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ON THE TIDAL EFFECTS IN THE MOTION OF EARTH SATELLITES AND THE LOVE PARAMETERS OF THE EARTH

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DECEMBER 1972

GSFC

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GREENBELT, MARYLAND

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NOTATIONS

t – time

e – the basis of natural logarithms

$j = \sqrt{-1}$

$\mathfrak{m}$ – the mass of the Earth

m' – the mass of the Moon

m'' – the mass of the Sun

λ', μ', ν' – the equatorial components of the unit vector directed from the center of the Earth toward the Moon

r' – distance of the Moon from the center of the Earth

a' – mean value of r' defined in such a way that the constant term in the trigonometrical expansion of a'/r' is equal to 1

λ'', μ'', ν'' – the equatorial components of the unit vectors directed from the center of the Earth toward the Sun

r'' – distance of the Sun from the center of the Earth

a'' – mean value of r'' defined in such a way that the constant term in the trigonometrical expansion of a''/r'' is equal to 1

a – semimajor axis of satellite's orbit

i – the inclination of satellite's orbital plane toward the equatorial plane

Ω – the right ascension of ascending node of satellite's orbital plane

Ω_1 – the mean motion of the ascending node of satellite's orbital plane

e – the eccentricity of satellite's orbit

n – the mean motion of the satellite

ω – the argument of the perigee of satellite's orbit

$\pi - \omega + \Omega$

M – the mean anomaly of the satellite

R – the equatorial radius of the Earth

$\alpha = R/a$

$\alpha' = R/a'$ – the lunar parallactic factor

$\alpha'' = R/a''$ – the solar parallactic factor

$\ell, \ell', F, D, \Gamma$ – the arguments of the lunar theory

$\ell_1, \ell_1', F_1, D_1, \Gamma_1$ – the mean motions of the lunar arguments

ON THE TIDAL EFFECTS IN THE MOTION OF EARTH SATELLITES AND THE LOVE PARAMETERS OF THE EARTH

INTRODUCTION

At the present time, the tidal effects in the motion of artificial satellites attract considerable attention from specialists in geophysics and celestial mechanics, mainly because from these effects one can determine the elastic properties of the Earth as they are observed from extraterrestrial space.

The disturbing potential pertaining to the problem is obtained as the analytical continuation of the tidal potential from the surface of the earth into outer space. This continuation must include parameters which characterize the earth's elastic response to tidal attraction by the moon and the Sun. There are in use at the present time several versions of the exterior tidal potential.

In setting a priori the tidal potential in the form of a functional linear in the elastic impulse responses of the earth we are being guided by our previous expansions and also by earlier works on the theory of tides. In our previous work (Musen and Estes, 1971), (Musen and Felsentreger, 1972) we expanded the exterior tidal potential into a sum of products of spherical harmonics in Maxwellian form. The first factor in each product depends upon the rectangular coordinates of the satellite and the second upon the coordinates of the disturbing body.

The elastic parameters of the earth were introduced in the form of factors (Love numbers) attached to harmonics in the expansion of the potential. The simplest assumption about the Love numbers is that they are the same for all spherical harmonics of a given degree. Such an approach was taken by Kozai (1965), Fisher and Felsentreger (1966), and Smith, Kolenkiewicz and Dunn (1971). As we stated before (Musen and Felsentreger, 1972) this approach is equivalent to introducting "global" Love numbers obtained by a double averaging process, as applied to the local Love numbers, performed over the whole earth.

A different approach was suggested by Kaula (1969). He pointed out that the dependence of the local Love numbers on latitude can produce noticeable long period effects in the motion of a satellite, whereas the dependence on longitude will produce predominantly small short period effects of the period of one day or less. If we are interested only in the long period tidal effects in the motion of a satellite, then the dependence of Love parameters upon latitude shall be taken into consideration before we start the analytical continuation of the tidal potential into outer space. Neglecting the dependence upon longitude is equivalent to averaging the local Love parameters along the parallels, and not over the whole earth. The resulting expansion of the exterior tidal potential is, as before, a sum of products of spherical harmonics. Each pair of harmonics of a given degree and order has the identical Love number.

It is of interest to note that this idea can be found already in the theory of tides by Laplace. Among the new approaches the one developed by Munk and Cartwright (1966) deserves special attention. The tides are represented by a sum of convolutions between the spherical harmonics and the earth's impulse responses. The impulse responses replace the Love numbers. The earth's tide is a very complicated process. Continuous masses are moving and the mutual influence of adjacent (in time and space) configurations takes place. The evidence of such "cross-effects" is supported by the observational material.

A theory should be devised to include the influence of these combined cross-effects on the motion of an artificial satellite. It is clear that the resulting potential cannot be obtained from the "first principles" of celestial mechanics.

There is a time lag in the exterior tidal potential acting on the satellite. This fact, together with the possibility of interpreting Love numbers as Dirac impulse functions leads to a representation of the tidal potential in the form of a convolution between the Love impulse responses and the spherical harmonics depending upon the coordinates of the moon (Sun). This is a general form of the exterior tidal potential which includes the influence of past configurations on the motion of the artificial satellite. The impulse responses which replace the delta functions mentioned previously are, in all probability, continuous, fast decreasing functions. They are determined from the condition that the mean quadratic error between the observed and the computed tidal effects shall be a minimum, which leads to a system of Fredholm integral equations. The transformation of these equations from the time domain to frequency domain provides us with simple relations which can serve as a check for the existence of Love numbers.

We formulate the integral equations and the corresponding relations in the frequency domain for the main problem only, considering harmonics of the second degree in the tidal potential. In the present exposition we resort, as in our earlier work, to an expansion into trigonometric series with arguments ℓ, ℓ', F, D, Γ of the lunar theory and Ω , π of the satellite.

TIDAL DISTURBING FUNCTION

Let λ' , μ' , ν' and λ'' , μ'' , ν'' be the equatorial components of the unit vectors directed from the center of the Earth toward the Moon and the Sun, respectively.

