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A STABILITY ANALYSIS FOR A

A SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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List of symbols

Roman letters: lower case

a	r
f	s
j	t
k	u
ℓ	v
n	w
p	x
q	z

Roman letters: upper case

A	S
B	T
C	U
D	V
F	X
H	Y
I	Z
R	

Greek letters: lower case

ζ zeta	ρ rho
η eta	σ sigma
μ mu	τ tau
ν nu	ϕ phi
ξ xi	ψ psi
π pi	ω omega

Greek letters: upper case

$\mathcal{X}$ chi

Mathematical symbols

$\times$ multiplication

$\sum$

$\rightarrow$

$\mapsto$

$\int$

∂

∞

ε

$\subseteq$

$\leq$

$\| \quad |\,|$

$\cap$

Abstract

We consider a parabolic partial differential equation $u_t = u_{xx} + f(u)$, where $-\infty < x < +\infty$ and $0 < t < +\infty$. Under suitable hypotheses pertaining to f, we exhibit a class of initial data $\phi(x)$, $-\infty < x < +\infty$, for which the corresponding solutions $u(x,t)$ approach zero as $t \to +\infty$. This convergence is uniform with respect to x on any compact subinterval of the real axis.

Introduction

Consider the following initial-value problem:

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (-\infty < x < +\infty,\ 0 < t < +\infty) \tag{1a}$$

$$u(x,0) = \phi(x) \qquad (-\infty < x < +\infty) \tag{1b}$$

Here f is a given function continuously mapping the real line R into itself; ϕ is any bounded continuous function taking R into R; and u is to be a suitably smooth function mapping $R \times [0,+\infty)$ into R.

Our interest in (1) centers on the problem of determining the behavior of $u(x,t)$ as $t \to +\infty$. Clearly, this behavior depends on detailed properties of f and on one's choice of the initial data ϕ. In this context, a natural assumption regarding f is that $f(0) = 0$. Under this hypothesis, of course, Eq. (1a) has a trivial solution $u(x,t) \equiv 0$. The question then arises, what are the stability properties of this zero solution?

Many authors have studied this type of problem for semilinear parabolic partial differential equations. In particular, we mention the works [1-3], [7,8], [10,11] and [14,15].

In the present work we shall exhibit a class of initial data ϕ for which the corresponding solutions $u(x,t)$ approach zero as $t \to +\infty$. In doing this we shall assume that f satisfies the following hypotheses.

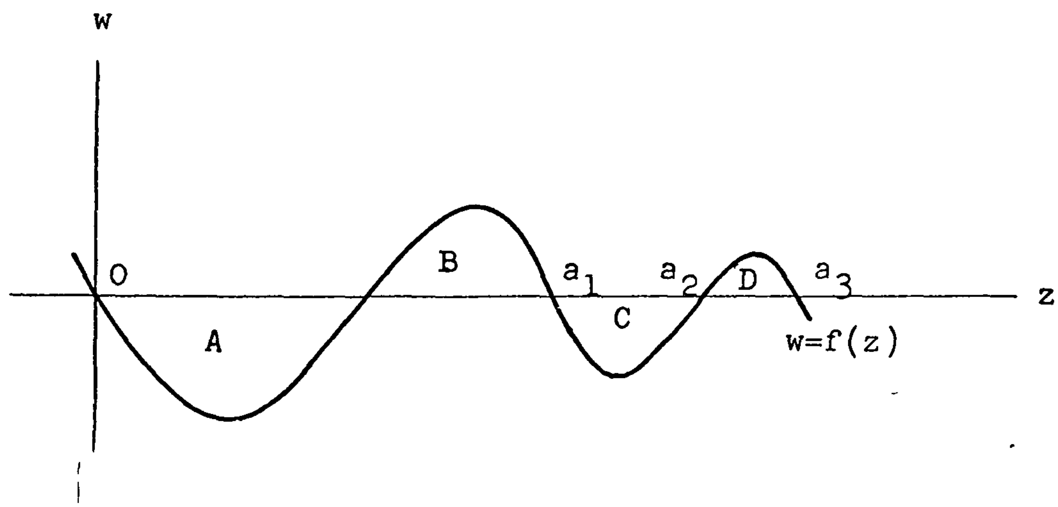
(H1) The derivatives f', f'', and f''' exist and are continuous everywhere on R.

(H2) $f(0) = 0$ and $f'(0) < 0$.

(H3) There exists a number $a_0 \varepsilon (0,+\infty)$ such that $f(a_0) < 0$ and

$$\int_0^z f(\zeta)d\zeta < 0 \qquad (0 \leq z \leq a_0) .$$

It may be instructive to consider (H3) in relation to the particular function f displayed in the accompanying figure. If the area A strictly exceeds in



magnitude the area B, then any number $a_0 \varepsilon (a_1, a_2)$ satisfies (H3). If, in addition, the area C strictly exceeds in magnitude the area D and if $f(z) < 0$ for all $z \varepsilon (a_3, +\infty)$, then any number $a_0 \varepsilon (a_3, +\infty)$ satisfies (H3).

The main results of this paper are stated in Theorems 4.4 and 4.5 below. We can summarize them in the following way. Let Y be the class of all differentiable functions $\phi: R \to R$ such that (i) ϕ and ϕ' are bounded and uniformly continuous on R; (ii) $\int_{-\infty}^{+\infty} \{\phi(x^2) + \phi'(x)^2\} dx < +\infty$; (iii) $0 \leq \phi(x) \leq a_0$ for all $x \varepsilon R$, where a_0 is as in (H3). Then, for any $\phi \varepsilon Y$, the corresponding solution $u(x,t)$ of (1) is defined for all $(x,t) \varepsilon R \times [0,+\infty)$ and $u(\cdot,t) \varepsilon Y$ for all $t \varepsilon [0,+\infty)$. Moreover, $u(x,t) \to 0$ as $t \to +\infty$ uniformly with respect to x on any compact subinterval of R. The partial derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ have this same convergence property.

Our proof of the preceding assertions is based upon techniques associated with the theory of Liapunov stability and dynamical systems (see [5], [6], [9]). The first step in this procedure is to interpret (1) as a flow in some suitable function space. We do this in Section 3 below. Also, in Section 3 we derive some geometric properties of our flow connected with the notion of an ω-limit set. These properties are set forth in Theorem 3.4.

Section 2 provides some necessary background material for Section 3.

In Section 4 we complete our analysis of (1). With the aid of an appropriate Liapunov functional (see Eq. (4.3)), we prove Theorems 4.4 and 4.5 mentioned above.

Now suppose that we modify Eq. (1) by restricting x to vary in some proper subinterval I of R and by imposing some suitable boundary condition on u at the endpoints of I. At the same time, we continue to assume that f satisfies (H1) - (H3). The question is, what are the appropriate analogues of Theorems 4.4 and 4.5 for this new problem?

In Section 5 we discuss two problems of this type. For the first we take $I = [0,+\infty)$ and the boundary condition $u_x(0,t) \equiv 0$. Our conclusions, stated in Theorem 5.1, are precisely analogous to Theorems 4.4 and 4.5. For the second problem we take $I_1 = [0,+\infty)$ and $u(0,t) \equiv 0$. Again, the conclusions are exact analogues of Theorems 4.4 and 4.5.

In Section 6 we discuss a third problem, one in which $I = [0,\pi]$ and $u_x(0,t) \equiv u_x(\pi,t) \equiv 0$. Our results appear in Theorem 6.1 and, in this instance, they do not exactly parallel Theorems 4.4 and 4.5.

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1. Notation

Let R denote the real number system and Z_0 the set of all integers $\ell \geq 0$. Let I be any closed interval, bounded or unbounded, in R.

