An Analysis of the Belinfante-Swihart Theory of Gravity*

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ABSTRACT

We show that the Belinfante-Swihart (BS) theory can be reformulated in a representation in which uncharged matter responds to gravity in the same way as in metric theories. The BS gravitationally modified Maxwell equations can also be put into metric form to first order in the deviations of the physical metric from flat space, but not to second order; consequently the theory is nonmetric except in first order. We also show that the theory violates the high precision Eötvös-Dicke experiment, but cannot be ruled out by the gravitational precession of gyroscopes.

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I. INTRODUCTION AND SUMMARY

This paper analyzes the most complete and extensively developed non-metric theory that exists: the 1957 theory of F. J. Belinfante and J. C. Swihart.[1,2,3] Belinfante and Swihart constructed their theory as a Lorentz symmetric[4] linear field theory which would be easily quantized. However, as we shall show, in terms of measurable quantities the theory has all the nonlinearities of typical "curved-spacetime" theories. Moreover it is nearly a metric[4] theory: We construct a new mathematical representation which has metric form to first order in deviations of the physical metric from flatness, but does not have metric form to higher orders.

Section II gives a brief summary of the original BS representation. Included are discussions of nonlinearities and the behavior of rods and clocks. Section III presents our new mathematical representation of the theory. Section IV gives a prescription for obtaining the post-Newtonian Limit[5,6] of the theory and Sec. V considers various experimental tests. Contrary to previous calculations[7] it is found that both the geodetic and the Lens-Thirring precessions of gyroscopes[8] cannot distinguish BS from General Relativity (for a particular choice of adjustable parameters). However, using results of another paper,[9] we show that the failure of the theory to be metric at second order causes a violation of the Eötvös-Dicke (ED)[10,11] experimental results. Our calculations confirm the Belinfante-Swihart conclusion that their theory agrees with the three classical tests of gravitation theories, (perihelion shift of Mercury, bending of light by the sun, and redshift of light), and also agrees with the weak equivalence principle[4] (WEP) to <u>first</u> order.

II. THE BELINFANTE-SWIHART REPRESENTATION OF THEIR THEORY

a. Lagrangian and Equations of Motion

The original representation of the BS theory is Lagrangian based[4] but is not in generally covariant form.[4] In this section we generalize, in a trivial manner, the original representation so that it is generally covariant. The dynamical equations are obtained by extremization of the following action:

$$I = \int \mathcal{L}_G d^4x + \int \mathcal{L}_M d^4x + \int \mathcal{L}_I d^4x \tag{1}$$

where

$$\mathcal{L}_G = -(16\pi)^{-1} \eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma}\left(a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta}\right)(-\eta)^{1/2} \tag{2}$$

$$\mathcal{L}_M = \sum_A \int \left[-m_A b_A + (\Pi_\mu + e_A A_\mu)\frac{dx^\mu}{d\lambda_A} - \Pi_\mu a^\mu\right]\delta^4\left(\underset{\sim}{x} - \underset{\sim}{z}_A(\lambda_A)\right) d\lambda_A$$
$$+ (4\pi)^{-1}\left(\frac{1}{4} H^{\nu\mu}H_{\nu\mu} - H^{\nu\mu}A_{\mu|\nu}\right)(-\eta)^{1/2} \tag{3}$$

$$\mathcal{L}_I = \frac{1}{2}\overline{T}^{\mu\nu} h_{\mu\nu} + \sum_A \int K m_A b_A h_\lambda{}^\lambda\, \delta^4\left(\underset{\sim}{x} - \underset{\sim}{z}_A(\lambda_A)\right) d\lambda_A \tag{4}$$

$$\overline{T}^{\mu\nu} = (4\pi)^{-1}\left(H^{\lambda\mu}H_\lambda{}^\nu - \frac{1}{4}\eta^{\mu\nu}H^{\alpha\beta}H_{\alpha\beta}\right)(-\eta)^{1/2} + \sum_A \int a_A{}^\mu \Pi_A{}^\nu \delta^4\left(\underset{\sim}{x} - \underset{\sim}{z}_A(\lambda_A)\right) d\lambda_A \tag{5a}$$

$$b_A \equiv a_{A\mu} a_A{}^\mu \quad . \tag{5b}$$

Equations (1-5) describe the interactions of a collection of charged particles (labeled by A) with the electromagnetic and gravitational fields. Conventions and definitions for the above are the following:

(i) We use units such that $c = G = 1$.

(ii) $\eta_{\alpha\beta}$ is a Riemann flat background metric (absolute gravitational field[4]). In some coordinate system it therefore takes on Minkowski values, $\eta_{\alpha\beta} = \text{diag.}(-1, 1, 1, 1)$. All tensorial indices occurring in Eqs. (1-5) are raised and lowered with $\eta_{\alpha\beta}$.

(iii) Greek and Latin indices run through 0-3, 1-3, respectively.

