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PERTURBED NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The existence of a solution defined for all t and possessing a type of boundedness property is established for the perturbed nonlinear system $\dot{y} = f(t,y) + F(t,y)$. The unperturbed system $\dot{x} = f(t,x)$ has a dichotomy in which some solutions exists and are well behaved as t increases to ∞ and some solution exists and are well behaved as t decreases to $-\infty$. A similar study is made for a perturbed nonlinear differential equation defined on a half line, say R^+, and the existence of a family of solutions with special boundedness properties is established. Finally, the ideas are applied to the study of integral manifolds. Examples are given.

1. INTRODUCTION

The following is a study of the system

$$\dot{y} = f(t,y) + F(t,y), \tag{1}$$

which is regarded as a perturbation of the nonlinear system

$$\dot{x} = f(t,x). \tag{2}$$

We impose hypotheses on (2) which guarantee the existence of a bounded solution (or a family of bounded solutions) and prove that, under conditions on the perturbation terms, such solutions are also present in (1). Results of this type have been obtained by May [11] and Harbertson and Struble [8] for nonlinear systems and Coppel [3], Hallum [6], [7] and Hale [4], [5] for perturbations of linear systems. The present work removes certain hypotheses from the past results and introduces function spaces which allow new types of behavior to be studied.

Theorem 1 concerns the case when (1) and (2) are defined for all t and give the existence of particular solutions defined on R. Theorem 2 covers the case when the systems are defined on some half line $t \geq \alpha$ and concern the presence of special solutions on this interval. Theorem 3 extends the above ideas to integral manifolds. Examples are discussed.

2. NOTATION AND GENERAL SETTING

Let p and q be nonnegative integers with $p + q = m > 0$, let I be an interval of the type $I = \{t \geq \alpha\}$ (α may be $-\infty$), let $D_1(t)$, $D_2(t)$ be continuous nonsingular $p \times p$ and $q \times q$ matrices respectively on I and for $\sigma > 0$ define

$$\Omega_1^\sigma = \{(t,x) \text{ in } I \times R^p : |D_1(t)x| \leq \sigma\},$$

$$\Omega_2^\sigma = \{(t,x) \text{ in } I \times R^q : |D_2(t)x| \leq \sigma\}.$$

We assume that (2) may be written in the form

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1), \\ \dot{x}_2 &= f_2(t,x_2), \end{aligned} \tag{3}$$

where for some $\sigma > 0$, f_1 is a continuous function on Ω_1^σ into R^p, f_2 is a continuous function on Ω_2^σ into R^q, where f_1 and f_2 have continuous partial derivatives in x_1 and x_2 respectively and $f_1(t,0) = 0$, $f_2(t,0) = 0$. For (τ, a_1) in Ω_1^σ, (τ, a_2) in Ω_2^σ we denote the solutions of (3) such that $x_1(\tau) = a_1$, $x_2(\tau) = a_2$ by $x_1(t,\tau,a_1)$, $x_2(t,\tau,a_2)$. We assume that for some $0 < \gamma \leq \sigma$ and (τ,a_1) in Ω_1^γ, (τ,a_2) in Ω_2^γ, $x_1(t,\tau,a_1)$ exists for $t \geq \tau$ and $x_2(t,\tau,a_2)$ exists for $\alpha \leq t \leq \tau$.

For convenience when a is in R^ℓ we will use norm

$|a| = \max_i \{|a_i|\}$ where $i = 1,2,\ldots,\ell$, a_i is the i^{th} component of a. Then for a_1 in R^p, $a_2 = R^q$, $a = \mathrm{col}(a_1,a_2)$ $|a| = \max\{|a_1|, |a_2|\}$. Let B_1^γ be the continuous functions y on I into R^p with $|D_1(t)y(t)| \le \gamma$ and B_2^γ be the continuous functions y on I into R^q with $|D_2(t)y(t)| \le \gamma$.

We denote the matrix $\frac{\partial x_i}{\partial a_i}(t,\tau,a_i)$ by $\Phi_i(t,\tau,a_i)$, $i = 1,2$, and assume that F_1 and F_2 are continuous functions on

$$\Omega^\gamma = \{(t,x_1,x_2) \text{ in } I \times R^m: |D_1(t)x_1| \le \gamma, |D_2(t)x_2| \le \gamma\}$$

into R^p and R^q respectively.

3. SYSTEMS DEFINED ON R

In this section we consider the case when $I = R$. We note the unperturbed system (3) has a solution in $B_1^\gamma \times B_2^\gamma$, viz. $x_1 = 0$, $x_2 = 0$. We seek hypotheses on the perturbed system

$$(4) \qquad \begin{aligned} \dot{y}_1 &= f_1(t,y_1) + F_1(t,y_1,y_2), \\ \dot{y}_2 &= f_2(t,y_2) + F_2(t,y_1,y_2), \end{aligned}$$

which guarantee the existence of a solution (y_1,y_2) in $B_1^\gamma \times B_2^\gamma$.