For any $\ell \varepsilon Z_0$ we let $X_\infty^{(\ell)}(I)$ be the space of all ℓ-times continuously differentiable functions $\phi: I \to R$ such that $\phi, \phi^{(1)}, \ldots, \phi^{(\ell)}$ are bounded and uniformly continuous on I. We norm $X_\infty^{(\ell)}(I)$ by setting

$$||\phi||_{\infty}^{(\ell)} \overset{\text{def}}{=} \sum_{j=0}^{\infty} \sup_{x\varepsilon I}|\phi^{(j)}(x)| \qquad (\phi\varepsilon X_{\infty}^{(\ell)}(I)) .$$

For any $p\varepsilon[1,+\infty)$ we let $X_p^{(\ell)}(I)$ be the space of all those $\phi\varepsilon X_{\infty}^{(\ell)}(I)$ such that $\phi,\phi^{(1)},\ldots,\phi^{(\ell)}$ are pth-power Lebesgue integrable on I. We define a norm $||\ ||_p^{(\ell)}$ on $X_p^{(\ell)}(I)$ by setting

$$||\phi||_p^{(\ell)} \overset{\text{def}}{=} ||\phi||_{\infty}^{(\ell)} + \sum_{j=0}^{\ell} \{\int|\phi^{(j)}(x)|^p dx\}^{1/p} \qquad (\phi\varepsilon X_p^{(\ell)}(I)) .$$

For every $\ell\varepsilon Z_0$ and $p\varepsilon[1,+\infty]$, the space $X_p^{(\ell)}(I)$ is a Banach space under $||\ ||_p^{(\ell)}$. Moreover, if $q\varepsilon[p,+\infty)$ then $X_p^{(\ell)}(I)\subseteq X_q^{(\ell)}(I)$. Where no ambiguity can arise, we will usually write $X_p^{(\ell)}$ instead of $X_p^{(\ell)}(I)$.

For any $\ell\varepsilon Z_0$, $p\varepsilon[1,+\infty]$, and $r\varepsilon(0,+\infty)$, we let $B_p^{(\ell)}(r)$ denote the open ball in $X_p^{(\ell)}$ centered at the origin and having radius r. By $\overline{B}_p^{(\ell)}(r)$ we mean the corresponding closed ball in $X_p^{(\ell)}$.

For any $\ell\varepsilon Z_0$ we introduce a Fréchet norm $||\ ||_*^{(\ell)}$ on $X_{\infty}^{(\ell)}$ by setting

$$||\phi||_*^{(\ell)} \overset{\text{def}}{=} \sum_{j=0}^{\ell} \sum_{n=0}^{+\infty} \frac{1}{2^n} \sup_{x\varepsilon I\,[-n,n]} \{\frac{|\phi^{(j)}(x)|}{1+|\phi^{(j)}(x)|} \qquad (\phi\varepsilon X_{\infty}^{(\ell)}) .$$

When we are considering $X_{\infty}^{(\ell)}$ under $||\ ||_*^{(\ell)}$ rather than $||\ ||_{\infty}^{(\ell)}$, we shall denote $X_{\infty}^{(\ell)}$ by $X_*^{(\ell)}$. $X_*^{(\ell)}$ is a metric linear space [13, pp. 154-155] but, if I is unbounded, then $X_*^{(\ell)}$ is not a Fréchet space [13, p. 157].

For any $\ell\varepsilon Z_0$ and $p\varepsilon[1,+\infty)$ we may consider the restriction of $||\ ||_*^{(\ell)}$ to $X_p^{(\ell)}$. In this connection, we note the following theorem.

<u>Theorem 1.1</u>. For a given $\ell\varepsilon Z_0$ and $p\varepsilon[1,+\infty]$, let S be a nonempty subset of $X_p^{(\ell)}$. Suppose that there exists an $r\varepsilon(0,+\infty)$ such that $S\subseteq B_p^{(\ell+1)}(r)$. Then, with respect to $||\ ||_*^{(\ell)}$, the set S is relatively compact in $X_p^{(\ell)}$.

The proof is an exercise involving the Ascoli selection theorem and Fatou's lemma.

2. The Semigroup $\{T(t)\}$

In this section and in Sections 3 and 4 below the underlying interval will be $I = R$.

We now define a family $\{T(t): 0 \leq t < +\infty\}$ of transformations taking $X_\infty^{(0)}$ into $X_\infty^{(0)}$ by setting

$$[T(t)\phi](x) \overset{\text{def}}{=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} \phi(x + 2\eta\sqrt{t})d\eta \qquad (\phi\varepsilon X_\infty^{(0)}, x\varepsilon R, t\varepsilon[0,+\infty)) . \qquad (2.1)$$

This family arises in connection with the classical heat equation $u_t = u_{xx}$. Our purpose in this section is to state some properties of $\{T(t)\}$ necessary for our later work. Except for the briefest indications, we shall omit the corresponding proofs.

We note the well-known formula

$$[T(t)\phi](x) = (4\pi t)^{-1/2} \int_{-\infty}^{+\infty} \exp[- \frac{(\xi-x)^2}{4t}]\phi(\xi)d\xi \quad (\phi\varepsilon X^{(0)}, x\varepsilon R, t\varepsilon(0,+\infty)). \quad (2.2)$$

Also, for any $p\varepsilon[1,+\infty)$, we have

$$|[T(t)\phi](x)|^p \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} |\phi(x + 2\eta\sqrt{t})|^p \, d\eta \qquad (\phi\varepsilon X_p^{(0)}, x\varepsilon R, t\varepsilon[0,+\infty)) . \qquad (2.3)$$

The relations (2.2) and (2.3) are useful in establishing the following theorem.

<u>Theorem 2.1</u>. For any $p\varepsilon[0,+\infty]$ and $\ell\varepsilon Z_0$, the family $\{T(t)\}$ is a strongly continuous semigroup of bounded linear transformations taking $X_p^{(\ell)}$ into itself. For each $t\varepsilon[0,+\infty)$ and $\phi\varepsilon X_p^{(\ell)}$, we have $||T(t)\phi||_p^{(\ell)} \leq ||\phi||_p^{(\ell)}$. Also, for each

$t\varepsilon(0,+\infty)$ and $\phi\varepsilon X_p^{(\ell)}$, we have $T(t)\phi\varepsilon X_p^{(\ell+1)}$ and $||T(t)\phi||_p^{(\ell+1)} \leq (1+t^{-1/2})||\phi||_p^{(\ell)}$.

Theorem 2.2. For any $p\varepsilon[1,+\infty)$, $\ell\varepsilon Z_0$, $\phi\varepsilon X_p^{(\ell)}$, and $t\varepsilon(0,+\infty)$, we have

$$\lim_{x_1\to+\infty} \{\int_{x_1}^{+\infty} |\frac{\partial^\ell}{\partial x^\ell}[T(t)\phi](x)|^p\, dx + \int_{-\infty}^{-x_1} |\frac{\partial^\ell}{\partial x^\ell}[T(t)\phi](x)|^p\, dx\} = 0 ,$$

and the convergence here is uniform with respect to t on any compact subinterval of $(0,+\infty)$.

To prove Theorem 2.2 it suffices to consider the case $\ell = 0$. One then uses (2.3).

Now fix $p\varepsilon[1,+\infty]$, $\ell\varepsilon Z_0$, and $t_1,t_2\varepsilon R$ with $t_1 < t_2$. Let v be any function taking $R\times[t_1,t_2]$ into R such that $v(\cdot,t)\varepsilon X_p^{(\ell)}$ for each $t\varepsilon[t_1,t_2]$ and such that the map $t\mapsto v(\cdot,t)$ from $[t_1,t_2]$ into $X_p^{(\ell)}$ is continuous. We define a function $w:R\times[t_1,t_2] \to R$ by setting

$$w(\cdot,t) = \int_{t_1}^{t} T(t-\tau)v(\cdot,\tau)d\tau \qquad (t_1 \leq t \leq t_2) \tag{2.4}$$

On the basis of Theorem 2.1, one can prove the following result.

