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A THEORETICAL ANALYSIS OF THE FREE VIBRATIONS OF RING- AND/OR STRINGER-STIFFENED ELLIPTICAL CYLINDERS WITH ARBITRARY END CONDITIONS

Volume I - Analytical Derivation and Applications

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16. Abstract

This paper presents the derivation and application of a Rayleigh-Ritz modal vibration analysis for ring and/or stringer stiffened noncircular cylindrical shells with arbitrary end conditions. Comparisons with previous results from experimental and analytical studies showed this method of analysis to be accurate for a variety of end conditions. Results indicate a greater effect of rings on natural frequencies than of stringers.

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SUMMARY

An analysis was made in this study to determine the natural frequencies and mode shapes of ring- and/or stringer-stiffened noncircular cylinders with arbitrary end conditions. Cases of circular, noncircular, unstiffened, and stiffened cylindrical shells with various end conditions were investigated and the following observations were made.

1) Comparisons with previous results from experimental and analytical studies of circular, noncircular, unstiffened, and stiffened cylindrical shells with arbitrary end conditions showed this method of analysis to be accurate.

2) The natural frequencies obtained in this study for a clamped-free circular cylinder were slightly higher (for the whole range of m and n) than those previously obtained experimentally. This discrepancy increases as the number of circumferential waves decreases.

3) Comparisons with analytical results obtained previously for stringer-stiffened, freely supported, circular shells showed that the

frequencies previously obtained (neglecting insurface inertias and employing Donnell's shell theory) were slightly higher than those of the present analysis. The discrepancies between the results decreased as the number of circumferential waves increased, which is a typical characteristic of Donnell's theory.

4) Comparisons with Forsberg's exact results for ring-stiffened circular shells, showed that the results of the present analysis were in error only by a maximum of 0.51% for zero-eccentricity rings and 1.75% for negative-eccentricity rings.

5) Comparisons with Al-Najafi and Warburton's finite element and experimental results (obtained for ring-stiffened circular shells) showed that the results for freely supported cylinders obtained from the present analysis were closer to their experimental results than their analytical results using the finite-element method. For the free-free case, of the six experimental results presented, the results of the present analysis were closer to the first three experimentally obtained frequencies, whereas their finite element results were closer to the next three frequencies.

6) The number of terms required in the displacement series for convergence of results for ring-stiffened shells differed from problem to problem. Shells with positive eccentricities needed more terms for convergence than those with zero or negative eccentricities.

INTRODUCTION

Discussion

The free vibrations of ring- and/or stringer-stiffened circular and noncircular cylindrical shells are of interest to designers of flight and marine structures. Frequently, fuselages of flight structures and hulls of submarines have noncircular cross-section due either to special internal storage requirements or to imperfections occurring during manufacture. The method of analysis developed in this report is capable of evaluating the free-vibrational characteristics of ring- and stringer-stiffened "singly" symmetric noncircular cylinders with arbitrary end conditions.

Background

Solutions for the vibrational characteristics of the special cases of unstiffened, circular, isotropic cylinders with specialized boundary conditions have been available for many years. Recent investigations have taken advantage of computers to analyze more complicated models of shell structures. One of the most general cases that can be analyzed is a stiffened, noncircular, anisotropic cylinder with arbitrary end conditions.

Great attention has been paid to the application of the finite element and finite difference methods of analysis because of their generality and adaptability to the computer. However, computer storage and the speed of execution are two important factors which have still prevented economically feasible studies of shell structures. The closely related and well-known Rayleigh-Ritz method was successfully employed in the present study to obtain the vibrational characteristics of stiffened, noncircular cylinders with arbitrary end conditions. This method may provide significant economical advantages over the finite element and finite difference methods. The limitation of the Rayleigh-Ritz method is that the accuracy of the results is dependent upon the assumed mode shapes. In cases such as stiffened, noncircular cylinders with arbitrary end conditions (for which the displacement functions can be approximated fairly accurately by a double finite series) the Rayleigh-Ritz method is certainly advantageous to use.

Studies of noncircular cylinders are relatively few compared to those of circular cylinders. The variable radius of curvature of the cylinder causes difficulties in obtaining analytical solutions. If finite trigonometric series are used to represent the components of the assumed displacement functions, there will be coupling of the circumferential terms due to noncircularity of the cross-section of the shell. Furthermore, the resulting set of simultaneous equations is sufficiently large that a digital computer must be used for the solution of the general problem.

Kempner (1) presented energy expressions and differential equations for cylindrical shells with arbitrary cross-sections. Kempner and his associates have used these equations to study a wide range of problems dealing with statics, buckling and postbuckling (2-7) of a special class of oval cylinders. Klosner and Pohle (8, 9, 10) studied the free and forced vibrations of the same class of oval cylinders, but with infinite length. An approximate method was used in which the frequencies of noncircular cylinders were determined by perturbation of the equivalent circular cylinder frequencies. Culberson and Boyd (11) obtained exact free vibrational characteristics of the same class of oval cylinders studied by Klosner and Pohle and showed that the approximate perturbation technique is accurate for small eccentricities.

The displacement functions used by Boyd (12) in a static analysis of noncircular panels subjected to uniform normal pressures were used in a free vibrational analysis of noncircular cylindrical panels by Kurt and Boyd (13).

Herrmann and Mirsky (14) investigated the longitudinal, torsional, and flexural vibrations of elliptical cylinders. Malkina (15) also studied the free vibrations of oval cylinders.

Sewall et al. (16, 17) carried out both analytical (by Rayleigh-Ritz) and experimental analyses of elliptical unstiffened cylinders with arbitrary end conditions.

Analyses of stiffened shell structures may be classified either as "smeared," or as "discrete" depending upon the treatment of the stiffeners. In the conventional smearing technique (which is reasonably effective if the stiffeners are closely spaced) the effects of the stiffeners are averaged out over the entire surface of the shell, thus effectively replacing a stiffened shell by an equivalent orthotropic shell. A discrete analysis (which is accurate irrespective of the number and location of the stiffeners) treats the stiffeners as discrete elastic structural elements.

The present analysis may be considered as an extension (to include noncircularity) of the work in Refs. (18 and (19) in which the free vibrational characteristics of ring- and stringer-stiffened noncircular cylinders with arbitrary end conditions were developed through the use of a Rayleigh-Ritz technique. The stiffeners may be arbitrarily located and all stiffeners need not possess the same geometric and material properties; however, the stiffeners are assumed to be uniform along their axes. The analysis considers the extension and flexure of the shell and extension, torsion, and flexure about both cross-section axes of the stiffeners. The stringers may have nonsymmetric cross-sections but the rings are assumed to have "singly" symmetric cross-sections. The rotary inertia of the shell is neglected.

The derivation of the energy expressions for noncircular cylinders is described in the Method of Analysis section of this report. The stiffener energies are presented in Appendix B. The compatibility relations used in these equations are derived in Appendix A. The elements of the mass and stiffness matrices are given in Appendix C. Documentation of the computer program developed for this analysis is given in Reference (20).

SYMBOLS

a	length of the shell (except as noted in Figure 1)
$A_{s\ell}, A_{rk}$	cross-sectional area of the ℓ^{th} stringer, k^{th} ring
D	isotropic plate flexural stiffness
$e_x, e_\theta, e_{x\theta}$	strains of shell (see eq. (1))
$(e_x)_s$, $(e_\theta)_r$	normal strains of stringer and ring, respectively
E_c	Young's modulus of shell
$E_{s\ell}$, E_{rk}	Young's modulus of ℓ^{th} stringer, k^{th} ring
$(GJ)_{s\ell}$, $(GJ)_{rk}$	the torsional stiffness of the ℓ^{th} stringer, k^{th} ring
h	thickness of shell
$IR1_1$ to $IR1_8$ $IR2_1$ to $IR2_{10}$ $IR3_1$ to $IR3_2$ $IR4_1$ to $IR4_5$ $IR5_1$ to $IR5_{18}$ $IR6_1$ to $IR6_{11}$	circumferential integrals of ring equations (see eq. C7)
IX_1 to IX_5	longitudinal integrals (see eq. C3)
$IS1_1$ to $IS1_9$ $IS2_1$ to $IS2_5$	circumferential integrals of shell equations (see eq. C2)
$I_{yys\ell}$, I_{xxrk}	the moment of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area, about y' and x' axes passing through their shear centers
$I_{yzs\ell}$	product of inertia of the ℓ^{th} stringer cross-sectional area about y' and z' axes passing through its shear center

$I_{zzs\ell}$, I_{zzrk}	the moment of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area about z′ axis
$I_{yycs\ell}$,I_{xxcrk}	the moment of inertia of the ℓ^{th} stringer, k^{th} ring cross-sectional area about axes parallel to y and x axes passing through its centroid
$I_{yzcs\ell}$	product of inertia of the ℓ^{th} stringer cross-sectional area about axes parallel to y and z axes passing through its centroid
$I_{zzcs\ell}$	the moment of inertia of the ℓ^{th} stringer cross-sectional area about an axis parallel to z axis passing through its centroid
K	total number of rings
L	total number of stringers
M^*	final value of m in the assumed displacement series
N^*	final value of n in the assumed displacement series
$q_{mn}(t)$	generalized coordinate
R	radius of curvature of the shell
R_{crk}	radius of the centroid of the k^{th} ring
t	time
T	kinetic energy
u,v,w	longitudinal, circumferential, and radial displacements of the middle surface of the shell, respectively (see fig. 1)
u_i,v_i,w_i	displacements of an arbitrary point in the cross-section of the i^{th} stiffener in the x, θ, and z directions
u_{sci},v_{sci},w_{sci}	displacements of the shear center of the i^{th} stiffener in the x, θ, z directions
u_{mn},v_{mn},w_{mn}	generalized coordinates for symmetric mode displacements u, v, and w, respectively
u'_{mn},v'_{mn},w'_{mn}	generalized coordinates for antisymmetric mode displacements u, v, and w, respectively
U	strain energy

$U_m(x)$, $V_m(x)$, $W_m(x)$	axial mode functions representing displacements in the x, θ, and z directions
x, θ, z	longitudinal, circumferential, and radial shell coordinates (see fig. 1)
x', y', z'	longitudinal, circumferential, and radial coordinates of the stiffener, measured from its shear center
$X_m(x)$	Bernoulli-Euler beam eigenfunctions
$y_{1s\ell}$	y-distance of the shear center of the ℓ^{th} stringer from the z axis passing through its point of attachment
$y_{2s\ell}$	y-distance of the centroid of the ℓ^{th} stringer from its shear center
$z_{1s\ell}, z_{1rk}$	z-distance of the shear center of the ℓ^{th} stringer, k^{th} ring from the middle surface of the shell
$z_{2s\ell}, z_{2rk}$	z-distance of the centroid of the ℓ^{th} stringer, k^{th} ring from its shear center
$\Phi_m(x)$	general axial mode function (see eq. 28)
$A_{mn,\bar{m}\bar{n}}, B_{mn,\bar{m}\bar{n}}$, $C_{mn,\bar{m}\bar{n}}, D_{mn,\bar{m}\bar{n}}$, $E_{mn,\bar{m}\bar{n}}, F_{mn,\bar{m}\bar{n}}$	elements of the stiffness matrix (see appendix C)
$N_{mn,\bar{m}\bar{n}}, NN_{mn,\bar{m}\bar{n}}$, $P_{mn,\bar{m}\bar{n}}, Q_{mn,\bar{m}\bar{n}}$, $R_{mn,\bar{m}\bar{n}}, S_{mn,\bar{m}\bar{n}}$	elements of the mass matrix (see appendix C)
ν	Poisson's ratio
ρ_c	mass density of the shell
$\rho_{s\ell}, \rho_{rk}$	mass density of ℓ^{th} stringer, k^{th} ring
ω	circular frequency

Subscripts:

a	antisymmetric
c	refers to cylinder; centroid

k	refers to the k^{th} ring
ℓ	refers to the ℓ^{th} stringer
$m, \bar{m}$	identifies m^{th} and $\bar{m}^{th}$ longitudinal modal components
$n, \bar{n}$	identifies n^{th} and $\bar{n}^{th}$ circumferential modal components
r	refers to rings
s	refers to stringers
sc	refers to shear center

Notes:

(1) A comma before a subscript denotes partial differentiation with respect to that subscript;

e.g., $u,_x$ denotes $\frac{\partial u}{\partial x}$ and $w,_{\theta\theta}$ denotes $\frac{\partial^2 w}{\partial \theta^2}$.

(2) Superscript T denotes transpose of a matrix.

(3) Dots over quantities denote differentiation with respect to time.

METHOD OF ANALYSIS

The analytical method employed in this analysis was the well-known Rayleigh-Ritz (i.e. "assumed modes") energy technique. At the outset the strain and kinetic energies of the shell, ring, and stringer were derived. The compatibility relations were developed to express the displacements of rings and stringers in terms of the displacements of the median surface of the shell. The total strain energy of the shell and that of rings and stringers were combined to obtain the total strain energy of the stiffened cylinder expressed in terms of displacements of the shell median surface. The total kinetic energy of the stiffened cylinder was similarly formulated. Finite series were assumed representing the circumferential, axial, and radial displacements of the median surface of the shell and satisfying the shell kinematic boundary conditions. Simple trigonometric functions were used to describe the circumferential displacement distributions and beam functions were chosen to describe distributions along the axis of the shell. The assumed displacement functions with undetermined coefficients were substituted into the total energy expressions of the structure, and the regular eigenvalue problem was formulated by minimizing the action integral.

Geometry

Strain-displacement relations: The classical theories of thin shells and beams were used to derive the energy expressions for the shell and the stiffeners, respectively. The geometry of the middle surface of a typical elliptical shell is illustrated by Figure 1. The three orthogonal coordinates x, θ, and z locate points within the structure and u, v, and w are the corresponding displacement components. The variable radius of curvature of the shell cross-section is expressed as a function of the θ coordinate. The following Kempner (1) relations were used to determine strains at points within the shell:

$$e_x = u,_x - zw,_{xx}$$

$$e_\theta = \frac{v,_\theta}{R} + \frac{1}{R+z}\left\{z\left[\left(v-w,_\theta\right)\left(\frac{1}{R}\right),_\theta - \frac{w,_{\theta\theta}}{R}\right] + w\right\} \tag{1}$$

$$e_{x\theta} = \frac{u,_\theta}{R+z} + \frac{(R+z)}{R} v,_x - \frac{z(2R+z)}{R(R+z)} w,_{x\theta}$$

where e_x, and e_θ are normal strains of x- and θ-oriented line elements, respectively, and $e_{x\theta}$ is the distortion angle between these two line elements. Furthermore, u, v, w, and R refer to middle surface (z=0) values.

For the stringers and rings the normal strains were expressed as

$$(e_x)_s = u_{s,x} \tag{2}$$

$$(e_\theta)_r = \frac{1}{R_r}\left(v_{r,\theta} + w_r\right) \approx \frac{1}{R_{cr}}\left(v_{r,\theta} + w_r\right) \tag{3}$$

where the subscripts s and r indicate arbitrary points in the stringer and ring, respectively. $(e_x)_s$ is the normal strain of the stringer in the x direction, and $(e_\theta)_r$ is the normal strain of ring in the θ direction. R_{cr} is the radius of the centroid of the ring.

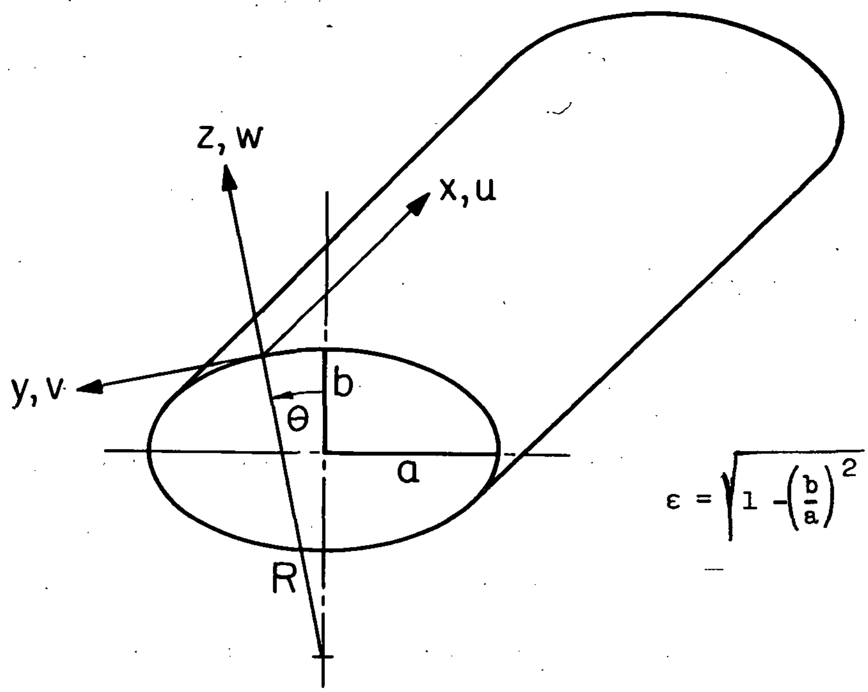


Figure 1. Geometry of an Elliptical Shell.

Compatibility relations: The geometric details of eccentric stiffeners are shown by Figures 2 and 3. The compatibility equations relating the displacements of any point in the stiffener cross-section to those of its shear center are presented in Appendix A. The following equations were derived to determine the displacements in the stiffeners;

For the stringers: $$u_s = u_{scs} - z' w_{scs,x} - y' v_{scs,x} \tag{4}$$

For the rings:
$$v_r = v_{scr} - \frac{x'}{R_{scr}} u_{scr,\theta} - \frac{z'}{R_{scr}} \left(w_{scr,\theta} - v_{scr} \right)$$

$$w_r = w_{scr} + x' w_{scr,x} \tag{5}$$

where the subscript sc identifies the shear center, and the coordinates x', y', and z' are measured from the shear center of the stiffener.

The following compatibility equations relating the displacements of the shear center of the stiffener to those of the shell's median surface were derived and are presented in Appendix A.

For the stringers:

$$u_{scs} = u - z_{1s} w_{,x} - y_{1s} v_{,x}$$

$$v_{scs} = v - z_{1s} \left(\frac{w_{,\theta}}{R} - \frac{v}{R} \right)$$

$$w_{scs} = w + y_{1s} \left(\frac{w_{,\theta}}{R} - \frac{v}{R} \right) \tag{6}$$

For the rings:

$$u_{scr} = u - z_{1r} w_{,x}$$

$$v_{scr} = \left(1 + \frac{z_{1r}}{R} \right) v - \frac{z_{1r}}{R} w_{,\theta}$$

$$w_{scr} = w \tag{7}$$

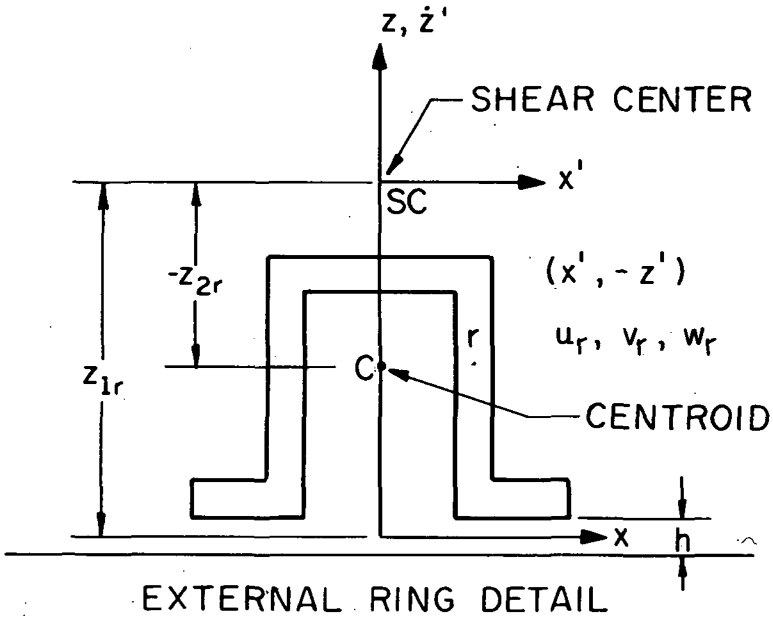
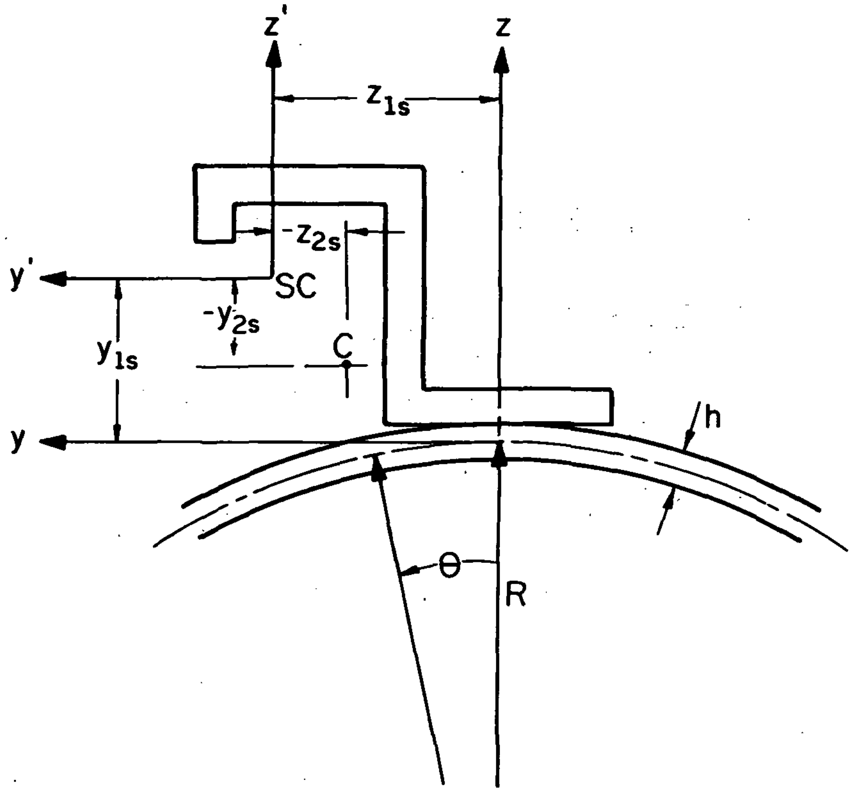


Figure 2. Geometric Details of an Eccentric Ring Stiffener



EXTERNAL STRINGER DETAIL

Figure 3. Geometric Details of an Eccentric Stringer Stiffener

Strain and Kinetic Energies

Shell energies: From Reference (1), the strain energy in an isotropic, elastic body subjected to small strains e_x, e_θ, and $e_{x\theta}$ is

$$U = \int_{vol} \frac{E}{2(1-\nu^2)} \left[e^2_x + e^2_\theta + 2\nu e_x e_\theta + \frac{(1-\nu)}{2} e^2_{x\theta} \right] d(vol) \tag{8}$$

For a shell of uniform thickness h, the above expression can be written as

$$U_c = \frac{E_c}{2(1-\nu^2)} \int_0^a \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[e^2_x + e^2_\theta + 2\nu e_x e_\theta + \frac{(1-\nu)}{2} e^2_{x\theta} \right] (R + z)\, dz\, d\theta\, dx \tag{9}$$

where E_c is Young's modulus and ν is Poisson's ratio of the shell. After substituting the Equation (1) into the Equation (9) and integrating over the thickness of the shell, we obtain the strain energy of the shell in terms of the displacements of its median surface; i.e.

$$\begin{aligned} U_c = \frac{12D}{h^2} \int_0^a \int_0^\pi & \left[Ru^2_{,x} + \frac{(1-\nu)}{2}\left(\frac{1}{R} + \frac{h^2}{12R^3}\right) u^2_{,\theta} + 2\nu u_{,x} v_{,\theta} \right. \\ & + (1-\nu)\, u_{,\theta} v_{,x} + 2\nu u_{,x} w + \frac{1}{R} v^2_{,\theta} + \frac{(1-\nu)}{2}\left(R + \frac{h^2}{4R}\right) v^2_{,x} \\ & \left. + \frac{2}{R} v_{,\theta} w + \left(\frac{1}{R} + \frac{h^2}{12R^3}\right) w^2 \right] d\theta\, dx + D \int_0^a \int_0^\pi \left[-2u_{,x} w_{,xx} \right. \\ & + \frac{(1-\nu)}{R^2} u_{,\theta} w_{,x\theta} - \frac{2\nu}{R} v_{,\theta} w_{,xx} - \frac{3(1-\nu)}{R} v_{,x} w_{,x\theta} + Rw^2_{,xx} + \frac{w^2_{,\theta\theta}}{R^3} \\ & + \frac{1}{R^3}\left(ww_{,\theta\theta} + w_{,\theta\theta} w\right) + \frac{\nu}{R}\left(w_{,xx} w_{,\theta\theta} + w_{,\theta\theta} w_{,xx}\right) \\ & \left. + \frac{2(1-\nu)}{R} w^2_{,x\theta} \right] d\theta\, dx \end{aligned}$$

$$+ D\int_0^a\int_0^\pi \left[\frac{1}{R}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 v^2 - 2\nu\left(\frac{1}{R}\right)_{,\theta} vw_{,xx} - \frac{2}{R}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 vw_{,\theta}\right.$$

$$- \frac{2}{R^2}\left(\frac{1}{R}\right)_{,\theta}\left(vw_{,\theta\theta} + vw\right) + \frac{1}{R}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 w_{,\theta}^2 + \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta}\left(w_{,\theta}w_{,\theta\theta}\right.$$

$$\left. + w_{,\theta\theta}w_{,\theta}\right) + \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta}\left(ww_{,\theta} + w_{,\theta}w\right) + \nu\left(\frac{1}{R}\right)_{,\theta}\left(w_{,xx}w_{,\theta}\right.$$

$$\left.\left. + w_{,\theta}w_{,xx}\right)\right] d\theta\ dx \tag{10}$$

where

$$D = \frac{E_c h^3}{12(1-\nu^2)}$$

The last integral in Equation (10) vanishes for constant R. The first two integrals are equivalent to those developed by Miller (21) and by Egle and Soder (19).

