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ANALYSIS OF SHAPE OF POROUS COOLED MEDIUM FOR AN IMPOSED SURFACE HEAT FLUX AND TEMPERATURE

by Robert Siegel

Lewis Research Center

Cleveland, Ohio 44135

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Robert Siegel

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16. Abstract
The surface of a porous cooled medium is to be maintained at a specified design temperature while being subjected to uniform heating by an external source. An analytical method is given for determining the shape of the medium surface that will satisfy these boundary conditions. The analysis accounts for temperature dependent variations of fluid density and viscosity and for temperature dependent matrix thermal conductivity. The energy equation is combined with Darcy's law in such a way that a potential can be defined that satisfies Laplace's equation. All of the heat-transfer and flow quantities are expressed in terms of this potential. The determination of the shape of the porous cooled region is thereby reduced to a free-boundary problem such as in inviscid free jet theory. Two illustrative examples are carried out: a porous leading edge with coolant supplied through a slot and a porous cooled duct with a rectangular outer boundary.

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ANALYSIS OF SHAPE OF POROUS COOLED MEDIUM FOR AN IMPOSED SURFACE HEAT FLUX AND TEMPERATURE

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SUMMARY

A two-dimensional porous cooled medium has part of its boundary open to a coolant reservoir and another part subjected to uniform heating by an external source. Coolant from the reservoir is forced through the medium and exits through the part of the boundary subjected to the heat load. For design purposes it is specified that the coolant exit surface is to be at a given temperature. This analysis provides a method to obtain the shape of the porous surface to meet these simultaneous constraints of a specified temperature and heat loading. The analysis includes temperature variations of fluid density, fluid viscosity, and matrix thermal conductivity. The solution is obtained by combining the energy equation with Darcy's law in such a way that a potential function can be introduced that satisfies Laplace's equation. All of the heat-transfer and coolant flow quantities can be expressed in terms of this potential, which is governed by simple boundary conditions. The determination of the shape of the porous cooled region is thereby reduced to a free-boundary problem such as in irrotational, inviscid free jet theory. Two illustrative examples are carried out: a porous leading edge with coolant supplied through a slot and a porous cooled duct with a rectangular outer boundary.

INTRODUCTION

Some of the advanced power producing devices such as fusion reactors and gaseous nuclear reactors will require advanced heat-transfer designs to cope with the large wall heat fluxes that are anticipated. One proposed cooling scheme is transpiration or porous wall cooling. A metal or ceramic wall is made in a porous form and coolant is forced through the wall so that it exits from the side of the wall exposed to the high heat flux. Any heat conducted into the wall is transferred to the coolant and carried back out. The result is an effective cooling technique. If part of the wall heat flux is provided by heat transfer from a convective boundary layer, the transpirant blowing out of the wall tends to push the boundary layer away from the wall, thus reducing the convective load.

The power required to pump the coolant through the wall represents a loss. Also, if the hot gas heating the porous wall is being used as the working fluid in a power producing device, the mixing of the coolant with the gas decreases the gas temperature and thus reduces the thermal efficiency of the device. In a device that is not ground based, such as a high speed aircraft, the coolant must be carried along, and this represents an additional weight penalty. These considerations illustrate that the coolant should be used as effectively as possible. A poor distribution of coolant flow will tend to overcool some portions of the surface and undercool others.

This report will consider a two-dimensional porous cooled region. The objective of the analysis will be to determine the shape of the region so that it will provide a specified heat-transfer performance. Since the region is two-dimensional, its length in the direction normal to the cross section must be large in comparison with a characteristic dimension of the cross section. A porous region along the leading edge of a wing or turbine blade would be an example of this type of configuration. The coolant enters the porous region through the part of its boundary that is open to a reservoir. The coolant exit surface has a heat flux specified along it. It is desired to maintain the coolant exit surface at a specified temperature, for example, at a value slightly below the melting point or one dictated by structural strength considerations. A method is presented here to obtain the shape of the porous surface such that a given surface temperature is maintained while the surface is being heated uniformly by an external source.

The analysis includes the effect of variable properties. The fluid viscosity can be a function of temperature, and, if the coolant is a gas, its density is found from the perfect gas law. The thermal conductivity of the solid matrix material can also be a function of temperature.

To obtain an analytical solution for a situation including all of these effects, some simplifying assumptions are also required. The coolant mass flow through the porous material is assumed to be at a pore Reynolds number small enough so that Darcy's law applies. The thermal resistance between the fluid and the porous matrix material is assumed small enough so that the local fluid and matrix temperatures are equal. As a result of this assumption a single energy equation can be written for the balance of energy transport by conduction in the matrix and by convection of the flowing coolant.

The variable fluid properties cause the flow and energy equations to be coupled. Because the coolant exit boundary of the porous medium has an unknown shape, a solution must be found that will simultaneously satisfy the flow and energy equations and at the same time determine the unknown boundary shape so that it is compatible with the specified boundary conditions. The solution is obtained by use of a suitably defined potential function that satisfies Laplace's equation. The temperature distribution and local heat flux can be directly related to this potential. This was shown in reference 1 for a porous region of specified shape. The analysis of reference 1 is extended here to include variable thermal conductivity of the matrix material and to account for the unknown shape of the coolant exit surface.

The shape of the unknown boundary is obtained by using conformal mapping such as in inviscid free jet theory. The author previously adapted the free-jet theory to determine the free-boundary configuration of a two-dimensional freezing layer (ref. 2). It will be shown that the results of the freezing problem can be directly carried over to the porous medium problem. To illustrate the application of the method, results are carried out for a two-dimensional leading edge with coolant introduced through part of its boundary and for a porous cooled duct with a rectangular outer boundary.

ANALYSIS

Geometry and Specified Conditions at Boundary

The general type of porous region being considered is shown in figure 1. The configuration is two-dimensional, that is, it is a cylinder of arbitrary cross section, infinitely long in the z direction, and with its axis parallel to that direction. A coolant reservoir at pressure and temperature p_∞, t_∞ is adjacent to a portion S_0 of the porous material that is open to the flow. Coolant flows through the porous medium and leaves through the surface S, which is exposed to a heat flux q_s. The remaining surface area S_i is impervious to the flow and is also insulated. Some specific geometries of this type will be discussed later (see figs. 3(a) and 7(a)) to illustrate the application of the analysis.

The surface S through which the coolant exits is specified as being at uniform temperature and also as having a uniform heat flux being transferred to it. This would appear to be an overspecified set of boundary conditions, but it is the shape of S that is not specified. The objective of the analysis is to obtain the shape of S that will allow the simultaneous specification of these two conditions. This would provide the optimum shape to accommodate the incident heat flux for the specified temperature. For example, a heat flux could be specified and the shape found that would cause the surface temperature to be within a given value below the melting point or be at a temperature fixed by structural considerations. If the porous material can sublime without disturbing the porosity, the shape would be that formed by sublimation as a result of being subjected to the specified heat loading. In this instance the t_s would be the sublimation temperature.

As the fluid in the reservoir at p_∞ approaches the coolant inlet surface S_0, the acceleration effects are neglected with regard to affecting the fluid pressure. Hence, the pressure along S_0 is equated to the reservoir value, $p_0 = p_\infty$. Since $p_\infty > p_s$, the coolant flows from S_0 to S. (Symbols are defined in the appendix.) Inside the porous medium, the fluid picks up the energy being conducted in at the surface S and then carries it back out of the medium. Since p_0 and p_s are both constants, the fluid velocities along both the inlet and exit surfaces are locally perpendicular to these surfaces. This fact enters the boundary conditions along these surfaces.

The effective thermal conductivity of the matrix material is k_m, which is a function of temperature and is based on the entire cross-sectional area rather than on only the area of the solid material. Because the conductivity of the matrix material is generally much larger than that of the coolant, the heat conduction in the coolant is neglected. The Darcy velocity $\vec{u}$, that will be used throughout the analysis is the local volume flow divided by the entire cross-sectional area rather than by the pore cross-sectional area.

Governing Equations

The size of the porous medium is usually quite large compared with the size of the individual pores, so that it is not necessary to deal with the individual pores and volume averaged continuum equations can be used. Then the following equations apply within the porous material for compressible and incompressible coolants:

Conservation of mass -

$$\nabla \cdot (\rho\vec{u}) = 0 \qquad \text{(compressible)} \tag{1a}$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{(incompressible)} \tag{1b}$$

Darcy's law (fluid inertia effects are neglected) -

$$\vec{u} = -\frac{\kappa}{\mu(t)} \nabla p \tag{2}$$

Conservation of energy (kinetic energy is neglected) -

$$\nabla \cdot \vec{q} = 0 \tag{3}$$

where

$$\vec{q} \equiv -k_m(t)\, \nabla t + \rho\vec{u}C_p t \tag{4}$$

Perfect gas law (for compressible case) -

$$p = \rho R t \tag{5}$$

Boundary Conditions

As the fluid in the reservoir approaches the porous medium, it receives any heat that is being conducted from the coolant inlet face S_0 back into the fluid. This raises the coolant temperature (and hence matrix temperature, since they are locally equal in this analysis) to t_0, which is an unknown quantity along the surface S_0 and which will be obtained later in the analysis and will be found to be independent of position along S_0. The thermal conductivity of the fluid is generally much smaller than that of the porous matrix material, and hence, as the fluid approaches the wall, the temperature rise from t_∞ to t_0 takes place in a thin layer compared with the thickness of the porous material. A boundary condition is obtained by balancing the heat conducted out of the surface with the energy carried back into the matrix by the fluid. The pressure along the boundary can be taken at the reservoir value since the pressure change from flow acceleration in the reservoir is small compared with the pressure drop through the porous material. Hence, the conditions along the coolant inlet face are

$$\left.\begin{aligned} k_m \hat{n}_0 \cdot \nabla t &= \rho C_p (t - t_\infty) \hat{n}_0 \cdot \vec{u} \\ p = p_0 &= p_\infty = \text{constant} \end{aligned}\right\} \quad \text{for } (x,y) \text{ on } S_0 \tag{6}$$

