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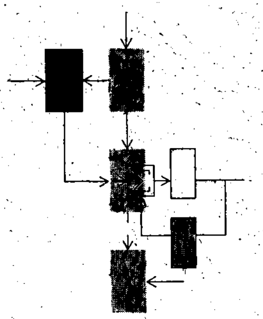

COMPUTATION OF OPTIMAL OUTPUT-FEEDBACK COMPENSATORS FOR LINEAR TIME-INVARIANT SYSTEMS

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This report is based on the unaltered thesis of Loren Platzman submitted in partial fulfillment of the requirements for the degree of Bachelor of Science at the Massachusetts Institute of Technology in May, 1972. This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory, with support extended by NASA/AMES under Grant NGL-22-009-124.

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I

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Submitted in Partial Fulfillment

of the Requirements for the

Degree of Bachelor of Science

at the

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Signature of Author . . . Loren Kerry Platzman . . . .
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ABSTRACT

The control of linear time-invariant systems with respect to a quadratic performance criterion is considered, subject to the constraint that the control vector be a constant linear transformation of the output vector. The optimal feedback matrix, $\underline{F}^*$, is selected to optimize the "expected" performance, given the covariance of the initial state.

It is first shown that the expected performance criterion can be expressed as the ratio of two multinomials in the elements of $\underline{F}$. This expression provides the basis for a feasible method of determining $\underline{F}^*$ in the case of single-input single-output systems.

A number of iterative algorithms are then proposed for the calculation of $\underline{F}^*$ for multiple input-output systems. For two of these, monotone convergence is proved, but they involve the solution of non-linear matrix equations at each iteration. Another is proposed involving the solution of Lyapunov equations at each iteration, and the gradual increase of the magnitude of a penalty function. Experience with this algorithm will be needed to determine whether or not it does, indeed, possess desirable convergence properties, and whether it can be used to determine the globally optimal $\underline{F}^*$.

Thesis Supervisor: Michael Athans
Title: Associate Professor of Electrical Engineering

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CONTENTS

CHAPTER I

INTRODUCTION

This thesis constitutes an attempt to overcome a hurdle which has hampered recent attempts to apply modern control theory to classical control problems: a class of simultaneous non-linear matrix equations. The output-feedback problem was selected as a framework, for it provides a fine illustration of the way these equations can prevent new theoretical results from being implementable in practice. In order to illustrate this better, a brief review of the history of this problem is in order.

The design of servomechanisms is generally acknowledged as the first control problem to be studied extensively, and to inspire the development of theoretical tools to aid in that design.[1,2] These theoretical tools involved frequency domain transformations, and were most effective in dealing with single-input single-output time-invariant linear systems. This general body of knowledge is known as classical control theory,[3] and although the theory was applied with some success to stochastic problems,[4] it has, thus far, been found inapplicable to time-varying problems.

Meanwhile, the application of linear algebra to similar problems led to reformulation of the servomechanism problem into the so-called "linear regulator problem." In particular, results obtained by R.E. Kalman[5,6] provided motivation for the development of modern control theory.[7,8,9]

There exists, however, a large gap between the current state-of-the-art of classical and modern control theories. For, whereas classical theory is most successful in dealing with single-input systems, modern control theory requires feedback of the entire state - a number of independent noise-free outputs equal to the order of the system. When these outputs are not available, the state must be reconstructed using a Kalman-Bucy filter, or an observer.[10] The physical implementation of either can be quite costly, and, occasionally unwarranted.

There has been considerable research in recent years, the purpose of which was to close the gap described above, but every attempt seems to reach the same point, and collapse. The problem can be described in the following way: Both the linear regulator and the Kalman-Bucy filter have solutions which apply to time-varying, terminal-time problems. The solution involves the backwards integration of the differential Riccati equation, starting at the terminal time.

The differential Riccati equation is shown below:

$$\underline{\dot{K}} = \underline{A}'\underline{K} + \underline{KA} + \underline{Q} - \underline{KBR}^{-1}\underline{B}'\underline{K} \tag{1.1}$$

where $\underline{A}$, $\underline{Q}$, $\underline{B}$, and $\underline{R}$ are known time-varying matrices.

This equation has two remarkable properties (for suitable values of $\underline{Q}$, $\underline{R}$, and K(T)),

1. The value of $\underline{K}(t)$ when integrated backwards remains bounded.
2. As $t \to -\infty$, $\underline{K}$ approaches a unique limiting value.

In particular, when a time-invariant problem is being solved, this unique limit can be used as the constant value of $\underline{K}$.

When we attempt to limit ourselves to output-feedback only, we get a variation of the Riccati equation:

$$\dot{K} = \underline{A}'\underline{K} + \underline{KA} + \underline{Q} - \underline{M}'\underline{KBR}^{-1}\underline{B}'\underline{KM} \tag{1.2}$$

where $\underline{M}$ is a messy expression.

The marvelous properties now disappear, and although a number of successful time-varying gains solutions have been found,[11,12,13,14] the time-invariant case degenerates sooner or later to a non-linear equation with many solutions (or no solutions) - an equation about which little is known.

There is reason to question the usefulness of the time-varying gains which may be obtained. Since the equations determining the values of these gains must be integrated backwards in time, the values must be calculated beforehand, and "read back" at run time, which is an expensive undertaking, and has inspired research on the computation of piecewise-constant gains for time-varying systems.[15] Furthermore, most problems can be modelled as time-invariant infinite-time with very little resulting error. However, if the reader desires a more complete survey of work in this area, reference 16 is highly recommended.

We would like to remark at this time that the optimal output-feedback gains for a time-invariant problem may well be time-varying, as is illustrated in the work of Levine.[11]

In spite of this, in the interest of simplicity in compensator design, we may be interested in a suboptimal constant output-feedback compensator, because of the cost involved in implementing the time-varying or state-feedback alternatives. In this thesis we will propose methods for solving this problem according to the following outline. In Chapter II, the problem we have selected is carefully formulated, and a canonical form is presented which simplifies the calculations involved in following chapters. In Chapter III we propose a method of "direct" algebraic expansion of the performance criterion, which can be used for single-input single-output systems. For higher orders systems, however, the treatment becomes too lengthy to be of use. In Chapter IV, we consider a number of iterative algorithms, the last of which we believe has great potential, but for which we were unable to prove convergence. In Chapter V we summarize the results we have obtained, and show how they may be applied to some related problems which involve the same non-linear equations in various forms.

CHAPTER II

THE CONSTANT OUTPUT-FEEDBACK COMPENSATOR PROBLEM

As we indicated in the introduction, the problem of designing limited-dimension (especially output-feedback) compensators has been treated in a number of ways in the literature recently. In this chapter, we will formulate this as a precise optimization problem, the one with which we will be concerned for the remainder of this thesis. We will then state some results obtained by others which are applicable to our problem. Finally, we present a canonical form for the parameters of the general problem, which we hope will simplify the treatment in following chapters.

2.1 Problem Formulation

Let the following represent a linear time-invariant deterministic system:

$$\underline{\dot{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t) \tag{2.1.1}$$

$$\underline{y}(t) = \underline{Cx}(t) \tag{2.1.2}$$

where: $\underline{A}$ is an n x n real constant matrix

$\underline{B}$ is an n x m real constant matrix

$\underline{C}$ is an r x n real constant matrix

$\underline{x}(t)$ is an n-vector (the state of the system)

$\underline{u}(t)$ is an m-vector (the system input)

$\underline{y}(t)$ is an r-vector (the system output)

Let the control be of the form

$$\underline{u}(t) = -\underline{Fy}(t) \tag{2.1.3}$$

where: $\underline{F}$ is an m x r real constant matrix.

We wish to select $\underline{F}$ so as to minimize a standard quadratic cost functional:

$$J_o \equiv \int_{t_o}^{\infty} [\underline{x}'(t)\underline{Q}\underline{x}(t) + \underline{u}'(t)\underline{R}\underline{u}(t)]\, dt \tag{2.1.4}$$

where: $\underline{Q}$ is an n x n symmetric positive semidefinite matrix

$\underline{R}$ is an m x m symmetric positive definite matrix

and $[\underline{A}, \underline{Q}^{1/2}]$ is observable.

It is easily seen that this in itself does not constitute a well-posed optimization problem, for the initial state is not known, and J_o cannot be calculated from Equation (2.1.4). Moreover the "initial condition" represents in fact a physical disturbance in the system which cannot be known a priori. A simple way to eliminate this problem is to require that the expected value of J_o be minimized, and to provide statistics describing the probabilistic distribution of the initial conditions. We now show that it is necessary only to provide the covariance of the initial condition.