Neglecting the contribution of rotary inertia, the shell kinetic energy may be written as

$$T_c = \rho_c h \int_0^a \int_0^\pi \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right] R\ d\theta\ dx \tag{11}$$

where ρ_c is the mass density of the shell and the dot represents the time derivative.

<u>Ring energies</u>: The ring is assumed to be subjected to normal strains and shearing strains due to twisting. The cross-section of the ring is assumed to be symmetric with respect to the outward normal to the shell surface through the line of attachment. The total strain energy in K rings due to normal strains is

$$U_r = \sum_{k=1}^{K} \frac{E_{rk}}{2}\int_0^{2\pi}\int_{A_{rk}} \left[(e_\theta)_r\right]^2_{x=x_k} dA_{rk}\ R_{cr}\ d\theta \tag{12}$$

Using the strain-displacement relation of the ring (Equation (3)) the above expression may be written as

$$U_r = \sum_{k=1}^{K} \frac{E_{rk}}{2} \int_0^{2\pi} \int_{A_{rk}} \frac{1}{R_{cr}} \left[v_{r,\theta}^2 + w_r^2 + v_{r,\theta} w_r + w_r v_{r,\theta} \right]_{x=x_k} dA_{rk}\, d\theta \qquad (13)$$

Substituting the first set of compatibility relations of the ring (Equations (5)) into Equation (13) and performing the integration over the cross-section of the ring, the strain energy of the ring due to extension (normal strain) may be written in terms of the displacements of its shear center as

$$U_{r_{ext}} = U_{r_{ext}} (u_{scr}, v_{scr}, w_{scr}) \qquad (14)$$

The function $U_{r_{ext}} (u_{scr}, v_{scr}, w_{scr})$ is given in Appendix B. Combining Equations (7) and (14) results in

$$U_{r_{ext}} = U_{r_{ext}} (u, v, w) \qquad (15)$$

The function $U_{r_{ext}} (u,v,w)$ is also given in Appendix B.

The strain energy due to twisting of the rings may be written as (Reference 27)

$$U_{r_{tor}} = \sum_{k=1}^{K} \frac{(GJ)_{rk}}{2} \int_0^{2\pi} \left[\frac{u_{scr,\theta}}{R_{cr}^2} + \frac{w_{,x\theta}}{R_{cr}} \right]_{x=x_k}^2 R_{cr}\, d\theta \qquad (16)$$

where $(GJ)_{rk}$ is the torsional stiffness of the k^{th} ring. Substitution of Equations (7) into Equation (16) results in

$$U_{r_{tor}} = U_{r_{tor}} (u,v,w) \qquad (17)$$

The function $U_{r_{tor}} (u,v,w)$ is given in Appendix B.

The kinetic energy of the ring is

$$T_r = \frac{1}{2}\sum_{k=1}^{K} \rho_{rk}\int_0^{2\pi}\int_{A_{rk}} \left[\mathring{u}_r^2 + \mathring{v}_r^2 + \mathring{w}_r^2\right]_{x=x_k} dA_{rk}\, R_{cr}\, d\theta \tag{18}$$

Substitution of Equations (5) into the above equation and integrating over the cross-section of the rings, and then substituting the Equations (7) into the resulting expression we have,

$$T_r = T_r(\mathring{u}, \mathring{v}, \mathring{w}) \tag{19}$$

The function $T_r(\mathring{u}, \mathring{v}, \mathring{w})$ is given in Appendix B. Note that Equation (19) includes both translation and rotation effects.

<u>Stringer energies</u>: The stringer is assumed to be subjected to both extension and twisting. The cross-section of the stringer may be non-symmetric. The strain energy due to normal strain in the stringer is

$$U_{s_{ext}} = \sum_{\ell=1}^{L} \frac{E_{s\ell}}{2}\int_0^{a}\int_{A_{s\ell}} \left[(e_x)_s\right]^2_{\theta=\theta_\ell} dA_{s\ell}\, dx \tag{20}$$

or, introducing Equation (2),

$$U_{s_{ext}} = \sum_{\ell=1}^{L} \frac{E_{s\ell}}{2}\int_0^{a}\int_{A_{s\ell}} \left[u_{s,x}\right]^2_{\theta=\theta_\ell} dA_{s\ell}\, dx \tag{21}$$

Substitution of Equation (4) into the above equation and integrating over the cross-section of the stringer, and then substituting Equations (6) into the resulting expression we obtain

$$U_{s_{ext}} = U_{s_{ext}}(u,v,w) \tag{22}$$

The function $U_{s_{ext}}(u,v,w)$ is given in Appendix B.

The strain energy due to twisting of the stringer may be written as

$$U_{s_{tor}} = \sum_{\ell=1}^{L} \frac{(GJ)_{s\ell}}{2}\int_0^{a} \left[\frac{w_{,\theta x}}{R} - \frac{v_{,x}}{R}\right]^2_{\theta=\theta_\ell} dx \tag{23}$$

where $(GJ)_{s\ell}$ is the torsional stiffness of the ℓ^{th} stringer. Thus,

$$U_{s_{tor}} = \sum_{\ell=1}^{L} \frac{(GJ)_{s\ell}}{2} \int_0^a \left[\frac{w_{,x\theta}^2}{R^2} + \frac{v_{,x}^2}{R^2} - 2\,\frac{v_{,x} w_{,x\theta}}{R^2} \right]_{\theta=\theta_\ell} dx \tag{24}$$

The kinetic energy of stringer is

$$T_s = \tfrac{1}{2} \sum_{\ell=1}^{L} \rho_{s\ell} \int_0^a \int_{A_{s\ell}} \left[\mathring{u}_s^2 + \mathring{v}_s^2 + \mathring{w}_s^2 \right]_{\theta=\theta_\ell} dA_{s\ell}\, dx \tag{25}$$

combining Equations (4, 6, and 25) and integrating the resulting expression over the cross-section of the stringer results in

$$T_s = T_s(\mathring{u}, \mathring{v}, \mathring{w}) \tag{26}$$

The function $T_s(\mathring{u}, \mathring{v}, \mathring{w})$ is given in Appendix B.

Displacement Functions

The displacements, u, v, and w were assumed to be double finite series. Each term of the series is a product of a circumferential and an axial modal function weighted by a time-dependent generalized coordinate (unknown amplitude coefficient). The assumed displacement functions were:

$$u(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (u_{mn} \cos n\theta + u'_{mn} \sin n\theta)\, U_m(x)\, e^{i\omega t}$$

$$v(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (v_{mn} \sin n\theta - v'_{mn} \cos n\theta)\, V_m(x)\, e^{i\omega t}$$

$$w(x, \theta, t) = \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} (w_{mn} \cos n\theta + w'_{mn} \sin n\theta)\, W_m(x)\, e^{i\omega t} \tag{27}$$

where $U_m(x)$, $V_m(x)$, and $W_m(x)$ are the axial mode functions which satisfy at least the kinematic boundary conditions of the stiffened shell. Also, u_{mn}, v_{mn}, and w_{mn} are unknown amplitude coefficients of the symmetric circumferential modes, and u'_{mn}, v'_{mn}, and w'_{mn} are those associated with the antisymmetric modes.

In this analysis the axial mode functions $U_m(x)$, $V_m(x)$, $W_m(x)$ were expressed by a single function $\Phi_m(x)$ such that

$$U_m(x) = \frac{d}{dx} \Phi_m(x) \tag{28a}$$

$$V_m(x) = \Phi_m(x)$$

$$W_m(x) = \Phi_m(x)$$

The following functions were implemented in this analysis.

Boundary Condition	Function Used	Eqn. No.
Freely supported:	$\Phi_m(x) = \sqrt{2}\ \mathrm{Sin}\ \frac{m\pi x}{a}$	(28b)
Clamped-free:	$\Phi_m(x) = X_m$, Characteristic function of a Clamped-free beam.	(28c)
Clamped-clamped:	$\Phi_m(x) = X_m$, Characteristic function of a Clamped-clamped beam.	(28d)
Free-free:	$\Phi_0(x) = 1$ $\Phi_1(x) = \frac{x}{a} - \frac{1}{2}$ $\Phi_m(x) = X_{m-1}$, Characteristic function of a Free-free beam. $(m \geq 2)$	(28e)

The characteristic functions X_m, their derivatives and eigenvalue properties are tabulated in Reference (22).

The Frequency Equation

The total strain energy of the stiffened shell was obtained by combining Equations (10, 15, 17, 22, and 24). Similarly, the total kinetic energy was obtained by combining Equations (11, 19, and 26). Substituting Equations (27 and 28) into the total energies of the stiffened shell, the strain energy expression becomes a positive definite quadratic function of the generalized coordinates u_{mn}, v_{mn}, w_{mn}, u'_{mn}, v'_{mn}, and w'_{mn}. Furthermore the kinetic energy expression becomes a positive definite quadratic function of the generalized velocities $\mathring{u}_{mn}$, $\mathring{v}_{mn}$, $\mathring{w}_{mn}$, $\mathring{u}'_{mn}$, $\mathring{v}'_{mn}$, $\mathring{w}'_{mn}$.

The total strain energy of the structure may be written as

$$U_{total} = \frac{1}{2} \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} \sum_{\bar{m}=0}^{M^*} \sum_{\bar{n}=0}^{N^*} K_{mn,\bar{m}\bar{n}}\, q_{mn} q_{\bar{m}\bar{n}} \tag{29}$$

where

$$\frac{\partial^2 U_{total}}{\partial q_{mn}\, \partial q_{\bar{m}\bar{n}}} = \frac{\partial^2 U_{total}}{\partial q_{\bar{m}\bar{n}}\, \partial q_{mn}} = K_{mn,\bar{m}\bar{n}} = K_{\bar{m}\bar{n},mn}$$

are known as elements of the stiffness matrix.

The total kinetic energy of the structure may be written as

$$T_{total} = \frac{1}{2} \sum_{m=0}^{M^*} \sum_{n=0}^{N^*} \sum_{\bar{m}=0}^{M^*} \sum_{\bar{n}=0}^{N^*} M_{mn,\bar{m}\bar{n}}\, \mathring{q}_{mn} \mathring{q}_{\bar{m}\bar{n}} \tag{30}$$

where $M_{mn,\bar{m}\bar{n}}$ are the elements of the mass matrix.

The mass and stiffness matrices obtained by the above operations were used together with Hamilton's principle to formulate the regular eigenvalue problem resulting in

$$\left[\begin{bmatrix} K_{ss} & K_{sa} \\ \hline K_{sa}^T & K_{aa} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{ss} & M_{sa} \\ \hline M_{sa}^T & M_{aa} \end{bmatrix}\right] \left\{\begin{matrix} q_s \\ \hline q_a \end{matrix}\right\} = 0 \tag{31}$$

where K, and M represent stiffness and mass matrices of size $3(M^*+1)(N^*+1)$, q_s and q_a denote the symmetric and antisymmetric mode vectors, respectively, and superscript T denotes the transpose of a matrix.

In Equation (31) the off-diagonal submatrices of both the stiffness and mass matrices vanish if the cross-section of the stiffened shell is symmetric with respect to the vertical axis (where $\theta = 0$). Thus, the above equation is uncoupled into two equations; one for symmetric, and the other for antisymmetric modes. The equation for the symmetric mode problem may be written as,

$$\left[\begin{bmatrix} A & D & E \\ D^T & B & F \\ E^T & F^T & C \end{bmatrix} - \omega^2 \begin{bmatrix} N & NN & P \\ NN^T & Q & R \\ P^T & R^T & S \end{bmatrix}\right] \left\{\begin{matrix} \bar{\bar{u}} \\ \bar{\bar{v}} \\ \bar{\bar{w}} \end{matrix}\right\} = 0 \tag{32}$$

Each letter in the stiffness and mass matrices represents a submatrix (presented in Appendix C) of order $(M^*+1)(N^*+1)$.

COMPUTER SOLUTION

General

A computer program (20) was developed to find the eigenvalues and eigenvectors of Equation (32). The mass and stiffness matrices were generated in this program and the frequencies and mode shapes were computed using the subroutine EIGENP (23). The Oklahoma State University IBM Model 360/65 computer was employed for this project.

The input data to the program may be categorized into four kinds. The first kind is general data. For example, the title of the problem, number of terms considered in the assumed displacement series, whether or not the cross-section of the shell is circular, the number of stiffeners, etc. The other three kinds of data are shell data, stringer data, and ring data.

The radius of curvature (R) of the shell was considered to be a function of the θ-coordinate. The expressions for R, $(\frac{1}{R})_{,\theta}$, and $(R)_{,\theta}$, were calculated (considering elliptical cross-section) in the function subprograms (RSHL), (RRRT), and (RSHLT), respectively. This procedure was used to make the computer program capable of analyzing arbitrary singly symmetric stiffened oval cylinders. However, only elliptical cylinders were considered in the present study.

Natural Frequencies and Mode Shapes

If the number of circumferential and axial terms considered in the assumed displacement series are M^* and N^*, respectively, (including $m = 0$, and $n = 0$, when needed) then the order of the stiffness and mass matrices is $3M^*N^*$. Equation (30) may be written as

$$\left[\begin{bmatrix} K \end{bmatrix} - \omega^2 \begin{bmatrix} M \end{bmatrix} \right] \begin{Bmatrix} \bar{\bar{u}} \\ \bar{\bar{v}} \\ \bar{\bar{w}} \end{Bmatrix} = 0 \qquad (33)$$

where

$$\left\{\bar{\bar{u}}\right\} = \begin{Bmatrix} u_{0\,0} \\ u_{0\,1} \\ u_{0\,2} \\ \vdots \\ u_{0\,N^*} \\ u_{1\,0} \\ u_{1\,1} \\ u_{1\,2} \\ \vdots \\ u_{M^*N^*} \end{Bmatrix} ; \quad \left\{\bar{\bar{v}}\right\} = \begin{Bmatrix} v_{0\,0} \\ v_{0\,1} \\ v_{0\,2} \\ \vdots \\ v_{0\,N^*} \\ v_{1\,0} \\ v_{1\,1} \\ v_{1\,2} \\ \vdots \\ v_{M^*N^*} \end{Bmatrix} ; \quad \left\{\bar{\bar{w}}\right\} = \begin{Bmatrix} w_{0\,0} \\ w_{0\,1} \\ w_{0\,2} \\ \vdots \\ w_{0\,N^*} \\ w_{1\,0} \\ w_{1\,1} \\ w_{1\,2} \\ \vdots \\ w_{M^*N^*} \end{Bmatrix}$$

K = Stiffness Matrix

M = Mass Matrix

ω = The natural frequencies from Equation (33) in radian/sec.

If the matrices K and M became singular due to the presence of zeroes in some of the rows and columns, the matrices were condensed by eliminating those rows and columns of zeroes. The subroutine called EIGENP (23), with double precision, was used to calculate the frequencies (ω^2) of Equation (33) and the resulting eigenvectors

$$\begin{Bmatrix} \bar{\bar{u}} \\ \bar{\bar{v}} \\ \bar{\bar{w}} \end{Bmatrix}$$

Once the eigenvalues and eigenvectors were obtained, the corresponding mode shapes were found.

NUMERICAL RESULTS

Introduction

The analysis described in this report was substantiated by comparing the results of this study with some of those obtained by previous investigators. Some parametric studies of stiffened noncircular cylinders were made and are also presented in this chapter.

Comparison With Known Solutions

This section presents the comparison of natural frequencies for (1) an unstiffened circular cylinder with various boundary conditions; (2) ring- and/or stringer-stiffened circular cylinders with various end conditions; (3) unstiffened noncircular shells with various end conditions; and, (4) ring- and stringer-stiffened elliptical cylinders.

Comparison of results for the unstiffened circular shells: Forsberg (24) presented exact frequencies for a freely supported unstiffened circular cylinder, obtained by solving the differential equations of motion. The results of this analysis and those of Forsberg's exact solution are compared in Table I. Both the analyses used the Flügge shell theory. As is evident from the Table I, good correlation exists between the frequencies of both the analyses. Such

TABLE I

COMPARISON OF ANALYTICAL FREQUENCIES OF A FREELY SUPPORTED UNSTIFFENED CIRCULAR CYLINDER, OBTAINED BY THE PRESENT ANALYSIS AND FORSBERG (Hz.)

n	m	PRESENT ANALYSIS	FORSBERG [a]
2	1	778	778
	2	2449	2449
	3	4253	4253
3	1	628	627
	2	1458	1458
	3	2682	2681
4	1	974	974
	2	1304	1303
	3	2021	2020
	4	2947	2946

a) Reference (24), figure 3(a).

type of accuracy was expected because the assumed mode functions satisfy the freely supported boundary condition exactly.

Comparisons were also made with the results of Reference (16) for the same boundary condition and m = 1 and 2. These are presented in Table II. In Reference (16), Sewall et al., using Sander's shell theory (25), applied the Rayleigh-Ritz method as in our analysis. As is evident from Table II, excellent comparisons were obtained.

Figure 4 shows a comparison between the analytical and experimental results of Reference (17) and those of the present analysis (for m = 1) considering a clamped-free, unstiffened, circular shell. The frequency curves reveal that this analysis yields results similar to those of Reference (17). The slight differences might be attributed to the difference in the shell theories. Comparisons were also made with the experimental results of Park, A. C. et al., (26) and the analytical results of Egle and Soder (19). These are presented in Table III. In this comparison four-place accuracy was obtained between the analytical results of Egle and Soder and the present analysis. The discrepancy between the analytical and experimental results increases as the number of circumferential waves decreases. Egle and Soder speculated in Reference (19) that the shell end may not have been absolutely fixed in the experiments.

The experimental and analytical results of Reference (16) for free-free circular shells were used to establish the validity of the present analysis for this boundary-condition case. Table IV shows the comparison of the results for m = 1 and 2. The present analysis yielded four-place accuracy.

TABLE II

COMPARISON OF ANALYTICAL FREQUENCIES OF A FREELY SUPPORTED UNSTIFFENED CIRCULAR CYLINDER[a], OBTAINED BY THE PRESENT ANALYSIS AND SEWALL (Hz.)

n	m = 1		m = 2	
	PRESENT ANALYSIS	SEWALL (Ref 16)	PRESENT ANALYSIS	SEWALL (Ref 16)
1	1565.3	1565.0	2309.3	2309.0
2	894.1	894.1	1782.4	1782.0
3	529.8	529.8	1314.9	1315.0
4	338.6	338.6	968.4	968.4
5	235.6	235.6	726.3	726.3
6	182.1	182.1	560.3	560.3
7	162.2	162.2	448.6	448.6
8	166.9	166.9	377.2	377.2
9	188.6	188.6	338.1	338.1
10	221.3	221.3	325.7	325.1
11	261.7	261.7	335.0	335.0
12	308.0	308.0	361.0	361.0
13	359.5	359.5	399.6	399.5
14	415.6	415.6	447.5	447.5

a) The geometry of the shell is given in Reference (16).

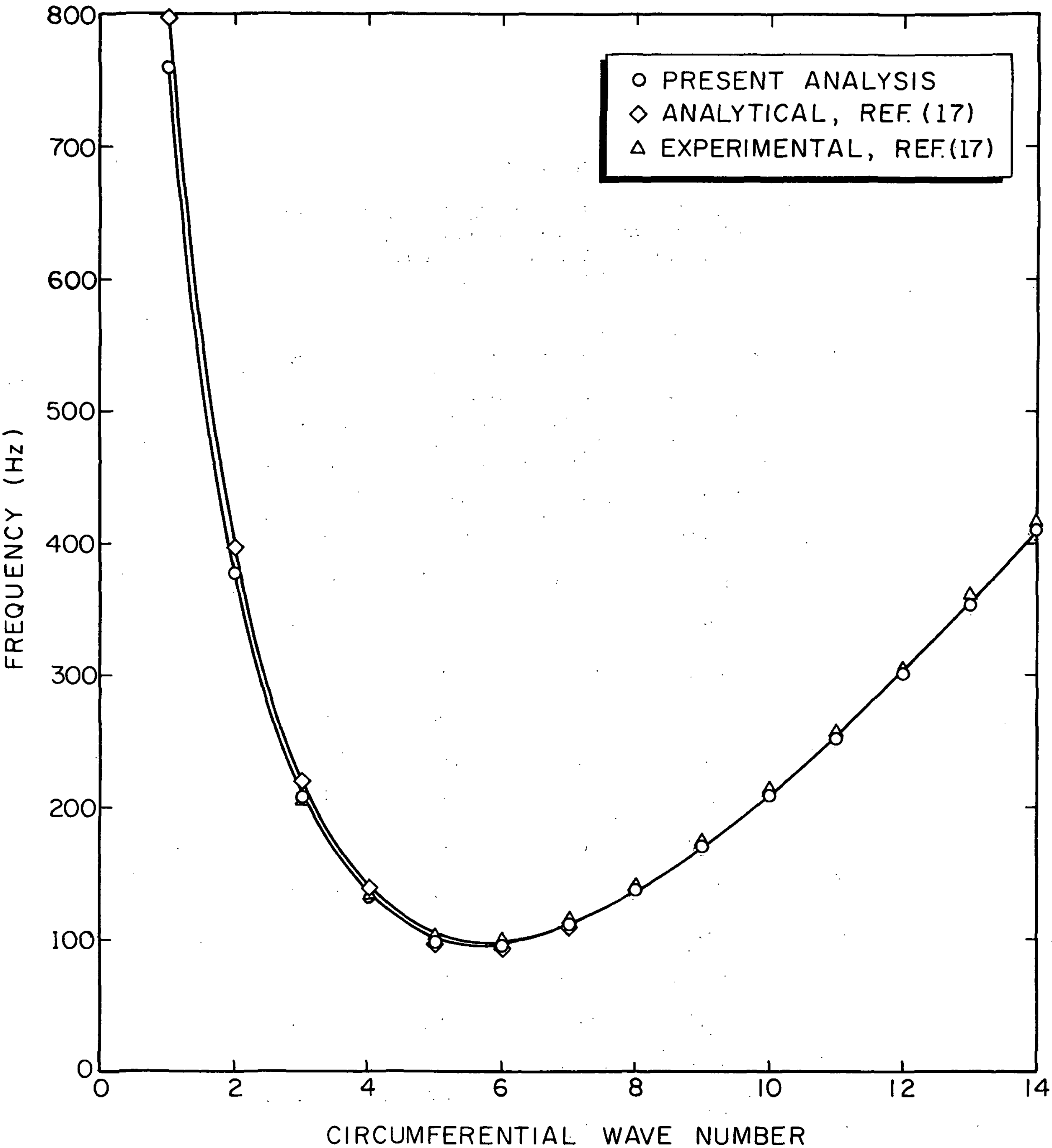


Figure 4: Comparison of Experimental and Analytical Frequencies of Clamped-Free Circular Cylindrical Shell (Hz).

TABLE III

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE UNSTIFFENED CIRCULAR CYLINDER[a]
(Hz.)

n	m = 1			m = 2		
	EGLE & SODER[b]	PRESENT ANALYSIS[b]	PARKS et al.[c]	EGLE & SODER[b]	PRESENT ANALYSIS[b]	PARKS et al.[c]
2	104.4	104.4	87.2 & 95.1	-	508.2	-
3	55.6	55.6	51.5	-	281.3	-
4	52.0	52.0	50.4	177.9	177.9	168.5 & 170.2
5	-	71.6	70.9	-	135.4	132.8
6	-	101.8	101.4	-	132.0	128.8 & 130.1
7	139.1	139.1	138.8	154.2	154.2	153.6
8	182.6	182.6	182.2	191.2	191.2	191.3

a) Reference (19), configuration 1, p. 28.

b) Flügge shell theory, insurface inertias included.

c) Reference (26), model 1.

