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COSMIC RAYS IN A RANDOM MAGNETIC FIELD: BREAKDOWN OF THE QUASILINEAR DERIVATION OF THE KINETIC EQUATION

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Abstract

We consider the problem of deriving a kinetic equation for the cosmic ray distribution function in a random magnetic field. A model is adopted which is mathematically simple but which contains the essential physics. We investigate the perturbation expansion upon which the quasilinear treatment employed by previous authors is based. As pointed out by Klimas and Sandri, the existence of resonant particles causes the breakdown of the adiabatic approximation frequently used in this theory. We find further that resonant particles cause a general secular growth of higher order terms in the expansion which invalidates the entire perturbative approach.

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COSMIC RAYS IN A RANDOM MAGNETIC FIELD: BREAKDOWN OF THE QUASILINEAR DERIVATION OF THE KINETIC EQUATION

I. Introduction.

During the past few years several authors have considered the problem of how best to describe the evolution of a distribution of cosmic ray particles in a random electromagnetic field. Certain of these investigations have employed a quasilinear analysis (Vedenov, Velikhov and Sagdeev 1961, 1962; Drummond and Pines 1962) of the Vlasov equation to derive a kinetic equation of the Fokker-Planck form (Hall and Sturrock 1967; Hasselmann and Wibberenz 1968; Kulsrud and Pearce 1969; Jokipii 1971, 1972). This approach treats the effect of the random field as a perturbation of the orbits of particles moving in an average background field. The distribution function is expressed as a power series in a small parameter which characterizes the strength of the random field. Quasilinear theory assumes that terms in this series of higher than second order are negligible. In order to obtain a Fokker-Planck equation it is further assumed that there are two time scales: the correlation time of the fluctuating field as seen by a particle moving along its unperturbed trajectory, and the much longer relaxation time of the distribution function.

Recently, these treatments have been criticized by Klimas and Sandri (1971) on the grounds that there are cosmic ray particles for which the correlation time can be arbitrarily long. In their analysis, this fact makes the time development of the distribution function non-Markovian, thereby precluding its description by a Fokker-Planck kinetic equation. They conclude, therefore, that the quasilinear kinetic equation must be dealt with in its full integro-differential form.

It is well known in plasma physics that resonance behavior can lead to terms in the kinetic equation that grow secularly with time, thus causing the quasilinear scheme to break down (Davidson 1972). Specifically, the breakdown is due to particles whose motion is not properly described in the approximation scheme; over the correlation time of the fluctuations, orbits of such particles deviate significantly from the trajectories in the average field. It was our suspicion that quasilinear theory would fail for just this reason in the case considered by Klimas and Sandri (1971).

To test this hypothesis we have investigated a model equation designed to include the relevant features of the Klimas and Sandri model, without all of the latter's mathematical complexity. We were specifically interested in comparing the behavior of higher order terms in the perturbation series with that of the quasilinear term. We found that the quasilinear term in our model has the same long time behavior as that of Klimas and Sandri, and that the non-vanishing term of next higher order, while important for even fairly short times, strongly dominates the quasilinear term for long times.

We are thus led to the conclusion that in the case of the cosmic ray problem, the behavior of resonant particles cannot be adequately handled by quasilinear theory, or indeed by any perturbation technique which assumes that the motion of particles deviates only slightly from the trajectories in the average field.

In §II we describe the details of our physical model, emphasizing its essential similarity to that of Klimas and Sandri. The continuity equation in the appropriate phase space is used in §III to derive a kinetic equation for the distribution function correct to fourth order in the strength of the random field, and in §IV we compare the fourth-order term with the second-order (quasilinear) term. An example with parameter values characteristic of the interplanetary region is discussed to show how quickly the higher order term dominates.

II. The Physical Model

We consider a two-dimensional distribution of monoenergetic particles free to move in the y-z plane. A static random magnetic field, δB, is oriented normal to this plane in the x-direction. The field is assumed to be a homogeneous stochastic function of z only, with Gaussian statistics and vanishing mean. Since any x-component of velocity as well as the magnitude of the velocity in the y-z plane are independent constants of the motion, extension of the model to three dimensional motion and a distribution of kinetic energies is trivial. Weak gradients in particle density in the z-direction are allowed, but the density is assumed to be uniform in the y-direction. The single-particle distribution function, f, then can be expressed as a function of time, t; position, z; and the angle, θ, between the velocity vector, $\vec{v}$, and the z-axis. The magnitude of $\vec{v}$ enters the analysis only as a parameter. We emphasize that the model, while obviously idealized, is nonetheless physically realizable.