On the surface S of unknown shape and through which the fluid leaves the porous material, the temperature and imposed heat flux are both specified as constant so that the boundary conditions are

$$\left.\begin{aligned} t &= t_s = \text{constant} \\ k_m \hat{n}_s \cdot \nabla t &= q_s = \text{constant} \\ p &= p_s = \text{constant} \end{aligned}\right\} \quad \text{for } (x,y) \text{ on } S \tag{7}$$

On the impervious surfaces the conditions are

$$\left.\begin{aligned} \hat{n}_i \cdot \vec{u} &= 0 \\ \hat{n}_i \cdot \vec{q} &= 0 \end{aligned}\right\} \quad \text{for } (x,y) \text{ on } S_i \tag{8}$$

Equations and Boundary Conditions in Dimensionless Form

The governing equations and boundary conditions will now be placed in dimensionless form. Note that the functions P and M differ for the compressible and incompressible flow cases; these differences will result in the two cases reducing to the same set of relations. The dimensionless variables are

$$X = \frac{x}{h_r}$$

$$Y = \frac{y}{h_r}$$

$$T = \frac{t}{t_\infty}$$

$$P = \begin{cases} \dfrac{p}{p_\infty} & \text{incompressible} \\[2ex] \left(\dfrac{p}{p_\infty}\right)^2 & \text{compressible} \end{cases} \tag{9}$$

$$M(T) = \begin{cases} \dfrac{1}{2} \dfrac{\mu(t)}{\mu_\infty} \dfrac{t_\infty}{t} & \text{incompressible} \\[2ex] \dfrac{\mu(t)}{\mu_\infty} & \text{compressible} \end{cases} \tag{10}$$

$$K_m(T) = \frac{k_m(t)}{k_{m,\infty}} \tag{11}$$

$$\vec{Q} = \frac{\vec{q} h_r}{k_{m,\infty} t_\infty} \tag{12a}$$

$$Q_s = \frac{q_s h_r}{k_{m,\infty} t_\infty} \tag{12b}$$

$$\lambda = \frac{\rho_\infty C_p \kappa p_\infty}{2\mu_\infty k_{m,\infty}} \tag{13}$$

$$\tilde{\nabla} = \hat{i}\,\frac{\partial}{\partial X} + \hat{j}\,\frac{\partial}{\partial Y}$$

For the compressible case equation (5) is first used to eliminate ρ from equations (1a) and (4). Then for both the compressible and incompressible cases, Darcy's law is used to eliminate $\vec{u}$. The result is that equations (1), (3), and (4) reduce to

$$\tilde{\nabla}^2 P = \frac{1}{MT}\,\tilde{\nabla}MT \cdot \tilde{\nabla}P \tag{14}$$

$$\tilde{\nabla} \cdot \vec{Q} = 0 \tag{15}$$

$$\vec{Q} = -K_m\,\tilde{\nabla}T - \frac{\lambda}{M}\,\tilde{\nabla}P \tag{16}$$

By a similar manipulation boundary conditions (6) to (8) become

$$\left.\begin{aligned} &\frac{K_m T}{T - 1}\,\hat{n}_0 \cdot \tilde{\nabla}T + \frac{\lambda}{M}\,\hat{n}_0 \cdot \tilde{\nabla}P = 0 \\ &P = 1 \end{aligned}\right\} \quad \text{for } (X, Y) \text{ on } S_0 \tag{17}$$

$$\left.\begin{aligned} &T = T_s \\ &K_m \hat{n}_s \cdot \tilde{\nabla}T = Q_s \\ &P = P_s \end{aligned}\right\} \quad \text{for } (X, Y) \text{ on } S \tag{18}$$

$$\left.\begin{aligned} &\hat{n}_i \cdot \tilde{\nabla}P = 0 \\ &\hat{n}_i \cdot \vec{Q} = 0 \end{aligned}\right\} \quad \text{for } (X, Y) \text{ on } S_i \tag{19}$$

Formulation in Terms of Potential Function

At this point we shall postulate, pending later verification that P is a function of T only. It will be found that with this condition a solution is obtained satisfying the governing equations and boundary conditions, and hence this is the required solution. It will now be shown that this assumption permits the energy flux to be expressed as the negative gradient of a potential Φ,

$$\vec{Q} = -\tilde{\nabla}\Phi \tag{20}$$

and then from equation (15)

$$\tilde{\nabla}^2\Phi = 0 \tag{21}$$

Hence the potential function is a solution to Laplace's equation; the boundary conditions for Φ will be given later.

Since P is only a function of T

$$\frac{1}{M(T)}\tilde{\nabla}P = \frac{1}{M(T)}\frac{dP}{dT}\tilde{\nabla}T$$

Hence, if we put

$$h(T) = \int_{T_0}^{T} \frac{1}{M(T)}\frac{dP}{dT}\,dT \tag{22a}$$

$$g(T) = \int_{T_0}^{T} K_m(T)dT \tag{22b}$$

it follows that

$$\tilde{\nabla}h(T) = \frac{dh}{dT}\tilde{\nabla}T = \frac{1}{M(T)}\frac{dP}{dT}\tilde{\nabla}T = \frac{1}{M(T)}\tilde{\nabla}P \tag{23a}$$

$$\tilde{\nabla}g(T) = \frac{dg}{dT}\tilde{\nabla}T = K_m(T)\tilde{\nabla}T \tag{23b}$$

Equation (16) can then be written as

$$\vec{Q} = -\tilde{\nabla}g(T) - \lambda\tilde{\nabla}h(T) = -\tilde{\nabla}[g(T) + \lambda h(T)] = -\tilde{\nabla}\Phi \tag{24}$$

where the potential function Φ is defined by

$$\Phi - \Phi_0 = g(T) + \lambda h(T) = \int_{T_0}^{T} \left[K_m(T) + \frac{\lambda}{M(T)}\frac{dP}{dT}\right] dT \tag{25}$$

and Φ_0 is an arbitrary constant that can be used to fix the level of the potential. Since P is only a function of T, Φ is a function of T only, and hence P and T are functions of Φ only.

Equations (15) and (16) have been expressed in terms of a potential as given by equations (21) and (20) with the potential given by equation (25). Now equation (14) will be considered. This can be written as

$$\tilde{\nabla}^2 P = \frac{1}{MT}\frac{d(MT)}{d\Phi}\tilde{\nabla}\Phi \cdot \frac{dP}{d\Phi}\tilde{\nabla}\Phi = \frac{1}{MT}\frac{d(MT)}{d\Phi}\frac{dP}{d\Phi}\left|\tilde{\nabla}\Phi\right|^2 \tag{26}$$

Now, $\tilde{\nabla}^2 P = \tilde{\nabla} \cdot \tilde{\nabla}P = \tilde{\nabla} \cdot \frac{dP}{d\Phi}\tilde{\nabla}\Phi$. In view of equation (21) this becomes[1]

$$\tilde{\nabla}^2 P = \frac{d^2P}{d\Phi^2}\left|\tilde{\nabla}\Phi\right|^2 \tag{27}$$

From equation (25)

$$\frac{d\Phi}{dT} = K_m + \lambda\frac{1}{M}\frac{dP}{dT} = K_m + \frac{\lambda}{M}\frac{dP}{d\Phi}\frac{d\Phi}{dT} \tag{28}$$

Equations (27) and (28) are used to eliminate P from equation (26) giving the following equation, which is a restatement of equation (14) in a form that determines T as a function of Φ,

$$\frac{d^2T}{d\Phi^2} + \frac{1}{K_m T}\frac{dT}{d\Phi}\left[1 - \left(K_m - T\frac{dK_m}{dT}\right)\frac{dT}{d\Phi}\right] = 0$$

[1]Note that $\tilde{\nabla} \cdot \frac{dP}{d\Phi}\tilde{\nabla}\Phi = \frac{dP}{d\Phi}\tilde{\nabla} \cdot \tilde{\nabla}\Phi + \tilde{\nabla}\Phi \cdot \tilde{\nabla}\frac{dP}{d\Phi} = \frac{dP}{d\Phi}\tilde{\nabla}^2\Phi + \tilde{\nabla}\Phi \cdot \frac{d^2P}{d\Phi^2}\tilde{\nabla}\Phi = \frac{dP}{d\Phi}\tilde{\nabla}^2\Phi + \frac{d^2P}{d\Phi^2}\left|\tilde{\nabla}\Phi\right|^2$

or equivalently

$$\frac{d}{d\Phi}\left[\frac{1}{T}\left(K_m \frac{dT}{d\Phi} - 1\right)\right] = 0 \tag{29}$$

Equation (29) can be integrated to obtain $T(\Phi)$, but before doing this the first boundary condition of equation (17) will be expressed in terms of Φ. It follows from equations (24) and (23) that

$$\frac{\lambda}{M} \tilde{\nabla} P = \tilde{\nabla}\Phi - K_m \tilde{\nabla} T$$

Substituting into the boundary condition gives

$$\frac{K_m T}{T - 1} \hat{n}_0 \cdot \tilde{\nabla} T + \hat{n}_0 \cdot (\tilde{\nabla}\Phi - K_m \tilde{\nabla} T) = 0$$

Using the relation $\tilde{\nabla} T = (dT/d\Phi)\tilde{\nabla}\Phi$ results in a new expression of the boundary condition:

$$\left(1 + \frac{K_m}{T - 1} \frac{dT}{d\Phi}\right) \hat{n}_0 \cdot \tilde{\nabla}\Phi = 0 \qquad \text{for } (X, Y) \text{ on } S_0 \tag{30}$$