Let r' and r'' be their distances from the center of the earth and a', a'' be the mean values of r' and of r'', respectively, defined in such a way that the constant terms in the trigonometrical expansions of the parallaxes a'/r' and a''/r'' are equal to 1.

The basic spherical functions which appear in the expansion of the exterior lunar and solar tidal disturbing functions associated with the "main problem" are (in Maxwellian form):

for the "lunar" disturbing function:

$$c'_0\ (t) = \left(\frac{a'}{r'}\right)^3 (1 - 3\,\nu'^2),$$

$$c'_{+1}\ (t) = \left(\frac{a'}{r'}\right)^3 \nu'\ (\mu' + j\,\lambda'),$$

$$c'_{-1}\ (t) = \left(\frac{a'}{r'}\right)^3 \nu'\ (\mu' - j\,\lambda') \tag{1'}$$

$$c'_{+2}\ (t) = \left(\frac{a'}{r'}\right)^3 (\mu' + j\,\lambda')^2$$

$$c'_{-2}\ (t) = \left(\frac{a'}{r'}\right)^3 (\mu' - j\,\lambda')^2$$

and, similarly, for the "solar" disturbing function:

$$c''_0\ (t) = \left(\frac{a''}{r''}\right)^3 (1 - 3\,\nu''^2),$$

$$c''_{+1}\ (t) = \left(\frac{a''}{r''}\right)^3 \nu''\ (\mu'' + j\,\lambda''),$$

$$c''_{-1}\ (t) = \left(\frac{a''}{r''}\right)^3 \nu''\ (\mu'' - j\,\lambda''), \tag{1''}$$

$$c''_{+2}\ (t) = \left(\frac{a''}{r''}\right)^3 (\mu'' + j\,\lambda'')^2,$$

$$c''_{-2}\ (t) = \left(\frac{a''}{r''}\right)^3 (\mu'' - j\,\lambda'')^2$$

Introducing the elastic impulse responses of the earth,

$$w_0\ (t),\ w_1\ (t),\ w_2\ (t),$$

we can rewrite the exterior tidal disturbing functions acting on the artificial satellite as given in our previous work (Musen and Estes, 1971) in the following compact form:

$$\begin{aligned} V' = n^2 a^2 \kappa' \Big\{ & \left(1 - \frac{3}{2} \sin^2 i\right) (c_0' * w_0) \\ & + [e^{+j\Omega} (c_{+1}' * w_1) + e^{-j\Omega} (c_{-1}' * w_1)] \cdot \sin i \cos i \\ & - [e^{+2j\Omega} (c_{+2}' * w_2) + e^{-2j\Omega} (c_{-2}' * w_2)] \cdot \sin^2 i \Big\} (1 - e^2)^{-3/2}, \end{aligned} \tag{2'}$$

$$\begin{aligned} V'' = n^2 a^2 \kappa'' \Big\{ & \left(1 - \frac{3}{2} \sin^2 i\right) (c_0'' * w_0) \\ & + [e^{+j\Omega} (c_{+1}'' * w_1) + e^{-j\Omega} (c_{-1}'' * w_1)] \cdot \sin i \cos i \\ & - [e^{+2j\Omega} (c_{+2}'' * w_2) + e^{-2j\Omega} (c_{-2}'' * w_2)] \cdot \sin^2 i \Big\} (1 - e^2)^{-3/2} \end{aligned} \tag{2''}$$

where

$$\kappa' = \frac{m'}{\mathfrak{M}} \alpha^2 \alpha'^3, \qquad \kappa'' = \frac{m''}{\mathfrak{M}} \alpha^2 \alpha''^3$$

and the asterisk designates the convolution, taking only the past into account:

$$c_k' * w_{|k|} = \int_0^\infty c_k' (t - \tau) w_{|k|} (\tau) \, d\tau,$$

$$c_k'' * w_{|k|} = \int_0^\infty c_k'' (t - \tau) w_{|k|} (\tau) \, d\tau$$

$$k = 0, \pm 1, \pm 2.$$

For combined luni-solar tidal effects the disturbing function becomes:

$$V = n^2 a^2 \kappa' \left\{ \left(1 - \frac{3}{2} \sin^2 i\right) (c_0 * w_0) \right.$$

$$+ [e^{+j\Omega}(c_{+1} * w_1) + e^{-j\Omega}(c_{-1} * w_1)] \sin i \cos i$$

$$\left. - [e^{+2j\Omega}(c_{+2} * w_2) + e^{-2j\Omega}(c_{-2} * w_2)] \sin^2 i \right\} (1 - e^2)^{-3/2}, \qquad (2)$$

where we set

$$c_s = c'_s + \kappa c''_s$$

and

$$\kappa = \frac{\kappa''}{\kappa'} = \frac{m''}{m'} \left(\frac{a'}{a''}\right)^3$$

All the functions

$$c'_s, \quad c''_s, \quad c_s \quad (s = 0, \pm 1, \pm 2)$$

are expanded into Fourier series with numerical coefficients and arguments

$$\ell, \ell', F, D, \Gamma, \Omega \text{ and } \pi.$$

The details of this expansion (by electronic computer) are described in our previous article already quoted. In (2') - (2) we retained only the long period and the secular terms. The short period terms, with periods equal to the period of revolution of the satellite or less, are removed by means of averaging the disturbing function over the instantaneous orbit of the satellite. In fact, we are interested only in the long period tidal effects in the elements of the satellite and therefore eliminate the secular effects in π and Ω by omitting the constant terms in the Fourier expansions of c'_0, c''_0 and c_0.