The essential feature of the Klimas and Sandri model is the existence of particles whose unperturbed orbits either remain within or periodically re-enter for arbitrarily long times a region throughout which values of the field strength are correlated. Such a class of particles, namely those for which $\theta \approx \pm\pi/2$, is present also in our model. We refer to them as <u>resonant</u> because the effect of the perturbing field adds coherently for arbitrarily long times.

Further similarities between the two models will be pointed out during the derivation of the kinetic equation for f in §III.

III. The Kinetic Equation

The continuity equation for f in the (z,θ) phase space is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z}\left(\frac{dz}{dt} f\right) + \frac{\partial}{\partial \theta}\left(\frac{d\theta}{dt} f\right) = 0 \quad . \qquad (1)$$

With dz/dt = v cos θ and

$$\frac{d\theta}{dt} = \frac{q\delta B(z)}{\gamma mc} \equiv \delta\omega(z) , \qquad (2)$$

where q is the particle charge and γ is the usual relativistic time dilation factor, Equation (1) can be written

$$\frac{\partial f}{\partial t} + v\cos\theta \frac{\partial f}{\partial z} + \delta\omega \frac{\partial f}{\partial \theta} = 0 \tag{3}$$

We next write Equation (3) in terms of the dimensionless variables $\tau = vt/z_c$, $\zeta = z/z_c$ and $\delta\phi = z_c\delta\omega/v$, z_c being the correlation length of the fluctuating field:

$$\frac{\partial f}{\partial \tau} + \cos\theta \frac{\partial f}{\partial \zeta} + \delta\phi \frac{\partial f}{\partial \theta} = 0 \tag{4}$$

Consider a statistical ensemble of systems, and denote the ensemble average of a quantity A by $\langle A \rangle$. The fluctuating part of A is $\delta A = A - \langle A \rangle$; clearly $\delta\langle A \rangle = 0$. With this convention Equation (4) can be written

$$\frac{\partial \langle f \rangle}{\partial \tau} + \cos\theta \frac{\partial \langle f \rangle}{\partial \zeta} + \delta\phi \frac{\partial \langle f \rangle}{\partial \theta} + \frac{\partial \delta f}{\partial \tau} + \cos\theta \frac{\partial \delta f}{\partial \zeta} + \delta\phi \frac{\partial \delta f}{\partial \theta} = 0 \tag{5}$$

The ensemble average of Equation (5) is

$$\frac{\partial \langle f \rangle}{\partial \tau} + \cos\theta \frac{\partial \langle f \rangle}{\partial \zeta} = - \left\langle \delta\phi \frac{\partial \delta f}{\partial \theta} \right\rangle \tag{6}$$

Subtracting Equation (6) from Equation (5) gives

$$\frac{\partial \delta f}{\partial \tau} + \cos\theta \frac{\partial \delta f}{\partial \zeta} = -\delta\phi \frac{\partial \langle f \rangle}{\partial \theta} - \delta\phi \frac{\partial \delta f}{\partial \theta} + \left\langle \delta\phi \frac{\partial \delta f}{\partial \theta} \right\rangle \tag{7}$$

At this point a quasilinear analysis proceeds by neglecting the last two terms on the right side of Equation (7). This approximation assumes that δf is proportional to $\delta\phi$ and that $\delta\phi$ is small compared to unity. We note that $\delta\phi = z_c/r_g$, where r_g is the gyro-radius of a particle in the fluctuating field. Thus, the fundamental assumption of the quasilinear analysis, as Klimas and Sandri point out, is that $z_c \ll r_g$. Corrections to the quasilinear treatment are obtained by iterating Equation (7) with respect to the $O(\delta\phi^2)$ terms. To $O(\delta\phi^n)$, δf is thus written as

$$\delta f = \delta f^{(1)} + \delta f^{(2)} + \ldots + \delta f^{(n)} \tag{8}$$

where $f^{(n)}$ is the n-th iterate and is thus of n-th order in $\delta\phi$. When the series, Equation (8), is substituted into Equation (6) the kinetic equation for $\langle f \rangle$, correct through $O(\delta\phi^{n+1})$, is obtained (Kaufman, 1968).