(iv) a, f, K are adjustable parameters.

(v) $h_{\mu\nu} = h_{\nu\mu}$ is a symmetric second rank dynamical gravitational field.[4]

(vi) The world line of particle A is parametrized by an arbitrary, monotonic parameter λ_A which varies from $-\infty$ to $+\infty$. Particle A is described by its coordinate $z_A{}^\mu$ and its "velocity and momentum variables" $a_A{}^\mu$ and $\Pi_A{}^\mu$, which are all functions of λ_A.

(vii) The electromagnetic field is described by the tensor fields A_μ and $H_{\mu\nu} = -H_{\nu\mu}$.

(viii) $\overline{T}^{\mu\nu}$ is a "stress-energy tensor" for particles and electromagnetic fields. (The bar above is used to distinguish it from a different "stress-energy tensor" definded in Sec. IV.)

(ix) Slashes denote covariant derivatives with respect to the flat background metric $\eta_{\alpha\beta}$.

(x) $\eta \equiv$ determinant of $\eta_{\alpha\beta}$.

Equations (5a) and (5b) are decomposition equations[4] for $\overline{T}^{\mu\nu}$ and b_A. The dynamical variables which one varies independently in the action are $h_{\mu\nu}(x)$, $z_A{}^\mu(\lambda_A)$, $a_A{}^\mu(\lambda_A)$, $\Pi_A{}^\mu(\lambda_A)$, $A_\mu(x)$, and $H_{\mu\nu}(x)$. Variation of the matter variables yields the following dynamical laws[12]

$$ma^\mu(1 - Kh) = b(\Pi^\mu - \frac{1}{2} h_\nu{}^\mu \Pi^\nu) \qquad [BS,I,(24)] \tag{6}$$

$$dz_A{}^\mu/d\lambda_A = a_A{}^\mu - \frac{1}{2} h_\nu{}^\mu(z_A) a_A{}^\nu \qquad [BS,I,(30)] \tag{7}$$

$$F_{\mu\nu} \equiv A_{\nu|\mu} - A_{\mu|\nu} = H_{\mu\nu}(1 - \frac{1}{2} h) + H_{\mu\lambda}h_\nu{}^\lambda - H_{\nu\lambda}h_\mu{}^\lambda \qquad [BS,II,(11)] \tag{8}$$

$$H^{\mu\nu}{}_{|\nu} = 4\pi \sum_A e_A \int (dz_A{}^\mu/d\lambda_A)\ \delta^4(\underset{\sim}{x} - \underset{\sim}{z}_A)\ d\lambda_A(-\eta)^{-1/2} \qquad [\mathrm{BS, II, (10)}] \tag{9}$$

$$\frac{d\Pi_A{}^\mu}{d\lambda_A} = e_A F_{\mu\nu} \frac{dz_A{}^\nu}{d\lambda_A} + \frac{1}{2} a_A{}^\rho \Pi_A{}^\sigma h_{\rho\sigma|\mu} + km_A b_A h_{|\mu} \qquad [\mathrm{BS, II, (5)}] \tag{10}$$

where $h \equiv h_\alpha{}^\alpha$.

Variation of $h_{\mu\nu}$ yields

$$a\Box h_{\alpha\beta} + f\eta_{\alpha\beta}\Box h = -4\pi\bar{T}_{\alpha\beta} - 8\pi K\eta_{\alpha\beta} \sum_A \int m_A b_A\ \delta^4(\underset{\sim}{x} - \underset{\sim}{z}_A) d\lambda_A \quad . \tag{11}$$

Here we have used the symbol $\Box h_{\mu\nu} \equiv \eta^{\alpha\beta} h_{\mu\nu|\alpha|\beta}$.

b. Nonlinearities in the Theory

Linear gravitational field equations do not preclude a nonlinear form for the response of particles to gravity. The BS theory is an example: Eqs. (16) and (17) endow the canonical variables $a_A{}^\mu$ and $\Pi_A{}^\mu$ with gravitational contributions. Consequently the equation of motion for a particle, Eq. (10), is nonlinear in the gravitational field $h_{\mu\nu}$. Indeed, although the BS theory is often called a "linear" theory, its "linear" first-order matter Lagrangian produces qualitatively many of the nonlinear effects of General Relativity (GRT) for example (see Secs. III and IV). Hence one should be cautious in the labelling of theories as "linear" or "nonlinear" on the mere basis of the linear forms of their gravitational equations.

c. Behavior of Rods and Clocks

In the third paper of their series[3] Belinfante and Swihart quantize the theory and obtain a gravitationally modified Dirac theory. We remind the reader that all nonmetric theories must exhibit explicitly the manner in which all the laws of physics are changed in the presence of gravity.