<u>Theorem 1.</u> Assume that we have

$$(5)\qquad \begin{aligned} &\int_{-\infty}^{t} |D_1(t)\Phi_1(t,s,y_1(s))F_1(s,y_1(s),y_2(s))|\,ds \le \gamma, \\ &\int_{t}^{\infty} |D_2(t)\Phi_2(t,s,y_2(s))F_2(s,y_1(s),y_2(s))|\,ds \le \gamma, \end{aligned}$$

for all t in R and (y_1, y_2) in $B_1^\gamma \times B_2^\gamma$. There is a solution $(y_1(t), y_2(t))$ of (4) defined for all t and (y_1, y_2) is in $B_1^\gamma \times B_2^\gamma$.

Proof: For each positive integer k, let B_k be the set of functions (y_1, y_2) where y_1 and y_2 map $[-k,k]$ continuously into R^p and R^q respectively. For $y = (y_1, y_2)$ in B_k let

$$|y| = \max\{\sup_t |D_1(t)y_1(t)|,\ \sup_t |D_2(t)y_2(t)|\},$$

then $(B_k, |\ |)$ is a Banach space and $B_1^\gamma \times B_2^\gamma$ is a closed convex subset. (The functions in $B_1^\gamma \times B_2^\gamma$ are here restricted to $[-k,k]$.) On $B_1^\gamma \times B_2^\gamma$ we define the transformation T by $Ty = u$ where

$$(6)\qquad \begin{aligned} u_1(t) &= \int_{-k}^{t} \Phi_1(t,s,y_1(s))F_1(s,y_1(s),y_2(s))\,ds, \\ u_2(t) &= -\int_{t}^{k} \Phi_2(t,s,y_2(s))F_2(s,y_1(s),y_2(s))\,ds, \end{aligned}$$

for $-k \le t \le k$. It is an easy exercise to check that Schauder's fixed point applies so there is a function y in $B_1^\gamma \times B_2^\gamma$ defined for $-k \le t \le k$ with $Ty = y$. We have

$$\dot{y}_1(t) = F_1(t,y_1(t),y_2(t)) + \int_{-k}^{t} H_1(t,s)F_1(s,y_1(s),y_2(s))\,ds,$$

$$\dot{y}_2(t) = F_2(t,y_1(t),y_2(t)) + \int_{k}^{t} H_2(t,s)F_2(s,y_1(s),y_2(s))\,ds,$$

where

$$H_i(t,s) = -\frac{\partial f_i}{\partial x_i}(t,x_i(t,s,y_i(s))\Phi_i(t,s,y_i(s)),$$

$i = 1,2$. Here we have used the well known ([3], page 22) theory of the variational equation. Also

$$f_1(t,y_1(t)) = \int_{-k}^{t} \frac{d}{ds}[f_1(t,x_1(t,s,y_1(s)))]ds,$$

$$= \int_{-k}^{t} H_1(t,s)[\dot{y}_1(s) - f_1(s,y_1(s))]ds;$$

and a similar expression holds for $f_2(t,y_2(t))$. For $i = 1,2$ and

$$w_i(t) = \dot{y}_i(t) - f_i(t,y_i(t)) - F_i(t,y_1(t),y_2(t)),$$

we obtain

(7)
$$w_1(t) = \int_{-k}^{t} H_1(t,s)w_1(s)\,ds,$$

$$w_2(t) = \int_{t}^{k} H_2(t,s)w_2(s)\,ds,$$

which implies $w_1(t) = 0$, $w_2(t) = 0$. Consequently $y = (y_1, y_2)$ is a solution of (4), $-k \le t \le k$ in $B_1^\gamma \times B_2^\gamma$.

Let $\{y(t,k)\}_{k=1}^\infty$ be a sequence of fixed points of T on $B_1^\gamma \times B_2^\gamma$ (restricted to $[-k,k]$). The following statements show there is a function y in $B_1^\gamma \times B_2^\gamma$ and a subsequence $\{y(t,n_k)\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} y(t,n_k) = y(t), \tag{8}$$

and the limit is uniform on compact t intervals.

There is a subsequence $\{y(t,n_{1k})\}$ converging uniformly on $[-1,1]$ since the original sequence is uniformly bounded and each function satisfies (4) on this interval. Similarly, there is a subsequence $\{y(t,n_{2k})\}_{k=1}^\infty$ of the sequence $\{y(t,n_{1k})\}_{k=1}^\infty$ converging uniformly on $[-2,2]$. In this way we obtain a chain of subsequences $\{y(t,n_{jk})\}_{k=1}^\infty$ converging uniformly on $[-j,j]$. Put $y(t,n_k) = y(t,n_{kk})$ to obtain (8). Since each $y(t,n_k)$ is a solution of (4) so is y.

