A NEW METHOD TO COMPUTE LUNISOLAR PERTURBATIONS IN SATELLITE MOTIONS
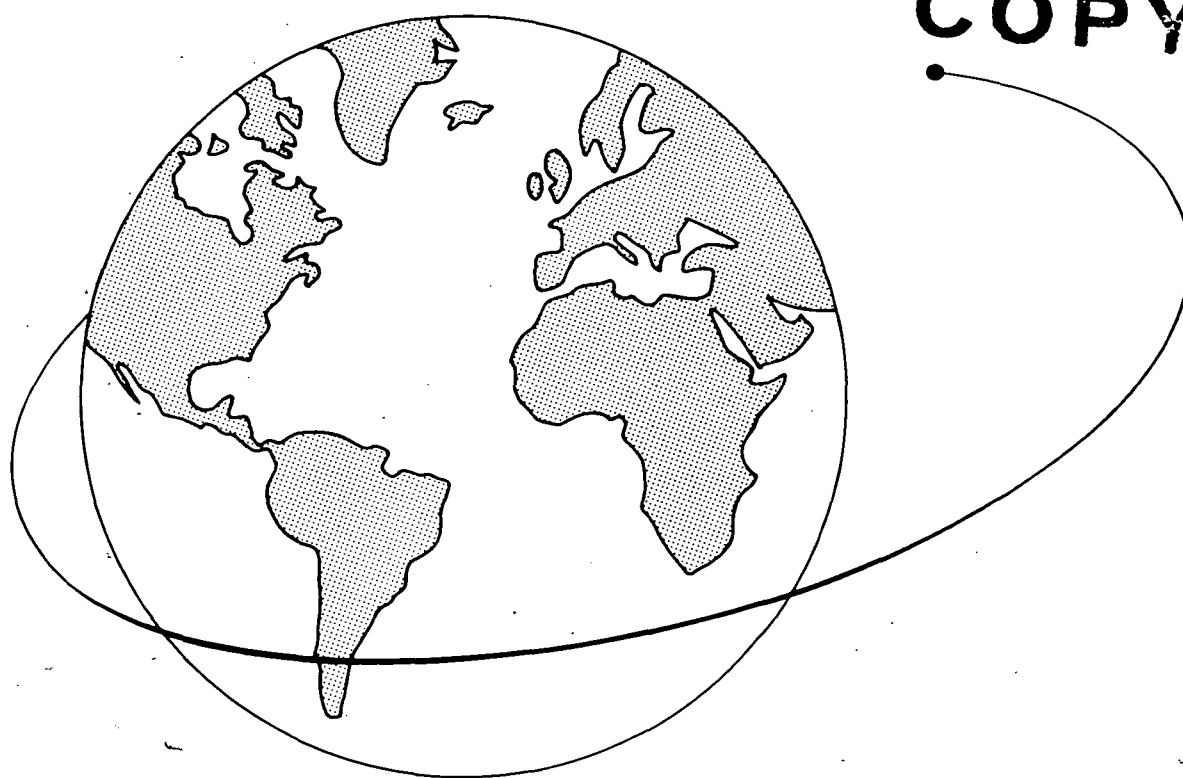
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IN SATELLITE MOTIONS

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ABSTRACT

A new method to compute lunisolar perturbations in satellite motion is proposed. The disturbing function is expressed by the orbital elements of the satellite and the geocentric polar coordinates of the moon and the sun. The secular and long-periodic perturbations are derived by numerical integrations, and the short-periodic perturbations are derived analytically. The perturbations due to the tides can be included in the same way.

In the Appendix, the motion of the orbital plane for a synchronous satellite is discussed; it is concluded that the inclination cannot stay below 7°.

RESUME

On propose une nouvelle méthode de calcul des perturbations luni-solaires dans le mouvement d'un satellite. La fonction perturbante est exprimée en fonction des éléments orbitaux du satellite et des coordonnées polaires géocentriques de la lune et du soleil. On déduit les perturbations séculaires et à longue période par intégrations numériques et les perturbations à courte période par l'analyse. Les perturbations dûes aux marées peuvent être incluses de la même manière.

On considère, dans l'Appendice, le mouvement du plan orbital pour un satellite synchrone; on conclut que l'inclinaison ne peut rester inférieure à 7°.

КОНСПЕКТ

Предлагается новый метод для вычисления лунно-солнечных возмущений. Функция возмущения выражается путем орбитальных элементов спутника и геоцентрических координат луны и солнца. Вековые и долго-периодические возмущения выводятся с помощью численных интегрований, а коротко-периодические выводятся аналитически. Возмущения вызываются приливами и могут быть включены подобным образом.

В приложении обсуждается движение орбитальной плоскости для синхронного спутника; приходится к выводу что наклон не может оставаться ниже 7° .

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Yoshihide Kozai

1. INTRODUCTION

In an earlier paper (Kozai, 1959), the principal secular and long-periodic terms of the disturbing function for the satellite motion due to the lunisolar gravitational attractions were given as a function of the orbital elements of the satellite, the sun, and the moon. The parallactic term in the disturbing function was later included by Musen, Bailie, and Upton (1961).

The disturbing factors are $1.63 \times 10^{-5}/n^2$ and $0.75 \times 10^{-5}/n^2$ for the moon and the sun, respectively; when the mean motion n of the satellite is expressed in revolutions per day, some of the amplitudes for the perturbation terms may become rather large, and they must be computed to 3 or 4 figures even if the satellite is close to the earth.

However, in order not to make the expressions too complicated, it is usually assumed that the orbital elements of the sun and the moon in the disturbing function are constant except for the lunar ascending node and perigee, which are assumed to move as linear functions of time.

It is a good assumption for computing the solar perturbations. However, that is not the case for the lunar perturbations, because it is known that the lunar eccentricity and the lunar inclination to the ecliptic change between 0.045 and 0.065 and between 5° 0!5 and 5° 17', respectively, and there are some large inequalities with amplitudes as large as a few tenths of a degree in the lunar mean longitude.

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Therefore, many celestial mechanicians conclude that they must depend on numerical integration methods if they need to compute precisely the lunisolar perturbations, particularly when the satellite is high.

In this paper, I propose a new method to compute the lunisolar perturbations. The disturbing function is expressed as a function of the orbital elements of the satellite and the polar coordinates of the sun and the moon; the latter can be found in any astronomical almanac. From the disturbing function of this form, the short-periodic terms can be eliminated analytically by taking their average with respect to the mean anomaly of the satellite.

