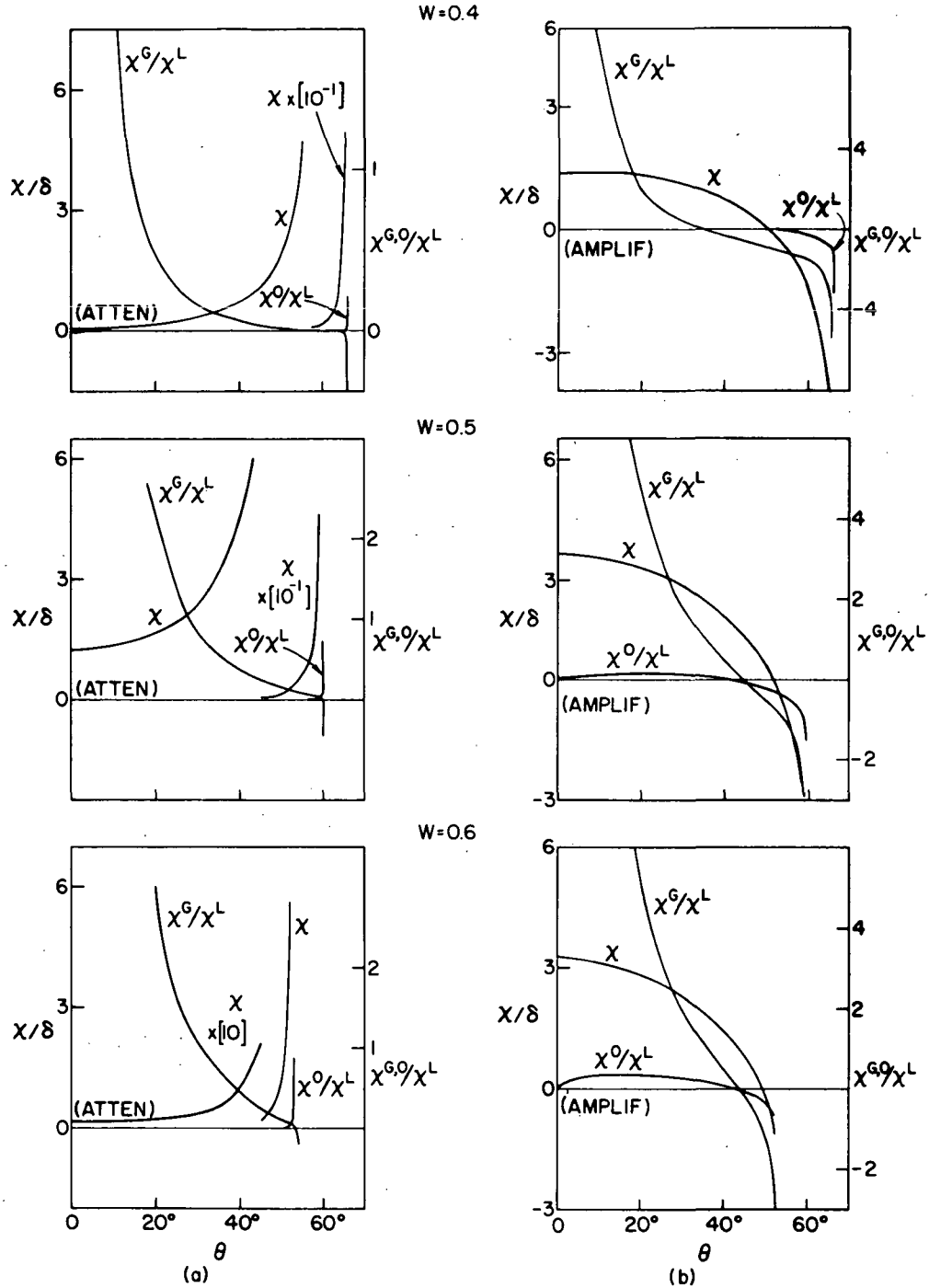


FIG. 3.5. Contributions of the Landau ( $\chi^L$ ), fundamental gyro ( $\chi^G$ ), and other ( $\chi^0$ ,  $m \neq 0, -1$ ) interactions to the oblique whistler imaginary refractive index ( $= -\chi$ ).

[ $w_p = 10$ ,  $q = 4.0$ ,  $p = 0.5$ ]. (a)  $U = 300$  eV.

(b)  $U = 10$  keV.

vicinity of the resonance cone, but in this region the approach used above to describe the cold plasma also fails; finite temperature effects should be taken into account [Stix, 1962].

As the discussion in Sections 3.3 and 3.4 might have led us to expect, the plots in Figure 3.5 show that the higher order interactions only contribute significantly to the overall growth rate when the resonance cone is approached. Naturally, as the mean energy of the hot population is increased, this situation is progressively, but slowly, altered, with  $\chi^0$  increasing in magnitude.

DEPENDENCE ON OBLIQUITY. Figure 3.5, for  $U = 300$  ev, shows the whistler growth decreasing as  $|\theta|$  increases, in contrast to the behavior obtained for  $U = 10$  kev. It is the former behavior,  $d\chi/d|\theta| > 0$ , that would be inferred from the increase in Landau damping brought about by the progressive wave normal deviation from the static magnetic field. The discussion in Sections 3.3 and 3.4 has shown, however, that the situation is more complicated; the overall growth depends upon the competing behavior of the two most important interactions and the cold plasma characteristics. Here, to explain the contrast between the sets of curves for  $U = 300$  ev and  $U = 10$  kev and the same cold plasma background, it is clear that we must consider the hot plasma distribution. An examination of both Figure 3.2, showing the dependence on  $\theta$  of the coefficients  $C_{\parallel}^L$ ,  $C_{\parallel}^G$ ,  $C_{\perp}^G$ , and Figure 3.3, displaying the velocity gradients  $F_{\parallel}$ ,  $F_{\perp}$  for 300 ev and 10 kev, clarifies the computed features of  $\chi(\theta)$ . We note that, with  $U = 300$  ev, the gradients of the velocity distribution are very strong functions of  $v_{\parallel}$ , increasing markedly with  $\theta$  (at  $\omega/\Omega = 0.5$ , both  $V_L$  and  $|V_G|$  decrease with  $\theta$ ). Yet, the variations that could lead to more favorable propagating conditions, i.e., the increases in  $F_{\parallel}(v_{\perp}, V_G)$  and the positive part of  $F_{\perp}(v_{\perp}, V_G)$ , occur in a domain of  $v_{\perp}$  where the coefficients  $C_{\parallel}^G$  and  $C_{\perp}^G$  have very small magnitude. The variations that yield substantial contributions to the positive dissipation, namely, the amplitude increases in  $F_{\parallel}(v_{\perp}, V_L)$  and the negative part of  $F_{\perp}(v_{\perp}, V_G)$ , cause whistler attenuation, so that the overall effect is a net algebraic increase in  $\chi$ , as the whistler becomes more oblique. When the mean kinetic energy of the hot electrons is increased to

$U = 10$  kev, the situation is very different: the velocity gradients become weak functions of  $v_{\parallel}$ ;  $|F_{\parallel}(v_{\perp}, v_{\parallel})|$  decreases with  $|v_{\parallel}|$  and both gradients are generally displaced to higher values of  $v_{\perp}$ . As a result, the variation in the dissipation is essentially determined by the  $\theta$ -dependence of the dissipation coefficients, bringing about a decrease in the dissipation, and  $\chi$ , as  $|\theta|$  increases.

We might wonder whether this behavior of  $\chi(\theta)$  depends strongly on the form of the assumed distribution function, and the choice of its parameters. The influence of the whistler frequency about  $\omega/\Omega = 0.5$ , and  $p$ ,  $q$ , and  $U$ , might be inferred from Figures 3.5 and 3.6. It is clear that the strongest factor in the qualitative behavior of the growth rate is the mean kinetic energy of the hot population. It is open to question whether the velocity distribution given by (3.30) represents a realistic energetic electron population in the equatorial magnetosphere for  $3 < L < 4$ , and it may be asked whether it overestimates the density of lower energy particles. We note, however, that the adoption of other distributions with a smaller content of low energy particles, or having a monochromatic energy spectrum, could enhance the features of the velocity gradients which are responsible for the behavior of  $\chi(\theta)$  shown in Figure 3.5 ( $U = 10$  kev). Indeed this behavior,  $d\chi/d|\theta| < 0$ , was obtained without resorting to distributions having a marked peak on the energy spectrum. In particular, we did not use Landau amplification brought about by humps in the parallel distribution,  $F_{\parallel}(v_{\perp}, v_{\parallel}) > 0$  [Thorne, 1968]. Isotropic ( $p = 0$ ) distributions already show maximum whistler attenuation for parallel propagation when  $U = 10$  kev.

**EXISTENCE OF MINIMAL PARALLEL GROWTH.** It is possible to determine for an arbitrary hot plasma distribution whether larger growths will be experienced by the whistler as its wave normal deviates from the static magnetic field. Noting that  $d\chi/d\theta$  is always zero for parallel propagation because  $\chi(\theta)$  is an even function of  $\theta$  [see (3.39) and recall the arguments of the Bessel functions], we conclude that a criterion for the existence of minimal growth at parallel propagation is to have  $\chi'' = d^2\chi/d\theta^2 < 0$  at  $\theta = 0$ . This condition does not ensure the existence of an absolute minimum for parallel growth; it is still possible to find a propagation angle

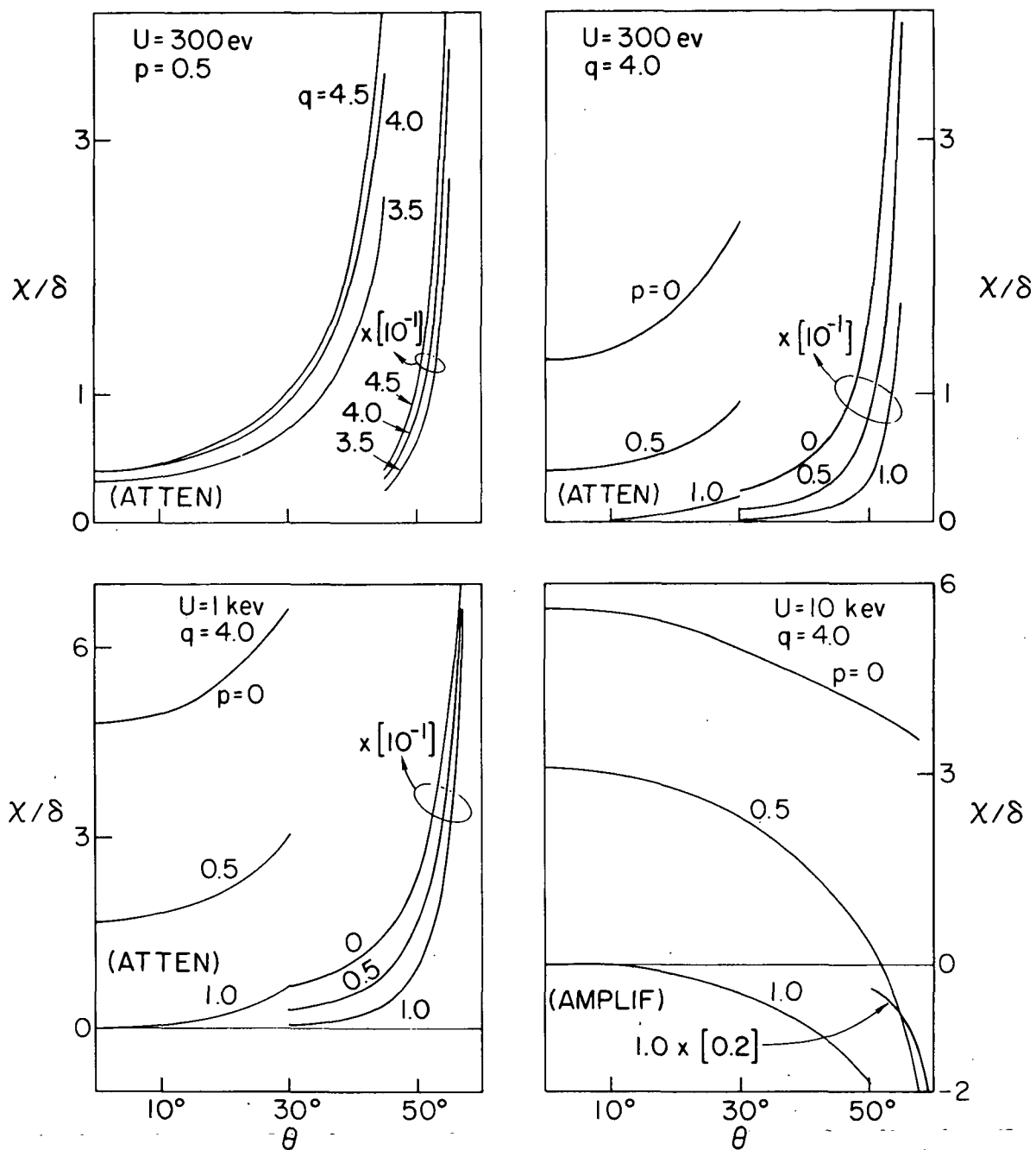


FIG. 3.6. Dependence of the oblique whistler imaginary refractive index ( $= -\chi$ ) on the hot plasma parameters. [ $w_p = 10$ ,  $w = 0.5$ ].

$\theta_m \neq 0$  where the growth is smaller than at  $\theta = 0$ . However, observation of Figures 3.5 and 3.6 suggests that, excluding the neighborhood of the resonance cone, the spatial growth rates are monotonic functions of  $\theta$ : the maxima and minima of parallel growth are generally absolute extreme values outside the resonance cone region.

Differentiating (3.39) with respect to  $\theta$ , using the quasi-parallel approximation for the refractive index, (3.27), and keeping only the terms due to the Landau and fundamental gyro interactions, we obtain

$$\begin{aligned} \chi''(0) = -\frac{1}{8\mu PD} \left\{ \Gamma_I^G \left\{ a_1 R [P(1+D/L) - 2D\omega/\Omega - R] (v_\perp/c) F_\parallel \right. \right. \\ - [a_1 PDR(\omega/\Omega)^2] (v_\perp/c)^3 F_\parallel + [-4DR + 2(\Omega/\omega) [PD(R+D/2)/L \\ - R^2 + P(D+R)]] F_\perp - [2PDR(\omega/\Omega)] (v_\perp/c)^2 F_\perp \\ + [PD(2+D/L)(1-\Omega/\omega)] v_\perp \frac{\partial F_\parallel}{\partial v_\parallel} + [PD(2+D/L)(\Omega/\omega) v_G] \frac{\partial F_\perp}{\partial v_\parallel} \Big\} \\ + \mu DR(\omega/\Omega) \Gamma_I^L [4 + P(\omega/\Omega)(v_\perp/c)^2] (v_\perp/c) F_\parallel \\ \left. + 4DR(1+D/L) \Lambda_I^L F_\parallel \right\}, \end{aligned} \quad (3.40)$$

where the right-hand side is evaluated at  $\theta = 0$  and the superscripts G and L denote the  $m = -1$  and  $m = 0$  terms in the sums defining  $\Gamma_I$  and  $\Lambda_I$ .

When the velocity distribution of the hot electron population is known, the sign of  $\chi''(0)$  will determine the  $\theta$ -dependence of the whistler growth outside the vicinity of the resonance cone ( $\theta \sim \theta_c$ ). We have particularized (3.40) to the hot population assumed in (3.30) and determined the threshold value(s) of the mean energy of the hot electrons yielding  $\chi''(0) = 0$ , as a function of whistler frequency. The results are shown in Figure 3.7; the influence of the parameter  $q$  is small and was omitted. (For a given frequency, an increase in  $q$  reduces the threshold energy  $U$ .) This figure illustrates clearly that, for low-frequency oblique whistlers,  $\omega \ll \Omega$ , the growth rates always maximize

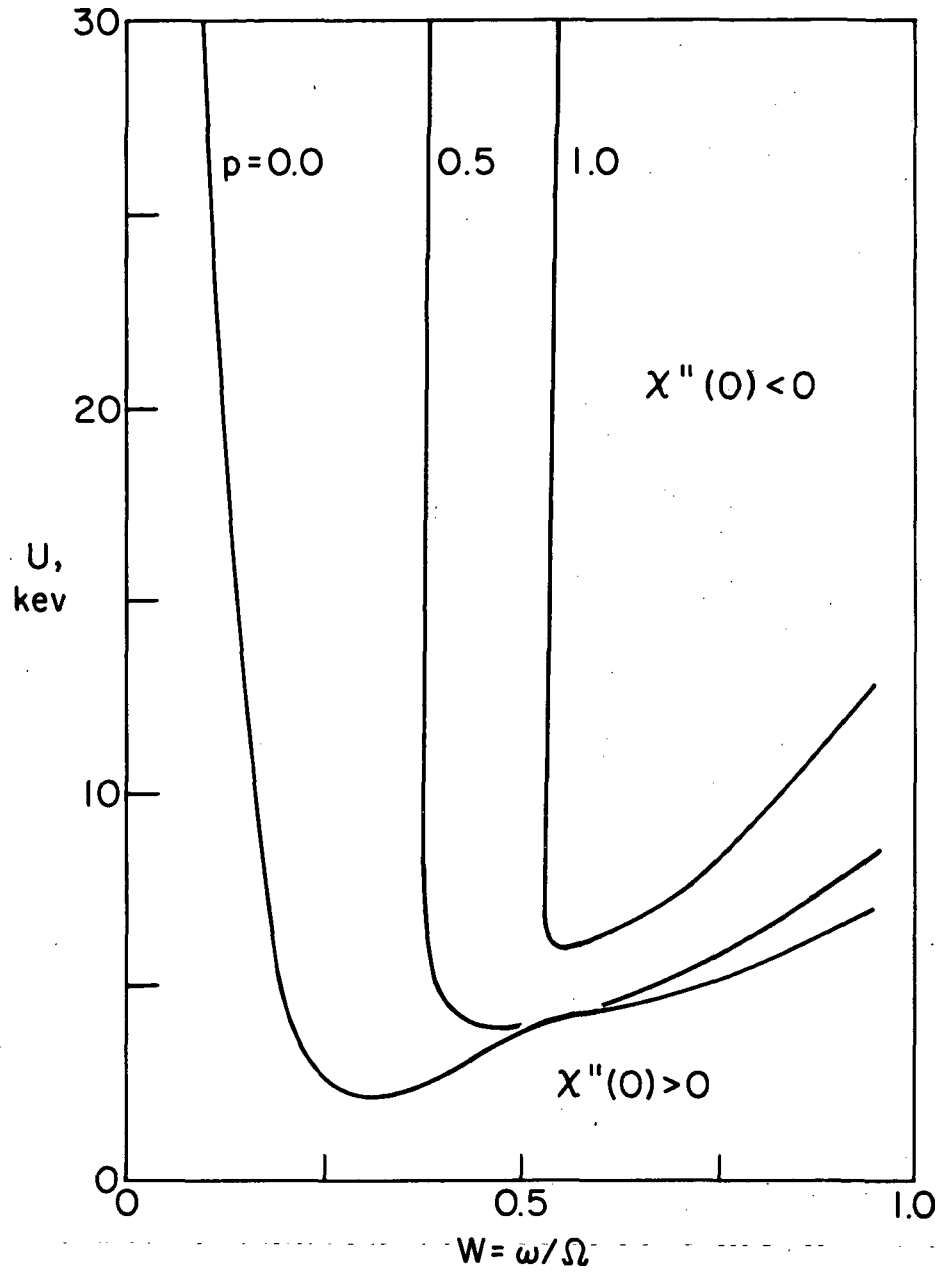


FIG. 3.7. Dependence of the limiting condition  $\chi''(0) = 0$  on the hot plasma parameters. A maximum (minimum) of the whistler growth occurs at parallel propagation when  $\chi''(0) > 0$  ( $< 0$ ). [ $w_p = 10$ ,  $q = 4.0$ ].

at parallel propagation, in agreement with the results of Kennel [1966]. In contrast, high-frequency oblique whistlers in dense plasmas ( $\omega_p^2 \gg \Omega^2$ ) with a hot electron population of a few keV will experience maximum attenuation at parallel propagation.

### 3.6 Discussion

The results obtained above on the importance of the wave-particle interactions (Section 3.4), and the dependence of the growth rate on  $\theta$  (Section 3.5), have shown features that are not deducible from previous analyses concerned with low-frequency whistlers. First, of the two most important interactions, the fundamental gyro may dominate not only for small obliquity, but also near the resonance cone (Figure 3.5). Second, depending mainly on the energy of the hot electron population, the high-frequency whistler may experience maximal amplification or attenuation at parallel propagation (Figures 3.5, 3.6 and 3.7). This result may be useful in the interpretation of satellite observations of nonducted high-frequency whistlers. We have shown that strong whistler activity for propagation at large angles to the geomagnetic field is compatible with the simultaneous existence of an energetic electron population without humps along  $v_{\parallel} > 0$ , since the 'Landau' damping at oblique propagation may be compensated by the fundamental gyro resonance growth.

With respect to the artificially stimulated emissions to be discussed in Chapter 4, we note that the relevance of the foregoing analysis is threefold. First, both the triggering and triggered signals are, in general, oblique whistlers, so that their linear stability characteristics are directly derivable from the results of this chapter. Second, the nonlinear evolution of the wave-particle interactions considered in (3.35) suggests a possibility for the (nonlinear) emission onset, as described in Brinca [1972b] and Section 4.1 below. Finally, the evolution of the stimulated emission after its onset is essentially determined by the relative importance of the Landau and fundamental gyro interactions during the triggering process; an estimate of the strengths of these interactions can be obtained from the foregoing analysis.

## 4. ARTIFICIALLY STIMULATED EMISSIONS

### 4.1 Background

The linear approximation for whistler propagation is adequate for small signals. As the whistler amplitude is raised progressively, we anticipate the occurrence of nonlinear phenomena. Among these, we shall be concerned in this chapter with the triggering of whistlers by large-amplitude signals.

Observations of very low frequency magnetospheric radio noise, made with ground-based receivers and satellites, have demonstrated the occurrence of almost monochromatic emissions that are stimulated by other discrete signals [Helliwell, 1965]. These emissions may be triggered by natural whistlers or man-made signals propagating in the whistler mode. We shall concentrate here on the latter type of triggering; the resulting emissions are termed artificially stimulated emissions (ASE). A typical spectrogram is sketched in Figure 4.1. The triggering signal is a Morse dash propagating in the whistler mode. The onset of the triggered emission is characterized by the triggering delay and the offset frequency. The triggering delay measures the time elapsed between the arrival of the leading edge of the whistler pulse and the beginning of the emissions. The offset frequency represents the initial difference in the central frequencies of the triggering signal and the triggered emission. We note that the offset frequency may be positive, as shown in Figure 1, or negative [Lee, 1968], and the main phase of the emission following its onset may have a variety of spectral shapes in addition to the 'riser' depicted in Figure 4.1. Detailed characteristics of the triggered emissions will be given in Section 4.2.

ASE were first observed at 0835 UT on 28 November 1959 on a synoptic tape recording made in Wellington, New Zealand [Helliwell et al., 1964]. The stimulating signal was emitted by station NPG (Jim Creek, Washington), and consisted of CW Morse code with central frequency 18.6 kHz. Since then, many other observations of ASE have been made [Helliwell, 1965], and several interpretations proposed [Helliwell, 1967; Das, 1968; Sudan and Ott, 1971; Matsumoto and Kimura, 1971; Dysthe, 1971; Nunn, 1971; Fung, 1972].

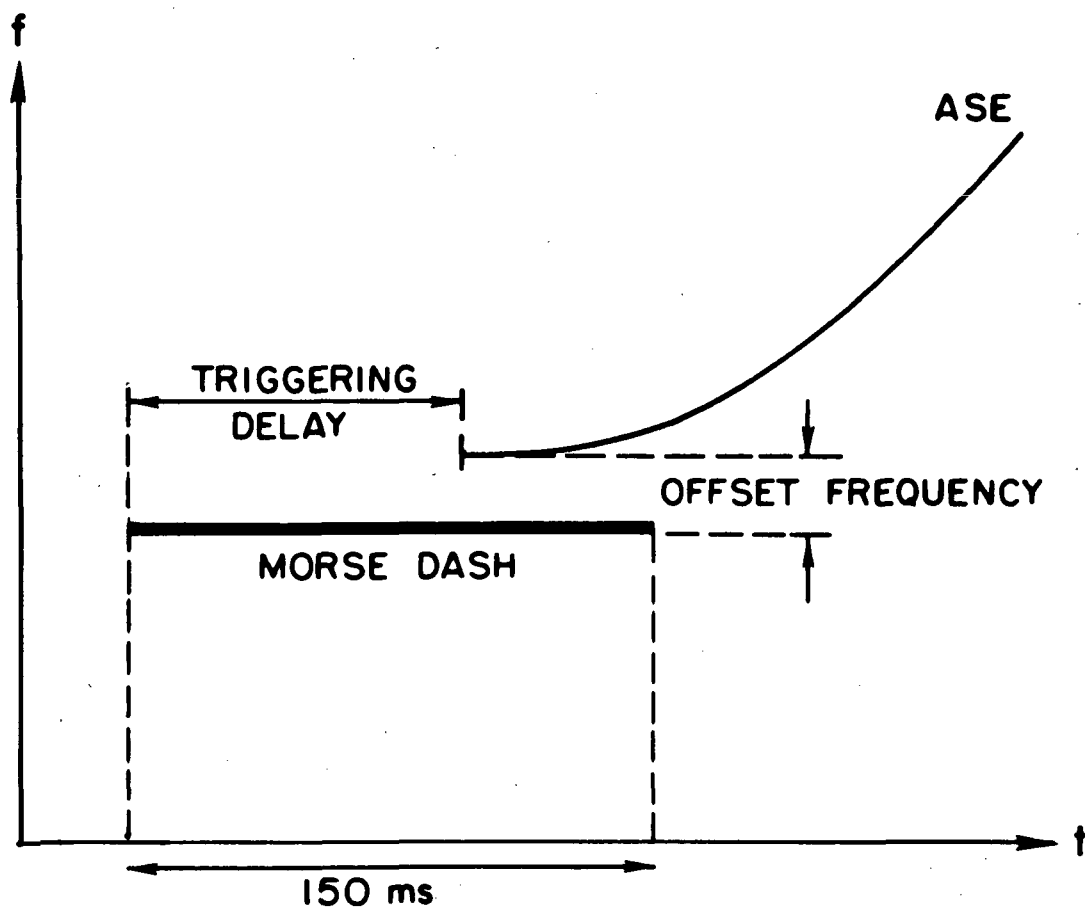


FIG. 4.1 Typical ASE spectrogram.

REVIEW OF PREVIOUS WORK. These theories assume a parallel-propagating stimulating whistler, considering only the gyroresonance interaction. The extension to triggering by obliquely propagating whistlers with a dominant Landau resonance was not contemplated. OGO-4 satellite observations have recently demonstrated, however, that ASE are also triggered by nonducted whistlers propagating at a large angle to the static magnetic field [Angerami and Bell, 1971]. Since Chapter 3 suggests that the Landau resonance may be the dominant interaction under these circumstances, we shall consider an onset mechanism valid for stimulating whistlers of arbitrary obliquity. Except for Helliwell [1967], the characteristics of the emission onset have not been analyzed in detail so far, and the observed emission spectral shapes cannot be explained by the mechanisms proposed for the main phase of the emission.

Helliwell [1967] has considered the interaction between the whistler and counter-streaming gyroresonant electrons in the inhomogeneous equatorial magnetosphere. He attributes the stimulated emission to the radiation arising from the current brought about by the phase-bunching of the particles. The temporal variation of the emitted frequency depends on the location of the interaction region and is proportional to the spatial gradient of the local geomagnetic field. The observed spectral shapes are explained by assuming that the interaction region drifts along geomagnetic field lines. Its location is chosen so that the temporal slope of the emitted frequency coincides with the observed slopes after dispersion and propagation effects are accounted for.

Dysthe [1971] and Sudan and Ott [1971] have analyzed the cyclotron interaction between a large amplitude whistler and an arbitrary electron velocity distribution. Dysthe obtained a nonlinear spatial growth rate for whistlers in agreement with the results of Palmedesso and Schmidt [1971], and studied the self-sustaining cyclotron mechanism first suggested by Brice [1963]. Sudan and Ott [1971] derive a constant growth rate for the test waves leading to the stimulated emission, though the time scales involved in the onset process, i.e. period of the triggering wave, bouncing period in the trapping rf magnetic well and triggering delay, suggest that the onset is a transient phenomenon.

None of these theories [Dysthe, 1971; Sudan and Ott, 1971] has the potentiality to explain negative frequency offsets.

The analyses of Matsumoto and Kimura [1971] and Fung [1972] use a quasilinear approach to study the gyroresonant interaction. These theories may be applicable to VLF emissions stimulated by broad band noise but they are not valid for the problem of concern here, i.e. triggering by an almost monochromatic signal (see Section 4.3.1).

The mechanism proposed by Nunn [1971] relies on the inhomogeneity of the geomagnetic field. The contributions of the 'second-order resonant' electrons to the distorted velocity distribution bring about the growth of a stimulated emission. Large growths, however, require the existence of 'second-order resonance' during time intervals that seem to be too large to obtain in the magnetosphere. Also, negative frequency offsets are not accounted for.

Das [1968] computes a distorted velocity distribution for the special case of an equilibrium sharp loss-cone distribution and uses it to study the growth rates experienced by whistler test waves. The approach is only valid under conditions discussed below with respect to Equation (4.30). The occurrence and spectral location of the unstable sidebands obtained by Das depend on the existence of a broad spectrum for the triggering whistler. Because the triggering Morse dashes are narrow band, his results are in conflict with observed magnitudes of the offset frequency.