We carry out the iterative solution of Equation (7) via the method of characteristics. The characteristic unperturbed trajectory is defined by

$$\zeta^*(\tau') = \zeta^*(\tau) + \cos\theta\,(\tau'-\tau)\ ,\quad \theta^*(\tau') = \theta\ ,\ \tau' \leq \tau \tag{9}$$

with $\zeta^*(\tau) \equiv \zeta$. The first three iterates of Equation (7), neglecting initial value terms, are

$$\delta f^{(1)}(\zeta,\theta,\tau) = -\int_0^{\tau} d\tau_1\ \delta\phi(1)\ \frac{\partial \langle f \rangle(1)}{\partial\theta} \tag{10}$$

$$\delta f^{(2)}(\zeta,\theta,\tau) = \int_0^{\tau} d\tau_1\,\delta\phi(1)\ \frac{\partial}{\partial\theta} \int_0^{\tau_1} d\tau_2\,\delta\phi(2)\ \frac{\partial \langle f \rangle(2)}{\partial\theta} - \int_0^{\tau} d\tau_1 \langle \delta\phi(1)\ \frac{\partial}{\partial\theta} \int_0^{\tau_1} d\tau_2\,\delta\phi(2) \rangle\ \frac{\partial \langle f \rangle(2)}{\partial\theta} \tag{11}$$

$$\begin{aligned}\delta f^{(3)}(\zeta,\theta,\tau) = &-\int_0^{\tau} d\tau_1\,\delta\phi(1)\frac{\partial}{\partial\theta} \int_0^{\tau_1} d\tau_2\,\delta\phi(2)\frac{\partial}{\partial\theta} \int_0^{\tau_2} d\tau_3\,\delta\phi(3)\frac{\partial \langle f \rangle(3)}{\partial\theta}\\ &+ \int_0^{\tau} d\tau_1\,\delta\phi(1)\frac{\partial}{\partial\theta} \int_0^{\tau_1} d\tau_2\ \langle \delta\phi(2)\ \frac{\partial}{\partial\theta} \int_0^{\tau_2} d\tau_3\,\delta\phi(3) \rangle \frac{\partial \langle f \rangle(3)}{\partial\theta}\\ &+ \int_0^{\tau} d\tau_1 \langle \delta\phi(1)\frac{\partial}{\partial\theta} \int_0^{\tau_1} d\tau_2\,\delta\phi(2)\frac{\partial}{\partial\theta} \int_0^{\tau_2} d\tau_3\,\delta\phi(3) \rangle\ \frac{\partial \langle f \rangle(3)}{\partial\theta}\end{aligned} \tag{12}$$

In obtaining Equation (12), the fact that $<\delta\phi> = 0$ was used. On the right side of Equations (10) - (12) an argument, n, means that the function is to be evaluated at the phase space point $[\zeta^*(\tau_n), \theta^*(\tau_n)]$, at time τ_n, where $[\zeta^*, \theta^*]$ is obtained from Equation (9) with $\tau' = \tau_n$ and $\tau = \tau_{n-1}$. Also, $\tau_o \equiv \tau$.

Substitution of Equations (8) - (12) in Equation (6) gives the kinetic equation for $<f>$ correct to fourth order in $\delta\phi$. This equation can be considerably simplified by exploiting the statistical properties we have assumed for $\delta\phi$, in particular

a) $$<\delta\phi> = 0 ,$$

b) $$<\delta\phi(\zeta)\,\delta\phi(\zeta')\,\delta\phi(\zeta'')> = 0 ,$$

c) $$<\delta\phi(\zeta)\delta\phi(\zeta')\delta\phi(\zeta'')\delta\phi(\zeta''')> = <\delta\phi(\zeta)\delta\phi(\zeta')><\delta\phi(\zeta'')\delta\phi(\delta''')>$$
$$+ <\delta\phi(\zeta)\delta\phi(\zeta'')>\ <\delta\phi(\zeta')\delta\phi(\zeta''')> + <\delta\phi(\zeta)\delta\phi(\zeta''')>\ <\delta\phi(\zeta')\delta\phi(\zeta'')>.$$