Belinfante and Swihart find that, in the case of a static spherically symmetric (SSS) gravitational source, the standard solutions to the unmodified Dirac equation are related to those in the presence of gravity in the following way:

$$\varphi_o(\underset{\sim}{x}_o, t) = N\varphi(\underset{\sim}{x}, t) \tag{12}$$

$$\underset{\sim}{x}_o = C\,\underset{\sim}{x} \qquad \text{[BS,III,(78)]} \tag{13}$$

$$t_o = (1 - U)t \tag{14}$$

$$N \equiv C^{-3/2} \equiv \left(\frac{1 - U}{1 - U/2a}\right)^{-3/2} \quad . \tag{15}$$

Here the subscripted quantities are those in the absence of gravity, φ is the electron wave function, U is the Newtonian gravitational potential for an SSS source and "a" is the previously mentioned adjustable parameter. The coordinate system is one in which $\eta_{\alpha\beta}$ = diag. (- 1, 1, 1, 1). The energy eigenvalues, i.e., E in $\varphi(\underset{\sim}{x}, t) = \varphi(\underset{\sim}{x}) \exp(- iEt/\hbar)$, are shifted in the presence of gravity:

$$E_o = (1 + U)E \qquad \text{[BS,III,(82)]} \tag{16}$$

— a result following essentially from Eq. (14). It is Eq. (16) which produces qualitatively the correct redshift. Equations (12) and (13) also indicate the effect of gravity on the coordinate sizes of atoms. Consider the expectation value of the coordinate size of an atom:

$$\langle r\rangle = \int |\varphi(\underset{\sim}{x}, t)|^2 r\, d^3x \quad . \tag{17}$$

Using Eqs. (12) and (13) we obtain

$$\langle r\rangle = \int N^2 |\varphi_o(\underset{\sim}{x}_o, t_o)|^2\, Cr_o C^3 d^3x_o = C\langle r_o\rangle$$

$$= \frac{(1 - U)}{(1 - U/2_a)} \langle r_o\rangle \approx \left[1 + U(\tfrac{1}{2}\, a^{-1} - 1)\right] \langle r_o\rangle \quad . \tag{18}$$

According to Eq. (16), the coordinate ticking rate of an atomic clock decreases in a gravitational field: $\omega = (1 - u)\omega_o$. According to Eq. (18) the coordinate size of a rod made of atoms increases in a gravitational field: $\ell = \left[1 + U(\frac{1}{2} a^{-1} - 1)\right] \ell_o$. Since $a \approx \frac{1}{4}$ to agree with the light bending experiment (see later sections) the above results are the same, to first order in U, as one obtains in GRT, using an "isotropic, post-Newtonian" coordinate system.[5]

III. ATTEMPTS TO PUT THE THEORY INTO METRIC FORM

The BS theory is a Lagrangian-based, relativistic theory of gravity.[4] Therefore, according to a theorem proved in Ref. 4, it is a metric theory if and only if the "nongravitational part" of its Lagrangian,

$$\mathcal{L}_{NG} \equiv \mathcal{L}_M + \mathcal{L}_I$$

can be put into universally coupled form.[4] Let us try to achieve universal coupling by a change of variables, i.e., by introducing a new mathematical representation of the theory.

a. Particle Part of Lagrangian

Begin with the terms in $\mathcal{L}_{NG}$ that refer only to particles and define the following tensors:

$$\bar{\Delta}_\nu{}^\mu \equiv \delta_\nu{}^\mu - \frac{1}{2} h_\nu{}^\mu \tag{19}$$

$$\Delta_\alpha{}^\beta \equiv (\bar{\Delta}_\alpha{}^\beta)^{-1} \quad ,\text{i.e.,}\ \bar{\Delta}_\alpha{}^\beta \Delta_\beta{}^\tau = \delta_\alpha{}^\tau \quad . \tag{20}$$

Then, from Eq. (7), obtain the relation

$$a^\nu = \Delta_\mu{}^\nu dz^\mu/d\lambda \quad . \tag{21}$$

Equation (21), which is obtained after variation of the Lagrangian, suggests that one define a new variable v^μ to replace a^μ in the Lagrangian:

$$a^\nu \equiv \Delta_\mu{}^\nu v^\mu \quad . \tag{22}$$

Then, the relation $v^\mu = dz^\mu/d\lambda$ will presumably turn out to be an Euler-Lagrange equation. Using Eqs. (19)-(21), bring the particle portion of the Lagrangian into the form

$$I_{\text{part.}} \equiv \left[\int \mathcal{L}_M d^4x + \int \mathcal{L}_I d^4x\right]_{\text{part.}} \tag{23}$$

$$= \sum_A \int \left[-(1 - Kh)m_A b_A + e_A A_\mu \frac{dz_A{}^\mu}{d\lambda_A} + \Pi_{A\mu}\left(\frac{dz_A{}^\mu}{d\lambda_A} - a_A{}^\mu + \frac{1}{2} h_\nu{}^\mu a_A{}^\nu\right)\right] d\lambda_A$$

$$= \sum_A \int \left\{- m_A[- (1 - Kh)^2 \Delta_\mu{}^\alpha \Delta_{\alpha\nu} v_A{}^\mu v_A{}^\nu]^{1/2} + e_A A_\mu \frac{dz_A{}^\mu}{d\lambda_A} + \Pi_{A\mu}\left(\frac{dz_A{}^\mu}{d\lambda_A} - v_A{}^\mu\right)\right\} d\lambda_A \ . \tag{24}$$

In obtaining Eq. (24) from Eq. (23) we have performed the integrations over d^4x and thus all of the spacetime functions should be evaluated at the particle position $z_A{}^\mu$.