Solution for Temperature Ratio in Terms of Potential Function

Equation (29) can be integrated twice to obtain

$$\Phi - \Phi_0 = \int_{T_0}^{T} \frac{K_m \, dT}{1 + C_1 T} \tag{31}$$

The constants C_1 and Φ_0 are to be evaluated from the boundary conditions. Thus inserting the solution (31) into equation (30) gives

$$(1 + C_1) \frac{T}{T - 1} \hat{n}_0 \cdot \tilde{\nabla}\Phi = 0 \qquad \text{for } (X, Y) \text{ on } S_0$$

This boundary condition is satisfied if $C_1 = -1$ so that equation (31) gives the relation between T and Φ

$$\Phi - \Phi_0 = \int_{T_0}^{T} \frac{K_m}{1 - T}\, dT \tag{32}$$

Relation Between Pressure Function and Temperature Ratio

The second of boundary conditions (17) involves P. It is necessary to relate P to T so that P can be related to Φ and the boundary condition formulated in terms of Φ. From equation (28)

$$\frac{dP}{dT} = \left(\frac{d\Phi}{dT} - K_m\right)\frac{M}{\lambda} \tag{33}$$

and from equation (32)

$$\frac{d\Phi}{dT} = \frac{K_m}{1 - T} \tag{34}$$

These relations are combined to obtain

$$\frac{dP}{dT} = \left(\frac{1}{1 - T} - 1\right)\frac{K_m M}{\lambda} = \frac{T}{1 - T}\,\frac{K_m M}{\lambda} \tag{35}$$

Integrating this from S_0 to an arbitrary position in the medium and taking into account the second boundary condition (17) gives the relation between P and T,

$$P - 1 = \frac{1}{\lambda}\int_{T_0}^{T} K_m M \frac{T}{1 - T}\, dT \tag{36}$$

The value T_0 is unknown in this integral. To evaluate it, the first and third boundary conditions of equation (18) are imposed to give

$$P_s - 1 = \frac{1}{\lambda} \int_{T_0}^{T_s} K_m M \frac{T}{1 - T} \, dT \tag{37}$$

This integral can be carried out once the thermal conductivity and viscosity variations are specified so that $K_m(T)$ and $M(T)$ are known. The quantity T_0 is thereby related to the specified quantities P_s and T_s.

Boundary Conditions for Determination of Potential Function

The temperature has been obtained in terms of a potential Φ (eq. (32)), and the pressure has been found in terms of the temperature (eqs. (36) and (37)). Hence all quantities will be known if the potential can be found as a function of position in the porous material. To obtain the simplest boundary conditions for the potential Φ, which is governed by Laplace's equation, the constant Φ_0 in equation (25) can be set equal to zero to give

$$\Phi = \int_{T_0}^{T} \left[K_m(T) + \frac{\lambda}{M(T)} \frac{dP}{dT} \right] dT \tag{38}$$

and since $T = T_0$ along S_0 the boundary condition is

$$\Phi = 0 \qquad \text{for } (X, Y) \text{ on } S_0 \tag{39}$$

With $\Phi_0 = 0$ equation (32) becomes

$$\Phi = \int_{T_0}^{T} \frac{K_m}{1 - T} \, dT \tag{40}$$

This is evaluated at the coolant exit face to obtain the boundary condition

$$\Phi = \Phi_s \equiv \int_{T_0}^{T_s} \frac{K_m}{1 - T} \, dT \qquad \text{for } (X, Y) \text{ on } S \tag{41}$$

The surface temperature T_s is specified and T_0 is calculated from equation (37). Hence, Φ_s is known.

It follows from equation (20) and the second condition in equation (19) that Φ must satisfy the boundary condition on the insulated impervious surfaces

$$\hat{n}_i \cdot \tilde{\nabla}\Phi = 0 \qquad \text{for } (X,Y) \text{ on } S_i \tag{42}$$

Since P is a function only of T, and T is a function only of Φ, the first boundary condition (19) is then automatically satisfied.

The remaining boundary condition to be considered is that the free boundary must have a shape such that the second condition of equation (18) will be satisfied. The condition can be written as

$$K_m \hat{n}_s \cdot \frac{dT}{d\Phi} \tilde{\nabla}\Phi = Q_s \qquad (X,Y) \text{ on } S$$

From equation (34) $(dT/d\Phi)_s = (1 - T)/K_m \big|_s$ so the boundary condition becomes

$$\hat{n}_s \cdot \tilde{\nabla}\Phi = \frac{Q_s}{1 - T_s} \qquad (X,Y) \text{ on } S \tag{43}$$

Equations (39) and (41) to (43) provide the required boundary conditions to solve equation (21) for Φ. In the solution of Laplace's equation for the potential function, it would be convenient to have the boundary potentials on S_0 and S go from 0 to 1 rather than from 0 to Φ_s, and to have the normal derivative on the coolant exit boundary be unity. To accomplish this, the potential is normalized as

$$\varphi(X,Y) = \frac{\Phi(X,Y)}{\Phi_s} \tag{44}$$

Also let $Q_s/(1 - T_s)\Phi_s = D$ and

$$\approx{\nabla} = \frac{\tilde{\nabla}}{D} = \hat{i}\frac{\partial}{\partial XD} + \hat{j}\frac{\partial}{\partial YD} = \hat{i}\frac{\partial}{\partial \overline{\overline{X}}} + \hat{j}\frac{\partial}{\partial \overline{\overline{Y}}}$$

Then φ is obtained from

$$\overset{\approx}{\nabla}{}^2 \varphi = 0 \tag{45}$$

with the boundary conditions

$$\varphi = 0 \qquad \text{for } (\overline{X}, \overline{Y}) \text{ on } S_0 \tag{46a}$$

$$\left.\begin{aligned} \varphi &= 1 \\ \hat{n}_s \cdot \approx{\nabla}\varphi &= 1 \end{aligned}\right\} \quad \text{for } (\overline{X}, \overline{Y}) \text{ on } S \tag{46b, 46c}$$

$$\hat{n}_i \cdot \approx{\nabla}\varphi = 0 \qquad \text{for } (\overline{X}, \overline{Y}) \text{ on } S_i \tag{46d}$$

These conditions are summarized in figure 1(b).

By eliminating T_0 from equations (40) and (41) the temperature distribution in terms of φ is given by

$$\varphi - 1 = -\frac{1}{\Phi_s} \int_T^{T_s} \frac{K_m}{1 - T} \, dT \tag{47}$$

Mass Flux at Coolant Exit Surface

A quantity of practical interest is the local coolant flux distribution leaving the porous medium,

$$\rho\vec{u} \cdot \hat{n}_s = -\frac{\kappa}{\mu(t)} \rho \nabla p \cdot \hat{n}_s \qquad \text{for } (x, y) \text{ on } S \tag{48}$$

First consider the case of a compressible flow. By using equation (5) and introducing dimensionless variables this becomes

$$\rho\vec{u} \cdot \hat{n}_s = -\frac{\kappa}{\mu(t)} \frac{p}{Rt} \nabla p \cdot \hat{n}_s = -\frac{k_{m,\infty}}{C_p h_r} \frac{\lambda}{MT} \tilde{\nabla} P \cdot \hat{n}_s \tag{49}$$

But upon noting that $\tilde{\nabla} P = (dP/dT)(dT/d\Phi)\tilde{\nabla}\Phi$ we find from equations (35) and (34) that

$$\tilde{\nabla} P = \frac{K_m M}{\lambda} \left(\frac{T}{1 - T}\right) \frac{(1 - T)}{K_m} \tilde{\nabla}\Phi = \frac{MT}{\lambda} \tilde{\nabla}\Phi$$

Inserting this in equation (49) gives

$$\frac{h_r C_p}{k_{m,\infty}} \rho\vec{u} \cdot \hat{n}_s = -\hat{n}_s \cdot \tilde{\nabla}\Phi \qquad \text{for } (X, Y) \text{ on } S \tag{50}$$

By use of equation (43) the dimensionless mass flux is given by

$$\frac{h_r C_p}{k_{m,\infty}} \rho\vec{u}_s \cdot \hat{n}_s = \frac{Q_s}{T_s - 1} \tag{51}$$

or

$$\rho\vec{u}_s \cdot \hat{n}_s = \frac{q_s}{C_p(t_s - t_\infty)} \tag{52}$$

A similar manipulation for incompressible flow shows that equation (52) also applies for that case.

Determination of Unknown Coolant Exit Boundary

The analysis up to this point has shown that the temperature distribution in the porous medium can be expressed in terms of a potential where the potential is a solution to Laplace's equation. The difficulty in solving the Laplace equation in the porous medium is that the coolant exit boundary has an unknown shape. In this problem the heat flux and temperature are both given along the coolant exit boundary. The relation between these quantities is what is usually desired in a porous medium heat-transfer analysis. The detailed temperature distribution in the medium is generally of lesser importance but is needed in thermal-stress calculations. Hence, the main result to be obtained here is the shape of the unknown coolant exit boundary - this type of problem is often called a free-boundary problem.

The solution of free-boundary problems has its foundation in inviscid free-jet theory where the velocity potential in the jet flow region is governed by Laplace's equation. In reference 2 a free-boundary theory was developed for use in freezing problems. The unknown shape was determined of a steady-state two-dimensional frozen region formed by cooling the frozen region along part of its boundary while subjecting another part to a convective heating condition. This method was extended to other frozen shapes in reference 3 and to a radiative heating condition in reference 4. (In ref. 4 the specified heat flux at the surface is nonuniform.)