Theorem 2.3. The function w defined by (2.4) has the properties that $w(\cdot,t)\varepsilon X_p^{(\ell+1)}$ for each $t\varepsilon[t_1,t_2]$ and that the map $t\mapsto w(\cdot,t)$ from $[t_1,t_2]$ into $X_p^{(\ell+1)}$ is continuous. Furthermore,

$$\begin{aligned} ||w(\cdot,t)||_p^{(\ell)} &\leq |t-t_1| \sup_{t_1\leq\tau\leq t} ||v(\cdot,\tau)||_p^{(\ell)} \\ ||w(\cdot,t)||_p^{(\ell+1)} &\leq (|t-t_1| + 2|t-t_1|^{1/2}) \sup_{t_1\leq\tau\leq t} ||v(\cdot,\tau)||_p^{(\ell)} \end{aligned} \qquad (t_1\leq t\leq t_2)$$

Finally, we state the following theorem.

Theorem 2.4. Let p, ℓ, t_1, t_2, v and w be as in connection with (2.4) and suppose that $p < +\infty$. Then for any $t\varepsilon[t_1,t_2]$ we have

$$\lim_{x_1\to+\infty} \left\{\int_{x_1}^{+\infty} \left|\frac{\partial^\ell w}{\partial x^\ell}(x,t)\right|^p dx + \int_{-\infty}^{-x_1} \left|\frac{\partial^\ell w}{\partial x^\ell}(x,t)\right|^p dx\right\} = 0$$

and the convergence here is uniform with respect to $t\varepsilon[t_1,t_2]$.

To prove Theorem 2.4 it suffices to consider the case $\ell = 0$. One uses Theorem 2.2 and the Lebesgue dominated convergence theorem.

3. The Semigroup $\{U(t)\}$

Throughout this section and Section 4 we shall let f be as in Hypotheses (H1) - (H3) stated in the Introduction. However, we remark that, in the present section, we actually only need to assume (H1) and the condition $f(0) = 0$.

We are going to consider the following initial-value problem. Given any $\phi\varepsilon X_2^{(1)}$, find a real-valued function u defined on a domain $\{(x,t):x\varepsilon R, t\varepsilon[0,s)\}$, $0 < s \le +\infty$, such that (i) $u(\cdot,t)\varepsilon X_2^{(1)}$ for every $t\varepsilon[0,s)$; (ii) the map $t \mapsto u(\cdot,t)$ from $[0,s)$ into $X_2^{(1)}$ is continuous on $[0,s)$; (iii) the partial derivatives u_{xx} and u_t exist and are continuous on $R\times(0,s)$; (iv) u satisfies the relations

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (x\varepsilon R,\ t\varepsilon(0,s)) \tag{3.1a}$$

$$u(s,0) = \phi(x) \qquad (x\varepsilon R) \tag{3.1b}$$

When speaking of a solution for (3.1), we shall mean a function u having the properties just specified.

For a given $\phi\varepsilon X_2^{(1)}$, suppose that u_1 and u_2 are solutions of (3.1) with domains of definition $R\times[0,s_1)$ and $R\times[0,s_2)$ respectively, $0 < s_1,s_2 \leq +\infty$. We say that u_2 is a continuation of u_1 if and only if $s_2 > s_1$ and $u_2(x,t) = u_1(x,t)$ for all $x\varepsilon R$, $t\varepsilon[0,s_1)$.

We say that a solution u of (3.1) is <u>noncontinuable</u> if and only if u has no continuation.

Using classical arguments of the type appearing in [12, pp. 139-145], one can prove the following assertion. Let u be a function mapping a domain $R\times[0,s)$, $0 < s \leq +\infty$, into R and suppose that u has the properties (i) and (ii) stated in connection with (3.1); then u is a solution of (3.1) if and only if

$$u(\cdot,t) = T(t)\phi + \int_0^t T(t-\tau)f(u(\cdot,\tau))d\tau \qquad (0 \leq t < s) . \tag{3.2}$$

Here, $\{T(t)\}$ is as in Section 2.

On the basis of Eq. (3.2) and Theorems 2.1 and 2.3, one can prove that, for any $\phi\varepsilon X_2^{(1)}$, Eqs. (3.1) have a unique noncontinuable solution $u(\phi)$. The reasoning here parallels arguments well known in the theory of ordinary differential equations.

The solution $u(\phi)$ is defined on a domain of the form $R\times[0,s(\phi))$, where $0 < s(\phi) \leq +\infty$. For any $x\varepsilon R$ and $t\varepsilon[0,s(\phi))$, we denote the value of $u(\phi)$ at (x,t) by $u(x,t;\phi)$. We define a semigroup $\{U(t)\}$ on $X_2^{(1)}$ by setting $U(t)\phi \overset{\text{def}}{=} u(\cdot,t;\phi)$ for all $\phi\varepsilon X_2^{(1)}$ and $t\varepsilon[0,s(\phi))$. This semigroup is strongly continuous on $X_2^{(1)}$ and in general is nonlinear.

For any $\phi\varepsilon X_2^{(1)}$ we let $\gamma(\phi)$ denote the <u>orbit</u> corresponding to ϕ, by which we mean $\gamma(\phi) \overset{\text{def}}{=} \{U(t)\phi : 0 \leq t < s(\phi)\}$.

We now point out some smoothing properties of $\{U(t)\}$.

Theorem 3.1. For any $\phi\varepsilon X_2^{(1)}$ and any $t\varepsilon(0,s(\phi))$, we have $U(t)\phi\varepsilon X_2^{(4)}$. Also, for any integer $j\varepsilon\{2,3,4\}$, the restriction of $\{U(t)\}$ to $X_2^{(j)}$ is a strongly continuous semigroup on $X_2^{(j)}$.

The proof of Theorem 3.1 is an exercise involving Hypothesis (H1) and Theorems 2.1 and 2.3. We omit all the details. Taking into account Eq. (3.1a), we obtain the following corollary of Theorem 3.1.

Corollary 3.1.1. For any $\phi\varepsilon X_2^{(1)}$ the corresponding solution $u(\phi)$ of (3.1) has a partial derivative $u_t(x,t;\phi)$ defined at each $(x,t)\varepsilon R\times(0,s(\varepsilon))$. Moreover, $u_t(\cdot,t;\phi)\varepsilon X_2^{(0)}$ for every $t\varepsilon(0,s(\phi))$ and the map $t\mapsto u_t(\cdot,t;\phi)$ from $(0,s(\phi))$ into $X_2^{(0)}$ is continuous. These same assertions are valid for the partial derivatives $u_x(\phi)$, $u_{xx}(\phi)$, $u_{tx}(\phi)$, $u_{tt}(\phi)$, $u_{xxx}(\phi)$, and $u_{xxt}(\phi)$.

Theorem 3.2. Let $\phi\varepsilon X_2^{(1)}$ and suppose that there exists an $r\varepsilon(0,+\infty)$ such that $\gamma(\phi)\subseteq B_2^{(0)}(r)\cap X_2^{(1)}$. Then, $s(\phi)=+\infty$ and, for any $t_1\varepsilon(0,+\infty)$, there exists an $r_1\varepsilon[r,+\infty)$ such that $U(t)\phi\varepsilon B_2^{(4)}(r_1)$ for all $t\varepsilon[t_1,+\infty)$.