Substituting Equations (2.1.2) and (2.1.3) into Equation (2.1.4) to eliminate $\underline{u}(t)$ yields

$$J_o = \int_{t_o}^{\infty} [\underline{x}'(t)\underline{Q}\underline{x}(t) + \underline{x}(t)'\underline{C}'\underline{F}'\underline{RFC}\underline{x}(t)]\, dt$$

$$= \int_{t_o}^{\infty} \underline{x}'(t)(\underline{Q} + \underline{C}'\underline{F}'\underline{RFC})\underline{x}(t)\, dt \tag{2.1.5}$$

Let the initial state be

$$\underline{x}_o \equiv \underline{x}(t_o) \tag{2.1.6}$$

Then, it is easily seen from Equations (2.1.1), (2.1.2) and (2.1.3) that

$$\underline{x}(t) = e^{[\underline{A} - \underline{BFC}]t}\underline{x}_o \tag{2.1.7}$$

Substituting Equation (2.1.7) into Equation (2.1.5), we have

$$J_o = \int_{t_o}^{\infty} \underline{x}_o' e^{[\underline{A} - \underline{BFC}]'t}(\underline{Q} + \underline{C}'\underline{F}'\underline{RFC})e^{[\underline{A} - \underline{BFC}]t}\underline{x}_o dt \tag{2.1.8}$$

Defining the <u>deterministic</u> matrix

$$\underline{K} \equiv \int_{t_o}^{\infty} e^{[\underline{A} - \underline{BFC}]'t}(\underline{Q} + \underline{C}'\underline{F}'\underline{RFC})e^{[\underline{A} - \underline{BFC}]t}dt \tag{2.1.9}$$

and observing that the integral sign commutes with a constant,

$$J_o = \underline{x}_o'\underline{K}\underline{x}_o \tag{2.1.10}$$

As J_o is a scalar, Equation (2.1.10) can also be written

$$J_o = tr[\underline{x}_o'\underline{K}\underline{x}_o] = tr[\underline{Kx_o x_o'}] \tag{2.1.11}$$

The expected value operator commutes with linear operators, so

$$J \equiv E\{J_o\} = tr[\underline{K}E\{\underline{x}_o\underline{x}_o'\}] = tr\ [\underline{KX}_o] \tag{2.1.12}$$

where

$$\underline{X}_o \equiv E\{\underline{x}_o\underline{x}_o'\} \tag{2.1.13}$$

Thus $\underline{X}_o$ may be specified in such a way as to reflect the relative magnitude of disturbances in various "directions" in state space. If this information is not available, $\underline{X}_o$ may be specified as $\underline{I}$ to reflect uniform distribution on the surface of the n-dimensional unit sphere.

It will later become necessary to require that $\underline{X}_o$ be <u>non-singular</u>.

Finally, we observe that t_o may be taken to be 0 without loss of generality.

To summarize, we formulate the optimization problem as follows:

Constant Output-feedback Compensator Problem

Given a time-invariant linear system

$$\underline{\dot{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t) \tag{2.1.14}$$

$$\underline{y}(t) = \underline{Cx}(t) \tag{2.1.15}$$

with control constrained to be of the form

$$\underline{u}(t) = -\underline{Fy}(t) \tag{2.1.16}$$

We wish to select a value of $\underline{F}$ for which the cost functional

$$J = \mathrm{tr}[\underline{KX_o}] \tag{2.1.17}$$

exists and its minimized. The matrices $\underline{K}$ and $\underline{X_o}$ are defined by

$$\underline{K} = {}_o\int^{\infty} e^{[\underline{A} - \underline{BFC}]'t}(\underline{Q} + \underline{C}'\underline{F}'\underline{RFC})e^{[\underline{A} - \underline{BFC}]t}dt \tag{2.1.18}$$

$$\underline{X_o} = E\{\underline{x_o}\underline{x_o}'\} = E\{\underline{x}(0)\underline{x}'(0)\} > 0 \tag{2.1.19}$$

We remark that $\underline{K}$ may also be defined as the solution of

$$\underline{0} = (\underline{A} - \underline{BFC})'\underline{K} + \underline{K}(A - \underline{BFC}) + \underline{Q} + \underline{C}'\underline{F}'\underline{RFC} \tag{2.1.20}$$

2.2 Background

In this section, we briefly review some results which will be of use in the sequel.

Fact 2.1 If the entire state is available for measurement ($\underline{C}$ is non-singular), the solution to the constant output-feedback problem is given by

$$\underline{F}^* = -\underline{R}^{-1}\underline{B}'\underline{\Pi}_{\infty}\underline{C}^{-1} \qquad (2.2.1)$$

where $\underline{\Pi}_{\infty}$ is the unique positive-definite solution of the algebraic Riccati equation.

$$\underline{0} = \underline{A}'\underline{\Pi}_{\infty} + \underline{\Pi}_{\infty}\underline{A} + \underline{Q} - \underline{\Pi}_{\infty}\underline{BR}^{-1}\underline{B}'\underline{\Pi}_{\infty} \qquad (2.2.2)$$

$\underline{\Pi}_{\infty}$ may also be defined as

$$\underline{\Pi}_{\infty} = \lim_{t\to-\infty} \underline{\Pi}(t) \qquad (2.2.3)$$

where:

$$\dot{\underline{\Pi}}(t) = \underline{A}'\underline{\Pi}(t) + \underline{\Pi}(t)\underline{A} + \underline{Q} - \underline{\Pi}(t)\underline{BR}^{-1}\underline{B}'\underline{\Pi}(t) \qquad (2.2.4)$$

and $\underline{\Pi}(0) = \underline{\Pi}_o = \underline{\Pi}_o' > 0$

This is commonly referred to as the Linear Regulator Problem.

Proof The entire subject is treated thoroughly in Section 23 of Brockett.[9]

Remark We observe that the solution is independant of $\underline{X}_o$.

Fact 2.2 (Levine, Johnson and Athans)

Let

$$\underline{L} \equiv \int_0^{\infty} e^{[\underline{A} - \underline{BFC}]t}\underline{X}_o e^{[\underline{A} - \underline{BFC}]'t} dt \qquad (2.2.5)$$

then a necessary condition for $\underline{F}$ to be optimal is that $\underline{F}$ satisfy

$$\underline{F} = \underline{R}^{-1}\underline{B}'\underline{KLC}'[\underline{CLC}']^{-1} \qquad (2.2.6)$$

Proof See References 11 and 16.

Fact 2.3 (Kleinman)

Let $\underline{K}_o \geq 0$ and $\underline{A} - \underline{BR}^{-1}\underline{B}'\underline{K}_o$ be stable. Then each term of the sequence described by

$$\underline{0} = (\underline{A} - \underline{BR}^{-1}\underline{BK}_N)'\underline{K}_{N+1} + \underline{K}_{N+1}(\underline{A} - \underline{BR}^{-1}\underline{B}'\underline{K}_N) + \underline{Q} + \underline{K}_N\underline{BR}^{-1}\underline{B}'\underline{K}_N \tag{2.2.7}$$

exists and satisfies $\underline{K}_{N+1} \geq 0$; $\underline{A} - \underline{BR}^{-1}\underline{B}'\underline{K}_{N+1}$ is stable. Moreover, the sequence converges monotonically to $\underline{\Pi}_\infty$, the solution of Equation (2.2.2).

Proof See Reference 17.

This method is commonly used to solve the linear regulator problem.

2.3 The Canonical Output Form

In the following chapters, we will present analytic tools and computational algorithms which involve a great deal of manipulation of the state-to-output transformation matrix $\underline{C}$. Having $\underline{C}$ in some preassigned form will contribute considerably to the lucidity of the former, as well as to the efficiency of the latter. With this in mind, we define a canonical form which requires each output component to be identical to one of the state components. This is accomplished by remodelling the state variable to conform to this requirement, leaving the input-output dynamics, as well as their relation to the performance criterion, unchanged. In other words, considering the system as a "black box," we replace it with a new "black box" indistinguishable from the first from the outside, but somewhat less complicated in the inside. We will call this form the Canonical Output Form.

A compensator is optimal for a given system if and only if it is optimal when applied to its canonical output form.

Consider a linear time-invariant system described by Equation (2.1.1) and Equation (2.1.2).

If $\underline{C}$ is not of full rank, at least one component of $\underline{y}$ is redundant, and may be ignored for purposes of compensator design. Thus, assuming $\underline{C}$ is of full rank r, we construct

$$\{\underline{\gamma}_1, \underline{\gamma}_2 \cdots \underline{\gamma}_{n-r}\} \text{ an orthonormal basis for the null space of } \underline{C} \tag{2.3.1}$$

(This can be done most easily by first constructing an orthonormal basis for the range of $\underline{C}'$ - applying Gram-Schmidt orthogonalization to the columns of $\underline{C}'$.)

We define the m x (m - r) <u>output complement matrix</u>

$$\underline{\Upsilon}' \equiv [\underline{\gamma}_1 \vdots \underline{\gamma}_2 \vdots \cdots \underline{\gamma}_{n-r}] \tag{2.3.2}$$

and note that by construction

$$\underline{\Upsilon}\,\underline{\Upsilon}' = \underline{I}_{n-r} \qquad \underline{C}\,\underline{\Upsilon}' = \underline{0} \tag{2.3.3}$$

Defining the n x n matrices

$$\underline{U} \equiv \begin{bmatrix} \underline{C} \\ \hline \underline{\Upsilon} \end{bmatrix} \qquad \underline{V} \equiv \left[\underline{C}'(\underline{CC}')^{-1} \vdots \underline{\Upsilon}' \right] \tag{2.3.4}$$

we note that

i) Since $\underline{C}$ is of full rank r, $(\underline{CC}')^{-1}$ exists

ii) $\underline{UV} = \underline{I}$ so $\underline{U} = \underline{V}^{-1}$

Defining:

$$\underline{\tilde{x}}(t) \equiv \underline{Ux}(t) \tag{2.3.5}$$

$$\underline{\tilde{A}} \equiv \underline{UAV} \tag{2.3.6}$$

$$\underline{\tilde{B}} \equiv \underline{UB} \tag{2.3.7}$$

$$\underline{\tilde{C}} \equiv \underline{CV} = [\underline{I}_r : \underline{0}] \tag{2.3.8}$$

$$\underline{\tilde{\Upsilon}} \equiv \underline{\Upsilon V} = [\underline{0} : \underline{I}_{n-r}] \tag{2.3.9}$$

we see that the system

$$\underline{\dot{\tilde{x}}}(t) = \underline{\tilde{A}\tilde{x}}(t) + \underline{\tilde{B}u}(t) \tag{2.3.10}$$

$$\underline{y}(t) = \underline{\tilde{C}\tilde{x}}(t) \tag{2.3.11}$$

is identical to the original system, from an input-output point of view.