If we now define an "effective metric"

$$g_{\alpha\beta} \equiv (1 - Kh)^2 \Delta_\alpha{}^\mu \Delta_{\mu\beta} = \eta_{\alpha\beta}(1 - 2Kh) + h_{\alpha\beta} + O(h^2) \tag{25}$$

Eq. (24) takes the universally coupled form, with $g_{\alpha\beta}$ being the only gravitational field occurring in $I_{\text{part.}}$. Variation of Π_μ then yields the desired relation

$$v^\mu = dz^\mu/d\lambda \quad . \tag{26}$$

To make our results look simpler, we explicitly introduce Eq. (26) into Eq. (24), thus eliminating Π_μ completely and obtaining

$$I_{\text{part.}} = \sum_A \int \left[- m_A\left(- g_{\alpha\beta} \frac{dz_A{}^\alpha}{d\lambda_A} \frac{dz_A{}^\beta}{d\lambda_A}\right)^{1/2} + e_A A_\mu \frac{dz^\mu}{d\lambda_A}\right] d\lambda_A \ . \tag{27}$$

Variation of Eq. (27) yields equations of motion which, by the use of Eqs. (6), (19)-(21), can be shown to be identical to the BS equations of motion, Eqs. (10). Equation (27) is the familiar "metric theory" action principle describing the interaction of charged particles with the gravitational field $g_{\mu\nu}$ and the electromagnetic field A_μ.

b. Electromagnetic Part of Lagrangian

It will now be shown that, to first order in $h_{\mu\nu}$, the electromagnetic Lagrangian can also be put into metric form. Change variables from $H_{\mu\nu}$ to an antisymmetric tensor $F_{\mu\nu}$ by

$$H_{\mu\nu} = F_{\mu\nu}(1 + \frac{1}{2} h + \frac{1}{4} h^2) + 2 F_{\lambda[\mu}h_{\nu]}{}^{\lambda} (1 + h) - 2F_{\alpha[\mu}h_{\nu]}{}^{\lambda} h^{\alpha}{}_{\lambda} - 2F_{\lambda\alpha}h^{\alpha}{}_{[\mu}h_{\nu]}{}^{\lambda} + O(Fh^3) \quad . \tag{28}$$

Equation (28) is simply the result of an inversion of Eq. (8). Square brackets around indicies denote antisymmetrization of indices (with the usual normalization of a factor of $\frac{1}{2}$). Variation of $F_{\mu\nu}$ in the new Lagrangian presumably will yield the relation

$$F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu} \quad . \tag{29}$$

Substitution of Eq. (28) into the electromagnetic portion of the action yields

$$L_{EM} \equiv (4\pi)^{-1} \int \left\{ \frac{1}{2} H_{\alpha\beta}H_{\nu\mu}[\frac{1}{2} \eta^{\beta\mu}(1 - \frac{1}{2} h) + h^{\beta\mu}]\eta^{\alpha\nu} - A_{\mu|\nu}H_{\alpha\beta}\eta^{\alpha\nu}\eta^{\beta\mu} \right\} (-\eta)^{1/2} d^4x$$

$$= (4\pi)^{-1} \int \left\{ \frac{1}{2}\left[F_{\alpha\beta}(1 + \frac{1}{2} h) + 2F_{\lambda[\alpha}h_{\beta]}{}^{\lambda}\right]\left[F_{\nu\mu}(1 + \frac{1}{2} h) + 2F_{\lambda[\nu}h_{\mu]}{}^{\lambda}\right]\eta^{\alpha\nu}\Gamma^{\beta\mu} - A_{\mu|\nu}\eta^{\alpha\nu}\eta^{\beta\mu}\left[F_{\alpha\beta}(1 + \frac{1}{2} h) + 2F_{\lambda[\alpha}h_{\beta]}{}^{\lambda}\right] \right\} (-\eta)^{1/2} d^4x \tag{31}$$

$$= (4\pi)^{-1} \int \left(A_{[\mu|\nu]} + \frac{1}{4} F_{\mu\nu}\right)F_{\alpha\beta}[\eta^{\alpha\mu}\eta^{\beta\nu}(1 + \frac{1}{2} h) - h^{\alpha\mu}\eta^{\beta\nu} - h^{\beta\nu}\eta^{\alpha\mu}](-\eta)^{1/2} d^4x + O(h^2) \tag{32}$$

where $$\Gamma^{\beta\mu} \equiv \frac{1}{2}\,\eta^{\beta\mu}(1 - \frac{1}{2}\,h) + h^{\beta\mu}$$