Proof. Using (3.2) one can show that there exists a $\sigma(r)\varepsilon(0,+\infty)$ such that for every $\psi\varepsilon B_2^{(0)}(r)\cap X_2^{(1)}$ we have $s(\psi)>\sigma(r)$. From this it follows that $s(\phi)=+\infty$.

Now given $t_1\varepsilon(0,+\infty)$, let $\phi_n \overset{\text{def}}{=} U(t_1+n)\phi$ for each integer $n\geq 1$. Then, for every $n\geq 1$, we have $s(\phi_n)=+\infty$ and

$$U(t)\phi_n = T(t)\phi_n + \int_0^t T(t-\tau)f([U(\tau)\phi_n](\cdot))d\tau \qquad (0\leq t<+\infty), \tag{3.3}$$

$$U(\tau)\phi_n = U(\tau+t_1+n)\phi \;\varepsilon\; B_2^{(0)}(r)\cap X_2^{(4)} \qquad (0\leq \tau<+\infty). \tag{3.4}$$

Using Eqs. (3.3) and (3.4) and Theorems 2.1 and 2.3, one can prove that there exists a number $r_0 \varepsilon [r,+\infty)$ such that $U(t)\phi_n \varepsilon B_2^{(4)}(r_0)$ for all $n \geq 1$ and $t \varepsilon [1,2]$. From this there follows the existence of the required number r_1, q.e.d.

Corollary 3.2.1. Given ϕ as in Theorem 3.2 and given any $t_1 \varepsilon (0,+\infty)$, there exists a number $\rho_1 \varepsilon (0,+\infty)$ such that $u_t(\cdot,t;\phi)\varepsilon B_2^{(0)}(\rho_1)$ for all $t\varepsilon[t_1,+\infty)$. The same assertion holds for $u_x(\phi)$, $u_{xx}(\phi)$, $u_{tx}(\phi)$, $u_{tt}(\phi)$, $u_{xxx}(\phi)$, and $u_{xxt}(\phi)$.

Corollary 3.2.1 follows from Theorem 3.2 and Eq. (3.1a). The following theorem involves in a restricted way the notion of continuity with respect to initial data.

Theorem 3.3. Let $\phi \varepsilon X_2^{(1)}$ be as in Theorem 3.2. Suppose that there exist an element $\psi \varepsilon X_2^{(2)}$ and a sequence $\{\tau_n\}_{n=1}^{\infty}$ in $(0,+\infty)$ such that $\tau_n \to +\infty$ and $||U(\tau_n)\phi-\psi||_*^{(2)} \to 0$ as $n \to +\infty$. Then $s(\psi) = +\infty$ and, for any $t\varepsilon[0,+\infty)$, we have $||U(\tau_n+t)\phi-U(t)\psi||_*^{(2)} \to 0$ as $n \to +\infty$. Moreover, the convergence here is uniform with respect to t on any compact subinterval of $[0,+\infty)$.

Proof. Choose any number $t_0\varepsilon(0,+\infty)$. For each integer $n \geq 1$ define a function $w_n: R\times[0,t_0] \to R$ by setting $w_n(x,t) \overset{\text{def}}{=} u(x,\tau_n+t;\phi)$ for all $x\varepsilon R$, $t\varepsilon[0,t_0]$.

Consider any subsequence $\{\tilde{w}_j\}_{j=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$. By Theorem 3.2 there exists an $r_0 > 0$ such that $\tilde{w}_j(\cdot,t)\varepsilon B_2^{(4)}(r_0)$ for all $t\varepsilon[0,t_0]$ and all $j \geq 1$. Taking into account Corollary 3.2.1, there exists a $\rho_0 > 0$ such that

$$|\tilde{w}_j(x_1,t_1)-\tilde{w}_j(x_2,t_2)| + |\frac{\partial\tilde{w}_j}{\partial x}(x_1,t_1) - \frac{\partial\tilde{w}_j}{\partial x}(x_2,t_2)| + |\frac{\partial^2\tilde{w}_j}{\partial x^2}(x_1,t_1) - \frac{\partial^2\tilde{w}_j}{\partial x^2}(x_2,t_2)|$$

$$\leq \rho_0|x_1-x_2| + \rho_0|t_1-t_2| \qquad (j\geq 1; x_1,x_2\varepsilon R; t_1,t_2\varepsilon[0,t_0]) . \qquad (3.5)$$

Using the Ascoli selection theorem, one can show that there exist a subsequence $\{\bar{w}_k\}_{k=1}^{\infty}$ of $\{\tilde{w}_j\}_{j=1}^{\infty}$ and a continuous function $w:R\times[0,t_0] \to R$ such that (a) w has continuous partial derivatives w_x and w_{xx} on $R\times[0,t_0]$; (b) the sequences $\{\bar{w}_k\}$, $\{\partial\bar{w}_k/\partial x\}$, and $\{\partial^2\bar{w}_k/\partial x^2\}$ converge to w, w_x and w_{xx} respectively on $R\times[0,t_0]$; (c) the convergence of each of these sequences is uniform with respect to (x,t) on any compact subset of $R\times[0,t_0]$.

It is easy to show that $w(\cdot,0) = \psi$. Also, using Fatou's lemma, one can show that

$$w(\cdot,t)\varepsilon B_2^{(2)}(r_0) \qquad (0 \leq t \leq t_0) . \tag{3.6}$$

Let us regard the map $t \mapsto w(\cdot,t)$ as a mapping from $[0,t_0]$ into $X_\infty^{(1)}$. Then, using (3.5), one can show that this map is continuous.

Now, for each $k \geq 1$, we have

$$\bar{w}_k(\cdot,t) = T(t)\bar{w}_k(\cdot,0) + \int_0^t T(t-\tau)f(\bar{w}_k(\cdot,\tau))d\tau \qquad (0 \leq t \leq t_0) .$$

Taking into account (2.1), one can show that

$$w(\cdot,t) = T(t)\psi + \int_0^t T(t-\tau)f(w(\cdot,\tau))d\tau \qquad (0 \leq t \leq t_0) . \tag{3.7}$$

Thus, we have a continuous function $w:R\times[0,t_0] \to R$ such that (i) $w(\cdot,t)\varepsilon X_\infty^{(1)}$ for each $t\varepsilon[0,t_0]$; (ii) the map $t \mapsto w(\cdot,t)$ from $[0,t_0]$ into $X_\infty^{(1)}$ is continuous; (iii) w satisfies (3.7). Using standard arguments, one can show that w is unique with respect to these three properties on $R\times[0,t_0]$.

But now, we observe that the solution $u(\psi)$ of (3.1) has the same properties (i) - (iii) on its domain $R\times[0,s(\psi))$. Therefore, w and $u(\psi)$ must agree on the intersection of their respective domains.

Suppose that $s(\psi) \leq t_0$. Then, by (3.6) and Theorem 3.2, we have $s(\psi) = +\infty$, which is a contradiction. Therefore, $s(\psi) > t_0$ and $u(x,t;\psi) = w(x,t)$ for all $x \varepsilon R$, $t \varepsilon [0,t_0]$.

Recall that $\{\bar{w}_k\}$ is a subsequence of $\{\tilde{w}_j\}$ which, in turn, is a subsequence arbitrarily selected from $\{w_n\}$. Also, recall that $w_n(\cdot,t) = U(\tau_n+t)\phi$ for all $t\varepsilon[0,t_0]$. Then, one sees that $||U(\tau_n+t)\phi-U(t)\psi||_*^{(2)} \to 0$ as $n \to +\infty$ uniformly with respect to t on $[0,t_0]$. Since t_0 was chosen arbitrarily, we now have the conclusions required by our theorem, q.e.d.