The initial conditions and performance criterion can be transformed similarly:

if $$\underline{\tilde{x}}_o \equiv \underline{Ux}_o \tag{2.3.12}$$

$$\underline{\tilde{X}}_o \equiv \underline{\tilde{x}_o\tilde{x}_o'} = \underline{Ux_o x_o'U'} = \underline{UX_oU'} \tag{2.3.13}$$

and if $$\underline{\tilde{Q}} \equiv \underline{V'QV} \tag{2.3.14}$$

$$\underline{\tilde{R}} \equiv \underline{R} \tag{2.3.15}$$

then i) $$\underline{Q} \geq 0 \rightarrow \underline{\tilde{Q}} \geq 0 \tag{2.3.16}$$

ii) $$\int_o^\infty \underline{x}'(t)[\underline{Q} + \underline{C}'\underline{F}'\underline{RFC}]\underline{x}(t)dt = \int_o^\infty \underline{\tilde{x}}'(t)[\underline{\tilde{Q}} + \underline{\tilde{C}}'\underline{F}'\underline{\tilde{R}F\tilde{C}}]\underline{\tilde{x}}(t)dt \tag{2.3.17}$$

and J is unchanged.

We will call a system for which the output matrix has the form

$$\underline{C} = [\underline{I}_r : \underline{0}] \tag{2.3.18}$$

a system in <u>canonical output form</u>.

Thus, it has been shown in this section that it is always possible to convert a time-invariant system compensator problem to canonical output form without loss of generality.

We also call attention to the output complement matrix, $\underline{\Upsilon}$, which is defined as "any matrix which satisfies Equation (2.3.3)." When a system is formulated in canonical output form, the canonical output complement matrix takes the form

$$\underline{\Upsilon} = [\underline{0} : \underline{I}_{n-r}] \tag{2.3.19}$$

Note that the output complement matrix is, in general, not unique.

CHAPTER III

ALGEBRAIC EXPANSION OF THE COST FUNCTIONAL FOR SINGLE-INPUT SINGLE-OUTPUT SYSTEMS

In the previous chapter we showed that the cost associated with the operation of a system with a constant output-feedback compensator may be viewed as a scalar function over feedback space. Naturally, it would be convenient to be able to derive a simple algebraic expression for J in terms of $\underline{F}$. Unfortunately, such an expression is in general too complicated to be of any use. Such an expression can, however, be derived for the scalar feedback applied to well behaved single-input single-output time-invariant linear systems.

It turns out that J can be expressed as the ratio of two polynomials in f, the scalar negative feedback gain. The optimal gain, f*, can be found by calculating the zeroes of a polynomial (computational algorithms are available which do this quite efficiently); but, more important is the insight this gives us into the nature of $J(\underline{F})$ in the case of a multi-input multi-output compensator.

3.1 Statement of the Problem

Consider a single-input single-output time-invariant system in canonical output form (as described in section 2.3):

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}u(t) \tag{3.1.1}$$

$$y(t) = \underline{C}\underline{x}(t) = [1\ 0 \ldots\ 0]\underline{x}(t) \tag{3.1.2}$$

The control is of the form

$$u(t) = -fy(t) \tag{3.1.3}$$

Let the performance criterion be defined by

$$J = \text{tr}[\underline{KX}_o] \tag{3.1.4}$$

where $\underline{K}$ is the solution of the Lyapunov equation

$$(\underline{A} - \underline{BfC})'\underline{K} + \underline{K}(\underline{A} - \underline{BfC}) + \underline{Q} + \underline{C}'f\underline{RfC} = 0 \tag{3.1.5}$$

J and $\underline{K}$ are functions of the scalar f.

This definition coincides with the definition in Equation (2.1.18) when $(\underline{A} - \underline{BfC})$ is stable, because $\underline{K}$ is then unique and is of the form shown in Equation (2.1.20). In this chapter, however, we extend the definition of J to hold whenever a solution to Equation (3.1.5) exists, although there is no physical interpretation for the values of J obtained in regions of f corresponding to unstable $(\underline{A} - \underline{BfC})$. This allows us to determine an expression for J(f) without knowing whether or not the system can be stabilized, or, if it can, in how many distinct regions of f stabilization is possible.

3.2 The Main Result

In this section, we derive an algebraic expression for J(f).

Step 1

We select a value of f for which a solution $\underline{K}$ to Equation (3.1.5) exists. A necessary and sufficient condition for this to occur is:

$$\lambda_i(\underline{A} - \underline{BfC}) \neq -\lambda_j(\underline{A} - \underline{BfC}) \tag{3.2.1}$$

$$f = f_o; \quad i, j = 1, 2, \ldots n$$

where $\lambda_i(\cdot)$ denotes the i^{th} eigenvalue of the argument. (See Reference 18, page 239, theorem 4.)

If no such f_o exists, J is undefined for all values of f, and the problem is meaningless. Moreover, this indicates that no f will stabilize the system, since the eigenvalues of a stable matrix have negative real parts, and therefore satisfy Equation (3.2.1).

Define the variables:

$$\hat{A} \equiv \underline{A} - \underline{B}f_o\underline{C} \tag{3.2.2}$$

$$\hat{f} \equiv f - f_o \tag{3.2.3}$$

$$P(\hat{f}) \equiv (\hat{f}^2 + 2f_o\hat{f} + f_o^2)R = f^2R \tag{3.2.4}$$

Substituting the new variables into Equation (3.1.5) we have:

$$(\underline{\hat{A}} - \underline{B}\hat{f}\underline{C})'\underline{K} + \underline{K}(\underline{\hat{A}} - \underline{B}\hat{f}\underline{C}) + \underline{Q} + P(\hat{f})\underline{C}'\underline{C} = 0 \tag{3.2.5}$$

<u>Step 2</u>

We need the following lemma:

<u>Lemma 3.1</u> Every solution $\underline{K}$ to Equation (3.2.5) can be expressed uniquely in the form

$$\underline{K} = \underline{K}_o + k_1\underline{K}_1 + k_2\underline{K}_2 + \dots k_n\underline{K}_n \tag{3.2.6}$$

where the $\underline{K}_i$ are solutions of

$$\underline{\hat{A}}'\underline{K}_o + \underline{K}_o\underline{\hat{A}} + \underline{Q} = 0 \tag{3.2.7}$$

$$\underline{\hat{A}}'\underline{K}_i + \underline{K}_i\underline{A} + \underline{E}_i = 0 \qquad 0 < i \le n \tag{3.2.8}$$

$$\underline{E}_1 \equiv \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & & & & 0 \end{bmatrix} \qquad \underline{E}_2 \equiv \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 0 & 0 & & & & 0 \end{bmatrix}$$

$$\cdots\cdots \qquad \underline{E}_n \equiv \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ 1 & 0 & & & & 0 \end{bmatrix} \tag{3.2.9}$$

and the k_i are functions of $\hat{f}$.

Proof Let

$$\underline{E} = \underline{E}' \equiv \underline{\hat{A}}'\underline{K} + \underline{K\hat{A}} + \underline{Q} \tag{3.2.10}$$

The output complement matrix (see section 2.3) takes the form:

$$\underline{\gamma} = [\underline{0}:\underline{I}_{n-1}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & & 1 \end{bmatrix} \tag{3.2.11}$$

Note that, by construction of $\underline{\gamma}$,

$$\underline{\gamma C}' = \underline{C\gamma}' = \underline{0} \tag{3.2.12}$$

Pre- and post-multiplying Equation (3.2.5) by $\underline{\gamma}$ and $\underline{\gamma}'$, respectively, and using Equation (3.2.12) we see that

$$\underline{\gamma}(\underline{\hat{A}}'\underline{K} + \underline{K\hat{A}} + \underline{Q})\underline{\gamma}' = \underline{0} \tag{3.2.13}$$

Substituting Equation (3.2.10) into (3.2.13)

$$\underline{\gamma E \gamma}' = \underline{0} \tag{3.2.14}$$

If we partition $\underline{E}$, isolating the first row and the first column

$$\underline{E} = \left[\begin{array}{c:c} \underline{\mathcal{E}}_{11} & \underline{\mathcal{E}}_{12} \\ \hdashline \underline{\mathcal{E}}_{21} & \underline{\mathcal{E}}_{22} \end{array}\right] \tag{3.2.15}$$

$\underline{\mathcal{E}}_{11}$ is 1 x 1 $\qquad$ $\underline{\mathcal{E}}_{12}$ is 1 x (n-1)

$\underline{\mathcal{E}}_{21}$ is (n-1) x 1 $\qquad$ $\underline{\mathcal{E}}_{22}$ is (n-1) x (n-1)

equation (3.2.14) becomes

$$\underline{0} = \underline{\gamma E \gamma'} = [\underline{0}:\underline{I}]\left[\begin{array}{c:c} \underline{\mathcal{E}}_{11} & \underline{\mathcal{E}}_{12} \\ \hdashline \underline{\mathcal{E}}_{21} & \underline{\mathcal{E}}_{22} \end{array}\right]\left[\begin{array}{c} \underline{0} \\ \hdashline \underline{I} \end{array}\right] = \underline{\mathcal{E}}_{22} \tag{3.2.16}$$

Therefore, we conclude that

1) $\underline{E}$ is everywhere $\underline{0}$ except on the first row and the first column

2) $\underline{E}$, being symmetric, can be uniquely described as a linear combination of the $\underline{E}_i$ defined in Equation (3.2.9).