The present solution has been shown to depend on a potential with boundary conditions given in figure 1(b). A comparison with the solution method as described in reference 2 reveals that these are the identical boundary conditions in the free-boundary freezing problem. Hence, this previously developed technique, and the results obtained with it, can be directly applied to the present problem. The general analytical technique will be briefly reviewed here. Then some examples will be given of porous configurations that can be applied in practice.

The analytical technique is based on the use of conformal mapping. Following reference 2 the negative potential is taken as the real part of an analytic function:

$$W = -\varphi + i\psi \tag{53}$$

The lines of constant φ and constant ψ form a two-dimensional orthogonal net. By using conformal transformations between the W plane (to be described) and the physical plane, the transformed functions will always be analytic. Hence in the physical plane the potential will satisfy Laplace's equation. Since a function that is analytic in a given region is completely determined by its boundary values, it is only necessary to require that the mapping satisfy the boundary conditions.

The potential W plane is shown in figure 2, and the porous region occupies a rectangle. The coolant inlet and outlet surfaces S_0 and S are lines of constant φ (eqs. (46a) and (46b)). Along the impermeable S_i surfaces the normal derivative of φ is zero (eq. (46d)); as a result of this orthogonality, these boundaries are lines of constant ψ. The height of the rectangle is unknown and will be related to the specified heat flux along S. If the mapping between the W plane and the physical plane can be obtained subject to the constraint in equation (46c), then the line 3-4 in the W plane can be transformed into the physical plane and the unknown free surface thus determined.

Let the coordinates in the dimensionless physical plane (fig. 1(b)) be designated by the complex variable $\overline{Z} = \overline{X} + i\overline{Y}$. To find the relation between the W and $\overline{Z}$ planes the fact is utilized that the derivative of an analytic function is independent of direction and hence the derivative with respect to $\overline{Z}$ can be written as

$$\frac{dW}{d\overline{Z}} = -\frac{\partial\varphi}{\partial\overline{X}} + i\,\frac{\partial\psi}{\partial\overline{X}} \tag{54}$$

Using the Cauchy-Riemann equation $\partial\psi/\partial\overline{X} = \partial\varphi/\partial\overline{Y}$ and defining a quantity ζ gives

$$\frac{dW}{d\overline{Z}} = -\frac{\partial\varphi}{\partial\overline{X}} + i\,\frac{\partial\varphi}{\partial\overline{Y}} \equiv \zeta \tag{55}$$

This is integrated to give

$$\bar{Z} = \int \frac{1}{\zeta} dW + C_2 \tag{56}$$

where C_2 is an integration constant. If ζ can be found as a function of W then the integration can be performed to obtain $\bar{Z}$ which thus relates $\bar{Z}$ to W.

To obtain a relation between ζ and W a potential derivative plane $(-\partial\varphi/\partial\bar{X}$ against $i\,\partial\varphi/\partial\bar{Y})$ is constructed from the known boundary conditions. In addition to the normal derivative boundary conditions shown in figure 1(b), there is the condition that $d\varphi/dl = 0$ along S_0 where l is a path along the boundary. This gives along S_0

$$\frac{d\varphi}{dl} = \frac{\partial\varphi}{\partial\bar{X}}\frac{d\bar{X}}{dl} + \frac{\partial\varphi}{\partial\bar{Y}}\frac{d\bar{Y}}{dl} = 0 \tag{57}$$

Since S_0 has a specified shape, the $d\bar{X}/dl$ and $d\bar{Y}/dl$ are known along this boundary. Hence, the curve representing this boundary can be drawn on the ζ plane. Along S the boundary condition $\hat{n}_s \cdot \bar{\bar{\nabla}}\varphi = 1$ means that this boundary will be part of a unit circle in the ζ plane. This behavior will be illustrated by a few examples.

After the representation in the ζ plane has been found, the conformal mapping between the ζ and W planes gives the required $\zeta(W)$ relation so that the integral in equation (56) can be evaluated.

To demonstrate the method and obtain some useful shapes for porous cooled media, two configurations will be analyzed: a leading-edge region with fluid supplied through a coolant slot and a porous duct with a rectangular outer boundary.

EXAMPLES ILLUSTRATING APPLICATION OF GENERAL METHOD

Leading-Edge Region Fed Through Coolant Slot

This geometry is shown in figure 3(a) and consists of a porous leading-edge region with coolant being supplied through a slot in an insulated wall that supports the porous region. The coolant exit surface S is to be maintained at uniform temperature t_s under the influence of a uniform imposed heat flux q_s, which could be supplied, for example, by convection from an external stream.

The region in dimensionless coordinates is shown in figure 3(b) along with the values of φ and its derivatives on the boundaries. From this point on, the analysis exactly parallels that for determining the configuration of a frozen layer in reference 2, and these previous results can be used with only a change in nomenclature. The regions in the potential and potential derivative planes are given in figure 4.

The mapping functions between these planes have been obtained in reference 2, and the integration in equation (56) carried out for $\overline{Z}(W)$. The result was evaluated along the coolant exit surface to yield its coordinates as

$$\left.\begin{aligned} \left.\frac{x}{a}\right|_s &= \frac{1}{2 \ln \sqrt{1 - b^2}} \ln\left(\frac{1 + \xi}{1 - \xi}\right) \\ \left.\frac{y}{a}\right|_s &= \frac{-1}{\ln \sqrt{1 - b^2}} \tan^{-1} \sqrt{\frac{b^2 - \xi^2}{1 - b^2}} \end{aligned}\right\} \tag{58}$$

where ξ is a variable with range $-b \leq \xi \leq b$. The b depends on the imposed physical conditions, and this relation will now be determined.

In the notation of reference 2 the boundary condition at the free surface was $\partial T/\partial N = 1$, and in the present analysis it is $\hat{n}_s \cdot \overset{\approx}{\nabla}\varphi = 1$. The T and φ are comparable quantities so that the only difference is in the nondimensionalizing factor in N compared with that in the $\overset{\approx}{\nabla}$. This shows that the quantity γ in reference 2 is to be replaced in the present analysis by $h_r/D = k_{m,\infty}(t_\infty - t_s)\Phi_s/q_s$. Then it directly follows from equation (32) in reference 2 that the parameter b is related to the physical quantities by

$$\frac{k_{m,\infty}(t_s - t_\infty)|\Phi_s|}{q_s a} = - \frac{b}{\ln \sqrt{1 - b^2}} K\left(\sqrt{1 - b^2}\right) \tag{59}$$

Another quantity of interest is the total heat (Q_{tot}) being conducted into the surface S, which is also equal to the energy transferred to and then carried out of the medium by the coolant. This is equal to the imposed heat flux integrated over the area of the free surface S. The flow rate per unit area into the medium along S_0 is given by $-\rho\vec{u} \cdot \hat{n}_0\big|_{S_0}$. From equation (50) it follows that this can be written for the present geometry as

$$\frac{h_r C_p \rho \vec{u} \cdot \hat{n}_0}{k_{m,\infty}} = \hat{n}_0 \cdot \tilde{\nabla}\Phi = \frac{\partial \Phi}{\partial Y} \qquad \text{for } (X, Y) \text{ on } S_0$$

To obtain the total flow, integrate over the inlet surface S_0 to obtain

$$2\int_4^3 \frac{h_r C_p \rho \vec{u} \cdot \hat{n}_0}{k_{m,\infty}} dx = 2h_r \int_4^3 \frac{\partial \Phi}{\partial Y} dX = 2\Phi_s h_r \int_4^3 \frac{\partial \varphi}{\partial \overline{Y}} d\overline{X}$$

(The integration limits refer to the numbers in fig. 3(b).) By use of the Cauchy-Riemann equations this can be integrated to obtain

$$2\int_4^3 \frac{h_r C_p \rho \vec{u} \cdot \hat{n}_0}{k_{m,\infty}} dx = 2\Phi_s h_r \int_4^3 \frac{\partial \psi}{\partial \overline{X}} d\overline{X} = 2\Phi_s h_r [\psi(3) - \psi(4)] = 2\Phi_s h_r [\psi(2) - \psi(1)]$$

The conformal mapping results (eqs. (24) and (31) of ref. 2) give

$$\psi(2) - \psi(1) = \frac{K(b)}{K\sqrt{1 - b^2})}$$

From an overall heat balance

$$Q_{tot} = -2\int_4^3 \rho \vec{u} \cdot \hat{n}_0 C_p (t_s - t_\infty) dx$$

Combining these relations and noting that Φ_s is negative, gives

$$\frac{Q_{tot}}{2k_{m,\infty}(t_s - t_\infty)|\Phi_s|} = \frac{C_p}{2k_{m,\infty}|\Phi_s|} G = \frac{K(b)}{K\left(\sqrt{1 - b^2}\right)} \tag{60}$$

The Q_{tot} is per unit length normal to the x-y plane in figure 3(a). The G is the total mass flow of coolant per unit time and unit length normal to the x-y plane, that is

$$G = -2\int_4^3 \rho \vec{u} \cdot n_0 \, dx$$

(The negative sign is present because the $\vec{u}$ is in a direction opposite to n_0.)

For a given set of physical conditions the left side of equation (59) is computed (note that the Φ_s is found from eqs. (41) and (37)). The b can then be found from equation (59), and the coordinates of surface S and the Q_{tot} obtained from equations (58) and (60). The results are shown in figures 5 and 6.

Porous Duct with Rectangular Outer Boundary

As a second example of the conformal mapping technique, results are given for a porous cooled rectangular duct (fig. 7(a)). An interior shape of the duct cross section is to be obtained that will maintain the interior surface at a given temperature when a uniform heat flux is being transferred to the interior surface by convection or radiation from a high-temperature gas. From symmetry, only the lower left quadrant of the duct need be considered.