In the setting provided by Eqs. (3.1), we now recall some definitions from the theory of dynamical systems [5], [6].

Let $\phi\varepsilon X_2^{(1)}$ and suppose that $s(\phi) = +\infty$. Then, by the <u>ω-limit set</u> of $u(\phi)$ with respect to $|| \ ||_*^{(2)}$ we mean the set

$$\omega(\phi) \overset{\text{def}}{=} \bigcap_{0<t<+\infty} cl[\{U(\tau)\phi : t \leq \tau < +\infty\}]_*^{(2)}$$

where $cl[\ ]_*^{(2)}$ denotes closure in $X_2^{(1)}$ with respect to $|| \ ||_*^{(2)}$. In general, $\omega(\phi)$ may be empty or nonempty. An element $\psi\varepsilon X_2^{(1)}$ belongs to $\omega(\phi)$ if and only if $\psi\varepsilon X_2^{(2)}$ and there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ in $(0,+\infty)$ such that $\tau_n \to +\infty$ and $||\psi-U(\tau_n)\phi||_*^{(2)} \to 0$ as $n \to +\infty$.

Let S be any nonempty subset of $X_2^{(1)}$. We say that S is <u>invariant</u> with respect to (3.1) if and only if, for any $\phi\varepsilon S$, there exists a function $\bar{u}: R\times R \to R$ such that (i) $\bar{u}(\cdot,0) = \phi$; (ii) $\bar{u}(\cdot,t)\varepsilon S$ for all $t\varepsilon R$; (iii) for each $t_0 \varepsilon R$ we have $s(\bar{u}(\cdot,t_0)) = +\infty$ and $U(t)\bar{u}(\cdot,t_0) = \bar{u}(\cdot,t+t_0)$ for all $t\varepsilon[0,+\infty)$.

Now we come to the main theorem of this section.

<u>*Theorem 3.4*</u>. *Let ϕ and r be as in Theorem 3.2 so that $\gamma(\phi) \subseteq B_2^{(0)}(r)$ and $s(\phi) = +\infty$. Then, with respect to $|| \ ||_*^{(2)}$, the solution $u(\phi)$ of (3.1) has a nonempty compact connected invariant ω-limit set $\omega(\phi) \subseteq \bar{B}_2^{(0)}(r) \cap X_2^{(2)}$. Also, $U(t)\phi \to \omega(\phi)$*

as $t \to +\infty$, the convergence here being with respect to $|| \ ||_*^{(2)}$.

Proof. By Theorem 3.2, there exists an $r_1 \varepsilon [r,+\infty)$ such that $U(t)\phi \varepsilon B_2^{(3)}(r_1)$ for all $t\varepsilon[1,+\infty)$. Hence, by Theorem 1.1, the set $\{U(t)\phi : 1 \leq t < +\infty\}$ is relatively compact in $X_2^{(2)}$ with respect to $|| \ ||_*^{(2)}$.

The proof can now be completed using arguments of the type appearing in [5], [6]. In particular, the invariance of $\omega(\phi)$ is established with the aid of Theorem 3.3.

4. A Stability Analysis for (3.1)

In this section we continue our study of Eqs. (3.1) under Hypotheses (H1) - (H3).

Lemma 4.1.1. Let $\phi \varepsilon X_2^{(1)}$ and let $t_1, t_2 \varepsilon (0, s(\phi))$ with $t_1 < t_2$. Then, for each $t\varepsilon[t_1,t_2]$, we have $u(x,t;\phi) \to 0$ and $u_x(x,t;\phi) \to 0$ as $|x| \to +\infty$, the convergence here being uniform with respect to $t\varepsilon[t_1,t_2]$.

Lemma 4.1.1 is proved using Theorems 2.2 and 2.4. We omit the details.

Let $a_0 \varepsilon (0,+\infty)$ be as in Hypothesis (H3). We introduce a set $Y \subseteq X_2^{(1)}$ by stipulating that an element $\phi \varepsilon X_2^{(1)}$ belongs to Y if and only if $0 \leq \phi(x) \leq a_0$ for all $x \varepsilon R$. Concerning Y we have the following theorem, which is similar to a result given by Yamaguti [14, p. 728, Proposition 2].

Theorem 4.1. For any $\phi \varepsilon Y$ we have $\gamma(\phi) \subseteq Y$.

Proof. First, we shall prove that $u(x,t;\phi) \leq a_0$ for all $x\varepsilon R$, $t\varepsilon[0,s(\phi))$.

Suppose the contrary; there exists a point $(x_1,t_1)\varepsilon R \times [0,s(\phi))$ such that $u(x_1,t_1;\phi) > a_0$. We shall derive a contradiction.

Since f is continuous on R and since $f(a_0) < 0$, there exists a number

$\bar{a}_0 \varepsilon (a_0,+\infty)$ such that $f(z) < 0$ for all $z\varepsilon[a_0,\bar{a}_0]$. Since the map $t \mapsto U(t)\phi$ from $[0,s(\phi))$ into $X_2^{(1)}$ is continuous on $[0,s(\phi))$ and since $\phi(x) \leq a_0$ for all $x\varepsilon R$, we can assume that $a_0 < u(x_1,t_1;\phi) < \bar{a}_0$ and that $u(x,t;\phi) < \bar{a}_0$ for all $x\varepsilon R$, $t\varepsilon[0,t_1]$. Moreover, there exists a number $t_0\varepsilon(0,t_1)$ such that $u(x,t;\phi) < u(x_1,t_1;\phi)$ for all $x\varepsilon R$, $t\varepsilon[0,t_0]$.

Now consider $u(\phi)$ on the domain $R\times[t_0,t_1]$. By Lemma 4.1.1, there exists a number $b\varepsilon(0,+\infty)$ such that $u(x,t;\phi) < a_0$ for all $x\varepsilon R$ with $|x| \geq b$ and all $t\varepsilon[t_0,t_1]$. Hence, there exist $x^*\varepsilon[-b,b]$ and $t^*\varepsilon[t_0,t_1]$ such that $u(x,t;\phi) \leq u(x^*,t^*;\phi)$ for all $x\varepsilon R$, $t\varepsilon[t_0,t_1]$. Now, we must have $a_0 < u(x^*,t^*;\phi) < \bar{a}_0$. Therefore

$$f(u(x^*,t^*;\phi)) < 0 \tag{4.1}$$

and $t^* > t_0$.

Now, (x^*,t^*) is an absolute maximum of $u(\phi)$ on $R\times[t_0,t_1]$ and, as we have just noted, $t^* > t_0$. Therefore,

$$u_t(x^*,t^*;\phi) - u_{xx}(x^*,t^*;\phi) \geq 0 \ .$$

By Eq. (3.1a) we have $f(u(x^*,t^*;\phi)) \geq 0$. This contradicts Eq. (4.1).

Thus, we must conclude that $u(x,t;\phi) \leq a_0$ for all $x\varepsilon R$, $t\varepsilon[0,s(\phi))$.

Similarly, we can show that $u(x,t;\phi) \geq 0$ for all $x\varepsilon R$, $t\varepsilon[0,s(\phi))$. This completes the proof of Theorem 4.1.

Now we define functions $F:R \to R$ and $V:X_2^{(1)} \to R$ by setting

$$F(z) \overset{\text{def}}{=} \int_0^z f(\zeta)d\zeta \qquad (z\varepsilon R) \tag{4.2}$$

$$V(\phi) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \{\frac{1}{2}\,\phi'(x)^2 - F(\phi(x))\}dx \qquad (\phi\varepsilon X_2^{(1)}) \ . \tag{4.3}$$

V is going to play the role of a Liapunov functional for Eqs. (3.1). This brings us to the following theorem.