TABLE IV

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A FREE-FREE UNSTIFFENED CIRCULAR CYLINDER (Hz.)

n	m = 1			m = 2		
	PRESENT ANALYSIS[a]	SEWALL ANALYSIS[b]	SEWALL EXPERIMENT[b]	PRESENT ANALYSIS	SEWALL ANALYSIS	SEWALL EXPERIMENT
1	2012.0[c]	2014.0[c]	-	2288.0	2293.0	-
2	7.5	7.5	7.7	1613.0	1616.0	-
3	19.0	19.0	18.9	1066.0	1068.0	-
4	34.2	34.2	35.7	716.9	717.8	-
5	53.4	53.4	53.0	504.4	504.8	-
6	76.6	76.7	76.4	375.4	375.6	377.3
7	104.1	104.1	103.8	299.8	299.9	299.1
8	135.7	135.7	135.3	262.6	262.2	257.4 & 262.1
9	171.4	171.5	170.7	253.6	253.4	248.8 & 249.3
10	211.4	211.5	210.2	266.5	266.3	268.8
11	255.6	255.7	253.0	294.8	294.7	290.9
12	303.9	304.1	305.5	333.9	334.0	327.6
13	356.5	356.7	352.0	381.2	381.1	-
14	413.3	413.5	412.5	434.7	434.7	436.6

a) Flügge shell theory; 6 even, and 6 odd axial mode functions considered.

b) Reference (16).

c) Extensional frequency.

Comparison of results for stringer-stiffened circular shells: Egle and Sewall (18) presented frequencies obtained for a stringer-stiffened, freely supported, circular cylinder using a method similar to that of the present analysis but using the Donnell shell theory and neglecting the insurface inertias of the stiffened shell. The shell theory used in the present analysis was modified to Donnell theory in order to compare the results of this analysis with those of Egle and Sewall. Table V gives the comparison between the frequencies for $m = 2$. The frequencies of Egle and Sewall are slightly higher than those of the present analysis, evidently attributable to their neglect of the inplane inertias. It is evident from Table V that the discrepancy between the results of both the theories decreases as the number of circumferential waves increases, which is a typical characteristic of Donnell theory.

Comparison of results with ring-stiffened circular shells: Forsberg (24) obtained exact solutions for the natural frequencies of ring-stiffened circular cylinders. Bushnell (27) obtained the natural frequencies of ring-stiffened segmented shells of revolution using an energy method in conjunction with the method of finite differences. The compatibility relations and the energy expressions used by Bushnell are similar to those of the present analysis. Table VI presents the frequencies obtained by Forsberg, Bushnell, and the present analysis for freely supported circular cylinders with three rings of both zero and negative eccentricity. The frequencies of this analysis which are presented in Table VI were obtained by considering 12 even and 13 odd axial mode functions in the assumed displacement series. The results of this analysis are in excellent agreement with the exact frequencies obtained by Forsberg and the approximate frequencies of Bushnell. The

TABLE V

COMPARISONS OF FREQUENCIES OF FREELY SUPPORTED CYLINDERS
WITH AND WITHOUT INSURFACE INERTIAS
(DONNELL THEORY)

m	n	THE PRESENT ANALYSIS (Insurface Inertias Included)			EGLE & SEWALL* (Insurface Inertias Neglected)		
		STRINGER STIFFENED INTERNALLY		UNSTIFFENED	STRINGER STIFFENED INTERNALLY		UNSTIFFENED
		Sym. Mode	Antisym. Mode		Sym. Mode	Antisym. Mode	
	3	555	555	568	591	591	602
	4	337	348	353	346	365	365
	5	236	235	246	241	241	251
	6	192	197	200	194	202	203
	7	189	189	194	191	191	196
2	8	208	213	216	209	217	218
	9	254	254	256	256	256	258
	10	295	303	308	297	306	309
	11	355	355	367	358	358	369
	12	421	427	435	424	430	436

* Reference (18).

TABLE VI

COMPARISON OF FREQUENCIES OF A FREELY SUPPORTED CIRCULAR CYLINDER[a] WITH THREE SYMMETRIC AND INTERNAL RING STIFFENERS, OBTAINED BY THE PRESENT ANALYSIS, BUSHNELL, AND FORSBERG (Hz.)

n	m	SYMMETRIC			INTERNAL		
		FORSBERG[b]	BUSHNELL[c]	PRESENT ANALYSIS[d]	FORSBERG	BUSHNELL	PRESENT ANALYSIS
2	1	788	787	787	999	987	994
	2	2219	2219	2219	2254	2264	2252
	3	3796	3802	3801	3710	3741	3711
3	1	1155	1152	1152	2087	2066	2081
	2	1661	1660	1660	2397	2382	2386
	3	2617	2619	2618	3073	3068	3066
4	1	1988	1982	1988	3161	3120	3142
	2	2132	2130	2141	3085	3023	3032
	3	2535	2539	2548	3014	3019	3030

a) Reference (24), figure 3(a).

b) Exact solution obtained by solving the equations of equilibrium.

c) Reference (27), an energy formulation is used in conjunction with the method of finite differences.

d) Energy expressions of ring are similar to those of Reference (27).

maximum discrepancy encountered for the case of zero eccentricity ring stiffener was 0.51% and 1.75% for the negative eccentricity, ring-stiffened case. The external ring-stiffened shell of Forsberg was also studied but the frequencies obtained did not converge for 12 even and 13 odd axial mode functions in the assumed displacement series; hence those results are not presented in this report.

Comparisons were also made with some of the results of Al-Najafi and Warburton (28), for freely supported and free-free ring-stiffened circular shells and are presented in Table VII. Their results were obtained using a finite element technique employing five elements per bay. Significant reduction in the order of the matrices was obtained in their study by considering the symmetry of the structures and neglecting insurface inertias. The results of the present analysis given in Table VII were obtained by considering circumferentially symmetric and 10 even and 10 odd axial mode functions in the assumed displacement series but including insurface inertias. The values for the frequencies converged for 15 even and 15 odd terms but the difference between the results for 10 terms and 15 terms was rather small. Hence, in order to compare on the basis of the order of the matrices, the results of 10 terms was chosen for comparison. It is evident from Table VII that the frequencies of the present analysis for the freely supported case are lower than those of the finite element method (except for m = 3) and are also closer to the experimental values. For the free-free case, the finite element results were observed to be closer to experimental values than the results of the present analysis, except for m = 1 and 2. In general, the agreement between the results of this analysis and those of the finite element and the experiment is good.

TABLE VII

COMPARISON OF FREQUENCIES OF RING-STIFFENED CYLINDERS, OBTAINED BY RAYLEIGH-RITZ AND FINITE ELEMENT METHODS (Hz.)
(n = 4); d = 0.25 in.

m	FREELY SUPPORTED			m	FREE-FREE		
	RAYLEIGH-RITZ[a]	FINITE ELEMENT[b]	EXPRTL.		RAYLEIGH-RITZ	FINITE ELEMENT	EXPRTL.
1	1867	1873	1867	0[c]	1550	1547	1551
2	2089	2091	2076	1[c]	1538	1537	1539
3	2651	2650	2600	2	1889	1895	1890
4	3415	3429	3355	3	2303	2290	2287
5	4239	4270	-	4	3075	3044	3044
6	4925	5022	-	5	3955	3920	3916
7	5846	-	-	6	4910	-	-
8	6585	-	-	7	5548	-	-
9	7330	-	-	8	6349	-	-
10	8079	-	-	9	7103	-	-

a) Present Analysis, number of terms considered in the displacement series is 10.

b) Reference (28).

c) Rigid body modes.

In order to show the rate of convergence of the results of this study, the frequencies were obtained with different assumed numbers of terms. These results are presented in Tables VIII and IX for the freely supported and free-free ring-stiffened shells studied by Al-Najafi and Warburton. Tables VIII and IX show that the rate of convergence of frequencies is rather rapid.

Comparison of results with ring- and stringer-stiffened circular shells: Park, A. C. et al. (26), presented a considerable amount of experimental information on the frequencies and mode shapes of stiffened and unstiffened circular and elliptical shells with clamped-free ends. Egle and Soder (19) compared their analytical results with those of Park's experimental results for a clamped-free circular cylinder with three equally spaced internal rings and sixteen internal stringers. The same shell was analyzed by the present analysis and comparisons are indicated in Table X. Because the cross-section of the stiffened shell was symmetric with respect to both the vertical and horizontal axes, the frequencies of even and odd circumferential modes were able to be evaluated separately. It is interesting to notice in Table X that the results of the present analysis are consistently lower than those of Egle and Soder. This improvement in the frequencies may be attributed to the improved stiffener theories of the present analysis. The fact that the discrepancy between the analytical and experimental frequencies decreases with the increase in wave numbers n and m suggests that the boundary conditions of the experiment and the theory may not match.

The results of the present analysis were obtained with 10 axial mode functions and 3 even and 3 odd circumferential mode functions.

TABLE VIII

SPEED OF CONVERGENCE OF FREQUENCIES OF FREELY SUPPORTED RING-STIFFENED CIRCULAR CYLINDER[a] (Hz.), n=4

m^c \ M^b	5	10	12	14	15
1	2032.29	1867.32	1853.29	1841.82	1841.83
2	2136.32	2089.33	2076.62	2067.81	2067.81
3	2682.82	2651.32	2640.59	2634.31	2634.30
4	3446.09	3414.67	3414.65	3409.95	3409.94
5	4263.22	4239.00	4238.98	4238.97	4235.32
6	4924.91	4924.59	4924.58	4924.57	4924.47
7	5877.52	5845.98	5845.97	5845.97	5841.54
8	6613.81	6585.41	6585.39	6580.90	6580.89
9	7348.25	7329.87	7321.17	7316.42	7316.41
10	8098.23	8079.40	8072.23	8067.25	8067.24

a) Reference (28), figure 2(c).

b) Number of terms considered in the displacement series.

c) Axial wave number.

TABLE IX

SPEED OF CONVERGENCE OF FREQUENCIES OF FREE-FREE RING-STIFFENED CIRCULAR CYLINDER[a]
(Hz.), n = 4

m^c \ M^b	5	10	12	14	15
1*	1591.53	1549.60	1546.82	1546.13	1544.91
2*	1585.73	1538.16	1537.45	1536.33	1535.35
3	2046.65	1888.92	1823.09	1816.19	1816.05
4	2380.46	2303.22	2300.84	2299.44	2299.35
5	3127.52	3075.50	3067.22	3066.92	3066.66
6	3979.47	3955.27	3952.06	3951.22	3950.53
7	4973.26	4909.71	4836.28	4833.91	4833.57
8	5595.02	5548.42	5542.69	5540.21	5539.64
9	6439.71	6348.83	6312.89	6309.63	6308.67
10	7189.93	7102.58	7096.81	7093.99	7091.25

* Rigid body modes.

a) Reference (28), figure 2(c).

b) Number of terms considered in the displacement series.

c) Axial wave number.

TABLE X

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE RING- AND STRINGER-STIFFENED CIRCULAR CYLINDER (Hz.)

n	m	PRESENT[a] ANALYSIS	EGLE & SODER[b]	PARK et al.[c]
2	1	100.2	105.8	80.2 & 88.2
	2	432.2	433.9	-
	3	907.0	-	-
4	1	207.6	216.9	184.6
	2	276.0	285.9	251.5
	3	437.2	447.1	397.0 & 430.4
6	1	308.5	315.0	-
	2	345.9	353.8	-
	3	402.6	414.0	-

a) n = 2, 4, 6; m = 1 to 10.

b) Reference (19).

c) Reference (26), model 1S.

The reason for considering fewer number of circumferential terms than the axial terms is that the coupling between the circumferential mode functions (due to the presence of stringers) is rather weak. This was also noticed experimentally by Scruggs et al. (29). The coupling between the axial mode functions (due to the presence of rings) is considerable; hence 10 terms were considered in the longitudinal direction. To determine whether or not 10 terms were sufficient for obtaining reasonably well-converged frequencies, M^* was increased to 30 and only one circumferential term was used. The comparison between these results is shown in Table XI. Since the difference in the results was found to be negligible, it was concluded that 10 terms were sufficient for convergence.

Comparison of results with unstiffened noncircular shells: Having established satisfactory results for stiffened and unstiffened circular shells of arbitrary end conditions, comparisons were then made for unstiffened noncircular shells. Sewall et al. (16, 17) presented analytical and experimental results for elliptical shells with arbitrary end conditions. Tables XII and XIII compare the analytical symmetric and antisymmetric frequencies for freely supported elliptical shells of eccentricities of 0.526 and 0.760 for $m = 1$. It is evident from Tables XII and XIII that the agreement between the results of both Sewall and the present analysis is generally satisfactory and is excellent for n less than 10.

Comparison of results obtained for elliptical shells with free-free and clamped-free end conditions were also made and are presented in Tables XIV and XV, respectively. The results of this analysis

TABLE XI

CONVERGENCE OF FREQUENCIES OF CLAMPED-FREE RING- AND STRINGER-STIFFENED CIRCULAR CYLINDER (Hz.)
(Circumferentially Symmetric)

n	m	a	b
	1	99.32	100.19
2	2	428.66	432.19
	3	903.77	906.96

a) $N^* = 2$, $M^* = 30$.

b) $N^* = 6$, $M^* = 10$.

TABLE XII

COMPARISON OF ANALYTICAL FREQUENCIES OF FREELY SUPPORTED ELLIPTICAL CYLINDERS[a] (Hz.)

n	$\epsilon = 0.526$, $m = 1$			
	SYMMETRIC		ANTISYMMETRIC	
	PRESENT ANALYSIS[b]	SEWALL[c]	PRESENT ANALYSIS	SEWALL
0	2550.2	2550.0	-	-
1	1439.7	1440.0	1685.7	1686.0
2	876.6	876.6	888.9	888.9
3	524.1	524.1	524.2	524.2
4	335.5	335.5	335.6	335.5
5	234.3	234.3	234.3	234.2
6	184.2	184.2	184.2	184.2
7	156.9	157.1	156.9	157.0
8	160.1	160.2	160.2	160.2
9	189.7	189.8	189.4	189.8
10	221.5	221.9	221.8	221.9
11	260.8	261.9	261.7	261.9
12	307.6	308.1	307.9	308.1
13	348.9	359.5	355.8	359.5
14	405.7	415.6	413.9	415.6

a) The geometric and material properties of the shells are given in Reference (16).

b) Number of terms used is 13.

c) Reference (16).

TABLE XIII

COMPARISON OF ANALYTICAL FREQUENCIES OF FREELY SUPPORTED ELLIPTICAL CYLINDERS[a] (Hz.)

n	ϵ = 0.760, m = 1			
	SYMMETRIC		ANTISYMMETRIC	
	PRESENT ANALYSIS[b]	SEWALL[c]	PRESENT ANALYSIS	SEWALL
0	2611.8	2612.0	-	-
1	1237.7	1238.0	1855.7	1856.0
2	785.1	785.2	858.5	858.5
3	491.1	491.1	492.5	492.4
4	319.8	319.4	318.9	319.4
5	-	-	226.6	226.9
6	-	-	-	-
7	138.5	138.5	138.5	138.5
8	140.0 & 177.8	140.1 & 178.3	140.1 & 178.5	140.1 & 178.3
9	182.3 & 226.1	184.1 & 226.9	184.0 -	184.1 -
10	221.7	223.9	223.5	223.9
11	261.6	263.6	259.2	263.6
12	310.6	307.3	296.9	307.3
13	378.4	359.4	338.6	359.4
14	464.8	417.1	399.6	417.1

a) The geometric and material properties of the shells are given in Reference (16).

b) Number of terms used is 13.

c) Reference (16).

TABLE XIV

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A FREE-FREE ELLIPTICAL CYLINDER (Hz.)
a = 12.95, b = 11.01, m = 0

n	SYMMETRIC			ANTISYMMETRIC			KLOSNER[c]	KLOSNER[d]
	PRESENT[a] ANALYSIS	SEWALL[b] ANALYSIS	SEWALL[b] EXPERIMENT	PRESENT ANALYSIS	SEWALL ANALYSIS	SEWALL EXPERIMENT		
2	5.62	5.62	5.6	5.68	5.68	5.6	5.92	5.56
3	15.89	15.89	16.1	15.89	15.89	16.2	16.30	15.4
4	30.52	30.52	30.9	30.52	30.52	30.8	31.2	31.1
5	49.42	49.41	50.1	49.42	49.41	50.1	50.3	50.45
6	72.54	72.54	74.8	72.54	72.54	74.4	73.8	71.6
7	99.87	99.87	102.4	99.87	99.87	102.4	101.5	98.9
8	131.42	131.40	134.6	131.42	134.4	-	133.5	133.2
9	167.18	167.20	171.5	167.17	167.2	171.7	169.8	168.1
10	207.13	207.10	212.5	207.10	207.1	212.8	210.3	204.8
11	251.34	251.30	258.8	251.26	251.3	258.4	255.2	251.1
12	299.46	299.60	312.1	299.72	299.6	-	304.2	302.8
13	351.95	352.20	363.8	353.19	352.2	362.3 & 363.0	357.5	351.9
14	408.14	409.00	423.2	411.17	409.0	-	415.2	405.8

a) $N^* = 20$, $M^* = 2$. b) Reference (16). c) Reference (9). d) Reference (10).

TABLE XV

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE ELLIPTICAL CYLINDER (Hz.)
a = 12.95, b = 11.01
m = 1

n	SYMMETRIC			ANTISYMMETRIC		
	PRESENT[a] ANALYSIS	SEWALL ANALYSIS	SEWALL EXPERIMENT	PRESENT ANALYSIS	SEWALL ANALYSIS	SEWALL EXPERIMENT
1	736.6	739.2	-	838.0	840.1	-
2	384.2	390.6	-	387.6	394.1	-
3	212.4	217.5	201.9 &	212.4	217.5	204.8
	-	-	201.1	-	-	-
4	133.6	136.4	129.5 &	133.6	-	134.0
	-	-	129.1	-	-	-
5	97.5	99.5	96.4	97.5	99.5	100.2
6	94.6	95.9	94.2 &	94.6	95.9	94.5
	-	-	93.1	-	-	-
7	113.2	114.2	115.1	113.2	114.2	116.5
8	138.4	139.6	141.8 &	-	136.4	142.3
	-	-	140.6	138.4	139.6	-
9	171.2	171.4	176.0	171.2	171.4	176.2
10	210.0	210.1	217.2 &	209.9	210.1	216.3
	-	-	217.1	-	-	-
11	253.6	253.7	260.4	253.5	253.7	260.8
12	301.4	301.7	309.5	301.6	301.7	310.6
13	353.8	354.1	365.0	354.9	354.1	-
14	409.8	410.7	423.6	412.8	410.7	-

a) $N^* = 20$, $M^* = 2$.

are similar to those obtained analytically by Sewall. Also included are Sewall's experimental results and analytical results obtained by Klosner (9, 10).

Comparison of results with ring- and stringer-stiffened elliptical shells: Park, A. C. et al. (26) presented experimental frequencies and mode shapes for a clamped-free elliptical cylinder with four equally spaced internal rings and sixteen internal stringers. This shell was also analyzed by the present analysis, and some comparisons are presented in Table XVI. Due to the symmetry of the cross-section with respect to both the vertical and horizontal axes, the frequencies of even and odd circumferential modes were evaluated separately. As is evident form Table XVI, the theoretical results are consistently higher than the experimental results. The discrepancy between the analytical and experimental frequencies may again be attributed to the possible difference in the boundary conditions of the experiment and the theory. However, storage limitations of the IBM 360/65 computer prevented the consideration of a sufficient number of terms in the displacement series to assure convergence of frequencies. The results of the present analysis were obtained with 5 axial mode functions and 6 even and 6 odd circumferential mode functions.

Studies of Stiffened Noncircular Cylinders

Having obtained satisfactory comparisons with known solutions of the circular, noncircular, unstiffened, and stiffened cylindrical shells, two studies of stiffened noncircular shells were made. This section presents the results of those studies.

TABLE I

COMPARISON OF ANALYTICAL AND EXPERIMENTAL FREQUENCIES OF A CLAMPED-FREE ELLIPTICAL CYLINDER[a] WITH FOUR RINGS AND TWELVE STRINGERS

n	m = 1		m = 2	
	PRESENT ANALYSIS[b]	PARK[c]	PRESENT ANALYSIS	PARK
1	177.92	163.5	-	-
2	92.08	60.8 & 79.7	-	-
3	151.75	141.1	242.64	226.7
4	-	-	377.68	352.6

a) The geometry of the stiffened shell is given in figure 32, model 4S, Park, A. C. et al., dynamics of shell-like lifting bodies, Part II, the experimental investigation. AFFDL-TR-65-17, Part II, June, 1965.

b) Rayleigh-Ritz method $N^* = 12$, $M^* = 5$.

c) Experimental results.

<u>Study of the effect of number of stringers</u>: Egle and Soder (19) studied the variation of the minimum frequency of a stringer-stiffened, circular cylinder with the number of stringers, keeping the total cross-sectional area ($L A_s$) and the total torsional stiffness ($L G J_s$) of the stringers constant. This is a reasonable approach for studying the explicit effect of the number of stringers. However, the implementation of "total" stringer properties being constant while the number of stringers is varied is more difficult in the experimental study than in the analytical study.

In order to avoid this difficulty in the present study, the cross-sectional properties of all the stringers were assumed to be the same while their total number varied. Table XVII presents the variation of the natural frequencies of various circumferential modes of an internal stringer-stiffened freely supported elliptical cylinder with the number of equally spaced stringers. The geometric and material properties of the stringers are given in the footnotes of Table XVII. In order to visualize the variation of the frequencies of various circumferential modes with the number of stringers, some of the results of Table XVII are plotted in Figure 5. As is evident from Figure 5, the overall effect of the stringers is a lowering of the frequencies. This effect is greater on the frequencies pertaining to lower circumferential wave numbers. The rate of decay of frequencies due to the presence of stringers is greater for small numbers of stringers and diminishes with an increase in the number of stringers.

TABLE XVII

STUDY OF THE EFFECT OF NUMBER OF STRINGERS ON THE FREQUENCIES OF A FREELY SUPPORTED ELLIPTICAL CYLINDER[a] (Hz.)
$\varepsilon = 0.760$, $m = 1$

n^b \ L^c	0	2	4	8	16
1	1238.0	1159.0	1090.0	984.5	831.3
3	491.1	470.4	448.3	450.2	433.7
7	139.5	121.1	121.1	122.7	114.5
9	183.5	184.2	184.3	145.7	141.9
	226.6	214.8	212.7	208.5	204.9
11	263.7	262.1	256.5	258.1	224.6
13	380.0	373.7	368.5	347.8	290.6

a) The geometry and material properties of the unstiffened shell are given in Reference (16).

b) Circumferential mode number.

c) Number of equidistant internal stringers. The properties of the stringers are:

$A_{s\ell} = 0.1037$ sq. in. $\qquad z_{1s\ell} = -0.0475$ in.

$I_{yys\ell} = 0.005957$ in.^{4} $\qquad z_{2s\ell} = -0.2340$ in.

$I_{zzs\ell} = 0.001285$ in.^{4} $\qquad y_{1s\ell} = 0.0$ in.

$I_{yzs\ell} = 0$ $\qquad y_{2s\ell} = 0.0$ in.