Then the equations of variations of the orbital elements must be integrated numerically to derive long-periodic perturbations. However, since the short-periodic terms have been eliminated, a half-day or even a day can be adopted as a step of the integration.

This method can be applied to compute the lunisolar perturbations for high satellites such as a synchronous one, and the short-periodic perturbations can be derived analytically.

The perturbations due to the tidal deformations of the earth can be included in the same way; their expressions are also given.

2. DISTURBING FUNCTION

The expression of the disturbing function due to the lunisolar gravitational attractions for the satellite motion can be derived in exactly the same way as that used to derive the disturbing function for the lunar motion due to the sun.

The equations of motion for the satellite under the attractions of the spherical earth and the spherical moon are written as

$$\frac{d^2\vec{r}}{dt^2} = G\frac{\partial}{\partial\vec{r}}\left[\frac{m}{r} + m'\frac{r^2}{r'^3}P_2(\cos S) + m'\frac{r^3}{r'^4}P_3(\cos S) + \cdots\right] , \qquad (1)$$

where $\vec{r}$ and r are the geocentric radius vector and distance of the satellite, r' is the geocentric distance of the moon, m and m' are the masses of the earth and the moon, G is the gravitational constant, $\cos S$ is the direction cosine between the radius to the satellite and that to the moon, and P_2, P_3, $\cdots$ are Legendre polynomials.

The first term in parentheses in (1) expresses the potential of the spherical earth; when the other terms are neglected, the equation becomes that for the two-body problem. The second term in parentheses gives the principal part of the disturbing function.

The disturbing factor, which is the ratio of the second term to the first, is expressed as

$$\frac{m'}{m}\left(\frac{r}{r'}\right)^3 \approx \frac{m'}{m}\left(\frac{a}{a'}\right)^3 , \qquad (2)$$

where a and a' are the semimajor axes of the satellite and the moon, respectively.

By use of Kepler's third law,

$$n^2 a^3 = Gm \quad ,$$

$$n'^2 a'^3 = G(m + m') \quad , \tag{3}$$

expression (2) becomes

$$\frac{m'}{m + m'} \left(\frac{n'}{n}\right)^2 \quad . \tag{4}$$

The ratio of the third term to the second is roughly equal to a/a' and is not a very small quantity.

The disturbing function due to the sun and the disturbing factor for the sun are derived in the same way. However, as Kepler's third law for the sun is written as

$$n'^2 a'^3 = Gm' \quad , \tag{5}$$

the disturbing factor becomes

$$(n'/n)^2 \quad . \tag{6}$$

And the ratio of the third term to the second is extremely small.

When Gm is replaced by $n^2 a^3$, the disturbing function can be written as

$$R = n'^2 \beta r^2 \left(\frac{a'}{r'}\right)^3 \left[P_2 (\cos S) + \frac{r}{r'} P_3 (\cos S) + \left(\frac{r}{r'}\right)^2 P_4 (\cos S) + \cdots \right] \quad , \tag{7}$$

where

$$\beta = \frac{m'}{m + m'} \qquad \text{for the moon} \quad ,$$

$$\beta = 1 \qquad \text{for the sun} \quad . \tag{8}$$

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The geocentric rectangular coordinates of the moon or the sun are expressed with the geocentric distance r', right ascension α, and declination δ as

$$x' = r' \cos \delta \cos \alpha \quad ,$$

$$y' = r' \cos \delta \sin \alpha \quad ,$$

$$z' = r' \sin \delta \quad , \tag{9}$$

whereas the coordinates of the satellite are expressed with the geocentric distance r, the argument of latitude L, the longitude of the ascending node Ω, and the orbital inclination i as

$$x = r \cos L \cos \Omega - r \cos i \sin L \sin \Omega \quad ,$$

$$y = r \cos L \sin \Omega + r \cos i \sin L \cos \Omega \quad ,$$

$$z = r \sin L \sin i \quad . \tag{10}$$

Then the direction cosine is expressed as

$$\cos S = (xx' + yy' + zz')/rr'$$

$$= \cos \delta \cos (\Omega - \alpha) \cos L$$

$$+ [- \cos \delta \cos i \sin (\Omega - \alpha) + \sin \delta \sin i] \sin L$$

$$= A \cos L + B \sin L \quad , \tag{11}$$

where

$$A = \cos \delta \cos (\Omega - \alpha) \quad ,$$

$$B = - \cos \delta \cos i \sin (\Omega - \alpha) + \sin \delta \sin i \quad . \tag{12}$$

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By use of the last expression for cos S in (11), the following formulas are derived:

$$\cos^2 S = \frac{A^2 + B^2}{2} + \frac{A^2 - B^2}{2} \cos 2L + AB \sin 2L \quad ,$$

$$\cos^3 S = \frac{3}{4}(A^2 + B^2)(A \cos L + B \sin L)$$

$$+ \frac{A^2 - 3B^2}{4} A \cos 3L + \frac{3A^2 - B^2}{4} B \sin 3L \quad ,$$

$$\cos^4 S = \frac{3}{8}(A^2 + B^2)^2 + \frac{1}{2}(A^2 + B^2)[(A^2 - B^2) \cos 2L + 2AB \sin 2L]$$

$$+ \frac{A^4 - 6A^2 B^2 + B^4}{8} \cos 4L + \frac{A^2 - B^2}{2} AB \sin 4L \quad ,$$

$$\cos^5 S = \frac{5}{8}(A^2 + B^2)(A \cos L + B \sin L) \quad . \tag{13}$$

In the expression of $\cos^5 S$, terms of $\frac{\sin}{\cos} 3L$ and $\frac{\sin}{\cos} 5L$ are neglected.