Thus, defining the k_i so that

$$\underline{E} = -\sum_{i=1}^{m} k_i \underline{E}_i \tag{3.2.17}$$

substituting Equation (3.2.10) into Equation (3.2.17)

$$\underline{\hat{A}}'\underline{K} + \underline{K\hat{A}} + \underline{Q} + \sum_{i=1}^{m} k_i \underline{E}_i = \underline{0} \tag{3.2.18}$$

by superposition, the solutions of Equation (3.2.7) and Equation (3.2.8) combine to yield Equation (3.2.6). Moreover, $\underline{\hat{A}}$ satisfies Equation (3.2.1), so all $\underline{K}_i$ exist, and are unique. QED

We now calculate the scalars

$$j_i \equiv \mathrm{tr}[\underline{K}_i \underline{X}_o] \qquad i = 0, 1, \ldots n \qquad (3.2.19)$$

and the vectors

$$\underline{g}_i \equiv \text{the first column vector of } [\underline{C}'\underline{B}'\underline{K}_i + \underline{K}_i\underline{BC}]$$
$$i = 0, 1, \ldots n \qquad (3.2.20)$$

This may be done by calculating a $\underline{K}_i$ from Equation (3.2.7) or (3.2.8), calculating the corresponding j_i and $\underline{g}_i$ as defined above, discarding that $\underline{K}_i$ and calculating the next $\underline{K}_i$. The fact that all $\underline{K}_i$ do not have to be retained at once makes this procedure programmable on a digital computer without requiring exorbitant amounts of storage.

Step 3

We now solve for the k_i as a function of $\hat{f}$.

Let

$$\underline{k} \equiv \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix} \qquad \underline{h} \equiv \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (3.2.21)$$

$$\underline{G} \equiv [\underline{g}_1 \vdots \underline{g}_2 \vdots \cdots \underline{g}_n] \qquad (3.2.22)$$

We claim that $\underline{k}$ is uniquely determined by:

$$[\underline{I} + \hat{f}\underline{G}]\underline{k} + \hat{f}\underline{g}_o - P(\hat{f})\underline{h} = \underline{0} \qquad (3.2.23)$$

Proof

(1) Rearranging Equation (3.2.5), we have:

$$-(\underline{\hat{A}}'\underline{K} + \underline{K\hat{A}} + \underline{Q}) + (\underline{B}\hat{f}\underline{C})'\underline{K} + \underline{K}(\underline{B}\hat{f}\underline{C}) - P(\hat{f})\underline{C}'\underline{C} = \underline{0} \qquad (3.2.24)$$

Substituting Equation (3.2.10) into Equation (3.2.24), we have

$$-\underline{E} + (\underline{B}\hat{f}\underline{C})'\underline{K} + \underline{K}(\underline{B}\hat{f}\underline{C}) - P(\hat{f})\underline{C}'\underline{C} = \underline{0} \qquad (3.2.25)$$

Decomposing $\underline{E}$ and $\underline{K}$ into components of magnitude k_i by use of Equations (3.2.17) and (3.2.6), we have

$$\sum_{i=1}^{n} k_i[E_i + (\underline{B}\hat{f}\underline{C})'\underline{K}_i + \underline{K}_i(\underline{B}\hat{f}\underline{C})] + (\underline{B}\hat{f}\underline{C})'\underline{K}_o + \underline{K}_o\underline{B}\hat{f}\underline{C} - P(\hat{f})\underline{C}'\underline{C} = \underline{0} \qquad (3.2.26)$$

Rearranging:

$$(\sum_{i=1}^{n}[E_i + \hat{f}(\underline{C}'\underline{B}'\underline{K}_i + \underline{K}_i\underline{BC})]k_i) + \hat{f}(\underline{C}'\underline{B}'\underline{K}_o + \underline{K}_o\underline{BC}) - P(\hat{f})\underline{C}'\underline{C} = \underline{0} \qquad (3.2.27)$$

Using Equations (3.2.9), (3.2.20), (3.2.21) and (3.2.22) we see that Equation (3.2.23) represents the <u>first column</u> of each term of Equation (3.2.27).

(2) Equation (3.2.23) represents n linear equations in n unknowns. The solution $\underline{k}$ exists and is unique when $[\underline{I} + \hat{f}\underline{G}]$ is non-singular. Clearly it is - unless

$$\hat{f} = -\frac{1}{\lambda(\underline{G})}$$

which can only occur for <u>at most</u> n isolated values of $\hat{f}$. QED

<u>Step 4</u>

Combining Equations (3.1.4), (3.2.6) and (3.2.19), we see that

$$J = j_o + \sum_{i=1}^{n} j_i k_i \qquad (3.2.28)$$

Thus, to obtain J as a function of $\hat{f}$, we might solve Equation (3.2.23) for each k_i as a function of $\hat{f}$. Since $\hat{f}$ is a parameter, not an unknown, we can use Cramer's rule to obtain each k_i as the ratio of two determinants. Although this approach lacks practical appeal, it allows us to prove the following result.

<u>Theorem 3.1</u>

$J(\hat{f})$ may always be expressed in the form

$$J(\hat{f}) = \frac{\hat{S}(\hat{f})}{\hat{T}(\hat{f})} \tag{3.2.29}$$

where: $\hat{S}(\cdot)$ is a polynomial of order <u>at most</u> n+1

$\hat{T}(\cdot)$ is a polynomial of order <u>at most</u> n

<u>Proof</u>

(1) Let $\hat{T}(\hat{f}) = \det(\underline{I} + \hat{f}\underline{G})$

- each element of the matrix $\underline{I} + \hat{f}\underline{G}$ is a polynomial in $\hat{f}$ of order not more than 1.

- each product of n elements of $\underline{I} + \hat{f}\underline{G}$ is a polynomial in $\hat{f}$ of order not more than n.

- $\hat{T}(\hat{f})$ consists of the sum of terms each of which is a polynomial in $\hat{f}$ of order not more than n.

<u>Therefore</u> $\hat{T}(\hat{f})$ is a polynomial in $\hat{f}$ of order not more than n.

(2) $$k_i = \frac{\det(\underline{M}_i)}{\hat{T}(\hat{f})} = \frac{\hat{S}_i(\hat{f})}{\hat{T}(\hat{f})} \tag{3.2.30}$$

where $\underline{M}_i$ is obtained by replacing a <u>single column</u> of $\underline{I} + \hat{f}\underline{G}$ by a vector, each element of which is a polynomial in $\hat{f}$ of order not more than 2.

- reasoning as we did in the case of $\hat{T}(\cdot)$, we see that $\hat{S}_i(\hat{f})$ is made up by the sum of polynomials, each of which is of order not more than n+1.

<u>Therefore</u> $\hat{S}_i(\hat{f})$ is a polynomial in $\hat{f}$ of order not more than n+1.

(3) From Equation (3.2.28), we have

$$J = j_o + j_1 k_1 + \dots j_n k_n$$

$$= \frac{j_o \hat{T}(\hat{f}) + j_1 S_1(\hat{f}) + \dots j_n S_n(\hat{f})}{\hat{T}(\hat{f})} \tag{3.2.31}$$

As the numerator is made up by the sum of polynomials in $\hat{f}$, each of which is of order not more than n+1, we see that in Equation (3.2.29) $\hat{S}(\hat{f})$ is a polynomial in $\hat{f}$ of order not more than n+1.

Corollary

J(f) may always be expressed in the form

$$J(f) = \frac{S(f)}{T(f)} \tag{3.2.32}$$

where: $S(\cdot)$ is a polynomial of order at most n+1

$T(\cdot)$ is a polynomial of order at most n

Proof

Substitution of f for $\hat{f}$ in Equation (3.2.29) by means of Equation (3.2.3) leaves the order of each polynomial unchanged.

In the interest of computational expediency, we propose the following method:

Let $\tilde{\underline{G}} = \underline{UGU}^{-1}$ (3.2.33)

be a real Jordan block form for $\underline{G}$.

Then defining:

$$\underline{j} \equiv \begin{bmatrix} j_1 \\ \cdot \\ \cdot \\ \cdot \\ j_n \end{bmatrix} \qquad \tilde{\underline{j}} = \underline{U}'\underline{j} = \begin{bmatrix} \tilde{j}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{j}_n \end{bmatrix} \tag{3.2.34}$$

C

$$\underline{\tilde{k}} = \underline{U}^{-1}\underline{k} = \begin{bmatrix} \tilde{k}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{k}_n \end{bmatrix} \qquad (3.2.35)$$

$$\underline{\tilde{g}}_o \equiv \underline{U}^{-1}\underline{g}_o \qquad \underline{\tilde{h}} = \underline{U}^{-1}\underline{h} \qquad (3.2.36)$$

We can convert Equations (3.2.23) and (3.2.28) to

$$(\underline{I} + \hat{f}\underline{\tilde{G}})\underline{\tilde{k}} + \hat{f}\underline{\tilde{g}}_o - P(\hat{f})\underline{\tilde{h}} = 0 \qquad (3.2.37)$$

$$J = j_o + \sum_{i=1}^{n} \tilde{j}_i \tilde{k}_i \qquad (3.2.38)$$

In this case, Equation (3.2.37) decouples into groups of V_ℓ simultaneous linear equations where V is the size of each Jordan block in $\underline{\tilde{G}}$. The determinants of size V_ℓ can be easily evaluated.