$(GJ)_{s\ell} = 912.5$ lb.-in.^{2} $\qquad \rho_{s\ell} = 0.0002588$ lbs.-sec.^{2}/in.^{4}

$$E_{s\ell} = 10.6 \times 10^7 \text{ lbs.-sec.}^2/\text{in.}^4$$

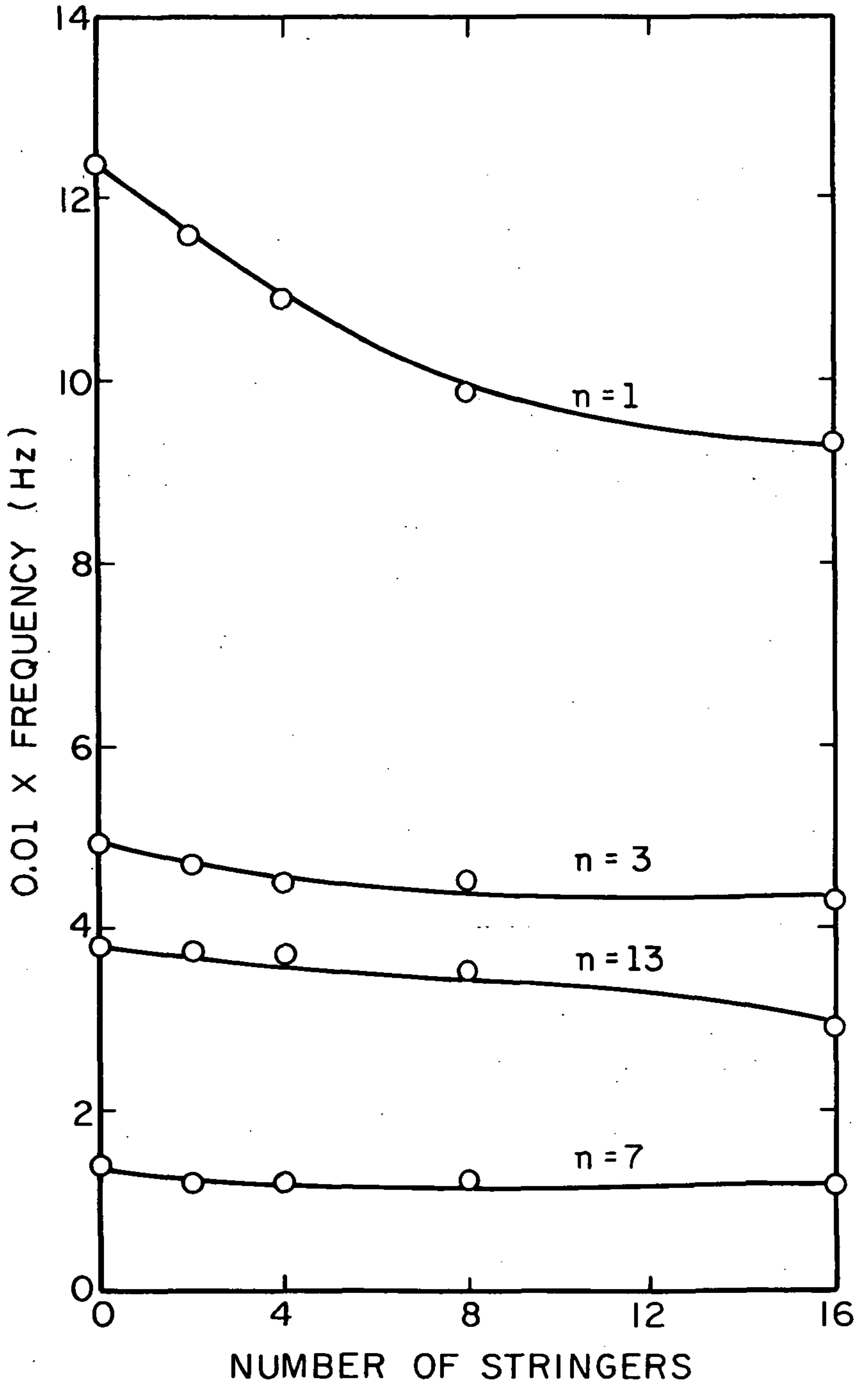


Figure 5. Study of the Effect of Number of Stringers on the Natural Frequencies of a Freely Supported Elliptical Cylinder with $\epsilon = 0.760$, $m = 1$.

Ring- and stringer-stiffened elliptical cylinders: This section presents results for a stiffened, noncircular freely supported cylinder with large numbers of rings and stringers. The frequencies of the unstiffened freely supported elliptical cylinder with $\epsilon = 0.760$ are presented in Table XVIII. To study the effect of large numbers of ring and stringer stiffeners, 16 internal stringers and 11 internal rings were added to the above elliptical shell. The geometric and material properties of the rings and stringers are assumed to be the same and are listed in the footnotes of Table XVII. The frequencies and the mode shapes of this shell were obtained using the present analysis. Table XVIII presents some of the frequencies. Figure 7 shows some of the axial mode shapes and Figure 8 shows some of the circumferential mode shapes. To visualize clearly the effect of the large number of rings and stringers on the natural frequencies, some of the frequencies presented in Tables XII, XIII, and XVIII are plotted in Figure 6. The results presented in Table XVIII were obtained with 5 axial mode functions and 6 even and 6 odd circumferential mode functions. It is quite evident from Figure 6 that the frequency curves of the ring- and stringer-stiffened shell under consideration, are more or less similar to those of the unstiffened shell; however, they are bodily shifted to the left. The minimum frequency of the stiffened shell is more than three times the minimum frequency of the unstiffened shell. The frequencies of the stiffened shell are consistently higher than those of the unstiffened shell. It should be noted that even though the ratio of number of rings to number of stringers in this problem is about 3:4, the effect of rings is predominant. Figure 6 reveals that the frequency curves for various m values tend to merge as n increases. The

TABLE XVIII

FREQUENCIES OF 16 STRINGER[a] AND 11 RING[a] INTERNALLY STIFFENED FREELY SUPPORTED ELLIPTICAL CYLINDER[b] WITH ϵ = 0.760 (Hz.)

n^c	m^d	
	1	3
1	741.0	1703.0
2	444.9	1303.0
3	437.9	974.3
4	743.7	973.5
5	1155.0	1340.0
6	1868.0	1998.0
7	2924.0	2959.0

a) The stringers and the rings have identical material and geometric properties which are given in the footnotes of Table XVII.

b) The geometric and material properties of the shell are given in Reference (16).

c) Circumferential mode number.

d) Axial mode number.

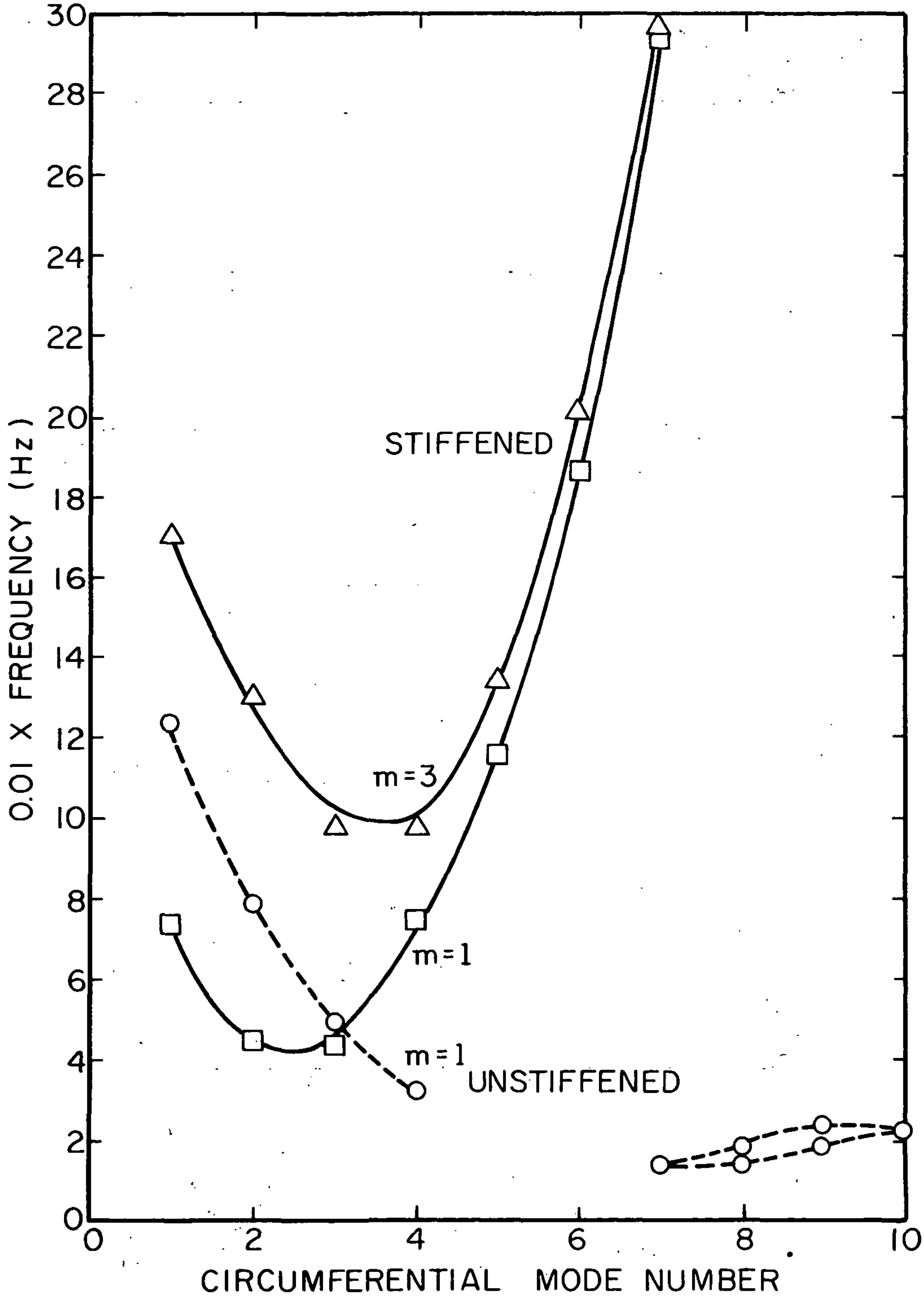


Figure 6. Comparison of Frequencies of Unstiffened, and Ring- and Stringer-Stiffened Freely Supported Elliptical Cylinder with ε = 0.760.

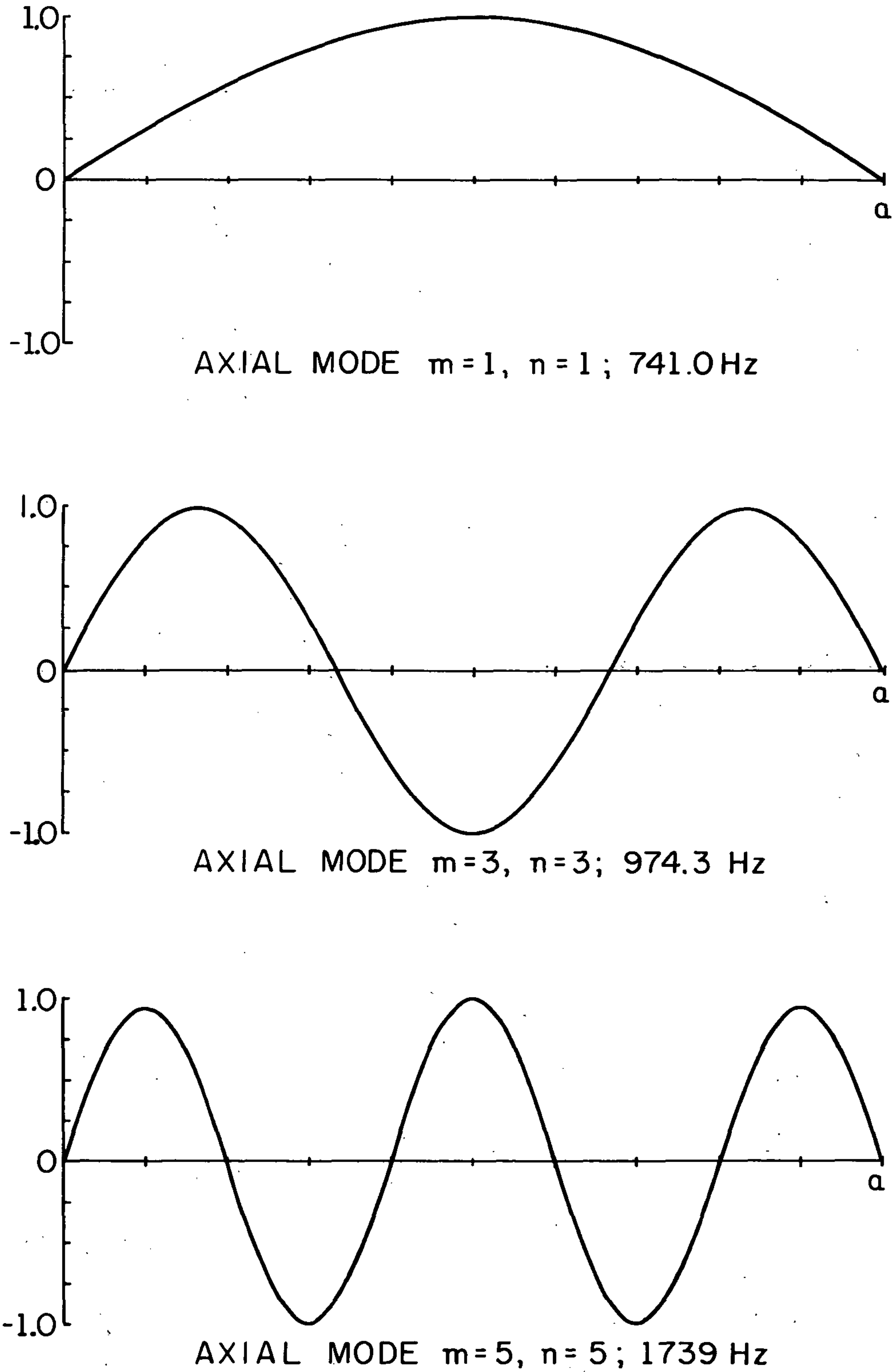


Figure 7. Axial Modes

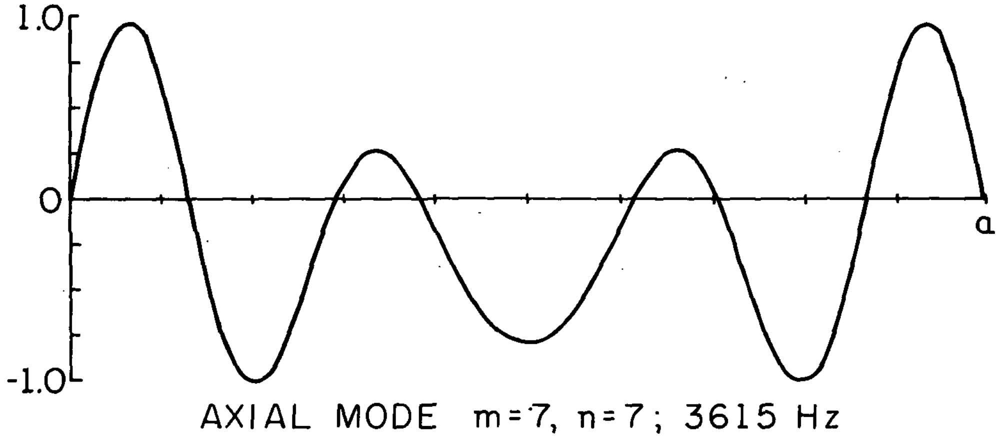


Figure 7. (Continued)

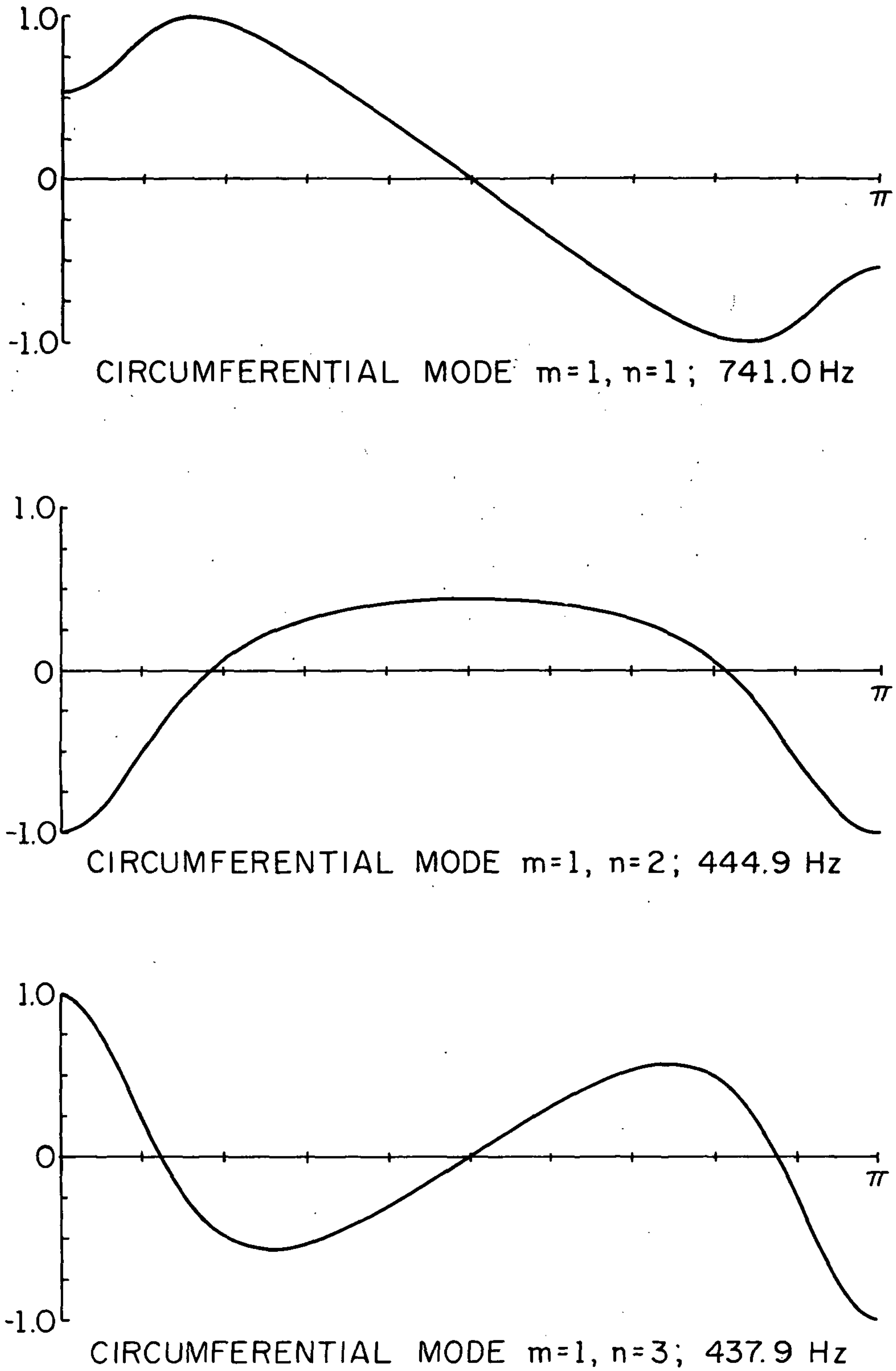


Figure 8. Circumferential Modes

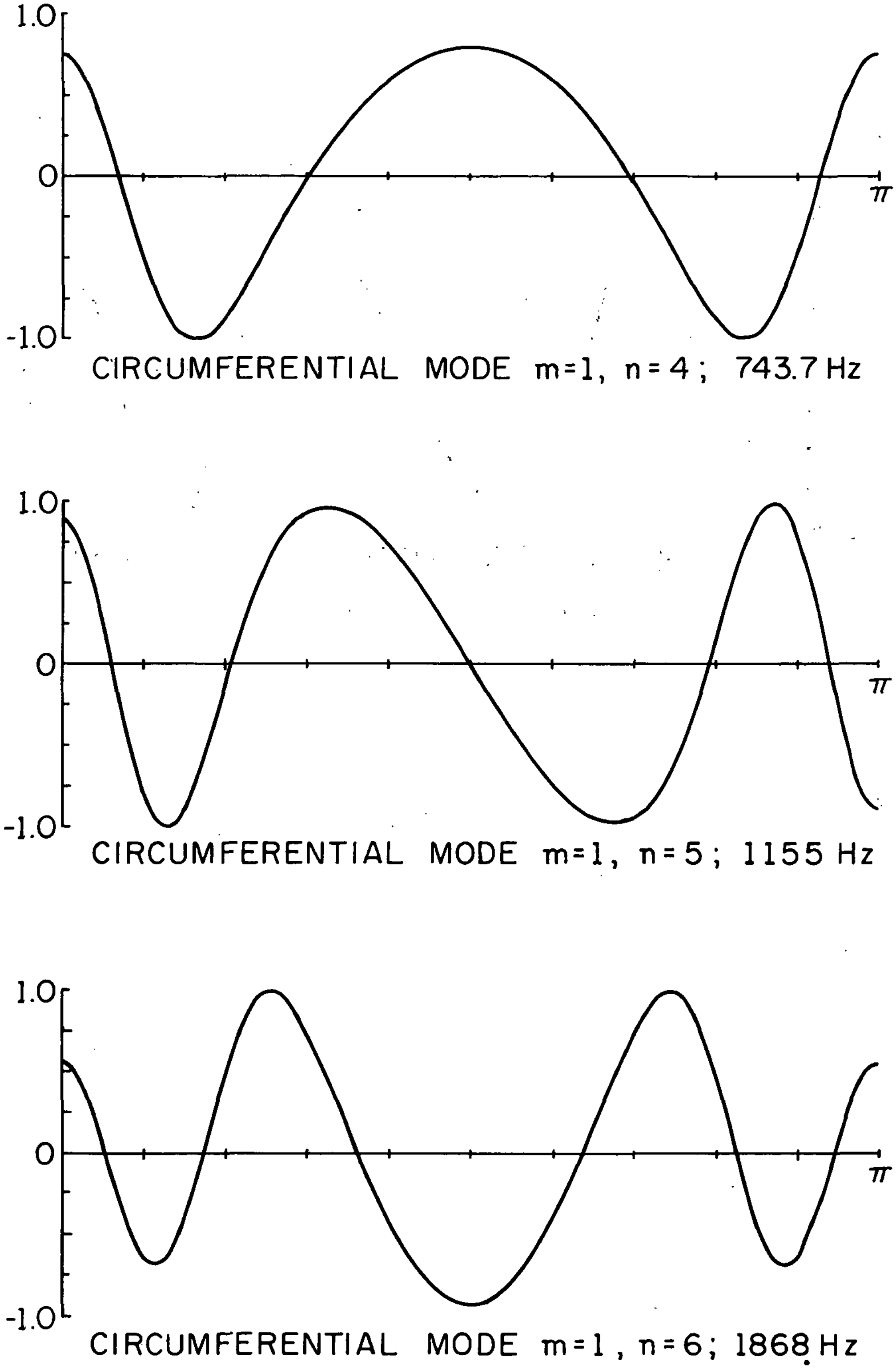


Figure 8. (Continued)

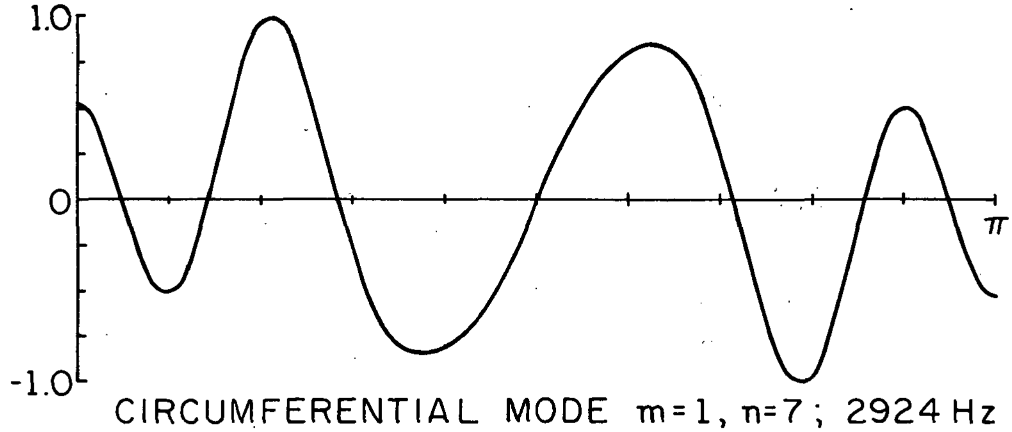


Figure 8. (Continued)

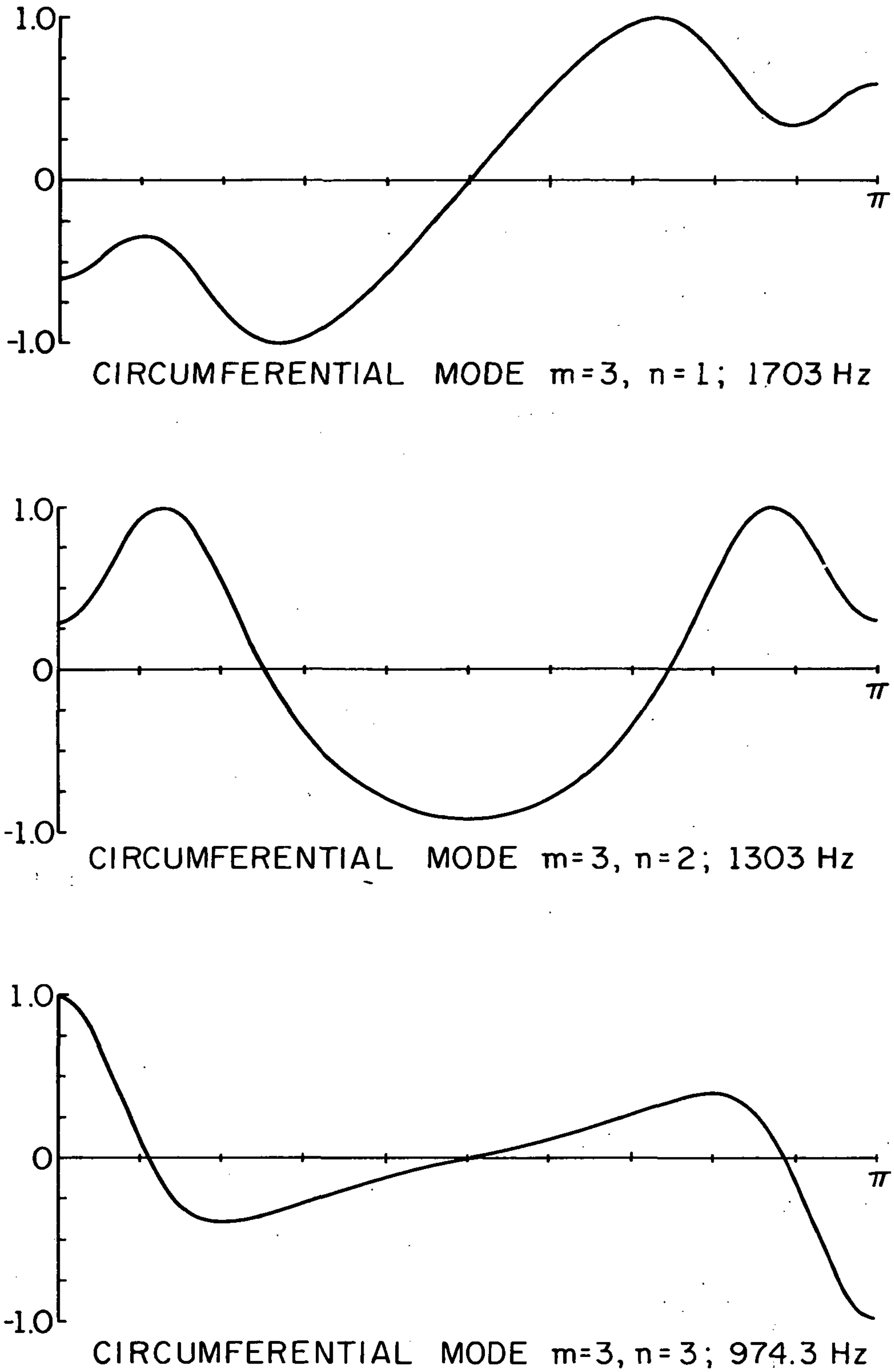


Figure 8. (Continued)