Then the expressions for Legendre polynomials are derived as

$$P_2(\cos S) = \frac{1}{4}[3(A^2 + B^2) - 2] + \frac{3}{4}[(A^2 - B^2) \cos 2L + 2AB \sin 2L] \quad ,$$

$$P_3(\cos S) = \frac{3}{8}[5(A^2 + B^2) - 4](A \cos L + B \sin L)$$

$$+ \frac{5}{8}[(A^2 - 3B^2) A \cos 3L + (3A^2 - B^2) B \sin 3L] \quad ,$$

$$P_4(\cos S) = \frac{3}{64}[35(A^2 + B^2)^2 - 40(A^2 + B^2) + 8]$$

$$+ \frac{5}{16}[7(A^2 + B^2) - 6][(A^2 - B^2) \cos 2L + 2AB \sin 2L]$$

$$+ \frac{35}{64}[(A^4 - 6A^2 B^2 + B^4) \cos 4L + 4(A^2 - B^2) AB \sin 4L] \quad ,$$

$$P_5(\cos S) = \frac{15}{64}[21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8](A \cos L + B \sin L) \quad . \tag{14}$$

Here also, terms of $\frac{\sin}{\cos} 3L$ and $\frac{\sin}{\cos} 5L$ are neglected in the expression of $P_5(\cos S)$.

Now the disturbing function is expressed by the orbital elements of the satellite and the polar coordinates of the moon and the sun.

3. SECULAR AND LONG-PERIODIC TERMS

Secular and long-periodic terms of the disturbing function are derived by taking their average with respect to the mean anomaly ℓ by the following formula:

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^n \cos mf \cdot d\ell = (-1)^m \frac{(n+2)(n+3)(n+m+1)}{m!} \left(\frac{e}{2}\right)^m$$

$$F\left(\frac{m-n}{2}, \frac{m-n-1}{2}, m+1, e^2\right) , \qquad (15)$$

where f and e are the true anomaly and the eccentricity, respectively, and F is the hypergeometric function.

If $(a/a')^2 e^4$, $(a/a')^3 e^3$, $(a/a')^3 e^5$, and $(a/a')^4$ are neglected, the secular and long-periodic terms of the disturbing function are written as

$$R_s \bigg/ \left[n'^2 a^2 \left(\frac{a'}{r'}\right)^3 \beta\right] = \frac{1}{8} [3(A^2 + B^2) - 2] (2 + 3e^2)$$

$$+ \frac{15}{8} e^2 [(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]$$

$$- \frac{5}{64} e\left(\frac{a'}{r'}\right)\left(\frac{a}{a'}\right) \{3(4 + 3e^2)[5(A^2 + B^2) - 4] (A \cos \omega + B \sin \omega)$$

$$+ 35e^2 [(A^2 - 3B^2) A \cos 3\omega + (3A^2 - B^2) B \sin 3\omega]\}$$

$$+ \frac{3}{64}\left(\frac{a'}{r'}\right)^2 \left(\frac{a}{a'}\right)^2 \{[35(A^2 + B^2)^2 - 40(A^2 + B^2) + 8] (1 + 5e^2)$$

$$+ 35e^2 [7(A^2 + B^2) - 6] [(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]\}$$

$$- \frac{105}{128} e\left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 [21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8] (A \cos \omega + B \sin \omega) ,$$

$$(16)$$

where ω is the argument of perigee for the satellite.

By use of Lagrange's equations,

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \ell} \quad ,$$

$$\frac{de}{dt} = \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial \ell} - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial \omega} \quad ,$$

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \Omega} \quad ,$$

$$\frac{d\omega}{dt} = - \frac{\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial e} \quad ,$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} \quad ,$$

$$\frac{d\ell}{dt} = n - \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} \quad , \tag{17}$$

the following equations are derived:

$$\frac{da}{dt} = 0 \quad , \tag{18}$$

$$\frac{de}{dt} \bigg/ \left[\frac{n'^2}{n} a^2 (a'/r')^3 \beta \sqrt{1 - e^2} \right]$$

$$= \frac{15}{4} e[(A^2 - B^2) \sin 2\omega - 2AB \cos 2\omega]$$

$$- \frac{5}{64} \left(\frac{a'}{r'}\right) \left(\frac{a}{a'}\right) \{ 3(4 + 3e^2)[5(A^2 + B^2) - 4] (A \sin \omega - B \cos \omega)$$

$$+ 105e^2[(A^2 - 3B^2) A \sin 3\omega - (3A^2 - B^2) B \cos 3\omega]\}$$

$$+ \frac{105}{32} \left(\frac{a'}{r'}\right)^2 \left(\frac{a}{a'}\right)^2 e[7(A^2 + B^2) - 6][(A^2 - B^2)\sin 2\omega - 2AB\cos 2\omega]$$

$$- \frac{105}{128} \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 [21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8](A\sin\omega - B\cos\omega) \ ,$$

<div align="right">(19)</div>

$$\left(\frac{di}{dt} + \frac{e}{1 - e^2}\cot i\, \frac{de}{dt}\right) \Big/ \left[-\frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3 \beta\, \frac{1}{\sqrt{1 - e^2}}\, \frac{1}{\sin i}\right]$$

$$= -\frac{3}{4}\left(A\frac{\partial A}{\partial \Omega} + B\frac{\partial B}{\partial \Omega}\right)(2 + 3e^2)$$

$$- \frac{15}{4} e^2 \left[\left(A\frac{\partial A}{\partial \Omega} - B\frac{\partial B}{\partial \Omega}\right)\cos 2\omega + \left(A\frac{\partial B}{\partial \Omega} + B\frac{\partial A}{\partial \Omega}\right)\sin 2\omega\right]$$

$$+ \frac{5}{64} e \left(\frac{a'}{r'}\right)\left(\frac{a}{a'}\right)\left(3(4 + 3e^2)\left\{\left[(15A^2 + 5B^2 - 4)\frac{\partial A}{\partial \Omega} + 10AB\frac{\partial B}{\partial \Omega}\right]\cos\omega\right.\right.$$

$$\left. + \left[(5A^2 + 15B^2 - 4)\frac{\partial B}{\partial \Omega} + 10AB\frac{\partial A}{\partial \Omega}\right]\sin\omega\right\}$$

$$+ 105e^2\left\{\left[(A^2 - B^2)\frac{\partial A}{\partial \Omega} - 2AB\frac{\partial B}{\partial \Omega}\right]\cos 3\omega\right.$$

$$\left.\left. + \left[(A^2 - B^2)\frac{\partial B}{\partial \Omega} + 2AB\frac{\partial A}{\partial \Omega}\right]\sin 3\omega\right\}\right)$$

$$- \frac{15}{32}\left(\frac{a'}{r'}\right)^2\left(\frac{a}{a'}\right)^2\left(2[7(A^2 + B^2) - 4]\left(A\frac{\partial A}{\partial \Omega} + B\frac{\partial B}{\partial \Omega}\right)(1 + 5e^2)\right.$$