Theorem 4.2. There exists a constant $k\varepsilon(0,+\infty)$ such that

$$V(\phi) \geq \frac{1}{2}\int_{-\infty}^{+\infty} \{\phi'(x)^2 + k\phi(x)^2\}dx \qquad (\phi\varepsilon Y). \qquad (4.4)$$

Proof. Since f' is continuous on R and since $f'(0) < 0$, there exists a number $z_1\varepsilon(0,a_0)$ such that $f'(z) < 0$ for all $z\varepsilon[0,z_1]$. Let

$$\mu_1 \overset{\text{def}}{=} -\sup\{f'(z) : 0 \leq z \leq z_1\}$$

$$\mu_2 \overset{\text{def}}{=} \inf\{|F(z)| : z_1 \leq z \leq a_0\}.$$

We note that $\mu_1 > 0$. Also, $\mu_2 > 0$ by virtue of (H3). Now choose $k\varepsilon R$ so that $0 < k < \min\{\mu_1, a_0^{-2}\mu_2\}$.

For any $z\varepsilon(0,z_1]$ we have

$$f'(\zeta) < -\mu_1 \qquad (0 \leq \zeta \leq z)$$

$$f(\zeta) < -\mu_1\zeta \qquad (0 \leq \zeta \leq z)$$

$$F(z) < -\frac{1}{2}\mu_1 z^2 < -\frac{1}{2}kz^2 .$$

For any $z\varepsilon[z_1,a_0]$ we have $F(z) \leq -\mu_2 \leq -ka_0^2 < -\frac{1}{2}ka_0^2 \leq \frac{1}{2}kz^2$. Hence, we have

$$F(z) + \frac{1}{2}kz^2 < 0 \qquad (0 \leq z \leq a_0) .$$

From this there follows (4.4), q.e.d.

Theorem 4.3. For any $\phi\varepsilon X_2^{(1)}$ the derivative $\dot{V}(U(t)\phi)$ exists at every $t\varepsilon(0,s(\phi))$ and

$$\dot{V}(U(t)\phi) = -\int_{-\infty}^{+\infty} |u_t(x,t;\phi)|^2\, dx \qquad (0 < t < s(\phi)) . \tag{4.5}$$

Proof. By (4.3),

$$V(U(t)\phi) = \int_{-\infty}^{+\infty} \{\tfrac{1}{2}\, u_x(x,t;\phi)^2 - F(u(x,t;\phi))\}dx \quad (0 \leq t \leq s(\phi)) . \tag{4.6}$$

Hence, $\dot{V}(U(t)\phi)$ exists at every $t\varepsilon(0,s(\phi))$ and

$$\dot{V}(U(t)\phi) = \int_{-\infty}^{+\infty} \{u_x(x,t;\phi)u_{xt}(x,t;\phi) - f(u(x,t;\phi))u_t(x,t;\phi)\}dx \tag{4.7}$$
$$(0 < t < s(\phi)) .$$

The differentiation under the integral sign in (4.6) can be justified using Eqs. (3.1a), (3.2) and Theorems 3.1, 2.2, 2.4.

The first term on the right-hand side of (4.7) can be integrated by parts. This together with Lemma 4.1.1 yields

$$\dot{V}(U(t)\phi) = -\int_{-\infty}^{+\infty} \{u_{xx}(x,t;\phi)u_t(x,t;\phi) + f(u(x,t;\phi))u_t(x,t;\phi)\}dx \tag{4.8}$$
$$(0 < t < s(\phi)) .$$

Eqs. (4.8) and (3.1a) lead us to (4.5), q.e.d.

Theorem 4.4. If $\phi\varepsilon Y$ then $s(\phi) = +\infty$ and there exists a number $r\varepsilon(0,+\infty)$ such that $\gamma(\phi) \subseteq Y \cap B_2^{(0)}(r)$.

Proof. By Theorem 4.1 we have $\gamma(\phi) \subseteq Y$. Hence, by Theorems 4.2 and 4.3, we have

$$V(\phi) \geq V(U(t)\phi) \geq \frac{k}{2}\int_{-\infty}^{+\infty} |u(x,t;\phi)|^2\, dx \qquad (0 \leq t < s(\phi)) . \tag{4.9}$$

Now set $r \overset{\text{def}}{=} a_0 + [2k^{-1}V(\phi)]^{1/2}$ and we see that $\gamma(\phi) \subseteq Y \cap B_2^{(0)}(r)$. By Theorem 3.2 we have $s(\phi) = +\infty$, q.e.d.

Lemma 4.5.1. If $\phi \varepsilon Y$ then

$$\lim_{t\to+\infty} \int_{-\infty}^{+\infty} |u_t(x,t;\phi)|^2 dx = 0 \ . \tag{4.10}$$

Proof. Let $\phi_1 \overset{\text{def}}{=} U(1)\phi$ and note that $s(\phi_1) = +\infty$. Define a function $\nu:[0,+\infty) \to [0,+\infty)$ by setting

$$\nu(t) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} |u_t(x,t;\phi_1)|^2 dx \qquad (0 \leq t < +\infty) \ .$$

By Theorem 4.3 we have $\nu(t) = V(U(t)\phi_1)$ for all $t\varepsilon[0,+\infty)$. This together with (4.9) implies

$$\int_0^{+\infty} \nu(t)dt \leq V(\phi_1) < +\infty. \tag{4.11}$$

Using Eqs. (3.1a), (3.2) and Theorems 2.2, 2.4 and 3.1, one can show that ν has a derivative $\dot{\nu}$ on $[0,+\infty)$ and

$$\dot{\nu}(t) = 2\int_{-\infty}^{+\infty} u_t(x,t;\phi_1)u_{tt}(x,t;\phi_1)dx \qquad (0 \leq t < +\infty) \ . \tag{4.12}$$

On the other hand, by Corollary 3.2.1, there exists an r_1 $(0,+\infty)$ such that $u_t(\cdot,t;\phi_1)$ and $u_{tt}(\cdot,t;\phi_1)$ belong to $B_2^{(0)}(r_1)$ for all t $[0,+\infty)$. From this and (4.12) it follows that

$$\dot{\nu}(t) \leq 2r_1 \qquad (0 \leq t < +\infty) \tag{4.13}$$

From (4.11) and (4.13) it follows that $\nu(t) \to 0$ as $t \to +\infty$. From this there follows (4.10), q.e.d.

Theorem 4.5. For any $\phi \varepsilon Y$ we have $||U(t)\phi||_*^{(2)} \to 0$ as $t \to +\infty$.

Proof. By Theorems 4.4 and 3.4, the solution $u(\phi)$ has, with respect to $|| \ ||_*^{(2)}$, a nonempty ω-limit set $\omega(\phi) \subset X_2^{(2)}$. This ω-limit set is compact connected and invariant and, with respect to $|| \ ||_*^{(2)}$, we have $U(t)\phi \to \omega(\phi)$ as $t \to +\infty$. Clearly, $\omega(\phi) \subseteq Y$.

Now consider any element $\psi \varepsilon \omega(\phi)$. Since $\omega(\phi)$ is invariant, the solution $u(\psi)$ has continuous partial derivatives $u_t(\psi)$ and $u_{xx}(\psi)$ defined on $R \times [0,+\infty)$ and

$$u_t(x,t;\psi) = u_{xx}(x,t;\psi) + f(u(x,t;\psi)) \qquad (x \varepsilon R,\ t \varepsilon [0,+\infty)) . \qquad (4.14)$$

Setting $t = 0$ in (4.14) we obtain

$$u_t(x,0;\psi) = \psi''(x) + f(\psi(x)) \qquad (x \varepsilon R) .$$

Clearly, $u_t(\cdot,0;\psi) \varepsilon X_2^{(0)}$.