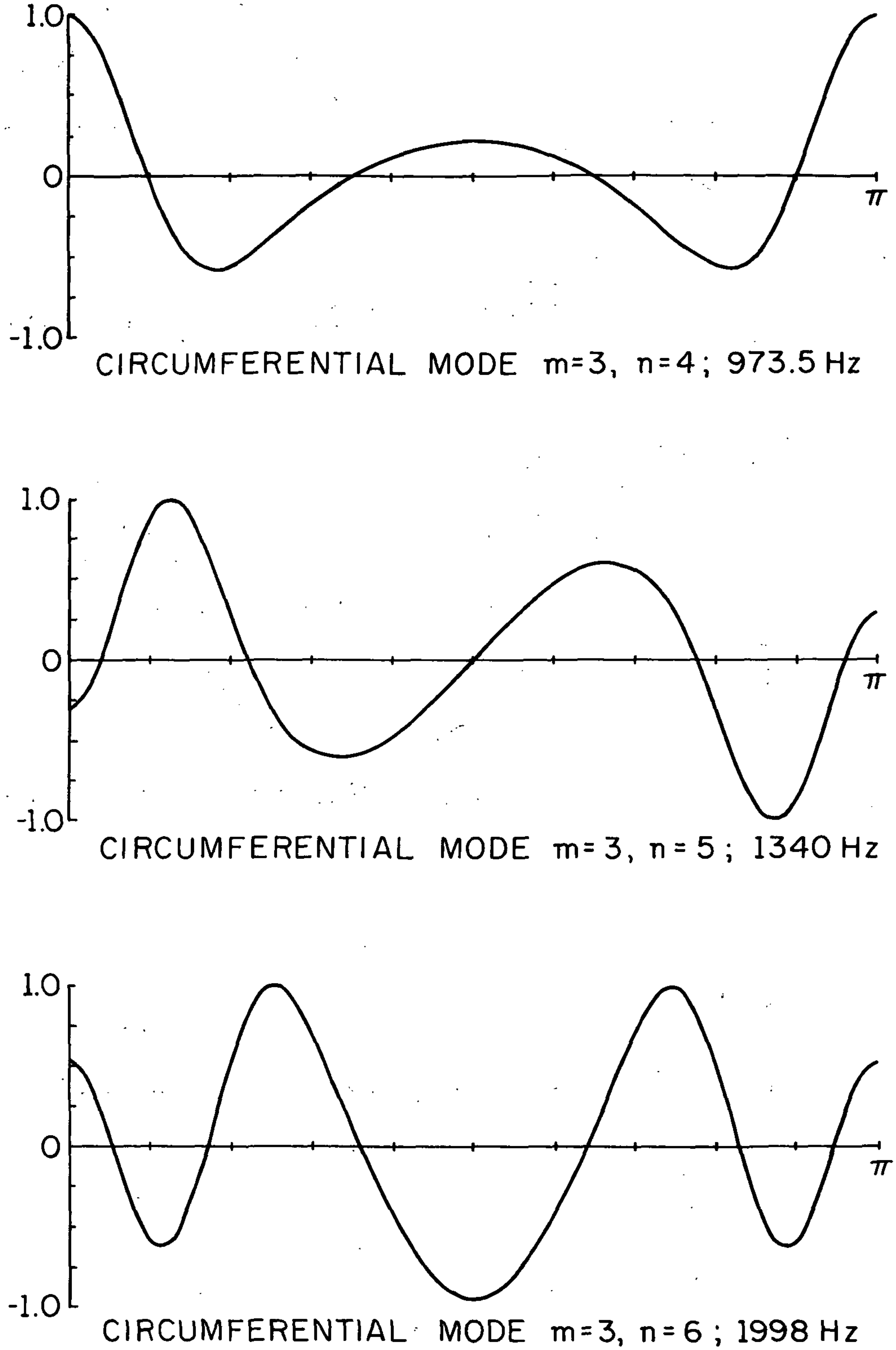


Figure 8. (Continued)

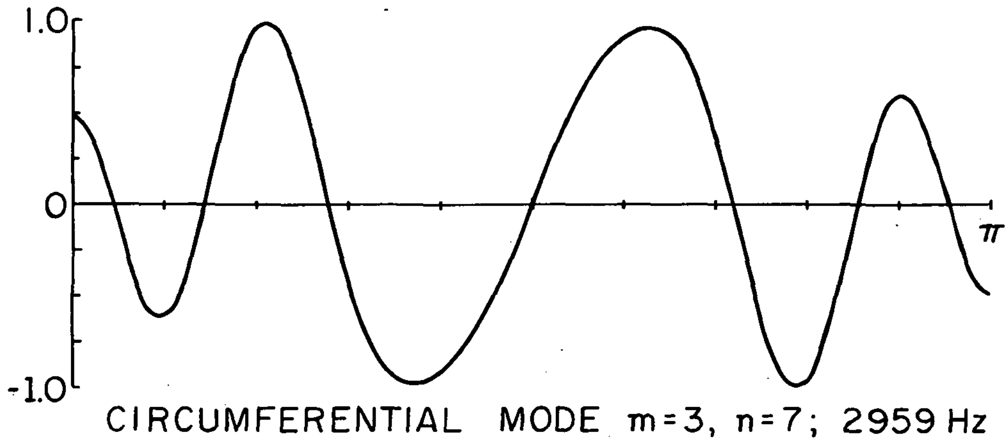


Figure 8. (Continued)

Conclusions

1) There is weak circumferential modal coupling due to the presence of stringers in both circular and noncircular cylinders.

2) The stringers contribute more to the total kinetic energy of the structure than to the strain energy. Therefore, the stringers tend to reduce the natural frequencies.

3) The rings contribute more to the strain energy than to the kinetic energy of the structure. Therefore, the rings tend to increase the natural frequencies. The influence due to the presence of rings is more than the stringers.

4) Reasonably accurate results for ring- and stringer-stiffened shells may be obtained by considering the same number of circumferential mode components as are necessary when the stringers are not present.

5) The reduction-of-frequencies effect due to the presence of stringers is greater on the frequencies associated with the lower circumferential wave numbers.

6) The rate of decay of frequencies due to the presence of stringers is greater for small numbers of stringers and diminishes with the increase of number of stringers.

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APPENDIX A

DERIVATION OF THE COMPATIBILITY RELATIONS

The compatibility relations of the stiffeners were derived based on the assumption that the stiffeners are attached to the shell along a line of attachment of infinitesimal width. This assumption is probably valid when the stiffeners are closely riveted with a single row of rivets.

The displacement vector of any point in the cross-section of the i^{th} stiffener can be written (in vector algebra notation) as

$$\left\{\bar{q}_i\right\} = \left\{\bar{q}_{sci}\right\} + \left\{\bar{\omega}\right\} \times \left\{\bar{R}_{i/sci}\right\}, \qquad i = \begin{cases} r \text{ for ring} \\ s \text{ for stringer} \end{cases} \tag{A1}$$

where $\bar{q}_i$ = The displacement vector of an arbitrary point in the cross-section of the stiffener;

$\bar{q}_{sci}$ = The displacement vector of the shear center of the stiffener;

$\bar{\omega}$ = The angle of rotation vector of the stiffener;

$\bar{R}_{i/sci}$ = The position vector of the point with reference to the shear center.

These vectors may be expanded as follows:

$$\bar{q}_i = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \quad ; \quad \bar{q}_{sci} = \begin{Bmatrix} u_{sci} \\ v_{sci} \\ w_{sci} \end{Bmatrix} \qquad i = r,s$$

$$\bar{\omega} = \begin{Bmatrix} \omega_{xi} \\ \omega_{\theta i} \\ \omega_{zi} \end{Bmatrix}$$

where (see for example, Reference (30))

$$\omega_{xi} = \frac{w_{sci,\theta}}{R_{sci}} - \frac{v_{sci}}{R_{sci}} \qquad\qquad i = r,s$$

$$\omega_{\theta i} = - w_{sci,x}$$

$$\omega_{zi} = \begin{cases} -\dfrac{u_{scr,\theta}}{R_{scr}} & \text{for rings} \\ v_{scs,x} & \text{for stringers} \end{cases}$$

Also, (see Figures 2 and 3)

$$\bar{R}_{r/scr} = \begin{Bmatrix} x' \\ 0 \\ z' \end{Bmatrix} \quad ; \qquad \bar{R}_{s/scs} = \begin{Bmatrix} 0 \\ y' \\ z' \end{Bmatrix}$$

where the vector components x', y', and z' are referenced to the shear center (sc).

Substituting the above equations into equation (A1), the compatibility relations of rings and stringers result.

For the rings:

$$\bar{q}_r \equiv \left\{q_r\right\} = \left\{q_{scr}\right\} + \begin{Bmatrix} -z' w_{scr,x} \\ \dfrac{-x'}{R_{scr}} u_{scr,\theta} - z' \left(\dfrac{w_{scr,\theta}}{R_{scr}} - \dfrac{v_{scr}}{R_{scr}}\right) \\ x' w_{scr,x} \end{Bmatrix} \tag{A2}$$

For the stringers:

$$\bar{q}_s \equiv \{q_s\} = \{q_{scs}\} + \begin{Bmatrix} -z' w_{scs,x} - y' v_{scs,x} \\ -z'\left(\dfrac{w_{scs,\theta}}{R_{scs}} - \dfrac{v_{scs}}{R_{scs}}\right) \\ y'\left(\dfrac{w_{scs,\theta}}{R_{scs}} - \dfrac{v_{scs}}{R_{scs}}\right) \end{Bmatrix} \tag{A3}$$

Another set of compatibility relations were obtained to relate the shear center displacements of the stiffeners to those of the shell at the line of attachment by replacing r by scr, q_{scr} by q, z' by z_{1r}, x' by x_{1r}, and R_{scr} by R in equation (A2) and s by scs, q_{scs} by q, z' by z_{1s}, y' by y_{1s}, and R_{scs} by R in equation (A3).

For the rings:

$$\{q_{scr}\} = \{q\} + \begin{Bmatrix} -z_{1r} w,_x \\ -\dfrac{x_{1r}}{R} u,_\theta - z_{1r}\left(\dfrac{w,_\theta}{R} - \dfrac{v}{R}\right) \\ x_{1r} w,_x \end{Bmatrix} \tag{A4}$$

The cross-section of the ring was assumed to be symmetric with respect to the normal to the shell surface. Hence, the above equation reduces to

$$\{q_{scr}\} = \{q\} + \begin{Bmatrix} -z_{1r} w,_x \\ -z_{1r}\left(\dfrac{w,_\theta}{R} - \dfrac{v}{R}\right) \\ 0 \end{Bmatrix} \tag{A5}$$

APPENDIX B

ENERGY EXPRESSIONS OF RINGS AND STRINGERS

Ring energy functions:

$$U_{\substack{r\\ext}}(u_{scr}, v_{scr}, w_{scr}) = \sum_{k=1}^{K} \frac{E_{rk}}{2} \int_0^{2\pi} \frac{1}{R_{cr}} \Big\langle I_{zzrk} \Big\{ \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big\}^2 u^2_{scr,\theta}$$

$$+ \frac{I_{zzrk}}{R^2_{scr}} u^2_{scr,\theta\theta} + \frac{I_{zzrk}}{R_{scr}} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big\{ u_{scr,\theta} u_{scr,\theta\theta} + u_{scr,\theta\theta} u_{scr,\theta} \Big\}$$

$$- I_{zzrk} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big\{ u_{scr,\theta} w_{scr,x} + w_{scr,x} u_{scr,\theta} \Big\} - \frac{I_{zzrk}}{R_{scr}} \Big\{ u_{scr,\theta\theta} w_{scr,x}$$

$$+ w_{scr,x} u_{scr,\theta\theta} \Big\} + \Big\{ A_{rk} + \frac{I_{xxrk}}{R^2_{scr}} + \frac{2}{R_{scr}} A_{rk} z_{2rk} \Big\} v^2_{scr,\theta}$$

$$+ I_{xxrk} \Big\{ \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big\}^2 v^2_{scr} + \Big\{ A_{rk} z_{2rk} + \frac{I_{xxrk}}{R_{scr}} \Big\} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big(v_{scr} v_{scr,\theta}$$

$$+ v_{scr,\theta} v_{scr} \Big) - \Big\{ \frac{A_{rk} z_{2rk}}{R_{scr}} + \frac{I_{xxrk}}{R^2_{scr}} \Big\} \Big(v_{scr,\theta} w_{scr,\theta\theta} + w_{scr,\theta\theta} v_{scr,\theta} \Big)$$

$$- \Big\{ A_{rk} z_{2rk} + \frac{I_{xxrk}}{R_{scr}} \Big\} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big(v_{scr,\theta} w_{scr,\theta} + w_{scr,\theta} v_{scr,\theta} \Big)$$

$$- \frac{I_{xxrk}}{R_{scr}} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big(v_{scr} w_{scr,\theta\theta} + w_{scr,\theta\theta} v_{scr} \Big)$$

$$- I_{xxrk} \Big\{ \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big\}^2 \Big(v_{scr} w_{scr,\theta} + w_{scr,\theta} v_{scr} \Big) + A_{rk} \Big(1$$

$$+ \frac{z_{2rk}}{R_{scr}} \Big) \Big(v_{scr,\theta} w_{scr} + w_{scr} v_{scr,\theta} \Big) + A_{rk} z_{2rk} \Big(\frac{1}{R_{scr}} \Big)_{,\theta} \Big(v_{scr} w_{scr}$$

$$+ w_{scr}v_{scr}\Big) + \frac{I_{xxrk}}{R^2_{scr}} w^2_{scr,\theta\theta} + I_{xxrk}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2 w^2_{scr,\theta}$$

$$+ \frac{I_{xxrk}}{R_{scr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}\left(w_{scr,\theta\theta}w_{scr,\theta} + w_{scr,\theta}w_{scr,\theta\theta}\right) + A_{rk}w^2_{scr}$$

$$- \frac{A_{rk}z_{2rk}}{R_{scr}}\left(w_{scr,\theta\theta}w_{scr} + w_{scr}w_{scr,\theta\theta}\right)$$

$$- A_{rk}z_{2rk}\left(\frac{1}{R_{scr}}\right)_{,\theta}\left(w_{scr,\theta}w_{scr} + w_{scr}w_{scr,\theta}\right)$$

$$+ I_{zzrk}w^2_{scr,x}\Big\rangle_{x=x_k} d\theta \qquad \text{(B1)}$$

where

$$I_{xxrk} = I_{xxcrk} + A_{rk}z^2_{2rk}$$

$$U_{\substack{r\\ext}}(u,\ v,\ w) = \sum_{k=1}^{K} E_{rk}\int_0^{\pi}\Big\langle \frac{I_{zzrk}}{R_{cr}R^2_{scr}} u^2_{,\theta\theta} - 2\frac{I_{zzrk}}{R_{cr}R_{scr}} u_{,\theta\theta}w_{,x} + \frac{A_{rk}}{R_{cr}} v^2_{,\theta}$$

$$+ \frac{I_{xxrk}}{R_{cr}R^2_{scr}} v^2_{,\theta} - 2\frac{I_{xxrk}}{R_{cr}R^2_{scr}} v_{,\theta}w_{,\theta\theta} + 2\frac{A_{rk}}{R_{cr}} v_{,\theta}w + \frac{I_{xxrk}}{R_{cr}R^2_{scr}} w^2_{,\theta\theta}$$

$$+ \frac{I_{zzrk}}{R_{cr}} w^2_{,x} + \frac{A_{rk}}{R_{cr}} w^2\Big\rangle_{x=x_k} d\theta$$

$$+ E_{rk}\int_0^{\pi}\Big\langle -2\frac{I_{zzrk}z_{1rk}}{R_{cr}R^3_{scr}} u_{,\theta\theta}w_{,x\theta\theta} + \left\{\frac{A_{rk}z^2_{1rk}}{R_{cr}R^2} + \frac{I_{xxrk}z^2_{1rk}}{R_{cr}R^2_{scr}R^2}\right.$$

$$\left. +2\frac{A_{rk}z_{1rk}}{R_{cr}R} + 2\frac{I_{xxrk}z_{1rk}}{R_{cr}R^2_{scr}R}\right\} v^2_{,\theta} - 2\left\{2\frac{I_{xxrk}z_{1rk}}{R_{cr}R^2_{scr}R} + \frac{A_{rk}z_{1rk}}{R_{cr}R} + \frac{A_{rk}z^2_{1rk}}{R_{cr}R^2}\right.$$

$$\left. + \frac{I_{xxrk}z^2_{1rk}}{R_{cr}R^2_{scr}R^2}\right\} v_{,\theta}w_{,\theta\theta} + 2\frac{A_{rk}z_{1rk}}{R_{cr}R} v_{,\theta}w + \left\{\frac{A_{rk}z^2_{1rk}}{R_{cr}R^2} + \frac{I_{xxrk}z^2_{1rk}}{R_{cr}R^2_{scr}R^2}\right.$$

$$+2\,\frac{I_{xxrk}z_{1rk}}{R_{cr}R_{scr}^2R}\Big\}\,w_{,\theta\theta}^2+\frac{I_{zzrk}z_{1rk}^2}{R_{cr}R_{scr}^2}\,w_{,x\theta\theta}^2+\frac{I_{zzrk}z_{1rk}}{R_{cr}R_{scr}}\Big\{w_{,x}w_{,x\theta\theta}$$

$$+w_{,x\theta\theta}w_{,x}\Big\}-\frac{A_{rk}z_{1rk}}{R_{cr}R}\Big\{ww_{,\theta\theta}+w_{,\theta\theta}w\Big\}\Big\rangle_{x=x_k}d\theta\;+E_{rk}\int_0^{\pi}\Big\langle\Big\{2\,\frac{A_{rk}z_{2rk}}{R_{cr}R_{scr}}$$

$$+2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R^2}+4\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}R}\Big\}\,v_{,\theta}^2\;-2\Big\{\frac{A_{rk}z_{2rk}}{R_{cr}R_{scr}}+3\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}R}$$

$$+2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R^2}\Big\}\,v_{,\theta}w_{,\theta\theta}+2\Big\{\frac{A_{rk}z_{2rk}}{R_{cr}R_{scr}}+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}R}\Big\}\,v_{,\theta}w$$

$$+\Big\{2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R^2}+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}R}\Big\}\,w_{,\theta\theta}^2-\Big\{\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}R}$$

$$+\frac{A_{rk}z_{2rk}}{R_{cr}R_{scr}}\Big\}\Big(ww_{,\theta\theta}+w_{,\theta\theta}w\Big)\Big\rangle_{x=x_k}\quad d\theta$$

$$+E_{rk}\int_0^{\pi}\Big\langle\frac{I_{zzrk}}{R_{cr}}\Big\{\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\}^2u_{,\theta}^2+\frac{I_{zzrk}}{R_{cr}R_{scr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\{u_{,\theta}u_{,\theta\theta}+u_{,\theta\theta}u_{,\theta}\Big\}$$

$$-2\,\frac{I_{zzrk}}{R_{cr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}u_{,\theta}w_{,x}+\frac{I_{xxrk}}{R_{cr}}\Big\{\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\}^2v^2$$

$$+\frac{I_{xxrk}}{R_{cr}R_{scr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\{v_{,\theta}v+vv_{,\theta}\Big\}-2\,\frac{I_{xxrk}}{R_{cr}R_{scr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}vw_{,\theta\theta}$$

$$-2\,\frac{I_{xxrk}}{R_{cr}R_{scr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}v_{,\theta}w_{,\theta}-2\,\frac{I_{xxrk}}{R_{cr}}\Big\{\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\}^2vw_{,\theta}$$

$$+\frac{I_{xxrk}}{R_{cr}}\Big\{\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\}^2w_{,\theta}^2+\frac{I_{xxrk}}{R_{cr}R_{scr}}\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\{w_{,\theta}w_{,\theta\theta}+w_{,\theta\theta}w_{,\theta}\Big\}\Big\rangle_{x=x_k}d\theta$$

$$+E_{rk}\int_0^{\pi}\Big\langle-2\,\frac{I_{zzrk}z_{1rk}}{R_{cr}}\Big\{\Big(\frac{1}{R_{scr}}\Big)_{,\theta}\Big\}^2u_{,\theta}w_{,x\theta}$$

$$-2\,\frac{I_{zzrk}\,z_{1rk}}{R_{cr}\,R_{scr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}\left\{u_{,\theta\theta}\,w_{,x\theta}+u_{,\theta}\,w_{,x\theta\theta}\right\}+\Bigg[\frac{A_{rk}\,z_{1rk}^{2}}{R_{cr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^{2}$$

$$+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}^{2}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^{2}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R^{2}}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^{2}$$

$$+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^{2}+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}\,R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Bigg]v^{2}+\Bigg[\frac{A_{rk}\,z_{1rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}^{2}}\left(\frac{1}{R}\right)_{,\theta}$$

$$+\frac{A_{rk}\,z_{1rk}^{2}}{R_{cr}\,R}\left(\frac{1}{R}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}^{2}\,R}\left(\frac{1}{R}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}\,R^{2}}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}\,R}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Bigg]\left(v_{,\theta}\,v+v\,v_{,\theta}\right)-2\Bigg[\frac{A_{rk}\,z_{1rk}^{2}}{R_{cr}\,R}\left(\frac{1}{R}\right)_{,\theta}$$

$$+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}^{2}\,R}\left(\frac{1}{R}\right)_{,\theta}+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}\,R}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}\,R^{2}}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}^{2}}\left(\frac{1}{R}\right)_{,\theta}\Bigg]v\,w_{,\theta\theta}-2\Bigg[\frac{A_{rk}\,z_{1rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}+\frac{A_{rk}\,z_{1rk}^{2}}{R_{cr}\,R}\left(\frac{1}{R}\right)_{,\theta}$$

$$+\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}^{2}}\left(\frac{1}{R}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}^{2}\,R}\left(\frac{1}{R}\right)_{,\theta}+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}\,R^{2}}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}\,R}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Bigg]v_{,\theta}\,w_{,\theta}-2\Bigg[\frac{A_{rk}\,z_{1rk}^{2}}{R_{cr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^{2}$$

$$+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R_{scr}^{2}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^{2}+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^{2}$$

$$+\frac{I_{xxrk}\,z_{1rk}^{2}}{R_{cr}\,R^{2}}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^{2}+2\,\frac{I_{xxrk}\,z_{1rk}}{R_{cr}\,R_{scr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{I_{xxrk}z_{1rk}^2}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]\, vw_{,\theta} + 2\,\frac{A_{rk}z_{1rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta} vw$$

$$+\frac{I_{zzrk}z_{1rk}^2}{R_{cr}}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2 w_{,x\theta}^2 + \frac{I_{zzrk}z_{1rk}^2}{R_{cr}R_{scr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big\{w_{,x\theta}w_{,x\theta\theta}$$

$$+ w_{,x\theta\theta}w_{,x\theta}\Big\} + \frac{I_{zzrk}z_{1rk}}{R_{cr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big\{w_{,x}w_{,x\theta} + w_{,x\theta}w_{,x}\Big\}$$