Step 5

Substitute f for $\hat{f}$ in the formulation $J(\hat{f})$ determined.

3.3 Example

Let $\underline{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\underline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{C} = [1 \quad 0]$ (3.3.1)

The system is in canonical output form.

Let the performance criterion be described by:

$$\underline{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 1 \qquad \underline{X}_o = \underline{I} \qquad (3.3.2)$$

Step 1

$\underline{A}$ is not stable, so we must select an f_o which stabilizes $(\underline{A} - \underline{B}f_o\underline{C})$. Let $f_o = 1$. Then

$$\hat{f} = f - f_o = f - 1 \qquad (3.3.3)$$

$$\underline{\hat{A}} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad P(\hat{f}) = \hat{f}^2 + 2\hat{f} + 1 \tag{3.3.4}$$

Step 2

Solving Equation (3.2.7) yields

$$\underline{K}_o = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \tag{3.3.5}$$

We retain $\underline{g}_o = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \qquad j_o = \frac{3}{2}$ (3.3.6)

Solving Equation (3.2.8) yields, in turn,

$$\underline{K}_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \underline{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad j_1 = 1 \tag{3.3.6}$$

$$\underline{K}_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \underline{g}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad j_2 = -1 \tag{3.3.8}$$

Step 3

$$\underline{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underline{h} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.3.9}$$

Step 4

G is already in Jordan form. Equation (3.2.23) becomes

$$\begin{bmatrix} 1 + \hat{f} & 0 \\ 0 & 1 \end{bmatrix} \underline{k} + \hat{f} \begin{bmatrix} -(\hat{f}^2 + 2\hat{f} + 1) \\ (1/2)\hat{f} \end{bmatrix} = 0 \tag{3.3.10}$$

$$J = \frac{3}{2} + \frac{\hat{f}^2 + \hat{f} + 1}{\hat{f} + 1} + \frac{1}{2}\hat{f}$$

$$= \frac{(3/2)\hat{f}^2 + 3\hat{f} + 5/2}{(\hat{f} + 1)} \tag{3.3.11}$$

Step 5

Substituting Equation (3.3.3) into Equation (3.3.11)

$$J = \frac{3/2f^2 + 1}{f} \tag{3.3.12}$$

3.4 Conclusions

We make the following claims based on the contents of this chapter.

Claim 3.1 The quadratic performance criterion, J, of an n^{th} order time-invariant system having m inputs and r outputs may be expressed in the form

$$J = \frac{S(f_{11}, f_{12} \ldots f_{mr})}{T(f_{11}, f_{12} \ldots f_{mr})} \tag{3.4.1}$$

where f_{ij} are elements of the m x r matrix $\underline{F}$

S is a multinomial of order at most n + 1

T is a multinomial of order at most n

Proof

If the Lyapunov equation Eqation (2.1.20) is viewed as n^2 simultaneous linear equations in n^2 unknowns (the elements of $\underline{K}$), the trace of $\underline{KX}_o$ may be expressed as the ratio of two determinants (each $n^2 \times n^2$).

Since each term of each determinant is a multinomial of order at most 2, J may be expressed as the ratio of multinomial of order at most 2^{n^2}.

However, any further constraint on the feedback of the form

$$\underline{F} = x\underline{F}_o \tag{3.4.2}$$

where $\underline{F}_o$ is a constant matrix, x a scalar parameter, will transform the system into a single-input single-output form and, by the corollary to theorem 3.1,

$$J = \frac{S_x(x)}{T_x(x)} \tag{3.4.3}$$

where the order of S_x may not exceed n + 1

and the order of T_x may not exceed n.

From this, we conclude that the order of multinomials in Equation (3.4.1) may not exceed the limits specified below Equation (3.4.1) for, if there existed a term in S or T of order exceeding the limits, there would exist some $\underline{F}_o$ which would map the term into a term of S_x or T_x of excessive order, violating the limits specified in Equation (3.4.3). QED

Remark - This reveals the complicated nature of $J(\underline{F})$ for systems of reasonable dimension, and eliminates the usefulness of the algebraic expansion of J for systems with more than one input or output.

Claim 3.2 The range of f may include at most $\frac{n+1}{2}$ ($\frac{n}{2}$ if n is even) distinct regions in which the system can be stabilized by output feedback.

Proof

In a stable region of f, J is continuous and finite. However, as the system approaches instability, $J \to \infty$. (This corresponds to the root locus crossing the imaginary axis). Selecting J_o "sufficiently large," we see that $J(f) = J_o$ must have two solutions for each distinct region of stability. But $J(f) = J_o$ is identical to

$$S(f) - J_o T(f) = 0 \tag{3.4.4}$$

which has at most n + 1 roots. Thus there can be at most $\frac{n+1}{2}$ regions of stability. (Note that as $f \to \pm\infty$, J becomes unbounded, and these count as "stability boundaries.") QED

Remark - This suggests a way of determining exactly the values of f at which a system can become unstable - clearly this may happen only when $T(f) = 0$, or $f = -\frac{1}{\lambda}$ where λ is a real eigenvalue of $\underline{G}$ (see Equations (3.2.20), (3.2.22), (3.2.23) and theorem 3.1).

CHAPTER IV

ITERATIVE APPROACHES TO THE PROBLEM

Since the "direct" algebraic expansion has been shown to be impractical when dealing with systems with multiple inputs or outputs, we now consider techniques involving iterative approximation. We will first consider a method proposed by Levine, and show that it converges monotonically to a local minimum in J. We then propose a penalty function type of algorithm which we hope will lead us to a global minimum. An example of the proposed algorithm is supplied.

4.1 A Method Due to Levine

We now consider an iterative algorithm due to Levine[11,16] and show that for any initial guess $\underline{F}_o$ which stabilizes the system, the algorithm will converge monotonically to a local minimum. Much of the development is identical to the original treatment by Levine, but he was unable to prove existance at each step, or convergence of the sequence.

Iterative Algorithm 1 (Levine)

Suppose $\underline{F}_o$ stabilizes $(\underline{A} - \underline{BF}_o\underline{C})$, let $\underline{K}_N$ be the solution of

$$\underline{0} = (\underline{A} - \underline{BF}_{N-1}\underline{C})'\underline{K}_N + \underline{K}_N(\underline{A} - \underline{BF}_{N-1}\underline{C}) + \underline{Q} + \underline{C}'\underline{F}'_{N-1}\underline{RF}_{N-1}\underline{C} \tag{4.1.1}$$

and $\underline{F}_N$, $\underline{L}_N$ are solutions of the simultaneous equations

$$\underline{F}_N = \underline{R}^{-1}\underline{B}'\underline{K}_N\underline{L}_N\underline{C}'(\underline{CL}_N\underline{C}')^{-1} \tag{4.1.2}$$

$$\underline{0} = (\underline{A} - \underline{BF}_N\underline{C})\underline{L}_N + \underline{L}_N(\underline{A} - \underline{BF}_N\underline{C})' + \underline{X}_o \tag{4.1.3}$$

If a solution exists such that $\underline{L}_n > 0$, $\underline{F}_N$ is used to initiate the next iteration. Also,

$$J_N = tr[\underline{K}_N \underline{X}_o] \tag{4.1.4}$$

<u>Theorem 4.1</u> Provided a stabilizing $\underline{F}_o$ can be found, the algorithm described above converges monotonically to a locally optimal $\underline{F}$.

<u>Outline of Proof</u>

A) If $\underline{F}_{N-1}$ stabilizes the system, $\underline{K}_N$ exists and is a continuous function of $\underline{F}_{N-1}$. Wherever $\underline{L}_N$, $\underline{F}_N$ exist, they are continuous functions of $\underline{F}_{N-1}$.

B) If $\underline{F}_N$ exists and stabilizes the system, $J_{N+1} \le J_N$.

C) The matrices $\underline{L}_N$, $\underline{F}_N$ exist for all stabilizing $\underline{F}_{N-1}$.

D) J converges monotonically.

E) As J approaches a constant value, so do $\underline{K}$, $\underline{L}$ and $\underline{F}$.

F) The limiting value of $\underline{F}$ corresponds to a local optimum.

<u>Proof</u>

A) Consider the space of all $\underline{F}_{N-1}$ for which $(\underline{A} - \underline{BF}_{N-1}\underline{C})$ is stable.

i) $\underline{K}_N$ exists, is unique, positive definite (see Reference 18, p. 239, theorem 4). It is easily seen that small changes in $\underline{F}_N$ induce small changes in $\underline{K}_N$, so long as consideration is limited to the set of stabilizing $\underline{F}_{N-1}$.

ii) Similarly, the pair $(\underline{F}_N, \underline{L}_N)$ is a continuous functions of $\underline{K}_N$, which is a continuous function of $\underline{F}_{N-1}$. The set of $\underline{F}_{N-1}$ for which solutions $\underline{F}_N$, $\underline{L}_N$ exist has at least one point in it - the global minimum in J. This set is also open, since small changes in $\underline{F}_{N-1}$ induce small changes in $(\underline{F}_N, \underline{L}_N)$, and $(\underline{A} - \underline{BF}_N\underline{C})$ is stable in an open set.