Since $\psi \varepsilon \omega(\phi)$ there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ in $(0,+\infty)$ such that $\tau_n \to +\infty$ and $||\psi - U(\tau_n)\phi||_*^{(2)} \to 0$ as $n \to +\infty$. Hence, $||u_t(\cdot,0;\psi) - u_t(\cdot,\tau_n;\phi)||_*^{(0)} \to 0$ as $n \to +\infty$. By Fatou's lemma and by Lemma 4.5.1,

$$0 \leq \int_{-\infty}^{+\infty} |u_t(x,0;\psi)|^2 dx \leq \liminf_{n \to +\infty} \int_{-\infty}^{+\infty} |u_t(x,\tau_n;\phi)|^2 dx = 0 .$$

Therefore, $u_t(\cdot,0;\psi) = 0$ and

$$0 = \psi''(x) + f(\psi(x)) \qquad (x \varepsilon R). \qquad (4.15)$$

From (4.15) and the condition $\psi \varepsilon X_2^{(2)}$ there follows

$$0 = \frac{1}{2} \psi'(x)^2 + F(\psi(x)) . \qquad (x \varepsilon R)$$

Using Hypothesis (H3) and the condition $\psi \varepsilon Y$, one can now show that $\psi = 0$.

Thus, $\omega(\phi) = \{0\}$. Therefore, $||U(t)\phi||_{*}^{(2)} \to 0$ as $t \to +\infty$, q.e.d.

5. Two Problems on the Interval $[0,+\infty]$

In this section we shall briefly discuss two initial-value problems similar to (3.1) but in which the variable x has as its domain the interval $[0,+\infty)$ rather than the real line R. Thus, throughout this section our underlying interval will be $I = [0,+\infty)$.

As in Sections 3 and 4, we shall let f be as in Hypotheses (H1) - (H3) stated in the Introduction.

Our first problem is as follows. Let $\phi \epsilon X_2^{(1)}[0,+\infty)$ have the property $\phi'(0) = 0$. Given any such ϕ, find a real-valued function u defined on a domain $\{(x,t): x\epsilon[0,+\infty), t\epsilon[0,s)\}$, $0 < s \leq +\infty$, such that (i) $u(\cdot,t)\epsilon X_2^{(1)}[0,+\infty)$ for every $t\epsilon[0,s)$; (ii) the mapping $t \mapsto u(\cdot,t)$ from $[0,s)$ into $X_2^{(1)}[0,+\infty)$ is continuous on $[0,s)$; (iii) the partial derivatives u_{xx} and u_t exist and are continuous on $R\times(0,s)$; (iv) u satisfies the relations

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (x\epsilon[0,+\infty),\ t\epsilon(0,s)) \tag{5.1a}$$

$$u_x(0,t) = 0 \qquad (t\epsilon[0,s)) \tag{5.1b}$$

$$u(x,0) = \phi(x) \qquad (x\epsilon[0,+\infty)). \tag{5.1c}$$

Now we state our second problem. Let $\phi \epsilon X_2^{(1)}[0,+\infty)$ have the property $\phi(0) = 0$. Given any such ϕ, find a real-valued function u defined on a domain $\{(x,t): x\epsilon R,\ t\epsilon[0,s)\}$, $0 < s \leq +\infty$, such that u has the properties (i) - (iii) stated in connection with (5.1) and such that

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (x\epsilon[0,+\infty),\ t\epsilon(0,s)) \tag{5.2a}$$

$$u(0,t) = 0 \qquad (t\epsilon[0,s)) \tag{5.2b}$$

$$u(x,0) = \phi(x) \qquad (x\varepsilon[0,+\infty)) . \tag{5.2c}$$

For each of the preceding two problems we can perform an analysis similar to that given for (3.1) in Sections 3 and 4. In the remaining part of this section we shall indicate how we do this for (5.1). The reasoning for (5.2) is similar.

Let $\mathfrak{X}_1$ be the closed linear subspace of $X_2^{(1)}[0,+\infty)$ consisting of all those $\phi\varepsilon X_2^{(1)}[0,+\infty)$ such that $\phi'(0) = 0$. $\mathfrak{X}_1$ plays the role of a phase space for (3.1) just as $X_2^{(1)}(R)$ plays the same role for (3.1).

For each $\phi\varepsilon\,\mathfrak{X}_1$ we let $\tilde{\phi}\varepsilon X_2^{(1)}(R)$ be the unique even extension of ϕ to R; i.e., $\tilde{\phi}(x) \overset{\text{def}}{=} \phi(|x|)$ for all $x\varepsilon R$.

Let u be any real-valued function defined on a domain $[0,+\infty)\times[0,s)$, $0 < s \leq +\infty$, and suppose that u satisfies the conditions (i) and (ii) stated in connection with (5.1). Also, suppose that $u(\cdot,t)\varepsilon\,\mathfrak{X}_1$ for all $t\varepsilon[0,s)$. Then u is a solution of (5.1) if and only if

$$u(\cdot,t) = T(t)\tilde{\phi} + \int_0^t T(t-\tau)\widetilde{[f(u(\cdot,\tau))]}d\tau \qquad (0 \leq t < s) . \tag{5.3}$$

Eq. (5.3) is our analogue to Eq. (3.2).

Using (5.3) one can prove that for any $\phi\varepsilon\,\mathfrak{X}_1$ Eqs. (5.1) have a unique non-continuable solution $u(\phi)$. This solution is defined on a domain $[0,+\infty)\times[0,s(\phi))$, $0 < s(\phi) \leq +\infty$. Thus, in the obvious manner, (5.1) induces a semigroup $\{U(t)\}$ on $\mathfrak{X}_1$. For any $\phi\varepsilon\,\mathfrak{X}_1$, we can speak of the orbit $\gamma(\phi) \overset{\text{def}}{=} \{U(t)\phi:0 \leq t < +\infty\}$.

Now one can state and prove analogues of Theorems 3.1 - 3.5 for Eqs. (5.1). We omit this part of the analysis.

Given a_0 as in (H3), let Y_1 be the set of all $\phi\varepsilon\mathfrak{X}_1$ such that $0 \leq \phi(x) \leq a_0$ for all $x\varepsilon[0,+\infty)$. As a Liapunov functional for (5.1) introduce $V: \mathfrak{X}_1 \to R$ by setting

$$V(\phi) \overset{\text{def}}{=} \int_0^{+\infty} \{\frac{1}{2}\phi'(x)^2 - F(\phi(x))\}dx \qquad (\phi\varepsilon\mathcal{X}_1) .$$

Here F is as in (4.2). With reference to (5.1), one can now repeat the reasoning contained in Section 4.

Our final result for Eqs. (5.1) is as follows.

Theorem 5.1. For any $\phi\varepsilon Y_1$ we have $s(\phi) = +\infty$, $\gamma(\phi) \subseteq Y_1$, and $||U(t)\phi||_*^{(2)} \to 0$ as $t \to +\infty$.

We leave it to the reader to formulate a similar theorem for (5.2).

6. A Problem in the Interval $[0,\pi]$.

We shall now consider the following problem. Let $\phi\varepsilon X_\infty^{(1)}[0,\pi]$ have the property $\phi'(0) = \phi'(\pi) = 0$. Given any such ϕ, find a real-valued function u defined on a domain $\{(x,t):x\varepsilon[0,\pi],t\varepsilon[0,s)\}$, $0 < s \leq +\infty$, such that (i) $u(\cdot,t)\varepsilon X_\infty^{(1)}[0,\pi]$ for all $t\varepsilon[0,s)$; (ii) the map $t \mapsto u(\cdot,t)$ from $[0,s)$ into $X_\infty^{(1)}[0,\pi]$ is continuous on $[0,s)$; (iii) the partial derivatives u_{xx} and u_t exist and are continuous on $[0,\pi]\times(0,s)$; (iv) u satisfies the relations

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)) \qquad (x\varepsilon[0,\pi],t\varepsilon(0,s)) \tag{6.1a}$$

$$u_x(0,t) = u_x(\pi,t) = 0 \qquad (t\varepsilon[0,s)) \tag{6.1b}$$

$$u(x,0) = \phi(x) \qquad (x\varepsilon[0,\pi]) \tag{6.1c}$$

Here we assume that f is as in Hypotheses (H1) - (H3) stated in the Introduction.