Properties (b) and (c) are true for any homogeneous Gaussian process for which (a) is true. Properties (a) and (b) imply that $\delta f^{(2)}$ and the third term of $\delta f^{(3)}$ [Equation (12)] do not contribute to the right side of Equation (6). Use of (c) allows Equation (6) finally to be put in the form

$$\frac{\partial <f>}{\partial \tau} + \cos\theta\,\frac{\partial <f>}{\partial \zeta} = \frac{\partial}{\partial\theta}\int_0^{\tau} d\tau_1 <\delta\phi\ \delta\phi(1)>\ \frac{\partial <f>(1)}{\partial\theta}$$
$$+ \frac{\partial}{\partial\theta}\int_0^{\tau} d\tau_1 <\delta\phi(1)\frac{\partial}{\partial\theta}\int_0^{\tau_1} d\tau_2\,\delta\phi(2)>\ \frac{\partial}{\partial\theta}\int_0^{\tau_2} d\tau_3 <\delta\phi\delta\phi(3)>\frac{\partial <f>(3)}{\partial\theta}$$
$$+ \frac{\partial}{\partial\theta}\int_0^{\tau} d\tau_1 <\delta\phi(1)\frac{\partial}{\partial\theta}\int_0^{\tau_1} d\tau_2\ <\delta\phi\delta\phi(2)>\ \frac{\partial}{\partial\theta}\int_0^{\tau_2} d\tau_3\,\delta\phi(3)>\frac{\partial <f>(3)}{\partial\theta} \qquad (13)$$

Our contention is that in the cosmic ray problem, effects of $\delta f^{(3)}$ [the second and third terms on the right side of Equation (13)] dominate those of the quasilinear term, $\delta f^{(1)}$, and the expansion procedure using the characteristic trajectories, Equation (9), becomes invalid over times of interest.

To facilitate evaluation of the integrals appearing in Equation (13) we assume that the two-point correlation function has the simple form

$$\langle \delta\phi(\zeta)\,\delta\phi(\zeta')\rangle = \langle \delta\phi^2\rangle\; S\,(|\zeta-\zeta'|-1) \tag{14}$$

where S is the unit step function

$$S(a-b) = \begin{cases} 1 \;, & a \leq b \\ 0 \;, & a > b \end{cases} \tag{15}$$

(Recall that all lengths have been scaled by the correlation length, z_C). Further following Klimas and Sandri (1971), we expand $\langle f\rangle$ in a series of Legendre polynomials

$$\langle f\rangle = \langle f\rangle_0 + \langle f\rangle_1 \cos\theta \tag{16}$$

$\langle f\rangle_0$ is the density averaged over θ, $\langle f\rangle_1$ measures the anisotropy and therefore is related to the particle flux, and terms of higher order than P_1 have been neglected. Finally, Equations (14) - (16) are substituted in Equation (13) and moments of the resulting equation taken with respect to $P_0 = 1$ and $P_1 = \cos\theta$ to obtain coupled equations for $\langle f\rangle_0$ and $\langle f\rangle_1$.

In evaluating the right side of Equation (13) it is assumed that spatial gradients in $\langle f\rangle$ are negligible.

The result is

$$\frac{\partial \langle f \rangle_0}{\partial \tau} + \frac{1}{2} \frac{\partial \langle f \rangle_1}{\partial \zeta} = 0 \tag{17}$$

$$\frac{1}{2} \frac{\partial \langle f \rangle_1}{\partial \tau} + \frac{1}{2} \frac{\partial \langle f \rangle_0}{\partial \zeta}$$

$$= - \langle \delta\phi^2 \rangle \int_0^\tau d\tau' [I_2(\tau') - \langle \delta\phi^2 \rangle I_4(\tau')] \langle f \rangle_1 (\tau - \tau') \tag{18}$$

where

$$I_2(\tau) = \frac{1}{2} S(\tau-1) + \frac{1}{\pi} \left[\frac{(\tau^2-1)^{1/2}}{\tau^2} + \sin^{-1}\left(\frac{1}{\tau}\right)\right] S(1-\tau) \tag{19}$$

$$I_4(\tau) = k_0(\tau) + k_1(\tau)\, S(1-\tau) + k_2(\tau)\, S(2-\tau) \tag{20}$$