$$+ 17e^2\left\{2\left[7\left(A^3\frac{\partial A}{\partial \Omega} - B^3\frac{\partial B}{\partial \Omega}\right) - 3\left(A\frac{\partial A}{\partial \Omega} - B\frac{\partial B}{\partial \Omega}\right)\right]\cos 2\omega\right.$$

$$+ \left[7\left(3A^2 B\frac{\partial A}{\partial \Omega} + A^3\frac{\partial B}{\partial \Omega} + 3AB^2\frac{\partial B}{\partial \Omega} + B^3\frac{\partial A}{\partial \Omega}\right)\right.$$

$$\left.\left.\left. - 6\left(A\frac{\partial B}{\partial \Omega} + B\frac{\partial A}{\partial \Omega}\right)\right]\sin 2\omega\right\}\right)$$

$$+ \frac{105}{128} e \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 \Bigg\{ 28[3(A^2 + B^2) - 2] \left(A \frac{\partial A}{\partial \Omega} + B \frac{\partial B}{\partial \Omega}\right)(A \cos \omega + B \sin \omega)$$

$$+ [21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8] \left(\frac{\partial A}{\partial \Omega} \cos \omega + \frac{\partial B}{\partial \Omega} \sin \omega\right)\Bigg\} \quad , \tag{20}$$

$$\frac{d\Omega}{dt} \Bigg/ \left[\frac{n'^2}{n} \left(\frac{a'}{r'}\right)^3 \beta \frac{1}{\sqrt{1 - e^2}} \cdot \frac{1}{\sin i} \frac{\partial B}{\partial i}\right]$$

$$= \frac{3}{4} B(2 + 3e^2) - \frac{15}{4} e^2 (B \cos 2\omega - A \sin 2\omega)$$

$$- \frac{15}{64} e \left(\frac{a'}{r'}\right) \left(\frac{a}{a'}\right) \{(4 + 3e^2)[10AB \cos \omega + (5A^2 + 15B^2 - 4) \sin \omega]$$

$$- 35e^2[2AB \cos 3\omega - (A^2 - B^2) \sin 3\omega]\}$$

$$+ \frac{15}{32} \left(\frac{a'}{r'}\right)^2 \left(\frac{a}{a'}\right)^2 \{2B[7(A^2 + B^2) - 4] (1 + 5e^2)$$

$$- 7e^2[2B(7B^2 - 3) \cos 2\omega - (7A^2 + 21B^2 - 6) A \sin 2\omega]\}$$

$$- \frac{105}{128} e \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 \{28AB[3(A^2 + B^2) - 2] \cos 2\omega$$

$$+ (21A^4 + 126A^2B^2 + 105B^4 - 28A^2 - 84B^2 + 8) \sin 2\omega\} \quad , \tag{21}$$

$$\left(\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt}\right) \Bigg/ \left[\frac{n'^2}{n} \left(\frac{a'}{r'}\right)^3 \beta \sqrt{1 - e^2}\right]$$

$$= \frac{3}{4} [3(A^2 + B^2) - 2] + \frac{15}{4} [(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]$$

$$- \frac{15}{64} \left(\frac{a'}{r'}\right) \left(\frac{a}{a'}\right) \frac{1}{e} \{(4 + 9e^2)[5(A^2 + B^2) - 4] (A \cos \omega + B \sin \omega)$$

$$+ 35e^2[(A^2 - 3B^2) A \cos 3\omega + (3A^2 - B^2) B \sin 3\omega]\}$$

$$+ \frac{15}{32} \left(\frac{a'}{r'}\right)^2 \left(\frac{a}{a'}\right)^2 \{35(A^2 + B^2)^2 - 40(A^2 + B^2) + 8$$

$$+ 7[7(A^2 + B^2) - 6][(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]\}$$

$$- \frac{105}{128} \frac{1}{e} \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 [21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8] (A \cos \omega + B \sin \omega) \, ,$$

$$(22)$$

$$\frac{d\ell}{dt} \Big/ \left[\frac{n'^2}{n} \left(\frac{a'}{r'}\right)^3 \beta\right]$$

$$= - \frac{1}{4} (7 + 3e^2)[3(A^2 + B^2) - 2]$$

$$- \frac{15}{4} (1 + e^2)[(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]$$

$$+ \frac{15}{64} \left(\frac{a'}{r'}\right) \left(\frac{a}{a'}\right) \frac{1}{e} \{(4 + 29e^2)[5(A^2 + B^2) - 4] (A \cos \omega + B \sin \omega)$$

$$+ 35e^2 [(A^2 - 3B^2) A \cos 3\omega + (3A^2 - B^2) B \sin 3\omega]\}$$

$$- \frac{3}{32} \left(\frac{a'}{r'}\right)^2 \left(\frac{a}{a'}\right)^2 \{3(3 + 5e^2)[35(A^2 + B^2)^2 - 40(A^2 + B^2) + 8]$$

$$+ 35e^2 [7(A^2 + B^2) - 6] [(A^2 - B^2) \cos 2\omega + 2AB \sin 2\omega]\}$$

$$+ \frac{105}{128} \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^3 \frac{1}{e} [21(A^2 + B^2)^2 - 28(A^2 + B^2) + 8] (A \cos \omega + B \sin \omega) \, ,$$

$$(23)$$

where

$$\frac{\partial A}{\partial \Omega} = - \cos \delta \sin (\Omega - \alpha) \quad ,$$

$$\frac{\partial B}{\partial \Omega} = - \cos \delta \cos i \cos (\Omega - \alpha) \quad ,$$

$$\frac{\partial R}{\partial t} = \cos \delta \sin i \sin (\Omega - \alpha) + \sin \delta \cos i \quad . \tag{24}$$

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These equations must be integrated numerically together with other terms, particularly the secular terms due to J_2, because the secondary effects in the secular motions due to the changes of the orbital elements caused by the lunisolar perturbations may become as large as the direct effects.

For close satellites, it is not necessary to include all the terms in equations (19) through (23). In fact, since Gm has been replaced by $n^2 a^3$, the expressions on the right-hand side of the equations are accurate up to J_2/a^2.