A comparable free-boundary problem for a solidified region inside a duct was analyzed in reference 3. These results can be directly used by only changing the nomenclature as in the previous example. The porous region in dimensionless coordinates is shown in figure 7(b). The mappings into the potential and potential derivative planes are given in figure 8. For detailed information on these mappings and the conformal transformation between the ζ and W planes the reader is referred to reference 3. A brief summary of results is given as follows:

The aspect ratio of the duct is related to two mapping parameters c and d by

$$\frac{a}{b} = \frac{K\left(\sqrt{\frac{d+1}{d+c}}\right) + K\left(\sqrt{\frac{d-1}{d+c}}\right)}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \qquad \begin{pmatrix} c > 1 \\ d > 1 \end{pmatrix} \tag{61}$$

where K is the complete elliptic integral of the first kind. This is used to find c,d pairs that correspond to various aspect ratios. Then the coordinates of the coolant exit surface of the porous medium can be found from

$$\left.\begin{aligned} \left.\frac{x}{b}\right|_S &= \frac{a}{b} - \frac{F\left[\sin^{-1}\sqrt{\frac{(c+d)(1+\xi)}{(1+d)(c+\xi)}},\ \sqrt{\frac{1+d}{c+d}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \\ \left.\frac{y}{b}\right|_S &= 1 - \frac{F\left[\sin^{-1}\sqrt{\frac{(c+d)(1-\xi)}{(1+c)(d-\xi)}},\ \sqrt{\frac{1+c}{c+d}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \end{aligned}\right\} \quad -1 < \xi < 1 \tag{62}$$

where F is the elliptic integral of the first kind.

The values of c and d can also be used to obtain the corresponding imposed physical conditions from the relation

$$\frac{k_{m,\infty}(t_s - t_\infty)|\Phi_s|}{q_s b} = \frac{\sqrt{\frac{2(c+d)}{(c+1)(d+1)}}\, K\left[\sqrt{\frac{(c-1)(d-1)}{(c+1)(d+1)}}\right]}{K\left(\sqrt{\frac{c+1}{c+d}}\right) + K\left(\sqrt{\frac{c-1}{c+d}}\right)} \tag{63}$$

The total heat transferred to the interior surface of the porous medium (equal to the energy carried away by the coolant) per unit length of the duct is given by

$$\frac{Q_{tot}}{4k_{m,\infty}(t_s - t_\infty)|\Phi_s|} = \frac{c_p}{4k_{m,\infty}|\Phi_s|} G = \frac{K\left[\sqrt{\frac{2(c+d)}{(c+1)(d+1)}}\right]}{K\left[\sqrt{\frac{(c-1)(d-1)}{(c+1)(d+1)}}\right]} \tag{64}$$

The configurations of the porous region for various aspect ratios are given in figure 9. Each of these figures gives curves for various values of the parameter containing the imposed physical conditions. Figure 10 gives the total heat transfer and coolant mass flow rates as a function of this parameter.

SUMMARY OF ANALYSIS

The geometric aspects of the solution are concerned with determining the unknown free boundary of a two-dimensional region in which Laplace's equation is valid (eq. (45)) and with boundary conditions (46). This can be accomplished by conformal mapping as described in relation to equation (56).

The mapping requires representing the porous region in the complex potential and potential derivative planes. In the complex potential plane the region is always a rectangle, and the derivative boundary conditions determine its shape in the potential derivative plane. The dimensionless heat flow rate as given for the examples by equations (60) and (64) involve Φ_s. This is obtained from equation (41) in which the quantity T_0 is found from equation (37).

The potential at any location in the porous region can be obtained from the result of the integration in equation (56), which relates the coordinates in the potential plane to those in the dimensionless physical plane. The temperature at that physical location can be found from equation (47), which gives the relation between the temperature and the potential. The T_0 in this relation is found from equation (37).

DISCUSSION

A method is developed for obtaining the heat-transfer characteristics of a two-dimensional porous cooled region having a free boundary. A free-boundary shape is found such that it will be at a specified temperature while being subjected to an imposed uniform heat flux. The solution was obtained by showing that all the heat-transfer quantities could be related to a potential function that is a solution to Laplace's equation in the porous region. Conformal mapping could then be used, as developed for free-boundary problems, to obtain the configuration of the unknown boundary. The procedure can be extended in principle to a nonuniform heat flux at the free boundary. In this instance the free boundary in the potential derivative plane would not be part of a circle, and the mapping into the potential plane would require more sophisticated analytical techniques or numerical methods. An example of this type of mapping by using a series expansion is given in reference 5. In reference 4 a particular nonuniform heat flux is considered as imposed by radiant heating.

Results are given for a porous leading edge region as shown in figure 3(a). This type of geometry might be used to cool the stagnation region along the leading edge of an airfoil. The shape of the porous region is shown in figure 5 as a function of the parameter $k_{m,\infty}(t_s - t_\infty)|\Phi_s|/q_s a$. Figure 5 gives the coolant total mass flow rate and the total heat transferred to the coolant exit surface of the porous region per unit length of the region. From an overall energy balance the Q_{tot} must be transferred to the coolant. Hence there is the relation

$$Q_{tot} = GC_p(t_s - t_\infty)$$

Because of the interaction of the various quantities, the results in figures 5 and 6 can be discussed in various ways. One approach is to consider a situation where a given pressure ratio is available and the surface of the porous medium is not to exceed a certain temperature t_s for a specified heat loading q_s. With these quantities given, the parameter $k_{m,\infty}(t_s - t_\infty)|\Phi_s|/q_s a$ is uniquely determined, and the shape of the porous region is found from figure 5. For the purpose of discussion, consider what would happen if q_s is kept constant and the porous region is made thinner. For simplicity ignore property variations so that Φ_s remains fixed. Then from figure 5 a thinner region corresponds to a smaller value of the parameter and hence t_s is decreased. It would appear then that an improved cooling situation would result by decreasing the thickness. However, it must be realized that according to figure 6 a decrease in the abscissa corresponds to an increased coolant flow. Hence decreasing the surface temperature to below what is really required results in wasted coolant. If the thickness of the porous region is made larger than given by the solution, the coolant flow will be decreased and the surface temperature will be higher than desired. Thus the solution gives the porous region shape that provides the minimum coolant flow with the correct flow distribution to cope with the imposed heat load and not let the surface temperature exceed the temperature limit that has been specified for it.

When the porous region in figure 5 becomes very thick compared with the coolant slot width, it approaches a circular shape. This would be expected from the uniform boundary conditions. The coolant is then flowing outward in a radial way from what is approximately a line source (normal to the x, y plane) at the origin of the region.

The second configuration treated is a porous cooled duct with rectangular outer shape as shown in figure 7(a). The coordinates and boundary conditions in the dimensionless system are shown in figure 7(b). The mappings into the potential and potential derivative planes are given in figure 8. This yields the porous shapes in figure 9 for ducts with aspect ratios from 1 to 5. For each aspect ratio a set of curves is given for various values of the physical parameter $k_{m,\infty}(t_s - t_\infty)|\Phi_s|/q_s b$. It should be noted that the mapping yields two roots for some values of the parameter. The group corresponding to very thick porous regions is shown dashed. The corresponding heat and mass flows are shown in figure 10.

The interesting feature of these results is the fact that for each aspect ratio there is a maximum value of the physical parameter beyond which no solution exists. To elaborate on this, consider a situation where t_s is fixed and q_s is being changed. For a large q_s the physical parameter is small, and the solution given by the solid curves in figure 9 gives a thin layer. From figure 10 the corresponding flow rate is large. If the q_s is decreased, the physical parameter is increased. Less coolant is required, and the thickness of the porous region is increased to decrease the flow and thus maintain the surface at the specified t_s. As q_s is further decreased, the thickness of the porous medium can be increased to reduce the flow and thus maintain the surface at t_s. However, as the thickness is increased, the inside surface area of the duct is reduced; thereby, the total heat loading on the surface is being reduced more rapidly than in direct proportion to the reduction of q_s. A point is reached for a very thick region where to maintain a given t_s the q_s must be increased with increased region thickness. If q_s were decreased, the t_s would decrease below the specified value. This is in the range of the dashed curves in figures 9 and 10.

CONCLUSIONS

An analytical solution was obtained to determine the optimum shape of a two-dimensional porous cooled region. The optimum shape provides the proper distribution of coolant flow through the region surface that is subjected to a uniform heat loading in order to maintain this surface at a specified uniform temperature. The analysis includes the effects of variable fluid viscosity and density, and it includes the effect of variable heat conductivity of the porous matrix material. The solution was found by combining the governing equations in such a way that a potential could be defined. This potential satisfies Laplace's equation in the porous region and is subject to simple boundary conditions. All of the heat-transfer quantities can be related to this potential. The potential and shape of the porous region are obtained by use of a conformal mapping technique analogous to the free streamline method of irrotational inviscid free-jet theory.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 15, 1972,
502-28.