**APPROACH.** We have divided our study of ASE into consideration of the triggering signal, the emission onset, and the stimulated whistler. The triggering signal is assumed to be an oblique whistler propagating near the equatorial plane. Its linear stability properties were studied in Chapter 3, together with the relative importance of the Landau and cyclotron resonances. In this chapter we analyze the onset and evolution of the emission itself.

The mechanism proposed for the emission onset is suggested by (3.35). For oblique whistlers in a plasma with a hot electron velocity distribution satisfying  $F(v_{\perp}, V_{G,L}) = -\text{sgn } V_{G,L}$ , the integrand in (3.35) shows the stabilizing Landau resonance, and the fundamental gyroresonance

(represented by the last two terms), whose stability character depends on the anisotropy of the hot electron velocity distribution. Assuming the existence of a triggering wave packet in the whistler mode near the equator, the Landau and fundamental gyro interactions will affect electrons with parallel velocities in opposite directions, since  $V_L > 0$  and  $V_G < 0$ . It has been argued elsewhere [Brinca, 1972a] for electrostatic waves, that at the early stages of the nonlinear evolution of the Landau wave-particle interaction, the velocity gradient near resonance,  $F_{\parallel}$ , is distorted, and that its value becomes large and positive for a narrow range of phase velocities slightly displaced from  $V_L$ . A similar evolution may occur in the oblique whistler case, due to the parallel component of the wave electric field; the first term in the integrand of (3.35),  $C_{\parallel}^L F_{\parallel}$ , then leads to large negative dissipation, i.e. growth of whistler noise in narrow sidebands of the triggering whistler. The nonlinear evolution of the cyclotron interaction, to be considered in Section 4.3 for  $\mathbf{k} \parallel \mathbf{B}_0$ , also leads to distortion of the velocity gradients near  $V_G$ , strongly enhancing the amplification of whistler noise in sidebands of the triggering frequency  $\omega_0$ . For pulses with a duration above a certain minimum depending on their amplitude, as derived in Section 4.3, this nonlinear evolution of the two wave-particle interactions will create within the triggering packet a region where the new distorted electron velocity distribution is strongly unstable to whistler perturbations. This mechanism may then be responsible for the onset of a stimulated emission.

The unstable region thus formed, being made up of resonant particles of two kinds, will move along geomagnetic field lines in a direction that depends on the relative importance of the two interactions. If the Landau electrons predominate, the displacement will take place in the direction of the stimulating signal. If the gyroresonance is stronger, the opposite movement will follow and the possibility of having a self-sustained process arises [Brice, 1963].

The description of the main phase of the emission using a self-sustained process was made by Helliwell [1967] and Dysthe [1971], and shall not be reviewed here. Instead, we shall speculate in Section 4.4

on non self-sustained, i.e. drift mechanisms that may possibly describe the main phase of most short duration emissions, particularly when the Landau interaction is responsible for their onset.

#### 4.2 Characteristics

The observations of ASE to date have been fortuitous to the extent that they did not arise from controlled experiments designed to study stimulated emissions. The majority of the cases reported have been excited by U.S. Navy VLF transmitters, during their regular CW pulse schedules. In particular, the NAA transmitter in Cutler, Maine, with an approximate output power of 1 MW and frequency of 14.7 kHz, and the network of Omega transmitters (in Aldra, Norway; Haiku, Hawaii; Forest Port, New York, and Trinidad, Trinidad and Tobago,) with output powers between 0.1 and 4 kW, and frequencies ranging from 10.2 to 13.6 kHz, have been especially successful in triggering emissions.

Because the experimental conditions vary from observation to observation, and no control is exerted on the transmitter parameters, it is difficult to categorize the results obtained. The most complete attempt in that direction is Lee's [1968] study of the amplitude and frequency spectra of VLF emissions. Another spectral analysis of triggered emissions is now being undertaken at Stanford by Stiles [1972], and it is hoped to start operating soon the first experiment designed specially to study stimulated emissions and related phenomena. The stations involved will be located in Siple, Antarctica, and Roberval, Canada, at magnetically conjugate points [Helliwell, 1972]. Below we shall summarize the relevant characteristics of ASE by describing the observed features of the triggering signal, onset region, and spectral shapes.

TRIGGERING SIGNAL. The location of the emitting ground station is not critical. The signal propagates in the earth-ionosphere waveguide and can excite magnetospheric ducts with different values of  $L$ . Here  $L$  is one of the two geomagnetic coordinates introduced by McIlwain [1961]; in a dipolar magnetic field, it represents the equatorial geocentric radial distance of the field line, expressed in earth's radii.

In contrast, the central frequency,  $\omega_0$ , of the stimulating CW pulse appears to be very important. Triggering of ASE occurs frequently along field lines with  $\omega_0 \sim \Omega_0/2$ , where  $\Omega_0$  represents the equatorial electron cyclotron frequency of the geomagnetic field [Carpenter et al., 1969; Angerami and Bell, 1971].

Observations show [Helliwell, 1965] that almost all ASE are associated with Morse dashes, rather than dots. This suggests that a minimal pulse duration is required for triggering.

On the other hand, the transmission power is not a fundamental triggering parameter: stimulating pulses have been emitted by both NAA (1 MW, 150 ms pulses) and Omega (100 W, 1 s pulses) [Kimura, 1968]. We note, however, that the conditions prevailing at the foregoing transmissions were not identical in frequency and space-time locations. Allowance should be made for possible variations in coupling efficiency between the ionosphere and troposphere, and propagation losses or gains along the magnetospheric paths.

ONSET REGION. OGO-1 satellite observations reported by Dunckel and Helliwell [1969] indicate that whistler mode emissions originate in the vicinity of the magnetic equator. The value of the optimal triggering frequency,  $\omega_0 \sim \Omega_0/2$ , mentioned above, suggests that a similar situation prevails for ASE. In this respect, the results obtained by Angerami and Bell [1971] are very significant: they report ASE triggered by signals from the Omega transmitters, and observed on the OGO-4 satellite. The satellite-detected Omega pulse, which is often as much as three times the duration of the original transmitted pulse, can be reproduced if a number of whistler paths are postulated. Only the paths representing the largest travel time appear to be connected with triggering, i.e. the satellite spectrograms show the emissions linked with the final portion of the pulse received. Remarkably, when these nonducted triggering paths cross the equatorial plane, the Omega frequency and the local geomagnetic electron gyrofrequency are related by  $\omega_0 \sim \Omega_0/2$ . Thus, a single experiment not only yields evidence supporting the equatorial location of the onset region and the magnitude of the optimal triggering frequency, but also demonstrates that nonducted oblique whistlers may trigger emissions.

SPECTRAL FORMS. A typical spectrogram of a stimulated emission is sketched in Figure 4.1. The very narrow band (tens of Hz) emission starts after an apparent triggering delay, and has an initial frequency that often differs from the pulse frequency by an offset frequency which, though usually positive, may be also negative [Lee, 1968]. It is important to note that the analysis of the onset is beset with fundamental difficulties associated with the transiency of the phenomenon. Frequently, this transiency prevents the determination of the signal frequency and thus precludes the distinction between the original signal and the emission, so that the concepts of triggering delay and offset frequency become meaningless [Stiles, 1972; private communication].

After the onset, and usually after the termination of the triggering pulse, the emission changes its central frequency in what seems to be an erratic manner, giving rise to a rich variety of spectral shapes. Their basic forms, with self-explanatory designations, have been classified and illustrated by Helliwell [1965]; they comprise risers, falling tones, hooks, inverted hooks and combinations thereof, with the riser being the most frequently observed type. The rate of frequency variation observed on the ground is typically 5 Hz/ms but may reach about 35 Hz/ms [Lee, 1968]. Note, however, that this is not the rate corresponding to the actual emission process occurring in the equatorial zone: dispersion and propagation effects must be taken into consideration. This point will be discussed further in Section 4.4. In general, it is found that triggering delays and offset frequency magnitudes have inverse dependency on the strength of the stimulating signal [Kimura, 1968].

#### 4.3 Onset: Whistler Sidebands

##### 4.3.1 Introduction

The understanding of the emission onset mechanism is a prerequisite for the analysis of the main phase of the emission. Here, we shall study the distortion caused by a large amplitude whistler to a dilute hot electron population permeating an effectively cold magnetoplasma, and demonstrate that whistler sidebands may be created as a result of nonlinear wave-particle interactions. This sideband

growth yields good agreement with the onset characteristics of triggered emissions described in Section 4.2, and thus suggests itself as the triggering mechanism.

Since the stimulating whistler is in general oblique, the analysis should consider the distortion of the energetic velocity distribution brought about by a pulse propagating at an angle to the static magnetic field. However, the algebraic complexity of the oblique whistler case, already evident in the linear analysis of Chapter 3, becomes unmanageable when this nonlinear problem is contemplated. To circumvent the difficulty, we shall first study the parallel whistler case. Then, in the light of the formal similarity between the creation of whistler sidebands and Landau sidebands [Brinca, 1972d], and of the knowledge gained from Chapter 3 that the only relevant wave-particle interactions are the fundamental cyclotron and Landau resonances, we shall discuss the characteristics to be expected in a general, oblique whistler, onset mechanism.

The analysis to be presented here of whistler sidebands is the outgrowth of a number of studies on large amplitude waves. Theoretical [O'Neil, 1965; Al'tshul and Karpman, 1966] and experimental [Malmberg and Wharton, 1967] investigations of large amplitude electrostatic Landau waves, and recent analytical [Palmedesso and Schmidt, 1971] and computer simulation [Ossakow et al., 1972] studies of large amplitude electromagnetic whistlers, have shown that physically different trapping mechanisms for resonant electrons lead to qualitatively similar temporal evolution in both cases: if the linear damping rate is much smaller than the oscillation frequency of particles in the bottom of the trapping well, after an initial linear damping followed by oscillations of decaying magnitude, the large amplitude waves approach a finite constant amplitude. Experiments on large amplitude Landau waves [Wharton et al. 1968] indicated that this behavior is accompanied by growth of sideband frequencies, and motivated analytical [Kruer et al., 1969; Goldman, 1970; Eldridge, 1970; Yagishita and Ichikawa, 1970; Bud'ko et al., 1971; Brinca, 1972a] and computer simulation [Kruer and Dawson, 1970] studies of this additional phenomenon. The similarity in the

evolution of large amplitude electrostatic and electromagnetic waves naturally poses the question of whether whistler sidebands are also to be expected. We will conclude with a qualified affirmative: parallel whistlers of large amplitude may create electromagnetic sidebands if the unperturbed velocity distribution of the magnetoplasma satisfies two conditions enunciated in Section 4.3.3. The suppression of whistler sidebands for other velocity distributions is mainly due to a phase mixing effect caused by the spread in perpendicular velocities.

Following an approach similar to the method used in the analysis of electrostatic sidebands [Brinca, 1972a], this section stresses the wave-particle interaction aspect of the problem; we concentrate on the transient initial period where the formation of sidebands occurs, and neglect wave coupling interactions. (The feasibility of wave-wave interactions to produce discrete emissions was considered, and rejected, by Harker and Crawford [1969].) We evaluate analytically the distribution function distorted in the resonance region by a large (constant) amplitude whistler (Section 4.3.2). Quasilinear theory is not used in this evaluation because, as shown by Roux and Solomon [1970], it is not applicable to almost monochromatic stimulating signals. After averaging this resonant distribution over one wavelength and one period of the original wave, we analyze the temporal evolution of whistler perturbations impressed on the plasma (Section 4.3.3). When the original electron velocity distribution satisfies the conditions specified in Section 4.3.3, test waves with frequencies in two narrow sidebands of the original wave experience consecutive growths of large magnitudes. The growth of these sidebands occurs during the early stages of the wave-particle interaction (within two periods of oscillation in the bottom of the trapping well) and, together with the creation of a fine jaggedness in the resonance region of the distorted velocity distribution, places an upper limit on the time interval over which the approach adopted is valid. In closing this analysis in Section 4.3.4, we shall assess the relevance of the results to the onset of ASE by adapting the initial value theory to the whistler wavepacket problem.

#### 4.3.2 Distorted Distribution

In determining the stability of whistler test waves coexisting with the original large amplitude whistler, we use below the averaged, slowly evolving, distorted distribution of the electron population. To apply Liouville's theorem in the solution of Vlasov's equation, we evaluate the trajectories in the original whistler field of test electrons in the cyclotron resonance region. Two types of trajectories are obtained, trapped and untrapped, and although the mathematical manipulations are identical to the electrostatic problem, the trapping mechanisms are physically different. Because the nonlinear temporal periodicities of the trajectories are much larger than the period of the original whistler, we obtain the desired slowly evolving, homogeneous, distorted distribution by averaging the solution of the Vlasov equation over one wavelength and one period of the large amplitude wave.

**TEST PARTICLE MOTION.** We shall calculate the nonlinear orbits of electrons in the field of a uniform whistler. Though inexact, the assumption of constant whistler amplitude used below is made plausible by the work of Palmadesso and Schmidt [1971], provided that the original velocity distribution,  $F_0(v_{\parallel}, v_{\perp})$ , yields a linear damping rate,  $\gamma_L$ , of magnitude much smaller than  $\langle \omega_{NL} \rangle$ , the oscillation frequency in the bottom of the trapping well of electrons with perpendicular speed equal to the rms value of  $v_{\perp}$ .

At time  $t = 0$ , a whistler wave of frequency  $\omega_0$ , wavenumber  $k_0$ , and with electric and magnetic field amplitudes  $E$  and  $B$ , is applied to an infinite, homogeneous magnetoplasma of stationary ions with the dc magnetic field along the  $z$ -axis,  $\underline{B}_0 = B_0 \hat{z}$ . We assume that the whistler fields evolve according to

$$\begin{aligned} \underline{B} &= B [\cos(\omega_0 t - k_0 z) \hat{x} + \sin(\omega_0 t - k_0 z) \hat{y}] , \\ \underline{E} &= E [\sin(\omega_0 t - k_0 z) \hat{x} - \cos(\omega_0 t - k_0 z) \hat{y}] . \end{aligned} \quad (4.1)$$

Defining the quantities

$$(\Omega, a) = \frac{e(B_0, B)}{m_e} , \quad V_{G0} = \frac{(\omega_0 - \Omega)}{k_0} , \quad \varphi = \tan^{-1} \left( \frac{v_y}{v_x} \right) , \quad (4.2)$$

where  $e$  and  $m_e$  are the charge (magnitude) and mass of the electron, the orbits of test particles in the fields  $\underline{E}$  and  $\underline{B}_0 + \underline{B}$  are governed by [Dysthe, 1971]

$$\dot{w} = -av_{\perp} \sin \psi, \quad \dot{v}_{\perp} = a \left( w - \frac{\Omega}{k_0} \right) \sin \psi, \quad (4.3)$$

$$\dot{\psi} = k_0 w + \frac{a}{v_{\perp}} \left( w - \frac{\Omega}{k_0} \right) \cos \psi, \quad (4.3)$$

where  $\psi = \varphi + k_0 z - \omega_0 t + \pi$  is the angle between  $-\underline{B}$  and  $\underline{v}_{\perp}$ , and  $w = v_{\parallel} - V_{G0}$ . Manipulation of (4.3) yields the following integral invariants of motion [Dysthe, 1971]

$$u^2 = v_{\perp}^2 + \left( w - \frac{\Omega}{k_0} \right)^2, \quad \chi = w^2 - 2 \frac{av_{\perp}}{k_0} \cos \psi. \quad (4.4)$$

The types and characteristics of the orbits defined by (4.3) and (4.4) have been studied for the general case (see references in Palmadesso and Schmidt [1971]). Here we concentrate on electrons in the resonance region,  $w \approx 0$ , and, for reasons given in Section 4.3.3, we exclude from consideration particles with small perpendicular speeds. Specifically, we restrict the analysis to the regions of velocity space that satisfy

$$\frac{k_0}{\Omega} |w| = o(\delta^{1/2}), \quad \frac{|w|}{v_{\perp}} = o(\delta^{1/2}), \quad (4.5)$$

using  $\delta = a/\Omega$ , with  $\delta^{1/2} \ll 1$ , and the notation  $k_1 = o(k_2)$  indicating that the infinitesimal  $k_1$  is of higher order than  $k_2$ . From (4.3) and (4.4) we can then write

$$v_{\perp} \approx \left[ u^2 - \left( \frac{\Omega}{k_0} \right)^2 \right]^{1/2}, \quad (4.6)$$

$$\ddot{\psi} \approx -\omega_{NL}^2 \sin \psi, \quad \omega_{NL}^2 = a k_0 v_{\perp},$$

i.e., each test particle satisfying (4.5) has a perpendicular velocity of constant magnitude whose angle with respect to  $-\underline{B}$  obeys the pendulum equation with a restoring force proportional to  $v_{\perp}$ , and the whistler amplitude and wavenumber. Introducing

$$\kappa^2 = \frac{4\omega_{NL}^2}{k_0^2 \chi + 2\omega_{NL}^2} \geq 0, \quad (4.7)$$

and

$$\sin(\xi/2) = \kappa \sin(\psi/2), \quad (4.8)$$

the integration of the pendulum equation for  $\psi$  yields

$$F(\psi/2, \kappa) - F(\psi_0/2, \kappa) = \pm \frac{\omega_{NL}}{\kappa} t \quad (\kappa^2 < 1), \quad (4.9)$$

$$F(\xi/2, 1/\kappa) - F(\xi_0/2, 1/\kappa) = \pm \omega_{NL} t \quad (\kappa^2 > 1), \quad (4.10)$$

where  $F(r, s)$  represents the elliptic integral of the first kind with argument  $r$  and modulus  $s$ . From (4.4) and (4.7), the definitions of  $\chi$  and  $\kappa^2$ , we obtain

$$w^2 = \frac{4\omega_{NL}^2}{k_0^2 \kappa^2} [1 - \kappa^2 \sin^2(\psi/2)], \quad (4.11)$$

showing that, for  $\kappa^2 > 1$ , the  $\psi$ -oscillations about zero have a limiting amplitude of  $2 \sin^{-1}(1/\kappa)$ . In this case, we say that the electrons are trapped (with respect to the  $\psi$  variation), in contrast to the situation where  $\kappa^2 < 1$ , and  $\psi$  may vary without bounds. The phase space trajectories of test particles are given by Dysthe [1971], relating  $w$  and  $\psi$  for varying  $\kappa$ , as shown in Figure 4.2.

**SOLUTION OF THE VLASOV EQUATION.** The electron distribution function in the presence of the large amplitude whistler,  $f(v_{\parallel}, v_{\perp}, z, t)$ , is determined by the Vlasov equation,

$$\frac{\partial f}{\partial t} + v_{\parallel} \frac{\partial f}{\partial z} - \frac{e}{m} [\underline{E} + \underline{v} \times (\underline{B}_0 + \underline{B})] \cdot \frac{\partial f}{\partial \underline{v}} = 0, \quad (4.12)$$

and the initial conditions,

$$f(v_{\parallel}, v_{\perp}, z, t = 0) = F_0(v_{\parallel}, v_{\perp}) + f_{10}(v_{\parallel}, v_{\perp}) \cos k_0 z, \quad (4.13)$$

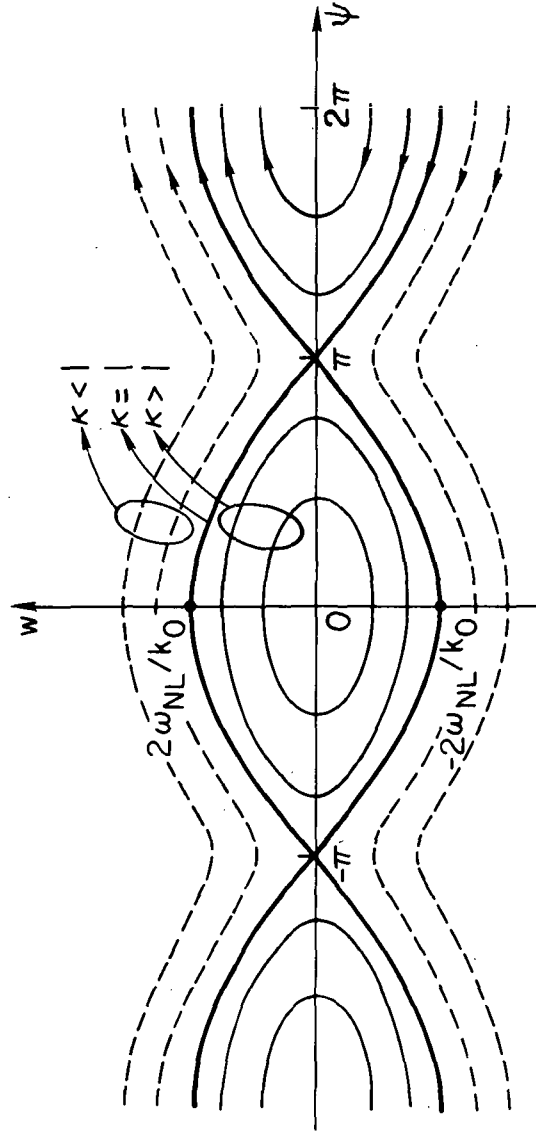


FIG. 4.2. Phase trajectories of untrapped ( $\kappa^2 < 1$ ) and trapped ( $\kappa^2 > 1$ ) electrons in the cyclotron resonance region.  
(Separatrix:  $\kappa^2 = 1$ .)

where the second term on the right-hand side of (4.13) applies the whistler wave at  $t = 0$ . The whistler fields are assumed to satisfy (4.1). Noting, from (4.6), that  $v_{\perp}$  is an invariant of motion under the conditions assumed in (4.5), the application of Liouville's theorem to (4.12) and (4.13) yields the solution

$$f(v_{\parallel}, v_{\perp}, z, t) = F_0[v_{\parallel 0}(v_{\parallel}, v_{\perp}, z, t), v_{\perp}] + f_{10}[v_{\parallel 0}(v_{\parallel}, v_{\perp}, z, t), v_{\perp}] \cos[k_0 z_0(v_{\parallel}, v_{\perp}, z, t)], \quad (4.14)$$

where  $z = z_0$  and  $v_{\parallel} = v_{\parallel 0}$  at  $t = 0$ . The second term on the right-hand side of (4.14) will be discarded since its contribution to the observables of our problem will rapidly tend to zero through phase mixing [Palmadesso and Schmidt, 1971]. Particularizing this solution to the resonance region,  $w = v_{\parallel} - V_{GO} \approx 0$ , allows us to write

$$f(V_{GO} + w, v_{\perp}, z, t) \approx F_0(V_{GO}, v_{\perp}) + w_0(w, v_{\perp}, z, t) F_{\parallel}(V_{GO}, v_{\perp}), \quad (4.15)$$

where  $F_{\parallel} = \partial F_0 / \partial v_{\parallel}$  and  $w = w_0$  at  $t = 0$ .

Since we are assuming the amplitude of the original wave to be constant, we may now use the results previously obtained for the electron orbits in a uniform whistler field. To derive an expression for  $w_0 = w_0(w, v_{\perp}, z, t)$ , we combine (4.11) with (4.9) and (4.10), obtaining

$$w_0 = \pm \frac{2\omega_{NL}}{k_0 \kappa} \operatorname{dn} \left[ F(\psi/2, \kappa) - \frac{\omega_{NL}}{\kappa} t, \kappa \right] \quad (\kappa^2 < 1), \quad (4.16)$$

for the untrapped electrons, and

$$w_0 = \pm \frac{2\omega_{NL}}{k_0 \kappa} \operatorname{cn} \left[ F(\xi/2, 1/\kappa) - \omega_{NL} t, 1/\kappa \right] \quad (\kappa^2 > 1), \quad (4.17)$$

for the trapped electrons, where  $\operatorname{dn}$  and  $\operatorname{cn}$  are Jacobian elliptic functions.

Looking at the explicit expressions for  $\psi$ ,  $\xi$  and  $\kappa$  we note that the distorted resonant distribution (4.15), with  $w_0$  given by (4.16)

or (4.17), shows the existence of a fine structure in time and space, depending on  $\omega_0 t - k_0 z$ , combined with a slow time evolution on the scale of  $\omega_{NL} t$ .

**AVERAGING.** Because in the introduction of Section 4.3.3 we conclude that the slowly evolving part of the distribution function determines the stability properties of small amplitude whistler waves with resonant velocities  $v_G = (\omega - \Omega)/k$  close to  $v_{G0}$ , we now smooth the distribution function (4.15) by averaging  $w_0$  over one period and one wavelength of the original whistler wave. Noting that the fine structure is introduced in the distorted velocity distribution through  $\psi$ , we will smooth (4.15) by averaging over this angle.

As a first step, we attribute to each combination of  $\kappa$  and  $v_\perp$  two symmetric values of  $w$ . The physical meaning of this operation is clear when examining the phase space diagram of Figure 4.2. For a given  $v_\perp$ , we note that each value of  $\kappa$  can indeed be associated, on the average, with two values  $\pm w(\kappa)$ . From (4.11), and for untrapped particles, we obtain

$$\begin{aligned} \langle w \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} w d\psi = \pm \frac{2\omega_{NL}}{\pi k_0 \kappa} \int_0^{\pi} [1 - \kappa^2 \sin^2(\psi/2)]^{1/2} d\psi \\ &= \pm \frac{4\omega_{NL}}{\pi k_0} \frac{E_1(\kappa)}{\kappa} \quad (\kappa^2 < 1), \quad (4.18) \end{aligned}$$

where  $E_1$  is the complete elliptic integral of the second kind. For trapped particles we recall that  $\psi$  oscillates between  $\pm \psi_p = \pm 2 \sin(1/\kappa)$ , so that we find

$$\begin{aligned} \langle w \rangle &= \frac{1}{2\pi} \int_{-\psi_p}^{\psi_p} w d\psi = \pm \frac{4\omega_{NL}}{\pi k_0 \kappa^2} \int_0^{\pi/2} \frac{\cos^2(\xi/2)}{[1 - \sin^2(\xi/2)/\kappa^2]^{1/2}} d(\xi/2) \\ &= \pm \frac{4\omega_{NL}}{\pi k_0} [E_1(1/\kappa) - (1 - 1/\kappa^2) K(1/\kappa)] \quad (\kappa^2 > 1), \quad (4.19) \end{aligned}$$

where  $K$  is the complete elliptic integral of the first kind and we have used standard integral forms [Gradshteyn and Ryzhik, 1965]. Recalling the definition of  $\omega_{NL}$ , we find for the separatrix,  $\kappa = 1$ ,

$$v_{\perp} = v_{\perp}^* = \frac{\pi^2}{16} \frac{k_0}{a} w^2 \quad (\kappa^2 = 1), \quad (4.20)$$

with the following interpretation: on the average, particles with given parallel velocities,  $v_{\parallel} = V_{GO} + w$ , will be trapped when  $v_{\perp} > v_{\perp}^*$ , and untrapped when  $v_{\perp} < v_{\perp}^*$ .

The averaging of  $w_0$  in (4.16) and (4.17) is obtained by first using the trigonometric expansions of the Jacobian elliptic functions [Gradshteyn and Ryzhik, 1965]

$$\text{dn}(x, m) = \frac{\pi}{2K(m)} + \frac{2\pi}{K(m)} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos[n\pi x/K(m)], \quad (4.21)$$

$$\text{cn}(x, m) = \frac{2\pi}{mK(m)} \sum_{n=1}^{\infty} \frac{q^{n-1/2}}{1+q^{2n-1}} \cos[(n-1/2)\pi x/K(m)], \quad (4.22)$$

where the nome is defined by

$$q = \exp \left\{ -\pi K[(1-m^2)^{1/2}] / K(m) \right\}.$$

Combining (4.16) and (4.21), and using Eq. (A3) of Palmadesso and Schmidt [1971],

$$\int_{-\pi/2}^{\pi/2} \cos \left[ \frac{n\pi F(y, m)}{K(m)} \right] dy = \frac{2\pi q^n}{1+q^{2n}},$$

we obtain for the untrapped particles,

$$\begin{aligned}\langle w_0 \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} w_0 d\psi \quad (\kappa^2 < 1) \\ &= \frac{\pi \omega_{NL}}{k_0 \kappa K(\kappa)} \left\{ 1 + 8 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1+q^{2n})^2} \cos \left[ \frac{n\pi}{\kappa K(\kappa)} \omega_{NL} t \right] \right\}, \quad (4.23)\end{aligned}$$

and, for the trapped particles, from (4.17), (4.22) and Eq. (A2) of Palmadesso and Schmidt [1971],

$$m \int_{-K(m)}^{K(m)} \text{cn}(y, m) \cos \left[ \frac{(2n+1)\pi y}{2K(m)} \right] dy = \frac{2\pi q^{n+1/2}}{1+q^{2n+1}},$$

we get

$$\begin{aligned}\langle w_0 \rangle &= \frac{1}{2\pi} \int_{-\psi_p}^{\psi_p} w_0 d\psi \quad (\eta^2 = 1/\kappa^2 < 1) \\ &= \frac{8\pi \omega_{NL}}{k_0 K(\eta)} \left\{ \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} \cos \left[ \frac{(n-1/2)\pi}{K(\eta)} \omega_{NL} t \right] \right\}. \quad (4.24)\end{aligned}$$

Both in (4.23) and (4.24) we have dropped the  $\pm$  signs of (4.16) and (4.17).

