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SMOOTHING PROPERTIES OF NEUTRAL EQUATIONS

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Smoothing Properties of Neutral Equations

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For given $r \geq 0$, $A \geq 0$, $x: [-r,A] \to E^n$ and any $t \in [0,A]$, define $x_t: [-r,0] \to E^n$ by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Also, let $C = C([-r,0],E^n)$ be the space of continuous functions mapping $[-r,0]$ into E^n with the topology of uniform convergence. If $f: C \to E^n$ is a given continuous function, then a retarded functional differential equation (RFDE) is a relation

$$\dot{x}(t) = f(x_t) \quad . \tag{1}$$

A solution $x = x(\varphi)$ through $\varphi \in C$ is a continuous function defined on $[-r,A)$ for some $A > 0$ which satisfies (1) on $[0,A)$ and coincides with φ on $[-r,0]$. If a solution x is defined on $[-r,A)$ with $A > r$, then x_t, $t \in [r,A)$ is a function which also has a continuous first derivative; that is, the solution of (1) is in general smoother than the initial data.

To generalize (1), suppose $D: C \to E^n$ is continuous, linear and atomic at zero; that is, there exists an $n \times n$ matrix function μ of bounded variation on $[-r,0]$ and a continuous non-negative function $\gamma(s)$, $s \geq 0$, $\gamma(0) = 0$, such that

$$\begin{cases} D\varphi = \varphi(0) - g(\varphi) , \\ g(\varphi) = \int_{-r}^{0} [d\mu(\theta)]\varphi(\theta) \\ \int_{-s}^{0} |d\mu(\theta)| \leq \gamma(s) , \quad 0 \leq s \leq r \quad . \end{cases} \tag{2}$$

By a neutral functional differential equation (NFDE), we mean a relation

$$(3) \qquad \frac{d}{dt} Dx_t = f(x_t)$$

with D linear, continuous, atomic at zero and f as before. A solution $x = x(\varphi)$ of (3) is defined as above. For equation (3), the solution generally is no smoother than the initial data after any finite number of steps. However, we define a more restrictive class of D-operators for which some smoothing takes place after an infinite number of steps. This result will say that a solution of (3) can be in an ω-limit set only if it corresponds to initial data which is "smooth".

With D as above, the space $C_D = \{\psi \in C: D\psi = 0\}$ can be considered as a Banach space with the topology of uniform convergence. On C_D, consider the equation

$$(4) \qquad Dy_t = 0 \quad , \quad y_0 = \psi \in C_D \quad .$$

There exist positive constants a, $K = K(a)$ such that

$$(5) \qquad |y_t(\psi)| \leq Ke^{at}|\psi| \quad , \quad t \geq 0 \ , \quad \psi \in C_D \quad .$$

Let $a_D = \inf\{a : \exists\, K = K(a) \text{ satisfying } (5)\}$. Following Cruz and Hale [1], we say D is <u>stable</u> if $a_D < 0$.

A result we need from [1] is the following.

Lemma 1. If D is stable, then there exist $b > 0$, $a > 0$ such that for all $h \in C([0,\infty),E^n)$, the solution $z(\psi,h)$ of

$$(6) \qquad Dz_t = h(t) \ , \quad w_o = \psi$$

satisfies

$$(7) \qquad |z_t(\psi,h)| \leq be^{-at} |\varphi| + b \sup_{o \leq u \leq t} |h(u)| , \ t \geq 0 \ .$$

Lemma 2. If $f: C \to E^n$ is continuous, takes bounded sets of C into bounded sets E^n, D is stable and the orbit $\gamma^+(\varphi) = \bigcup_{t \geq o} x_t(\varphi)$ of the solution of (3) through φ is bounded, then there exist constants $M > 0$, $\alpha > 0$, such that

$$(8) \qquad |x_{t+\tau}(\varphi) - x_t(\varphi)| \leq Me^{-\alpha t} |x_\tau - \varphi| + M\tau$$

for all $t \geq 0$, $\tau \geq 0$.