However, for high satellites such as a synchronous one, for which a/a' is 0.11 for the moon, and J_2/a^2 is 2.5×10^{-5}, all the terms must be included. For this case, a more accurate expression for Kepler's third law for the moon,

$$1.002723 \, n'^2 a'^3 = G(m + m') \quad , \tag{25}$$

must be used. If the value 1/81.302 is adopted for m'/m, β for the moon takes the following value:

$$\beta = 0.0121835 \quad . \tag{26}$$

4. EXPRESSIONS FOR SMALL ECCENTRICITY AND INCLINATION

As most of the existant and planned synchronous satellites have orbits that are nearly circular and nearly equatorial, expressions (19) through (23) are simplified by assuming that e and i are very small, although the inclination cannot remain very small, as explained in the Appendix.

When $\sin^3 i$, $e^2 \sin^2 i$, $(a/a')\, e \sin^2 i$, $(a/a')\, e^3$, $(a/a')^2 \sin^2 i$, $(a/a')^2 e^2$, $(a/a')^3 e^3$, and $(a/a')^3 e \sin i$ are neglected, expression (16) becomes

$$
R_s = n'^2 a^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ \frac{1}{16} \left[3 \cos^2 \delta \left(2 - \sin^2 i\right) + 6 \sin^2 i \sin^2 \delta - 4 \right. \right.
$$

$$
\left. - 6 \sin i \sin 2\delta \sin(\Omega - \alpha) + 3 \sin^2 i \cos^2 \delta \cos 2(\Omega - \alpha)\right] \cdot \left(2 + 3e^2\right)
$$

$$
+ \frac{15}{8} e^2 \left[\cos^2 \delta \cos 2(\omega + \Omega - \alpha) + \sin i \sin 2\delta \sin(2\omega + \Omega - \alpha)\right]
$$

$$
- \frac{15}{32} \left(\frac{a'}{r'}\right)\left(\frac{a}{a'}\right) e \left[2 \cos \delta \left(5 \cos^2 \delta - 4\right) \cos(\omega + \Omega - \alpha)\right.
$$

$$
\left. + 4 \sin i \sin \delta \left(5 \cos^2 \delta - 2\right) \sin \omega - 5 \sin i \sin 2\delta \cos \delta \sin(\omega + 2\Omega - 2\alpha)\right]
$$

$$
+ \frac{3}{64} \left(\frac{a}{a'}\right)^2 \left(\frac{a'}{r'}\right)^2 \left[35 \cos^4 \delta - 40 \cos^2 \delta + 8 - 10 \sin i \sin 2\delta \left(7 \cos^2 \delta - 4\right)\right.
$$

$$
\left. \sin(\Omega - \alpha)\right]
$$

$$
- \frac{105}{128} e \left(\frac{a}{a'}\right)^3 \left(\frac{a'}{r'}\right)^3 \left(21 \cos^4 \delta - 28 \cos^2 \delta + 8\right) \cos \delta \cos(\omega + \Omega - \alpha) \left.\vphantom{\frac{1}{16}}\right\} .
$$

$$
(27)
$$

Then the equations of variations of the orbital elements are derived with sufficient accuracy as follows:

$$\frac{de}{dt} = \frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3 \beta \sqrt{1-e^2} \left\{ \frac{15}{4} \, e\,[\cos^2\delta \, \sin 2\,(\omega + \Omega - \alpha) \right.$$

$$- \sin i \, \sin 2\delta \, \cos\,(2\Omega + \Omega - \alpha)]$$

$$- \frac{15}{32}\left(\frac{a}{a'}\right)\left(\frac{a'}{r'}\right)\,[2 \cos\delta \,(5\cos^2\delta - 4)\,\sin\,(\omega + \Omega - \alpha)$$

$$- 4 \sin i \, \sin \delta \,(5\cos^2\delta - 2)\cos\omega + 5 \sin i \, \sin 2\delta \, \cos\delta \, \cos\,(\omega + 2\Omega - 2\alpha)]$$

$$\left. - \frac{105}{128}\left(\frac{a}{a'}\right)^3\left(\frac{a'}{r'}\right)^3 (21\cos^4\delta - 28\cos^2\delta + 8)\cos\delta \, \sin\,(\omega + \Omega - \alpha) \right\} \quad ,$$

$$(28)$$

$$\frac{di}{dt} = \frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3 \beta \frac{1}{\sqrt{1-e^2}} \left\{ \frac{3}{4}\,[\sin 2\delta \, \cos\,(\Omega - \alpha) + \sin i \, \cos^2\delta \, \sin 2\,(\Omega - \alpha)] \right.$$

$$\left. + \frac{15}{32}\left(\frac{a}{a'}\right)^2\left(\frac{a'}{r'}\right)^2 \sin 2\delta \,(7\cos^2\delta - 4)\cos\,(\Omega - \alpha) \right\} \quad , \qquad (29)$$

$$\frac{d\Omega}{dt} = \frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3 \beta \frac{1}{\sqrt{1-e^2}} \left\{ \frac{3}{4}\,[-\cos i \, \cos^2\delta + 2\cos i \, \sin^2\delta \right.$$

$$- \cot i \, \sin 2\delta \, \sin\,(\Omega - \alpha) + \cos i \, \cos^2\delta \, \cos 2\,(\Omega - \alpha)]$$

$$\left. - \frac{15}{32}\left(\frac{a}{a'}\right)^2\left(\frac{a'}{r'}\right)^2 \cot i \cdot \sin 2\delta \,(7\cos^2\delta - 4)\sin\,(\Omega - \alpha) \right\} \quad , \qquad (30)$$

$$\frac{d\omega}{dt} = -\frac{d\Omega}{dt}\cos i$$

$$+ \frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3 \beta \sqrt{1-e^2} \left\{ \frac{3}{4}\,[3\cos^2\delta - 2 - 3\sin i \, \sin 2\delta \, \sin\,(\Omega - \alpha)] \right.$$

$$+ \frac{15}{4}\,[\cos^2\delta \, \cos 2\,(\omega + \Omega - \alpha) + \sin i \, \sin 2\delta \, \sin\,(2\omega + \Omega - \alpha)]$$

$$-\frac{15}{32}\left(\frac{a'}{r'}\right)\left(\frac{a}{a'}\right)\frac{1}{e}\,[2\cos\delta\,(5\cos^2\delta-4)\cos(\omega+\Omega-\alpha)$$

$$+4\sin i\,\sin\delta\,(5\cos^2\delta-2)\sin\omega-5\sin i\,\sin 2\delta\,\cos\delta\,\sin(\omega+2\Omega-2\alpha)]$$