We shall continue, however, to use the same notations, c_0', c_0'', c_0, for the corresponding series with the constant parts omitted. We set:

$$a_s'(t) = \int e^{+js\Omega} c_s'(t)\,dt = \alpha_s'(t) + j\,\beta_s'(t),$$

$$a_s''(t) = \int e^{+js\Omega} c_s''(t)\,dt = \alpha_s''(t) + j\,\beta_s''(t). \tag{3}$$

$$a_s(t) = \int e^{js\Omega} c_s(t)\,dt = \alpha_s(t) + j\,\beta_s(t).$$

where

$$\alpha_s = \alpha_s' + \kappa\,\alpha_s'', \qquad \beta_s = \beta_s' + \kappa\,\beta_s''$$

$$\alpha_s = \alpha_{-s}, \qquad \beta_s = -\beta_{-s}$$

From (1') we deduce:

$$\alpha_{+1}' = \alpha_{-1}' = \int \left(\frac{a'}{r'}\right)^3 \nu'\,(\mu' \cos\Omega - \lambda' \sin\Omega)\,dt,$$

$$\beta_{+1}' = -\beta_{-1}' = \int \left(\frac{a'}{r'}\right)^3 \nu'\,(\mu' \sin\Omega + \lambda' \cos\Omega)\,dt, \tag{4}$$

$$\alpha_{+2}' = \alpha_{-2}' = \int \left(\frac{a'}{r'}\right)^3 [(\mu'^2 - \lambda'^2)\cos 2\Omega - 2\lambda'\mu' \sin 2\Omega]\cdot dt,$$

$$\beta_{+2}' = -\beta_{-2}' = \int \left(\frac{a'}{r'}\right)^3 [(\mu'^2 - \lambda'^2)\sin 2\Omega + 2\lambda'\mu' \cos 2\Omega]\cdot dt.$$

Similar expressions can be deduced for $\alpha''_{\pm 1}$, $\beta'_{\pm 1}$, $\alpha''_{\pm 2}$ and $\beta''_{\pm 2}$. The integration of Fourier series in (3) and (4) is performed in a purely formal manner, without adding constants of integration. As a result, Fourier series for a'_s, a''_s, a_s contain only purely periodic terms.

We deduce after some easy transformations:

$$\int e^{js\Omega} (c'_s * w_{|s|}) \, dt = a'_s * \left(w_{|s|} \cdot e^{js\Omega_1 t}\right),$$

$$\int e^{js\Omega} (c''_s * w_{|s|}) \, dt = a''_s * \left(w_{|s|} \cdot e^{js\Omega_1 t}\right),$$

$$\int e^{js\Omega} (c_s * w_{|s|}) \, dt = a_s * \left(w_{|s|} \cdot e^{js\Omega_1 t}\right)$$

As before the convolutions are computed taking only the past into consideration and the integration with respect to time is performed in a formal manner.

PERTURBATIONS IN THE ELEMENTS DUE TO TIDES

Making use of the Lagrange equations

$$\frac{d\,\delta i}{dt} = - \frac{1}{n a^2 \sqrt{1 - e^2} \sin i} \frac{\partial V}{\partial \Omega},$$

$$\frac{d\,\delta \Omega}{dt} = + \frac{1}{n a^2 \sqrt{1 - e^2} \sin i} \frac{\partial V}{\partial i},$$

$$\frac{d\,\delta\,\pi}{d\,t} = + \frac{\operatorname{tg}\frac{i}{2}}{n\,a^2\sqrt{1-e^2}}\,\frac{\partial V}{\partial i} + \frac{\sqrt{1-e^2}}{n\,a^2\,e}\,\frac{\partial V}{\partial e},$$

$$\frac{d\,\delta\,M}{d\,t} = -\frac{1-e^2}{n\,a^2\,e}\,\frac{\partial V}{\partial e} - \frac{2}{n\,a^2}\left(a\,\frac{\partial V}{\partial a}\right), \tag{5}$$

We deduce for the tidal perturbations in the frame of the main problem:

$$\frac{d\,\delta\,i}{d\,t} = -\frac{n\,\kappa'\,j}{(1-e^2)^2}\,\{[e^{+j\Omega}(c_{+1} * w_1) - e^{-j\Omega}(c_{-1} * w_1)]\cdot\cos i$$

$$- 2\,[e^{+2j\Omega}(c_{+2} * w_2) - e^{-2j\Omega}(c_{-2} * w_2)]\cdot\sin i\}$$

$$\frac{d\,\delta\,\Omega}{d\,t} = +\frac{n\,\kappa'}{(1-e^2)^2}\,\{-3\,(c_0 * w_0)\cos i$$

$$+ [e^{+j\Omega}(c_{+1} * w_1) + e^{-j\Omega}(c_{-1} * w_1)]\cdot(\operatorname{cosec} i - 2\sin i) \tag{7}$$

$$- 2\,[e^{+2j\Omega}(c_{+2} * w_2) + e^{-2j\Omega}(c_{-2} * w_2)]\cdot\cos i\},$$

$$\frac{d\,\delta\,\pi}{d\,t} = +\frac{n\,\kappa'}{(1-e^2)^2}\left\{+\frac{3}{2}\,(5\cos^2 i - 2\cos i - 1)\,(c_0 * w_0)\right.$$

$$+ (5\cos^2 i + 3\cos i - 1)\operatorname{tg}\frac{i}{2}\,[e^{+j\Omega}(c_{+1} * w_1) + e^{-j\Omega}(c_{-1} * w_1)]$$

$$\left. - (5\cos i + 3)(1-\cos i)\,[e^{+2j\Omega}(c_{+2} * w_2) + e^{-2j\Omega}(c_{-2} * w_2)]\right\} \tag{8}$$

$$\frac{d\,\delta M}{d\,t} = + \frac{3}{n\,a^2} V$$

$$= \frac{n\,\kappa'}{(1-e^2)^{3/2}} \left\{ \left(1 - \frac{3}{2}\sin^2 i\right) (c_0 * w_0) \right.$$

$$+ \left[e^{+j\Omega} (c_{+1} * w_1) + e^{-j\Omega} (c_{-1} * w_1)\right] \sin i \cos i$$

$$\left. - \left[e^{+2j\Omega} (c_2 * w_2) + e^{-2j\Omega} (c_{-2} * w_2)\right] \sin^2 i \right\}$$