B) Let us assume that $\underline{F}_N$, $\underline{L}_N$ exist for some $\underline{F}_{N-1}$.

Then, K_{N+1} exists, and is the solution of

$$0 = (A - BF_N C)'K_{N+1} + K_{N+1}(A - BF_N C) + Q + C'F_N'RF_N C \qquad (4.1.4)$$

Adding terms to Equation (4.1.1) yields

$$\begin{aligned} 0 = {} & (A - BF_{N-1}C)'K_N + K_N(A - BF_{N-1}C) + Q + C'F_{N-1}'RF_{N-1}C \\ & - (BF_N C)'K_N + (BF_N C)'K_N - K_N(BF_N C) + K_N(BF_N C) \\ & + K_N BR^{-1}B'K_N - K_N BR^{-1}B'K_N \end{aligned} \qquad (4.1.5)$$

$$\begin{aligned} 0 = {} & (A - BF_N C)'K_N + K_N(A - BF_N C) + Q + C'F_{N-1}'RF_{N-1}C \\ & - (BF_{N-1}C)'K_N + (BF_N C)'K_N - K_N(BF_{N-1}C) + K_N(BF_N C) \\ & + K_N BR^{-1}B'K_N - K_N BR^{-1}B'K_N \end{aligned} \qquad (4.1.6)$$

Define:

$$W_N^{(1)} \equiv [C'F_N' - K_N BR^{-1}]R[F_N C - R^{-1}B'K_N] \qquad (4.1.7)$$

$$W_N^{(2)} \equiv [C'F_{N-1}' - K_N BR^{-1}]R[F_{N-1}C - R^{-1}B'K_N] \qquad (4.1.8)$$

$$\delta K_N = K_{N+1} - K_N \qquad (4.1.9)$$

Subtracting Equation (4.1.6) from Equation (4.1.4) yields

$$0 = (A - BF_N C)'\delta K_N + \delta K_N(A - BF_N C) + W_N^{(1)} - W_N^{(2)} \qquad (4.1.10)$$

We recall that solutions of Lyapunov equations Equation (4.1.3) and Equation (4.1.10) take the form

$$L_N = \int_0^\infty e^{[A - BF_N C]t} X_o e^{[A - BF_N C]'t} dt \qquad (4.1.11)$$

$$\delta \underline{K}_N = {}_o\int^{\infty} e^{[\underline{A} - \underline{BF}_N\underline{C}]'t}(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})e^{[\underline{A} - \underline{BF}_N\underline{C}]t}dt \qquad (4.1.12)$$

Factor the symmetric positive definite matrix $\underline{L}_N$ into

$$\underline{\phi}_N\underline{\phi}_N = \underline{L}_N \qquad (4.1.13)$$

Let $\{\underline{u}_1 \dots \underline{u}_r\}$ be an orthonormal basis for the range of $[\underline{\phi}_N\underline{C}']$

$\{\underline{v}_1 \dots \underline{v}_{n-r}\}$ be an orthonormal basis for the null space of $[\underline{C\phi}_n]$

and let

$$\underline{U}' \equiv [\underline{u}_1, \underline{u}_2 \cdots \underline{u}_r] \qquad (4.1.14)$$

$$\underline{V}' \equiv [\underline{v}_1, \underline{v}_2 \cdots \underline{v}_{n-r}] \qquad (4.1.15)$$

$\underline{U}$ is an r x n matrix

$\underline{V}$ is an (n-r) x n matrix

By construction

$$\left[\frac{\text{U}}{\text{V}}\right][\underline{U}' \vdots \underline{V}'] = \underline{I} \qquad (4.1.16)$$

Recalling that the trace operator is invariant to cyclic permutations, we have

$$\begin{aligned} J_{N+1} - J_N &= \text{tr}[\underline{K}_{N+1}\underline{X}_o] - \text{tr}[\underline{K}_N\underline{X}_o] \\ &= \text{tr}[\delta\underline{K}_N\underline{X}_o] \\ &= \text{tr}[(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{L}_N] \\ &= \text{tr}[\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N] \\ &= \text{tr}[\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N[\underline{U}' \vdots \underline{V}']\left[\frac{\text{U}}{\text{V}}\right]] \end{aligned}$$

$$= \mathrm{tr}\left[\begin{bmatrix} \underline{U} \\ \underline{V} \end{bmatrix} \underline{\phi}_N (\underline{W}_N^{(1)} - \underline{W}_N^{(2)}) \underline{\phi}_N [\underline{U}' : \underline{V}'] \right]$$

$$= \mathrm{tr}\begin{bmatrix} \underline{U}\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N\underline{U}' & \underline{U}\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N\underline{V}' \\ \underline{V}\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N\underline{U}' & \underline{V}\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N\underline{V}' \end{bmatrix} \tag{4.1.17}$$

But $\underline{V}\underline{\phi}_N\underline{C} = \underline{0}$ by construction, so

$$\underline{V}\underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N\underline{V}' = \underline{V}\underline{\phi}_N(\underline{K}_N\underline{BR}^{-1}\underline{B}'\underline{K}_N - \underline{K}_N\underline{BR}^{-1}\underline{B}'\underline{K}_N)\underline{\phi}_N\underline{V}'$$

$$= \underline{0} \tag{4.1.18}$$

Moreover,

$$\underline{U}\underline{\phi}_N\underline{C}'\underline{F}_N' = \underline{U}\underline{\phi}_N\underline{C}'(\underline{C}\underline{L}_N\underline{C}')^{-1}\underline{C}\underline{L}_N\underline{K}_N\underline{BR}^{-1}$$

$$= \underline{U}\underline{\phi}_N\underline{C}'(\underline{C}\underline{\phi}_N\underline{\phi}_N\underline{C}')^{-1}\underline{C}\underline{\phi}_N\underline{\phi}_N\underline{K}_N\underline{BR}^{-1} \tag{4.1.19}$$

Since range $(\underline{U})$ = range $(\underline{\phi}\underline{C}')$

$$\underline{U}\underline{\phi}_N\underline{C}'(\underline{C}\underline{\phi}_N\underline{\phi}_N\underline{C}')^{-1}\underline{C}\underline{\phi}_N = U \tag{4.1.20}$$

(See Reference 19, page 579, theorem 5.)

So, $$\underline{U}\underline{\phi}_N\underline{C}'\underline{F}_N = \underline{U}\underline{\phi}_N\underline{K}_N\underline{BR}^{-1} \tag{4.1.21}$$

and $$\underline{U}\underline{W}_N^{(1)} = 0 \tag{4.1.22}$$

Thus, using Equations (4.1.18) and (4.1.22), we have

$$\begin{bmatrix} \underline{U} \\ \underline{V} \end{bmatrix} \underline{\phi}_N(\underline{W}_N^{(1)} - \underline{W}_N^{(2)})\underline{\phi}_N[\underline{U}'\ \ \underline{V}'] = -\begin{bmatrix} \underline{U}\underline{\phi}_N\underline{W}_N^{(2)}\underline{\phi}_N\underline{U}' & \underline{U}\underline{\phi}_N\underline{W}_N^{(2)}\underline{\phi}_N\underline{V}' \\ \underline{V}\underline{\phi}_N\underline{W}_N^{(2)}\underline{\phi}_N\underline{U}' & \underline{0} \end{bmatrix} \tag{4.1.23}$$

Substituting this into Equation (4.1.17),

$$J_{N+1} - J_N = -\,\mathrm{tr}[\underline{U}\underline{\phi}_N \underline{W}_N^{(2)} \underline{\phi}_N \underline{U}'] \leq 0 \tag{4.1.24}$$

since $\underline{W}_N^{(2)}$ is a quadratic form.

C) Consider again the pair $(\underline{F}_N, \underline{L}_N)$, solutions of Equations (4.1.2) and (4.1.3). In (A), we showed that $\underline{F}_N$ is a continuous function of $\underline{F}_{N-1}$ when it exists. As $\underline{F}_{N-1}$ approaches a value for which there is no solution $\underline{F}_N$, $\underline{F}_N$ approaches a value which does not stabilize $(\underline{A} - \underline{BF}_N\underline{C})$. However, as this happens, $J_{N+1} \leq J_N$ when $\underline{F}_N$ does exist.

Thus, as $\underline{F}_{N-1}$ approaches a value for which $\underline{F}_N$ does not exist, J_N increases without bound and $\underline{F}_{N-1}$ approaches a value for which $\underline{A} - \underline{BF}_{N-1}\underline{C}$ is unstable. Thus, for every stabilizing $\underline{F}_{N-1}$, a stabilizing $\underline{F}_N$ exists.

D) $J_N > 0$ for all N, and $J_{N+1} \leq J_N$. These are the conditions for monotone convergence of J.