$$+\Big[\frac{A_{rk}z_{1rk}^2}{R_{cr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 + \frac{I_{xxrk}z_{1rk}^2}{R_{cr}R_{scr}^2}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 + \frac{I_{xxrk}z_{1rk}^2}{R_{cr}R^2}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2$$

$$+2\,\frac{I_{xxrk}z_{1rk}^2}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta} + 2\,\frac{I_{xxrk}z_{1rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{I_{xxrk}z_{1rk}}{R_{cr}R}\left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2\Big]\, w_{,\theta}^2 + \Big[\frac{A_{rk}z_{1rk}^2}{R_{cr}R}\left(\frac{1}{R}\right)_{,\theta} + \frac{I_{xxrk}z_{1rk}^2}{R_{cr}R_{scr}^2 R}\left(\frac{1}{R}\right)_{,\theta}$$

$$+\frac{I_{xxrk}z_{1rk}^2}{R_{cr}R_{scr}R^2}\left(\frac{1}{R_{scr}}\right)_{,\theta} + 2\,\frac{I_{xxrk}z_{1rk}}{R_{cr}R_{scr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+\frac{I_{xxrk}z_{1rk}}{R_{cr}R_{scr}^2}\left(\frac{1}{R}\right)_{,\theta}\Big]\Big\{w_{,\theta}w_{,\theta\theta} + w_{,\theta\theta}w_{,\theta}\Big\} - \frac{A_{rk}z_{1rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}\Big\{ww_{,\theta}$$

$$+ w_{,\theta}w\Big\}\Big\rangle_{x=x_k} d\theta$$

$$+ E_{rk}\int_0^{\pi}\Big\langle\Big[2\,\frac{A_{rk}z_{1rk}^2 z_{2rk}}{R_{cr}R_{scr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 + 2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$+2\,\frac{A_{rk}z_{1rk}^2 z_{2rk}}{R_{cr}R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]\, v^2 + \Big[2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta}$$

$$+2\,\frac{A_{rk}z_{1rk}^2 z_{2rk}}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta} + \frac{A_{rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R_{scr}}\right)_{,\theta} + \frac{A_{rk}z_{1rk}^2 z_{2rk}}{R_{cr}R^2}\left(\frac{1}{R_{scr}}\right)_{,\theta}$$

$$
\begin{aligned}
&+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]\left(vv_{,\theta}+v_{,\theta}v\right)-2\Big[2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta} \\
&+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R^2}\left(\frac{1}{R_{scr}}\right)_{,\theta} \\
&+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta}\Big]vw_{,\theta\theta}-2\Big[2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta} \\
&+2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta}+\frac{A_{rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R^2}\left(\frac{1}{R_{scr}}\right)_{,\theta} \\
&+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]v_{,\theta}w_{,\theta}-2\Big[2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 \\
&+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}+2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]vw_{,\theta} \\
&+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta}+\frac{A_{rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]vw \\
&+\Big[2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}}\left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2+2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta} \\
&+2\,\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R}\right)_{,\theta}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]w_{,\theta}^2+\Big[2\,\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R_{scr}R}\left(\frac{1}{R}\right)_{,\theta} \\
&+\frac{A_{rk}z_{1rk}^2z_{2rk}}{R_{cr}R^2}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta} \\
&+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta}\Big]\left\{w_{,\theta}w_{,\theta\theta}+w_{,\theta\theta}w_{,\theta}\right\}-\Big[\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R_{scr}}\left(\frac{1}{R}\right)_{,\theta} \\
&+\frac{A_{rk}z_{1rk}z_{2rk}}{R_{cr}R}\left(\frac{1}{R_{scr}}\right)_{,\theta}+\frac{A_{rk}z_{2rk}}{R_{cr}}\left(\frac{1}{R_{scr}}\right)_{,\theta}\Big]\left\{ww_{,\theta}+w_{,\theta}w\right\}\Big\rangle_{x=x_k}d\theta
\end{aligned}
\tag{B2}
$$

$$U_{r_{tor}} = \sum_{k=1}^{K} (GJ)_{rk} \int_0^{\pi} \left[\frac{u_{,\theta}^2}{R_{scr}^4} + 2\,\frac{u_{,\theta} w_{,x\theta}}{R_{scr}^3} + \frac{w_{,x\theta}^2}{R_{scr}^2} \right]_{x=x_k} R_{cr}\, d\theta$$

$$+ (GJ)_{rk} \int_0^{\pi} \left[-2\,\frac{z_{1rk}}{R_{scr}^4} u_{,\theta} w_{,x\theta} + \frac{z_{1rk}^2}{R_{scr}^4} w_{,x\theta}^2 - 2\,\frac{z_{1rk}}{R_{scr}^3} w_{,x\theta}^2 \right]_{x=x_k} R_{cr}\, d\theta \tag{B3}$$

$$T_r = \tfrac{1}{2} \sum_{k=1}^{K} 2\rho_{rk} \int_0^{\pi} \Big\langle A_{rk} R_{cr} \mathring{u}^2 + \frac{I_{zzrk} R_{cr}}{R_{scr}^2} \mathring{u}_{,\theta}^2 + A_{rk} R_{cr} \mathring{v}^2 + \frac{I_{xxrk} R_{cr}}{R_{scr}^2} \mathring{v}^2$$

$$-2\,\frac{I_{xxrk} R_{cr}}{R_{scr}^2} \mathring{v}\mathring{w}_{,\theta} + I_{xxrk} R_{cr} \mathring{w}_{,x}^2 + \frac{I_{xxrk} R_{cr}}{R_{scr}^2} \mathring{w}_{,\theta}^2 + A_{rk} R_{cr} \mathring{w}_r^2$$

$$+ I_{zzrk} R_{cr} \mathring{w}_{,x}^2 \Big\rangle_{x=x_k} d\theta$$

$$+2\rho_{rk} \int_0^{\pi} \Big\langle -2\, A_{rk} z_{1rk} R_{cr} \mathring{u}\mathring{w}_{,x} - 2\,\frac{I_{zzrk} z_{1rk} R_{cr}}{R^2} \mathring{u}_{,\theta} \mathring{w}_{,x\theta}$$

$$+ \left[A_{rk} R_{cr} \left(\frac{z_{1rk}^2}{R^2} + 2\,\frac{z_{1rk}}{R} \right) + \frac{I_{xxrk} R_{cr}}{R_{scr}^2} \left(\frac{z_{1rk}^2}{R^2} + 2\,\frac{z_{1rk}}{R} \right) \right] \mathring{v}^2$$

$$-2 \left[\frac{A_{rk} z_{1rk} R_{cr}}{R} \left(1 + \frac{z_{1rk}}{R} \right) + \frac{I_{xxrk} z_{1rk}}{R_{scr}^2 R} \left(2 + \frac{z_{1rk}}{R} \right) R_{cr} \right] \mathring{v}\mathring{w}_{,\theta}$$

$$+ A_{rk} z_{1rk}^2 R_{cr} \mathring{w}_{,x}^2 + \left[\frac{A_{rk} z_{1rk}^2 R_{cr}}{R^2} + 2\,\frac{I_{xxrk} z_{1rk} R_{cr}}{R_{scr}^2 R} + \frac{I_{xxrk} z_{1rk}^2 R_{cr}}{R_{scr}^2 R^2} \right] \mathring{w}_{,\theta}^2$$

$$+ \frac{I_{zzrk} R_{cr} z_{1rk}^2}{R_{scr}^2} \mathring{w}_{,x\theta}^2 \Big\rangle_{x=x_k} d\theta$$

$$+2\rho_{rk} \int_0^{\pi} \Big\langle -2\, A_{rk} z_{2rk} R_{cr} \mathring{u}\mathring{w}_{,x} + 2\, A_{rk} z_{2rk} \left(\frac{1}{R_{scr}} + \frac{z_{1rk}^2}{R^2 R_{scr}} \right.$$

$$+2 \frac{z_{1rk}}{RR_{scr}}\Big) R_{cr} \mathring{v}^2 - 2 A_{rk} z_{2rk} R_{cr} \left(\frac{1}{R_{scr}} + 3 \frac{z_{1rk}}{RR_{scr}} + 2 \frac{z_{1rk}^2}{R^2 R_{scr}} \right) \mathring{v} \mathring{w}_{,\theta}$$

$$+2 A_{rk} z_{1rk} z_{2rk} R_{cr} \mathring{w}_{,x}^2 + 2 A_{rk} z_{2rk} R_{cr} \left(\frac{z_{1rk}}{RR_{scr}} + \frac{z_{1rk}^2}{R^2 R_{scr}} \right) \mathring{w}_{,\theta}^2 \Big\rangle_{x=x_k} d\theta$$

(B4)

Stringer Energy Functions:

$$U_{s_{ext}}(u,v,w) = \sum_{\ell=1}^{L} \frac{E_{s\ell}}{2} \int_0^a \Big\langle A_{s\ell} u_{,x}^2 + I_{zzs\ell} v_{,xx}^2 + I_{yys\ell} w_{,xx}^2 \Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \frac{E_{s\ell}}{2} \int_0^a \Big\langle -2 A_{s\ell} z_{1s\ell} u_{,x} w_{,xx} + \left\{ \frac{I_{zzs\ell} z_{1s\ell}^2}{R^2} + 2 \frac{I_{zzs\ell} z_{1s\ell}}{R} \right\} v_{,xx}^2$$

$$-2 \left\{ \frac{I_{zzs\ell} z_{1s\ell}^2}{R^2} + \frac{I_{zzs\ell} z_{1s\ell}}{R} \right\} v_{,xx} w_{,xx\theta} + A_{s\ell} z_{1s\ell}^2 w_{,xx}^2$$

$$+ \frac{I_{zzs\ell} z_{1s\ell}^2}{R^2} w_{,xx\theta}^2 \Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \frac{E_{s\ell}}{2} \int_0^a \Big\langle -2 A_{s\ell} z_{2s\ell} u_{,x} w_{,xx} + 2 A_{s\ell} z_{1s\ell} z_{2s\ell} w_{,xx}^2 \Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \frac{E_{s\ell}}{2} \int_0^a \Big\langle 2 \left(-A_{s\ell} y_{1s\ell} + \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}}{R} \right) u_{,x} v_{,xx}$$

$$-2 A_{s\ell} z_{2s\ell} \frac{y_{1s\ell} u_{,x} w_{,xx\theta}}{R} + \left(A_{s\ell} y_{1s\ell}^2 + \frac{I_{yys\ell} y_{1s\ell}^2}{R^2} \right.$$

$$\left. -2 \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}^2}{R} \right) v_{,xx}^2 + 2 \left(A_{s\ell} y_{1s\ell} z_{1s\ell} - \frac{I_{yys\ell} y_{1s\ell}}{R} \right.$$

$$\left. - \frac{A_{s\ell} z_{1s\ell} z_{2s\ell} y_{1s\ell}}{R} + A_{s\ell} z_{2s\ell} y_{1s\ell} \right) v_{,xx} w_{,xx} + 2 \left(- \frac{I_{yys\ell} y_{1s\ell}^2}{R^2} \right.$$

$$+ \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}^2}{R}\Big) v,_{xx} w,_{xx\theta} + \frac{I_{yys\ell} y_{1s\ell}^2}{R^2} w,_{xx\theta}^2 + \Big(\frac{I_{yys\ell} y_{1s\ell}}{R}$$

$$+ A_{s\ell} z_{1s\ell} z_{2s\ell} \frac{y_{1s\ell}}{R}\Big)\Big(w,_{xx} w,_{xx\theta} + w,_{xx\theta} w,_{xx}\Big)\Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \frac{E_{s\ell}}{2}\int_0^a \Big\langle -2\Big(A_{s\ell} y_{2s\ell} + \frac{A_{s\ell} z_{1s\ell} y_{2s\ell}}{R}\Big) u,_x v,_{xx}$$

$$+2\,\frac{A_{s\ell} z_{1s\ell} y_{2s\ell}}{R} u,_x w,_{xx\theta} + \Big\{2\,A_{s\ell} y_{1s\ell} y_{2s\ell} + 2\,\frac{A_{s\ell} z_{1s\ell} y_{1s\ell} y_{2s\ell}}{R}\Big\} v,_{xx}^2$$

$$-2\,\frac{A_{s\ell} z_{1s\ell} y_{1s\ell} y_{2s\ell}}{R} v,_{xx} w,_{xx\theta} + 2\Big(A_{s\ell} z_{1s\ell} y_{2s\ell}$$

$$+ \frac{A_{s\ell} z_{1s\ell}^2 y_{2s\ell}}{R}\Big) v,_{xx} w,_{xx} - \frac{A_{s\ell} z_{1s\ell}^2 y_{2s\ell}}{R}\Big(w,_{xx} w,_{xx\theta} + w,_{xx\theta} w,_{xx}\Big)\Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \frac{E_{s\ell}}{2}\int_0^a \Big\langle -2\Big(\frac{I_{yzs\ell} y_{1s\ell}}{R} + \frac{I_{yzs\ell} z_{1s\ell} y_{1s\ell}}{R^2}\Big) v,_{xx}^2 + 2\Big(I_{yzs\ell}$$

$$+ \frac{I_{yzs\ell} z_{1s\ell}}{R}\Big) v,_{xx} w,_{xx} + 2\Big(\frac{I_{yzs\ell} y_{1s\ell}}{R} + 2\,\frac{I_{yzs\ell} z_{1s\ell} y_{1s\ell}}{R^2}\Big) v,_{xx} w,_{xx\theta}$$

$$- \frac{I_{yzs\ell} z_{1s\ell}}{R}\Big(w,_{xx} w,_{xx\theta} + w,_{xx\theta} w,_{xx}\Big) - 2\,\frac{I_{yzs\ell} z_{1s\ell} y_{1s\ell}}{R^2} w,_{xx\theta}^2 \Big\rangle_{\theta=\theta_\ell} dx$$

(B5)

where:

$$I_{zzs\ell} = I_{zzcs\ell} + A_{s\ell} y_{2s\ell}^2$$

$$I_{yys\ell} = I_{yycs\ell} + A_{s\ell} z_{2s\ell}^2$$

$$I_{yzs\ell} = I_{yzcs\ell} + A_{s\ell} y_{2s\ell} z_{2s\ell}$$

$$T_s(\mathring{u},\mathring{v},\mathring{w}) = \tfrac{1}{2}\sum_{\ell=1}^{L} \rho_{s\ell} \int_0^a \Big\langle A_{s\ell} \mathring{u}^2 + I_{zzs\ell} \mathring{v},_x^2 + \Big(A_{s\ell} + \frac{I_{yys\ell}}{R^2} + \frac{I_{zzs\ell}}{R^2}\Big) \mathring{v}^2$$

$$- \frac{2}{R^2}\left(I_{yys\ell} + I_{zzs\ell}\right) \mathring{v}\mathring{w},_{\theta} + I_{yys\ell}\mathring{w},_{x}^{2}$$

$$+ \left(\frac{I_{yys\ell}}{R^2} + \frac{I_{zzs\ell}}{R^2}\right) \mathring{w},_{\theta}^{2} + A_{s\ell}\mathring{w}^{2} \Big\rangle_{\theta=\theta_{\ell}} dx$$

$$+ \rho_{s\ell} \int_{0}^{a} \Big\langle -2\, A_{s\ell} z_{1s\ell} \mathring{u}\mathring{w},_{x} + \left(\frac{I_{zzs\ell} z_{1s\ell}^{2}}{R^2} + 2\, \frac{I_{zzs\ell} z_{1s\ell}}{R}\right) \mathring{v}^{2},_{x}$$

$$+ \left(\frac{A_{s\ell} z_{1s\ell}^{2}}{R^2} + 2\, \frac{A_{s\ell} z_{1s\ell}}{R}\right) \mathring{v}^{2} - 2\left(\frac{I_{zzs\ell} z_{1s\ell}^{2}}{R^2} + \frac{I_{zzs\ell} z_{1s\ell}}{R}\right) \mathring{v},_{x}\mathring{w},_{x\theta}$$

$$-2\left(\frac{A_{s\ell} z_{1s\ell}^{2}}{R^2} + \frac{A_{s\ell} z_{1s\ell}}{R}\right) \mathring{v}\mathring{w},_{\theta} + A_{s\ell} z_{1s\ell}^{2} \mathring{w},_{x}^{2} + \frac{I_{zzs\ell} z_{1s\ell}^{2}}{R^2}\, w,_{x\theta}^{2}$$

$$+ \frac{A_{s\ell} z_{1s\ell}^{2}}{R^2}\, \mathring{w},_{\theta}^{2} \Big\rangle_{\theta=\theta_{\ell}} dx$$

$$+ \rho_{s\ell} \int_{0}^{a} \Big\langle -2\, A_{s\ell} z_{2s\ell} \mathring{u}\mathring{w},_{x} + 2\left(\frac{A_{s\ell} z_{2s\ell}}{R} + \frac{A_{s\ell} z_{1s\ell} z_{2s\ell}}{R^2}\right) \mathring{v}^{2}$$

$$-2\left(\frac{A_{s\ell} z_{2s\ell}}{R} + 2\, \frac{A_{s\ell} z_{1s\ell} z_{2s\ell}}{R^2}\right) \mathring{v}\mathring{w},_{\theta} + 2\, \frac{A_{s\ell} z_{1s\ell} z_{2s\ell}}{R^2}\, \mathring{w},_{\theta}^{2}$$

$$+2\, A_{s\ell} z_{1s\ell} z_{2s\ell} \mathring{w},_{x}^{2} \Big\rangle_{\theta=\theta_{\ell}} dx$$

$$+ \rho_{s\ell} \int_{0}^{a} \Big\langle 2\left(-A_{s\ell} y_{1s\ell} + \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}}{R}\right) \mathring{u}\mathring{v},_{x} - 2\, \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}}{R}\, \mathring{u}\mathring{w},_{x\theta}$$

$$+ \left(A_{s\ell} y_{1s\ell}^{2} + \frac{I_{yys\ell} y_{1s\ell}^{2}}{R^2} - 2\, \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}^{2}}{R}\right) \mathring{v},_{x}^{2} + \frac{A_{s\ell} y_{1s\ell}^{2}}{R^2}\, \mathring{v}^{2}$$

$$+2\left(A_{s\ell} z_{1s\ell} y_{1s\ell} - \frac{I_{yys\ell} y_{1s\ell}}{R} - \frac{A_{s\ell} z_{1s\ell} z_{2s\ell} y_{1s\ell}}{R}\right.$$

$$+ A_{s\ell} z_{2s\ell} y_{1s\ell} \Big) \mathring{v}_{,x} \mathring{w}_{,x} + 2 \left(- \frac{I_{yys\ell} y_{1s\ell}^2}{R^2} + \frac{A_{s\ell} z_{2s\ell} y_{1s\ell}^3}{R} \right) \mathring{v}_{,x} \mathring{w}_{,x\theta}$$

$$-2 \frac{A_{s\ell} y_{1s\ell}^2}{R^2} \mathring{v}\mathring{w}_{,\theta} - 2 \frac{A_{s\ell} y_{1s\ell}}{R} \mathring{v}\mathring{w} + \frac{I_{yys\ell} y_{1s\ell}^2}{R^2} \mathring{w}_{,x\theta}^2 + \Big(\frac{I_{yys\ell} y_{1s\ell}}{R}$$

$$+ \frac{A_{s\ell} z_{1s\ell} z_{2s\ell} y_{1s\ell}}{R} \Big) \left(\mathring{w}_{,x} \mathring{w}_{,x\theta} + \mathring{w}_{,x\theta} \mathring{w}_{,x} \right) + \frac{A_{s\ell} y_{1s\ell}^2}{R^2} \mathring{w}_{,\theta}^2$$

$$+ \frac{A_{s\ell} y_{1s\ell}}{R} \left(\mathring{w}_{,\theta} \mathring{w} + \mathring{w}\mathring{w}_{,\theta} \right) \Big\rangle_{\theta=\theta_\ell} dx$$

$$+ \rho_{s\ell} \int_0^a \Big\langle -2 \left(A_{s\ell} y_{2s\ell} + \frac{A_{s\ell} z_{1s\ell} y_{2s\ell}}{R} \right) \mathring{u}\mathring{v}_{,x} + 2 \frac{A_{s\ell} z_{1s\ell} y_{2s\ell}}{R} \mathring{u}\mathring{w}_{,x\theta}$$

$$+ \Big(2 A_{s\ell} y_{1s\ell} y_{2s\ell} + 2 \frac{A_{s\ell} z_{1s\ell} y_{1s\ell} y_{2s\ell}}{R} - 2 \frac{I_{yzs\ell} y_{1s\ell}}{R}$$

$$-2 \frac{I_{yzs\ell} y_{1s\ell} z_{1s\ell}}{R^2} \Big) \mathring{v}_{,x}^2 + 2 \frac{A_{s\ell} y_{1s\ell} y_{2s\ell}}{R^2} \mathring{v}^2 + 2 \Big(A_{s\ell} z_{1s\ell} y_{2s\ell}$$

$$+ \frac{A_{s\ell} z_{1s\ell}^2 y_{2s\ell}}{R} + I_{yzs\ell} + \frac{I_{yzs\ell} z_{1s\ell}}{R} \Big) \mathring{v}_{,x} \mathring{w}_{,x} + 2 \Big(- \frac{A_{s\ell} z_{1s\ell} y_{1s\ell} y_{2s\ell}}{R}$$

$$+ \frac{I_{yzs\ell} y_{1s\ell}}{R} + 2 \frac{I_{yzs\ell} y_{1s\ell} z_{1s\ell}}{R^2} \Big) \mathring{v}_{,x} \mathring{w}_{,x\theta} - 2 \frac{A_{s\ell} y_{2s\ell}}{R} \mathring{v}\mathring{w}$$

$$-2 \frac{A_{s\ell} y_{1s\ell} y_{2s\ell}}{R^2} \mathring{v}\mathring{w}_{,\theta} - \Big(\frac{A_{s\ell} z_{1s\ell}^2 y_{2s\ell}}{R}$$

$$+ \frac{I_{yzs\ell} z_{1s\ell}}{R} \Big) \left(\mathring{w}_{,x} \mathring{w}_{,x\theta} + \mathring{w}_{,x\theta} \mathring{w}_{,x} \right) - 2 \frac{I_{yzs\ell} y_{1s\ell} z_{1s\ell}}{R^2} \mathring{w}_{,x\theta}^2$$

$$+ 2 \frac{A_{s\ell} y_{1s\ell} y_{2s\ell}}{R^2} \mathring{w}_{,\theta}^2 + \frac{A_{s\ell} y_{2s\ell}}{R} \left(\mathring{w}\mathring{w}_{,\theta} + \mathring{w}_{,\theta} \mathring{w} \right) \Big\rangle_{\theta=\theta_\ell} dx \qquad \text{(B6)}$$

APPENDIX C

MATRIX ELEMENTS AND INTEGRALS

The matrix elements of Equation (32) and the circumferential and longitudinal integrals involved in these elements are presented in this appendix. The closed-form expressions for the longitudinal integrals were obtained with the help of a table of formulas for integrals derived by Felgar (31). The circumferential integrals were evaluated numerically using the 8-point Gaussian quadrature method with four subintervals.

The elements of the mass and stiffness matrices of a ring- and stringer-stiffened noncircular shell may be written as follows:

Contribution of the Noncircular Shell

$$A_{mn,\bar{m}\bar{n}} = S_1 IS1_1 IX_1 + (S_2 IS1_2 + S_3 IS1_3)\, n\bar{n}\, IX_2$$

$$D_{mn,\bar{m}\bar{n}} = S_4\, \bar{n} IS1_5 IX_3 - S_2 n IS1_6 IX_2$$

$$E_{mn,\bar{m}\bar{n}} = S_4\, IS1_5 IX_3 - S_5\, IS1_5 IX_1 + S_3\, n\bar{n} IS1_7\, IX_2$$

$$B_{mn,\bar{m}\bar{n}} = S_1 n\bar{n} IS1_8 IX_5 + (S_2 IS1_9 + S_6\, IS1_2) IX_2 + S_5\, IS2_1 IX_5$$

$$F_{mn,\bar{m}\bar{n}} = S_1 n IS1_8 IX_5 - S_7 n IS1_8 IX_4 + S_6 \bar{n} IS1_2 IX_2 - S_7 IS2_3 IX_4$$
$$-S_5 IS2_2\, (1-\bar{n}^2) IX_5 + S_5 \bar{n} IS2_1 IX_5$$

$$C_{mn,\bar{m}\bar{n}} = (S_1 IS1_8 + S_5 IS1_4) IX_5 + S_5 \left\{ IS1_1 IX_1 + (n^2\bar{n}^2 - n^2 - \bar{n}^2) IS1_4 IX_5 \right\}$$
$$-S_7 IS1_8 (\bar{n}^2 IX_3 + n^2 IX_4) + S_6 n\bar{n} IS1_2 IX_2 + S_5 \left\{ n\bar{n} IS2_1 + (n\bar{n}^2 - n) IS2_2 + (n^2\bar{n} - \bar{n}) IS2_4 \right\} IX_5 - S_7 (\bar{n} IS2_5\, IX_3 + n IS2_3\, IX_4)$$

$$N_{mn,\bar{m}\bar{n}} = 2\, \rho_c h\, IS1_1\, IX_2$$

$$Q_{mn,\bar{m}\bar{n}} = 2\, \rho_c h\, IS1_9\, IX_5$$

$$S_{mn,\bar{m}\bar{n}} = 2\,\rho_c h\, IS1_1\, IX_5 \tag{C1}$$

where $IS1_1$ to $IS2_3$ are circumferential integrals, IX_1 to IX_5 are longitudinal integrals, and S_1 to S_8 are constants defined in Appendix D.