$$-\frac{105}{128}\left(\frac{a'}{r'}\right)^3\left(\frac{a}{a'}\right)^3\frac{1}{e}\,[(21\cos^4\delta-28\cos^2\delta+8)\cos\delta\,\cos(\omega+\Omega-\alpha)]\bigg\}\quad,$$

$$(31)$$

$$\frac{d\ell}{dt}=\frac{n'^2}{n}\left(\frac{a'}{r'}\right)^3\beta\,\bigg\{-\frac{7}{4}\,[3\cos^2\delta-2-3\sin i\,\sin 2\delta\,\sin(\Omega-\alpha)]$$

$$-\frac{15}{4}\,[\cos^2\delta\,\cos 2(\omega+\Omega-\alpha)+\sin i\,\sin 2\delta\,\sin(2\omega+\Omega-\alpha)]$$

$$+\frac{15}{32}\frac{1}{e}\left(\frac{a}{a'}\right)\left(\frac{a'}{r'}\right)[2\cos\delta\,(5\cos^2\delta-4)\cos(\omega+\Omega-\alpha)$$

$$+4\sin i\,\sin\delta\,(5\cos^2\delta-2)\sin\omega-5\sin i\,\sin 2\delta\,\cos\delta\,\sin(\omega+2\Omega-2\alpha)]$$

$$-\frac{3}{8}\left(\frac{a}{a'}\right)^2\left(\frac{a'}{r'}\right)^2(35\cos^4\delta-40\cos^2\delta+8)$$

$$+\frac{105}{128}\left(\frac{a}{a'}\right)^3\left(\frac{a'}{r'}\right)^3\frac{1}{e}\,(21\cos^4\delta-28\cos^2\delta+8)\cos\delta\,\cos(\omega+\Omega-\alpha)\bigg\}\quad.$$

$$(32)$$

5. SHORT-PERIODIC PERTURBATIONS

For some satellites, especially high ones, it is necessary to include, for precise orbit computations, short-periodic perturbations due to the lunisolar attractions.

If only P_2 (cos S) is included, the short-periodic term of the disturbing function is written as

$$
R_p = n'^2 a^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ \frac{1}{4} [3(A^2 + B^2) - 2] \left[\left(\frac{r}{a}\right)^2 - 1 - \frac{3}{2} e^2 \right] \right.
$$
$$
+ \frac{3}{4} (A^2 - B^2) \left[\left(\frac{r}{a}\right)^2 \cos 2L - \frac{5}{2} e^2 \cos 2\omega \right]
$$
$$
\left. + \frac{3}{2} AB \left[\left(\frac{r}{a}\right)^2 \sin 2L - \frac{5}{2} e^2 \sin 2\omega \right] \right\} \; . \tag{33}
$$

When the relations

$$
\left(\frac{r}{a}\right)^2 = 1 + \frac{3}{2} e^2 - 2e \cos \ell - \frac{e^2}{2} \cos 2\ell + \cdots \; ,
$$

$$
\left(\frac{r}{a}\right)^2 \cos 2f = \frac{5}{2} e^2 - 3e \cos \ell + \left(1 - \frac{5}{2} e^2\right) \cos 2\ell + e \cos 3\ell + e^2 \cos 4\ell + \cdots \; ,
$$

$$
\left(\frac{r}{a}\right)^2 \sin 2f = -3e \sin \ell + \left(1 - \frac{5}{2} e^2\right) \sin 2\ell + e \sin 3\ell + e^2 \sin 4\ell + \cdots \; ,
$$

$$\tag{34}$$

are used and when e^2 is neglected, the expression of R_p becomes

$$R_p = n'^2 a^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ e \left[1 - \frac{3}{2} (A^2 + B^2) \right] \cos \ell \right.$$

$$+ \frac{3}{4} (A^2 - B^2) \left[\cos 2 (\ell + \omega) - 3e \cos (\ell + 2\omega) + e \cos (3\ell + 2\omega) \right]$$

$$\left. + \frac{3}{2} AB \left[\sin 2 (\ell + \omega) - 3e \sin (\ell + 2\omega) + e \sin (3\ell + 2\omega) \right] \right\} \quad . \tag{35}$$

Then Lagrange's equations are easily derived. When the right-hand sides of the equations are integrated, it is assumed that all the variables except the mean anomaly ℓ of the satellite are constant. By this approximation, n'/n is neglected in the results. Also, terms with e as factors are neglected.

The results are as follows:

$$\frac{da}{a} = \frac{3}{2} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left[(A^2 - B^2) \cos 2 (\ell + \omega) + 2AB \sin 2 (\ell + \omega) \right] \quad , \tag{36}$$

$$de = \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ \left[1 - \frac{3}{2} (A^2 + B^2) \right] \cos \ell \right.$$

$$+ \frac{1}{4} (A^2 - B^2) \left[9 \cos (\ell + 2\omega) + \cos (3\ell + 2\omega) \right]$$

$$\left. + \frac{1}{2} AB \left[9 \sin (\ell + 2\omega) + \sin (3\ell + 2\omega) \right] \right\} \quad , \tag{37}$$

$$ed\ell = \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ - \left[1 - \frac{3}{2} (A^2 + B^2) \right] \sin \ell \right.$$

$$+ \frac{1}{4} (A^2 - B^2) \left[9 \sin (\ell + 2\omega) - \sin (3\ell + 2\omega) \right]$$

$$\left. - \frac{1}{2} AB \left[9 \cos (\ell + 2\omega) - \cos (3\ell + 2\omega) \right] \right\} \quad , \tag{38}$$

$$d(\ell + \omega) = \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ -\frac{21}{8} \left[(A^2 - B^2) \sin 2 (\ell + \omega) - 2AB \cos 2 (\ell + \omega)\right] \right.$$

$$\left. + \frac{3}{4} \cot i \frac{\partial B}{\partial i} \left[B \sin 2 (\ell + \omega) + A \cos 2 (\ell + \omega)\right] \right\} \quad , \tag{39}$$

$$d\Omega = -\frac{3}{4} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \frac{1}{\sin i} \frac{\partial B}{\partial i} \left[B \sin 2 (\ell + \omega) + A \cos 2 (\ell + \omega)\right] \quad , \tag{40}$$

$$di = \frac{3}{4} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \frac{1}{\sin i} \left\{ \cos i \left[(A^2 - B^2) \cos 2 (\ell + \omega) + 2AB \sin 2 (\ell + \omega)\right] \right.$$

$$\left. - \left(A \frac{\partial A}{\partial \Omega} - B \frac{\partial B}{\partial \Omega}\right) \sin 2 (\ell + \omega) + \left(A \frac{\partial B}{\partial \Omega} + B \frac{\partial A}{\partial \Omega}\right) \cos 2 (\ell + \omega) \right\} \quad . $$

$$\tag{41}$$

To derive expression (39) for $d(\ell + \omega)$, the fact must be taken into account that the mean motion changes by $-\frac{3}{2} n \frac{da}{a}$, which is computed from (36).