E) We need the following lemma:

Let $\underline{P}$ be a symmetric positive semidefinite matrix with elements $[\underline{P}]_{ij}$. Then

$$\text{I)}\quad [\underline{P}]_{ii} \leq \mathrm{tr}[\underline{P}] \tag{4.1.25}$$

$$\text{II)}\quad [\underline{P}]_{ij}^2 \leq [\underline{P}]_{ii} \cdot [\underline{P}]_{jj} \tag{4.1.26}$$

<u>Proof</u>

$$\text{I)}\quad \mathrm{tr}[\underline{P}] = \sum_{j=1}^{n} [\underline{P}]_{jj}, \text{ all } [\underline{P}]_{jj} > 0 \tag{4.1.27}$$

II) $\underline{e}_i$, $\underline{e}_j$ are elements of the natural basis

$\begin{bmatrix} \underline{e}_i \\ \underline{e}_j \end{bmatrix} \underline{P} \begin{bmatrix} \underline{e}_i \\ \underline{e}_j \end{bmatrix}'$ is a quadratic form,

so

$$\det\left(\begin{bmatrix} \underline{e}_i \\ \underline{e}_j \end{bmatrix} \underline{P}[\underline{e}_i' \vdots \underline{e}_j']\right) \geq 0 \tag{4.1.28}$$

but $$\det(\cdot) = [\underline{P}]_{ii}[\underline{P}]_{jj} - [\underline{P}]_{ij}^2 \geq 0 \tag{4.1.29}$$

or $$[\underline{P}]_{ij}^2 \leq [\underline{P}]_{ii}[\underline{P}]_{jj} \qquad \text{QED} \tag{4.1.30}$$

Since J converges monotonically, for any $\varepsilon > 0$, there exists an $M(\varepsilon)$ such that $J_N - J_{N+1} < \varepsilon$ for all $N > M(\varepsilon)$

From Equation (4.1.24), we see that

$$\text{tr}[\underline{U\emptyset_N W_N^{(2)} \emptyset_N U'}] < \varepsilon \;; \quad N > M(\varepsilon) \tag{4.1.31}$$

Furthermore, every element of

$$\underline{V\emptyset W_N^{(2)} \emptyset_N V'} = \underline{V\emptyset_N K_N BR^{-1} BK_N \emptyset_N V} \tag{4.1.32}$$

is bounded, for the $\text{tr}[\underline{K_N X_o}] = J_N$ bounds the elements of $\underline{K_N}$. From this we see that every element of

$$\left[\begin{array}{c:c} \underline{U\emptyset_N W_N^{(2)} \emptyset_N U'} & \underline{U\emptyset_N W_N^{(2)} \emptyset_N V'} \\ \hdashline \underline{V\emptyset_N W_N^{(2)} \emptyset_N U'} & \underline{0} \end{array}\right]$$

can be made arbitrarily small.

From Equation (4.1.23), noting that $\left[\frac{U}{V}\right]$ and $\underline{\emptyset}$ are invertible, we see that as $N \to \infty$, $[\underline{W}_N^{(1)} - \underline{W}_N^{(2)}] \to \underline{0}$.

From Equation (4.1.12), we see that as $N \to \infty$, $\delta\underline{K} \to \underline{0}$.

As $\underline{K}$ approaches a constant value, so do $\underline{F}$ and $\underline{L}$, being functions of $\underline{K}$.

F) A limiting value of $\underline{F}$ satisfies the necessary condition (Fact 2.2). It also stabilizes the system, so it is a local minimum.

This ends the proof of theorem 4.1.

We now propose another, similar way of solving the same problem.

Iterative Algorithm 2

Suppose $\underline{F}_o$ stabilizes $(\underline{A}-\underline{BF}_o\underline{C})$; let $\underline{L}_N$ be the solution of

$$\underline{0} = (\underline{A} - \underline{BF}_{N-1}\underline{C})\underline{L}_N + \underline{L}_N(\underline{A} - \underline{BF}_{N-1}\underline{C})' + \underline{X}_o \tag{4.1.33}$$

and $\underline{F}_N$, $\underline{K}_N$ are solutions of the simultaneous equations

$$\underline{F}_N = \underline{R}^{-1}\underline{B}'\underline{K}_N\underline{L}_N\underline{C}'(\underline{CL}_N\underline{C})^{-1} \tag{4.1.34}$$

$$\underline{0} = (\underline{A} - \underline{BF}_N\underline{C})'\underline{K}_N + \underline{K}_N(\underline{A} - \underline{BF}_N\underline{C}) + \underline{Q} + \underline{CF}_N\underline{RF}_N\underline{C} \tag{4.1.35}$$

If a solution exists such that $\underline{K}_N > 0$, $\underline{F}_N$ is used to initiate the next iteration. Also,

$$J_N = \mathrm{tr}[\underline{K}_N\underline{X}_o] \tag{4.1.36}$$

Theorem 4.2 - Provided a stabilizing $\underline{F}_o$ can be found, the algorithm described above converges monotonically to a locally optimal $\underline{F}$.

Proof

The proof duplicates the proof of theorem 4.1 with the following changes:

1) In (B) existance of the pair $(\underline{F}_N, \underline{K}_N)$ is postulated, and in (C) proved (rather than $(\underline{F}_N, \underline{L}_N)$ in theorem 4.1).

2) In (B) a few indices change.

$$\delta\underline{K}_N \equiv \underline{K}_N - \underline{K}_{N-1} \tag{4.1.37}$$

and
$$\underline{0} = (\underline{A} - \underline{BF}_{N-1}\underline{C})'\delta\underline{K}_N + \delta\underline{K}_N(\underline{A} - \underline{BF}_{N-1}\underline{C}) + \underline{W}_N^{(1)} - \underline{W}_N^{(2)} \tag{4.1.38}$$

However, $\underline{L}_N$ is now defined as the integral of $(\underline{A} - \underline{BF}_{N-1}\underline{C})$, (compare Equations (4.1.3) and (4.2.1)), and the indices of $\underline{F}$ in Equations (4.1.11) and (4.1.12) should be reduced accordingly. The remainder of the proof applies to this theorem unaltered.

Each of the two methods so far proposed has the important drawbacks:

1) A non-linear matrix equation must be solved at each iteration, and no method of solution immediately presents itself as applicable to the particular non-linear equations to be solved.

2) An "initial guess", $\underline{F}_o$ is required. Such a stabilizing feedback gains matrix may not be available to initiate the algorithm. In fact, it is in no way guaranteed that such a matrix does, in fact, exist; i.e. there may not exist any output-feedback gains which stabilize the system, in which case, of course, there is no "problem."

3) The choice of the "initial guess" determines whether the algorithm will converge to a local minimum, or to the global minimum in J. Thus, one has no way of knowing whether or not a solution obtained using either algorithm is in fact the desired solution to the problem.

4.2 Iterative Refinement of Values "Sufficiently Close" to the Solution

The similarity between Iterative Algorithms 1 and 2, and the rigid convergence properties they possess lead us to propose the following:

Iterative Algorithm 3

Suppose $\underline{F}_o$ stabilizes $(\underline{A} - \underline{B}\underline{F}_o\underline{C})$; let $\underline{L}_N$ and $\underline{K}_N$ be solutions of

$$\underline{0} = (\underline{A} - \underline{B}\underline{F}_N\underline{C})\underline{L}_N + \underline{L}_N(\underline{A} - \underline{B}\underline{F}_N\underline{C})' + \underline{X}_o \qquad (4.2.1)$$

$$\underline{0} = (\underline{A} - \underline{B}\underline{F}_N\underline{C})'\underline{K}_N + \underline{K}_N(\underline{A} - \underline{B}\underline{F}_N\underline{C}) + \underline{Q} + \underline{C}'\underline{F}_N'\underline{R}\underline{F}_N\underline{C} \qquad (4.2.2)$$

Then $\underline{F}_{N+1}$ may be used to initiate another iteration, where $\underline{F}_{N+1}$ is given

$$\underline{F}_{N+1} = \underline{R}^{-1}\underline{B}'\underline{K}_N\underline{L}_N\underline{C}'(\underline{C}\underline{L}_N\underline{C}')^{-1} \qquad (4.2.3)$$

We emphasize that convergence of this algorithm is only conjectured, but appears likely. The conjecture may be supported by considering the proof of theorem 4.1. If $\underline{L}_{N+1}$ is assumed to be sufficiently close to $\underline{L}_N$, the proof still holds in the case of Iterative Algorithm 3. This would seem to be a valid assumption if $\underline{F}_o$ is "sufficiently" close to the solution, $\underline{F}^*$.

4.3 A Penalty Function Type of Algorithm

Consider a linear time-invariant system (such as the one described by Equations (2.1.1) and (2.1.2)) and let $\underline{D}$ be an arbitrary full-rank $(n-r) \times n$ matrix such that

1) $\begin{bmatrix} \underline{C} \\ \underline{D} \end{bmatrix}$ is non-singular (4.3.1)

2) $\underline{CD}' = \underline{0}$ (4.3.2)

Now consider a system described by

$$\dot{\underline{x}}(t) = \underline{Ax}(t) + \underline{B}(\underline{u}_1(t) + \underline{u}_2(t)) \tag{4.3.3}$$

$$\underline{y}_1(t) = \underline{Cx}(t) \tag{4.3.4}$$

$$\underline{y}_2(t) = \underline{Dx}(t) \tag{4.3.5}$$

Let the control be of the form

$$\begin{bmatrix} \underline{u}_1(t) \\ \underline{u}_2(t) \end{bmatrix} = - \begin{bmatrix} \underline{F}_{11} & \underline{F}_{12} \\ \underline{F}_{21} & \underline{F}_{22} \end{bmatrix} \begin{bmatrix} \underline{C} \\ \underline{D} \end{bmatrix} \underline{x}(t) = -\underline{F} \begin{bmatrix} \underline{C} \\ \underline{D} \end{bmatrix} \underline{x}(t) \tag{4.3.6}$$

and the cost functional given by

$$J = {}_0\int^{\infty} \underline{x}'(t)\underline{Q}\underline{x}(t) + \underline{u}_1'(t)\underline{R}\underline{u}_1(t) + \underline{u}_2'(t)\underline{R}\underline{u}_1(t) + \underline{u}_1'(t)\underline{R}\underline{u}_2(t)$$

$$+ (1+w^2)\underline{u}_2'(t)\underline{R}\underline{u}_2(t)dt \qquad (4.3.7)$$

where w is a positive constant parameter.