and

$$k_0(\tau) = \frac{1}{2} \tau^2 ,$$

$$k_1(\tau) = \frac{1}{\pi} \{ (\frac{2}{3}\tau^2 - \frac{51}{4}) (\tau^2-1)^{\frac{1}{2}} - (\frac{9}{2}\tau^2 - 6) \cos^{-1}(\frac{1}{\tau})$$

$$+ 10\tau \ln [\tau + (\tau^2-1)^{\frac{1}{2}}] \} ,$$

$$k_2(\tau) = \frac{1}{\pi} \{ -(\frac{7}{6}\tau^2 - \frac{50}{3}) (\frac{\tau^2}{4}-1)^{\frac{1}{2}} + (\frac{7}{2}\tau^2 - 6) \cos^{-1}(\frac{2}{\tau})$$

$$- 10\tau \ln [\frac{\tau}{2} + (\frac{\tau^2}{4} - 1)^{\frac{1}{2}}] \} , \tag{21}$$

The degree to which our simple model is successful in simulating the more complicated Klimas and Sandri model is apparent from Equations (18) and (19). As mentioned earlier, our expansion parameter $\langle \delta\phi^2 \rangle = (z_c/r_g)^2$ is identical to theirs. Furthermore, the second-order contribution to our kernel has the long-time behavior $I_2(\tau) \sim \tau^{-1}$, which is exactly the long-time form of the quasilinear kernel obtained by Klimas and Sandri. Just as in their model, the long range of I_2 is caused by resonant particles, for which the approximation of unperturbed orbits breaks down.

IV. Failure of Quasilinear Theory

For large values of τ the second and fourth-order contributions to the integrand on the right side of Equation (18) have, respectively, the forms

$$I_2(\tau) \rightarrow I_2^\infty(\tau) = \frac{2}{\pi}\,\frac{1}{\tau}, \qquad \tau \gg 1,$$

$$\langle\delta\phi^2\rangle\, I_4(\tau) \rightarrow \langle\delta\phi^2\rangle\, I_4^\infty(\tau) = \frac{\langle\delta\phi^2\rangle}{12\pi}\,\tau^3, \qquad \tau \gg 1.$$

I_4 and I_4^∞ are plotted as functions of the dimensionless time variable, τ, in Figure 1. From the graph it is clear that $I_4^\infty(\tau)$ provides a lower bound for $I_4(\tau)$ for all values of τ. It is simply verified that $I_2^\infty(\tau)$ **similarly provides** an <u>upper</u> bound for $I_2(\tau)$. Therefore, the ratio of the fourth and second-order contributions is

$$\frac{I_4(\tau)}{I_2(\tau)}\langle\delta\phi^2\rangle > \frac{I_4^\infty(\tau)}{I_2^\infty(\tau)}\langle\delta\phi^2\rangle = \frac{\tau^4}{24}\langle\delta\phi^2\rangle$$

This ratio becomes appreciable — and the quasilinear approximation consequently begins to break down — on the time scale $\tau_4 \equiv (24/\langle\delta\phi^2\rangle)^{1/4}$, which is comparable to or shorter than what would be the natural time scale of Equation (18) if I_4 were neglected, $\tau \simeq \langle\delta\phi^2\rangle^{-1}$. There is every reason to believe that higher order terms in the perturbation series will become important for times $\lesssim \tau_4$, so that quasilinear theory applies, if at all, only for rather short times.

τ_4 depends, of course, on the characteristics of the random field and the energy of the particles. For protons in the interplanetary magnetic field, with $z_c \simeq 2 \times 10^{11}$ cm and $\langle \delta B^2 \rangle^{1/2} \cong 2.5 \times 10^{-5}$ gauss (Jokipii 1971),

$$\langle \delta\phi^2 \rangle \simeq \frac{2.5}{(T+1)^2-1} , \tag{22}$$

where T is the particle kinetic energy in GeV. Since $\langle \delta\phi^2 \rangle$ must be small compared to unity for the perturbation series to be sensible, Equation (22) implies that the perturbation solution can be valid only in the energy range $T \gtrsim 1$ GeV. For T = 10 GeV, one finds that $\tau_4 \simeq 6$, corresponding to a time $t_4 = \tau_4 \, z_c/v \simeq 40$ sec. This is an order of magnitude shorter than the time scale $\langle \delta\phi^2 \rangle^{-1} z_c/v \simeq 350$ sec, illustrating our contention that the validity of the quasilinear approximation is restricted to time scales much shorter than those of interest.