May [11] gives a similar theorem when $D_1(t) = I$, $D_2(t) = I$; however he requires the additional hypothesis of a Lipschitz condition in z on $\Phi(t,s,z)F(s,z)$. The essential difference in the proofs is that we define T using a finite limit k and use functions on intervals $[-k,k]$. The proof that the fixed point y satisfies (4) reduces to the fact (7) implies $w_1, w_2 = 0$, an easy observation when k is finite.

Hallum [6], [7] and Hartman and Onuchic [9] introduce the matrix $D(t) = g(t)I$ in studies of a perturbed linear system on a half line. Here $g(t)$ is a continuous nonnegative functions.

Condition (5) is restrictive on the linear parts in x of the f_1, f_2 functions. Suppose, for example, $f_1(t,x_1) = A_1(t)x_1 + h(t,x_1)$ where $h(t,x_1) = o(|x_1|)$ uniformly in t as $x_1 \to 0$. Let $X_1(t)$ be a fundamental matrix of solutions for $\dot{x}_1 = A_1(t)x_1$, then since $\Phi(t,s,0)$ is a solution of $\dot{z} = f_x(t,0)z$, $z(\tau) = I$, we see (4) implies, in particular,

$$\int_{-\infty}^{t} |D_1(t)X_1(t)X_1^{-1}(s)F_1(s,0,0)|\, ds \le \gamma.$$

(Theory concerning systems with hypotheses $\int_{-\infty}^{t} |X(t)X^{-1}(s)|\, ds \le K$ is given by Coppel [3].) If, in addition, $A_1(t) = 0$ we see $F_1(s,0,0)$ must be integrable.

As an example take $p = 1$, $q = 0$ and consider the system

$$\dot{y} = a(t)y - \sum_{i=1}^{\infty} b_i(t)y^{2i+1} + F(t,y), \tag{9}$$

where $a(t)$ is a continuous function on R to be further restricted later, the $b_i(t)$, $i = 1,2,\ldots$, are continuous nonnegative functions on R, the series $\sum b_i(t)y^{2i+1}$ converges to a continuous function with continuous partial derivative in y given by $\sum b_i(t)y^{2i}$, and the function F is continuous on R^2 and will be further specified later. We take as the unperturbed system,

$$\dot{x}(t) = a(t)x - \sum_{i=1}^{\infty} b_i(t)x^{2i+1},$$

then the variation equation is

$$\dot{z} = [a(t) - \sum b_i(t)x^{2i}(t)]z.$$

Consequently,

$$|\Phi(t,\tau,\gamma)| \leq e^{\int_\tau^t a(s)ds} .$$

Case 1. Suppose $a(t) = -1$ (or any negative constant). Then for $D(t) = 1$ we have that if there is a $\sigma > 0$ such that for continuous functions $y(t)$ with $|y(t)| \leq \sigma$,

$$\int_{-\infty}^{t} e^{-(t-s)}|F(s,y(s))|ds \leq \sigma,$$

then (9) has a solution y^* defined for all t with $|y^*(t)| \leq \sigma$. We notice here the nonlinearities $\sum b_i(t)y^{2i+1}$ are "harmless". This observation shows the nonlinear theory allows much larger bounds for some systems than the corresponding linear perturbation theory gives (Coppel [3], page 137). It is interesting to note that for $a(t)$ as given in Coppel [3], page 73, the above conclusions also hold. In this case the basic linear system is not exponentially stable.

Case 2. Sometimes the form of the linear term dictates the $D(t)$ function. Suppose $a(t)$ is given by

$$a(t) = \begin{cases} \dfrac{2t}{t^2 + 1}, & t \leq -1, \\ \dfrac{-1}{t + 2}, & -1 < t. \end{cases}$$

Then it is easy to see there is a $k > 0$ so that for $D(t) = k/(|t| + 2)$, $\int_{-\infty}^{t} D(t)\Phi(t,s,y(s))ds$ is bounded, say by M, for all t and continuous functions y. If there is a $\sigma > 0$. So y in B_1^σ implies $|F(t,y(t)| \leq \sigma/M$ then (8) has a solution y^* defined for all t with $|y^*(t)| \lesssim \sigma(|t| + 2)/k$. Other choices for $a(t)$ will give decreasing $D(t)$ functions as $t \to \infty$.