After the integration we obtain:

$$\delta\, i = - \frac{n\,\kappa'\,j}{(1-e^2)^2} \left\{ \left[a_{+1} * \left(w_1\, e^{+j\Omega_1 t}\right) - a_{-1} * \left(w_1\, e^{-j\Omega_1 t}\right)\right] \cos i \right.$$

$$\left. - 2 \left[a_{+2} * \left(w_2\, e^{+2j\Omega_1 t}\right) - a_{-2} * \left(w_2\, e^{-2j\Omega_1 t}\right)\right] \sin i \right\}, \qquad (10)$$

$$\delta\, \Omega = + \frac{n\,\kappa'}{(1-e^2)^2} \left\{ - 3\,(a_0 * w_0) \cos i \right.$$

$$+ \left[a_{+1} \left(w_1\, e^{+j\Omega_1 t}\right) + a_{-1} * \left(w_1\, e^{-j\Omega_1 t}\right)\right] (\operatorname{cosec} i - 2 \sin i)$$

$$\left. - 2 \left[a_{+2} * \left(w_2\, e^{+2j\Omega_1 t}\right) + a_{-2} * \left(w_2\, e^{-2j\Omega_1 t}\right)\right] \cos i \right\}, \qquad (11)$$

$$\delta\, \pi = + \frac{n\,\kappa'}{(1-e^2)^2} \left\{ \frac{3}{2} (a_0 * w_0) (5 \cos^2 i - 2 \cos i - 1) \right.$$

$$+ \left[a_{+1} * \left(w_1\, e^{+j\Omega_1 t}\right) + a_{-1} * \left(w_1\, e^{-j\Omega_1 t}\right)\right] (5 \cos^2 i + 3 \cos i - 1) \operatorname{tg} \frac{i}{2}$$

$$\left. - 2 \left[a_{+2} * \left(w_2\, e^{+2j\Omega_1 t}\right) + a_{-2} * \left(w_2\, e^{-2j\Omega_1 t}\right)\right] (5 \cos i + 3)(1 - \cos i) \right\} \qquad (12)$$

$$\delta M = \frac{n\kappa'}{(1-e^2)^{3/2}} \left\{ \left(1 - \frac{3}{2}\sin^2 i\right) (a_0 * w_0) \right.$$

$$+ \left[a_{+1} * \left(w_1\, e^{+j\Omega_1}\right) + a_{-1} * \left(w_1\, e^{-j\Omega_1 t}\right)\right] \sin i \cos i$$

$$\left. + \left[a_{+2} * \left(w_2\, e^{+2j\Omega_1 t}\right) + a_{-2} * \left(w_2\, e^{-2j\Omega_1 t}\right)\right] \sin^2 i \right\}$$

INTEGRAL EQUATIONS FOR THE IMPULSE RESPONSES

We determine the impulse responses from the condition that the mean quadratic error between the observed and the computed tidal perturbations in the elements shall be a minimum. The application of this principle leads to a system of Fredholm integral equations of the first kind for the impulse responses. As we stated before all functions with which we operate in the computation of the tidal effect and in the formation of the integral equations are given in the form of Fourier series with numerical coefficients.

Such a form facilitates greatly the computation of mean values of the form

$$E\,[f(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)\, dt$$

and of the cross- and autocorrelations

$$a_{p,q}(\tau) = E\,[a_p(t+\tau)\, a_q^*(t)]$$

$$= E\,[a_p(t)\, a_q^*(t-\tau)],$$

which we introduce in the exposition. In the process of computation of $a_{pq}(\tau)$ only terms with the identical arguments in $a_p(t)$ and $a_q^*(t)$ produce a non-zero term in the cross-correlation. One of us (R. E.) developed a program for the computation of a_{pq} in the form of Fourier series. The arguments in the output are linear combinations of the arguments ℓ, ℓ', F, D, Γ, π and Ω, but with t replaced by τ and with the constants of the phase omitted. For example, making use of (10) we deduce integral equations for w_1 and w_2 by minimizing the quadratic functional:

$$J = E \left| \delta i + j s \int_0^\infty \left\{ \left[a_{+1} (t - \tau) e^{+j \Omega_1 \tau} - a_{-1} (t - \tau) e^{-j \Omega_1 \tau} \right] w_1 (\tau) \cos i \right. \right.$$
$$\left. \left. - 2 \left[a_{+2} (t - \tau) e^{+2j \Omega_1 \tau} - a_{-2} (t - \tau) e^{-2j \Omega_1 \tau} \right] w_2 (\tau) \sin i \right\} d\tau \right|^2, \tag{14}$$

where

$$s = \frac{n \kappa'}{(1 - e^2)^2}$$

and $\delta i(t)$ are the observed tidal perturbations in inclination. After some easy transformations and taking

$$a_k^* (t) = a_{-k} (t)$$

into account, we obtain assuming that the order of averaging and of the integration can be interchanged:

$$J = E |\delta i|^2 + 2 s \int_0^\infty [\ell_{10} (\tau) w_1 (\tau) \cos i + \ell_{20} (\tau) w_2 (\tau) \sin i] \, d\tau$$
$$- s^2 \int_0^\infty \int_0^\infty [\ell_{11} (\tau, \nu) w_1 (\tau) w_1 (\nu) \cos^2 i$$
$$- 2 \ell_{12} (\tau, \nu) w_1 (\tau) w_2 (\nu) \sin i \cos i$$
$$+ \ell_{22} (\tau, \nu) w_2 (\tau) w_2 (\nu) \sin^2 i] \, d\tau \, d\nu \tag{15}$$

Taking the gradients with respect to w_1 and w_2 and setting them equal to zero, we obtain:

$$\ell_{10}(\tau) - s \int_0^{+\infty} [\ell_{11}(\tau, \nu)\, w_1(\nu) \cos i - \ell_{12}(\tau, \nu)\, w_2(\nu) \sin i]\, d\nu = 0 \quad (16)$$

$$\ell_{20}(\tau) - s \int_0^{\infty} [-\ell_{12}(\nu, \tau)\, w_1(\nu) \cos i + \ell_{22}(\tau, \nu)\, w_2(\nu) \sin i] \cdot d\nu = 0, \quad (17)$$

As we will see later

$$\ell_{12}(\tau, \nu) \equiv 0, \quad (17')$$

and we obtain a pair of uncoupled integral equations

$$\ell_{10}(\tau) - s \cos i \int_0^{\infty} \ell_{11}(\tau, \nu)\, w_1(\nu)\, d\nu = 0 \quad (18)$$

$$\ell_{20}(\tau) - s \sin i \int_0^{\infty} \ell_{22}(\tau, \nu)\, w_2(\nu)\, d\nu = 0 \quad (19)$$

It is of interest to note the presence of the orbital inclination in these equations.