If one now uses the inverse of Eq. (25), i.e.,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + 2Kh^{\alpha\beta} + O(h^2)$$

and

$$(-\,g)^{1/2} = (-\,\eta)^{1/2}\,[1 + h(\frac{1}{2} - 4K)] + O(h^2)$$

one finds Eq. (32) can be written as

$$L_{EM} = (4\pi)^{-1} \int \left(A_{[\mu|\nu]} + \frac{1}{4}\,F_{\mu\nu}\right) F_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}(-\,g)^{1/2}\,d^4x + L_{CORR.} \tag{33}$$

where

$$L_{corr} \equiv (4\pi)^{-1} \int F_{\mu\nu}\left(\frac{1}{4}\,F_{\alpha\beta} - A_{[\alpha|\beta]}\right)\Gamma^{\mu\alpha\nu\beta}d^4x = O(h^2) \tag{34a}$$

and

$$\Gamma^{\mu\alpha\nu\beta} \equiv \frac{1}{8}\,h^2\eta^{\mu\alpha}\eta^{\nu\beta} - h\eta^{\mu\alpha}h^{\nu\beta} + \frac{3}{2}\,\eta^{\mu\alpha}h^{\nu\sigma}h_{\sigma}^{\ \beta} + 3h^{\mu\alpha}h^{\nu\beta} \quad . \tag{34b}$$

Thus L_{EM} has universally coupled form at $O(h)$; at $O(h^2)$ deviations occur, arising from the L_{corr} term in Eq. (33). Variation of $F_{\mu\nu}$ in Eq. (33) yields the desired relation between $F_{\mu\nu}$ and A_μ, i.e., Eq. (29). Completely equivalent equations are obtained if Eq. (29) is now substituted into Eq. (33), yielding

$$L_{EM} = -\,(16\pi)^{-1} \int F_{\alpha\beta}F_{\mu\nu}g^{\alpha\beta}g^{\mu\nu}(-\,g)^{1/2}\,d^4x + L_{CORR.} \tag{35a}$$

$$\equiv -\,(16\pi)^{-1} \int F_{\alpha\beta}F^{\alpha\beta}(-\,g)^{1/2}\,d^4x + L_{CORR.} \quad . \tag{35b}$$

The relation given in Eq. (29) is now understood to hold in Eqs. (35). Since we now have constructed a second metric $g_{\alpha\beta}$ (the "physical metric"), indices on all quantities except the constituents of $g_{\alpha\beta}$ ($\eta_{\alpha\beta}$, $h_{\alpha\beta}$, $\Delta_{\alpha\beta}$) henceforth will be raised and lowered with $g_{\alpha\beta}$. Equation (35), aside from the $O(h^2)$ correction term, is recognized as the electromagnetic Lagrangian for metric theories. Thus the BS theory is a metric theory at first order but nonmetric at all higher orders (in h).

c. Summary of Our New Representation

Our new representation of the BS theory is summarized succinctly in Table I. In particular one sees that for uncharged particles the theory is metric to all orders in h, with $g_{\alpha\beta}$ playing the role of the "physical" metric.[4] When electromagnetic phenomena are included, and when one goes beyond first order in h, the theory is nonmetric (cf. $\mathcal{L}_{CORR}$ in Table I).

IV. THE POST-NEWTONIAN LIMIT OF THE THEORY

We now proceed to calculate the post-Newtonian (PN) limit of the theory. The PN limit is a perturbation solution of the gravitational field equations — expanding in the small quantities occurring in the solar system, e.g.,

$$v^2 \equiv (\text{macroscopic velocities of bodies})^2 = O(\epsilon^2)$$

$$U \equiv \text{Newtonian gravitational potential} = O(\epsilon^2)$$

$$p/\rho \equiv (\text{pressure})/(\text{proper density of rest mass}) = O(\epsilon^2)$$

$$\Pi \equiv (\text{internal energy density})/(\text{rest mass density}) = O(\epsilon^2) \quad .$$

We refer the reader to Ref. 5 for further details of the expansion scheme.