Case 3. The form of the perturbation term may dictate a $D(t)$ function. Suppose $a(t) = -1$ and $F(t,y) = h(t)k(t,y)$ where

$$h(t) = \begin{cases} e^{t} & t < 0, \\ e^{-s} & t \geq 0. \end{cases}$$

Then

$$\int_{-\infty}^{t} e^{-(t-s)} h(s)\,ds = \begin{cases} e^{t}/2, & t < 0, \\ \frac{1}{2} + te^{-t}, & t \geq 0, \end{cases}$$

consequently a natural choice for $D(t)$ is

$$D(t) = \begin{cases} 2e^{-t}, & t < 0, \\ \dfrac{2}{1 + 2te^{-t}}, & t \geq 0. \end{cases}$$

In most instances the solutions of (3) and consequently Φ_1, Φ_2 are not known precisely. However, as in the example above, it may be possible to obtain information which implies the hypothesis of Theorem 1. Consider the following situation in which, for convenience, we assume $q = 0$, and suppress the subscript 1 notation. Suppose $D(t)$ is given and there are positive numbers σ, K, a set $\Omega \subset R^p$ and a continuous real valued function λ on R such that:

(a) for s in R, $|\gamma| \leq |D^{-1}(s)|\sigma$, $x(t,s,\gamma)$ lies in Ω for $t \geq s$;

(b) $\mu[f_x(t,x(t))] \leq \lambda(t)$ for all continuous functions x from R to Ω;

(c) $D(t) \int_{-\infty}^{t} \exp \int_{s}^{t} \lambda(u)\,du\,ds \leq K.$

We notice that for y in B^σ, $|y(s)| \leq |D^{-1}(s)|\sigma$, hence

$x(t,s,y(s))$ is in Ω for $t \geq s$. By a well known result (Coppel [3], page 58) condition (b) implies

$$|\Phi(t,s,y(s))| \leq \exp \int_s^t \mu(f_x[u,x(u,s,y(s))])du;$$

hence condition (c) together with a boundedness assumption on F will imply inequality (5) in Theorem 1.

4. SYSTEMS DEFINED ON A HALF LINE

In this section we consider the case α finite. We note the unperturbed system may have a family of bounded solutions on $t \geq \alpha$. We seek hypotheses on the perturbed system (4) which guarantees the existence of a family of solutions in $B_1^\gamma \times B_2^\gamma$.

<u>Theorem 2</u>. Assume that we have

$$\int_\alpha^t |D_1(t)\Phi_1(t,s,y_1(s))F_1(s,y_1(s),y_2(s))|\,ds \leq \gamma/2,$$

(10)

$$\int_t^\infty |D_2(t)\Phi_2(t,s,y_2(s))F_2(s,y_1(s),y_2(s))|\,ds \leq \gamma,$$

for all t in R and (y_1,y_2) in $B_1^\gamma \times B_2^\gamma$. Further assume that for some $\Delta > 0$ and $|a_1| \leq \Delta$, a_1 in R^p we have

(11) $$D_1(t)x_1(t,\alpha,a_1)| \leq \gamma/2.$$

Then for $|a_1| \leq \Delta$, a_1 in R^p there is a solution $(y_1(t), y_2(t))$ of (4) defined for $t \geq \alpha$, (y_1, y_2) is in $B_1^\gamma \times B_2^\gamma$ and $y_1(\alpha) = a_1$.

Proof: For each positive integer $k > \alpha$, let B_k be the set of functions (y_1, y_2) where y_1 and y_2 map $[\alpha, k]$ continuously into R^p and R^q respectively. On $B_1^\gamma \times B_2^\gamma$ restricted to $[\alpha, k]$ we define the transformation T by $Ty = u$ where

$$u_1(t) = x_1(t, \alpha, a_1) + \int_\alpha^t \Phi_1(t, s, y_1(s)) F_1(s, y_1(s), y_2(s)) ds,$$

$$u_2(t) = -\int_t^k \Phi_2(t, s, y_2(s)) F_2(s, y_1(s), y_2(s)) ds,$$

for $\alpha \leq t \leq k$. As before such a transformation has a fixed point which is a solution of (4). The family of such fixed points $(\alpha < k < \infty)$ has a convergent subfamily which converges to a function (y_1, y_2) satisfying the conclusions of the theorem.

Suppose that, in addition to line 2 of inequality (10), we have a system such that

$$\int_\alpha^t |D_1(t)\Phi_1(t, s, y_1(s))| ds \leq \gamma^* \tag{12}$$

for all y_1 in B_1^γ. Then appropriate boundedness conditions on F_1 will give line 1 of inequality (10). Inequality (12) for $D_1(t) = I$ gives a type of exponential stability to $x(t, \alpha_1, a_1)$ if $f_1(t, x_1) = A_1(t)x_1$. Coppel [3], page 68, proves that there is a constant N such that

$$|X_1(t)X^{-1}(\alpha)| \leq Ne^{\frac{1}{\gamma^*}(t-\alpha)}, \quad \alpha + 1 \leq t.$$

In this case inequality (11) is not an added restriction but merely defines notation. Corresponding theorems are given for general $D(t)$ by modifying Hallum [6], page 255-6. Brauer [1], gives a theorem in the direction for nonlinear functions $f_1(t,x_1)$ which could again be modified to include $D(t)$.