APPENDIX - SYMBOLS

A,B dimensionless half widths of long and short sides of rectangular duct

a half width of coolant slot; half width of long side of rectangular duct

b parameter in mapping; half width of short side of rectangular duct

C_p specific heat at constant pressure

C_1, C_2 constants of integration

c,d mapping parameters for rectangular duct

D dimensionless quantity, $Q_s/(1 - T_s)\Phi_s$

F elliptic integral of the first kind, $F(\eta,k) = \int_0^\eta d\eta/\sqrt{1 - k^2 \sin^2 \eta}$

G mass flow rate of coolant per unit length normal to x,y plane

h_r reference dimension of porous region

$\hat{i},\hat{j}$ unit vectors in x and y directions, respectively

K complete elliptic integral of the first kind, $K(k) = \int_0^{\pi/2} d\eta/\sqrt{1 - k^2 \sin^2 \eta}$

K_m thermal conductivity ratio, $k_m/k_{m,\infty}$

k_m effective thermal conductivity of porous region

l path along boundary S_0 in $\bar{X},\bar{Y}$ plane

M for incompressible case $M = (1/2)(\mu/\mu_\infty)(t_\infty/t)$; for compressible case $M = \mu/\mu_\infty$

$\bar{N}$ dimensionless outward normal coordinate in $\bar{X},\bar{Y}$ plane

$\hat{n}$ unit outward normal

P for incompressible case $P = p/p_\infty$; for compressible case $P = (p/p_\infty)^2$

p pressure

Q_s dimensionless heat flux imposed at interface $q_s h_r/k_{m,\infty} t_\infty$

Q_{tot} heat conducted into porous surface (equal to heat removed by coolant) per unit length normal to x,y plane

$\vec{Q}$ dimensionless energy flux vector, $\vec{q} h_r/k_{m,\infty} t_\infty$

q_s heat flux imposed at porous surface S

$\vec{q}$ energy flux vector, $-k_m \nabla t + \rho \vec{u} C_p t$

R perfect gas constant

S coolant exit surface of porous medium

S_i insulated and impervious surface

S_0 coolant inlet surface of porous medium

T temperature ratio t/t_∞

t absolute temperature

$\vec{u}$ velocity vector

W complex potential $-\varphi + i\psi$

X, Y dimensionless coordinates, x/h_r, y/h_r

$\bar{\bar{X}}, \bar{\bar{Y}}$ dimensionless coordinates, XD, YD

x, y coordinates in physical plane

$\bar{\bar{Z}}$ complex variable $\bar{\bar{X}} + i\bar{\bar{Y}}$

ζ complex potential derivative, $-\frac{\partial \varphi}{\partial \bar{\bar{X}}} + i \frac{\partial \varphi}{\partial \bar{\bar{Y}}}$

κ permeability of porous material

λ parameter, $\rho_\infty C_p \kappa p_\infty / 2\mu_\infty k_{m,\infty}$

μ fluid viscosity

ξ dummy variable in mapping

ρ fluid density

Φ potential defined by equation (25)

Φ_s potential along coolant exit surface

φ dimensionless potential Φ/Φ_s

ψ function orthogonal to φ

$\tilde{\nabla}$ dimensionless gradient $\hat{i} \frac{\partial}{\partial X} + \hat{j} \frac{\partial}{\partial Y}$

$\approx\!\!\nabla$ dimensionless gradient $\hat{i} \frac{\partial}{\partial \bar{\bar{X}}} + \hat{j} \frac{\partial}{\partial \bar{\bar{Y}}}$

Subscripts:

i insulated and impervious surface

s on surface where coolant exits from porous medium

0 on surface where coolant enters porous medium

∞ at coolant reservoir condition

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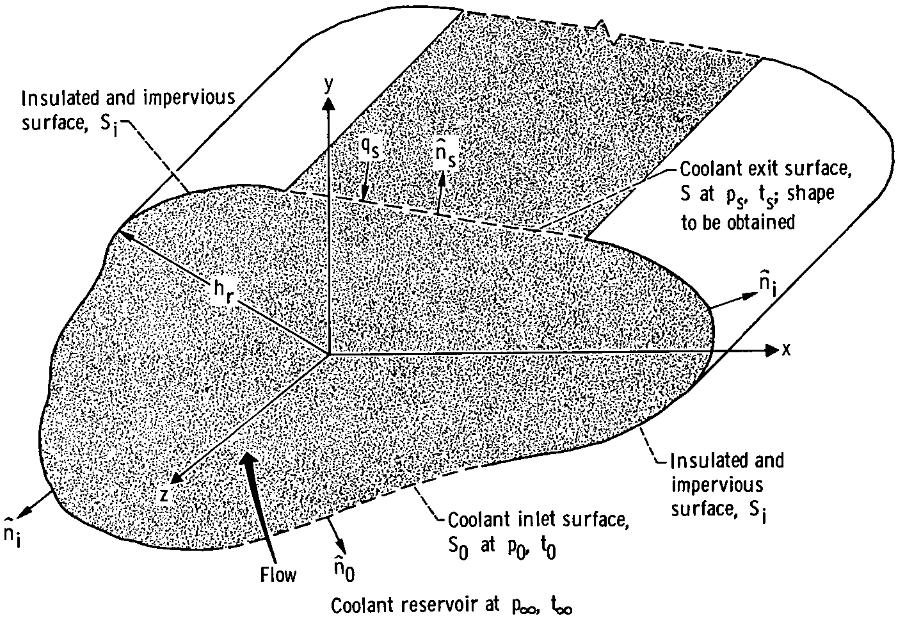
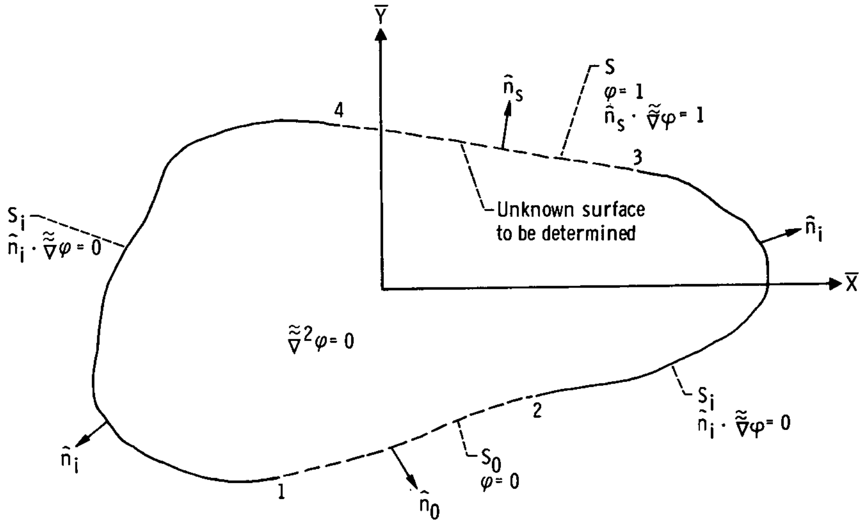


(a) Physical coordinates and imposed conditions.



(b) Dimensionless coordinates with boundary conditions in terms of potential.

Figure 1. - Two-dimensional porous region with unknown shape of coolant exit surface; $p_\infty > p_s$, $t_s > t_\infty$.

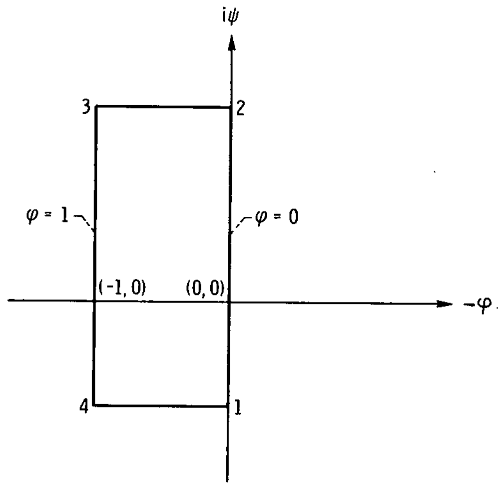
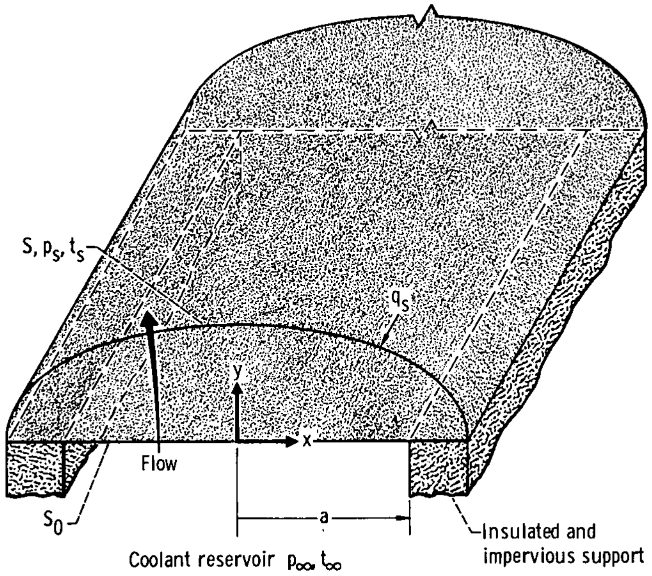
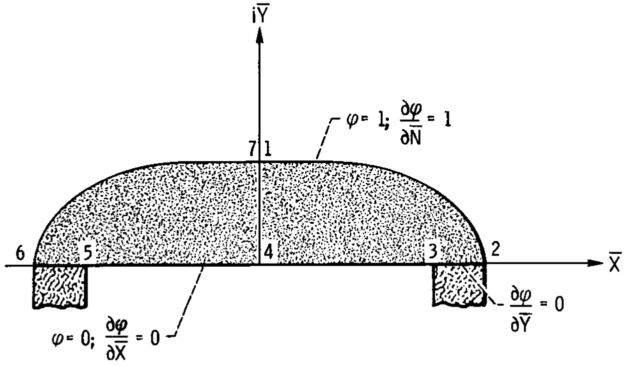


Figure 2. - Porous region mapped into potential plane, $W = -\varphi + i\psi$.

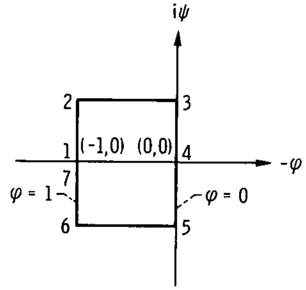


(a) Physical coordinates and imposed conditions.