The series defining  $\langle w_0 \rangle$  can also be summed in closed form by averaging  $w_0$  directly from (4.16) and (4.17), without resorting to the trigonometric expansions (4.21) and (4.22). We find [Brinca, 1972a]

$$\begin{aligned}\langle w_0 \rangle &= \frac{4\omega_{NL}}{\pi k_0 \kappa} \text{dn}(\omega_{NL}/\kappa, \kappa) \left\{ K(\kappa) - \frac{\cos \beta}{\sin \beta \text{dn}(\omega_{NL}/\kappa, \kappa)} [K(\kappa)E(\beta, \kappa) - E_1(\kappa)F(\beta, \kappa)] \right\} \\ &\quad (\kappa^2 < 1), \quad (4.25)\end{aligned}$$

$$\langle w_0 \rangle = \frac{4\omega_{NL}}{\pi k_0} \text{cn}(\omega_{NL} t, \eta) \left\{ \eta^2 K(\eta) - \frac{\text{dn}(\omega_{NL} t, \eta)}{\sin \alpha \cos \alpha} [K(\eta) E(\alpha, \eta) - E_1(\eta) F(\alpha, \eta)] \right\}$$

$$(\eta^2 = 1/\kappa^2 < 1) , \quad (4.26)$$

where  $\beta = \sin^{-1} [|\text{sn}(\omega_{NL} t/\kappa, \kappa)|]$  ;  $\alpha = \sin^{-1} [|\text{sn}(\omega_{NL} t, \eta)|]$  , and  $E(r, s)$  is the elliptic integral of the second kind with argument  $r$  and modulus  $s$  .

The smooth resonant distribution, slowly evolving in time due to the presence of the large amplitude whistler wave, is thus given by

$$\langle f \rangle \approx F_0(V_{G0}, v_{\perp}) + \langle w_0(w, v_{\perp}, \omega_{NL} t) \rangle F_{\parallel}(V_{G0}, v_{\perp}) , \quad (4.27)$$

where  $\langle w_0 \rangle$  is obtained from (4.23) or (4.25) for untrapped particles ( $|w| > w^*$ ), and from (4.24) or (4.26) for trapped particles ( $|w| < w^*$ ), with

$$w^* = \frac{4\omega_{NL}}{\pi k_0} . \quad (4.28)$$

Expressing the results in terms of  $\omega_{NL}$  , we plot  $\langle w_0 \rangle$  in Figure 4.3 as a function of  $w$  for several instants of normalized time,  $\tau = \omega_{NL} t$  . Using (4.27), the averaged resonant distribution,  $\langle f \rangle$  , is readily pictured from these curves; we note that the  $v_{\perp}$ -dependence is contained in  $\omega_{NL}$  . As  $v_{\perp}$  increases, the distribution (via  $\langle w_0 \rangle$ ) will have a more rapid temporal evolution and larger critical velocities  $w^*$  . This dependence on  $v_{\perp}$  yields strong phase mixing in all observables requiring an integration over  $v_{\perp}$  . In particular, the growth rates experienced by test waves with  $V_G \approx V_{G0}$  , and computed in Section 4.3.3, will demonstrate the influence of this phase mixing effect. Figure 4.3 also shows that, as time progresses, the distorted distribution becomes increasingly jagged, in correspondence with the phase space movement of the trapped, and nearly trapped, electrons. As pointed out by Sagdeev and Galeev [1969] in the electrostatic problem, this jaggedness will invalidate the approach taken here for large times (a few  $2\pi/\langle \omega_{NL} \rangle$ ); the analysis presented in Section 4.3.4 will show, however, that the creation of sidebands may occur within the first two nonlinear periods ( $t < 4\pi/\langle \omega_{NL} \rangle$ ).

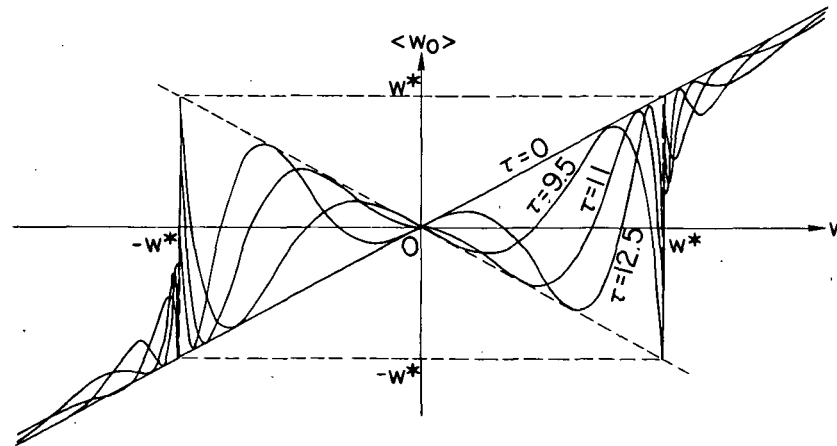
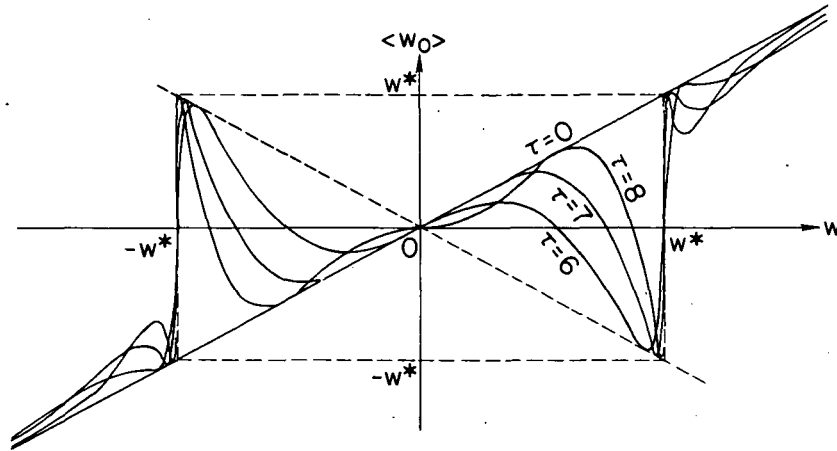
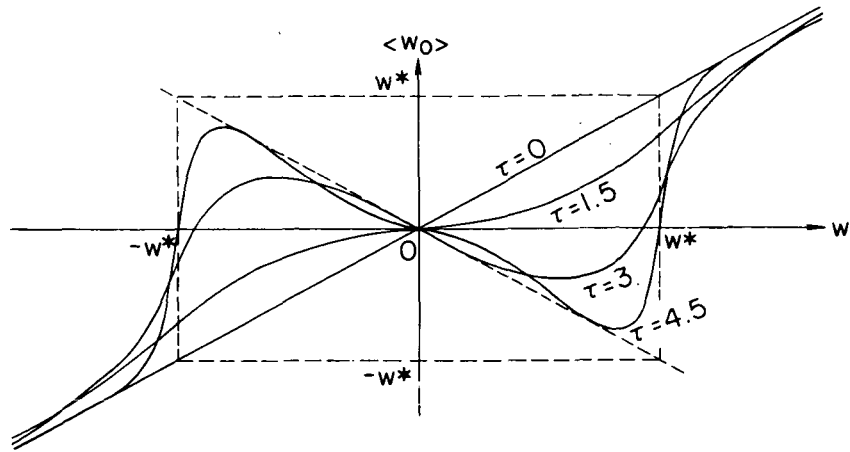


FIG. 4.3. Temporal evolution of  $\langle w_0 \rangle$ . The distorted, averaged, electron velocity distribution in the resonance region is  $\langle f \rangle \approx F_0(V_{GO}, v_\perp) + \langle w_0 \rangle F_\parallel(V_{GO}, v_\perp)$ . Note that both  $\tau (= \omega_{NL} t)$  and  $w^*(= 4\omega_{NL}/\pi k_0)$  depend on  $v_1 (\omega_{NL}^2 = a k_0 v_1)$ .

### 4.3.3 Sideband Stability

THEORY. We demonstrate the possible formation of whistler sidebands by studying the evolution of test waves  $(\omega, k)$  impressed on the plasma perturbed by the large amplitude wave  $(\omega_0, k_0)$ . This plasma displays fine variations in time and space  $(\omega_0 t - k_0 z)$ , together with slow temporal variations  $(\langle \omega_{NL} \rangle t)$ . We expect the test waves to have also a fine structure, and a slow temporal variation that can be described by their growth rate  $\gamma(\langle \omega_{NL} \rangle t)$ . Thus, to characterize the test waves, we need to relate (real)  $\omega$  with  $k$ , and determine their slow amplitude variation,  $\exp \gamma t$ . For the evaluation of the growth rate we use the distorted distribution averaged over its fine structure  $(\omega_0 t - k_0 z)$ , as obtained in Section 4.3.2. We note that the averaged distribution,  $\langle f \rangle$ , takes into consideration the presence of the original wave, and reflects the existence of trapped and untrapped particles. Once  $\langle f \rangle$  is known, and for  $\omega \gg \langle \omega_{NL} \rangle$ , so that  $\langle f \rangle$  is almost constant with respect to the test wave, the linear whistler stability theory is valid to study the growth rates experienced by (infinitesimal) test waves  $(\omega, k)$  impressed on the evolving plasma, provided the wave-particle interactions predominate over the mode-coupling interactions. The velocity distribution distorted by the large amplitude whistler is given in (4.15), and can be expressed as

$$f(z, v_{\parallel}, v_{\perp}, t) = \langle f(v_{\parallel}, v_{\perp}, \omega_{NL} t) \rangle + f_1(z, v_{\parallel}, v_{\perp}, t), \quad (4.29)$$

where  $f_1$  defines the fine structure of the distorted distribution, and  $\langle f \rangle$  was evaluated in Section 4.3.2. Denoting by  $g(z, v_{\parallel}, v_{\perp}, t)$  the perturbation in  $f$  ( $\gg g$ ) caused by the test wave with fields  $\underline{E}_t(z, t)$  and  $\underline{B}_t(z, t)$ , the linearized Vlasov equation for  $g$  reads

$$\begin{aligned} \frac{\partial g}{\partial t} + v_{\parallel} \frac{\partial g}{\partial z} - \frac{e}{m} \underline{v} \times \underline{B}_0 \cdot \frac{\partial g}{\partial \underline{v}} - \frac{e}{m} [\underline{E}_t + \underline{v} \times \underline{B}_t] \cdot \frac{\partial \langle f \rangle}{\partial \underline{v}} \\ = \frac{e}{m} [\underline{E}_t + \underline{v} \times \underline{B}_t] \cdot \frac{\partial f_1}{\partial \underline{v}} + \frac{e}{m} [\underline{E} + \underline{v} \times \underline{B}] \cdot \frac{\partial g}{\partial \underline{v}}. \end{aligned} \quad (4.30)$$

The right-hand side of (4.30) relates to mode-coupling interactions, in contrast to the last term on the left-hand side which describes the direct interaction between the wave and the plasma particles [Yagishita

and Ichikawa, 1970]. Here we neglect the mode-coupling interactions: a crude estimate of orders of magnitude indicates that the right-hand side is of higher order than the left-hand side at the early stages of evolution. (For the electrostatic problem, a detailed study [Bud'ko et al., 1971] has shown that the parametric interactions only become significant for perturbations with frequencies  $\omega$  such that  $|\omega - \omega_0|$  is much smaller than  $\omega_{NL}$ .) Equation (4.30) with the right-hand side set equal to zero, and Maxwell's equations

$$\nabla \times \underline{E}_t = -\frac{\partial \underline{B}_t}{\partial t}, \quad \nabla \times \underline{B}_t = \frac{1}{c^2} \frac{\partial \underline{E}_t}{\partial t} - \frac{e}{\epsilon_0 c^2} \int g \underline{v} \, dv$$

form the basis of linear whistler stability analysis in a magnetoplasma with an 'equilibrium' distribution  $\langle f(v_{\parallel}, v_{\perp}, \omega_{NL} t) \rangle$  slowly evolving in time [Stix, 1962]. Thus, assuming plane wave propagation as  $\exp[\gamma t + i(\omega t - kz)]$ ,  $|\gamma| \ll \omega$ , the growth rate experienced by these whistler test waves is determined by [Vedenov et al., 1962]

$$\frac{\gamma}{\Omega} = -\left[\pi\left(1 - \frac{\omega}{\Omega}\right)\right]^2 \int_0^{\infty} \left[2 \frac{\Omega}{k} \langle f \rangle - v_{\perp}^2 \frac{\partial \langle f \rangle}{\partial w}\right]_{w=V_G - V_{GO}} v_{\perp} \, dv_{\perp}, \quad (4.31)$$

or, substituting for  $\langle f \rangle$ ,

$$\begin{aligned} -\left[\pi\left(1 - \frac{\omega}{\Omega}\right)\right]^{-2} \frac{\gamma}{\Omega} &= 2 \frac{\Omega}{k} \int_0^{\infty} f_0(v_{GO}, v_{\perp}) v_{\perp} \, dv_{\perp} \\ &+ 2 \frac{\Omega}{k} \int_0^{\infty} \langle w_0 \rangle \left| F_{\parallel}(v_{GO}, v_{\perp}) v_{\perp} \, dv_{\perp} - \int_0^{\infty} \frac{\partial \langle w_0 \rangle}{\partial w} \right|_{w=V_G - V_{GO}} F_{\parallel}(v_{GO}, v_{\perp}) v_{\perp}^3 \, dv_{\perp}. \end{aligned} \quad (4.32)$$

For a given test wave defined by the value of  $w = V_G - V_{GO}$ , the range of integration over  $v_{\perp}$  is conveniently divided into  $(0, v_{\perp}^*)$  and  $(v_{\perp}^*, \infty)$ , corresponding to untrapped ( $\kappa < 1$ ) and trapped ( $\kappa > 1$ ) particles,

respectively. Using (4.18) and (4.19) we change the variable of integration into  $\kappa$  and  $\eta = 1/\kappa$ ,

$$v_{\perp} = \frac{v_{\perp}^* \kappa^2}{[E_1(\kappa)]^2} \quad (v_{\perp} < v_{\perp}^*), \quad (4.33)$$

$$v_{\perp} = \frac{v_{\perp}^*}{[E_1(\eta) - (1-\eta^2)K(\eta)]^2} \quad (v_{\perp} > v_{\perp}^*), \quad (4.34)$$

and utilize (4.23) and (4.24) to express the growth rate of whistler perturbations in the form:

$$\begin{aligned} & -2 \left[ \pi \left( 1 - \frac{\omega}{\Omega} \right) \right]^{-2} \frac{\gamma}{\Omega} = \frac{\Omega}{k} \int_0^{\infty} f_0(v_{GO}, v_{\perp}) v_{\perp} dv_{\perp} \\ & + (v_{\perp}^*)^2 \int_0^1 \frac{\kappa^3 K}{E_1^5} F_{\parallel}[v_{GO}, v_{\perp}(\kappa)] \left\{ \frac{\pi}{K} \left[ \frac{2\Omega}{k k_0} \frac{\omega_{NL}(\kappa)}{\kappa} - \frac{\pi v_{\perp}^2(\kappa) E_1}{4(1-\kappa^2)K^2} \right] \right. \\ & \times [1 + 8A(\kappa)] + \frac{2v_{\perp}^2(\kappa)}{\kappa(1-\kappa)^2} \left( \frac{\pi}{K} \right)^4 B(\kappa) \left. \right\} d\kappa + (v_{\perp}^*)^2 \int_0^1 \frac{\eta K}{[E_1 - (1-\eta^2)K]^5} \\ & \times F_{\parallel}[v_{GO}, v_{\perp}(\eta)] \left\{ \frac{2\pi}{K} \left[ \frac{8\Omega}{k k_0} \omega_{NL}(\eta) + \frac{\pi v_{\perp}^2(\eta)}{\eta^2(1-\eta^2)} \frac{E_1 - (1-\eta^2)K}{K^2} \right] c(\eta) \right. \\ & \left. - \frac{v_{\perp}^2(\eta)}{\eta^2(1-\eta^2)} \left( \frac{\pi}{K} \right)^4 D(\eta) \right\} d\eta, \quad (4.35) \end{aligned}$$

using

$$\begin{aligned}
A(\kappa) &= \sum_{n=1}^{\infty} \frac{p_u}{(1+p_u)^2} \cos \alpha_u, \\
B(\kappa) &= \sum_{n=1}^{\infty} \frac{n p_u}{(1+p_u)^2} \left[ \kappa \left( \frac{1-p_u}{1+p_u} \right) \cos \alpha_u + \frac{E_1}{\pi} \omega_{NL}(\kappa) t \sin \alpha_u \right], \\
\alpha_u &= \frac{n\pi}{\kappa K} \omega_{NL}(\kappa) t, \quad p_u = q^{2n},
\end{aligned} \tag{4.36}$$

for the untrapped electrons, and

$$\begin{aligned}
C(\eta) &= \sum_{n=1}^{\infty} \frac{p_t}{(1+p_t)^2} \cos \alpha_t, \\
D(\eta) &= \sum_{n=1}^{\infty} \frac{(2n-1)p_t}{(1+p_t)^2} \left[ \left( \frac{1-p_t}{1+p_t} \right) \cos \alpha_t + \frac{E_1 - (1-\eta^2)K}{\pi} \omega_{NL}(\eta) t \sin \alpha_t \right], \\
\alpha_t &= \frac{(2n-1)\pi}{2K} \omega_{NL}(\eta) t, \quad p_t = q^{2n-1}
\end{aligned} \tag{4.37}$$

for the trapped electrons, where the nome  $q$  was defined for the expansions (4.21) and (4.22).

The fine structure of the test waves  $(\omega, k)$ , can be characterized by (4.43), the whistler cold dense plasma dispersion relation, because the magnetospheric plasma is made up of a cold dense component (which we assume to define the real frequency and wavenumber of test waves) permeated by a dilute hot electron population (which is distorted in the cyclotron resonance region by the original whistler and determines the growth rate of test waves).

The main features of the test wave stability can be predicted from examination of Figure 4.3 and the expressions for the growth rates. Using  $\partial \langle w_0(w) \rangle / \partial w = \partial \langle w_0(-w) \rangle / \partial w$  in (4.32), we note that perturbations in symmetric sidebands with respect to  $V_{G0}$  display almost the same behavior; the differences in the growth rates for two symmetric sidebands arise from the second term on the right-hand side of (4.32), and will be

analyzed in the discussion of the numerical results. The first term in (4.31),  $I_1 = 2(\Omega/k) \int_0^\infty \langle f \rangle v_\perp dv_\perp$ , always contributes to whistler damping ( $\gamma < 0$ ); the second term,  $I_2 = \int_0^\infty v_\perp^3 (\partial \langle f \rangle / \partial w) dv_\perp$ , depending on the value of the derivative,  $\partial \langle f \rangle / \partial w$ , may enhance or reduce this damping, bringing about growth when  $I_2 > I_1$ . Figure 4.3 shows that the value of this derivative (and hence  $I_2$ ) is strongly influenced by the large amplitude whistler wave, in contrast to  $\langle f \rangle$  (and hence  $I_1$ ) which are only slightly affected by the original wave. It follows that the effect of the large amplitude whistler on the growth rates of the test waves will be negligible if  $I_1 \gg |I_2|$ . Thus, a necessary condition for the creation of sidebands of the original wave is that  $I_1$  is of the same order of magnitude (or smaller than)  $I_2$  at  $t = 0$ . For initial distributions with no humps along  $v_\parallel < 0$ , i.e.,  $F(v_{G0}, v_\perp) > 0$ , one has  $I_2(t = 0) > 0$  so that in a magnetoplasma yielding linear damping for the original whistler,  $\gamma_L < 0$ , the foregoing necessary condition becomes  $I_1 \gtrsim I_2$ . In contrast, when linear growth occurs,  $\gamma_L > 0$ , this necessary condition is always satisfied.

The normalized time parameter  $\tau$  in Figure 4.3 is simply proportional to time  $t$  when the unperturbed velocity distribution is monochromatic in  $v_\perp$ . This figure then suggests that whistler perturbations with resonant velocities close to  $V_{G0}$ ,  $w \approx 0$ , will experience an enhancement of the damping as time progresses, in contrast to perturbations with  $w \approx \pm w^*$ , (where  $\partial \langle f \rangle / \partial w$  increases during the early stages of the evolution) which may experience growth. If the unperturbed distribution is not monochromatic in  $v_\perp$ , the time parameter  $\tau$  is proportional to both time  $t$  and  $v_\perp^{1/2}$ . The temporal evolution of the distribution depends upon the value of  $v_\perp$  under consideration; for a fixed time, the dependence of the integrand of  $I_2$  (and  $I_1$ ) on the perpendicular velocity will bring about a phase mixing that will smear out the behavior observed for the monochromatic distribution in  $v_\perp$ . We thus expect that a second necessary condition for the creation of whistler sidebands will be the existence of a certain degree of monochromaticity in  $v_\perp$  for the unperturbed velocity distribution. The

two necessary conditions are interrelated (the degree of  $v_1$ -monochromaticity influences the value of  $I_1/I_2$ ) and can only be presented in a quantitative form when the characteristics of the whistler noise, which will eventually grow to form sidebands, are known. By 'noise' we mean both the thermal spectrum arising from charge density fluctuations in the magnetoplasma, and the frequency spectrum of the original wave which occupies a finite band about  $\omega_0$  due to the amplitude modulation experienced by the large amplitude whistler in its temporal evolution, as noted in Section 4.3.1, and the finite duration of the original wave train.

To analyze the creation of whistler sidebands, we study the temporal behavior of the growth rates for test waves with resonant velocities  $V_G = V_{GO} + w$  close to  $(\omega_0 - \Omega)/k_0 = V_{GO}$ . The time interval of interest goes from  $t = 0$  to  $t \approx 4\pi/\langle\omega_{NL}\rangle$ ; for larger times, the jaggedness of the distorted distribution mentioned at the end of Section 4.3.2 makes our approach invalid. We shall demonstrate, however, that the creation of sidebands may occur for times smaller than  $4\pi/\langle\omega_{NL}\rangle$ .

In analyzing  $\gamma(t)$  for each test wave, we look for the intervals of time  $(t_i, t_{i+1})$  where growth occurs ( $\gamma > 0$ ), compute the consecutive logarithmic growth experienced in those intervals,  $\Lambda_i = \int_{t_i}^{t_{i+1}} \gamma(t) dt$ , and determine the maximal consecutive growth experienced by the test wave under consideration,  $\Lambda_M = \max(\Lambda_i)$ . The simple cumulative growth,

$\int_0^t \gamma(t) dt$ , is not physically meaningful in evaluating the growth of sidebands originated from noise: for example, even if the perturbation  $(\omega, k)$  is consistently attenuated from  $t = 0$  to  $t = t_1$ , the random nature of the fluctuations makes it possible that the amplitude of  $(\omega, k)$  will recover its original ( $t = 0$ ) value at  $t = t_1$ , regardless of the value of the cumulative growth from  $t = 0$  to  $t = t_1$ . Stated differently, rather than considering cumulative growth, we should look for the maximal consecutive growth experienced by each perturbation.

In what follows, to study the temporal evolution of the growth rates for each test wave in plasmas with arbitrary distributions, we introduce a normalized time parameter that is always proportional to time. We define  $\theta = \bar{\omega}_{NL} t$ , with  $\bar{\omega}_{NL} = (a k_0 \bar{v}_\perp)^{1/2}$ , where  $\bar{v}_\perp$  is a constant value of the perpendicular speed to be defined conveniently in the applications.

**RESULTS.** To particularize the theory developed above, we choose an equilibrium distribution that fulfills the two necessary conditions for the creation of sidebands when appropriate parametrization is used. We assume that the large amplitude whistler wave is applied at time  $t = 0$  to a cold dense plasma neutralized with stationary ions and permeated by a dilute hot electron population of fractional number density  $\beta (\ll 1)$ . The electron plasma frequency is  $\omega_p (\gg \Omega)$  and we use, for the hot plasma distribution,

$$f_0(v_\parallel, v_\perp) = \beta A_N \exp(-v_\parallel^2/v_p^2) \exp[-(v_\perp - \bar{v}_\perp)^2/v_c^2], \quad (4.38)$$

with the normalization,

$$2\pi \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} f_0 v_\perp dv_\perp = \beta \quad (4.39)$$

defining  $A_N$ . The mean square velocities of the energetic electron population are given by

$$\langle v_\parallel^2 \rangle = \frac{v_p^2}{2}, \quad \langle v_\perp^2 \rangle = 2\pi^{3/2} v_p A_N B_N, \quad (4.40)$$

with

$$B_N = \int_0^{\infty} v_\perp^3 \exp[-(v_\perp - \bar{v}_\perp)^2/v_c^2] dv_\perp. \quad (4.41)$$

The linear growth rates for small amplitude whistlers with resonant velocities  $V_G = (\omega - \Omega)/k$  are obtained from (4.31) and (4.38),

$$\frac{\gamma_L}{\Omega} = -\beta \left[ \pi \left( 1 - \frac{\omega}{\Omega} \right) \right]^2 \left( \frac{2A_N^B V_G}{v_p^2} + \frac{\Omega}{\pi^{3/2} k v_p} \right) \exp \left( -v_G^2 / v_p^2 \right), \quad (4.42)$$

with the whistler real frequency and wavenumber satisfying the cold dense plasma dispersion relation [Dysthe, 1971]

$$k^2 c^2 = \frac{\omega^2 \omega_p}{\Omega - \omega} \quad (\omega_p^2 \gg \Omega^2), \quad (4.43)$$

where  $c$  is the free space speed of light.

It is clear from (4.38) that a sufficiently large value of the ratio  $\bar{v}_\perp / v_c$  will satisfy the requirement for the creation of sidebands related to the  $v_\perp$ -monochromaticity of  $f_0$ . To comply with the other necessary condition,  $|I_2/I_1|$  of order unity or larger, we have computed this ratio for several combinations of  $\bar{v}_\perp$  and  $v_p$  (and  $\bar{v}_\perp / v_c$ ), with  $\omega_p / \Omega = 10$  and  $\omega_0 / \Omega = 0.5$ , so that  $V_{G0}/c = -0.05$ . (This choice of  $\omega_p$  and  $\omega_0$  is due to our interest in triggered whistler emissions; we wish to model a region of the plasmasphere near the equatorial plane, between three and four earth radii.) The results thus obtained show that the velocities  $\bar{v}_\perp / c = v_p / c = 0.025$  yield ratios  $|I_2/I_1|$  of order unity for the frequency range of interest to us (neighborhood of  $\omega_0$ ); these values for  $\bar{v}_\perp$ ,  $v_p$ ,  $\omega_p$  and  $\omega_0$  will be used in the applications. Figure 4.4 displays the dependence on  $\bar{v}_\perp / v_c$  of  $|I_2/I_1|$ ,  $\gamma_L$  and the rms of  $v_\perp$ . As  $\bar{v}_\perp / v_c$  decreases to approach unity, the character of the linear behavior is drastically changed. In particular, the linear stability of the original whistler changes from damping to growth. Since we wish to stress that the creation of sidebands may occur in magnetoplasmas yielding linear damping for the large amplitude whistler wave, the applications will use  $\bar{v}_\perp / v_c \geq 2$ ; in this domain the mean kinetic energy of the hot electron population is about 250 ev. We note, however, that the creation of sidebands, other conditions being identical, is obtained more easily when  $|I_2/I_1| > 1$  at  $t = 0$ , i.e., when the original whistler is linearly unstable.

Using an intensity for the original whistler wave of  $a/\Omega = 4 \times 10^{-4}$  (the 'high field' case referred to by Dysthe [1971]), we find  $\bar{\omega}_{NL}/\Omega = 10^{-2}$  and  $\bar{\omega}_{NL}/ck_0 = 10^{-3}$ . We recall that, in order to use the results of Palmalessio and Schmidt [1971] on the behavior of the original whistler, we have assumed that  $|\gamma_L|$  is much smaller than  $\langle \omega_{NL} \rangle$ ; since Figure 4.4 shows that  $|\gamma_L/\Omega| \approx 10^{-2} \beta$  and  $\langle \omega_{NL} \rangle$  is larger than  $\bar{\omega}_{NL}$ , the foregoing assumption is satisfied when  $\beta \ll 1$ .

The discussion of Figure 4.3 for  $v_{\perp}$ -monochromatic distributions has suggested that large consecutive growth may occur for whistler test waves with resonant velocities  $V_G \approx V_{GO} \pm \langle w^* \rangle$ , where  $\langle w^* \rangle = (4/\pi)(\langle \omega_{NL} \rangle/k_0)$ . Since for large values of  $\bar{v}_{\perp}/v_c$  we have  $\langle \omega_{NL} \rangle \approx \bar{\omega}_{NL}$ , we will explore the behavior of whistler perturbations with values of  $w = V_G - V_{GO}$  close to  $\pm \langle w^* \rangle = \pm (4/\pi) 10^{-3} c$ .