5. INTEGRAL MANIFOLDS

In this section we consider the system

$$(13) \qquad \begin{aligned} \dot{\theta} &= h(\theta,t,z) + H(\theta,t,y,z,\epsilon), \\ \dot{y} &= f(t,y) + F(\theta,t,y,z,\epsilon), \\ \dot{z} &= \epsilon g(z) + \epsilon G(\theta,t,y,z,\epsilon), \end{aligned}$$

where (θ,y,z) is in $R^1 \times R^m \times R^n$ and where hypotheses will be introduced to insure the existence, for small ϵ, of an integral manifold of solutions. The form of the system and the hypotheses given are motivated by previous work by Hale [4], [5] and Harbertson and Struble [8]. Such problems arise in the "method of averaging" introduced by Kryloff and Bogoliubov [10] (see also [2]) and studied extensively by many.

The ideas introduced in the previous sections of this paper are applied to the study of (13). A treatment of such a system without the D matrix has been given in [8]; however our treatment improves the allowable bounds for the perturbation. In addition, the form of the θ equation has been changed to allow a larger class of examples. Such an example is given at the end of this section.

It is possible to present this theory for the case where f is split into two functions, $f = \text{column}(f_1, f_2)$ as was done in the preceding sections. Correspondingly the g function (i.e., the $\dot{z}$ equation) can also be split into two pieces, one which is well behaved as $t \to \infty$, the other well behaved as $t \to -\infty$ (see [8]). In order to present these ideas without unnecessary clutter we will not make these decompositions of the $\dot{y}$ and $\dot{z}$ equations. It will be clear how the hypotheses must be altered to obtain a corresponding theory with the dichotomies present in the $\dot{y}$ and $\dot{z}$ equations.

Let $D(t)$ and $E(t)$ be continuous nonsingular $m \times m$ and $n \times n$ matrices respectively on R and for $\sigma = (\sigma_1, \sigma_2)$ in $R^+ \times R^+$ we define

$$\Omega^\sigma = \{(t,y,z) \text{ in } R^{1+m+n}\colon |D(t)y| \leq \sigma_1,\ |E(t)z| \leq \sigma_2\},$$

$$\Omega_1^\sigma = \{(t,y) \text{ in } R^{1+m}\colon |D(t)y| \leq \sigma_1\},$$

$$\Omega_2^\sigma = \{(t,z) \text{ in } R^{1+n}\colon |E(t)z| \leq \sigma_2\},$$

and consider the following hypotheses. Let σ in $R^+ \times R^+$ and $\epsilon_o > 0$ be given.

(1) h is a continuous function from $R^{\ell} \times \Omega_2^{\sigma}$ into R^{ℓ}. H, F and G are continuous functions from $R^{\ell} \times \Omega^{\sigma} \times [0,\epsilon_o]$ into R^{ℓ}, R^m and R^n respectively. f is continuous on Ω_1^{σ}, has a continuous derivative in y and $f(t,0) = 0$. g is continuous on R^n and has continuous derivatives in z. There is an $\omega = (\omega_1,\omega_2,\ldots,\omega_m)$ in R^m such that h, H, F and G have period ω_i in θ_i. H, F, and G vanish for $\epsilon = 0$ and $|h| \leq C_1$, $|H| \leq C_2$ on their domains.

(2) The solutions of $\dot{y} = f(t,y)$, $y(\tau) = a$, (τ,a) in Ω_1^{σ}, are denoted by $y(t,\tau,a)$ and we assume for some $0 < \sigma_1^* \leq \sigma$, $y(t,\tau,a)$ exists for $t \geq \tau$ when (τ,a) is in $\Omega_1^{\sigma^*}$. We denote by $\Phi(t,\tau,a)$ the matrix $\frac{\partial y}{\partial a}$.

(3) The solution of $\dot{z} = g(z)$, $z(0) = b$, b in R^n, is denoted by $z(t,b)$ and we assume $z(t,b)$ exists for $t \geq 0$. We denote by $\Lambda(t,b)$ the matrix $\frac{\partial z}{\partial b}$.