We wish to select that $\underline{F}$ that minimizes J. When w = 0, this corresponds to the linear regulator problem, and is known to have a unique solution. We will show that as $w \to \infty$, either J diverges or $\underline{F}$ approaches a solution to the output-feedback problem; selection of different $\underline{D}$ could cause the algorithm to converge to different local minima. We also propose an algorithm for the minimization of J for any finite w which involves the solution of linear equations at each iteration.

First, we place the system in canonical output form. As $\underline{C}$ and $\underline{D}$ are complementary, we can require that they take the form

$$\underline{C} = [\underline{I}r \vdots \underline{0}] \qquad (4.3.8)$$

$$\underline{D} = [\underline{0} \vdots \underline{I}_{n-r}] = \underline{Y} \qquad (4.3.9)$$

We remark that each new selection of $\underline{D}$ requires a new transformation of the system to canonical output form.

Let

$$\underline{T} \equiv \begin{bmatrix} \underline{0} & \vdots & \underline{0} \\ \cdots & \cdots & \cdots \\ \underline{0} & \vdots & w\underline{I}_{n-r} \end{bmatrix} \qquad (4.3.10)$$

where w is the parameter defined in Equation (4.3.7).

We now define $\underline{L}$ and $\underline{K}$, solutions of

$$\underline{0} = (\underline{A} - \underline{BF})\underline{L} + \underline{L}(\underline{A} - \underline{BF})' + \underline{X}_o \qquad (4.3.11)$$

$$\underline{0} = (\underline{A} - \underline{BF})'\underline{K} + \underline{K}(\underline{A} - \underline{BF}) + \underline{Q} + \underline{F}'\underline{RF} + \underline{TF}'\underline{RFT} \qquad (4.3.12)$$

It is easily seen that this definition is compatible with the previous definitions.

Using a derivation similar to that of Levine, we show that a necessary condition for optimality of $\underline{F}$ is

$$\underline{F} = \underline{R}^{-1}\underline{B}'\underline{KL}(\underline{L} + \underline{TLT})^{-1} \qquad (4.3.13)$$

The derivation is straight-forward but lengthy, and will not be given.

Theorem 4.3 - As $w \to \infty$, the necessary condition Equation (4.3.13) approaches the necessary condition for output-feedback optimality, Equation (2.2.5). If J remains bounded as $w \to \infty$, $\underline{F}$ approaches a locally optimal output-feedback solution.

Proof

It is necessary to show that

$$\lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1} = \underline{C}'(\underline{CLC}')^{-1}\underline{C} \qquad (4.3.14)$$

provided J remains bounded (i.e. provided the limit exists).

1) If the limit exists,

$$[\underline{LC}' \vdots \underline{\gamma}'] = \begin{bmatrix} \underline{CLC}' & \vdots & \underline{0} \\ \cdots & \vdots & \cdots \\ \underline{\gamma LC}' & \vdots & \underline{I}_{n-r} \end{bmatrix} \text{ is non singular}$$

2) Noting that by construction of $\underline{I}$, $\underline{TC}' = \underline{0}$ and, so, $\underline{TLTC}' = \underline{0}$, hence,

$$\lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1}\underline{LC}' = \lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1}(\underline{L} + \underline{TLT})\underline{C}'$$

$$= \underline{C}' = [\underline{C}'(\underline{CLC}')^{-1}\underline{C}]\underline{LC}' \qquad (4.3.15)$$

$$\lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1}\underline{Y}' = \lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1}\frac{(\underline{L} + \underline{TLT})}{w^2}\,\underline{Y}'(\underline{YLY}')^{-1}$$

$$= \lim_{w\to\infty}\frac{1}{w^2}\,\underline{Y}'(\underline{YLY}')^{-1} = \underline{0}$$

$$= [\underline{C}'(\underline{CLC}')^{-1}\underline{C}]\underline{Y}' \qquad (4.3.16)$$

Thus $$\lim_{w\to\infty}(\underline{L} + \underline{TLT})^{-1}[\underline{LC}' \vdots \underline{Y}'] = \underline{C}'(\underline{CLC}')^{-1}\underline{C}[\underline{LC}' \vdots \underline{Y}'] \qquad (4.3.17)$$

and, as $[\underline{LC}' \vdots \underline{Y}']$ is non-singular, Equation (4.3.14) holds, and the theorem is proved.

The similarity between the problem of finding the optimal $\underline{F}$ for a given w, and the optimal output-feedback problem is immediately recognized, and we can immediately propose three algorithm analogous to the Iterative Algorithms 1 through 3. However, it is now possible to ensure that the initial guess will be arbitrarily close to the solution by making the increase in w sufficiently small, and solving for the optimal $\underline{F}$ for each value of w. We, therefore, can immediately propose an algorithm based on Iterative Algorithm 3.

<u>Iterative Algorithm 4</u>

Suppose $\underline{F}_o$ stabilizes $(\underline{A} - \underline{BF}_o)$; let $\underline{L}_N$ and $\underline{K}_N$ be solutions of

$$\underline{0} = (\underline{A} - \underline{BF}_N)\underline{L}_N + \underline{L}_N(\underline{A} - \underline{BF}_N)' + \underline{X}_o \qquad (4.3.18)$$

$$\underline{0} = (\underline{A} - \underline{BF}_N)'\underline{K}_N + \underline{K}_N(\underline{A} - \underline{BF}_N) + \underline{Q} + \underline{F}_N'\underline{RF}_N + \underline{TF}_N'\underline{RF}_N\underline{T} \qquad (4.3.19)$$

Then $\underline{F}_{N+1}$ may be used to initiate another iteration, where $\underline{F}_{N+1}$ is given by

$$\underline{F}_{N+1} = \underline{R}^{-1}\underline{K}_N\underline{L}_N(\underline{L}_N + \underline{TL}_N\underline{T})^{-1} \qquad (4.3.20)$$

We emphasize that convergence of this algorithm is only <u>conjectured</u>, but appears likely if $\underline{F}_o$ is sufficiently close to the solution $\underline{F}^*$ (for some particular w).

If Iterative Algorithm 4 does, in fact, possess desirable convergence properties, it may be used to determine the optimal output-feedback compensator, eliminating most of the drawbacks listed for Iterative Algorithms 1 and 2. By selecting a number of differently weighted $\underline{D}$, it is possible to obtain a reliable "solution profile" - that is:

1) If none of the $\underline{D}$ selected caused the algorithm to converge, there <u>probably</u> is no solution;

2) If most of the $\underline{D}$ selected converge to various local minima, but selection of new $\underline{D}$ causes the algorithm to converge to local minima already known, these are <u>probably</u> the only local minima.

CHAPTER V

CONCLUSIONS

In the previous chapters, we have presented two approaches to the computation of optimal output-feedback conpensators for linear time-invariant systems. We will discuss each of these in turn.

In the third chapter, we showed that the performance of a single-input single-output system can be expressed as the ratio of two polynomials in the feedback gain. The computation of the coefficients of these polynomials, when solving the Lyapunov equations by the method due to R.A. Smith,[20] involves the inversion of one n x n matrix, a tolerable number of matrix multiplications and additions, and the diagonalization of another: a relatively small task. This compares favorably with the Nyquist plot or root locus methods of determining the "stability boundary values" - values of the gain at which the system becomes unstable, and has the advantage of yielding these values directly (algebraically, rather than graphically). In addition, a number of properties of the stability boundary as a manifold in feedback space may be deduced.

In the fourth chapter, we proved monotone convergence for a method proposed by Levine,[11,16] and, in turn proposed a penalty function type of algorithm to handle this problem. Once again, the implementation of the algorithm on a digital computer may be accomplished quite efficiently, as it involves only the solution of Lyapunov equations and a matrix inversion at each step. The order of convergence of the algorithm is yet to be determined, and the author hopes to determine this experimentally in the near future.

Should the penalty function approach prove successful, it could be applied to a variety of other problems:

1) The linear compensator of limited dimension proposed by T.L. Johnson[11,21] may be designed, using an algorithm almost identical to the one proposed for output-feedback design.

2) The globally optimal time-invariant solution could provide a reliable "first guess" in computing the optimal <u>time-varying</u> output-feedback gains or limited dimension compensator, using algorithms proposed by Levine[11] or Axsater.[12]

3) The design of a time-invariant limited-dimension Kalman-Bucy filter, or solution of the limited-dimension compensator problem where measurement noise is present, could be accomplished.

4) The design of piecewise constant compensators for time-varying problems based on the approach proposed by Kleinman,[15] could be attempted.

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