From the dispersion relation (4.43), the real frequency of the perturbation,  $\omega$ , can be related to  $w$ . When  $|w| \ll |V_{GO}|$ , we find

$$w = \left(1 + \frac{\Omega}{2\omega_0}\right) \left(\frac{\omega - \omega_0}{k_0}\right), \quad (4.44)$$

or, for  $\omega_0/\Omega = 0.5$  and  $\omega_p/\Omega = 10$ ,

$$\frac{\omega}{\Omega} \approx 0.5 + \frac{5w}{c}. \quad (4.45)$$

In Figures 4.5 - 4.7, together with typical curves of  $\gamma(t)$ , we represent the maximal consecutive logarithmic growths,  $\Lambda_M$ , of test waves near  $\langle w^* \rangle$ , in the interval of time between  $\theta = 0$  and  $\theta = 9$ , for several values of  $\bar{v}_{\perp}/v_c$ . Also shown is the total (logarithmic) linear attenuation,  $\Lambda_L$ , suffered by the test waves if the large amplitude whistler wave were not present, during the same period of time in which the maximal consecutive growth takes place. Perturbations with symmetric values of  $V_G$  with respect to  $V_{GO}$ ,  $w < 0$ , would experience similar evolution, except for the small effect introduced by the second term on the right-hand side of (4.32). At the early stages of the interaction ( $\theta \approx 0$ ), and for the parameters used above, this effect

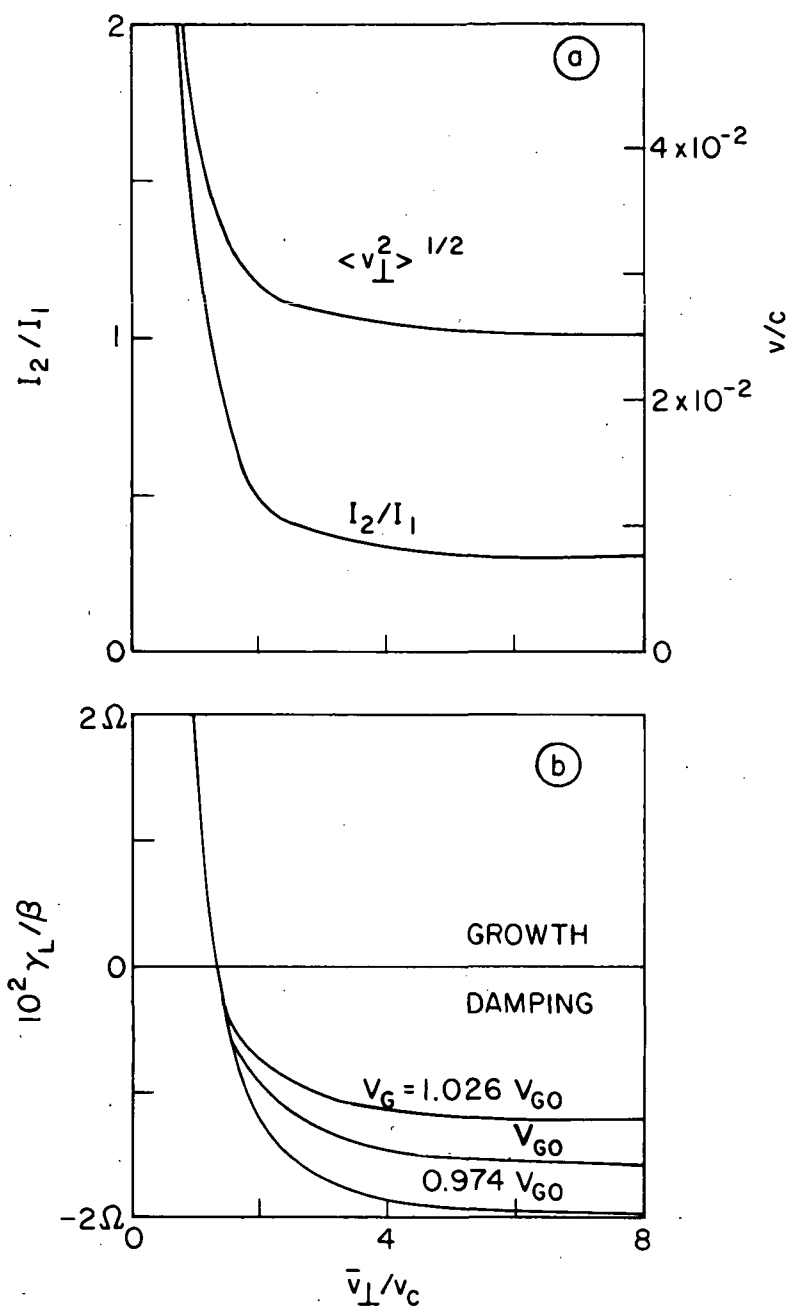


FIG. 4.4. Influence of the  $v_\perp$ -monochromaticity on the linear hot plasma characteristics. (a) Root mean square perpendicular speed and  $I_2/I_1$ . (b) Whistler growth rates.

slightly enhances the damping of the upper sideband (see Figures 4.3 and 4.4); when the original whistler is linearly unstable, the effect is even smaller because the last term on the right-hand side of (4.32) predominates. The outcome of the competition between the growth of the two sidebands will certainly be decided by further nonlinear effects depending on the finite amplitude of the sidebands. The results shown in Figures 4.5, 4.6 and 4.7 were obtained from (4.35) using the first eight terms of the series in (4.36) and (4.37), after particularizing for the hot electron distribution defined in (4.38); the growth rates were evaluated at  $\theta = 0, 1, 2, \dots, 9$ . The truncation of this series is valid everywhere except in a very narrow band of  $\kappa$  (and hence  $v_{\perp}$ ) centered at  $\kappa = 1$  (separatrix); for example, the first five terms of (4.23) and (4.24) reproduce the exact expressions (4.25) and (4.26) up to the sixth significant figure outside the interval  $0.9975 < \kappa < 1.0025$ , and in the range of  $\omega_{NL} t$  used above.

A quantitative discussion of the creation of sidebands requires knowledge of both  $\beta$  and whistler noise characteristics, and will be made in Section 4.3.4, in connection with the conditions prevailing at the onset of artificially stimulated emissions. It is possible, however, to infer from Figures 4.5, 4.6 and 4.7 that if sideband growth occurs, the sidebands will be very narrow in frequency. Indeed, the maximal growth affects test waves with resonant velocities within a range of width  $\Delta\omega \approx 4 \times 10^{-4} c$  or, using (4.45), in a frequency band of  $\Delta\omega/\Omega \approx 2 \times 10^{-3}$ .

#### 4.3.4 Discussion

**SIDEBANDS IN THE MAGNETOSPHERE.** Sideband growth in parallel whistlers requires satisfaction by the unperturbed energetic velocity distribution of the two conditions stated in Section 4.3.3. It seems possible that these necessary conditions may be met in the magnetosphere. Indeed, the condition requiring  $|I_2/I_1|$  to be of order unity, or larger, can be easily met. Linearly (whistler) unstable magnetoplasmas always satisfy this condition; for stable distributions, it either requires an original wave with adequate frequency, or certain kinetic (perpendicular and parallel) energies for the nonthermal electrons. In Section 4.3.3 we

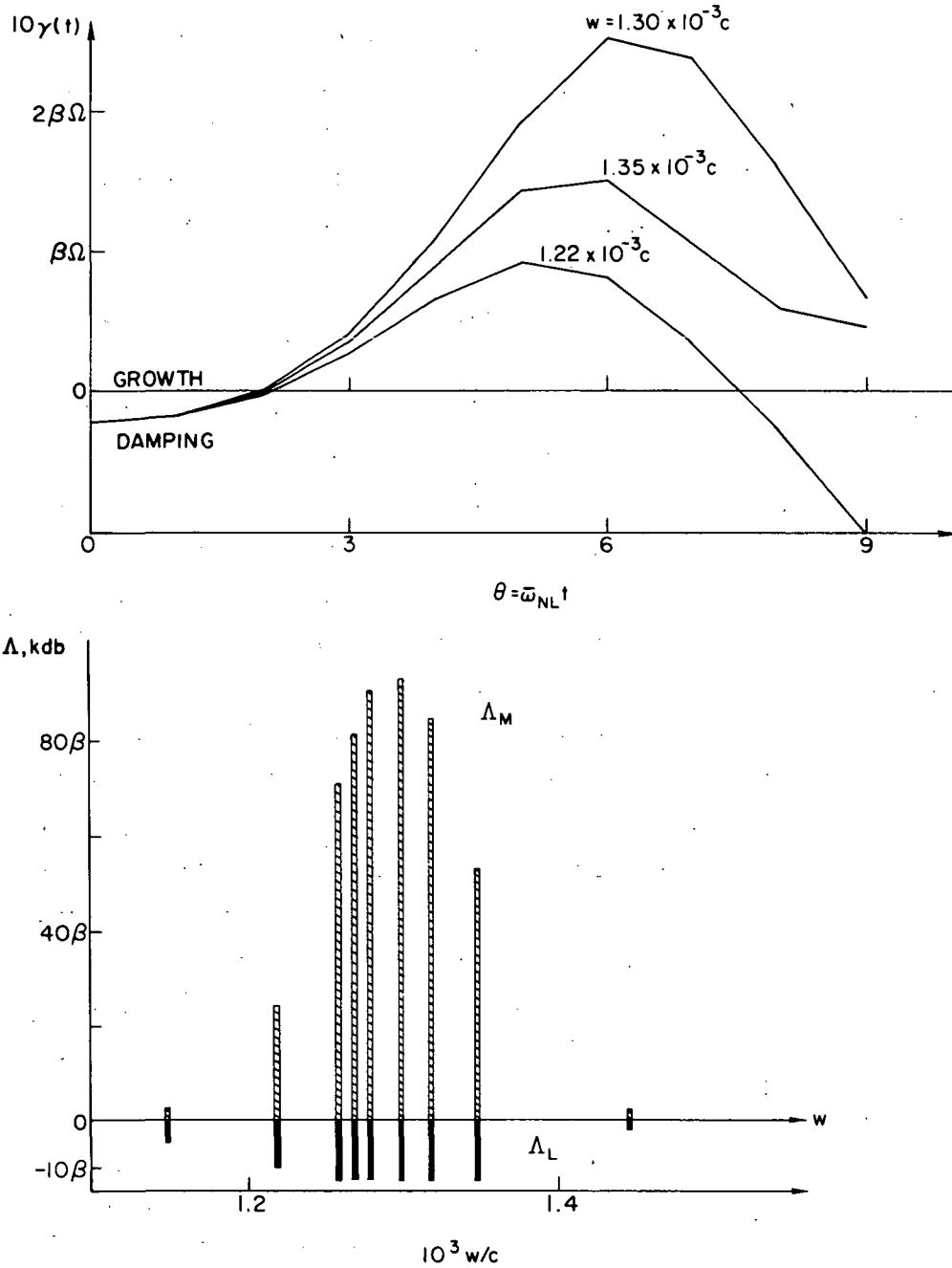


FIG. 4.5. Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Delta_M$ ), and linear attenuation ( $\Delta_L$ ) in the absence of the large amplitude whistler ( $\beta$  is the fractional number density of the hot plasma). [ $\bar{v}_\perp/v_c = 8$ ].

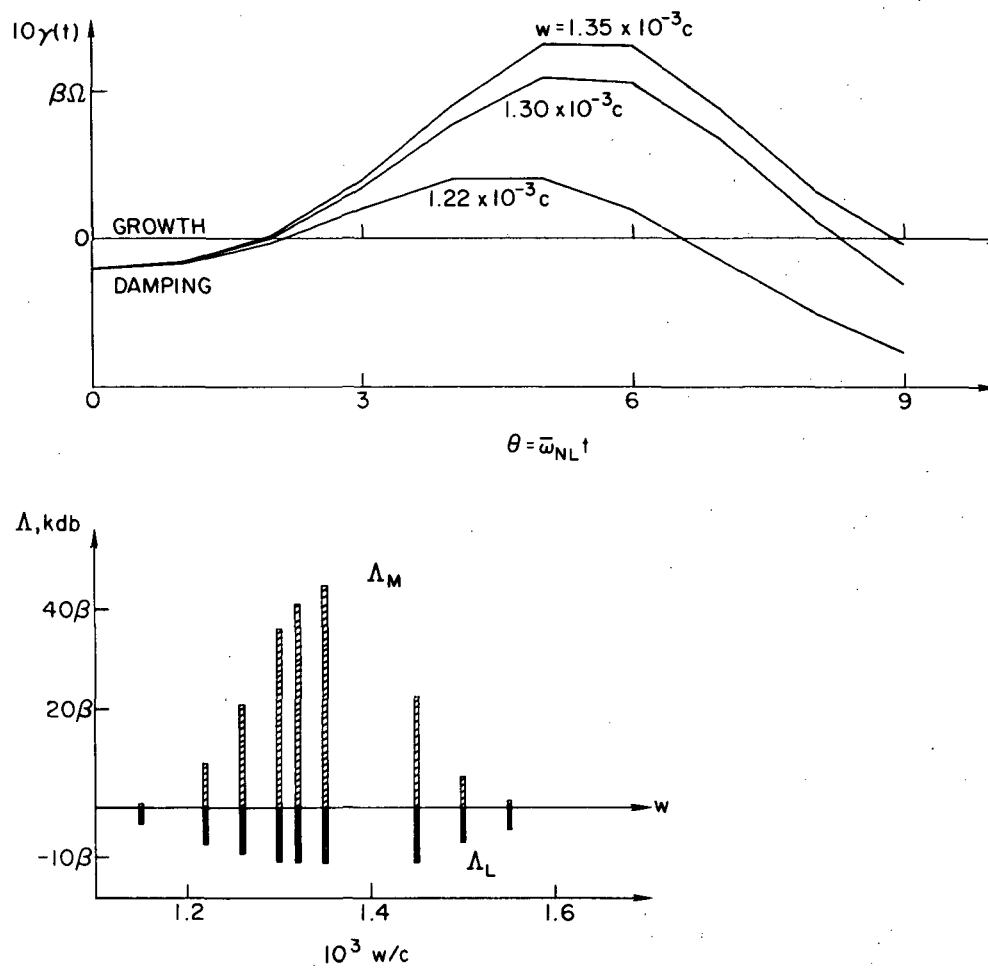


FIG. 4.6. Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Delta_M$ ), and linear attenuation ( $\Delta_L$ ) in the absence of the large amplitude whistler ( $\beta$  is the fractional number density of the hot plasma).  $[\bar{v}_\perp/v_c = 4]$ .

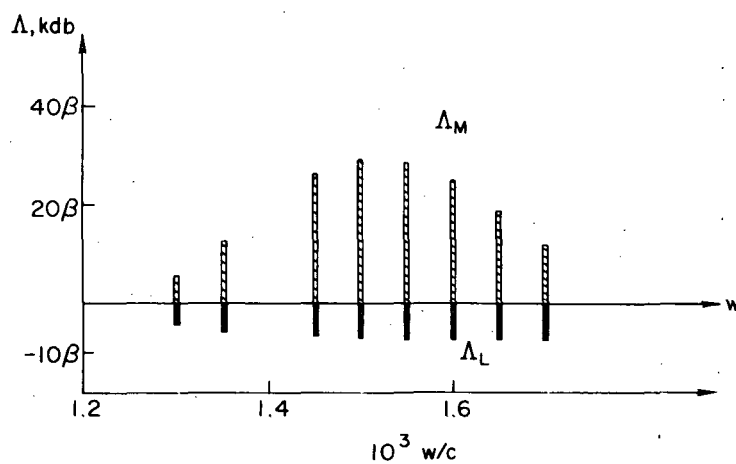
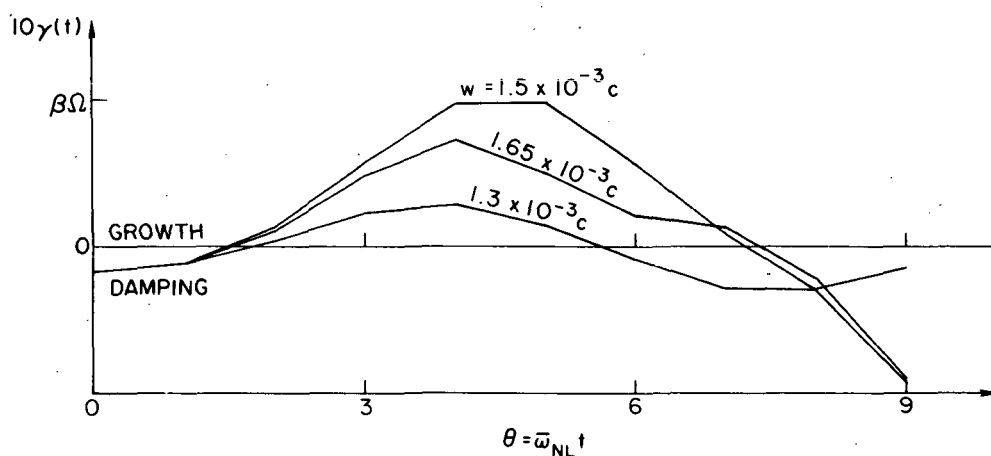


FIG. 4.7. Characteristics of test waves in the resonance region: growth rates ( $\gamma$ ), maximal consecutive growths ( $\Lambda_M$ ), and linear attenuation ( $\Lambda_L$ ) in the absence of the large amplitude whistler ( $\beta$  is the fractional number density of the hot plasma).  $[\bar{v}_\perp/v_c = 2]$ .

have used a mean total kinetic energy around 250 ev, and almost equal perpendicular and parallel 'temperatures'. Satellite measurements withOGO-3 reported by Schild and Frank [1970] for electron densities at low magnetic latitudes between three and ten earth radii, suggest not only that electrons in that energy range exist, but also that their number density,  $n_h$ , may frequently exceed one thousandth of  $n_0$ , the cold plasma number density ( $\beta = n_h/n_0$ ).

As to the need for a 'monochromatic' distribution in  $v_\perp$ , we note that a maximum in  $v_\perp$  is a naturally occurring situation in magnetic mirror configurations, where the particles with small  $v_\perp$  enter the loss cone, and the large values of  $v_\perp$  are restricted by the energy cutoff. Fredricks [1971] cites recent satellite measurements on ATS-5 which show that distributions with a peak at  $v_\perp > 0$  are possible in the magnetosphere. The degree of  $v_\perp$ -monochromaticity achieved by these distributions is certainly not equivalent to  $\bar{v}_\perp/v_c = 8$ , but Figure 4.7 shows that even a modest  $\bar{v}_\perp/v_c \approx 2$  may yield very large consecutive growths for  $\beta > 10^{-3}$ .

Finally, the level of whistler noise available in the unstable frequency bands seems to be appropriate for the creation of sidebands. Figures 4.5, 4.6 and 4.7 indicate the availability of gains around 90 db in bandwidths of about 60 Hz for  $\Omega/2\pi \approx 30$  kHz (see end of Section 4.3.3), so that whistler noise magnetic intensities as low as  $4 \times 10^{-7} \gamma \text{ Hz}^{-1/2}$  ( $1\gamma = 10^{-5}$  G) would be sufficient to give rise to sideband intensities of  $0.1 \gamma$ , already a 'high field' case whistler, as referred to by Dysthe [1971]. (Satellite observations of magnetospheric whistler-mode noise reported by Dunckel and Helliwell [1969] do not cover the range of parameters of interest to us here, but they suggest that whistler noise magnetic intensities higher than  $4 \times 10^{-7} \gamma \text{ Hz}^{-1/2}$ , and with frequencies  $\sim \Omega/2$ , may be common in the equatorial triggering region.) We stress, however, that the creation of whistler sidebands may also occur in 'noiseless' backgrounds since the large amplitude whistler has a non-monochromatic frequency spectrum due to the following two factors. First, the small amplitude modulation experienced by the original whistler with a frequency of about  $\langle \omega_{NL} \rangle$  (see Figure 4 of Palmedesso and Schmidt [1971]); this may excite almost resonantly the unstable

frequency bands found in Section 4.3.3 and a minute modulation index of  $10^{-3}$  would only require a gain of 60 db to create sidebands with amplitudes of the same order as the original whistler. Second, the finite duration of the signal; e.g., for the parameters used in Figures 4.5, 4.6 and 4.7, a CW Morse dash (see the discussion on stimulated emissions below) of 150 ms with a central frequency of 15 kHz has a magnetic intensity in the unstable frequency bands about 40 db below the level in the vicinity of the central frequency. A 'noiseless' magnetoplasma with appropriate energetic electron distribution is thus suitable for the creation of electromagnetic sidebands when excited by a large amplitude whistler wave.

**EMISSION ONSET VIA PARALLEL WHISTLERS.** To assess the relevance of the creation of parallel whistler sidebands to the onset of artificially stimulated emissions in the magnetosphere, we consider the triggering signal to be a CW Morse dash of (central) frequency  $\omega_0$  propagating in the whistler mode, near the equator, with group velocity  $v_{g0} > 0$  along a geomagnetic field line. The resonant electrons have parallel velocities in a narrow range centered on  $V_{G0} < 0$ , and thus slip through the packet of length  $L$  in a time  $\hat{t} \approx L/(|V_{G0}| + v_{g0})$ .

The theory developed above was obtained for the initial value problem, whereas now we are concerned with a spatial wave propagation experiment. A similar situation was studied for the electrostatic case by Lee and Schmidt [1970] who concluded that the temporal analysis is applicable to the boundary value problem: spatial and temporal growth rates are related by the wave group velocity, and time transforms into distance through the (electrostatic) wave phase velocity. In the whistler packet case, some consideration will show that the growth rates are again related by the group velocity, whereas time transforms into distance (originated in the packet front, and measured along  $-z$ ) via the velocity  $|V_{G0}| + v_{g0}$ .

Within the packet, from  $s = 0$  (front) to  $s = L$  (rear), we find different, but constant (on the average), resonant velocity distributions which are equivalent to those obtained in (4.27) at  $t = (s/L)\hat{t}$ . (The coordinate  $s$  measures distances in the packet frame, starting from

the packet front, along  $-z$ .) Behind the packet,  $s > L$ , there exists a spatially uniform, distorted, resonant distribution with average characteristics identical to those found in (4.27) at  $t = \hat{t}$ . If the energetic velocity distribution meets the conditions enunciated in Section 4.3.3, and the whistler packet is large enough to yield  $\hat{t} \gtrsim 4\pi/\langle\omega_{NL}\rangle$ , we note that in the rear of the packet (and behind it), there are velocity distributions which correspond to large whistler growth in narrow sidebands of  $\omega_0$ . Since the unstable range of frequencies  $\omega$  is very close to  $\omega_0$ , and thus  $v_g(\omega) \approx v_{g0}$ , a perturbation originated inside (or behind) the packet will be exposed to amplification during a large period of time. As the packet propagates away from the equator, the inhomogeneity of the magnetosphere will eventually quench the sideband growth. The outcome of the growth competition between the two sidebands will be decided, presumably, by further nonlinear effects related to the finite amplitude of the newly created sidebands.

Regarding the sidebands as having been 'triggered' by the Morse dash, we note that the 'offset frequency'  $[\sim 5(\langle w^* \rangle/c)\Omega]$  is about  $\pm (20/\pi)(\Omega/ck_0) \bar{\omega}_{NL}$ ; the 'triggering delay' can be roughly decomposed into the minimal packet duration yielding strongly unstable distorted distributions ( $\hat{t} \sim 4\pi/\bar{\omega}_{NL}$ , i.e., a duration of  $8\pi/\bar{\omega}_{NL}$  since  $v_{g0} \sim |v_{g0}|$ ) and the growth time  $t_g$  required to amplify the sideband from noise level to a detectable amplitude. These expressions explain qualitatively the observed dependence on the transmitter power of the triggering delay and offset frequency (Section 4.2), recalling that  $\bar{\omega}_{NL}$  is proportional to the square root of the amplitude of the stimulating signal. Quantitatively, using  $\Omega/2\pi = 30$  kHz,  $a/\Omega = 10^{-5}$ ,  $\omega_p/\Omega = 10$ ,  $\omega_0/\Omega = 0.5$ , and  $\bar{v}_1/c = 0.05$ , the offset frequency and the triggering delay are about  $\sim 35$  Hz and  $\sim 60$  ms +  $t_g$ . This offset frequency is somewhat small. However, we note here that our later discussion of the main phase of the stimulated emission in Section 4.4 suggests that ASE are most likely triggered by Landau, and not cyclotron, interaction. As shown below, the Landau onset yields adequate magnitudes for both the triggering delay and the frequency offset.

EMISSION ONSET VIA OBLIQUE WHISTLERS. For oblique stimulating whistlers, the two important wave-particle interactions are the fundamental cyclotron and Landau resonances discussed in Chapter 3. The cyclotron interaction is identical to the mechanism analyzed above. In contrast, the Landau interaction arises from the component  $E_{\parallel}$  of the whistler electric field along  $B_0$  and, qualitatively, is similar to the electrostatic Landau interaction analyzed elsewhere [Brinca, 1972a]. The parallel electric field may trap electrons with parallel velocities in the vicinity of  $V_L = \omega_0/k_{0\parallel}$ , regardless of their perpendicular velocities. The periodicity of the resulting nonlinear motion ( $\omega_{NL}^2 \approx ek_{0\parallel} E_{\parallel}/m$ ) is also independent of  $v_{\perp}$ , and thus no phase mixing occurs due to the spread in  $v_{\perp}$ . It follows that whistler sidebands due to Landau interaction do not require the existence of an equilibrium distribution satisfying the conditions discussed in Section 4.3.3, and may therefore occur more frequently.

By analogy with the results obtained in the electrostatic case [Brinca, 1972a], we expect the distorted energetic distribution to become unstable for oblique whistlers with parallel phase velocities,  $\omega/k_{\parallel} \sim (\omega_0 \pm \omega_{NL})/k_{0\parallel}$ . Given the oblique whistler dispersion relation used in (3.29), we find that the corresponding offset frequency of the stimulated emission is

$$\Delta \omega \approx \pm \frac{2V_L \omega_{NL}}{c^2 k_0} \frac{\omega_p^2}{(\Omega \cos \theta - 2\omega_0)}. \quad (4.46)$$

For  $\theta = 30^\circ$ , and using again  $\Omega/2\pi = 30$  kHz,  $a/\Omega = 10^{-5}$ ,  $\omega_p/\Omega = 10$ ,  $\omega_0/\Omega = 0.5$ , the offset frequency and triggering delay are now  $\sim 300$  Hz and  $\sim 60$  ms +  $t_g$ , numbers that are well within observed values [Lee, 1968].

OPTIMAL TRIGGERING FREQUENCY. The foregoing theory of the emission onset does not explain the existence of an optimal triggering frequency,  $\omega_0 \sim \Omega/2$ . As pointed out by Helliwell [1969], this phenomenon seems to be related to the shape of the whistler refractive index surface when  $\omega_0 = \Omega/2$ . Figure 3.1 shows that oblique whistlers with not too

large  $|\theta|$  have their group velocities along  $B_0$  if  $\omega_0 = \Omega/2$ . This behavior facilitates the wave-particle interactions responsible for the onset, because the triggering oblique whistler is forced to propagate in the equatorial region along constant  $B_0$ . If the whistler group velocity were not along  $B_0$ , the stimulating signal, albeit in the equatorial region, would propagate across an inhomogeneous magnetoplasma, reducing the effective length of the interaction region [Dysthe, 1971].

SIDEBANDS AND 'PULSATIONS'. The occurrence of whistler sidebands in the magnetosphere may be important in contexts other than artificially stimulated emissions. Because the frequency of the sidebands is very close to the stimulating frequency, especially for cyclotron interactions, magnetospheric phenomena characterized by low frequency amplitude modulation ('pulsations') may involve the coexistence of sidebands with the original signal. The amplitude and frequency modulation of whistler mode signals reported by Bell and Helliwell [1971], and by Likhter et al. [1971], may provide examples of this situation. Bud'ko et al. [1971] explain the observations in terms of the creation of whistler sidebands, whereas Istomin and Karpman [1972] invoke the nonlinear amplitude oscillations mentioned in Section 4.3.1 to interpret the experiments. We shall come back to this phenomenon in Chapter 5 where we discuss the possible role of another mechanism, modulational instability, in the production of these 'pulsations'.

#### 4.4 Main Phase

##### 4.4.1 Background

Following the closing considerations of Section 4.1, we envision two possible mechanisms, which are not necessarily mutually exclusive, to describe the main phase of the stimulated emission; we may term them 'feedback' and 'drift' mechanisms.

The basic idea underlying the feedback process was formulated by Brice [1963], and relies on electron gyroresonance. Because the triggered whistler wavelets and the gyroresonant electrons move in opposite directions along the geomagnetic field, it is conceivable that in a given interaction region the newly arrived wavelets phase-organize arriving gyroresonant electrons, as discussed in Section 4.3.2 above.

Since the transverse current due to organized electrons radiates as an end-fire array, the establishment of a self-sustained process is plausible. In fact, by computer simulation and using gyroresonant electron beams monochromatic in  $v_{\parallel}$  and  $v_{\perp}$ , Helliwell and Crystal [1972] have demonstrated that a whistler pulse may indeed trigger a self-sustained discrete emission. They are currently working on this feedback mechanism. The observed spectral shapes may be retrieved by postulating that the region where the interaction takes place moves along the field lines in a possibly oscillatory manner. The details of this motion are not properly understood to date, and the mechanism relies entirely on the capability of the cyclotron interaction to phase-organize the resonant electrons. For triggering oblique whistlers, this phase-bunching has not been demonstrated yet, and the results obtained in Chapter 3 suggest that the Landau resonance may dominate the interactions for certain ranges of obliquity. We thus speculate on another mechanism for the main phase of the emission suggested by the onset analysis of Section 4.3 which obtains for both Landau and cyclotron interactions, and thus for triggering by whistlers of arbitrary obliquity.

The drift mechanism is based on the radiation emitted by strongly unstable 'slabs' of energetic electrons, almost monochromatic in  $v_{\parallel}$ , evolving in the geomagnetic mirror. This unstable character is caused by the triggering oblique whistler pulse. Through nonlinear wave-particle interaction, the original velocity distribution is distorted, near the equator in two resonance regions centered about  $V_L = \omega_0/k_{0\parallel} > 0$  and  $V_G = (\omega_0 - \Omega)/k_{0\parallel} < 0$ . We neglect the other, higher-order, wave-particle interactions since, as shown in Chapter 3, the Landau and fundamental cyclotron resonances are the most important ones. These unstable slabs move in opposite directions along geomagnetic field lines and, in the process, radiate whistler wavelets. In general, depending on the obliquity of the stimulating whistler and characteristics of the energetic electron population, one of these interactions will be dominant (Chapter 3). Thus we may analyze separately the characteristics of the wavelets emitted by the Landau and gyro slabs. Since these characteristics will clearly depend on the properties of the inhomogeneous medium where the

slab movement takes place, we shall describe next the magnetospheric model to be used in our further discussion.

#### 4.4.2 Model of the Equatorial Magnetosphere

Because we wish to provide possible mechanisms to explain qualitatively the observed spectral shapes of ASE, a sophisticated model of the magnetosphere is unnecessary. It will suffice to account for the most relevant features of the equatorial zone of the magnetic shell  $L \sim 3$ .