(4) There is an $N > 0$ such that on its domain

$$|h(\theta,t,z) - h(\theta^*,t,z^*)| \leq N(|z-z^*| + |\theta-\theta^*|).$$

There is a continuous function $\ell(t,\epsilon)$ on $R \times [0,\epsilon_o]$ decreasing to 0 as $\epsilon \to 0$ such that on its domain

$$|H(\theta,t,y,z,\epsilon) - H(\theta^*,t,y^*,z^*,\epsilon)|$$
$$\leq \ell(t,\epsilon)(|\theta-\theta^*| + |y-y^*| + |z-z^*|).$$

There is a continuous function $\nu_o(\epsilon)$ on $[0,\epsilon_o]$, $\nu_o(0) = 0$ and a continuous function $\Delta(t,s)$ for $s \leq t$ such that on its domain

$$|\Phi(t,s,y)F(\theta,s,y,z,\epsilon) - \Phi(t,s,y^*)F(\theta^*,s,y^*,z^*,\epsilon)|$$
$$\leq \nu_o(\epsilon)\Delta(t,s)(|\theta-\theta^*| + |y-y^*| + |z-z^*|).$$

There is a continuous function $\delta(t,s)$, $s \leq t$, such that on its domain

$$|\Lambda(\epsilon(t-s),z)G(\theta,s,y,z,\epsilon) - \Lambda(\epsilon(t-s),z^*)G(\theta^*,s,y^*,z^*,\epsilon)|$$
$$\leq \delta(t,s)(|\theta-\theta^*| + |y-y^*| + |z-z^*|).$$

(5) There are positive constants M, K_1, K_2 and ϵ_1 $(0 < \epsilon_1 \leq \epsilon_o)$ such that

$$\int_{-\infty}^{t} \Delta(t,s)\exp\int_{s}^{t} L(u,\epsilon_1)du\, ds \leq K_1,$$

$$\int_{-\infty}^{t} \delta(t,s)\exp\int_{s}^{t} L(u,\epsilon_1)du\, ds \leq K_2, \qquad t \text{ in } R,$$

where $L(u,\epsilon) = (2M+1)\ell(u,\epsilon) + (M+1)N.$

For $0 < \gamma_1 \leq \sigma_1^*$, $0 < \gamma_2 \leq \sigma_2$, $\gamma = (\gamma_1,\gamma_2)$ we define S^γ to be all $v = (v_1,v_2)$ where $v_1(\theta,t)$ is continuous from $R^{\ell+1}$ into R^m with $|D(t)v_1(\theta,t)| \leq \gamma_1$, where $v_2(\theta,t)$ is continuous from $R^{\ell+1}$ into R^n with $|E(t)v_2(\theta,t)| \leq \gamma_2$ and where v_1, v_2

have period ω in θ and satisfy Lipschitz conditions in θ with constant M. For v in S^γ we define $H^v(\theta,t,\epsilon) = H(\theta,t,v_1(\theta,t), v_2(\theta,t),\epsilon)$ and F^v, G^v similarily. We assume for some γ the following conditions hold:

(6) There are continuous functions $\nu_1(\epsilon),\nu_2(\epsilon)$ on $[0,\epsilon_1]$, $\nu_1(0) = \nu_2(0) = 0$ so that

$$\int_{-\infty}^{t} |D(t)\Phi(t,s,v_1(\theta(s),s)F^v(\theta(s),s,\epsilon)|\, ds \le \nu_1(\epsilon),$$

$$\epsilon \int_{-\infty}^{t} |E(t)\Lambda(\epsilon(t-s),\ v_2(\theta(s),s))G^v(\theta(s),s,\epsilon)|\, ds \le \nu_2(\epsilon)$$

for all v in S^γ and continuous functions θ from R into R^m.

(7) For any continuous function θ from R into R^m and v in S^γ let

$$p_{\theta v}(t) = \int_{-\infty}^{t} \Phi(t,s,v_1(\theta(s),s))F^v(\theta(s),s,\epsilon)\,ds$$

$$q_{\theta v}(t) = \int_{-\infty}^{t} \Lambda(\epsilon(t-s),\ v_2(\theta(s),s))G^v(\theta(s),s,\epsilon)\,ds.$$

For any ϵ in $[0,\epsilon_1]$, fixed we assume

$$|p_{\theta v}(t) - p_{\theta v}(t^*)|,\quad |q_{\theta v}(t) - q_{\theta v}(t^*)|$$

are $o(|t-t^*|)$ uniformly for v in S^γ and θ continuous from R into R^m.

Theorem 3. Assume that conditions (1) - (7) are satisfied. For ϵ sufficiently small there are functions $(v_1(\theta,t), v_2(\theta,t)$ in S^γ such that $y = v_1(\theta,t)$, $z = v_2(\theta,t)$ is an integral manifold of (13).