a. The Metric-Theory Approximation

Using Table I, we write the field equations as

$$\begin{aligned}\frac{\delta\mathcal{L}_G}{\delta h_{\mu\nu}} = -\frac{\delta\mathcal{L}_{NG}}{\delta h_{\mu\nu}} &= -\left(\frac{\delta\mathcal{L}_{Metric}}{\delta h_{\mu\nu}} + \frac{\delta\mathcal{L}_{CORR.}}{\delta h_{\mu\nu}}\right) \\ &= -\left(\frac{\delta\mathcal{L}_{Metric}}{\delta g_{\mu\nu}}\frac{\partial g_{\mu\nu}}{\partial h_{\mu\nu}} + \frac{\delta\mathcal{L}_{CORR.}}{\delta h_{\mu\nu}}\right) \\ &- \left(\frac{(-g)^{1/2}}{2} T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial h_{\mu\nu}} + \frac{\delta\mathcal{L}_{CORR.}}{\delta h_{\mu\nu}}\right) \qquad (36)\end{aligned}$$

where we have used the usual definition (as in metric theories)

$$T^{\mu\nu} = \frac{2}{(-g)^{1/2}} \frac{\delta \mathcal{L}_{Metric}}{\delta g_{\mu\nu}} \quad . \tag{37}$$

To PN order, the first term on the right-hand side of Eq. (36) is of order

$$\text{First Term} \approx (\text{Total energy density}) \cdot \epsilon^2$$

while the second term is of order (see Table I)

$$\text{Second Term} \approx (\text{Electromagnetic energy density}) \cdot \epsilon^2 \quad .$$

Since the electromagnetic energy of a substance is typically smaller than the total mass-energy by a factor $\lesssim 10^{-3}$, the second source term in Eq. (36) can be neglected at PN order, by comparison with the first. Similarly, one can make a metric-theory approximation for the response of matter to gravity. For metric-theory (i.e., universally-coupled[4]) Lagrangians one always has

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{38}$$

when the matter field equations are satisfied, where the semicolan denotes covariant differentiation with respect to the physical metric $g_{\alpha\beta}$. In the BS case

$$T^{\mu\nu}{}_{;\nu} = O\left(\frac{\delta \mathcal{L}_{CORR.}}{\delta h_{\mu\nu}} h \, h_{,\nu}\right) \quad ; \tag{39}$$

so again one can conclude that effects resulting from the deviation in the matter response equation from Eq. (38) will be $\lesssim 10^{-3}$ of PN effects. Thus for all PN phenomena we can neglect $\mathcal{L}_{CORR.}$ and treat the BS theory as a metric theory.

b. From Point Particles to Perfect Fluid

In one of their original papers[1] Belinfante and Swihart, when solving their gravitational field equations with the sun as the external source, use an ad hoc perfect-fluid stress energy-tensor for $\overline{T}^{\mu\nu}$, rather than the expression given in Eq. (5). Their $\overline{T}_{\mu\nu}$ is precise enough to yield an adequate treatment of the "three classical gravitation tests" but not precise enough to adequately handle such effects as the effective gravitational mass of gravitational energy (cf., "Nordtvedt effect" in Ref. 5). To avoid such problems, and to ensure self-consistency of the theory when dealing with gravitating sources in the solar system, we will build up the fluid BS stress energy tensor, $T^{\mu\nu}$, as an average over charged point particles and their electromagnetic fields [cf., Eq. (27) and Table I].

The kinetic-theory procedure for constructing a perfect fluid out of interacting particles is the same in any metric theory as in general relativity — and the same in general relativity as in special relativity ("equivalence principle").[13] By following that standard procedure and by neglecting the resulting non-perfect fluid terms, we obtain the standard stress energy tensor:

$$T^{\mu\nu} = (\epsilon + p)\, u^{\mu}u^{\nu} + pg^{\mu\nu} \quad . \tag{40}$$

Here u^{μ} is a suitable macroscopic average of the microscopic particel 4-velocities; ϵ is the density of total mass-energy (rest mass plus kinetic energy of particles plus electromagnetic energy) as measured in the macroscopic rest frame; and p is the similarly measured averaged pressure.

c. The Parametrized Post-Newtonian (PPN) Formalism

References 5 and 6 present a "parametrized post-Newtonian formalism" in which the PN limit of every metric theory is summarized by the coefficients of various integral functions in its metric. These coefficients, the so called PPN parameters, are obtained by the previously mentioned perturbation solution (PN limit) of the gravitational field equations. We have constructed such a solution for our new mathematical representation of the BS theory, using Eqs. (36), (40), and Table I. The details are spelled out in Ref. 14. (Actually Ref. 14 is the presentation of an exact gravitation theory closely related to the metric-theory approximation of the BS theory.) We refer the reader to Ref. 14 and here quote only the PPN parameters of the BS theory.

$$\begin{array}{lll} \gamma = \bar{\gamma} + O(w) & \zeta_2 = 0 & \alpha_1 = O(w) \\ \beta = \bar{\beta} + O(w) & \zeta_3 = 0 & \alpha_2 = O(w) \\ \zeta_1 = 0 & \zeta_4 = 0 & \alpha_3 = 0 \quad . \end{array} \tag{41}$$