**MAGNETIC FIELD.** The geomagnetic field is assumed to be dipolar. Denoting the equatorial field at the earth's surface by  $B_{0e}$ , we have [Chapman, 1964]

$$B_0(r, \lambda) = B_{0e} \left( \frac{r_e}{r} \right)^3 (1 + 3 \cos^2 \lambda)^{1/2}, \quad (4.47)$$

where, Figure 4.8, the (magnetic) colatitude  $\lambda$ , and the geocentric distance  $r$ , define a field line through

$$r = L r_e \sin^2 \lambda, \quad (4.48)$$

and  $r_e$  is the earth's radius. For the equatorial zone,  $\lambda \sim \pi/2$ , we can use [Helliwell, 1967]

$$B_0(z) = B_0(0) \left[ 1 + 4.5 \left( \frac{z}{L r_e} \right)^2 \right], \quad (4.49)$$

where  $z$  represents the distance along the field line, increasing from (geographic) North to South with  $z = 0$  at the equator. The actual geomagnetic field is then  $\underline{B}_0 = -B_0 \hat{z}$ , the minus sign being required since the North magnetic pole is at the South geographic pole, and we assume that the whistlers propagate from geographic North to South, so that  $k_{\parallel} = k_z > 0$ . Comparison of (4.47) and (4.49) shows that the latter is a good approximation for  $|z| < 9,000$  km,  $|\lambda - 90^\circ| < 25^\circ$  at  $L \sim 3$ .

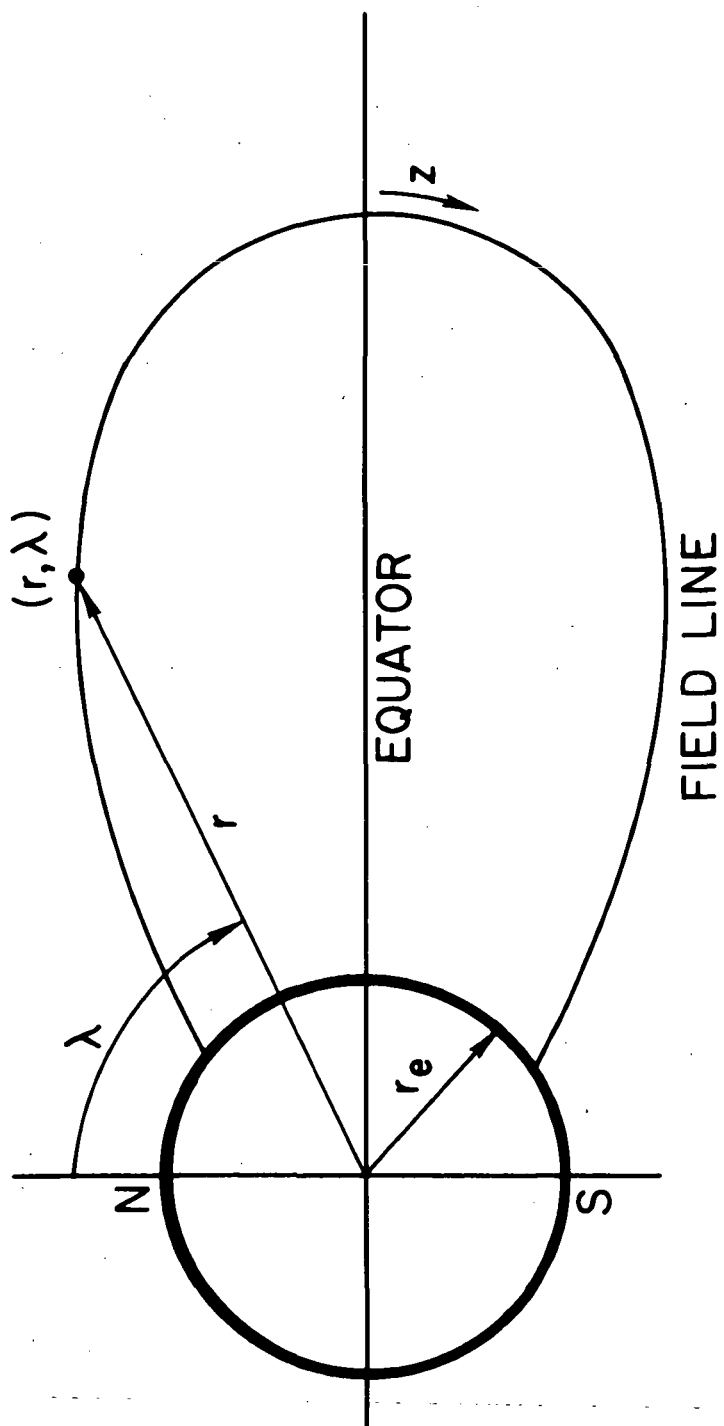


FIG. 4.8. Geomagnetic dipolar field line coordinates.

PLASMA DENSITY. The equatorial region of interest is in the plasma-sphere, more precisely in the protonosphere, i.e. the outer region of the plasmasphere with ionized atomic hydrogen. The possible proximity of the plasmopause, where a large density gradient occurs, may play an important role in the paths of the unducted stimulated emissions, as speculated in Section 4.4.4. We take  $\omega_p/\Omega = 10$  at the equator ( $z = 0, L \sim 3$ ), and adopt the gyrofrequency model,  $\omega_p^2(z) \propto \Omega(z)$  [Helliwell, 1965], to describe density variations along the field line. Whenever ray tracing is required, we use the program developed by Walter [1969], with the possible inclusion of field-aligned ducts of enhanced ionization with gaussian profiles [Angerami, 1970]. These ducts are characterized by their field line location,  $L$ , relative increase of the center density,  $C$ , and semithickness  $\Delta L$ .

#### 4.4.3 Spectral Shapes

Our aim here is to determine the frequency and obliquity characteristics of the whistler radiation emitted by the unstable slabs of the drift mechanism described in Section 4.4.1.

Assuming that the interaction leading to the emission onset takes place at  $z = z_0$  with a whistler of frequency  $\omega_0$  and obliquity  $\theta = \theta_0$ , the unstable Landau and gyro slabs have initial velocities

$$v_{L0} = \frac{\omega_0}{k(z_0)\cos\theta_0}, \quad v_{G0} = \frac{\omega_0 - \Omega(z_0)}{k(z_0)\cos\theta_0}, \quad (4.50)$$

where the wavenumber  $k$  satisfies the dispersion relation

$$\frac{c^2 k^2(z)}{\omega^2} = \frac{\omega_p^2(z)}{\omega[\Omega(z)\cos\theta(z) - \omega]}. \quad (4.51)$$

Denoting the rms perpendicular speed of the nonthermal electron population by  $\bar{v}_\perp$ , and making use of the first adiabatic invariant and the constancy of the particle kinetic energy, the slab velocities along the geomagnetic mirror field will be

$$v_{G,L}^2(z) = v_{G,L}^2(z_0) - \bar{v}_1^2(z_0) \left[ 1 - \frac{\Omega(z_0)}{\Omega(z)} \right] \left[ \frac{\Omega(z)}{\Omega(z_0)} \right]^{1/2}. \quad (4.52)$$

Here, as in Section 4.3, the distribution in  $v_1$  has been assumed to be reasonably monochromatic. Curves showing the spatial and temporal evolution of the slabs for several ratios of  $|\bar{v}_1/v_{\parallel}|$ , and a constant  $|v_{\parallel}|$  at the equator, are shown in Figure 4.9.

For each slab, the combination of its known parallel velocity with the local dispersion characteristics establishes a relation that should be satisfied by the emitted radiation. We find

$$v_G(z) = c \left( \frac{\omega - \Omega}{\omega_p \cos\theta} \right) \left( \frac{\Omega \cos\theta - \omega}{\omega} \right)^{1/2}, \quad (4.53)$$

$$v_L(z) = c \frac{1}{\omega_p \cos\theta} [\omega(\Omega \cos\theta - \omega)]^{1/2}. \quad (4.54)$$

In each case, we know  $\Omega(z)$  and  $\omega_p(z)$ , together with the slab velocity, and wish to obtain  $\omega(z)$  and  $\theta(z)$ . Figures 4.10 and 4.11 represent solutions of (4.53) and (4.54) for the cyclotron and Landau slabs, at  $z = 0$  and  $|z| = 5,000$  km. At each point of the field line, where  $v_G(z)$  is known from (4.52), the cyclotron slab can radiate in a band of frequencies associated with a corresponding  $\theta$ -range. In contrast, the Landau slab can radiate at every position in two frequency bands associated with the same  $\theta$ -range.

The termination of the emission may be explained by several mechanisms. Assuming that phase organization of the particles could be conserved, reflection of the slab at the geomagnetic mirror would yield whistler radiation in the opposite direction. Most likely, the progressive parallel velocity mixing brought about by the spread of perpendicular speeds in the slab would ultimately quench its unstable character. Also, the evolution of the emission curves  $(\theta, \omega)$  along  $z$  might lead to forbidden domains, as defined by Figures 4.10 and 4.11. Finally, an emission might occur with  $(\theta, \omega)$  characteristics that preclude its observation on the ground or at a given satellite.

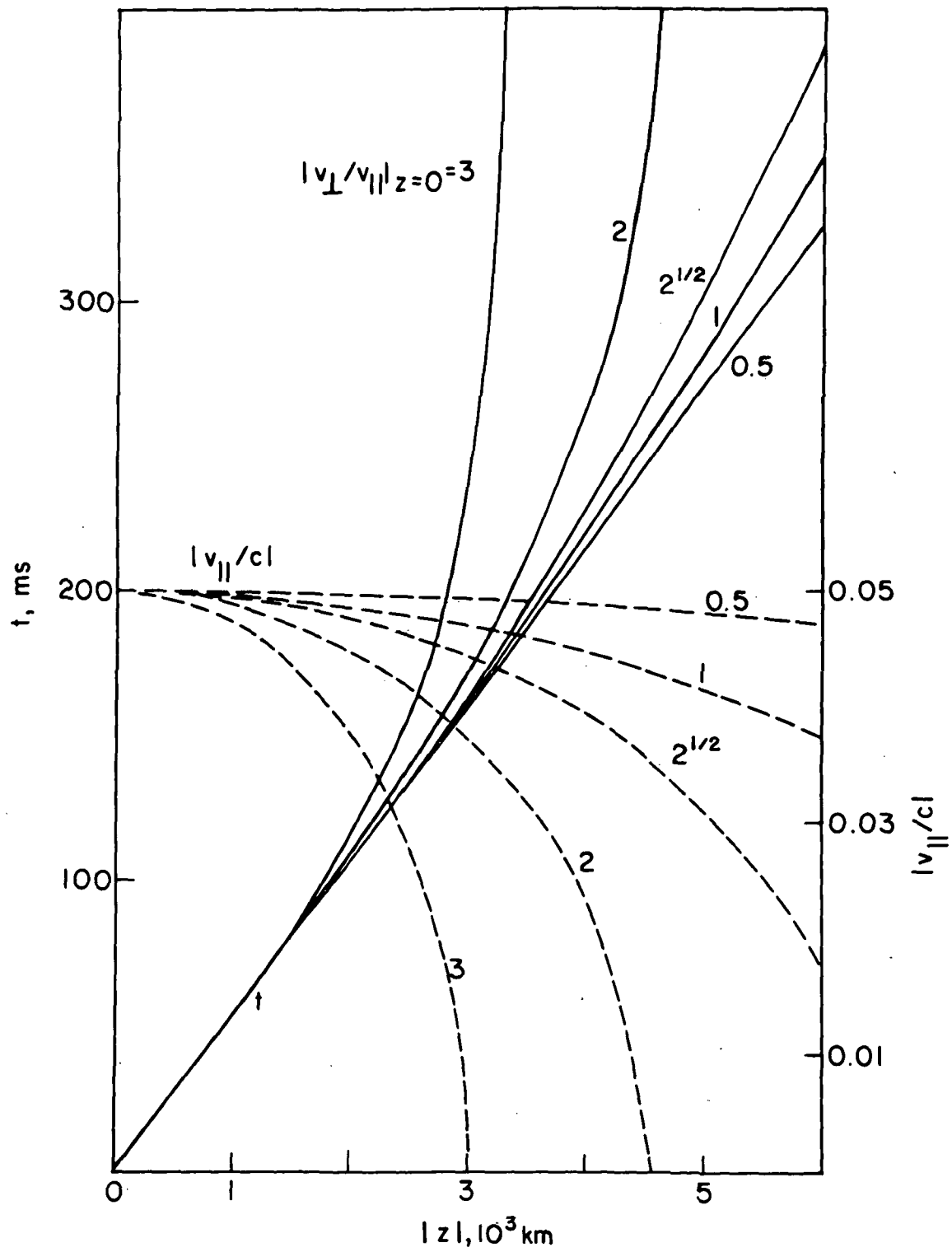


FIG. 4.9. Electron motion in the geomagnetic mirror.

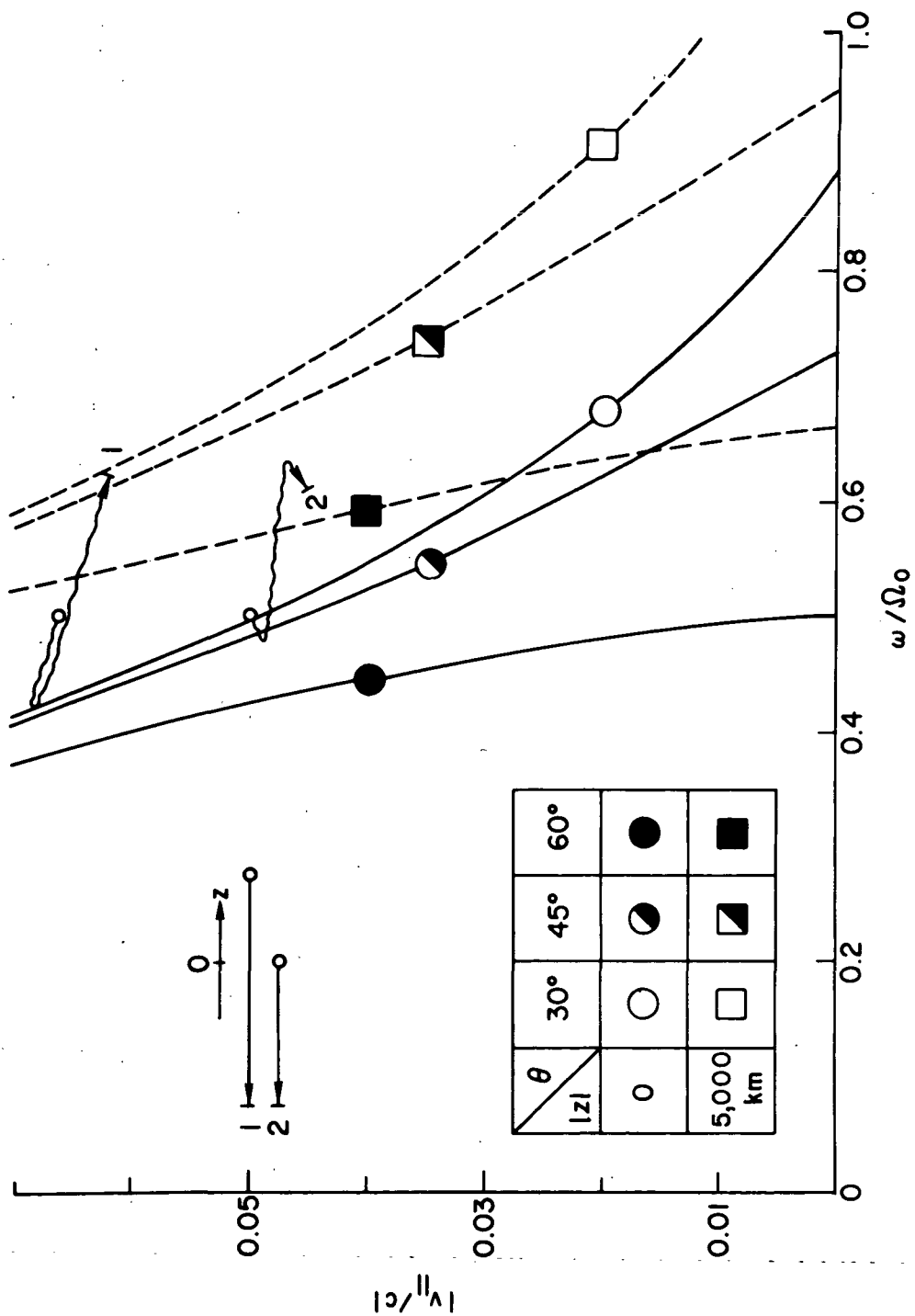


FIG. 4.10. Allowed frequency and obliquity emission bands for the gyro slab as a function of its position and parallel velocity. Numbers 1 and 2 identify possible emissions. [ $L = 3$ ,  $\Omega_0 = \Omega(z = 0)$ ,  $v_p(\theta = 0) = 0.05c$  at  $z = 0$  for  $\omega/\Omega_0 = 0.5$ ].

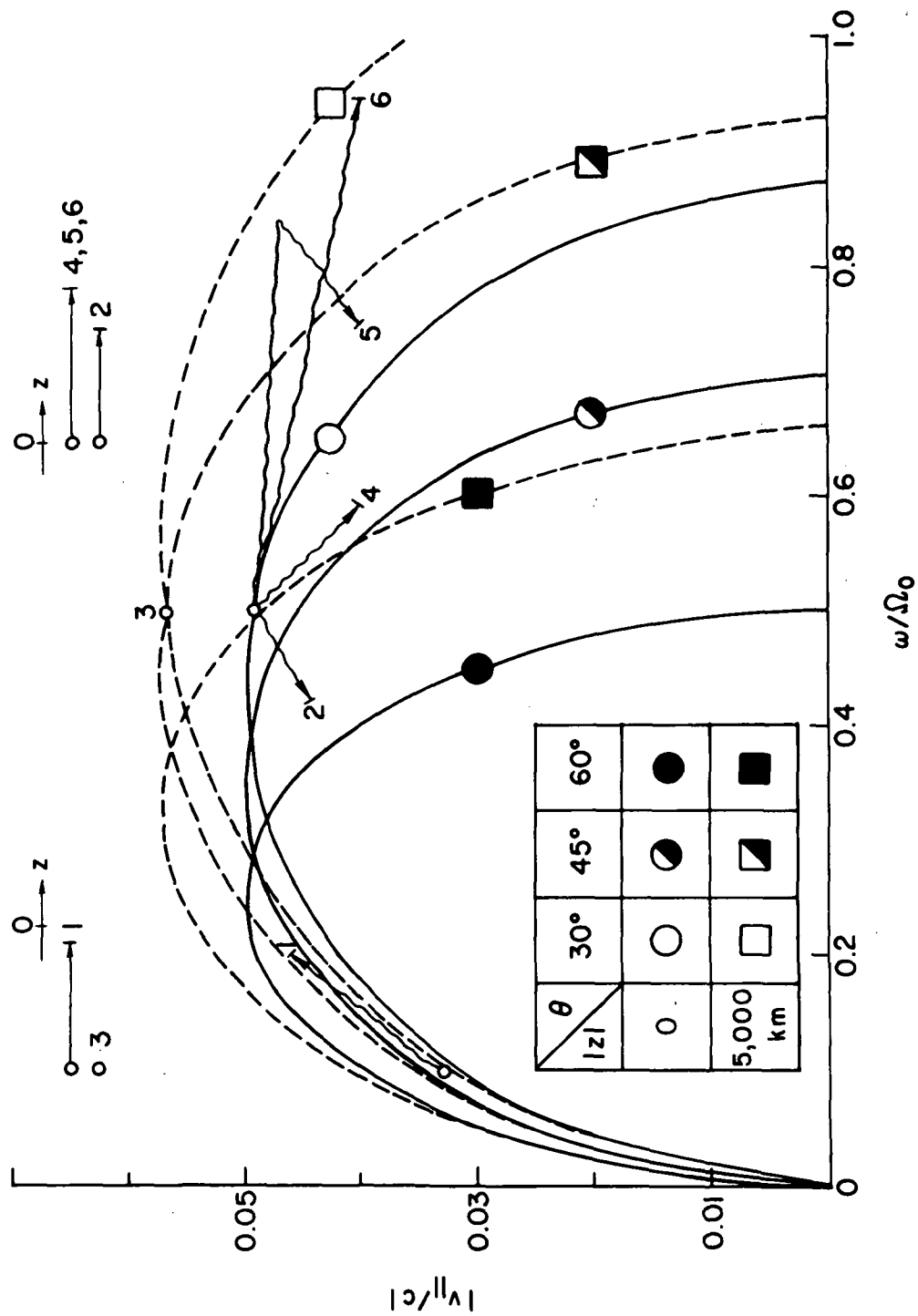


FIG. 4.11. Allowed frequency and obliquity emission bands for the Landau slab as a function of its position and parallel velocity. Numbers 1-6 identify possible emissions. [ $L = 3$ ,  $\Omega_0 = \Omega(z = 0)$ ,  $v_p(\theta = 0) = 0.05c$  at  $z = 0$  for  $\omega/\Omega_0 = 0.5$ ].

The procedure to determine possible spectral shapes of discrete emissions is illustrated for the gyro slab in Figure 4.12. It can be described for both types of slabs as follows:

(i) Choose the location  $z_0$  of the interaction region where the stimulating pulse of frequency  $\omega_0$  and obliquity  $\theta_0$  is assumed to have distorted the nonthermal velocity distribution, and created the two unstable slabs. This enables  $V_G(z_0)$  and  $V_L(z_0)$  to be computed from (4.50), and defines the geomagnetic field line  $L$ . We have used  $L = 3$ ,  $z_0 = 3,000$  km,  $\omega_0 = \Omega/2$  and  $\theta_0 = 0$  in Figure 4.12.

(ii) Determine the space-time evolution of the slabs along the field line assuming the value of  $\bar{v}_\perp(z_0)$  to be known, i.e., obtain  $V_{G,L}(z)$  and  $z_{G,L}(t)$  from (4.52) and Figure 4.9. We have used  $\bar{v}_\perp = 2|V_G|$  at  $z = 0$  in Figure 4.12.

(iii) One at a time, follow the movement of the slabs and, at each position, assume that a wavelet of frequency  $\omega$  and obliquity  $\theta$  is emitted, consistent with curves of the types shown in Figures 4.10 and 4.11. Several possible emissions are sketched in these figures. We have used  $\theta = 0$  throughout in Figure 4.12.

(iv) Ray trace the wavelets emitted along the field line and, recalling the different emission times, determine their arrival times at a given colatitude  $\lambda = \lambda_a$ , already outside the equatorial region and such that the dispersion beyond  $\lambda_a$  may be neglected. We have used  $\lambda_a = 116^\circ$ , corresponding to  $z_a = 9,000$  km, in Figure 4.12.

(v) Construct the spectrogram,  $\omega(t)$ , of the emission received at  $\lambda = \lambda_a$ . Because the dispersion beyond  $\lambda_a$  is negligible, the constructed spectrogram closely represents the spectral shape observed on the ground. The propagation aspects of this approach will be discussed in Section 4.4.4. In Figure 4.12, where ducted propagation has been assumed, the spectral shape obtained is a hook.

It is apparent from the above procedure that the final shape of the emission spectrogram depends on the choice of several parameters. Assuming that the characteristics of the cold magnetoplasma and triggering pulse are known, we are left with the sometimes arbitrary choices of  $z_0$ ,  $\bar{v}_\perp(z_0)$  and  $\theta(z)$ . The location of the onset,  $z_0$ , ought to be

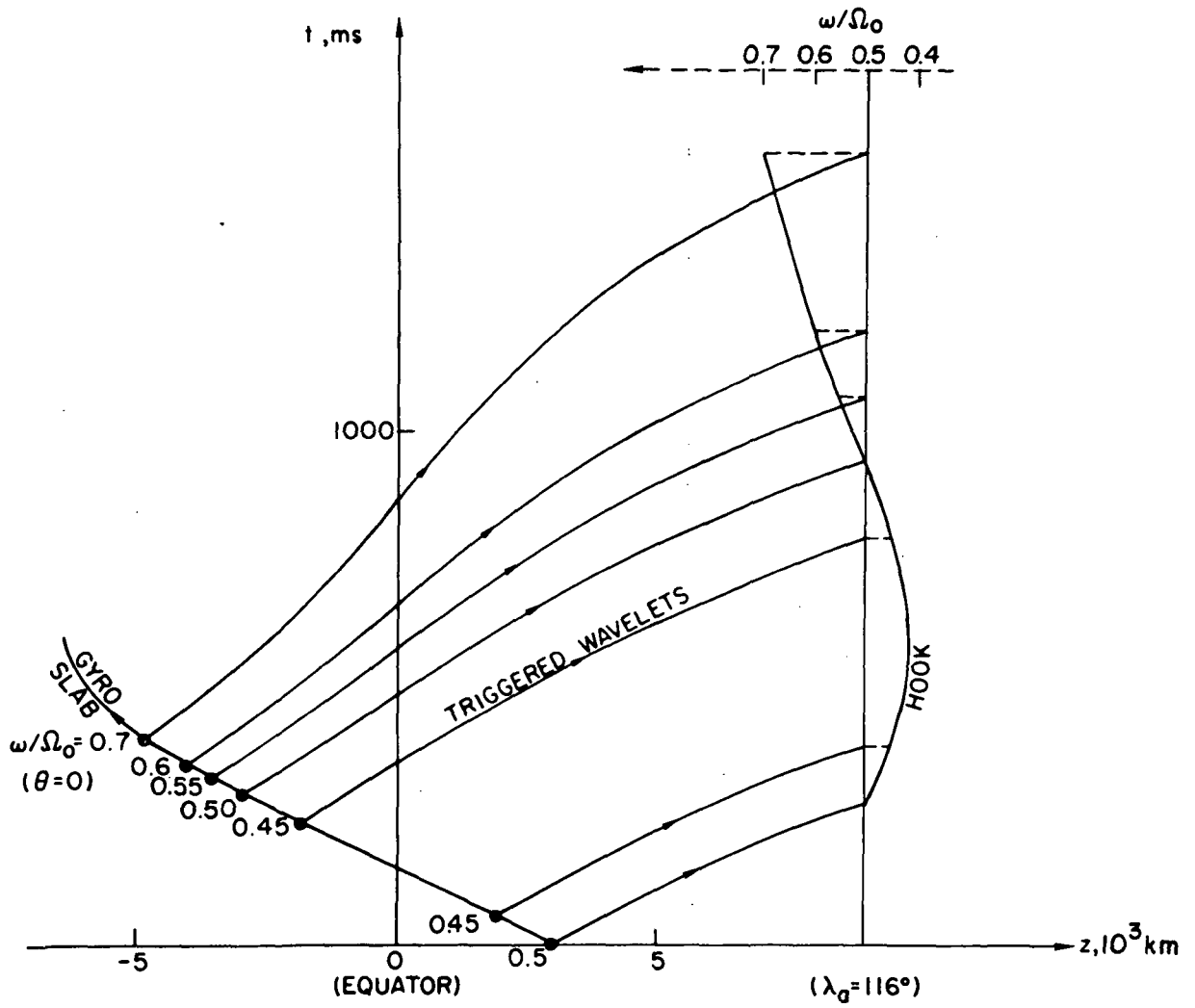


FIG. 4.12. Ducted hook radiated by gyro slab.

$[L = 3, z_0 = 3,000 \text{ km}, |v_G|/\bar{v}_1 = 0.5 \text{ at } z = 0].$

in the equatorial region (see Section 4.2), and  $\bar{v}_\perp$  should be compatible with observed pitch angle anisotropies of the nonthermal population. However, the choice of  $\theta(z)$  is rich in possibilities. An analysis of the  $\theta$ -dependence of the unstable growth rate of the slabs,  $\gamma$ , could reduce this uncertainty, though the results obtained in Chapter 3 suggest that  $\gamma(\theta)$  will certainly not be sufficiently peaked to yield a unique, preferred obliquity. Most likely, the propagation characteristics required to observe the emission on the ground or at a given satellite will impose more stringent limits on  $\theta(z)$  and, hence, on the observed frequency.

To facilitate the discussion of the possible spectral shapes, we think of  $\omega(t)$  as being made up of  $\omega(z)$  and  $z(t)$ . The dependence of  $t$  on  $z$  consists of two parts: the slab drifting time,  $t_d$ , which is the elapsed time between onset at  $z = z_0$  and arrival at  $z$ , and the wavelet propagation time,  $t_p$ , from  $z$  to the observation point or to a given colatitude  $\lambda = \lambda_a$ . We thus have

$$t = t_d + t_p = \int_{z_0}^z \frac{dz}{v_{\parallel}(z)} + \int_{s[\lambda(z)]}^{s[\lambda_a]} \frac{ds}{v_g(\omega, \theta)}, \quad (4.55)$$

where  $s$ , defined along the wavelet path, is not necessarily along a field line. The slab drifting time,  $t_d$ , is obtained from curves of the type shown in Figure 4.9, after determination of  $v_{G,L}(z_0)$  and choice of  $\bar{v}_\perp(z_0)$ . The wavelet propagation time,  $t_p$ , can be estimated for ducted propagation along a field line by taking for the group velocity its value at  $\theta = 0$  [Smith, 1961]. Here generally, for nonducted propagation,  $t_p$  is obtained by ray tracing with the program cited in Section 4.4.2, and is discussed in Section 4.4.4. In terms of these definitions, we may now speculate on possible shapes of  $\omega(z)$  for the two slab types, using the emission examples sketched in Figures 4.10 and 4.11.

**GYRO SLAB:** Examination of Figure 4.10 shows that, for given  $z$  and  $v_G(z)$ , the allowed emission frequency band is relatively narrow. The

slab radiated frequency follows roughly the variations of the static magnetic field. The regions where it is conceivable to have  $(d\omega/d|z|)(dB_0/d|z|) < 0$  are small, and only come about by strongly varying  $\theta$ , as discussed below with respect to Emission 2.

Emission 1 represents a possible hook initiated at  $z_0 \sim 3,000$  km. The gyro slab increases its speed when approaching the equator and, in the process, radiates decreasing frequencies. Beyond the equator ( $z < 0$ ) the slab is decelerated and the emission frequency increases. The sketch assumes  $\theta \sim 0$  throughout.