Our purpose in this section is to acquire information about the asymptotic behavior of solutions of (6.1) as $t \to +\infty$. We shall obtain a result not quite analogous to Theorem 5.1.

Clearly, we have $I = [0, \pi]$. As our phase space for (6.1) we take the subspace

$\mathcal{X}_2 \subset X_\infty^{(1)}[0,\pi]$ consisting of all $\phi \varepsilon X_\infty^{(1)}[0,\pi]$ such that $\phi'(0) = \phi'(\pi) = 0$. Given a_0 as in (H3), we let Y_2 be the set of all $\phi \varepsilon \mathcal{X}_2$ such that $0 \le \phi(x) \le a_0$ for all $x \varepsilon [0,\pi]$.

Proceeding as in Sections 3 or 5, one can prove that, for each $\phi \varepsilon \mathcal{X}_2$, Eqs. (6.1) have a unique noncontinuable solution $u(\phi)$. This solution is defined on a domain $[0,\pi]\times[0,s(\phi))$, $0 < s(\phi) \le +\infty$. Thus, for (6.1) we have a semigroup $\{U(t)\}$ on $\mathcal{X}_2$ given by $U(t)\phi \overset{\text{def}}{=} u(\cdot,t;\phi)$ for all $\phi \varepsilon \mathcal{X}_2$ and $t \varepsilon [0,s(\phi))$. For any $\phi \varepsilon \mathcal{X}_2$ we have the orbit $\gamma(\phi) \overset{\text{def}}{=} \{U(t)\phi : 0 \le t < s(\phi)\}$.

Our discussion of (6.1) continues in this manner, paralleling the treatment of (3.1) contained in Sections 3 and 4. As our Liapunov functional for (6.1) we take the functional $V: \mathcal{X}_2 \to R$ defined by

$$V(\phi) = \int_0^\pi \{\tfrac{1}{2}\,\phi'(x)^2 - F(\phi(x))\}dx \qquad (\phi \varepsilon \mathcal{X}_2) .$$

At the final stage of our analysis we have the following assertions.

If $\phi \varepsilon Y_2$ then $s(\phi) = +\infty$ and $\gamma(\phi) \subseteq Y_2$. Also, with respect to $\|\ \|_*^{(2)}$, the solution $u(\phi)$ has a nonempty compact connected invariant ω-limit set $\omega(\phi) \subseteq Y_2 \cap X_\infty^{(2)}[0,\pi]$. With respect to $\|\ \|_*^{(2)}$, we have $U(t)\phi \to \omega(\phi)$ as $t \to +\infty$.

Since the interval $[0,\pi]$ is bounded in R, the norms $\|\ \|_\infty^{(2)}$ and $\|\ \|_*^{(2)}$ generate the same topology on $X_\infty^{(2)}[0,\pi]$. Therefore, in the preceding paragraph we may replace $\|\ \|_*^{(2)}$ by $\|\ \|_\infty^{(2)}$.

Now consider $\omega(\phi)$. Arguing as in Section 4, one can show that any $\psi \varepsilon \omega(\phi)$ satisfies the relations

$$\begin{aligned} &0 = \psi''(x) + f(\psi(x)) \qquad (0 \le x \le \pi) \\ &\psi'(0) = \psi'(\pi) = 0 . \end{aligned} \tag{6.2}$$

Eqs. (6.2) are the analogue of (4.15). Thus, to characterize $\omega(\phi)$ we want to find all those $\psi \varepsilon Y_2$ which satisfy (6.2). The crucial observation is this: if $\psi \varepsilon Y_2$ satisfies (6.2), then we do not necessarily have $\psi = 0$. Of course, the converse is valid; if $\psi = 0$, then $\psi \varepsilon Y_2$ and ψ satisfies (6.2). Nevertheless, we are still left with the possibility that $\omega(\phi) \neq \{0\}$.

An interesting problem is: find all those solutions ψ of (6.2) which belong to Y_2. An approach to solving this problem is to use a phase-energy diagram. However, to obtain a definitive answer, one must have more information about f than is contained in (H1) - (H3). We shall not pursue this matter any further here.

Thus, our final results for (6.1) are as follows.

Theorem 6.1. For any $\phi \varepsilon Y_2$ we have $s(\phi) = +\infty$ and $\gamma(\phi) \subseteq Y_2$. Also, with respect to $\|\ \|_\infty^{(2)}$, the solution $u(\phi)$ has a nonempty compact connected invariant ω-limit set $\omega(\phi) \subseteq Y_2 \cap X_\infty^{(2)}[0,\pi]$ and $U(t)\phi \to \omega(\phi)$ as $t \to +\infty$. Finally, each $\psi \varepsilon \omega(\phi)$ is a solution of (6.2).

In closing we remark that any solution of (6.2) is a steady-state solution of (6.1a,b).

References

1. Arima, R., and Y. Hasegawa, On global solutions for mixed problem of a semi-linear differential equation, Proc. Japan Acad. (10) 39 (1963), 721-725.

2. Auchmuty, J.F.G., Lyapunov methods and equations of parabolic type, Proceedings Batelle Summer Institute on Applications of Non-Linear Analysis, 1972, to be published in Springer-Verlag Lecture Notes.

3. Chafee, N., and E.F. Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, Applicable Anal., to appear.

4. Friedman, A., Partial Differential Equations of Parabolic Type, Prentice-Hall, New Jersey, 1964.

5. Hale, J.K., and E.F. Infante, Extended dynamical systems and stability theory, Proc. Nat. Acad. Sci. U.S.A., vol. 58, no. 2 (1967), 405-409.

6. Hale, J.K., Dynamical systems and stability, J. Math. Anal. Appl. 26 (1969), 39-59.

7. Kanel, Ya. I., On the stability of solutions for a Cauchy problem involving an equation encountered in the theory of combustion, Mat. Sbornik 59 (1962), 245-288.

8. Kolmogoroff, A., Petrovsky, I, and N. Piscounoff, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bulletin de l'Université d'Etat à Moscou, Série Internationale, vol. I, 1937.

9. LaSalle, J.P. An invariance principle in the theory of stability, Int. Symp. Diff. Eqs. Dyn. Sys., ed. by J.K. Hale and J.P. LaSalle, Academic Press, New York, 1967, pp. 277-286.

10. Nagumo, J., Arimoto, S., and S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 50 (1962), 2061-2070.

11. Nagumo, J., Yoshizawa, S., and S. Arimoto, Bistable transmission lines, IEEE Transactions on Circuit Theory CT-12 (1965), no. 3, 400-412.

12. Sobolev, S.L., Partial Differential Equations of Mathematical Physics, Pergamon, New York, 1964.

13. Taylor, A., Introduction to Functional Analysis, John Wiley & Sons, Inc., New York, 1958.

14. Yamaguti, M., The asymptotic behaviour of the solution of a semi-linear partial differential equation related to an active pulse transmission line, Proc. Japan Acad. (10) 39 (1963), 726-730.

15. Yoshizawa, S., Population growth process described by a semilinear parabolic equation, Math. Biosci. 7 (1970), 291-303.