Here $\bar{\gamma}$ and $\bar{\beta}$ are given implicitly in terms of a and f by

$$a = 1/(2\bar{\gamma} + 2) \tag{42}$$

$$f = \frac{10\bar{\beta} + 6\bar{\gamma}\bar{\beta} - 7\bar{\gamma}^2 - 8\bar{\gamma} - 6}{2(\bar{\gamma} + 1)(3\bar{\gamma} + 5 - 4\bar{\beta})^2} \tag{43}$$

and to obtain the correct Newtonian limit, one must require

$$\frac{16K^2 a - 4aK + a + 3f}{a(a + 4f)} = 2 \quad . \tag{44}$$

By $O(w)$, we denote terms involving the cosmological boundary values of $h_{\mu\nu}$ (see Ref. 14 for further details). Imposing Eq. (44) reduces the number of arbitrary parameters to two (a and f for example); so we may regard $\bar{\gamma}$ and $\bar{\beta}$ as being arbitrary. For comparison, General Relativity has no arbitrary parameters and its only nonzero parameters are $\gamma = \beta = 1$.

V. EXPERIMENTAL CONSEQUENCES AND TESTS OF THE THEORY

In his 1972 Varenna Lectures, Will[8] summarizes, within the PPN framework, the constraints which may be placed on a metric theory's parameters by current solar system gravitation experiments. As has been indicated in the previous section, the difference between the BS theory and a metric theory for PPN-type experiments is less than one part in 10^3. For most experiments the microscopic internal energies play a minor role — e.g., it is the macroscopic rotation of the earth which produces the macroscopic Lens-Thirring precession of gyroscopes. For such experiments the BS theory is effectively a metric theory to a much higher accuracy than indicated above. In summary, so far as PN experiments are concerned, to the precision of the technology of the 1970's, the BS theory is accurately summarized by the values of its PPN parameters, Eqs. (41). We refer the reader to Ref. 8 for the experimental consequences of those values. Here we merely point out a few salient features.

Perhaps the most important feature is this: If the O(w) terms in the parameters are sufficiently small, and if the arbitrary parameters are chosen so that $\overline{\gamma} = \overline{\beta} = 1$, then the PN predictions of the metric-theory approximation to BS are the same as the PN predictions of general relativity. In particular, the predictions for the "three classical tests" are the same as Belinfante and Swihart[1] themselves deduced by complicated calculations.

a. Preferred Frame-Effects

For the coordinate system in which $\underset{\sim}{\eta}$ is Minkowskian, it is natural to set the boundary values of $\underset{\sim}{h}$ to zero when treating the solar system, as was done originally by Belinfante and Swihart. However, the correct way to determine the boundary values of $\underset{\sim}{h}$ is through the solution of the cosmological problem. If the solution produces nonzero cosmological boundary values of $\underset{\sim}{h}$, then those values will effect certain of the PPN parameters [cf., O(w) terms in Eqs. (41)]. In the case of the BS theory the presence of such terms is a direct consequence of the presence of the "absolute gravitational field" $\underset{\sim}{\eta}^4$ (cf., Table I), and leads to various preferred-frame effects[8] such as anamolous solid earth tides and contributions to the perihelion shift of mercury. We refer the reader to Ref. 14 for a more complete discussion of the derivation of such effects in the BS theory.

b. Precession of Gyroscopes

We specifically mention this experimental test only because there seems to be some confusion[15] as to the prediction of the BS theory. Using formulas from Ref. 8 and the BS PPN parameters, Eq. (41), one obtains for the precession of the spin $\underset{\sim}{S}$ of a gyroscope orbiting the earth

$$\frac{d\underset{\sim}{S}}{ds} = \underset{\sim}{\Omega} \times \underset{\sim}{S} \tag{45}$$

where

$$\underset{\sim}{\Omega} = \underset{\sim}{\Omega}_{LENS-THIRRING} + \underset{\sim}{\Omega}_{GEODETIC} \tag{46a}$$

$$\Omega_{L-T} = \frac{1}{8}[4\,\overline{\gamma} + 4 + O(w)][.05'' \text{ of arc/year}] \tag{46b}$$

$$\Omega_G = \frac{1}{3}[1 + 2\,\overline{\gamma} + O(w)][7'' \text{ of arc/year}] \quad . \tag{46c}$$

Thus the results of the upcoming (to be launched before 1977) Everitt-Fairbank[16] gyroscope experiment can only place upper limits on the cosmological boundary volues of $h_{\mu\nu}$ [cf., O(w) terms in Eqs. (46)] for a given choice of $\overline{\gamma}$.

