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A SUCCESSIVE OVERRELAXATION ITERATIVE TECHNIQUE FOR AN ADAPTIVE EQUALIZER

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16. Abstract

This study deals with an adaptive strategy for the equalization of pulse-amplitude-modulated signals in the presence of intersymbol interference and additive noise. The successive overrelaxation iterative technique is used as the algorithm for the iterative adjustment of the equalizer coefficients during a training period for the minimization of the mean square error. With 2-cyclic and non-negative Jacobi matrices substantial improvement was demonstrated in the rate of convergence over the commonly used gradient techniques. The Jacobi theorems were also extended to non-positive Jacobi matrices. Numerical examples strongly indicate that the improvements obtained for the special cases are possible for general channel characteristics. The technique was analytically demonstrated to decrease the mean square error (norm) at each iteration for a large range of parameter values for light or moderate intersymbol interference and for small intervals for general channels. Again, numerical examples indicate that the norm-decreasing property is valid for a much larger parameter range for all types of intersymbol interference. Analytically, convergence of the relaxation algorithm was proven in a noisy environment and the coefficient variance was demonstrated to be bounded. Numerical simulations conducted indicate that the relaxation algorithm consistently converged much faster than the gradient techniques; hence, it requires much less time in the training period than do the gradients.

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INTRODUCTION

This study is concerned with an adaptive strategy for a receiver to improve reception of pulse-amplitude-modulated signals (PAM) in the presence of intersymbol interference and additive noise.

As a result of imperfect channel characteristics, the pulses, representing transmitted information, arrive at the receiver smeared out in time. If the rate of transmission is high enough, successive pulses overlap, causing what is known as intersymbol interference. The number of detectable amplitude levels and the rate of transmission have very often been limited by this intersymbol interference rather than by the noise.

Currently used adaptive equalizers for the minimization of the mean-square error commonly use a fixed step-size gradient procedure. Because of the slow rate of convergence, various other techniques have been investigated yielding limited success—considerable improvement for moderate intersymbol interference, but little improvement for large intersymbol interference. To improve the rate of convergence, the successive overrelaxation algorithm is proposed in this study for the iterative adjustment of the equalizer parameters. The resultant convergence rates provide considerable improvement for all types of intersymbol interference. In noisy environments, the resultant variance is of the same order as the variance for the fixed step-size gradient. The overall net result is that the successive overrelaxation method provides vast improvement over existing adaptive equalization schemes in the rate of convergence with no degradation in noisy environment.

Historical Background

The data transmission system considered in this study is shown in figure 1. The message $\{a_n\}$, a random sequence of real numbers belonging to a discrete set of possible amplitude levels, is amplitude modulated. The transmitted signal is given by

$$s(t) = \sum_n a_n g(t - nT) \tag{1}$$

where $g(t)$ is the impulse response of the modulator. The modulation function $g(t)$ is such that $g(kt) = 0$ for all $k \neq 0$, with T the time separation between samples. Thus if the channel were totally

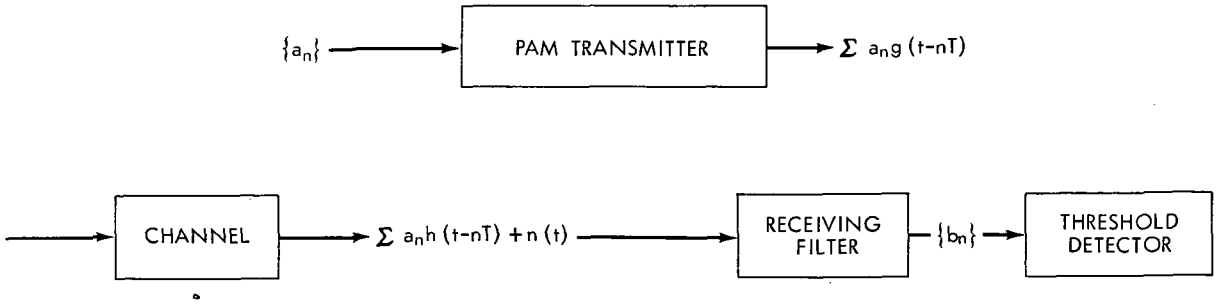

Figure 1.—Data transmission system.