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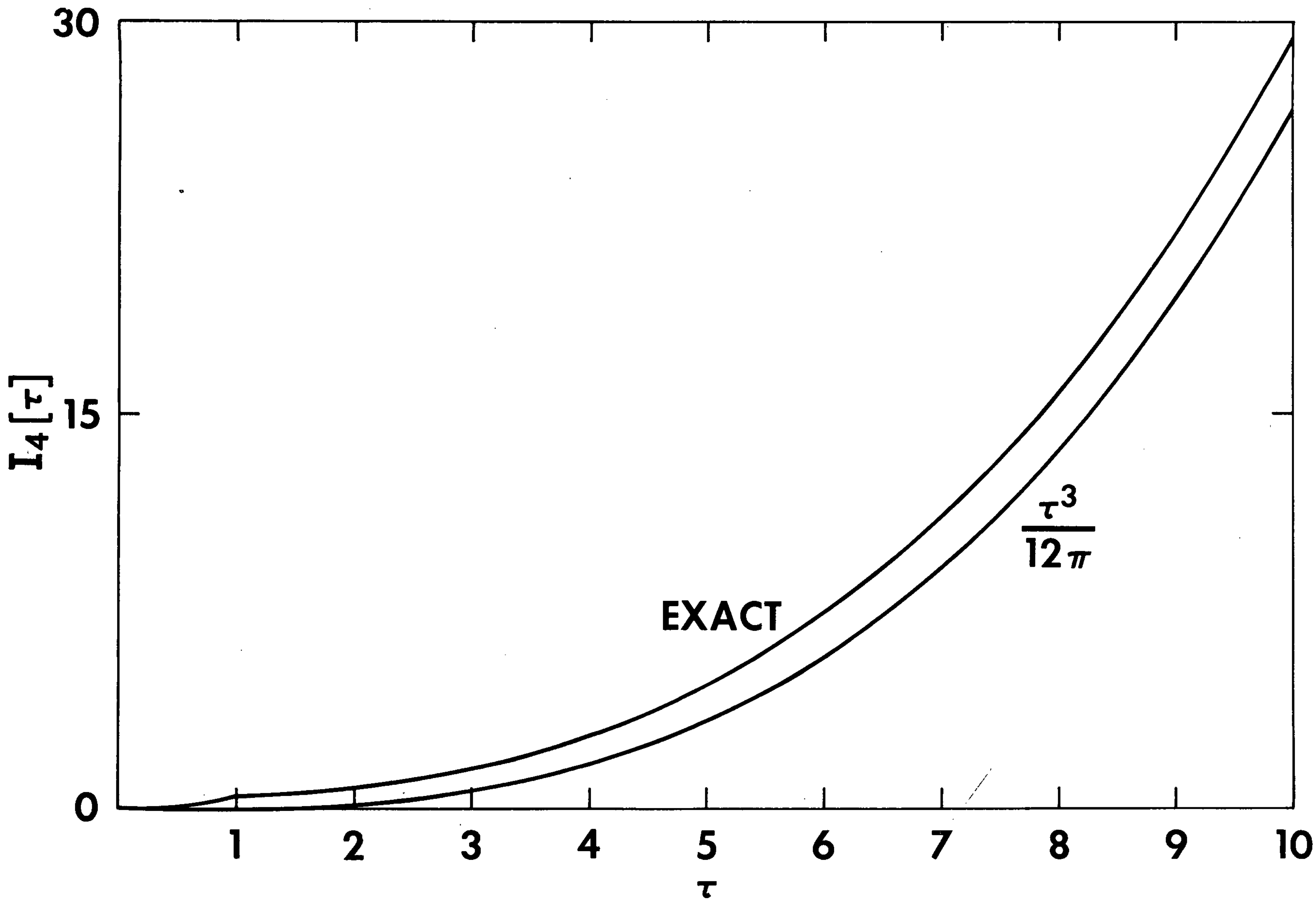
Figure Caption

Fig. 1. Time dependence of fourth-order contribution to the right side of equation (18), both the exact expression, I_4, and the approximation for long times, $I_4^{\infty} = \tau^3/12\pi$.



30
15
0
$I_4[\tau]$
1
2
3
4
5
6
7
8
9
10
τ
EXACT
$\frac{\tau^3}{12\pi}$