Emission 2 starts at the equator with  $\theta \sim 0$  and, by substantially increasing  $\theta$ , may radiate a decreasing frequency as it is decelerated along  $-z$ . However, this process cannot last long; as the slab enters regions of increasing  $|z|$ , the allowed emission frequencies increase and it will eventually have to stop emitting, or radiate with  $d\omega/d|z| > 0$ . If this latter radiation is made with small  $\theta$ , again it will be possible to emit decreasing frequencies for a short while by increasing  $\theta$ .

Anticipating considerations of Section 4.4.4, we conclude from these examples that the gyro slab emissions do not have, in general, large  $|d\omega/dt|$ . This is plausible, recalling the relative movement of the slab and wavelets. The spectral shapes radiated by drifting gyro slabs do not differ substantially from parts of the hook. As an example, we have shown above in Figure 4.12 a hook obtained by assuming ducted propagation along  $L = 3$ , and initiated at  $z_0 = 3,000$  km, with  $|V_G| = \bar{v}_1/2$  at the equator,  $z = 0$ .

LANDAU SLAB. We first note from Figure 4.11 that, the lower of the two allowed emission bands for each pair  $(V_L, z)$  covers frequencies which are too small to play a role in emissions initiated at  $\omega \sim \Omega_0/2$ . This band may, however, be involved in the main phase of emissions triggered by the low frequency part of natural whistlers. As sketched in Emission 1, for  $z_0 = -5,000$  km and  $\omega/\Omega_0 \sim 0.1$ , the accelerating Landau slab will radiate increasing frequencies until the evolution of  $\theta$  and  $V_L(z)$  eliminates the emission band.

Emission 3 illustrates that a high frequency onset ( $\omega/\Omega_0 \sim 0.5$ ) at  $z < 0$  cannot yield a Landau emission: the slab accelerates toward the equator, but there are no allowed emission bands corresponding to  $|z| \rightarrow 0$  as the slab speed increases.

Landau emissions initiated at the equator may display a wide variety of  $\omega(z)$  forms. One can have risers with small or large  $d\omega/dz$ , as illustrated by Emissions 4 and 6, respectively; risers followed by falling tones, as illustrated by Emission 5, and simple falling tones, for which Emission 2 constitutes an example.

Note that here the designations 'riser' and 'falling tone' are being used with respect to the curve  $\omega(z)$  and not  $\omega(t)$ . Falling tones are somewhat difficult to achieve, if initiated at  $\omega \sim \Omega_0/2$ ; see for example Emission 2. They can only last for a short distance because the evolution of  $\theta$  and  $V_L(z)$  eliminates the allowed emission band. In contrast, falling tones that occur as continuation of risers, e.g. Emission 5, are obtained easily.

The Landau slab thus has the potentiality to yield almost any type of curve  $\omega(z)$ . To find out whether this flexibility carries over to the observed spectral forms  $\omega(t)$ , the propagation problem has to be discussed.

#### 4.4.4 Propagation

The study of ASE raises propagation questions that have not been explained to date. Accepting the view that most whistlers observed on the ground propagate in ducts of enhanced ionization located along geomagnetic field lines [Helliwell, 1965], the first propagation question centers on the stimulating signal. When crossing the equator, its frequency  $\omega \sim \Omega_0/2$  is on the threshold of untrapping. Testing the much invoked assumption of Section 4.1, that the triggering signal crosses the equator with  $\theta = 0$ , we have ray traced parallel whistlers launched from the equator at different altitudes, but within the same duct centered at  $L = 3$ , and with  $C = 0.4$ ,  $\Delta L = 0.04$ . We have used the ray tracing program mentioned in Section 4.4.2 and have found that trapping is a very sensitive function of altitude, as shown in Figure 4.13. It occurs only for launching in the close vicinity of  $L = 3.015$ ; whistlers

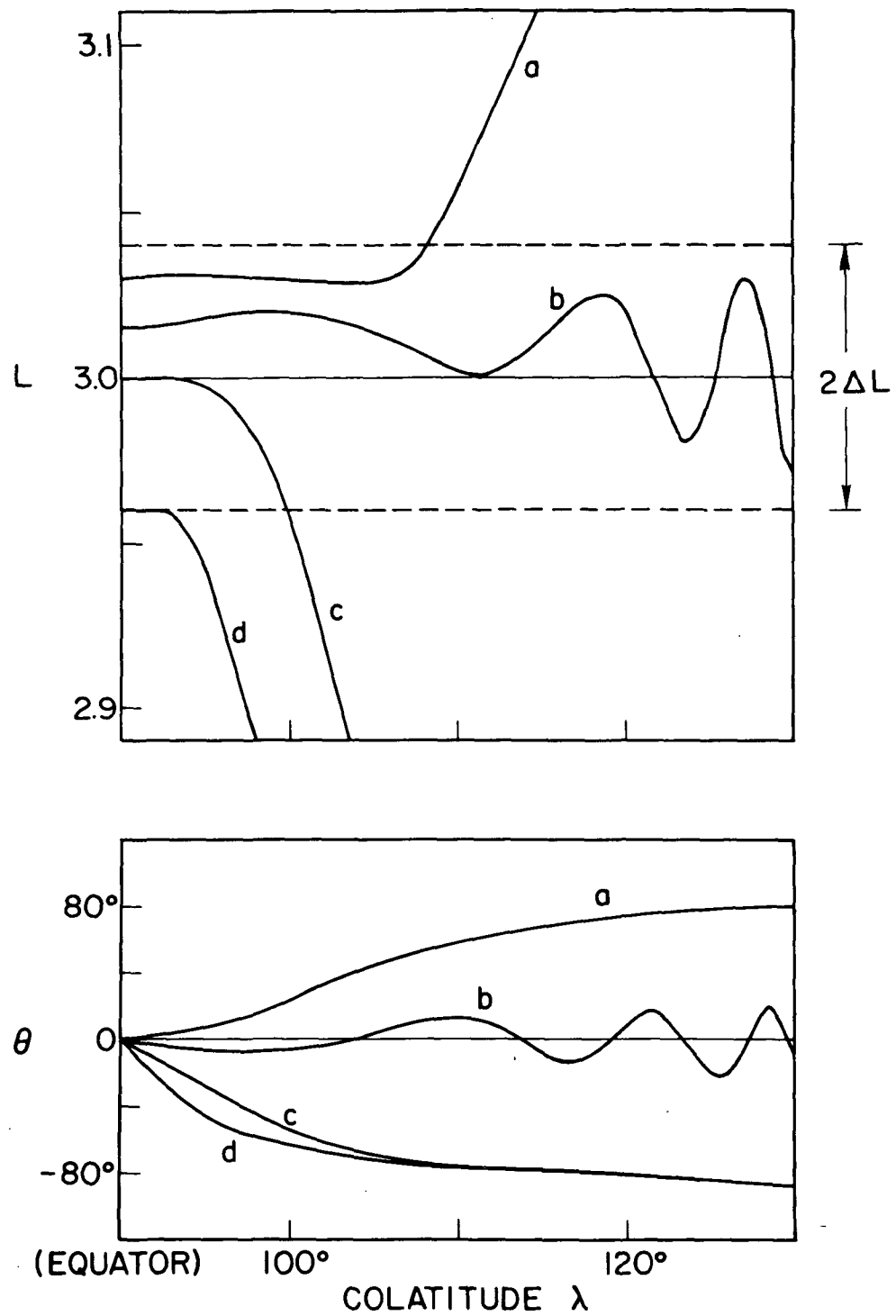


FIG. 4.13. Path and obliquity of parallel whistlers launched in the equatorial plane at different altitudes.

launched with  $L = 3$  and  $L = 3.03$  are unducted. Because we have used a large enhancement  $C$ , which facilitates the occurrence of trapping, these results suggest that the triggering signal is, in general, oblique and locally unducted when crossing the equator; some of the ways in which retrapping may occur are mentioned below.

The next problem posed by ASE is the propagation of hook emissions. Available explanations assume that the rising part of the hook is generated at  $z < 0$ , implying that frequencies larger than  $\Omega_0/2$  crossed the equator and were observed on the ground. These problems are not confined to ASE; some nose whistlers also have frequencies larger than  $\Omega_0/2$ . Ducting theory [Smith, 1961] precludes the occurrence of trapping in the equatorial region but this does not mean that the signal will be unobservable on the ground. For example, a signal with  $\omega = 0.6 \Omega_0$  emitted at  $L = 3$ ,  $z = -1,000$  km,  $\theta = 31^\circ$ , in the duct specified above, but with  $C = 0.2$ , becomes ducted, after crossing the equator, at about  $\lambda \sim 105^\circ$ . It is important to recognize that, due to the curvature of the geomagnetic field lines, whistlers with symmetric values of  $\theta$  have different paths; we use positive  $\theta$  when the wavenormal is inside the field line.

Other retrapping possibilities include the influence of the nearby plasmopause, since crossing of steep density gradients in the proper direction may reduce  $|\theta|$ , and the existence of other ducts. For example, an emission with  $\omega/\Omega_0 = 0.59$  at  $L = 3.15$ , launched on  $L = 3$  at  $\lambda = 93^\circ$  and  $\theta = 52^\circ$ , is trapped by a duct centered about  $L = 3.15$ , with  $C = 0.2$  and  $\Delta L = 0.04$ . Finally, the nonducted increase in  $|\theta|$  that precludes the transmission of the untrapped whistler from the magnetosphere to the troposphere may sometimes be corrected by the large density gradients met by the whistler as it approaches the ionosphere from above [Kimura, 1971; private communication].

These considerations indicate that it is possible to observe a whistler on the ground which has propagated in part of its path as an unducted mode. The propagation time consumed in this untrapped portion is a very strong function of the obliquity at the launching of the wave. As shown in Figure 4.14 for  $\omega/\Omega_0 = 0.6$  at  $L = 3$ , the propagation time

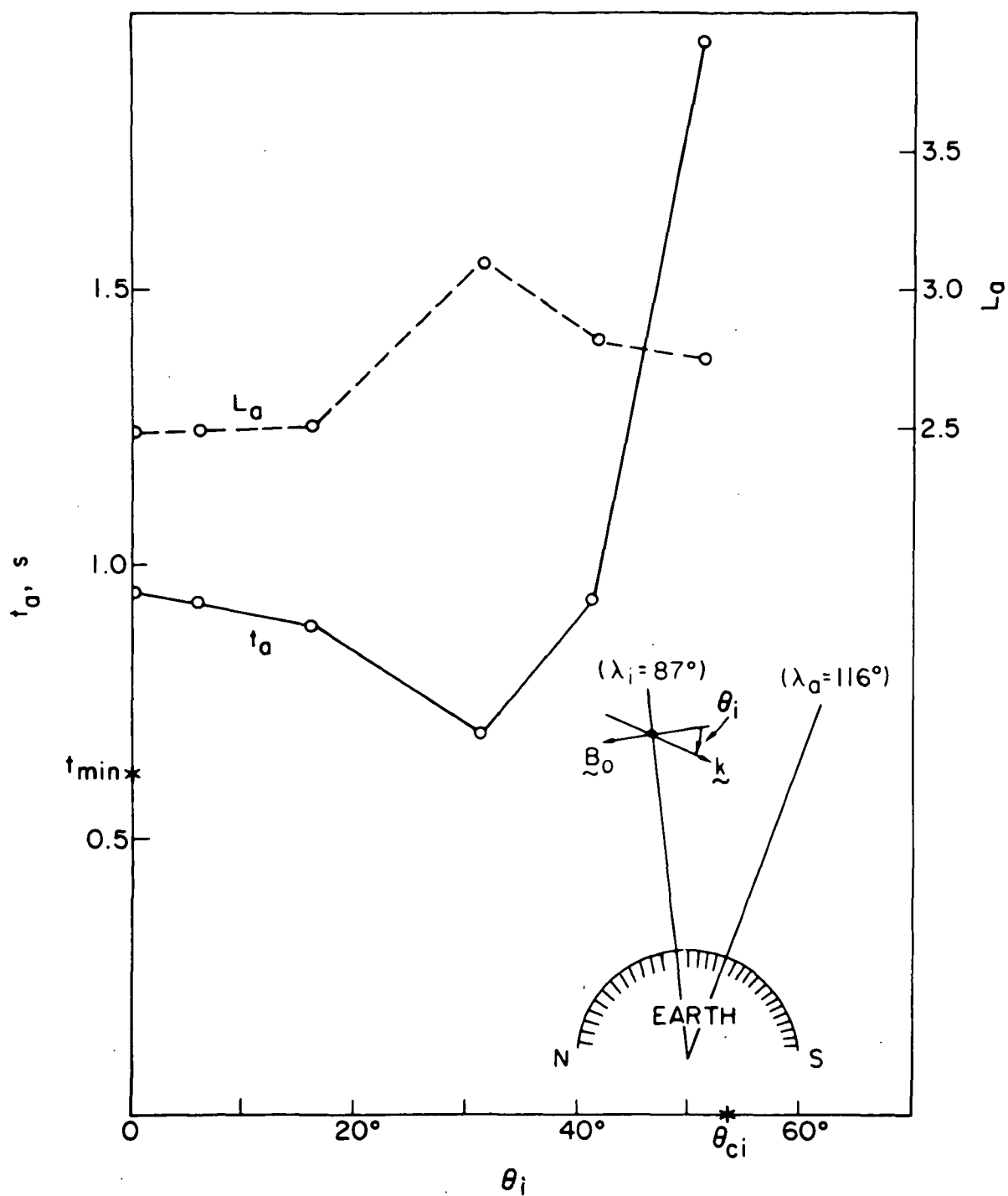


FIG. 4.14. Whistler propagation time between  $\lambda_i = 87^\circ$ ,  $L = 3$  and  $\lambda_a = 116^\circ$ ,  $L_a$  as a function of initial obliquity  $\theta_i$ . [Ducted propagation takes  $t_{\min}$  seconds].

between ( $\lambda_i = 87^\circ$ ,  $L = 3$ ) and ( $\lambda_a = 116^\circ$ ,  $L_a$ ) varies from 690 ms to 1950 ms by choosing the initial obliquity in the range  $\theta_i = 0 - 51.2^\circ$ , where we note that the resonance cone semi-angle at the launching point is  $\theta_{ci} = 53.55^\circ$ . Parallel propagation along  $L = 3$  between the same colatitudes would take  $t_{\min} = 620$  ms.

This use of obliquity to obtain different propagation times, and hence different spectral forms of stimulated emissions may occur naturally. Indeed, its existence is strongly implied by the observations reported by Angerami and Bell [1971], and described in Section 4.2. The distinguishing feature of the paths involved in the production of the satellite signals is the whistler obliquity and not the path length.

The flexibility already obtained in the emission frequency curve  $\omega(z)$ , especially for the Landau slab, is reinforced by the wide range of possible propagation delays. The drift mechanism has thus the potentiality to explain the variety of spectral shapes of short ASE,  $\omega(t)$ , observed on the ground and in satellites, and triggered by oblique whistlers. Long-enduring emissions, however, will certainly involve the self-sustained process described by the feedback mechanism. We note that the two mechanisms are not incompatible. It has already been pointed out that the gyro slab drift mechanism is a limiting case of the feedback mechanism when the region of interaction is allowed to move along field lines [Helliwell, 1967]. Conversely, an oblique emission initiated by the slab mechanism might become nearly parallel due to propagation effects and continue as a self-sustained emission.

#### 4.5 Discussion

In this chapter we have studied artificially stimulated emissions, a transient, nonlinear phenomenon taking place in an inhomogeneous medium. Because of the subtleties involved, the analysis undertaken could not explain all the observed features of the emissions.

We have not explained why only one of the two possible sidebands is generally excited, though some comfort might be drawn from the fact that the, extensively studied, equivalent electrostatic problem (see references in Brinca [1972a]) faces similar difficulties [Kruer et al., 1969]. In this respect, the recent work of Nunn [1972] on the sideband

instability of electrostatic waves in an inhomogeneous medium might be relevant. Because of the complexity of the analysis he had to adopt an approach which is not self-consistent, but his results show that when the amplitude of the sidebands is taken into account in an inhomogeneous medium, only one of the sidebands grows.

The possibly related problem of the transition between the transient emission onset, and the main phase was not analyzed. The need to consider, simultaneously, nonlinear wave-wave and wave-particle interactions involving oblique whistlers in inhomogeneous media, is certainly a strong justification for the omission. The understanding of this transition may explain when and why the self-sustained cyclotron feedback mechanism dominates over the drifting radiating slabs.

On the more optimistic side, we note that the characteristics of the proposed onset theory seem to be in good agreement with spectral measurements of stimulated emissions [Lee, 1968], particularly when the possibility of a Landau onset is taken into account, as described in Section 4.3.4. Also, the interpretation of observed spectral shapes in terms of the drift mechanism analyzed in Section 4.4 provides an explanation of the main phase of the emission. This interpretation may be of special importance when the obliquity of the triggering whistler precludes the dominance of the cyclotron resonance over the Landau wave-particle interaction, and thus the conditions assumed by the cyclotron feedback mechanism do not obtain.

## 5. MODULATIONAL INSTABILITY

### 5.1 Background

Recent observations of the amplitude spectrum of whistlers originally emitted as constant amplitude monochromatic waves have revealed the occurrence of marked amplitude modulation [Bell and Helliwell, 1971; Likhter et al., 1971; Stiles, 1972]. In this chapter, we shall study the theory, and analyze the possible relevance to the observations, of a self-action effect known as 'modulational instability' previously contemplated in the context of the magnetospheric [Litvak, 1970; Brinca, 1972c] and solar wind [Hasegawa, 1972a] plasmas.

Propagation of finite amplitude waves in nonlinear dispersive media gives rise to self-focusing [Akhmanov et al., 1966] and self-trapping [Chiao et al., 1964] effects caused by the dependence of the dielectric properties on the wave fields. The application of these self-action effects to plasma waves has been primarily [Taniuti and Washimi, 1968; Tam, 1969; Taniuti and Washimi, 1969; Litvak, 1970; Hasegawa, 1970; Hasegawa, 1972b] but not exclusively [Kakutani et al., 1967; Dysthe, 1968; Tang and Sivasubramanian, 1971; Dewar et al., 1972] concentrated on cyclotron waves.

Here we shall study the modulational instability (self-trapping) of whistlers in cold and hot dense plasmas. This instability, as sketched in Figure 5.1, may bring about large amplitude modulation in an initially constant amplitude wave packet. Previous work on this problem is restricted to cold plasmas and either neglects ion motion [Tam, 1969; Tang and Sivasubramanian, 1971], or disregards relativistic effects [Taniuti and Washimi, 1968; Tam, 1969; Hasegawa, 1970; Hasegawa, 1972b]. It will be demonstrated for the cold plasma case that the simultaneous consideration of these two factors alters the modulational stability spectrum of the wave trains in a fundamental way. The results obtained for hot plasmas indicate that, when  $B/B_0 \ll 1$ , the influence of an energetic electron population dominates, and modifies, the instability characteristics derived for cold plasmas.

In Section 5.2 we obtain the equations obeyed by the complex amplitude of the wave train envelope, assuming that the (linear and

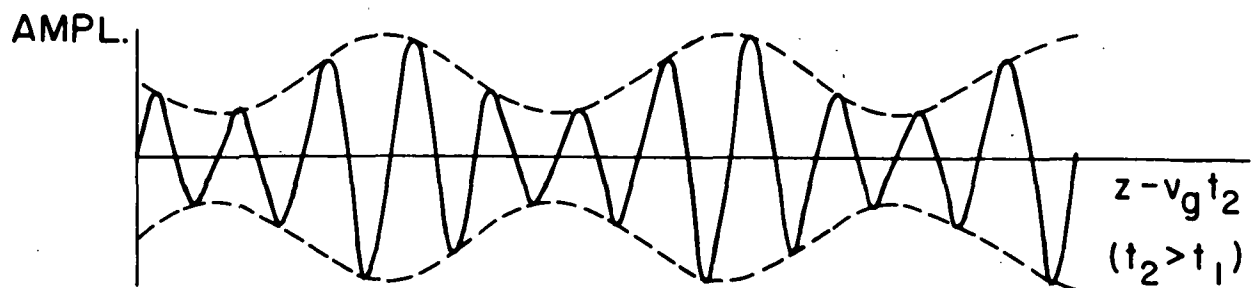
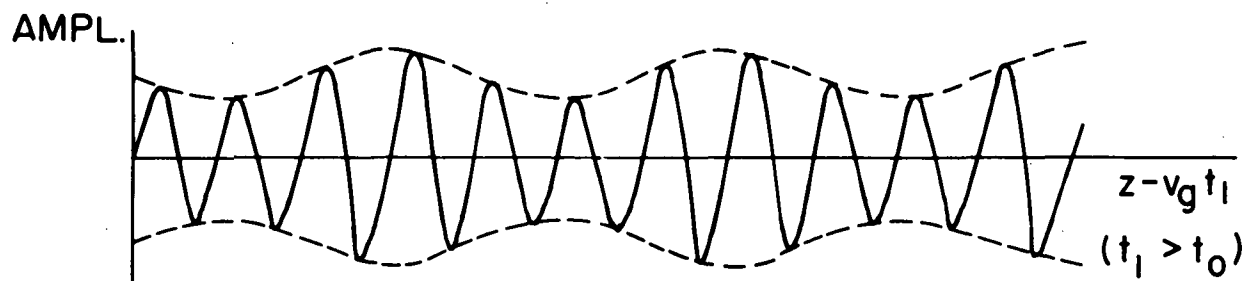
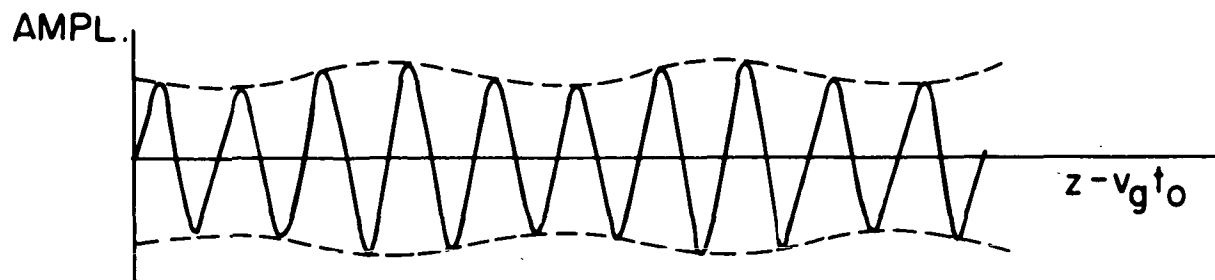


FIG. 5.1. Possible evolution of a wave packet with group velocity  $v_g$  in a modulationally unstable medium.

nonlinear) dispersive properties of the medium are known. These equations are used to establish the conditions required for the occurrence of modulational instability. Sections 5.3 and 5.4 apply the results to whistlers in cold and hot plasmas. Section 5.5 discusses the possible occurrence of the modulational instability in magnetospheric whistlers, speculating on the relevance of other mechanisms to some observed 'pulsations' [Bell and Helliwell, 1971].

## 5.2 Theory

We consider one-dimensional propagation of a plasma mode in a nonlinear dispersive medium that can be characterized, to lowest order in the wave amplitude  $a$ , by the dispersion relation ( $\beta > 0$ )

$$\omega = \Omega + \mu a^\beta, \quad (5.1)$$

where  $\omega = \Omega(k)$  is the linear dispersion relation and

$$\mu(k) = \left( \frac{\partial \omega}{\partial a^\beta} \right)_{a=0} \quad (5.2)$$

characterizes the frequency shift caused by the dependence of the average properties of the medium on the small wave amplitude.

We assume the existence of an equilibrium state consisting of the propagation along the  $z$ -axis of a wave with amplitude  $a_0$ , frequency  $\omega_0$ , and wavenumber  $k_0$ , satisfying

$$\omega_0 = \Omega_0 + \mu_0 a_0^\beta, \quad (5.3)$$

with  $\Omega_0 \equiv \Omega(k_0)$  and  $\mu_0 = \mu(k_0)$ . To study the stability of this equilibrium with respect to a modulational perturbation, we derive the equation satisfied by the complex amplitude of the wave train.

The perturbed wave train may be represented at  $t = 0$  by

$$\psi(z, 0) = \int_{-\infty}^{\infty} dk \phi(k) \exp(-ikz), \quad (5.4)$$

where  $\phi(k)$ , the spatial Fourier transform of the initial wave, is concentrated about  $k_0$ . The subsequent temporal evolution of the wave train is then obtained from

$$\psi(z, t) = \int_{-\infty}^{\infty} dk \phi(k) \exp[i(\omega t - kz)] , \quad (5.5)$$

with  $\omega = \omega(k)$  given by (5.1). Using the first three terms of the Taylor expansion of  $\Omega(k)$  about  $k_0$ , we can write (5.5) as

$$\psi(z, t) = \varphi(z, t) \exp[i(\omega_0 t - k_0 z)] , \quad (5.6)$$

where the complex amplitude of the envelope is ( $\kappa = k - k_0$ )

$$\varphi(z, t) = a(z, t) \exp i\theta(z, t)$$

$$= \int_{-\infty}^{\infty} d\kappa \phi(\kappa + k_0) \exp i \left\{ \left[ \kappa v_{g0} + \frac{1}{2} \kappa^2 v'_{g0} + \mu_0 (a^\beta - a_0^\beta) \right] t - \kappa z \right\} , \quad (5.7)$$

and

$$v_{g0} = \frac{\partial \Omega(k_0)}{\partial k} , \quad v'_{g0} = \frac{\partial^2 \Omega(k_0)}{\partial k^2} . \quad (5.8)$$

Differentiation of  $\varphi$  with respect to  $t$  and  $z$  shows that the envelope amplitude satisfies the equation

$$- i \left( \frac{\partial \varphi}{\partial t} + v_{g0} \frac{\partial \varphi}{\partial z} \right) + \frac{1}{2} v'_{g0} \frac{\partial^2 \varphi}{\partial z^2} - \mu_0 (a^\beta - a_0^\beta) \varphi = 0 , \quad (5.9)$$

or, going to the packet wave frame with the introduction of new variables,  $\xi = z - v_{g0} t$  and  $\tau = t$ ,

$$- i \frac{\partial \varphi}{\partial \tau} + \frac{1}{2} v'_{g0} \frac{\partial^2 \varphi}{\partial \xi^2} - \mu_0 (a^\beta - a_0^\beta) \varphi = 0 . \quad (5.10)$$

This equation has been derived previously by other methods, and for  $\beta = 2$ , by Karpman and Krushkal' [1969], Taniuti and Yajima [1969], and Dysthe [1970]. It is identical to the Schrödinger equation with a nonlinear potential term (note that  $a = |\varphi|$ ) and, as assumed at the outset, admits  $\varphi = a_0$  as a solution. Separating real and imaginary parts, we obtain

$$\begin{aligned} -\frac{\partial a^\beta}{\partial \tau} + v'_{g0} \frac{\partial a^\beta}{\partial \xi} \frac{\partial \theta}{\partial \xi} + v'_{g0} \frac{\beta a^\beta}{2} \frac{\partial^2 \theta}{\partial \xi^2} &= 0, \\ a \frac{\partial \theta}{\partial \tau} + \frac{1}{2} v'_{g0} \frac{\partial^2 a}{\partial \xi^2} - \frac{1}{2} v'_{g0} a \left( \frac{\partial \theta}{\partial \xi} \right)^2 - \mu_0 a (a^\beta - a_0^\beta) &= 0. \end{aligned} \quad (5.11)$$

A modulational perturbation about the equilibrium  $a = a_0$  and  $\theta = 0$ , of the form

$$\begin{aligned} a^\beta - a_0^\beta &= a_1 \exp i(\bar{\omega} \tau - \bar{k} \xi) & (a_1 \ll a_0^\beta), \\ \theta &= \theta_1 \exp i(\bar{\omega} \tau - \bar{k} \xi) & (\theta_1 \ll 1), \end{aligned} \quad (5.12)$$

yields the linear dispersion relation

$$\begin{aligned} \bar{\omega}^2 &= \left( \frac{\bar{k} v'_{g0}}{2} \right)^2 (\bar{k}^2 - \bar{k}_L^2), \\ \bar{k}_L^2 &= - \frac{2 \beta a_0^\beta \mu_0}{v'_{g0}}. \end{aligned} \quad (5.13)$$

When  $\mu_0/v'_{g0} < 0$ , i.e. when the potential in the nonlinear Schrödinger equation becomes attractive, we note that the perturbations will be linearly unstable ( $\bar{\omega}^2 < 0$ ) if the modulation wave number satisfies  $\bar{k}^2 < \bar{k}_L^2$ . The maximal temporal (linear) growth rate is

$$\bar{\omega}_{iM} = - \frac{\beta}{2} |\mu_0| a_0^\beta, \quad (5.14)$$

and occurs for

$$\bar{k} = \bar{k}_M = \left( \frac{\bar{k}_L^2}{2} \right)^{1/2}. \quad (5.15)$$

We thus expect that wave trains in media satisfying  $\mu_0/v'_{g0} < 0$  will be unstable when modulated by perturbations of sufficiently large wavelength. The initial (linear) evolution of the instability will tend to increase the depth of modulation, but the subsequent (nonlinear) behavior of the unstable wave train must be followed numerically. This has been done in a few cases for  $\beta = 2$  [Karpman and Krushkal', 1969; Hasegawa, 1970; Hasegawa, 1972b], and sometimes indicates the breaking up of the original wave into a number of solitary waves (solitons) of stationary (envelope) amplitude in the wave train frame. With respect to this evolution, it is interesting to note that the soliton-type expressions

$$a^\beta = A^2 \operatorname{sech}^2 \left\{ \beta A \left[ - \frac{\mu_0}{(\beta+2)v'_{g0}} \right]^{1/2} (\xi - \xi_0) \right\}, \quad (5.16)$$

$$\theta = \mu_0 \left[ \frac{2A^2}{\beta+2} - a_0^\beta \right] \tau + \theta_0,$$

where  $A$ ,  $\theta_0$  and  $\xi_0$  are real constants, represent exact solutions of (5.10) or (5.11). These solutions were obtained by Chiao et al. [1964], and Karpman and Krushkal' [1969] for the case  $\beta = 2$  but, as stressed in Section 5.5 below, observable modulational instabilities in magnetospheric whistlers will probably occur for  $\beta = 1/2$ .