Proof: Since $\gamma^+(\varphi)$ is bounded and f takes bounded sets into bounded sets, there is a constant N such that $|f(x_t(\varphi))| \leq N$, $t \geq 0$. Since $D(x_{t+\tau}-x_t) = \int_t^{t+\tau} f(x_s)ds$ for all t, $\tau \geq 0$, the result now follows immediately from Lemma 1.

Theorem 1. If $f: C \to E^n$ is continuous, takes bounded sets of C into bounded sets of E^n, D is stable, the solutions of (3) depend continuously on initial data, and $\gamma^+(\varphi)$ is bounded, then the ω-limit set $\omega(\varphi)$ of φ consists of equilipschitz functions; that is, there is a constant $k = k(\varphi)$ such that for any $\psi \in \omega(\varphi)$, we have $|\psi(\theta_1) - \psi(\theta_2)| \leq k|\theta_1 - \theta_2|$ for all $\theta_1,\theta_2 \in [-r,0]$.

Proof: If $\psi \in \omega(\varphi)$, then there exists a sequence of real $t_k \to \infty$ such that $x_{t_k}(\varphi) \to \psi$ as $k \to \infty$. But $x_{t_k+\tau}(\varphi) \to x_\tau(\psi)$ as $k \to \infty$ for every $\tau \geq 0$. Thus, using Lemma 2 for $t = t_k$, we obtain

$$|x_{t_k+\tau}(\varphi) - x_{t_k}(\varphi)| \leq Me^{-\alpha t_k} |x_\tau(\varphi) - \varphi| + M\tau \quad .$$

Taking the limit as $k \to \infty$, it follows that $|x_\tau(\psi) - \psi| \leq M\tau$ for all $\tau \geq 0$. This proves the theorem.

Theorem 1 shows that functions φ which are in the ω-limit set of bounded orbits of (3) must have a derivative almost everywhere and the derivatives are equibounded. It is also shown in [1] that for such φ, there must be a solution x of (3) on $(-\infty, 0]$ with $x_0 = \varphi$. With this remark, an even stronger conclusion for a special case is the following

Theorem 2. Suppose $f: C \to E^n$ is continuous, takes bounded sets of C into bounded sets of E^n, $D\varphi = \varphi(0) - A\varphi(-1)$, $|A| < 1$. Then any solution x of (3) which is defined and bounded on $(-\infty, 0]$ must have a continuous uniformly bounded first derivative.

Proof: If $\tau \geq 0$, $y(t) = x(t+\tau) - x(t)$, $h(t) = \int_t^{t+\tau} f(x_s)ds$, then

$$\begin{aligned} y(t) &= Ay(t-1) + h(t) \\ &= A^2 y(t-2) + h(t) + Ah(t-1) \\ &= --- \\ &= A^N y(t-N) + \sum_{k=0}^{N-1} A^k h(t-k) \quad . \end{aligned}$$

Since f takes bounded sets into bounded sets and $|A| < 1$, the series on the right is absolutely and uniformly convergent on $(-\infty,0]$ and $y(t) = \sum_{k=0}^{\infty} A^k h(t-k)$. Now, one can verify that

$$\lim_{\tau \to 0} \frac{1}{\tau} \sum_{k=0}^{\infty} A^k h(t-k) = \sum_{k=0}^{\infty} A^k f(x_{t-k}) .$$

This shows that the right hand derivative of $x(t)$ exists and is bounded on $(-\infty,0]$ and equal to $\sum_{k=0}^{\infty} A^k f(x_{t-n})$. But, Lemma 2 implies x_t is uniformly continuous on $(-\infty,0]$. Thus, the right hand derivative of x is continuous. Since x is continuous, we have the derivative of x exists and is continuous. This proves the theorem.

It is certainly reasonable to conjecture that the conclusion of Theorem 2 is true under only the hypothesis that D is stable.

[1] Cruz, M. A. and J. K. Hale, Stability of functional differential equations of neutral type. J. Differential Eqns. 7(1970), 334-355.