The explicit form of $\ell_{10}(\tau)$, $\ell_{20}(\tau)$, $\ell_{11}(\tau, \nu)$, $\ell_{12}(\tau, \nu)$ and $\ell_{22}(\tau, \nu)$ are:

$$\ell_{10}(\tau) = j \left[+ e^{+j\Omega_1 \tau} k_{+1}(\tau) - e^{-j\Omega_1 \tau} k_{-1}(\tau)\right], \quad (20)$$

$$\ell_{20}(\tau) = 2j \left[- e^{+2j\Omega_1 \tau} k_{+2}(\tau) + e^{-2j\Omega_1 \tau} k_{-2}(\tau)\right], \quad (21)$$

where

$$k_{-1}(\tau) = E[\delta i(t)\, a_{-1}(t - \tau)], \quad (22)$$

$$k_{+1}(\tau) = E[\delta i(t)\, a_{+1}(t - \tau)], \quad (23)$$

$$k_{-2}(\tau) = E[\delta i(t)\, a_{-2}(t - \tau)], \quad (24)$$

$$k_{+2}(\tau) = E[\delta i(t)\, a_{+2}(t - \tau)], \quad (25)$$

and

$$
\begin{aligned}
\ell_{11}(\tau,\nu) = &+ e^{+j\Omega_1(\tau+\nu)}\, a_{+1,-1}(\tau-\nu) \\
&- e^{-j\Omega_1(\tau-\nu)}\, a_{+1,+1}(\tau-\nu) \\
&- e^{+j\Omega_1(\tau-\nu)}\, a_{-1,-1}(\tau-\nu) \\
&+ e^{-j\Omega_1(\tau+\nu)}\, a_{-1,+1}(\tau-\nu)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\frac{1}{2}\ell_{12}(\tau,\nu) = &+ e^{+j\Omega_1(\tau+2\nu)}\, a_{+2,-1}(\tau-\nu) \\
&- e^{-j\Omega_1(\tau-2\nu)}\, a_{+2,+1}(\tau-\nu) \\
&- e^{+j\Omega_1(\tau-2\nu)}\, a_{-2,-1}(\tau-\nu) \\
&+ e^{-j\Omega_1(\tau+2\nu)}\, a_{-2,+1}(\tau-\nu)
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\frac{1}{4}\ell_{22}(\tau,\nu) = &+ e^{+2j\Omega_1(\tau+\nu)}\, a_{+2,-2}(\tau-\nu) \\
&- e^{-2j\Omega_1(\tau-\nu)}\, a_{+2,+2}(\tau-\nu) \\
&- e^{+2j\Omega_1(\tau-\nu)}\, a_{-2,-2}(\tau-\nu) \\
&+ e^{-2j\Omega_1(\tau+\nu)}\, a_{-2,+2}(\tau-\nu)
\end{aligned}
\tag{28}
$$

Substituting

$$a_k = \alpha_k + j\,\beta_k, \quad k = \pm 1, \pm 2$$

into (26)-(28) and defining:

$$
\begin{aligned}
p_{11}(\tau) &= E\,[\alpha_1(t+\tau)\,\alpha_1(t) - \beta_1(t+\tau)\,\beta_1(t)], \\
q_{11}(\tau) &= E\,[\alpha_1(t+\tau)\,\beta_1(t) + \beta_1(t+\tau)\,\alpha_1(t)], \\
r_{11}(\tau) &= E\,[\alpha_1(t+\tau)\,\alpha_1(t) + \beta_1(t+\tau)\,\beta_1(t)], \\
s_{11}(\tau) &= E\,[\alpha_1(t+\tau)\,\beta_1(t) - \beta_1(t+\tau)\,\alpha_1(t)],
\end{aligned}
\tag{29}
$$

$$p_{12}(\tau) = E\,[\alpha_2(t+\tau)\,\alpha_1(t) - \beta_2(t+\tau)\,\beta_1(t)],$$

$$q_{12}(\tau) = E\,[\beta_2(t+\tau)\,\alpha_1(t) + \alpha_2(t+\tau)\,\beta_1(t)],$$

$$r_{12}(\tau) = E\,[\alpha_2(t+\tau)\,\alpha_1(t) + \beta_2(t+\tau)\,\beta_1(t)],$$

$$s_{12}(\tau) = E\,[\alpha_2(t+\tau)\,\beta_1(t) - \beta_2(t+\tau)\,\alpha_1(t)], \tag{30}$$

$$p_{22}(\tau) = E\,[\alpha_2(t+\tau)\,\alpha_2(t) - \beta_2(t+\tau)\,\beta_2(t)],$$

$$q_{22}(\tau) = E\,[\alpha_2(t+\tau)\,\beta_2(t) + \beta_2(t+\tau)\,\alpha_2(t)],$$

$$r_{22}(\tau) = E\,[\alpha_2(t+\tau)\,\alpha_2(t) + \beta_2(t+\tau)\,\beta_2(t)],$$

$$s_{22}(\tau) = E\,[\beta_2(t+\tau)\,\alpha_2(t) - \alpha_2(t+\tau)\,\beta_2(t)], \tag{31}$$

we deduce:

$$\begin{aligned}\ell_{11}(\tau,\nu) = +\,2\,[&p_{11}(\tau-\nu)\cos\Omega_1(\tau+\nu) - q_{11}(\tau-\nu)\sin\Omega_1(\tau+\nu)\\ &- r_{11}(\tau-\nu)\cos\Omega_1(\tau-\nu) + s_{11}(\tau-\nu)\sin\Omega_1(\tau-\nu)]\end{aligned} \tag{32}$$