The circumferential integrals are defined as follows:

$$IS1_1 = \int_0^{\pi} R \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IS1_2 = \int_0^{\pi} \frac{1}{R} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS1_3 = \int_0^{\pi} \frac{1}{R^3} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS1_4 = \int_0^{\pi} \frac{1}{R^3} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IS1_5 = \int_0^{\pi} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IS1_6 = \int_0^{\pi} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS1_7 = \int_0^{\pi} \frac{1}{R^2} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS1_8 = \int_0^{\pi} \frac{1}{R} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IS1_9 = \int_0^{\pi} R \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS2_1 = \int_0^{\pi} \frac{1}{R} \left\{ \left(\frac{1}{R}\right)_{,\theta} \right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IS2_2 = \int_0^{\pi} \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IS2_3 = \int_0^{\pi} \left(\frac{1}{R}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IS2_4 = \int_0^{\pi} \frac{1}{R^2}\left(\frac{1}{R}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IS2_5 = \int_0^{\pi} \left(\frac{1}{R}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta \qquad \text{(C2)}$$

The matrix elements of the antisymmetric mode equations for the shell are identical in form to the above equations and are obtained by interchanging Sine terms with Cosine terms and vice versa. Furthermore, $\left(\frac{1}{R}\right)_{,\theta}$ must be replaced by $-\left(\frac{1}{R}\right)_{,\theta}$. It was found that if the cross-section of the shell is symmetric with respect to the horizontal axis of the shell, there is no coupling between the even and odd terms of n and $\bar{n}$. Thus, in the analysis of elliptical cylinders, two computations must be made in both the cases of symmetric and antisymmetric modes (with respect to the vertical axis); one with all even terms of n and $\bar{n}$, and the other with all odd terms of n and $\bar{n}$.

The longitudinal integrals may be defined by a general axial mode function

$$\Phi_m$$

as follows:

$$IX_1 = \int_0^{a} \Phi''_m \Phi''_{\bar{m}} \, dx$$

$$IX_2 = \int_0^{a} \Phi'_m \Phi'_{\bar{m}} \, dx$$

$$IX_3 = \int_0^a \Phi''_m \Phi_{\bar{m}} \, dx$$

$$IX_4 = \int_0^a \Phi_m \Phi''_{\bar{m}} \, dx$$

$$IX_5 = \int_0^a \Phi_m \Phi_{\bar{m}} \, dx \qquad \text{(C3)}$$

Substituting Equations (28b to 28e) into the above equations, the longitudinal integrals for various boundary conditions may be written as:

For freely supported cylinders:

$$\left.\begin{aligned} IX_1 &= \frac{m^4 \pi^4}{2a^3} \\ IX_2 &= -IX_3 = -IX_4 = \frac{m^2 \pi^2}{2a} \\ IX_5 &= \frac{a}{2} \end{aligned}\right\} \quad \text{For } m = \bar{m}$$

$$IX_1 \text{ to } IX_5 = 0 \quad \Big\} \quad \text{For } m \neq \bar{m} \qquad \text{(C4a)}$$

For clamped-free cylinders:

$$IX_1 = \begin{cases} \beta_m^4 a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

$$IX_2 = \begin{cases} \alpha_m \beta_m (2 + \alpha_m \beta_m a) & m = \bar{m} \\ \dfrac{4\beta_m \beta_{\bar{m}}}{\beta_m^4 - \beta_{\bar{m}}^4} \left[(-1)^{m+\bar{m}} (\alpha_{\bar{m}} \beta_m^3 - \alpha_m \beta_{\bar{m}}^3) - \beta_{\bar{m}} \beta_m (\alpha_m \beta_m - \alpha_{\bar{m}} \beta_{\bar{m}}) \right] & m \neq \bar{m} \end{cases}$$

$$IX_3 = \begin{cases} \alpha_m \beta_m (2 - \alpha_m \beta_m a) & m = \bar{m} \\ \dfrac{4\beta_m^2 (\alpha_{\bar{m}} \beta_{\bar{m}} - \alpha_m \beta_m)}{\beta_{\bar{m}}^4 - \beta_m^4} \left[(-1)^{m+\bar{m}} \beta_m^2 + \beta_{\bar{m}}^2 \right] & m \neq \bar{m} \end{cases}$$

$$IX_4 = \begin{cases} \alpha_m\beta_m(2 - \alpha_m\beta_m a) & m = \bar{m} \\ \dfrac{4\beta_{\bar{m}}^2(\alpha_m\beta_m - \alpha_{\bar{m}}\beta_{\bar{m}})}{\beta_m^4 - \beta_{\bar{m}}^4}\left[(-1)^{m+\bar{m}}\beta_{\bar{m}}^2 + \beta_m^2\right] & m \neq \bar{m} \end{cases}$$

$$IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases} \qquad \text{(C4b)}$$

For clamped-clamped cylinders:

$$IX_1 = \begin{cases} \beta_m^4 a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases}$$

$$IX_2 = -IX_3 = -IX_4 = \begin{cases} \alpha_m\beta_m(\alpha_m\beta_m a - 2) & m = \bar{m} \\ \dfrac{4\beta_{\bar{m}}^2\beta_m^2(\alpha_{\bar{m}}\beta_{\bar{m}} - \alpha_m\beta_m)}{\beta_m^4 - \beta_{\bar{m}}^4}\left[(-1)^{m+\bar{m}}+1\right] & m \neq \bar{m} \end{cases}$$

$$IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases} \qquad \text{(C4c)}$$

For free-free cylinders:

<u>m = 0</u>

$$\left.\begin{array}{l} IX_1 = IX_2 = IX_3 = IX_4 = 0 \\ IX_5 = a \end{array}\right\} \quad \bar{m} = 0$$

$$IX_1 = IX_2 = IX_3 = IX_4 = IX_5 = 0 \quad \bar{m} = 1$$

$$IX_1 = IX_2 = IX_3 = IX_5 = 0 \quad \bar{m} \geq 2$$

$$IX_4 = \begin{cases} 4\alpha_{\bar{m}-1}\beta_{\bar{m}-1} & \bar{m} \geq 2 \text{ even only} \\ 0 & \bar{m} > 2 \text{ odd only} \end{cases}$$

<u>m = 1</u>

$$IX_1 = IX_2 = IX_3 = IX_4 = IX_5 = 0 \quad \bar{m} = 0$$

$$\left.\begin{array}{l} IX_1 = IX_3 = IX_4 = 0 \\ IX_2 = \dfrac{1}{a}; \quad IX_5 = \dfrac{a}{12} \end{array}\right\} \quad \bar{m} = 1$$

$$IX_1 = IX_3 = IX_5 = 0 \quad \bar{m} \geq 2$$

$$IX_2 = \begin{cases} -\dfrac{4}{a} & \bar{m} > 2 \text{ odd only} \\ 0 & \bar{m} \geq 2 \text{ even only} \end{cases}$$

$$IX_4 = \begin{cases} \dfrac{4}{a} - 2\alpha_{\bar{m}-1}\beta_{\bar{m}-1} & \bar{m} > 2 \text{ odd only} \\ 0 & \bar{m} \geq 2 \text{ even only} \end{cases}$$

<u>$m \geq 2$</u>

$$\left.\begin{array}{l} IX_1 = IX_2 = IX_4 = IX_5 = 0 \\ IX_3 = \begin{cases} 4\beta_{m-1}\alpha_{m-1} & m \geq 2 \text{ even only} \\ 0 & m > 2 \text{ odd only} \end{cases} \end{array}\right\} \bar{m} = 0$$

$$\left.\begin{array}{l} IX_1 = IX_4 = IX_5 = 0 \\ IX_2 = \begin{cases} -\dfrac{4}{a} & m > 2 \text{ odd only} \\ 0 & m \geq 2 \text{ even only} \end{cases} \\ IX_3 = \begin{cases} \dfrac{4}{a} - 2\alpha_{m-1}\beta_{m-1} & m > 2 \text{ odd only} \\ 0 & m \geq 2 \text{ even only} \end{cases} \end{array}\right\} \bar{m} = 1$$

$$\left.\begin{array}{l} IX_1 = \begin{cases} \beta_{m-1}^4 a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases} \\ IX_2 = \begin{cases} \alpha_{m-1}\beta_{m-1}(\alpha_{m-1}\beta_{m-1}a + 6) & m = \bar{m} \\ \dfrac{4\beta_{m-1}\beta_{\bar{m}-1}(\alpha_{m-1}\beta_{\bar{m}-1}^3 - \alpha_{\bar{m}-1}\beta_{m-1}^3)}{\beta_{\bar{m}-1}^4 - \beta_{m-1}^4}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \\ IX_3 = \begin{cases} \alpha_{m-1}\beta_{m-1}(2 - \alpha_{m-1}\beta_{m-1}a) & m = \bar{m} \\ \dfrac{4\beta_{m-1}^4(\alpha_{\bar{m}-1}\beta_{\bar{m}-1} - \alpha_{m-1}\beta_{m-1})}{\beta_{\bar{m}-1}^4 - \beta_{m-1}^4}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \end{array}\right\} \bar{m} \geq 2$$

$$IX_4 = \begin{cases} \alpha_{m-1}\beta_{m-1}(2 - \alpha_{m-1}\beta_{m-1}a) & m = \bar{m} \\ \dfrac{4\beta_{\bar{m}-1}^4(\alpha_{m-1}\beta_{m-1} - \alpha_{\bar{m}-1}\beta_{\bar{m}-1})}{\beta_{m-1}^4 - \beta_{\bar{m}-1}^4}\left[1 + (-1)^{m+\bar{m}-2}\right] & m \neq \bar{m} \end{cases} \quad m \geq 2$$

$$IX_5 = \begin{cases} a & m = \bar{m} \\ 0 & m \neq \bar{m} \end{cases} \tag{C4d}$$

The longitudinal integrals in Equations (C4a, C4c, and C4d) vanish if $m \pm \bar{m}$ is odd and are nonzero if $m + \bar{m}$ is even.

Contributions of Stringers

$$A_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left(SS_1 \cos n\theta \cos \bar{n}\theta\right)_{\theta=\theta_\ell}$$

$$D_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle\left(-SS_{11} + \frac{SS_{12}}{R} - SS_{20} - \frac{SS_{21}}{R}\right) \cos n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_1$$

$$E_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \Big\langle - SS_4 \cos n\theta \cos \bar{n}\theta - SS_9 \cos n\theta \cos \bar{n}\theta$$
$$+ SS_{12}\, \bar{n} \frac{\cos n\theta \sin \bar{n}\theta}{R} - \frac{SS_{21}}{R} \bar{n} \cos n\theta \sin \bar{n}\theta \Big\rangle_{\theta=\theta_\ell} IX_1$$

$$B_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \Big\langle\Big(SS_2 + \frac{SS_5}{R^2} + \frac{SS_8}{R} + SS_{13} + \frac{SS_{14}}{R^2} - \frac{SS_{15}}{R} + SS_{22}$$
$$+ \frac{SS_{23}}{R} - \frac{SS_{26}}{R} - \frac{SS_{27}}{R^2}\Big) \sin n\theta \sin \bar{n}\theta \Big\rangle_{\theta=\theta_\ell} IX_1$$
$$+ (GJ)_{s\ell} \frac{\sin n\theta \sin \bar{n}\theta}{R^2} IX_2 \Big|_{\theta=\theta_\ell}$$

$$F_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \Bigg\langle \Bigg\langle \left(\frac{SS_5}{R^2} + \frac{SS_7}{R}\right) \bar{n} \sin n\theta \sin \bar{n}\theta + \left(SS_{16} - \frac{SS_{17}}{R} - \frac{SS_{18}}{R}\right.$$

$$\left. + SS_{12}\right) \sin n\theta \cos \bar{n}\theta + \left(\frac{SS_{14}}{R^2} - \frac{SS_{19}}{R}\right) \bar{n} \sin n\theta \sin \bar{n}\theta$$

$$+ \frac{SS_{24}}{R} \bar{n} \sin n\theta \sin \bar{n}\theta + \left(SS_{21} + \frac{SS_{25}}{R}\right) \sin n\theta \cos \bar{n}\theta$$

$$+ \left(SS_{28} + \frac{SS_{29}}{R}\right) \sin n\theta \cos \bar{n}\theta$$

$$-\left(\frac{SS_{30}}{R} + \frac{SS_{27}}{R^2}\right) \bar{n} \sin n\theta \sin \bar{n}\theta \Bigg\rangle_{\theta=\theta_\ell} IX_1$$

$$+ \Bigg\langle \left(GJ\right)_{s\ell} \bar{n} \frac{\sin n\theta \sin \bar{n}\theta}{R^2} \Bigg\rangle_{\theta=\theta_\ell} IX_2$$

$$C_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \Bigg\langle SS_3 \cos n\theta \cos \bar{n}\theta + SS_6 \cos n\theta \cos \bar{n}\theta + \frac{SS_5}{R^2} n\bar{n} \sin n\theta \sin \bar{n}\theta$$

$$+ SS_{10} \cos n\theta \cos \bar{n}\theta + \frac{SS_{14}}{R^2} n\bar{n} \sin n\theta \sin \bar{n}\theta - \left(\frac{SS_{17}}{R}\right.$$

$$\left. + \frac{SS_{18}}{R}\right)\left(\bar{n} \cos n\theta \sin \bar{n}\theta + n \sin n\theta \cos \bar{n}\theta\right)$$

$$+ \frac{SS_{25}}{R}\left(\bar{n} \cos n\theta \sin \bar{n}\theta + n \sin n\theta \cos \bar{n}\theta\right)$$

$$+ \frac{SS_{29}}{R}\left(\bar{n} \cos n\theta \sin \bar{n}\theta + n \sin n\theta \cos \bar{n}\theta\right)$$

$$- \frac{SS_{27}}{R^2} n\bar{n} \sin n\theta \sin \bar{n}\theta \Bigg\rangle_{\theta=\theta_\ell} IX_1$$

$$+ \Bigg\langle (GJ)_{s\ell} n\bar{n} \frac{\sin n\theta \sin \bar{n}\theta}{R^2} \Bigg\rangle_{\theta=\theta_\ell} IX_2$$

$$N_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle T_1 \cos n\theta \cos \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2$$

$$NN_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle \left(-T_{16} + \frac{T_{17}}{R} - T_{29} + \frac{T_{30}}{R} \right) \cos n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2$$

$$P_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle - \left(T_6 + T_{13} \right) \cos n\theta \cos \bar{n}\theta + \left(\frac{T_{17}}{R} \right.\right.$$

$$\left.\left. - \frac{T_{30}}{R} \right) \bar{n} \cos n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2$$

$$Q_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle \left(T_2 + \frac{T_7}{R^2} + \frac{T_8}{R} + T_{18} + \frac{T_{19}}{R^2} - \frac{T_{20}}{R} + T_{31} + \frac{T_{32}}{R} - \frac{T_{33}}{R} \right.\right.$$

$$\left.\left. - \frac{T_{34}}{R^2} \right) \sin n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2 + \left\langle \left(T_1 + \frac{T_4}{R^2} + \frac{T_9}{R^2} + \frac{T_6}{R} \right.\right.$$

$$\left.\left. + \frac{T_{13}}{R} + \frac{T_{14}}{R^2} + \frac{T_{18}}{R^2} + \frac{T_{31}}{R^2} \right) \sin n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_5$$

$$R_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle \left(\frac{T_5}{R^3} + \frac{T_{12}}{R^2} + \frac{T_6}{R} + \frac{T_{13}}{R} + \frac{T_{15}}{R^2} + \frac{T_{25}}{R^2} \right.\right.$$

$$\left.\left. + \frac{T_{39}}{R^2} \right) \bar{n} \sin n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_5 + \left\langle \left(\frac{T_{11}}{R^2} + \frac{T_8}{R} + \frac{T_{24}}{R^2} \right.\right.$$

$$\left.\left. - \frac{T_{20}}{R} + \frac{T_{32}}{R} - \frac{T_{33}}{R} - \frac{T_{38}}{R^2} \right) \bar{n} \sin n\theta \sin \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2$$

$$+ \left\langle \left(T_{21} - \frac{T_{22}}{R} - \frac{T_{23}}{R} + T_{17} + T_{30} + \frac{T_{35}}{R} + T_{36} \right.\right.$$

$$\left.\left. + \frac{T_{37}}{R} \right) \sin n\theta \cos \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2 - \left\langle \left(\frac{T_{16}}{R} \right.\right.$$

$$\left.\left. + \frac{T_{29}}{R} \right) \sin n\theta \cos \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_5$$

$$S_{mn,\bar{m}\bar{n}} = \sum_{\ell=1}^{L} \left\langle \left(T_3 + T_9 + T_{14} \right) \cos n\theta \cos \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_2 + \left\langle \left(T_4 + T_9 \right. \right.$$

$$\left. + T_{14} + T_{18} + T_{31} \right) \frac{n\bar{n}}{R^2} \sin n\theta \sin \bar{n}\theta \Big\rangle_{\theta=\theta_\ell} IX_5$$

$$+ \left\langle T_1 \cos n\theta \cos \bar{n}\theta \right\rangle_{\theta=\theta_\ell} IX_5 + \left\langle \left(T_7 + T_{19} \right. \right.$$

$$\left. - T_{34} \right) \frac{n\bar{n}}{R^2} \sin n\theta \sin \bar{n}\theta \Big\rangle_{\theta=\theta_\ell} IX_2 - \left\langle \left(\frac{T_{28}}{R} \right. \right.$$

$$\left. + \frac{T_{42}}{R} \right) \left(\bar{n} \cos n\theta \sin \bar{n}\theta + n \sin n\theta \cos \bar{n}\theta \right) \Big\rangle_{\theta=\theta_\ell} IX_5$$

$$+ \left\langle \left(T_{40} + T_{41} - T_{26} - T_{27} \right) \left(\bar{n} \frac{\cos n\theta \sin \bar{n}\theta}{R} \right. \right.$$

$$\left. \left. + n \frac{\sin n\theta \cos \bar{n}\theta}{R} \right) \right\rangle_{\theta=\theta_\ell} IX_2$$

(C5)

where SS_1 to SS_{30} and T_1 to T_{42} are constants defined in Appendix D.

Contributions of Rings

$$A_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} C_1 n^2 \bar{n}^2 \, IR1_1 X_1 + C_1 n\bar{n} IR4_1 X_1 + C_1 (IR4_2 \, n\bar{n}^2 + n^2 \bar{n} \, IR4_3) X_1$$

$$+ C_{21} \, n\bar{n} \, IR1_5 X_1$$

$$E_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} C_1 n^2 IR1_2 X_1 - C_4 n^2 \bar{n}^2 \, IR1_1 X_1 + C_1 \, n IR4_4 X_1 - C_4 \, n\bar{n} \, IR4_1 X_1$$

$$- C_4 n^2 \bar{n} \, IR4_3 X_1 - C_4 n\bar{n}^2 IR4_2 X_1 + C_{21} n\bar{n} \, IR1_6 X_1$$

$$- C_{25} \, n\bar{n} \, IR1_5 X_1$$

$$B_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} \left(C_2 n\bar{n} \, IR1_3 + C_3 n\bar{n} \, IR1_1 \right) X_2 + \left(C_5 \, IR2_1 + C_6 \, IR2_2 + C_7 \, IR2_3 \right.$$

$$+C_6\, IR2_4\Big)n\bar{n}X_2 + \Big(C_{11}\, IR1_2 + C_{12}IR3_1 + C_{13}IR3_2\Big)\, n\bar{n}X_2$$

$$+ C_3\, IR4_1X_2 + C_3\Big(IR4_2\bar{n} + IR4_3n\Big)\, X_2 + \Big(C_5\, IR5_{18}$$

$$+ C_6\Big\{IR5_1 + IR5_2\Big\} + C_8\Big\{IR5_3 + IR5_4\Big\} + C_{18}\, IR5_5\Big)X_2$$

$$+ \Big\{C_9\Big(nIR5_7 + \bar{n}IR5_6\Big) + C_{19}\Big(nIR5_9 + \bar{n}IR5_8\Big) + C_5\Big(\bar{n}IR5_{10}$$

$$+ nIR5_{11}\Big) + C_6\Big(\bar{n}IR5_{12} + nIR5_{13}\Big) + C_6\Big(nIR5_{15} + \bar{n}IR5_{14}\Big)$$

$$+ C_8\Big(nIR5_{17} + \bar{n}IR5_{16}\Big)\Big\}X_2 + \Big\{C_{12}\, IR6_1 + C_{17}\, IR6_2$$

$$+ C_{12}\, IR6_3\Big\}X_2 + \Big\{C_{17}\Big(\bar{n}IR6_4 + nIR6_5 + \bar{n}IR6_{10} + nIR6_{11}\Big)$$

$$+ C_{12}\Big(\bar{n}IR6_6 + nIR6_7\Big) + C_{14}\Big(\bar{n}IR4_4 + nIR4_5\Big) + C_{20}\Big(\bar{n}IR6_8$$

$$+ nIR6_9\Big)\Big\}\, X_2$$

$$F_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K}\Big\langle C_3n\bar{n}^2\, IR1_1 + C_2\, nIR1_3 + \Big(C_8IR2_4 + C_9IR2_3 + C_5IR2_1$$

$$+ C_6IR2_2\Big)\, n\bar{n}^2 + C_9IR2_3n + \Big(C_{14}IR1_2 + C_{15}IR3_2$$

$$+ C_{12}IR3_1\Big)n\bar{n}^2 + \Big(C_{14}IR1_2 + C_{16}IR3_2\Big)n + C_3\bar{n}^2\, IR4_2$$

$$+ C_3IR4_3n\bar{n} + C_3\bar{n}IR4_1 + C_5IR5_{10} + C_6\Big(IR5_{12} + IR5_{14}\Big)$$

$$+ C_8IR5_{16} + C_{19}IR5_8 + \Big\{C_9\, IR5_7 + C_5\, IR5_{11} + C_{19}IR5_9$$

$$+ C_6\Big(IR5_{13} + IR5_{15}\Big) + C_8IR5_{17}\Big\}\, n\bar{n} + \Big\{C_5IR5_{18} + C_6\Big(IR5_1$$

$$+ IR5_2\Big) + C_8\Big(IR5_3 + IR5_4\Big) + C_{18}IR5_5\Big\}\, \bar{n} + C_9IR5_6$$

$$+ \Big\{C_{12}IR6_6 + C_{20}IR6_8 + C_{16}\Big(IR6_{10} + IR6_4\Big)\Big\}\, \bar{n}^2 + \Big\{C_{17}\Big(IR6_5$$

$$+ IR6_{11}\Big) + C_{12}IR6_7 + C_{14}IR4_5 + C_{20}IR6_9\Big\}\, n\bar{n} + \Big\{C_{12}IR6_1$$

$$+ C_{17}IR6_2 + C_{12}IR6_3\Big\}\, \bar{n} + C_{16}\Big(IR6_4 + IR6_{10}\Big)$$

$$+ C_{14}IR4_4\Big\rangle X_2$$

$$C_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} C_3 n^2 \bar{n}^2 IR1_1 X_2 + C_1 IR1_3 X_1 + C_2 IR1_3 X_2 + \Big(C_5 IR2_1$$

$$+ C_6 IR2_2 + C_8 IR2_4\Big) n^2 \bar{n}^2 X_2 + C_{10} IR1_1 n^2 \bar{n}^2 X_1 - C_4 (n^2$$

$$+ \bar{n}^2) IR1_3 X_1 + C_9 (n^2 + \bar{n}^2) IR2_3 X_2 + \Big(C_{12} IR3_1 + C_{17} IR3_2\Big) n^2 \bar{n}^2 X_2$$

$$+ \Big(C_{16} IR3_2 + C_{14} IR1_2\Big) n^2 + \bar{n}^2 X_2 + C_3 n\bar{n} IR4_1 X_2 + C_3 \Big(IR4_2 n\bar{n}^2$$

$$+ n^2 \bar{n}\, IR4_3\Big) X_2 + C_{10} IR4_1 n\bar{n} X_1 + C_{10}\Big(n\bar{n}^2 IR4_2 + n^2 \bar{n}\, IR4_3\Big) X_1$$

$$- C_4 \Big(\bar{n} IR4_5 + n IR4_4\Big) X_1 + \Big\{C_5 IR5_{18} + C_6 \Big(IR5_1 + IR5_2\Big)$$