c. The Weak Equivalence Principle and Eötvös-Dicke Type Experiments

We conclude by considering the Eötvös-Dicke (ED) type experiments,[10,11] which test gravity so precisely that they fall outside of the PN realm of precision. Braginsky's[11] recent version of the ED experiment shows that the difference in accelerations of test bodies of aluminum and platinum in the gravitational field of the sun is smaller than one part in 10^{12}. Such a result represents a strong validation of the weak equivalence principle[4] (WEP). Consider the contribution of electromagnetic energy at order F^2h^2 (see bottom of Table I) to the gravitational mass and acceleration $\underset{\sim}{a}$ of a test body:

$$\left|\frac{\underset{\sim}{a}}{\underset{\sim}{g}}\right| \sim \left|\frac{1}{\underset{\sim}{g}} \nabla\left[\left(\frac{\text{electromagnetic energy}}{\text{total energy}}\right)h^2\right]\right| \approx \frac{\text{E.M. energy}}{\text{total mass}}\, U \qquad (47)$$

where $h^2 \approx U^2$ and $\underset{\sim}{g} \equiv \nabla U$. For platinum, the following relation holds:

$$(\text{E.M. energy/total mass}) \approx 10^{-3}$$

and the Newtonian potential due to the sun at the earth is

$$U \approx 10^{-8} \,.$$

Equation (47) and the above numerical estimates indicate that the ED experiment can distinguish between the BS theory and its metric-theory approximation (cf., $\mathcal{L}_{\text{CORR.}}$ in Table I). All metric theories satisfy WEP identically. The BS theory, however, as is shown in Ref. 9, predicts

$$\left|\frac{\langle\underset{\sim}{a}\rangle_{Pt} - \langle\underset{\sim}{a}\rangle_{Al}}{\underset{\sim}{g}}\right| \approx 6 \times 10^{-11} \approx \left(\frac{\text{E.M. energy}}{\text{total mass}}\right) U \qquad (48)$$

in clear violation of the Dicke[10] and Braginsky[11] versions of the experiment. The reader is referred to Ref. 9 for complete details as to the derivation of

Eq. (48) from considerations of particles interacting with gravity and electromagnetism.

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TABLE I

A New Mathematical Representation of the Belinfante-Swihart Theory

1. Gravitational fields:

a. Absolute field . . . . $\underset{\sim}{\eta}$

b. Dynamical symmetric second rank tensor . . . . $\underset{\sim}{h}$

c. "Physical" metric . . . . $\underset{\sim}{g}$

2. Nongravitational variables:

a. Particle coordinates . . . . $\underset{\sim}{z}_A$

b. Electromagnetic vector potential . . . . $\underset{\sim}{A}$

c. Affine parameter of particle world lines . . . . λ_A

3. Gravitational field equations:

a. Flatness of $\underset{\sim}{\eta}$: Riemann $(\underset{\sim}{\eta}) = 0$

b. Field equations for $\underset{\sim}{h}$ obtained by variation of $h_{\alpha\beta}$ in Lagrangian below

c. Decomposition equation for g: $g_{\alpha\beta} = (1 - Kh)^2 \Delta_\alpha{}^\mu \Delta_{\mu\beta}$ where we have defined $\Delta_\alpha{}^\beta(\delta_\beta{}^\tau - \frac{1}{2} h_\beta{}^\tau) \equiv \delta_\alpha{}^\tau$, K is an arbitrary constant, $h = \eta^{\alpha\beta} h_{\alpha\beta}$, and indices are raised and lowered on $h_{\alpha\beta}$, $\Delta_{\alpha\beta}$ with $\eta_{\alpha\beta}$.

4. Influence of gravity on matter:

Equations for $\underset{\sim}{A}$, $\underset{\sim}{z}_A$, obtained by variation of those quantities in Lagrangian

5. Lagrangian density:

a. $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{NG}$

b. $\mathcal{L}_G = - (16\pi)^{-1} \left(a h^{\mu\sigma}{}_{|\alpha} h_{\mu\sigma}{}^{|\alpha} + f h_{|\alpha} h^{|\alpha} \right) (-\eta)^{1/2}$

TABLE I (continued)

$$\text{c. } \mathcal{L}_{NG} = \sum_A \int \left[- m_A \left(- g_{\alpha\beta} \frac{dz_A^{\ \alpha}}{d\lambda_A} \frac{dz_A^{\ \beta}}{d\lambda_A} \right)^{1/2} + e_A A_\mu \frac{dz_A^{\ \mu}}{d\lambda_A} \right] d\lambda_A \delta^4(\underset{\sim}{x} - \underset{\sim}{z}_A)$$

$$- (16\pi)^{-1} F_{\alpha\beta} F^{\alpha\beta} (- g)^{1/2} + \mathcal{L}_{CORR.} = \mathcal{L}_{METRIC} + \mathcal{L}_{CORR.}$$

where $\mathcal{L}_{CORR.}$, the "correction term" in the Lagrangian, which represents the amount by which the purely electromagnetic portion of the Lagrangian fails to have metric form, satisfies

$$\mathcal{L}_{CORR.} = O(F^2 h^2) \qquad \text{[see Eqs. (34)]}$$

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