$$\begin{aligned}\ell_{12}(\tau,\nu) = +\,4\,[&p_{12}(\tau-\nu)\cos\Omega_1(\tau+2\nu) - q_{12}(\tau-\nu)\sin\Omega_1(\tau+2\nu)\\ &- r_{12}(\tau-\nu)\cos\Omega_1(\tau-2\nu) + s_{12}(\tau-\nu)\sin\Omega_1(\tau-2\nu)]\end{aligned} \tag{33}$$

$$\begin{aligned}\ell_{22}(\tau,\nu) = 8\,[&p_{22}(\tau-\nu)\cos\Omega_1(\tau+2\nu) - q_{22}(\tau-\nu)\sin\Omega_1(\tau+2\nu)\\ &- r_{22}(\tau-\nu)\cos\Omega_1(\tau-2\nu) - s_{22}(\tau-\nu)\sin\Omega_1(\tau-2\nu)]\end{aligned} \tag{34}$$

The algebraic computations performed on the machine by one of us (R. E.) yield

$$\begin{aligned}p_{11}(\tau) = q_{11}(\tau) = p_{12}(\tau) = q_{12}(\tau) = r_{12}(\tau) = s_{12}(\tau)\\ = p_{22}(\tau) = q_{22}(\tau) = r_{22}(\tau) \equiv 0\end{aligned}$$

and the equations (32)-(34) become

$$\ell_{11}(\tau, \nu) = -2\, r_{11}(\tau-\nu), \cos \Omega_1 (\tau - \nu) + 2\, s_{11}(\tau - \nu) \sin \Omega_1 (\tau - \nu),$$

$$\ell_{12}(\tau, \nu) \equiv 0,$$

$$\ell_{22}(\tau, \nu) = -8\, s_{22}(\tau - \nu) \sin \Omega_1 (\tau - 2\nu).$$

The equations (18)-(19) can be put into a simpler form by transforming them to the frequency domain. In the actual computations with the assumed numerical accuracy the kernels of the integral equations, $\ell_{11}(\tau, \nu)$ and $\ell_{22}(\tau, \nu)$, also $\ell_{20}(\tau)$, $\ell_{10}(\tau)$, as well as $a_{pq}(\tau - \nu)$, $a_p(\tau)$ ($p = 0, \pm 1, \pm 2$) are trigonometric polynomials with arguments linear in τ and ν and, consequently, they are almost periodic. In transforming an almost periodic function g(t) to the frequency domain the averaging operator E_t takes the place of the Fourier transform,

$$G(f) = E_t\, [e^{-jft}\, g(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} e^{-jft}\, g(t)\, dt,$$

and G(f) is different from zero only if f is a Fourier exponent of g(t).

We are not presupposing, however, that $w_1(\nu)$, $w_2(\nu)$ are almost periodic. In the transformation of (18)-(19) to frequency domain the standard Fourier transforms

$$W_i(f) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-jf\nu}\, w_i(\nu)\, d\nu$$

will appear in the final result.

From (26)-(28) it is evident that the integrals in the equations (18)-(19) are sums of terms of the form:

$$h_{p,q}^{m,n}(\tau) = \int_0^\infty e^{+j\Omega_1(m\tau+n\nu)}\, a_{p,q}(\tau - \nu)\, w(\nu)\, d\nu$$

$$w = w_1,\ w_2.$$

Taking into account that $h_{p,q}^{m,n}(\tau)$ is an almost periodic function of τ and assuming that the operators of averaging and of integration commute, we have:

$$H_{p,q}^{m,n}(f) = \int_0^\infty E_\tau \left[e^{-j\tau(f-m\Omega_1)}\, a_{p,q}(\tau - \nu)\right] e^{+jn\Omega_1\nu}\, W(\nu)\, d\nu$$

and, after the change of variable under the E_τ-sign,

$$H_{p,q}^{m,n}(f) = 2\pi E_\tau \left[e^{-j\tau(f-m\Omega_1)}\, a_{p,q}(\tau)\right] W[f - (m+n)\Omega_1]$$

$$= 2\pi E_\tau \left\{ e^{-j\tau(f-m\Omega_1)}\, E_t\, [a_p(t+\tau)\, a_{-q}(t)] \cdot W[f-(m+n)\Omega_1]\right\}$$

where W(f) is the Fourier transform of $w(\nu)$.

In the case of almost periodic functions the order of averaging can be interchanged and, after another change of variable, we deduce:

$$H_{p,q}^{m,n}(f) = 2\pi A_p(+f - m\Omega_1)\, A_{-q}(-f + m\Omega_1)\, W[f - (m+n)\Omega_1]. \tag{35}$$

Similarly we obtain for

$$k_p(\tau) = E_t\, [\delta\, i\,(t+\tau)\, a_p(t)]$$

the relation:

$$E_\tau \left[e^{-j f \tau + j f m \Omega_1 \tau} k_p (\tau) \right] = \phi (+ f) A_p (- f)$$

where

$$\phi (f) = E_\tau [e^{-j f t} \delta i (t)].$$

By applying (35) and (36) to (18)-(19) and taking (20)-(28) into account we deduce, instead of integral equations, the following relations in the frequency domain:

$$\begin{aligned}
&+ A_{+1} (+ f - \Omega_1) A_{+1} (- f + \Omega_1) W_1 (+ f - 2 \Omega_1) \\
&- A_{+1} (+ f + \Omega_1) A_{-1} (- f - \Omega_1) W_1 (+ f) \\
&- A_{-1} (+ f - \Omega_1) A_{+1} (- f + \Omega_1) W_1 (+ f) \\
&+ A_{-1} (+ f + \Omega_1) A_{-1} (- f - \Omega_1) W_1 (+ f + 2 \Omega_1) \\
&= \frac{1}{2 \pi j s \cos i} \phi (f) [A_{-1} (- f) - A_{+1} (- f)],
\end{aligned} \tag{37}$$