Proof: For v in S^γ let $\xi^v(t,\tau,\theta_o)$ be the solution of

$$(14) \qquad \dot{\theta} = h(\theta,t,v_2(\theta,t)) + H^v(\theta,t,\epsilon), \qquad \theta(\tau) = \theta_o.$$

The right side of the equation is Lipschitz in θ with constant $L(t,\epsilon)$ and is bounded by $C_1 + C_2$. A routine use of Gronwall's inequality gives

$$(15) \qquad |\xi^v(t,\tau,\theta_o) - \xi^v(t,\tau^*,\theta_o^*)| \leq (|\theta_o - \theta_o^*| + (C_1 + C_2)|\tau-\tau^*|)\exp \int_t^\tau L(u,\epsilon_1)du$$

for $\tau \geq t$ and $0 < \epsilon \leq \epsilon_1$. Let k be a positive integer and let B_k be the functions $(v_1(\theta,t), v_2(\theta,t))$ mapping $R^\ell \times [-k,k]$ continuously into (R^m,R^n) with period ω in θ. Then S^γ restricted to $[-k,k]$ is a closed convex subset of B_k. For v in S^γ define $Tv(\theta,t) = (w_1(\theta,t), w_2(\theta,t))$ by

$$w_1(\theta,t) = \int_{-k}^{t} \Phi(t,s,v_1(\xi^v(s,t,\theta),s)F^v(\xi^v(s,t,\theta),s,\epsilon)ds$$

$$w_2(\theta,t) = \epsilon \int_{-k}^{t} \Lambda(\epsilon(t-s), v_2(\xi^v(s,t,\theta),s)G^v(\xi^v(s,t,\theta),s,\epsilon)ds$$

for θ in R^{ℓ}, $-k \leq t \leq k$. Tv exists by (6) since D,E are nonsingular. Since the right side of (14) has period ω in θ we note $\xi^{v}(s,t,\theta+\omega) = \xi^{v}(s,t,\theta) + \omega$; consequently Tv has period ω in θ. For ϵ sufficiently small $\nu_1(\epsilon) \leq \gamma_1$, $\nu_2(\epsilon) \leq \gamma_2$ and hence $|D(t)w_1(\theta,t)| \leq \gamma_1$, $|E(t)w_2(\theta,t)| \leq \gamma_2$. We have

$$|w_1(\theta,t) - w_1(\theta^*,t^*)| \\ \leq \nu_o(\epsilon) \int_{-k}^{t} \Delta(t,s)[2M+1]|\xi^{v}(s,t,\theta) - \xi^{v}(s,t^*,\theta^*)|ds$$

and a similar equation for $w_2(\theta,t)$. By (15) and condition (5) we have the family TS^{γ} is equicontinuous in (θ,t) and for $t = t^*$ above for ϵ sufficiently small $|w_i(\theta,t) - w_i(\theta,t)| \leq M|\theta-\theta^*|$, $i = 1,2$. Note the condition on ϵ is independent of k. Schauder's fixed point implies there is a fixed point of T. Let v be such a point and τ,θ_o be fixed. A repetition of the argument in Theorem 1 shows $\theta(t) = \xi^{v}(t,\tau,\theta_o)$, $y(t) = v_1(\xi^{v}(t,\tau,\theta_o),t)$, $z(t) = v_2(\xi^{v}(t,\tau,\theta_o),t)$ is a solution of (13).

For each positive k let v^k be a fixed point of T and let $\theta^k(t,\tau,\theta_o)$, $y^k(t,\tau,\theta_o)$, $z^k(t,\tau,\theta_o)$ be the corresponding solutions of (13). If Z is the set of positive integers, $\{v^k(\theta,t): k \text{ in } Z\}$ is uniformly bounded and equicontinuous (by condition (7) on $R^{\ell} \times [-1,1]$) so there is a $Z_1 \subset Z$ so that $\{v^k(\theta,t): k \text{ in } Z_1\}$ converges uniformly on $R^{\ell} \times [-1,1]$. In this way we obtain a decreasing sequence $Z_j = \{k_{j1}, k_{j2}, \ldots\}$ such that

$\{v^k(\theta,t): k$ in $Z_j\}$ converges uniformly on $R^\ell \times [-j,j]$. Let $k_i = k_{ii}$, $i = 1,2,\ldots$, $Z^* = \{k_i\}_{i=1}^\infty$, then $\{v^k(\theta,t); k$ in $Z^*\}$ converges uniformly on sets of the form $R^\ell \times C$, C compact in R. Let $v(\theta,t)$ be the limit function. The corresponding sequence θ^k, y^k, z^k also converges uniformly on compact sets in R; thus the limit functions are solutions of (13). Consequently

$$y = v_1(\theta,t), \qquad z = v_2(\theta,t)$$

is an integral manifold of (13).