### 5.3 Cold Plasma Case

For one-dimensional propagation along  $z$  in a two-component cold magnetoplasma with constant magnetic field  $\underline{B}_0 = B_0 \hat{z}$ , the equations of motion, Maxwell, and continuity, determine the evolution of the system:

$$m_j \left( \frac{\partial}{\partial t} + v_{\parallel j} \frac{\partial}{\partial z} \right) \left( \gamma_j \underline{v}_j \right) = q_j \underline{E} + q_j \underline{v}_j \times (\underline{B}_0 + \underline{B}) ,$$

$$\nabla \times \underline{B} = \frac{1}{\epsilon_0 c^2} \sum_{j=i,e} q_j n_j \underline{v}_j + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} , \quad (5.17)$$

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} ,$$

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial z} (n_j v_{\parallel j}) = 0 .$$

The wave fields are  $\underline{B}$  and  $\underline{E}$ ; subscript  $j$  distinguishes the electron and ion variables;  $\epsilon_0$  and  $c$  represent the free space permittivity and speed of light; we use  $q_e = -e$ ,  $v_{\parallel} = v_z$ , and  $\gamma_j = (1 - v_j^2/c^2)^{-1/2}$ . To proceed, we make several simplifications. First, the displacement current is neglected and charge neutrality is assumed. This amounts to dropping the last term of the second equation in (5.17), using  $n_i = n_e = n$  and (continuity equation)  $v_{\parallel i} = v_{\parallel e} = v_{\parallel}$ . The range of validity of these approximations was analyzed by Kakutani et al. [1967] and includes our domain of interest. The displacement current can be disregarded for dense plasma,  $\omega_{pe}^2 \gg \Omega_e^2$ , while charge neutrality is a good approximation for  $v_A \ll c$ ; the electron plasma and cyclotron frequencies,  $\omega_{pe}$  and  $\Omega_e$ , and the Alfvén velocity,  $v_A$ , are defined by

$$\omega_{pe}^2 = \frac{n_0 e^2}{\epsilon_0 m_e} , \quad \Omega_{e,i} = \frac{|q_{e,i}| B_0}{m_{e,i}} , \quad v_A^2 = \frac{\epsilon_0 B_0^2 c^2}{n_0 m_i} , \quad (5.18)$$

where  $n_0$  is the unperturbed number density. The expression for  $v_A^2$  makes use of  $m_i \gg m_e$ ; this inequality will be utilized below and, when combined with our interest in right-hand circularly polarized waves of frequencies  $\omega$  such that  $\Omega_e > \omega \gg \Omega_i$ , justifies disregarding the

relativistic ion mass correction, i.e. we shall take  $\gamma_i = 1$ .

Utilization of these simplifications in (5.17), and elimination of the wave electric field and the ion velocity, leads to the following system of equations

$$\begin{aligned}
\frac{dH}{dt} + H \frac{\partial v_{\parallel}}{\partial z} - i \frac{v_A}{\Omega_e} \frac{\partial}{\partial z} \left[ \frac{d}{dt} (\gamma_e u) \right] - v_A \frac{\partial u}{\partial z} &= 0, \\
\frac{du}{dt} - v_A \frac{n_0}{n} \frac{\partial H}{\partial z} + i \frac{v_A}{\Omega_i} \frac{d}{dt} \left( \frac{n_0}{n} \frac{\partial H}{\partial z} \right) &= 0, \\
\frac{\partial n}{\partial t} + \frac{\partial}{\partial z} (n v_{\parallel}) &= 0, \\
\frac{dv_{\parallel}}{dt} + \frac{v_A^2}{2} \frac{n_0}{n} \frac{\partial |H|^2}{\partial z} &= 0,
\end{aligned} \tag{5.19}$$

with

$$H = \frac{B_x + iB_y}{B_0}, \quad u = \frac{v_x + iv_y}{v_A}, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + v_{\parallel} \frac{\partial}{\partial z}.$$

A possible solution of this system is

$$\begin{pmatrix} v_{\parallel} \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ n_0 \end{pmatrix}, \quad \begin{pmatrix} H \\ u \end{pmatrix} = \begin{pmatrix} H_0 \\ u_0 \end{pmatrix} \exp i(\omega_0 t - k_0 z), \tag{5.20}$$

provided  $\omega_0$  and  $k_0$  satisfy the relativistic dispersion relation,

$$\left( \frac{\omega_0}{k_0 v_A} \right)^2 = \left( 1 + \frac{\omega_0}{\Omega_i} \right) \left( 1 - \frac{\gamma_e \omega_0}{\Omega_e} \right), \tag{5.21}$$

with  $\gamma_e = [1 - (v_A |u_0|/c)^2]^{-1/2}$ . Monochromatic whistlers of arbitrary

amplitude are exact solutions of (5.19) if their frequency and wavenumber satisfy (5.21). We have thus found an equilibrium state similar to the one used in Section 5.2 with  $\beta = 2$ . In particular, expanding  $\gamma_e$ ,  $\gamma_e \approx 1 + |u_0|^2 v_A^2 / 2c^2$ , noting from (5.19) that  $|u_0|^2 = (k_0 v_A / \omega_0)^2 (1 + \omega_0 / \Omega_i)^2 |H_0|^2$  and considering whistlers with frequencies  $\omega_0 \gg \Omega_i$ , we can write the relativistic dispersion relation (5.21) as

$$\omega_0 = \Omega(k_0) - \left( \frac{\omega_0^3}{2\omega_{pe}^2} \right) \frac{|H_0|^2}{(1 - \omega_0 / \Omega_e)^2}, \quad (5.21a)$$

where the linear dispersion relation,

$$\omega = \Omega(k) = \frac{k^2 v_A^2}{\Omega_i} \left( 1 - \frac{\omega}{\Omega_e} \right) = \frac{c^2 k^2}{\omega_{pe}^2} (\Omega_e - \omega), \quad (5.22)$$

defines

$$v_{g0} = 2 \frac{\omega_0}{k_0} \left( 1 - \frac{\omega_0}{\Omega_e} \right), \quad v'_{g0} = 2 \frac{\omega_0}{k_0^2} \left( 1 - 4 \frac{\omega_0}{\Omega_e} \right) \left( 1 - \frac{\omega_0}{\Omega_e} \right). \quad (5.23)$$

The relativistic frequency shift is characterized by

$$\mu_{0r} = \frac{\partial \omega}{\partial |H_0|^2} = - \frac{\omega_0^3}{2\omega_{pe}^2} \frac{1}{(1 - \omega_0 / \Omega_e)^2}, \quad (5.24)$$

showing that, if no other nonlinear effects were involved, the whistler mode would be unstable to modulational perturbations ( $v'_{g0} / \mu_{0r} < 0$ ) for frequencies  $\omega < \Omega_e / 4$ . This result is at variance with the conclusions of Tang and Sivasubramanian [1971] who did not take into account the

dependence of  $\text{sgn } v'_{g0}$  on  $\omega/\Omega_e$ , and derived a relativistic frequency shift that seems to be incorrect: for dense plasmas,  $\omega_{pe}^2 \gg \Omega_e^2$ , it is independent of  $(1-\omega_0/\Omega_e)$ , ignoring the influence of the cyclotron resonance on the electron velocity,  $|u_0| \propto (1-\omega_0/\Omega_e)^{-1}$ .

However, the relativistic correction just considered is not the only nonlinear effect contributing to the total frequency shift experienced by the whistler train of finite amplitude. Although (5.19) admits (5.20) as exact solutions subject to the dispersion relation (5.21), we note [Taniuti and Washimi, 1968] that a superposition of two (or more) waves satisfying (5.20), with different frequencies and wavenumbers obeying (5.21), is not a solution of (5.19). We are thus led to look for wave train solutions of the form  $\varphi(z,t) \exp i(\omega_0 t - k_0 z)$ , where  $\varphi$  is a slowly varying function of time and space. Following the method of Taniuti and Yajima [1969], it is found in the nonrelativistic case ( $\gamma_e = \gamma_i = 1$ ) that the (nonlinear) perturbations in the number density  $n$ , and the creation of a nonzero  $v_{\parallel}$ , give rise to a further frequency shift characterized by [Hasegawa, 1972b]

$$\mu_{0i} = \frac{\partial \omega}{\partial |H_0|^2} = \frac{k_0 v_A^2}{4 v_{g0}} = \frac{\Omega_i}{8(1-\omega_0/\Omega_e)^2} . \quad (5.25)$$

In contrast to the relativistic one, this component of the frequency shift is positive, and becomes zero when the ion motion is neglected ( $m_i \rightarrow \infty$ ,  $\Omega_i \rightarrow 0$ ). Hence, previous analyses [Tam, 1969] neglecting relativistic effects and ion motion were led to attribute a zero frequency shift, and modulational stability, to whistlers propagating along  $B_0$ . When ion motion is considered, but relativistic effects are ignored [Taniuti and Washimi, 1968; Hasegawa, 1972a], the resulting positive frequency shift leads to the conclusion that modulational instability occurs for whistler frequencies,  $\omega$ , satisfying  $\Omega_e/4 < \omega < \Omega_e$ . The actual modulational stability spectrum of the whistler train is obtained by combining these two effects, ion motion and relativistic dynamics. The total frequency shift is then characterized by

$$\mu_0 = \mu_{0r} + \mu_{0i} = \frac{(\Omega_i/4 - \omega_0^3/\omega_p^2)}{2(1 - \omega_0/\Omega_e)^2} . \quad (5.26)$$

Recalling the expression given in (5.23) for  $v'_{g0}$ , and defining

$$\bar{\omega}_p^2 = \frac{\Omega_e^3}{16 \Omega_i} , \quad \bar{\omega} = \left( \frac{\Omega_i \omega_{pe}^2}{4} \right)^{1/3} < 1 , \quad (5.27)$$

we find the following modulational stability spectrum for whistler trains of frequency  $\omega (\gg \Omega_i)$  in cold dense plasmas ( $\omega_p^2 \gg \Omega_e^2$ ). The unstable band for  $\omega_{pe} > \bar{\omega}_p$  is given by  $\bar{\omega} > \omega > \Omega_e/4$ , whereas for  $\omega_{pe} < \bar{\omega}_p$ , instability arises in the band  $\Omega_e/4 > \omega > \bar{\omega}$ . When  $\omega_{pe} = \bar{\omega}_p$ , no modulational instability occurs.

The importance of considering simultaneously the influence of ion motion and relativistic dynamics when studying the whistler modulational instability in cold plasmas is stressed in Figure 5.2, where the stability spectra obtained using simplifications are contrasted with the actual cold plasma spectrum.

#### 5.4 Hot Plasma Case

BACKGROUND. Previous work on the propagation of large amplitude whistlers in hot plasmas [Palmadesso and Schmidt, 1971] shows that, when  $|\gamma_L| \ll \bar{\omega}_{NL}$ , the waves approach a finite constant amplitude after an initial linear damping followed by oscillations of decaying magnitude. Here,  $\gamma_L$  is the whistler linear growth rate, and  $\bar{\omega}_{NL}$  represents the oscillation frequency of electrons with the rms perpendicular speed in the bottom of the rf magnetic trapping well. This nonlinear evolution indicates that the complex whistler frequency obtained in the solution of the initial value problem,  $\omega(t) = \omega_r(t) + i\gamma(t)$ , has an imaginary component satisfying  $\gamma(t \rightarrow \infty) = \hat{\gamma} = 0$ . We use  $\hat{\phantom{x}}$  to denote time-asymptotic values. However, as shown in Section 5.2, the relevant quantity to study the modulational stability of the asymptotic state is the real frequency shift  $\delta\Omega = \omega_r(0) - \bar{\omega}_r$ . In obtaining  $\delta\Omega$  we shall use a method similar to the approach adopted by Morales and O'Neil [1972]

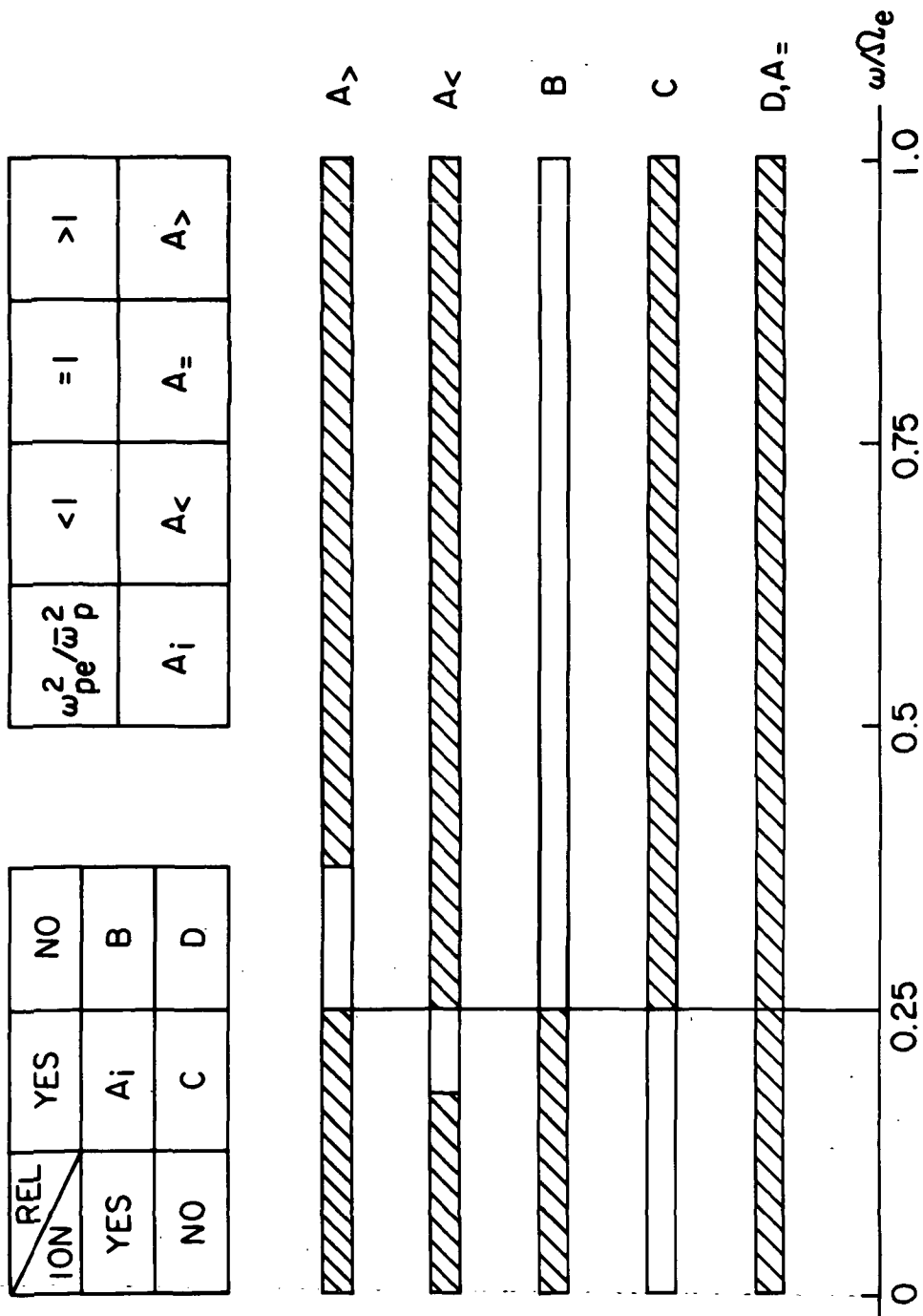


FIG. 5.2. Whistler modulational stability spectrum in cold plasmas (stable bands are hatched). (REL - Relativistic force equation used; ION - Ion motion considered.)  $[\omega \gg \Omega_i, \omega_{pe}^2 \gg \Omega_e^2]$ .

in the determination of the frequency shift of electrostatic Landau plasma waves. The whistler frequency shift in hot plasmas, as derived below, is proportional to  $(B/B_0)^{1/2}$ . Because our interest concerns situations where  $B/B_0 \ll 1$ , we shall neglect, in this section, the ion motion and relativistic effects: they yield frequency shifts proportional to  $(B/B_0)^2$ .

**THE FREQUENCY SHIFT.** We consider a parallel whistler ( $\underline{k} \parallel B_0 \hat{z}$ ) in a dense magnetoplasma ( $\omega_{pe}^2 \gg \Omega_e^2$ ), so that the displacement current becomes negligible. The wave fields satisfy

$$\nabla \times \underline{B} = - \frac{en_0}{c^2 \epsilon_0} \int \underline{v} d\underline{v} \quad , \quad \nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (5.28)$$

where  $F = F_0 + f$ ; the perturbed (exact) electron velocity distribution, satisfies the Vlasov equation

$$\frac{\partial F}{\partial t} + \underline{v} \parallel \frac{\partial F}{\partial z} - \frac{e}{m_e} \left[ \underline{E} + \underline{v} \times (\underline{B} + \underline{B}_0) \right] \cdot \frac{\partial F}{\partial \underline{v}} = 0 \quad (5.29)$$

The equilibrium electron velocity distribution, normalized to unity, is

$F_0 = F_0(v_{\parallel}, v_{\perp})$ , and  $n_0$  represents the electron number density.

Using

$$H = B_x + iB_y \quad , \quad u = v_x + iv_y = v_{\perp} \exp i\phi, \quad ( \quad )_k = \int_{-\lambda/2}^{\lambda/2} ( \quad ) \exp ikz d(z/\lambda), \quad (5.30)$$

(note that  $k = 2\pi/\lambda$  and, in contrast to Section 5.3,  $H$  and  $u$  are not normalized to  $B_0$  and  $v_A$ ), and assuming that

$$H = B \exp i \left( \int_0^t \omega(t') dt' - kz \right) \quad , \quad (5.31)$$

we find

$$kH_k(t) = - \frac{en_0}{c^2 \epsilon_0} \int u f_k d\underline{v} \quad (5.32)$$

Adapting the electrostatic approach of Morales and O'Neil [1972] to this electromagnetic problem, we subtract from both sides of this equation the linear current density consistent with the exact whistler fields to obtain

$$\left\{ k - \frac{\omega^2(t)}{kc^2} \epsilon[k, \omega(t)] \right\} H_k(t) = - \frac{en_0}{c^2 \epsilon_0} \int u (f_k - f_k^L) d\underline{v} , \quad (5.33)$$

where  $f_k^L$  is the solution of the Vlasov equation linearized about  $F_0$ , and  $\epsilon$  is the linear whistler equivalent permittivity.

In the linear regime, we have  $f_k \approx f_k^L$ , and the linear dispersion relation  $c^2 k^2 / \omega^2 = \epsilon$  determines the linear complex frequency for the initial value problem,  $\omega(t) = \omega_L$ . Whistlers of large amplitude cause a nonlinear correction to the current density, bringing about a small nonlinear complex frequency shift  $\delta\omega = \omega(t) - \omega_L$ . To determine the desired real frequency shift  $\delta\Omega$ , we expand the coefficient of  $\hat{H}_k$  in the time asymptotic form of (5.33) about  $\omega_L = \omega_r(0) + i\gamma_L \approx \omega_r(0) \equiv \omega_0$ , obtaining

$$\frac{\omega_0}{c^2 k} \left[ 2 \epsilon(\omega_0) + \omega_0 \frac{\partial \epsilon(\omega_0)}{\partial \omega} \right] \hat{H}_k \delta\Omega = \frac{en_0}{c^2 \epsilon_0} \int u (\hat{f}_k - \hat{f}_k^L) d\underline{v} , \quad (5.34)$$

where

$$\begin{aligned} \hat{f}_k^L = & -i \frac{e}{2m_e} \left[ \left( \frac{\hat{\omega}}{k} - v_{\parallel} \right) \frac{\partial F_0}{\partial v_{\perp}} + v_{\perp} \frac{\partial F_0}{\partial v_{\parallel}} \right] \exp(-i\phi) \\ & \times \int_0^t \hat{H}_k(t') \exp \left[ -i(\Omega_e + kv_{\parallel})(t' - t) \right] dt' \end{aligned} \quad (5.35)$$

is the solution to the linearized Vlasov equation

$$\frac{\partial \hat{f}_k^L}{\partial t} + \Omega_e \frac{\partial \hat{f}_k^L}{\partial \phi} - ikv_{\parallel} \hat{f}_k^L = -i \frac{e}{2m_e} \left[ \left( \frac{\hat{\omega}}{k} - v_{\parallel} \right) \frac{\partial F_0}{\partial v_{\perp}} + v_{\perp} \frac{\partial F_0}{\partial v_{\parallel}} \right] \exp(-i\phi) \hat{H}_k(t) , \quad (5.36)$$

the whistler equivalent permittivity is

$$\epsilon = -k \left( \frac{m_e}{e} \right) \left( \frac{\omega}{\hat{\omega}} \right)^2 \int u \frac{\hat{f}_k^L}{\hat{H}_k} d\underline{v} , \quad (5.37)$$

and  $\hat{f}$  is determined in Appendix C.

Performing the  $\phi$ ,  $v_{\parallel}$  and  $z$  integrations on the right-hand side of (5.34) as indicated in Appendix C, yields

$$\int u (\hat{f}_k - \hat{f}_k^L) d\underline{v} = -1.63 \pi \exp i\hat{\omega}t \int_0^{\infty} dv_{\perp} v_{\perp}^2 \left( \frac{\omega_{NL}}{k} \right)^3 \left( \frac{\partial^2 F_0}{\partial v_{\parallel}^2} \right)_{v_{\parallel}=V_G} , \quad (5.38)$$

with

$$\omega_{NL}^2 = \frac{kv_{\perp} eB}{m_e} , \quad V_G = \frac{(\hat{\omega} - \Omega_e)}{k} \approx \frac{(\omega_0 - \Omega_e)}{k} , \quad (5.39)$$

so that the asymptotic (real) frequency shift becomes

$$\delta\Omega = \mu_0 \left( \frac{B}{B_0} \right)^{1/2} , \quad (5.40)$$

$$\mu_0 = -1.63\pi \frac{\omega_0^2}{\omega_0} \left[ 2\epsilon(\omega_0) + \omega_0 \frac{\partial \epsilon(\omega_0)}{\partial \omega} \right]^{-1} \left( \frac{\Omega_e}{k} \right)^{1/2} \int_0^{\infty} dv_{\perp} v_{\perp}^{7/2} \left( \frac{\partial^2 F_0}{\partial v_{\parallel}^2} \right)_{v_{\parallel}=V_G} .$$

This frequency shift has  $\beta = 1/2$ , and depends on the characteristics of the hot electron population.

MAGNETOSPHERE. In the magnetosphere a cold plasma is permeated by a dilute energetic electron population of fractional number density  $\alpha$ , and equilibrium distribution  $f_0(v_{\parallel}, v_{\perp})$ . Using the energetic velocity distribution given by (3.30) in  $\mu_0$ , and adopting for  $\epsilon$  the cold dense plasma whistler permittivity value, we find

$$\mu_0 = -0.23 \alpha \frac{\Gamma(p+2.25) \Gamma(q-p-1.25)}{(q+1)\Gamma(p+1) \Gamma(q-p-1.5)} \left( \frac{v_0^2}{v_G^2 + v_0^2} \right)^{q-p-1.25} \quad (5.41)$$

$$\times \left\{ (q-p-1.25) \frac{[(2q+1)v_G^2 - v_0^2]}{(v_G^2 + v_0^2)} - (p+2.25) \right\} \Omega_e \left( 1 - \frac{\omega_0}{\Omega_e} \right)^2 \left( \frac{\Omega_e}{kv_0} \right)^{1/2}.$$

From Section 5.2 we know that the modulational instability arises when  $\text{sgn}(\mu_0 v_{g0}') = -1$ . Thus, using (5.23) and (5.41), we can determine the whistler stability spectrum in hot or magnetospheric plasmas. The results shown in Figure 5.3, for nonthermal electron populations with varying energies and anisotropies, sharply contrast with the cold plasma stability spectrum of Figure 5.2. Since the frequency shifts, and hence the growth rates, are proportional to  $(B/B_0)^2$  and  $(B/B_0)^{1/2}$  in cold ( $\beta = 2$ ) and hot ( $\beta = 1/2$ ) plasmas respectively, we conclude that for magnetospheric whistlers, where  $B/B_0 \ll 1$ , the hot plasma modulational instability is dominant.

### 5.5 Discussion

COLD PLASMA. The possible observation of the modulational instability in magnetospheric whistlers requires the existence of an unstable band, and a growth rate large enough to ensure the development of initial large wavelength amplitude perturbations into marked amplitude modulation. The instability is most likely to occur in the equatorial portion of the whistler path, due to both homogeneity and slower wave velocity. Since  $\bar{\omega}_p = 11.2 \Omega_e$  for an electron-proton plasma, most magnetospheric L-shells have  $\omega_{pe} < \bar{\omega}_p$ : the narrow unstable band in cold plasmas

|                                                     | U =<br>300 ev | U =<br>1 kev | U =<br>10 kev |
|-----------------------------------------------------|---------------|--------------|---------------|
| $T_{\perp} = T_{\parallel} \text{ (} p=0 \text{)}$  | ○             | △            | □             |
| $T_{\perp} = 2T_{\parallel} \text{ (} p=1 \text{)}$ | ●             | ▲            | ■             |

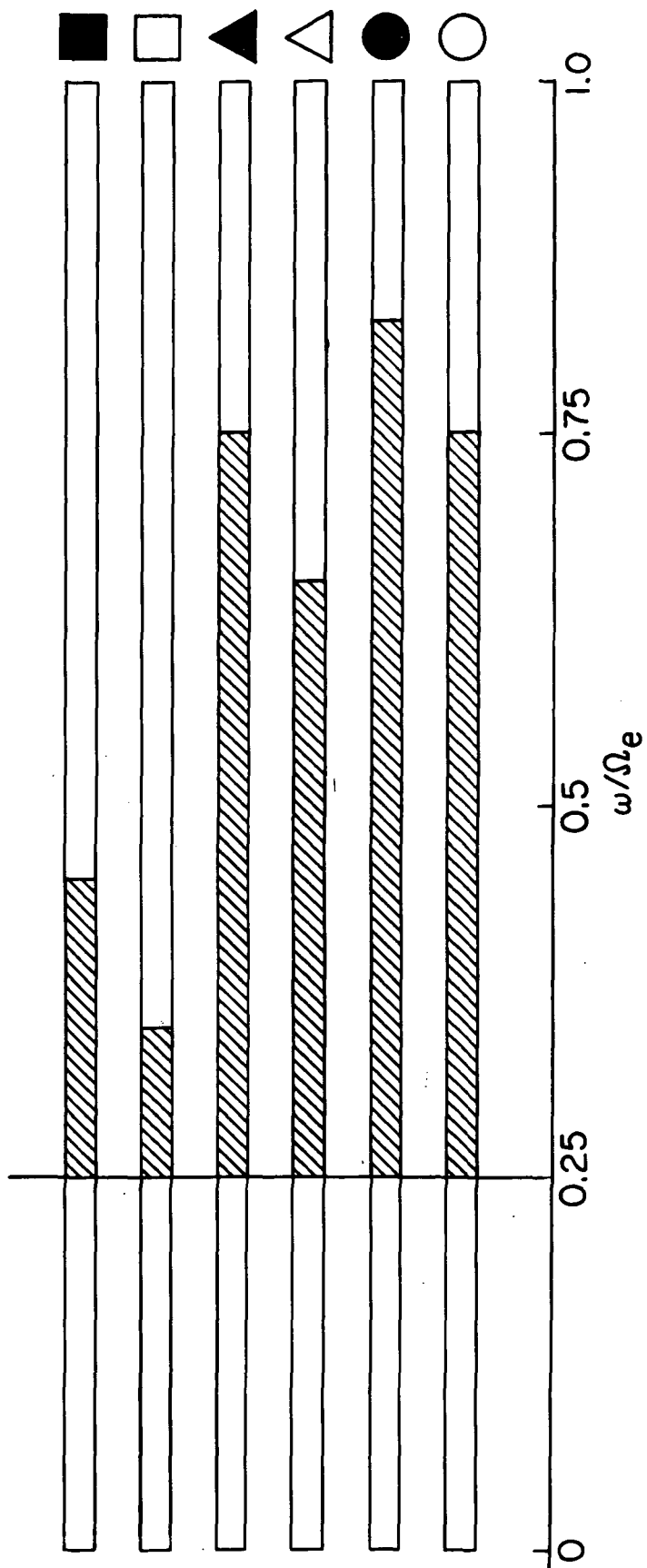


FIG. 5.3. Whistler modulational stability spectrum in hot plasmas (stable bands are hatched).  
 $[\omega_{pe}/\Omega_e = 10]$

will be  $\Omega_e/4 > \omega > \underline{\omega}$ . Denoting by  $\delta\Omega (= \mu_0 a_0^\beta)$  the frequency shift experienced by the whistler in the initial value problem, we find, from (5.14) that  $|\bar{\omega}_{iM}| = |\delta\Omega|/2$ . (For CW key-down transmission, the frequency, rather than the wave number, is fixed and thus the shift occurs in  $k$ .) In the cold plasma case  $\mu_0$  is defined by (5.26), and  $a_0^\beta = (B/B_0)^2$ . Using  $\omega_0/\Omega_e \sim 0.25$  and  $\omega_p^2/\Omega_e^2 \sim 50$ , we obtain  $|\delta\Omega| \sim 1.6 \times 10^{-4} \Omega_e (B/B_0)^2$ . The growth rate,  $|\bar{\omega}_{iM}| = |\delta\Omega|$ , is negligible since the amplitude of a 'high-field' whistler [Dysthe, 1971] is  $B/B_0 \sim 10^{-4}$ . It is reasonable to conclude that the whistler modulational instability due to the cold plasma (nonlinear and dispersive) properties may not be observed in the magnetosphere.