Expressions (36) through (39) can be combined to express the perturbations in the radius dr, in the argument of latitude dL, in the radial component of the velocity $\dot{r}$, and in the time derivative of the true anomaly $\dot{f}$, by the following relations:

$$dr = \frac{r}{a} da - a \cos f \, de + \frac{ae \sin f}{\sqrt{1 - e^2}} d\ell \quad ,$$

$$dL = d\omega + \left(\frac{a}{r} + \frac{1}{1 - e^2}\right) \sin f \cdot de + \frac{a^2 \sqrt{1 - e^2}}{r^2} d\ell \quad ,$$

$$d\dot{f} = -\frac{3}{2} \frac{n}{a} \left(\frac{a}{r}\right)^2 \sqrt{1 - e^2} \, da - 2ne \sin f \left(\frac{a}{r}\right)^3 d\ell$$

$$+ n \left(\frac{a}{r}\right)^2 \frac{1}{\sqrt{1 - e^2}} (2 \cos f + e \cos 2f) \, de \quad ,$$

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$$\dot{r} = -\frac{1}{2} \frac{ne}{\sqrt{1 - e^2}} \sin f \, da + ane \left(\frac{a}{r}\right)^2 \cos f \cdot d\ell$$

$$+ \frac{an}{\sqrt{1 - e^2}} \sin f \left[e\left(1 + \frac{r}{p}\right) \frac{a}{r} \cos f + \frac{1}{1 - e^2} \right] de \quad . \tag{42}$$

If the eccentricity is neglected, the results are expressed as follows:

$$\frac{dr}{a} = -\left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left[1 - \frac{3}{2}(A^2 + B^2) + (A^2 - B^2) \cos 2\,(\ell + \omega) \right.$$

$$\left. + 2AB \sin 2\,(\ell + \omega) \right] \quad , \tag{43}$$

$$dL = \frac{1}{8} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left\{ 11(A^2 - B^2) \sin 2\,(\ell + \omega) - 22AB \cos 2\,(\ell + \omega) \right.$$

$$\left. + 6 \cot i \frac{\partial B}{\partial i} \, [B \sin 2\,(\ell + \omega) + A \cos 2\,(\ell + \omega] \right\} \quad , \tag{44}$$

$$\dot{r} = 2na \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \, [(A^2 - B^2) \sin 2\,(\ell + \omega) - 2AB \cos 2\,(\ell + \omega)] \quad ,$$

$$\tag{45}$$

$$\dot{f} = \frac{1}{4} n \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \{ 4[2 - 3\,(A^2 + B^2)]$$

$$+ 11\,(A^2 - B^2) \cos 2\,(\ell + \omega) + 22AB \sin 2\,(\ell + \omega) \} \quad . \tag{46}$$

If the inclination is also small, these results can be written as

$$\frac{dr}{a} = -\left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \left[1 - \frac{3}{2} \cos^2 \delta + \cos^2 \delta \cos 2\,(L + \Omega - \alpha) \right] \quad , \tag{47}$$

$$dL = \frac{1}{8} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \, [11 \cos^2 \delta \sin 2 \, (L + \Omega - \alpha)$$

$$+ \, 3 \cot i \sin 2\delta \cos 2 \, (L + \Omega - \alpha)] \quad , \tag{48}$$

$$d\dot{r} = 2na \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \cos^2 \delta \sin 2 \, (L + \Omega - \alpha) \quad , \tag{49}$$

$$d\dot{f} = \frac{1}{4} n \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \, [4(2 - 3 \cos^2 \delta) + 11 \cos^2 \delta \cos 2 \, (L + \Omega - \alpha)] \quad . \tag{50}$$

Also for the small-inclination case, expressions (40) and (41) are written as

$$d\Omega = -\frac{3}{8} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \, \frac{\sin 2\delta}{\sin i} \cos 2 \, (L + \Omega - \alpha) \quad , \tag{51}$$

$$di = \frac{3}{8} \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{r'}\right)^3 \beta \sin 2\delta \sin (2L + \Omega - \alpha) \quad . \tag{52}$$

6. PERTURBATIONS DUE TO THE TIDAL DEFORMATIONS

Besides the direct lunisolar perturbations, the motion of the satellite is disturbed by the tidal deformations of the earth due to the attractions of the moon and the sun (Kozai, 1965).

Owing to the tidal potential

$$G m' \frac{a_e^2}{r'^3} P_2 \cos S) \quad , \tag{53}$$

where a_e is the earth's radius, the geoid is deformed by

$$k_2 \frac{m'}{m} \frac{a_e^4}{r'^3} P_2 (\cos S) \quad , \tag{54}$$

where k_2 is Love's number. And owing to the friction, the rotation carries the tidal bulge forward; the tide is high not when the moon is directly overhead but at some time delay Δt.

The geopotential is also changed, by

$$\frac{G m' a_e^5}{r^3 r'^3} k_2 P_2 (\cos S) \quad . \tag{55}$$

Therefore, the disturbing function due to the tidal deformation is written as

$$R = n'^2 \beta r^2 \left(\frac{a'}{r'}\right)^3 k_2 P_2 (\cos S) \quad . \tag{56}$$

However, owing to the time lag of the tide, the right ascensions of the sun and the moon should be shifted by $n_\oplus \cdot \Delta t$, where $n_\oplus$ is the rotational angular velocity of the earth.