$$\begin{aligned}
&+ A_{+2} (+ f - 2 \Omega_1) A_{+2} (+ f - 2 \Omega_1) W_2 (+ f - 4 \Omega_1) \\
&- A_{+2} (+ f + 2 \Omega_1) A_{-2} (- f - 2 \Omega_1) W_2 (+ f) \\
&- A_{-2} (+ f - 2 \Omega_1) A_{+2} (- f + 2 \Omega_1) W_2 (+ f) \\
&+ A_{-2} (+ f + 2 \Omega_1) A_{-2} (- f - 2 \Omega_1) W_2 (+ f + 4 \Omega_1) \\
&= \frac{1}{4 \pi j s \sin i} \phi (f) [A_{+2} (- f) - A_{-2} (- f)],
\end{aligned} \tag{38}$$

where $A_p (\pm f + m \Omega_1)$ is the Fourier coefficient of the term $\exp [j(\pm f + m \Omega_1)t]$ in the trigonometrical expansion of a (t).

If the impulse responses $w_1(t)$ and $w_2(t)$ degenerate into the delta-functions

$$w_1(\tau) = w_1\,\delta(\tau - \tau_1),$$
$$w_2(\tau) = w_2\,\delta(\tau - \tau_2) \tag{39}$$

the integral equations become

$$\ell_{10}(\tau) - s\,w_1\,\ell_{11}(\tau, \tau_1)\cos i = 0.$$
$$\ell_{20}(\tau) - s\,w_2\,\ell_{22}(\tau, \tau_2)\sin i = 0 \tag{40}$$

For the Fourier transforms we have:

$$W_1(f) = \frac{w_1}{2\pi}\,e^{-jf\tau_1}$$

and

$$W_2(f) = \frac{w_2}{2\pi}\,e^{-jf\tau_2}$$

and the relations (37)-(38) become:

$$\begin{aligned} w_1 \Big\{ \Big[& A_{+1}(+f - \Omega_1)\,A_{+1}(-f + \Omega_1)\,e^{+2j\Omega_1\tau_1} \\ & + A_{-1}(+f + \Omega_1)\,A_{-1}(-f - \Omega_1)\,e^{-2j\Omega_1\tau_1}\Big] \\ & - [A_{+1}(+f + \Omega_1)\,A_{-1}(-f - \Omega_1) + A_{-1}(+f - \Omega_1)\,A_{+1}(-f + \Omega_1)]\Big\} \\ & = \frac{e^{+jf\tau_1}}{s\,j\cos i}\,\phi(f)\,[A_{-1}(-f) - A_{+1}(-f)], \end{aligned} \tag{41}$$

and

$$\begin{aligned} w_2 \Big\{ \Big[& A_{+2}(+f - 2\,\Omega_1)\,A_{+2}(-f + 2\,\Omega_1)\,e^{+4j\Omega_1\tau_2} \\ & + A_{-2}(+f + 2\,\Omega_1)\,A_{-2}(-f - 2\,\Omega_1)\,e^{-4j\Omega_1\tau_2}\Big] \end{aligned}$$

$$- \Big[A_{+2}(+f + 2\Omega_1)\, A_{-2}(-f - 2\Omega_1)$$

$$+ A_{-2}(+f - 2\Omega_1)\, A_{+2}(-f + 2\Omega_1)\Big]\Big\}$$

$$= \frac{e^{+jf\tau_2}}{s\, j \sin i}\, \phi(f)\, [A_{+2}(-f) - A_{-2}(-f)] \tag{42}$$

These equations show the existence of relations between the Love numbers, time lags and Fourier coefficients in the expansion of the tidal effects.

We deduce the integral equation for $w_0(t)$ by minimizing the quadratic functional:

$$J' = E \Big| \delta\Omega(t) - s \Big\{ -3\, a_0 * w_0 \cos i$$
$$+ \Big[a_{+1} * \Big(w_1 e^{+j\Omega_1 t}\Big) - a_{-1} * \Big(w_1 e^{-j\Omega_1 t}\Big)\Big] (\csc i - 2 \sin i)$$
$$- 2 \Big[a_{+2} * \Big(w_2 e^{+2j\Omega_1 t}\Big) - a_{-2} * \Big(w_2 e^{-2j\Omega_1 t}\Big)\Big] \cos i \Big\} \Big|^2$$

which represents the mean quadratic error between the observed and computed tidal perturbations in Ω. After some easy transformations and taking (17') and (26)-(28) into account we can rewrite the last equation in the form:

$$J' = E(|\delta\Omega(t)|^2) + 6\, s \cos i \int_0^{+\infty} k_{00}(\tau)\, w_0(\tau)\, d\tau$$

$$+ 9\, s^2 \cos^2 i \int_0^{+\infty}\int_0^{+\infty} a_{00}(\tau - \nu)\, w_0(\tau)\, w_0(\nu)\, d\tau\, d\nu$$

$$- s^2 \int_0^{\infty}\int_0^{\infty} [(\operatorname{cosec} i - 2 \sin i)^2\, \ell_{11}(\tau, \nu)\, w_1(\tau)\, w_1(\nu)$$

$$+ 4 \cos^2 i\, \ell_{22}(\tau, \nu)\, w_2(\tau)\, w_2(\nu)]\, d\tau\, d\nu, \tag{43}$$

where we set

$$k_{00}(\tau) = E\,[\delta\,\Omega(t)\,a_0(t-\tau)].$$

Taking the gradient of (43) with respect to w_0 and setting it equal to zero, we obtain the integral equation for w_0:

$$k_{00}(\tau) + 3\,s\,\cos\,i \int_0^{\infty} a_{00}(\tau-\nu)\,w_0(\nu)\,d\nu = 0 \qquad (44)$$