**MAGNETOSPHERIC PLASMA.** Consideration of the dilute energetic electron population strongly enhances the growth rate of the whistler modulational instability when  $B \ll B_0$ . We have computed the frequency shift for several nonthermal populations with different energies (100 ev - 10 kev) and anisotropies ( $T_\perp/T_\parallel = 1 - 3$ ) and found that  $|\delta\Omega| \sim 5 \times 10^{-2} \alpha \Omega_e (B/B_0)^{1/2}$ . Noting from (5.14) that the maximal growth rate is  $|\bar{\omega}_{iM}| = |\delta\Omega|/4$ , we realize that for typical magnetospheric parameters around  $L \sim 3$  the growth process is still rather weak, yielding  $|\bar{\omega}_{iM}| \sim 3 \times 10^{-2} \text{ s}^{-1}$ , whereas the time taken by a whistler to cross the equatorial region is well below 1 s. It is conceivable, however, that in multiple-hop whistlers a cumulative effect may take place at each equatorial crossing, with observable amplitude modulation as the end result. (The depth of the amplitude modulation will cease increasing with each hop when the nonlinear evolution cited at the end of Section 5.2 is attained.) If this effect were indeed observable, it would certainly occur at larger values of  $L$  where the ratio  $B/B_0$  may become larger and hence more favorable.

**OBSERVATIONS.** To facilitate the eventual interpretation of amplitude modulation in terms of the modulational instability, we establish a relation between measurable parameters and theoretical growth rates. . . . The measured spatial and temporal periodicities of the amplitude modulation define spatial and temporal frequencies which, if created by the modulational instability, should approximate  $\bar{k}_M$  and  $\bar{\omega}_{rM} (= \bar{k}_M v_{g0})$ .

Using (5.13)-(5.15) and (5.22), we find

$$|\bar{\omega}_{iM}| = \frac{\bar{\omega}_{rM}^2}{4\omega_0} \frac{|1 - 4\omega_0/\Omega_e|}{(1 - \omega_0/\Omega_e)} . \quad (5.42)$$

As an example, the 'pulsations' reported by Bell and Helliwell [1971] had, near the equator,  $\omega_0/\Omega_e \sim 0.5$ ,  $\omega_0/2\pi \sim 15$  kHz and  $\bar{\omega}_{rM}/2\pi \sim 8$  Hz, yielding  $|\bar{\omega}_{iM}| \sim 1.3 \times 10^{-2} \text{ s}^{-1}$ . Noting that the time required to cross the equatorial region is about  $\sim 0.3$  s, we conclude that this growth rate is too small to explain the observed pulsations. No gradual increase in the amplitude of the pulsations was observed at the beginning of the 'key-down' emission: their 'steady' amplitude was attained after one hop, thus ruling out the possible existence of a cumulative effect.

The observed pulsations may result from the variations of the nonlinear growth rate affecting the original large amplitude whistler. Palmadesso and Schmidt [1971] have shown for parallel whistlers that the nonlinear growth rate in the initial value problem indeed oscillates with a frequency  $\bar{\omega}_{NL}$  defined by (4.6), before phase mixing to zero. Their derivation, which parallels the work of O'Neil [1965] for large amplitude electrostatic waves, is valid when  $\bar{\omega}_{NL}$  is much larger than the magnitude of the linear growth rate affecting the original whistler. It is conceivable that, when this condition is relaxed, the oscillatory behavior of the nonlinear growth rate increases to the point of creating a marked amplitude modulation in the original whistler. A similar suggestion has been made by Istomin and Karpman [1972] and confirmed by computer simulations which have used an idealized energetic electron population with monochromatic parallel and perpendicular velocities, and obtained large amplitude oscillations in the original whistler [Helliwell and Crystal, 1971, private communication]. In this respect, it should be noted that the observed frequency of the pulsations,  $\bar{\omega}_{rM}$ , is not the trapping frequency  $\bar{\omega}_{NL}$ , obtained in the initial value problem. The two frequencies are related by [Brinca, 1972c; Istomin and Karpman, 1972]

$$\omega_{rM} = \bar{\omega}_{NL} \frac{v_{g0}}{v_{g0} + |V_{G0}|} , \quad (5.43)$$

where  $V_{G0} [= (\omega_0 - \Omega_e)/k_0]$  is the gyroresonant velocity.

Finally, taking into consideration that the original whistler will be oblique in general, we realize that trapping wave-particle interactions might not only be due to the gyroresonance but also due to the Landau resonance, as discussed in Chapter 3. The ratio of the frequencies of the two possible pulsations thus originated in the initial value problem is then [Brinca, 1972d; Palmadesso, 1972]

$$\frac{\omega^2_{\text{(Landau)}}}{\omega^2_{\text{(Gyro)}}} \approx \frac{\omega_0^2 \sin \theta}{k_0 \bar{v}_\perp (\Omega_e \cos \theta - \omega_0)} , \quad (5.44)$$

where  $\bar{v}_\perp$  is the rms perpendicular speed of the energetic electron population, and  $\theta$  is the wave normal obliquity of the whistler.

## 6. CONCLUSIONS

The unifying theme of this thesis is the phenomenon of artificially stimulated emissions in the magnetosphere. In Chapters 2 and 3 we studied the stability characteristics of the triggering whistler, and evaluated the relative strengths of the several wave-particle interactions possible in the equatorial region. This analysis suggested the existence of a mechanism for the emission onset which was analyzed in detail in Chapter 4, and predicted the creation of sidebands of the stimulating whistler pulse. The two strong resonances involved in this onset affected energetic electrons moving in opposite directions along the geomagnetic field lines. By following the evolution of these two distinct groups of resonant particles in the geomagnetic mirror, it was possible to speculate on the main phase of the emissions, and to derive a radiation mechanism potentially able to reproduce most of the observed emission spectral shapes, though detailed calculations for realistic magnetospheric conditions were too complex to be included within the scope of our work. Finally, the recent discovery that both triggering and triggered signals display amplitude modulation, [Stiles, 1972] motivated the study presented in Chapter 5 of a possible causative mechanism, the whistler modulational instability.

All of these studies represent extensions of fundamental work initiated by other researchers, and are themselves open to further criticism and extension. In particular, we stress that the mechanisms developed to describe the onset and main phase of the stimulated emissions cannot be decisively confirmed, or rejected, by the inconclusive observations available to date. The controlled wave-injection experiments that will take place in the near future between Siple and Roberval may be invaluable in this respect [Helliwell, 1972]. For example, the dependence of the triggering delay and offset frequency on the wave amplitude may be compared with the predictions of the onset theory.

The questions raised in the course of this research provide ample ground for future investigations. A more quantitative analysis of the onset for oblique whistlers might benefit from recent work by Palmadesso [1972]. The transition between the onset and the main phase of the

emission, together with the occurrence of self-sustained mechanisms capable of explaining long-duration emissions, should be analyzed. In both cases the utilization of numerical simulations should be useful. They are now being pursued at Stanford [Helliwell and Crystal, 1972; Matsuda, 1972, private communication] and elsewhere [Nunn, 1972; Ossakow et al., 1972; Denavit, 1972, private communication]. The study of the problems associated with slab radiation and, in particular, the propagation characteristics of triggered wavelets, may transform the present speculations on spectral shapes into more strongly based explanations.

Two possible applications of an improvement in the understanding of magnetospheric emissions involving the energetic population in the earth's radiation belts can be contemplated. First, the interpretation of the observed emission characteristics may provide a means of monitoring the energetic population. For example, Dowden [1971] has utilized Helliwell's theory [Helliwell, 1967] to analyze spectrograms of VLF emissions and determine the energy spectrum of the energetic electron population. Second, by regulating particle acceleration and particle dumping in the ionosphere through plasma or wave injection, the content of the radiation belts may possibly be controlled. This would permit reduction of some of the background noise in VLF and satellite communications; radio astronomy experiments would benefit from a lower level of noise originating from the belts by synchrotron radiation, and astronauts would be exposed to smaller doses of radiation.

## APPENDIX A

### THE CAUCHY (HILBERT TRANSFORM) INTEGRAL

The Cauchy integral is of the form

$$G(\xi) = \int_{-\infty}^{\infty} dv \frac{g(v)}{v - \xi} \quad , \quad \xi = \xi_r + i\xi_i \quad . \quad (A.1)$$

It represents a sectionally analytic function  $G(\xi)$  [Roos, 1969] when the sufficient, but not necessary condition is satisfied that the function  $g(v)$  is absolutely integrable over the real axis. That is, the functions

$$G^P(\xi) = \int_{-\infty}^{\infty} dv \frac{g(v)}{v - \xi} \quad (\xi_i > 0) \quad , \quad (A.2)$$

$$G^N(\xi) = \int_{-\infty}^{\infty} dv \frac{g(v)}{v - \xi} \quad (\xi_i < 0) \quad , \quad (A.3)$$

are analytic in the upper and lower complex ( $\xi$ ) half-planes, respectively, but not necessarily on the real axis. In general, each is not the analytic continuation of the other, and it is doubtful whether they may be analytically continued across the real axis boundary. One can show [Greenstein, 1960] that if, and only if,  $g(v)$  is real valued and analytic on the interval  $(a,b)$  of the real axis, the function  $G^P(\xi)$  can be analytically continued across the interval  $(a,b)$  into the lower half-plane, yielding

$$\tilde{G}^P(\xi) = [G^P(\xi^*)]^* + 2\pi i g(\xi) \quad (\xi_i < 0) \quad . \quad (A.4)$$

Clearly,  $\tilde{G}^P(\xi)$  depends on the analytic function  $g(\xi)$ . In particular, since  $G^P(\xi)$  is analytic for  $\xi_i > 0$ , singularities in the lower half-plane of  $\tilde{G}^P(\xi)$  originate from singularities of  $g(\xi)$ . (Note that

these singularities must exist, i.e.  $\tilde{G}^P(\xi)$  must have singularities for  $\xi_i < 0$ ; otherwise Liouville's theorem would ensure that  $G^P(\xi)$ , and its analytic continuation  $\tilde{G}^P(\xi)$ , were equal to a constant.)

Assuming that both  $G^P(\xi)$  and  $G^N(\xi)$  can be defined in the whole complex plane by analytic continuation, we realize that  $G(\xi)$  is a double-valued function. Its Riemann surfaces have a (real axis) cut along which the two sheets go through one another. As shown in Figure A.1, there is no branch point in the finite complex  $\xi$ -plane since the cut goes from  $-\infty$  to  $+\infty$ . [If in (A.1) the limits of integration were 0 and  $\infty$ , the origin would be a logarithmic branch point.]

For  $g(v)$  a square-integrable function, one can obtain more explicit expressions for  $G^P(\xi)$  and  $G^N(\xi)$ . Titchmarsh's theorem [Titchmarsh, 1937] states that a square-integrable function  $g(v_r)$  may be written as a sum of two uniquely defined functions,  $g^+(v)$  and  $g^-(v)$ , with analytic continuations free of singularities into the half-planes  $v_i > 0$  and  $v_i < 0$ , that tend to zero for  $v_i \rightarrow +\infty$  and  $v_i \rightarrow -\infty$ , respectively. Specifically, we have

$$\begin{aligned}
 g(v) &= g^+(v) + g^-(v) & (v_i = 0), \\
 g^+(v) &= \frac{1}{2\pi} \int_0^{\infty} ds \, g(s) \exp isv & (v_i \leq 0), \\
 g^-(v) &= \frac{1}{2\pi} \int_{-\infty}^0 ds \, g(s) \exp isv & (v_i \geq 0),
 \end{aligned} \tag{A.5}$$

with the transform

$$g(s) = \int_{-\infty}^{\infty} dv \, g(v) \exp (-isv) . \tag{A.6}$$

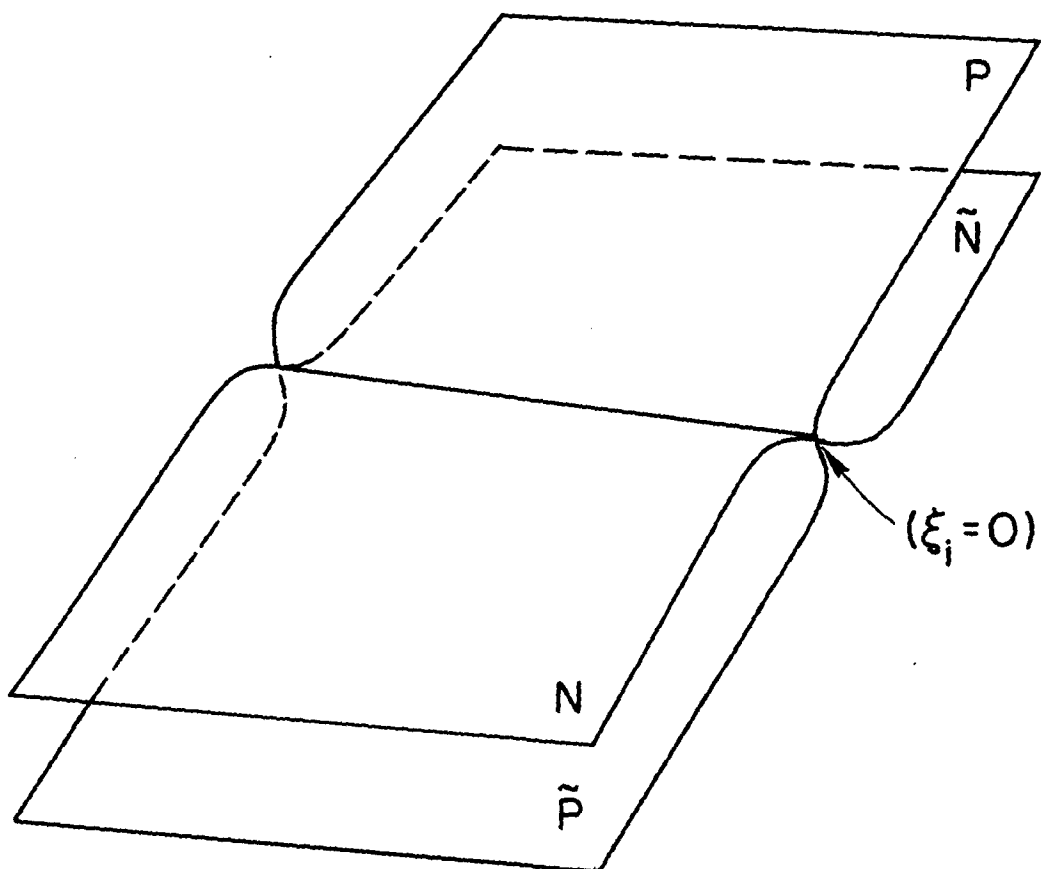


FIG. A.1. Two-sheet Riemann surface of  $G(\xi)$ .  $G^{P,N}(\xi)$  are analytic on  $P, N$  and  $\tilde{G}^{P,N}(\xi)$  have singularities on  $\tilde{P}, \tilde{N}$ .

Integrating (A.2) and (A.3) with the help of Jordan's lemma and Cauchy's integral theorem yields

$$\begin{aligned} G^P(\xi) &= 2\pi i g^+(\xi) & (\xi_i > 0), \\ G^N(\xi) &= -2\pi i g^-(\xi) & (\xi_i < 0), \end{aligned} \quad (A.7)$$

with the following analytic continuations,

$$\begin{aligned} \tilde{G}^P(\xi) &= \left[ G^P(\xi^*) \right]^* + 2\pi i g(\xi) & (\xi_i < 0), \\ \tilde{G}^N(\xi) &= \left[ G^N(\xi^*) \right]^* - 2\pi i g(\xi) & (\xi_i > 0). \end{aligned} \quad (A.8)$$

To simplify the discussions involving Cauchy integrals, we will denote by  $G_c^{P,N}(\xi)$  the functions  $G^{P,N}(\xi)$  and their analytic continuations, that is

$$G_c^P(\xi) = \begin{cases} G^P(\xi) = \int_{-\infty}^{\infty} C(v, \xi) dv = 2\pi i g^+(\xi) & (\xi_i > 0), \\ \oint C(v, \xi) dv + \pi i g(\xi) & (\xi_i = 0), \\ \tilde{G}^P(\xi) = \left[ G^P(\xi^*) \right]^* + 2\pi i g(\xi) & (\xi_i < 0), \end{cases} \quad (A.9)$$

$$G_c^N(\xi) = \begin{cases} G^N(\xi) = \int_{-\infty}^{\infty} C(v, \xi) dv = -2\pi i g^-(\xi) & (\xi_i < 0), \\ \oint C(v, \xi) dv - \pi i g(\xi) & (\xi_i = 0), \\ \tilde{G}^N(\xi) = \left[ G^N(\xi^*) \right]^* - 2\pi i g(\xi) & (\xi_i > 0), \end{cases} \quad (A.10)$$

with  $\oint$  denoting the Cauchy principal value and

$$C(v, \xi) = \frac{g(v)}{v - \xi} . \quad (A.11)$$

In conclusion, we can think of the Cauchy integral (A.1) as defining two distinct functions,  $G_c^P(\xi)$  and  $G_c^N(\xi)$ , which are analytic in  $\xi_i > 0$  and  $\xi_i < 0$ , respectively, and are defined over the whole  $\xi$ -plane.

**SYMMETRY RELATIONS.** With regard to applications, it is important to note that when  $g(v)$  is a real valued function of  $v_r$ , the definitions of  $g^+(v)$  and  $g^-(v)$  show that the following useful relations hold [see Figure A.2(a)]:

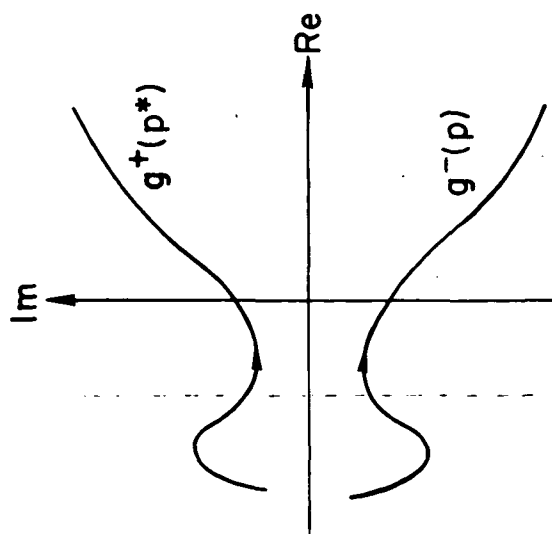
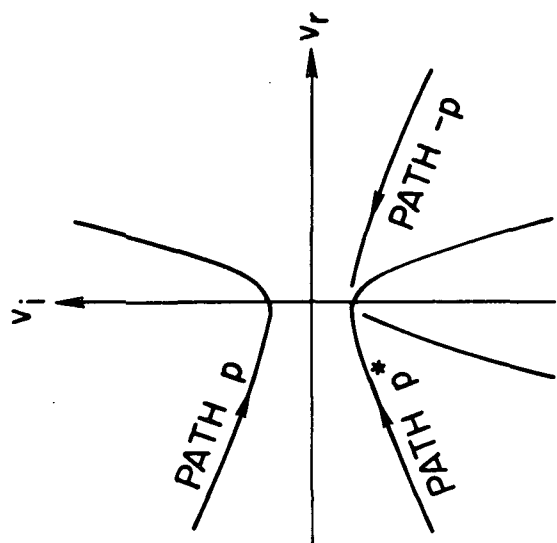
$$\begin{aligned} g(-s) &= [g(s)]^* , \\ g^+(v^*) &= [g^-(v)]^* . \end{aligned} \quad (A.12)$$

If  $g(v_r)$  is an even function, we have [see Figure A.2(b)]

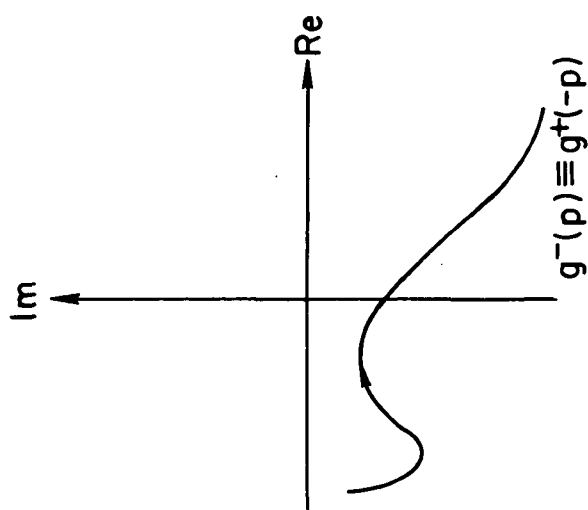
$$\begin{aligned} [g^+(v)]^* &= g^+(-v^*) = g^-(v^*) , \\ [g^-(v)]^* &= g^-(-v^*) = g^+(v^*) . \end{aligned} \quad (A.13)$$

If it is odd, we have [see Figure A.2(c)]

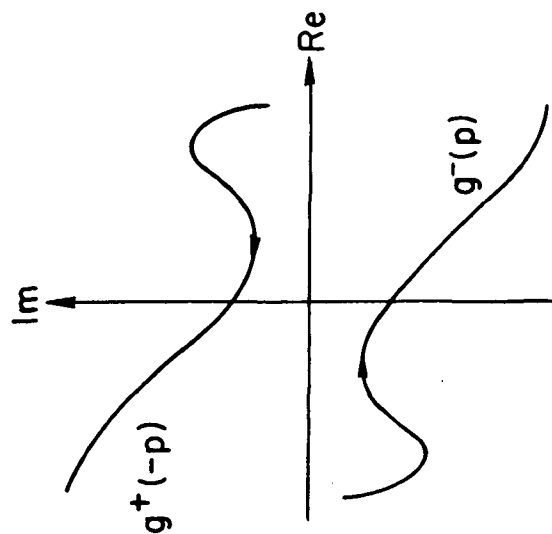
$$\begin{aligned} [g^+(v)]^* &= -g^+(-v^*) = g^-(v^*) , \\ [g^-(v)]^* &= -g^-(-v^*) = g^+(v^*) . \end{aligned} \quad (A.14)$$



(a)



(b)



(c)

FIG. A.2. Symmetry relations between the positive,  $g^+(v)$ , and negative,  $g^-(v)$ , frequency parts of  $g(v)$ , a real valued function of  $v_r$ . (a) Arbitrary  $g(v_r)$ . (b) Even  $g(v_r)$ . (c) Odd  $g(v_r)$ .

# APPENDIX B.

## SOME PROPERTIES OF $D^{P,N}(\omega, k)$

The two dispersion functions  $D^P(\omega, k)$  and  $D^N(\omega, k)$ , defined in (2.15) and (2.16), have the following properties:

(B.I) There exists a positive real number  $\alpha$  such that for  
(i)  $\omega_i < -\alpha < 0$  and (ii)  $k = k_r < 0$ , we have

$$D^P(\omega, k) \neq 0. \quad (B.1)$$

(B.II) There exists a positive real number  $\beta$  such that for  
(i)  $\omega_i < -\beta < 0$  and (ii)  $k = k_r > 0$ , we have

$$D^N(\omega, k) \neq 0. \quad (B.2)$$

To prove (B.I), we define  $1/\omega^2 = a + ib$  and obtain from (2.15), with  $k = k_r$ ,

$$\begin{aligned} D_r^P &= 1 - (k^2 c^2 + \omega_p^2) a - 2\pi \omega_p^2 \left[ b \left( \frac{\Omega}{k} S_r^+ - T_r^+ \right) + a \left( \frac{\Omega}{k} S_i^+ - T_i^+ \right) \right], \\ D_i^P &= - (k^2 c^2 + \omega_p^2) b + 2\pi \omega_p^2 \left[ a \left( \frac{\Omega}{k} S_r^+ - T_r^+ \right) - b \left( \frac{\Omega}{k} S_i^+ - T_i^+ \right) \right], \end{aligned} \quad (B.3)$$

or

$$D_r^P - \frac{a}{b} D_i^P = 1 - 2\pi \omega_p^2 \left( b + \frac{a^2}{b} \right) \left( \frac{\Omega}{k} S_r^+ - T_r^+ \right). \quad (B.4)$$

Since, by Titchmarsh's theorem [Titchmarsh, 1937],

$$\lim_{\substack{\omega_r = \text{const.} \\ \omega_i \rightarrow -\infty}} \left( b + \frac{a^2}{b} \right) = 0, \quad \lim_{\substack{k = k_r < 0 \\ \omega_i \rightarrow -\infty}} (S^+, T^+) = 0, \quad (B.5)$$

it is always possible to find a real positive  $\alpha$  such that for  $\omega_i < -\alpha < 0$ , we have  $D_r^P - \frac{a}{b} D_i^P - \frac{a}{b} D_i^P \approx 1$  and so, indeed, obtain (B.I).

The proof of (B.II) is similar.

## APPENDIX C

### ASYMPTOTIC DISTRIBUTION AND QUADRATURE

ASYMPTOTIC DISTRIBUTION. Electrons moving in a whistler field where  $k|w|/\Omega_e$  and  $|w|/v_\perp$  are infinitesimals of higher order than  $(B/B_0)^{1/2} \ll 1$ , have a constant perpendicular speed, and display phase trajectories in the plane  $\psi - w$  ( $\psi = \phi + kz - \hat{\omega}t + \pi$ ,  $w = v_\parallel - V_G$ ) that are identical to those of electrons in electrostatic waves [Palmadesso and Schmidt, 1971; O'Neil, 1965; Brinca, 1972c]. For whistlers of moderate amplitude,  $(f_k - f_k^L)$  in (5.33) only deviates from zero near  $v_\parallel \sim V_G$  and we may expand the velocity distributions about  $V_G$ . Following Appendix B of O'Neil [1965], but taking one more term in the Taylor expansion, we find that the asymptotic value of the exact distribution satisfying (5.29) is

$$\begin{aligned} \hat{F} &= \int_{-\pi}^{\pi} F_0(v_\perp, v_\parallel) \Delta w(\psi) d\psi \left[ \int_{-\pi}^{\pi} \Delta w(\psi) d\psi \right]^{-1} \\ &= F_0(v_\perp, V_G) + \hat{w} \frac{\partial F_0(v_\perp, V_G)}{\partial v_\parallel} + \frac{\hat{w}^2}{2} \frac{\partial^2 F_0(v_\perp, V_G)}{\partial v_\parallel^2} + \dots, \end{aligned} \quad (C.1)$$

where we have for trapped ( $\kappa^2 > 1$ ), and untrapped ( $\kappa^2 < 1$ ), particles

$$\hat{w} = \begin{cases} \frac{\pi \omega_{NL}}{K(\kappa)k\kappa} & (\kappa^2 < 1) \\ 0 & (\kappa^2 > 1) \end{cases}, \quad (C.2)$$

$$\hat{w}^2 = \begin{cases} \frac{4E_1(\kappa)\omega_{NL}^2}{K(\kappa)k^2\kappa^2} & (\kappa^2 < 1) \\ \frac{4\omega_{NL}^2}{K(1/\kappa)k^2} \left[ E_1(1/\kappa) - (1-1/\kappa^2)K(1/\kappa) \right] & (\kappa^2 > 1) \end{cases}. \quad (C.3)$$

Here  $\kappa$  denotes the trapping parameter [Brinca, 1972c], with  
 $\text{sgn } \kappa = \text{sgn } w$ ,

$$\frac{1}{\kappa^2} = \frac{1}{2} (1 - \cos \psi) + \left( \frac{kw}{2\omega_{NL}} \right)^2, \quad \omega_{NL}^2 = \frac{e B k v_{\perp}}{m_e} \quad (C.4)$$

and  $K$  and  $E_1$  denote the complete elliptic integrals of the first and second kinds.

The asymptotic value of the exact perturbation in the distribution is then

$$\begin{aligned} \hat{f} &= \hat{F} - F_0(v_{\perp}, v_{\parallel}) \\ &= (\hat{w} - w) \frac{\partial F_0(v_{\perp}, v_G)}{\partial v_{\parallel}} + \frac{(\hat{w}^2 - w^2)}{2} \frac{\partial^2 F_0(v_{\perp}, v_G)}{\partial v_{\parallel}^2} + \dots, \end{aligned} \quad (C.5)$$

where

$$w = \frac{2\omega_{NL}}{k\kappa} \left[ 1 - \kappa^2 \sin^2(\psi/2) \right]^{1/2}. \quad (C.6)$$

INTEGRATION. The quadrature in (5.34) is

$$\int u(\hat{f}_k - \hat{f}_k^L) d\mathbf{v} = \int_0^{\infty} v_{\perp}^2 dv_{\perp} \int_0^{2\pi} \exp i\phi d\phi \int_{-\infty}^{\infty} dw \int_{-\lambda/2}^{\lambda/2} (\hat{f} - \hat{f}^L) \exp ikz d(z/\lambda). \quad (C.7)$$

Replacing  $z$  by  $\psi$ , and performing the  $\phi$  integration, yields a quadrature over  $w$  and  $\psi$  formally similar to the one implicitly made by Morales and O'Neil [1972] (over  $v$  and  $x$ ), when computing the asymptotic frequency shift of electrostatic waves. Using their result

$$\int_{-\infty}^{\infty} dv \int_{-\lambda/2}^{\lambda/2} d(x/\lambda) \exp ikx (\hat{f} - \hat{f}^L) = 0.815 \left( \frac{\omega_{NL}}{k} \right)^3 F_0'' \left( \frac{\omega_0}{k} \right) \exp i\hat{\omega}t, \quad (C.8)$$

yields (5.38).

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