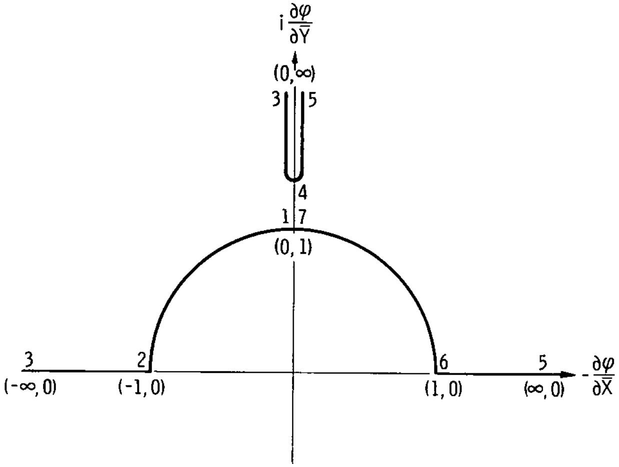


(b) Dimensionless coordinates with boundary conditions in terms of a potential.

Figure 3. - Porous leading-edge region with coolant supplied through slot.

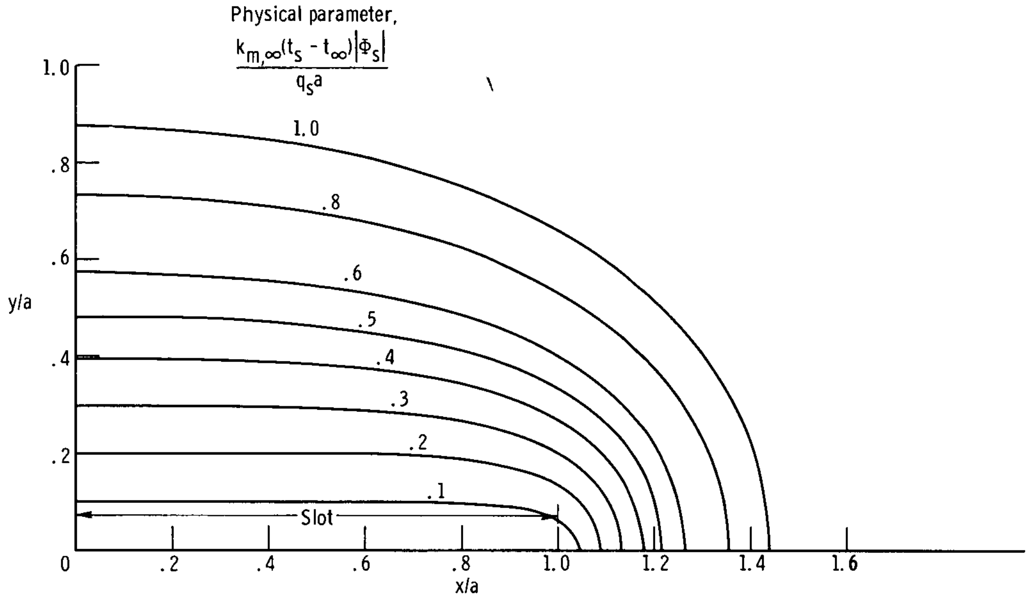


(a) Region in potential plane, $W = -\varphi + i\psi$.

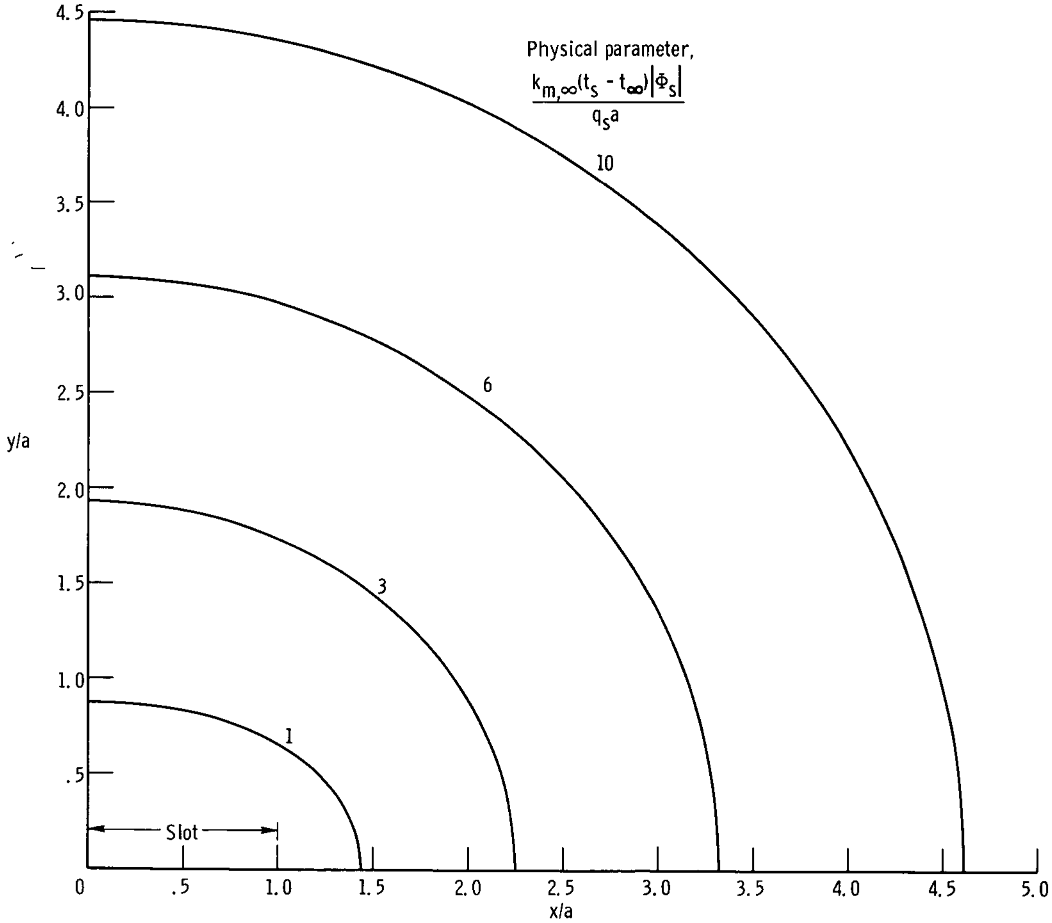


(b) Region in potential derivative plane, $\zeta = -(\partial\varphi/\partial\bar{X}) + i(\partial\varphi/\partial\bar{Y})$.

Figure 4. - Auxiliary planes used in mapping procedure.



(a) Physical parameter ≤1.



(b) Physical parameter ≥1.

Figure 5. - Configurations of porous leading-edge region for various values of imposed physical parameter.

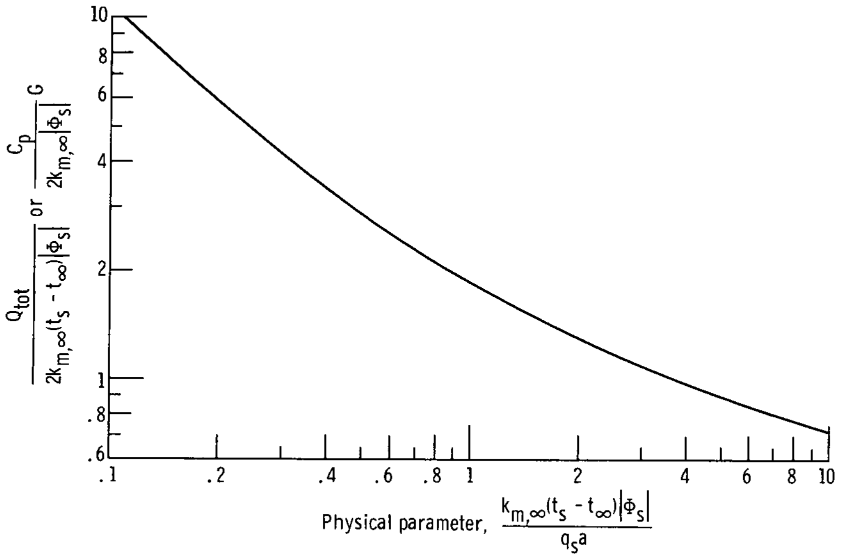
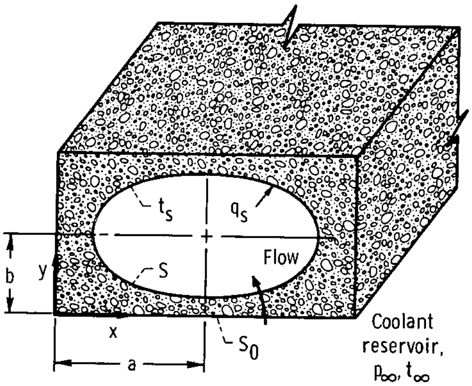
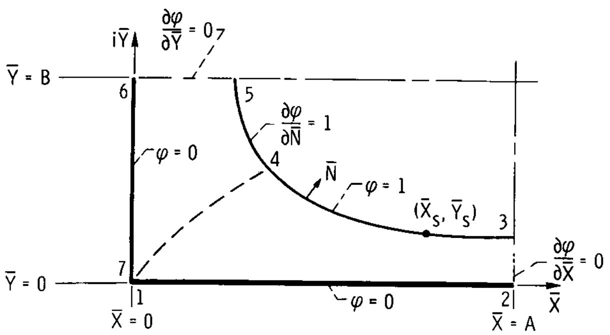


Figure 6. - Dimensionless heat and coolant mass flow rates at surface of porous leading edge region as function of imposed physical parameter.