The usual form for the θ equation in (13) is $\theta = \eta + H(\theta,t,y,z,\epsilon)$ where η is a constant. We include an example which may occur naturally (say by using Newton's equations) in a mathematical model and show how these equations can be reduced to the form (13) featuring an h term in the θ equation which is not constant. Consider a weakly coupled nonlinear system

$$\ddot{x} + \beta x^3 = -\epsilon\dot{x} + \epsilon^2 X(t,x,\dot{x},y)$$

$$\dot{y} = a(t)y + b(t,y) + \epsilon F_1(t,x,\dot{x},y)$$

where the $\dot{y}$ equation for $\epsilon = 0$ was described in section 3, β is a positive number and F_1, X are continuous on R^4. For $\epsilon = 0$ the first equation can be solved in terms of elliptic functions sn, dn and cn with modulus $1/\sqrt{2}$. Let $x = \rho \text{ cn } \theta$, $\dot{x} = -\rho^2\sqrt{\beta} \text{ sn } \theta \text{ dn } \theta$, then equation above takes the form

$$\dot{\theta} = \sqrt{\beta}\ \rho - \epsilon H_1(\theta, t, y, \rho),$$

$$\dot{y} = a(t)y + b(t,y) + \epsilon F_1(\theta, t, y, \rho),$$

$$\dot{\rho} = -\epsilon\rho\ \mathrm{sn}^2\theta\ \mathrm{dn}^2\theta - \epsilon^2 G_1(\theta, t, y, \rho).$$

Let

$$\gamma = \frac{1}{4K}\int_0^{4K} \mathrm{sn}^2 s\ \mathrm{dn}^2 s\ ds,$$

$$v(\theta, z) = z\int_0^{\theta} (\mathrm{sn}^2 s\ \mathrm{dn}^2 s - \gamma)ds$$

where $4K$ is the real period of the elliptic functions, then under the coordinate transformation given in the method of averaging ([5], Chapters 14-17), $\rho = z + \epsilon v(\theta, z)$. the differential equations become

$$\dot{\theta} = \sqrt{\beta}\ z + \epsilon H_2(\theta, t, y, z, \epsilon),$$

$$\dot{y} = a(t)y + b(t,y) + \epsilon F_2(\theta, t, y, z, t),$$

$$\dot{z} = -\epsilon\gamma z + \epsilon^2 G(\theta, t, y, z, \epsilon).$$

Relatively mild hypotheses on X, F_1 give the required smoothness conditions on H_2, F_2, G, etc.

6. CONCLUSION

The use of the $D(t)$ matrices in forming the underlying function spaces a) allows the asymptotic character of the solutions of the unperturbed system to be applied to the perturbed systems as in Case 2 of the example in section 3 and b) allows special time dependence of the perturbation to be taken into account as in Case 3 of the section 3 example. In comparison to the version of Theorem 1 given by May [11] our method of proof results in an improvement of the allowable size of the perturbation; however, our proof does not reveal any periodic or almost periodic character and does not give unique solutions

No Lipschitz conditions were used in section 4, thus no study was made of the differences $y(t,\alpha,a_1) - y(t,\alpha,a_1^*)$. (Here $y(t,\alpha,a_1)$ is a solution furnished by the conclusion of Theorem 2 with $y_1(\alpha,\alpha,a_1) = a_1$.) If such conditions are imposed then asymptotic estimates of these differences can be made. Harbertson and Struble [8], (Theorem 2) and Hale [4], (Lemma 2.3) give such estimates for integral manifolds. Even if Lipschitz conditions are imposed, our method of proof gives slightly larger bounds for the size of the perturbation at the expense of uniqueness.

We remark here that the hypotheses on the $\Lambda(\nu(\epsilon)(t-s),y)G(\theta,s,y,z)$ term in Harbertson and Struble [8], page 271, can include $\nu(\epsilon)$ as a multiplier on the left. This will allow a larger class of examples.

REFERENCES

1. F. Brauer, Perturbations of nonlinear systems of differential equations, IV, J. Math. Anal. Appl. 37(1972), 214-222.

2. N. N. Bogoliubov and Y. A. Mitropolski, Asymptotic Methods in the Theory of Nonlinear Oscillations, Hindustan Publishing Corporation, Delhi, 1961.

3. W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1961.

4. J. K. Hale, Integral manifolds of perturbed differential systems, Ann. of Math., 73(1961), 496-531.

5. J. K. Hale, Oscillations in Nonlinear Systems, McGraw-Hill Book Company, Inc., New York, 1963.

6. T. G. Hallum, On the asymptotic growth of the solutions of a system of non-homogeneous linear differential equations, J. Math. Anal. Appl., 25(1969), 254-265.

7. T. G. Hallum, On asymptotic equivalence of the bounded solutions of two systems of differential equations, Mich. Math. J., 16(1969), 353-363.

8. N. C. Harbertson and R. A. Struble, Integral manifolds for perturbed nonlinear differential equations, Applicable Analysis, 1(1971), 241-278.

9. P. Hartman and N. Onuchic, On the asymptotic integration of ordinary differential equations, Pac. J. Math., 13(1963), 1193-1207.

10. N. M. Krylov and N. N. Bogoliubov, Introduction to Nonlinear Mechanics, Annals of Mathematic Studies, Princeton University Press, Princeton, N. J., 1947.

11. L. E. May, Perturbations in fully nonlinear systems, SIAM J. Math. Anal., 1(1970), 376-391.