$$+ C_{18} IR5_5 + C_8 \Big(IR5_4 + IR5_3\Big)\Big\} n\bar{n} X_2 + \Big\{C_5 \Big(n\bar{n}^2 IR5_{10}$$

$$+ n^2 \bar{n}\, IR5_{11}\Big) + C_6 \Big(n\bar{n}^2 IR5_{12} + n^2 \bar{n}\, IR5_{13} + n\bar{n}^2 IR5_{14}$$

$$+ n^2 \bar{n}\, IR5_{15}\Big) + C_8 \Big(n\bar{n}^2 IR5_{16} + n\bar{n}^2 IR5_{17}\Big) + C_{19}\Big(n\bar{n}^2 IR5_8$$

$$+ n^2 \bar{n}\, IR5_9\Big)\Big\} X_2 + C_9 \Big(\bar{n} IR5_7 + n IR5_6\Big) X_2 + \Big\{C_{12}\Big(IR6_1$$

$$+ IR6_3\Big) + C_{17} IR6_2\Big\} X_2 n\bar{n} + \Big\{C_{12}\Big(n\bar{n}^2 IR6_6 + n^2 \bar{n}\, IR6_7\Big)$$

$$+ C_{20}\Big(n\bar{n}^2 IR6_8 + n^2 \bar{n}\, IR6_9\Big) + C_{16}\Big(n\bar{n}^2 IR6_{10} + n^2 \bar{n}\, IR6_{11}$$

$$+ n\bar{n}^2 IR6_4 + n^2 \bar{n}\, IR6_5\Big)\Big\} X_2 + \Big\{C_{16}\Big(\bar{n} IR6_5 + n IR6_4 + \bar{n} IR6_{11}$$

$$+ n IR6_{10}\Big) + C_{14}\Big(\bar{n} IR4_5 + n IR4_4\Big)\Big\} X_2 + \Big(C_{21} IR1_7 + C_{26} IR1_5$$

$$- C_{27} IR1_6\Big) n\bar{n} X_1$$

$$N_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} \Big(C_{22} IR1_4 + C_{23} IR1_7 n\bar{n}\Big) X_1$$

$$P_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} \Big(-C_{28} IR1_4 - C_{29} n\bar{n} IR1_7 - C_{35} IR1_4\Big) X_1$$

$$Q_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} \Big(C_{22} IR1_8 + C_{24} IR1_7 + C_{30} IR2_7 + C_{28} IR2_8 + C_{32} IR2_9$$

$$+ C_{33}IR2_{10} + C_{35}IS1_{8} + C_{37}IS1_{7} + C_{18}IS1_{2}\Big)X_2$$

$$R_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} \Big(2C_{24}\bar{n}IR1_7 + 2C_{31}\bar{n}IR2_8 + 2C_{30}\bar{n}IR2_7 + 2C_{33}\bar{n}IR2_{10}$$

$$+ 2C_{32}\bar{n}IR2_9 + C_{35}IS1_8\bar{n} + C_{39}IS1_2\bar{n} + 2C_{37}IS1_7\bar{n}\Big) X_2$$

$$S_{mn,\bar{m}\bar{n}} = \sum_{k=1}^{K} C_{24}IR1_4X_1 + C_{24}n\bar{n}IR1_7X_2 + C_{22}IR1_4X_2 + C_{23}IR1_4X_1$$

$$+ C_{30}IR1_4X_1 + \Big\{C_{30}n\bar{n}IR2_7 + C_{33}n\bar{n}IR2_{10} + C_{32}n\bar{n}IR2_9\Big\} X_2$$

$$+ C_{34}n\bar{n}IR1_7X_1 + C_{40}\,IR1_4X_1 + \Big(C_{40}\,IS1_2 + C_{37}IS1_7\Big)n\bar{n}X_2 \tag{C6}$$

where $IR1_1$ to $IR6_{11}$ are circumferential integrals and $X_1 = \Phi'_m\Phi'_{\bar{m}}\big|_{x=x_k}$ and $X_2 = \Phi_m\Phi_{\bar{m}}\big|_{x=x_k}$ and C_1 to C_{40} are constants defined in Appendix D.

The circumferential integrals are defined as follows:

$$IR1_1 = \int_0^{\pi} \frac{1}{R_{cr}R_{scr}^2} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR1_2 = \int_0^{\pi} \frac{1}{R_{cr}R_{scr}} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR1_3 = \int_0^{\pi} \frac{1}{R_{cr}} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR1_4 = \int_0^{\pi} R_{cr} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR1_5 = \int_0^{\pi} \frac{1}{R_{cr}R_{scr}^3} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR1_6 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR1_7 = \int_0^{\pi} \frac{1}{R_{cr}} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR1_8 = \int_0^{\pi} R_{cr} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_1 = \int_0^{\pi} \frac{1}{R_{cr} R^2} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR2_2 = \int_0^{\pi} \frac{1}{R_{cr} R^2 R_{scr}^2} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR2_3 = \int_0^{\pi} \frac{1}{R_{cr} R} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR2_4 = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}^2} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR2_5 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}^2} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_6 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_7 = \int_0^{\pi} \frac{R_{cr}}{R^2} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_8 = \int_0^{\pi} \frac{R_{cr}}{R} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_9 = \int_0^{\pi} \frac{1}{R_{cr}R^2} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR2_{10} = \int_0^{\pi} \frac{1}{R_{cr}R} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR3_1 = \int_0^{\pi} \frac{1}{R_{cr}R^2 R_{scr}} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR3_2 = \int_0^{\pi} \frac{1}{R_{cr}R R_{scr}} \cos n\theta \cos \bar{n}\theta \, d\theta$$

$$IR4_1 = \int_0^{\pi} \frac{1}{R_{cr}} \left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR4_2 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR4_3 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR4_4 = \int_0^{\pi} \frac{1}{R_{cr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR4_5 = \int_0^{\pi} \frac{1}{R_{cr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_1 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}^2} \left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_2 = \int_0^{\pi} \frac{1}{R_{cr}R^2} \left\{\left(\frac{1}{R_{scr}}\right)_{,\theta}\right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_3 = \int_0^{\pi} \frac{1}{R_{cr} R} \left\{ \left(\frac{1}{R_{scr}} \right)_{,\theta} \right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_4 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left(\frac{1}{R} \right)_{,\theta} \left(\frac{1}{R_{scr}} \right)_{,\theta} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_5 = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}} \left(\frac{1}{R} \right)_{,\theta} \left(\frac{1}{R_{scr}} \right)_{,\theta} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_6 = \int_0^{\pi} \frac{1}{R_{cr}} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_7 = \int_0^{\pi} \frac{1}{R_{cr}} \left(\frac{1}{R} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_8 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}^2} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_9 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}^2} \left(\frac{1}{R} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_{10} = \int_0^{\pi} \frac{1}{R_{cr} R} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_{11} = \int_0^{\pi} \frac{1}{R_{cr} R} \left(\frac{1}{R} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_{12} = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}^2} \left(\frac{1}{R} \right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_{13} = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}^2} \left(\frac{1}{R} \right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_{14} = \int_0^{\pi} \frac{1}{R_{cr} R^2 R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_{15} = \int_0^{\pi} \frac{1}{R_{cr} R^2 R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_{16} = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR5_{17} = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR5_{18} = \int_0^{\pi} \frac{1}{R_{cr}} \left\{\left(\frac{1}{R}\right)_{,\theta}\right\}^2 \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_1 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left\{\left(\frac{1}{R}\right)_{,\theta}\right\} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_2 = \int_0^{\pi} \frac{1}{R_{cr}} \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_3 = \int_0^{\pi} \frac{1}{R_{cr} R} \left(\frac{1}{R}\right)_{,\theta} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_4 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left(\frac{1}{R}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR6_5 = \int_0^{\pi} \frac{1}{R_{cr} R_{scr}} \left(\frac{1}{R}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_6 = \int_0^{\pi} \frac{1}{R_{cr} R R_{scr}} \left(\frac{1}{R}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR6_7 = \int_0^{\pi} \frac{1}{R_{cr}RR_{scr}} \left(\frac{1}{R}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_8 = \int_0^{\pi} \frac{1}{R_{cr}R^2} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR6_9 = \int_0^{\pi} \frac{1}{R_{cr}R^2} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta$$

$$IR6_{10} = \int_0^{\pi} \frac{1}{R_{cr}R} \left(\frac{1}{R_{scr}}\right)_{,\theta} \sin n\theta \cos \bar{n}\theta \, d\theta$$

$$IR6_{11} = \int_0^{\pi} \frac{1}{R_{cr}R} \left(\frac{1}{R_{scr}}\right)_{,\theta} \cos n\theta \sin \bar{n}\theta \, d\theta \tag{C7}$$

The quantities X_1 and X_2 for different boundary conditions are defined as follows:

For freely supported cylinders:

$$X_1 = 2 \frac{m\bar{m}\pi^2}{a^2} \cos \frac{m\pi x_k}{a} \cos \frac{\bar{m}\pi x_k}{a}$$

$$X_2 = 2 \sin \frac{m\pi x_k}{a} \sin \frac{\bar{m}\pi x_k}{a} \tag{C8a}$$

For clamped-free cylinders:

$$X_1 = \beta_m \beta_{\bar{m}} \left\{\sinh \beta_m x_k + \sin \beta_m x_k - \alpha_m \left(\cosh \beta_m x_k - \cos \beta_m x_k\right)\right\}\left\{\sinh \beta_{\bar{m}} x_k + \sin \beta_{\bar{m}} x_k - \alpha_{\bar{m}} \left(\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k\right)\right\}$$

$$X_2 = \left\{\cosh \beta_m x_k - \cos \beta_m x_k - \alpha_m \left(\sinh \beta_m x_k - \sin \beta_m x_k\right)\right\}\left\{\cosh \beta_{\bar{m}} x_k - \cos \beta_{\bar{m}} x_k - \alpha_{\bar{m}} \left(\sinh \beta_{\bar{m}} x_k - \sin \beta_{\bar{m}} x_k\right)\right\} \tag{C8b}$$

For clamped-clamped cylinders:

Expression is same as clamped-free but α_m's and β_m's are different.

For free-free cylinders:

<u>m = 0</u>

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= 1 \end{aligned}\right\} \qquad \bar{m} = 0$$

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= \left(\frac{x_k}{a} - \frac{1}{2}\right) \end{aligned}\right\} \qquad \bar{m} = 1$$

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= \cosh \beta_{\bar{m}-1} x_k + \cos \beta_{\bar{m}-1} x_k - \alpha_{\bar{m}-1}\Big(\sinh \beta_{\bar{m}-1} x_k \\ &\quad + \sin \beta_{\bar{m}-1} x_k\Big) \end{aligned}\right\} \qquad \bar{m} \geq 2$$

<u>m = 1</u>

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= \frac{x_k}{a} - \frac{1}{2} \end{aligned}\right\} \qquad \bar{m} = 0$$

$$\left.\begin{aligned} X_1 &= \frac{1}{a^2} \\ X_2 &= \frac{x_k^2}{a^2} + \frac{1}{4} - \frac{x_k}{a} \end{aligned}\right\} \qquad \bar{m} = 1$$

$$\left.\begin{aligned} X_1 &= \frac{\beta_{\bar{m}-1}}{a}\Big\{\sinh \beta_{\bar{m}-1} x_k - \sin \beta_{\bar{m}-1} x_k \\ &\quad - \alpha_{\bar{m}-1}\left(\cosh \beta_{\bar{m}-1} x_k + \cos \beta_{\bar{m}-1} x_k\right)\Big\} \\ X_2 &= \left\{\frac{x_k}{a} - \frac{1}{2}\right\}\Big\{\cosh \beta_{\bar{m}-1} x_k + \cos \beta_{\bar{m}-1} x_k \\ &\quad - \alpha_{\bar{m}-1}\left(\sinh \beta_{\bar{m}-1} x_k + \sin \beta_{\bar{m}-1} x_k\right)\Big\} \end{aligned}\right\} \qquad \bar{m} \geq 2$$

$\underline{m \geq 2}$

$$\left.\begin{aligned} X_1 &= 0 \\ X_2 &= \cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k \\ &\quad - \alpha_{m-1}\left(\sinh \beta_{m-1}x_k + \sin \beta_{m-1}x_k\right) \end{aligned}\right\} \quad \bar{m} = 0$$

$$\left.\begin{aligned} X_1 &= \frac{\beta_{m-1}}{a}\Big\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k \\ &\quad - \alpha_{m-1}\left(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k\right)\Big\} \\ X_2 &= \left\{\frac{x_k}{a} - \frac{1}{2}\right\}\Big\{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k \\ &\quad - \alpha_{m-1}\left(\sinh \beta_{m-1}x_k + \sin \beta_{m-1}x_k\right)\Big\} \end{aligned}\right\} \quad \bar{m} = 1$$

$$\left.\begin{aligned} X_1 &= \beta_{m-1}\beta_{\bar{m}-1}\Big\{\sinh \beta_{m-1}x_k - \sin \beta_{m-1}x_k \\ &\quad - \alpha_{m-1}\left(\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k\right)\Big\}\Big\{\sinh \beta_{\bar{m}-1}x_k \\ &\quad - \sin \beta_{\bar{m}-1}x_k - \alpha_{\bar{m}-1}\left(\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k\right)\Big\} \\ X_2 &= \Big\{\cosh \beta_{m-1}x_k + \cos \beta_{m-1}x_k - \alpha_{m-1}\Big(\sinh \beta_{m-1}x_k \\ &\quad + \sin \beta_{m-1}x_k\Big)\Big\}\Big\{\cosh \beta_{\bar{m}-1}x_k + \cos \beta_{\bar{m}-1}x_k \\ &\quad - \alpha_{\bar{m}-1}\left(\sinh \beta_{\bar{m}-1}x_k + \sin \beta_{\bar{m}-1}x_k\right)\Big\} \end{aligned}\right\} \quad \bar{m} \geq 2$$

(C8c)

APPENDIX D

CONSTANTS OF MATRIX ELEMENTS

This appendix contains the constants used in equations (C1, C5, and C6) of Appendix C. These are various combinations of the stiffner properties given in the list of symbols.

$$S_1 = \frac{24D}{h^2}$$

$$S_2 = \frac{12D(1-\nu)}{h^2}$$

$$S_3 = D(1-\nu)$$

$$S_4 = \frac{24D\nu}{h^2}$$

$$S_5 = 2D$$

$$S_6 = 3D(1-\nu)$$

$$S_7 = 2D\nu$$

$$S_8 = 4D(1-\nu)$$

$$SS_1 = E_{s\ell}A_{s\ell}$$

$$SS_2 = E_{s\ell}I_{zzs\ell}$$

$$SS_3 = E_{s\ell}I_{yys\ell}$$

$$SS_4 = E_{s\ell}A_{s\ell}z_{1s\ell}$$

$$SS_5 = E_{s\ell}I_{zzs\ell}z^2_{1s\ell}$$

$$SS_6 = 2\,E_{s\ell}I_{zzs\ell}z_{1s\ell}$$

$$SS_7 = E_{s\ell}I_{zzs\ell}z_{1s\ell}$$

$$SS_8 = E_{s\ell}A_{s\ell}z_{1s\ell}^2$$

$$SS_9 = E_{s\ell}A_{s\ell}z_{2s\ell}$$

$$SS_{10} = 2\ E_{s\ell}A_{s\ell}z_{1s\ell}z_{2s\ell}$$

$$SS_{11} = E_{s\ell}A_{s\ell}y_{1s\ell}$$

$$SS_{12} = E_{s\ell}A_{s\ell}z_{2s\ell}y_{1s\ell}$$

$$SS_{13} = E_{s\ell}A_{s\ell}y_{1s\ell}^2$$

$$SS_{14} = E_{s\ell}I_{yys\ell}y_{1s\ell}^2$$

$$SS_{15} = 2\ E_{s\ell}A_{s\ell}z_{2s\ell}y_{1s\ell}^2$$

$$SS_{16} = E_{s\ell}A_{s\ell}y_{1s\ell}z_{1s\ell}$$

$$SS_{17} = E_{s\ell}I_{yys\ell}y_{1s\ell}$$

$$SS_{18} = E_{s\ell}A_{s\ell}z_{2s\ell}z_{1s\ell}y_{1s\ell}$$

$$SS_{19} = E_{s\ell}A_{s\ell}z_{2s\ell}y_{1s\ell}^2$$

$$SS_{20} = E_{s\ell}A_{s\ell}y_{2s\ell}$$

$$SS_{21} = E_{s\ell}A_{s\ell}y_{2s\ell}z_{1s\ell}$$

$$SS_{22} = 2\ E_{s\ell}A_{s\ell}y_{2s\ell}y_{1s\ell}$$

$$SS_{23} = 2\ E_{s\ell}A_{s\ell}y_{2s\ell}y_{1s\ell}z_{1s\ell}$$

$$SS_{24} = E_{s\ell}A_{s\ell}y_{2s\ell}y_{1s\ell}z_{1s\ell}$$

$$SS_{25} = E_{s\ell}A_{s\ell}y_{2s\ell}z_{1s\ell}^2$$

$$SS_{26} = 2\ E_{s\ell}I_{yzs\ell}y_{1s\ell}$$

$$SS_{27} = 2\ E_{s\ell}I_{yzs\ell}y_{1s\ell}z_{1s\ell}$$

$$SS_{28} = E_{s\ell}I_{yzs\ell}$$

$$SS_{29} = E_{s\ell}I_{yzs\ell}z_{1s\ell}$$

$SS_{30} = E_{s\ell} I_{yzs\ell} y_{1s\ell}$

$T_1 = \rho_{s\ell} A_{s\ell}$

$T_2 = \rho_{s\ell} I_{zzs\ell}$

$T_3 = \rho_{s\ell} I_{yys\ell}$

$T_4 = \rho_{s\ell}(I_{zzs\ell} + I_{yys\ell})$

$T_5 = 2\ T_4$

$T_6 = 2\ \rho_{s\ell} A_{s\ell} z_{1s\ell}$

$T_7 = \rho_{s\ell} I_{zzs\ell} z_{1s\ell}^2$

$T_8 = 2\ \rho_{s\ell} I_{zzs\ell} z_{1s\ell}$

$T_9 = \rho_{s\ell} A_{s\ell} z_{1s\ell}^2$

$T_{10} = \rho_{s\ell} I_{zzs\ell} z_{1s\ell}$

$T_{11} = 2\ \rho_{s\ell} I_{zzs\ell} z_{1s\ell}^2$

$T_{12} = 2\ \rho_{s\ell} A_{s\ell} z_{1s\ell}^2$

$T_{13} = 2\ \rho_{s\ell} A_{s\ell} z_{2s\ell}$

$T_{14} = 2\ \rho_{s\ell} A_{s\ell} z_{2s\ell} z_{1s\ell}$

$T_{15} = 4\ \rho_{s\ell} A_{s\ell} z_{2s\ell} z_{1s\ell}$

$T_{16} = 2\ \rho_{s\ell} A_{s\ell} y_{1s\ell}$

$T_{17} = 2\ \rho_{s\ell} A_{s\ell} z_{2s\ell} y_{1s\ell}$

$T_{18} = \rho_{s\ell} A_{s\ell} y_{1s\ell}^2$

$T_{19} = \rho_{s\ell} I_{yys\ell} y_{1s\ell}^2$

$T_{20} = 2\ \rho_{s\ell} A_{s\ell} z_{2s\ell} y_{1s\ell}^2$

$T_{21} = 2\ \rho_{s\ell} A_{s\ell} y_{1s\ell} z_{1s\ell}$

$$T_{22} = 2\,\rho_{s\ell} I_{yys\ell}\, y_{1s\ell}$$

$$T_{23} = 2\,\rho_{s\ell} A_{s\ell}\, z_{2s\ell}\, z_{1s\ell}\, y_{1s\ell}$$

$$T_{24} = 2\,\rho_{s\ell} I_{yys\ell}\, y_{1s\ell}^{2}$$

$$T_{25} = 2\,\rho_{s\ell} A_{s\ell}\, y_{1s\ell}^{2}$$

$$T_{26} = \rho_{s\ell} I_{yys\ell}\, y_{1s\ell}$$

$$T_{27} = \rho_{s\ell} A_{s\ell}\, z_{2s\ell}\, z_{1s\ell}\, y_{1s\ell}$$

$$T_{28} = \rho_{s\ell} A_{s\ell}\, y_{1s\ell}$$

$$T_{29} = 2\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}$$

$$T_{30} = 2\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, z_{1s\ell}$$

$$T_{31} = 2\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, y_{1s\ell}$$

$$T_{32} = 2\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, y_{1s\ell}\, z_{1s\ell}$$

$$T_{33} = 2\,\rho_{s\ell} I_{yzs\ell}\, y_{1s\ell}$$

$$T_{34} = 2\,\rho_{s\ell} I_{yzs\ell}\, y_{1s\ell}\, z_{1s\ell}$$

$$T_{35} = 2\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, z_{1s\ell}^{2}$$

$$T_{36} = 2\,\rho_{s\ell} I_{yzs\ell}$$

$$T_{37} = 2\,\rho_{s\ell} I_{yzs\ell}\, z_{1s\ell}$$

$$T_{38} = 4\,\rho_{s\ell} I_{yzs\ell}\, y_{1s\ell}\, z_{1s\ell}$$

$$T_{39} = 4\,\rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, y_{1s\ell}$$

$$T_{40} = \rho_{s\ell} A_{s\ell}\, y_{2s\ell}\, z_{1s\ell}^{2}$$

$$T_{41} = \rho_{s\ell} I_{yzs\ell}\, z_{1s\ell}$$

$$T_{42} = \rho_{s\ell} A_{s\ell}\, y_{2s\ell}$$

$$C_{1} = 2\, E_{rk} I_{zzrk}$$

$C_2 = 2\, E_{rk}A_{rk}$

$C_3 = 2\, E_{rk}I_{xxrk}$

$C_4 = 2\, E_{rk}I_{zzrk}z_{1rk}$

$C_5 = 2\, E_{rk}A_{rk}z^2_{1rk}$

$C_6 = 2\, E_{rk}I_{xxrk}z^2_{1rk}$

$C_7 = 4\, E_{rk}A_{rk}z_{1rk}$

$C_8 = 4\, E_{rk}I_{xxrk}z_{1rk}$

$C_9 = 2\, E_{rk}A_{rk}z_{1rk}$

$C_{10} = 2\, E_{rk}I_{zzrk}z^3_{1rk}$

$C_{11} = 4\, E_{rk}A_{rk}z_{2rk}$

$C_{12} = 4\, E_{rk}A_{rk}z^2_{1rk}z_{2rk}$

$C_{13} = 8\, E_{rk}A_{rk}z_{1rk}z_{2rk}$

$C_{14} = 2\, E_{rk}A_{rk}z_{2rk}$

$C_{15} = 6\, E_{rk}A_{rk}z_{1rk}z_{2rk}$

$C_{16} = 2\, E_{rk}A_{rk}z_{1rk}z_{2rk}$

$C_{17} = 4\, E_{rk}A_{rk}z_{1rk}z_{2rk}$

$C_{18} = 4\, E_{rk}I_{xxrk}z^3_{1rk}$

$C_{19} = 2\, E_{rk}I_{xxrk}z_{1rk}$

$C_{20} = 2\, E_{rk}A_{rk}z^3_{1rk}z_{2rk}$

$C_{21} = 2\, (GJ)_{rk}$

$C_{22} = 2\, \rho_{rk}A_{rk}$

$C_{23} = 2\, \rho_{rk}I_{zzrk}$

$C_{24} = 2 \rho_{rk} I_{xxrk}$

$C_{25} = 2 \ (GJ)_{rk} z_{1rk}$

$C_{26} = 2 \ (GJ)_{rk} z^2_{1rk}$

$C_{27} = 4 \ (GJ)_{rk} z_{1rk}$

$C_{28} = 4 \ \rho_{rk} A_{rk} z_{1rk}$

$C_{29} = 4 \ \rho_{rk} I_{zzrk} z_{1rk}$

$C_{30} = 2 \ \rho_{rk} A_{rk} z^2_{1rk}$

$C_{31} = 2 \ \rho_{rk} A_{rk} z_{1rk}$

$C_{32} = 2 \ \rho_{rk} I_{xxrk} z^2_{1rk}$

$C_{33} = 4 \ \rho_{rk} I_{xxrk} z_{1rk}$

$C_{34} = 2 \ \rho_{rk} I_{zzrk} z^2_{1rk}$

$C_{35} = 4 \ \rho_{rk} A_{rk} z_{2rk}$

$C_{36} = 2 \ \rho_{rk} A_{rk} z_{2rk}$

$C_{37} = 4 \ \rho_{rk} A_{rk} z^2_{1rk} z_{2rk}$

$C_{38} = 8 \ \rho_{rk} A_{rk} z_{1rk} z_{2rk}$

$C_{39} = 12 \ \rho_{rk} A_{rk} z_{1rk} z_{2rk}$

$C_{40} = 4 \ \rho_{rk} A_{rk} z_{1rk} z_{2rk}$

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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