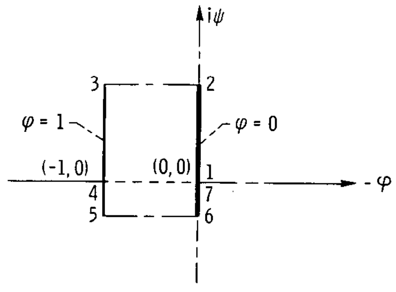


(a) Physical coordinates and imposed conditions

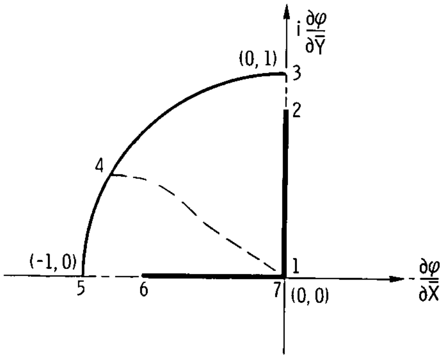


(b) Dimensionless coordinates with boundary conditions in terms of potential.

Figure 7. - Porous cooled rectangular duct.

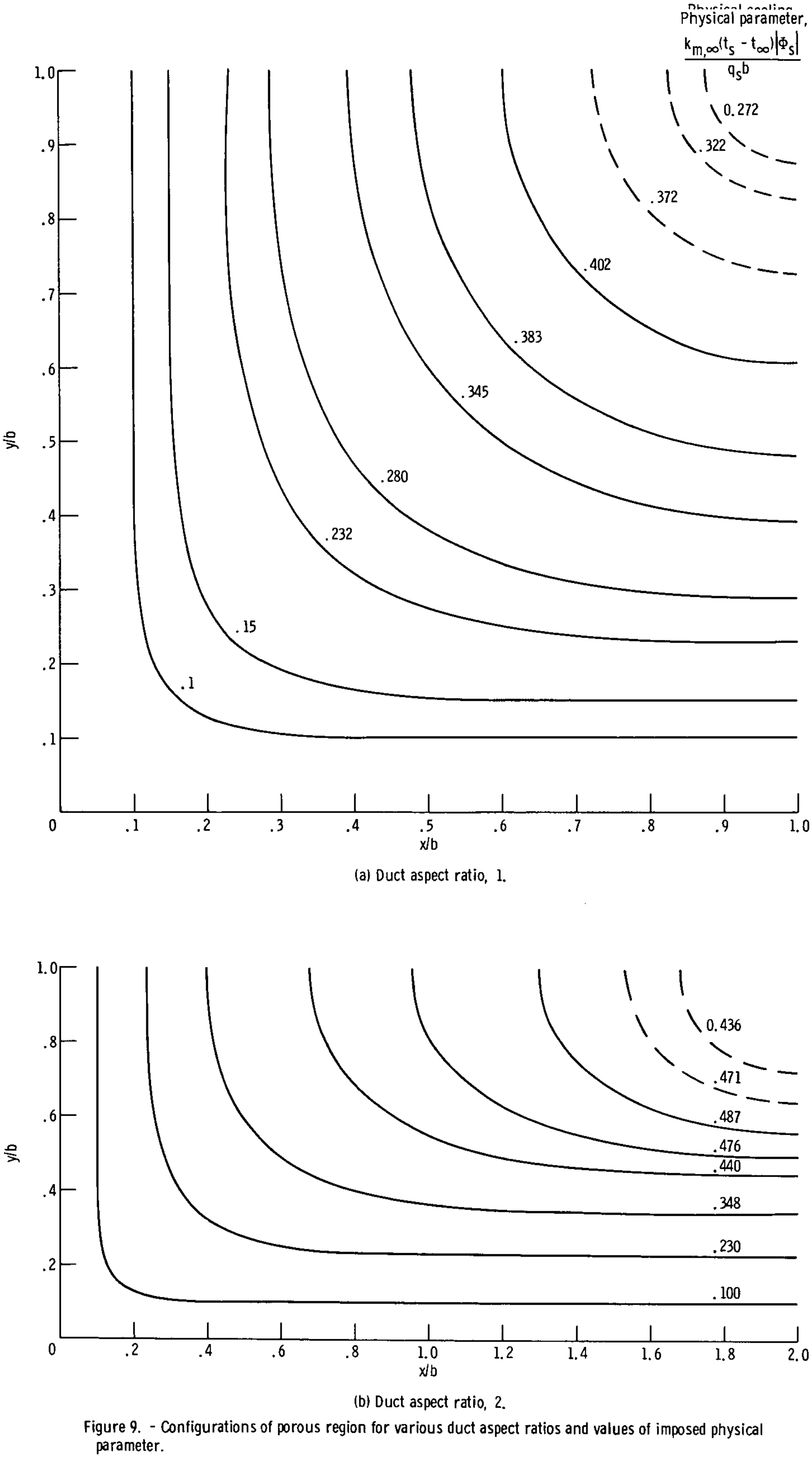


(a) Region in potential plane, $W = -\varphi + i\psi$.



(b) Region in potential derivative plane, $\zeta = -(\partial\varphi/\partial\bar{X}) + i(\partial\varphi/\partial\bar{Y})$.

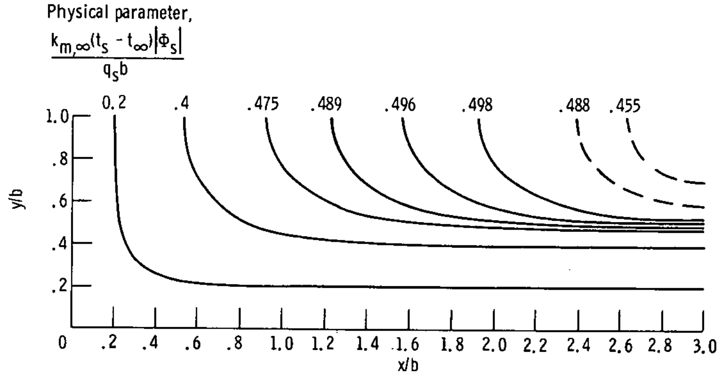
Figure 8. - Auxiliary planes used in mapping procedure.



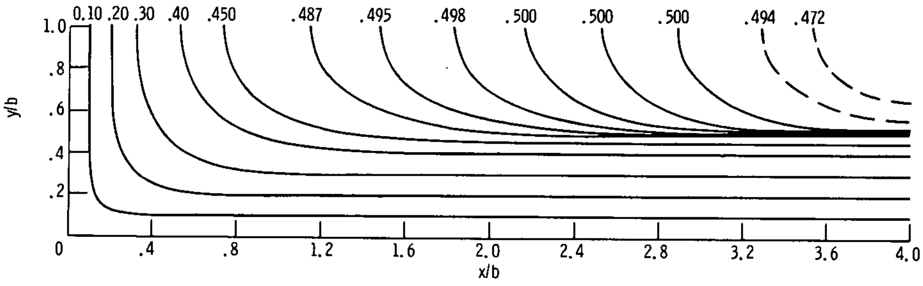
(a) Duct aspect ratio, 1.

(b) Duct aspect ratio, 2.

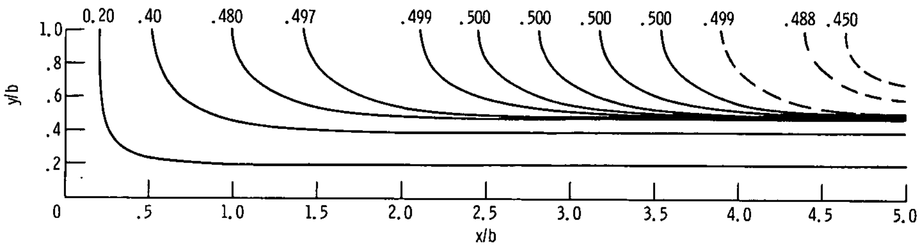
Figure 9. - Configurations of porous region for various duct aspect ratios and values of imposed physical parameter.



(c) Duct aspect ratio, 3.



(d) Duct aspect ratio, 4.



(e) Duct aspect ratio, 5.

Figure 9. - Concluded.

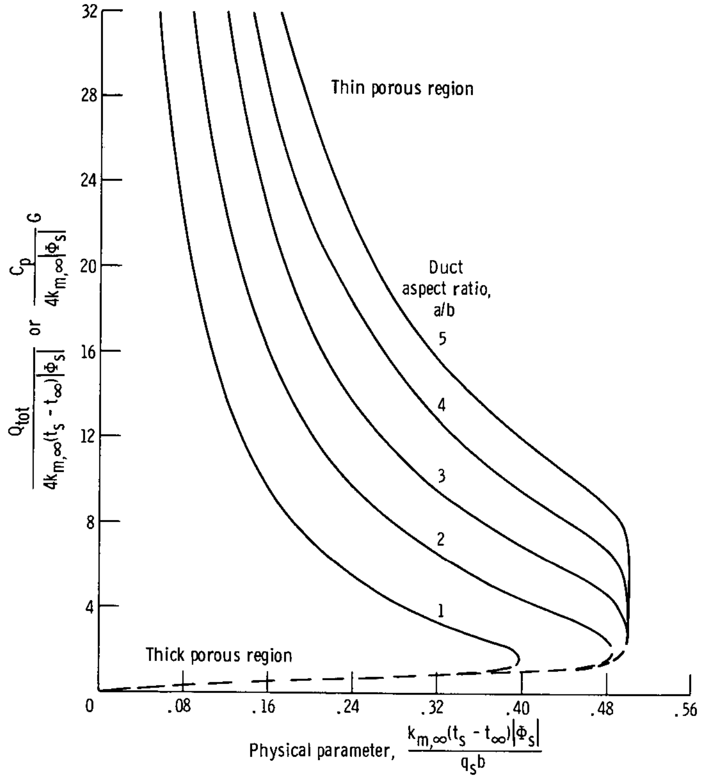


Figure 10. - Dimensionless heat and coolant mass flow rates at interior surface of porous duct as function of imposed physical parameter.

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