When this effect is taken into account, cos S can be written as

$$A^* = \cos\delta\,\cos(\Omega - a^*) \quad,$$

$$B^* = -\cos\delta\,\cos i\,\sin(\Omega - a^*) + \sin i \cdot \sin\delta \quad, \tag{57}$$

where

$$a^* = a + n_\oplus \cdot \Delta t \quad. \tag{58}$$

As the relations

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 d\ell = (1 - e^2)^{-3/2} \quad,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 \cos 2f \cdot d\ell = 0 \tag{59}$$

hold, then after the short-periodic terms are eliminated, the disturbing function is written as

$$R_s = n'^2\,\beta\,\left(\frac{a'}{r'}\right)^3\frac{a_e^5}{a^3}\,k_2(1 - e^2)^{-3/2}\left[\frac{3}{4}(A^{*2} + B^{*2}) - \frac{1}{2}\right] \quad. \tag{60}$$

Lagrange's equations for this case are then easily derived as

$$\frac{da}{dt} = 0 \quad,$$

$$\frac{de}{dt} = 0 \quad,$$

$$\frac{di}{dt} = -\frac{3}{2}\,F\,\frac{1}{\sin i}\left(A^*\frac{\partial A^*}{\partial \Omega} + B^*\frac{\partial B^*}{\partial \Omega}\right) \quad,$$

$$\frac{d\Omega}{dt} = \frac{3}{2} F \frac{1}{\sin i} B^* \frac{\partial B^*}{\partial i} \quad ,$$

$$\frac{d\omega}{dt} = - \cos i \frac{d\Omega}{dt} + 3F \left[\frac{3}{4} (A^{*2} + B^{*2}) - \frac{1}{2} \right] \quad ,$$

$$\frac{d\ell}{dt} = 3F \left[\frac{3}{4} (A^{*2} + B^{*2}) - \frac{1}{2} \right] \sqrt{1 - e^2} \quad ,$$

(61)

where

(62)

$$F = \frac{n'^2}{n} \beta \left(\frac{a'}{r'} \right)^3 \left(\frac{a_e}{a} \right)^5 k_2 (1 - e^2)^{-2} \quad .$$

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7. REFERENCES

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APPENDIX

MOTION FOR THE ORBITAL PLANE OF A SYNCHRONOUS SATELLITE

As a synchronous satellite has the large value of 6.65 equatorial radii for the semimajor axis, the disturbing factors due to J_2, the moon, and the sun are of the same order of magnitude:

$$J_2/a^2 = 2.46 \times 10^{-5},$$

$$\beta(n'/n)^2 = 1.63 \times 10^{-5} \text{ for the moon },$$

$$\beta(n'/n)^2 = 0.75 \times 10^{-5} \text{ for the sun } . \tag{63}$$

When both the eccentricity and the inclination are very small for the satellite, the main term of the disturbing function is written as (Kozai, 1959)

$$R = n^2 a^2 \frac{J_2}{a^2} \left(\frac{1}{2} - \frac{3}{2} \sin^2 i \right) \left(1 + \frac{3}{2} e^2 \right)$$

$$+ n'^2 a^2 \beta \left(1 + \frac{3}{2} e^2 \right) (1 - e'^2)^{-3/2} \left[\frac{1}{4} \left(1 - \frac{3}{2} \sin^2 i \right) \left(1 - \frac{3}{2} \sin^2 i' \right) \right.$$

$$\left. + \frac{3}{16} \sin 2i \cdot \sin 2i' \cos (\Omega - \Omega') + \frac{3}{16} \sin^2 i \sin^2 i' \cos 2 (\Omega - \Omega') \right] ,$$

$$\tag{64}$$

where primed letters express the orbital elements referred to the equator of the moon and the sun, and terms due to the sun and the moon must be added.

When the longitude of the ascending node of the moon referred to the ecliptic is expressed by N, the following relations hold:

for the moon,

$$\sin^2 i' = 0.164 + 0.068 \cos N \quad ,$$

$$\sin i' \cos \Omega' = 0.396 + 0.086 \cos N \quad ,$$

$$\sin i' \sin \Omega' = 0.086 \sin N \quad ;$$

for the sun,

$$\sin i' = 0.399 \quad ,$$

$$\sin^2 i' = 0.158 \quad ,$$

$$\Omega' = 0 \quad . \tag{65}$$

If the new variables

$$p = \sin i \cos \Omega \quad ,$$

$$q = \sin i \sin \Omega \quad , \tag{66}$$

are adopted instead of i and Ω, the equations describing the motion of the orbital plane are derived as

$$\frac{dp}{dt} = - \frac{1}{na^2} \frac{\partial R}{\partial q} = n \times 10^{-5}(5.06q + 0.14q + 0.09 \sin N) \quad ,$$

$$\frac{dq}{dt} = \frac{1}{na^2} \frac{\partial R}{\partial p} = - n \times 10^{-5}(5.06p - 0.14p + 0.09 \cos N - 0.646) \quad . \tag{67}$$

As $\dot{N} = - 1.47 \times 10^{-4} n$, the solutions of equations (67) are derived by introducing two integration constants, γ and t_0, as,

$$p = 1.06 \, \gamma \cos \left[-5.06 \times 10^{-5} \, n(t - t_0)\right] + 0.128 + 0.008 \cos N \quad ,$$

$$q = \gamma \sin \left[5.06 \times 10^{-5} \, n(t - t_0)\right] + 0.008 \sin N \quad . \tag{68}$$

If $\begin{smallmatrix} \cos \\ \sin \end{smallmatrix}$ N terms are neglected and γ is equal to zero, a stationary solution can be obtained:

$$\Omega = 0°$$

$$i = 7°3 \quad . \tag{69}$$

Other solutions are given approximately by ellipses around the stationary one on the (p, q)-plane with a period of 54 years. Therefore, the inclination cannot remain very small even if the initial inclination is very small. To keep the inclination below, say, one degree for a long interval of time, the initial orbital elements chosen should be $i = 1°$ and $\Omega = 270°$.

However, the eccentricity can remain below 0.001 because a term similar to $\cos \Omega$ that produces 0.128 in p comes from the parallactic term with factor e'.

BIOGRAPHICAL NOTE

YOSHIHIDE KOZAI received the M. S. and D. S. degrees from Tokyo University in 1951 and 1958, respectively. He has been associated with the Tokyo Astronomical Observatory since 1952 and has held concurrent positions as staff astronomer with that observatory and consultant to SAO since 1958.

Dr. Kozai specializes in celestial mechanics, his research at SAO being primarily in the determination of zonal harmonics coefficients in the earth's gravitational potential by using precisely reduced Baker-Nunn observations. He is also interested in the seasonal variability of the earth's potential.

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NOTICE

This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory.

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