If $w_0(\nu)$ degenerates into a delta-function,

$$w_0(\tau) = w_0\,\delta(\tau-\tau_0)$$

then (44) takes the form

$$k_{00}(\tau) + 3\,w_0\,s\,\cos\,i\,a_{00}(\tau-\tau_0) = 0, \qquad (45)$$

which can be used to obtain the Love number w_0. By transforming (44) to the frequency domain we have

$$A_0(-f)\,A_0(+f)\,W_0(f) = \frac{1}{2\pi}\,\theta(f)\,A_0(-f)$$

or, taking

$$A_0(-f) = A_0(+f)$$

into consideration, we obtain a simpler relation

$$A_0(f)\,W_0(f) = \frac{1}{2\pi}\,\theta(f) \qquad (44')$$

where

$$\theta(f) = E_t \left[e^{-jft} \, \delta \, \Omega(t) \right].$$

The equation corresponding to (45) is:

$$w_0 A_0(f) = e^{+jf\tau_0} \, \theta(f) \tag{45'}$$

and it gives the connection between the Love number w_0, the time lag τ_0 and the Fourier coefficients in the expansions of the tidal effects in the node of satellite's orbit. The relations (40) and (45) or (41), (42) and (45') can be helpful in checking the existence of Love numbers from the observations of the long period tidal effects in the motion of satellites.

CONCLUSIONS

The tidal effects represent a superposition of a large number of periodic terms. In the case of nearly equal periods the analytical expansion can easily establish the precise form of arguments and the relative significance of amplitudes.

Long period and resonance terms are especially important. For example, the rotation of the lunar orbital plane produces a term of 18 years period in tidal perturbations of the ascending node of the satellite's orbit. If these effects are not properly taken into account they will contaminate the coefficients of the zonal harmonics in the geopotential.

We do not want to oversimplify our statements. The present theory is not a proof, or a disproof, of the existence of Love numbers. However, it must be stated, at the same time, that no empirical fit, independently of how successful it is at the moment, can be considered as a complete replacement for theoretical thought or a final solution. The long term observations will require a more subtle approach, approximately along the lines presented here. We have to expect that, because of the rotation of the lunar orbital plane, the tidal effects in the motion of some satellites will increase.

These larger values of the tidal effects and the prolonged observations will stimulate a further cooperation between theoretical and numerical directions of work in the coming years.

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Table I. $r_{11}(\tau)$

Coef × 10^4 of cosine	$l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
2	(0	1	1	1	1	1)	16.259
49	(0	1	−1	1	1	1)	83.392
4	(1	−1	1	−1	−1	1)	40.660
1243	(0	2	0	2	2	−1)	11.758
1	(0	2	−2	2	2	−1)	86.578
3	(1	−2	0	−2	−2	1)	20.511
1	(1	0	0	−2	0	1)	23.106
93	(1	0	0	0	0	1)	40.905
14	(0	1	0	0	0	1)	09.809
89	(0	2	0	0	2	1)	57.019

Coef × 10^4 of cosine	$l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
44	(0	1	1	1	1	−1)	11.738
2368	(0	1	−1	1	1	−1)	65.489
4	(0	2	0	2	2	1)	16.298
6489	(0	2	0	0	2	−1)	57.736
22	(1	2	0	2	2	−1)	8.241
126785	(0	0	0	0	0	1)	84.428
5	(1	0	0	−2	0	−1)	51.046
24	(1	0	0	0	0	−1)	20.774
5	(0	1	0	0	0	−1)	68.577
17	(0	3	0	0	2	−1)	49.855

Table II. $S_{11}(\tau)$

Coef $\times 10^4$ of sine	Multiples of $l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
-2	(0	1	1	1	1	1)	16.259
-49	(0	1	-1	1	1	1)	83.392
-4	(1	-1	1	-1	-1	1)	40.660
1243	(0	2	0	2	2	-1)	11.758
1	(0	2	-2	2	2	-1)	86.578
-3	(1	-2	0	-2	-2	1)	20.511
-1	(1	0	0	-2	0	1)	23.106
-93	(1	0	0	0	0	1)	40.905
-14	(0	1	0	0	0	1)	09.809
-89	(0	2	0	0	2	1)	57.019

Coef $\times 10^4$ of sine	Multiples of $l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
44	(0	1	1	1	1	-1)	11.738
2388	(0	1	-1	1	1	-1)	85.489
-4	(0	2	0	2	2	1)	16.298
6489	(0	2	0	0	2	-1)	57.736
22	(1	2	0	2	2	-1)	8.241
-126785	(0	0	0	0	0	1)	84.428
5	(1	0	0	-2	0	-1)	51.046
24	(1	0	0	0	0	-1)	20.774
5	(0	1	0	0	0	-1)	68.577
17	(0	3	0	0	2	-1)	49.855

Table III. $S_{22}(\tau)$

Coef × 10^4 of sine	Multiples of $l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
-22252	(0	2	0	2	2	-2)	10.321
-53168	(0	2	0	0	2	-2)	34.288
-3	(1	2	0	0	2	-2)	15.277
-432	(1	2	0	2	2	-2)	7.508
-5	(2	2	0	2	2	-2)	5.901
-6	(1	0	0	-2	0	-2)	29.102
-3	(1	0	0	0	0	-2)	16.672
-535	(0	1	-1	1	1	-2)	42.478
-1	(1	1	-1	-1	1	-2)	26.696
-152	(0	3	0	0	2	-2)	31.345

Coef × 10^4 of sine	Multiples of $l_1\tau$	$l'_1\tau$	$F_1\tau$	$D_1\tau$	$\Gamma_1\tau$	$\Omega_1\tau$	Period in Days
-7	(0	2	0	4	2	-2)	6.075
-6	(0	2	-2	2	2	-2)	42.745
17	(1	-2	0	-4	-2	2)	7.793
45	(1	-2	0	-2	-2	2)	16.502
5960	(0	0	0	0	0	2)	42.214
66	(1	0	0	0	0	2)	79.348
-31	(0	1	1	1	1	-2)	10.305
12	(1	-1	1	-1	-1	2)	78.433
-5	(0	1	0	0	2	-2)	37.840