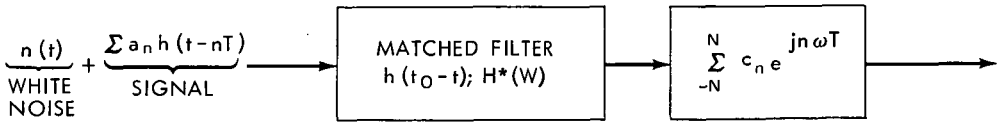

Figure 2.—Optimum receiving filter for PAM.

distortionless, the received signal at sample time jT would be equal to the j data bit transmitted; i.e.,

$$s(jT) = a_j$$

The channel is represented by a time-invariant linear system for which the response to $g(t)$ is $h(t)$ and an additive noise source. The channel output, i.e., the received signal, has the form

$$\sum_n a_n h(t - nT) + n(t) \tag{2}$$

where the channel noise $n(t)$ is a white gaussian random process with an autocorrelation function given by

$$R(\tau) = \sigma^2 \delta(\tau) \tag{3}$$

The receiver consists of a linear filter, the output of which is sampled at kT, and a threshold detector that determines in which decision region the sample lies.

The optimum linear receiver that minimizes the probability of error was derived by Aaron and Tufts (ref. 1), under the assumption that the channel dispersion of a single pulse is limited to $2N + 1$ samples. The receiver consists of two filters in cascade (fig. 2). The first portion is a filter matched to the received pulse $h(t)$ and the second has the transfer function

$$\sum_{n=-N}^{N} c_n e^{jn\omega T} \tag{4}$$

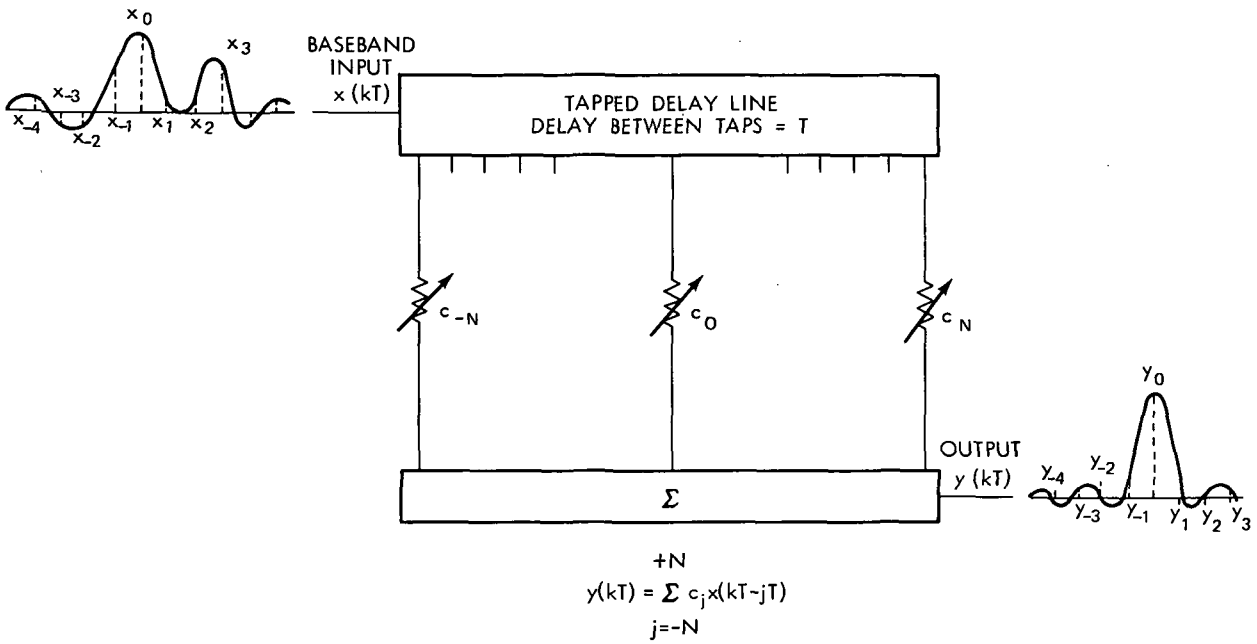

Figure 3.—Transversal filter.

This response can be obtained by using the transversal filter of figure 3 or by sampling the incoming signal at T-second intervals and using a digital filter having coefficients c_n. The transversal filter consists of a continuous delay line tapped at T-second intervals. Each tap has a variable gain associated with it, and the filter output is the sum of the tap gain outputs. The tap gains or the filter coefficients are the solutions of a system of $2N + 1$ linear equations (whose coefficients are not readily available).

Smith (ref. 2) and Tufts (ref. 3) have shown that the optimum linear filter for a mean-square-error criterion has the same structure except for the values of the filter coefficients. Other authors using different criteria have also arrived at the identical structure.

The matched filter portion of the optimum linear receiver of figure 2 increases the noise immunity of the receiver whereas the transversal or digital filter compensates for the distortions introduced by the channel—hence the name "equalizer."

A fundamental assumption, made in all of the aforementioned studies, is that the channel characteristics are known a priori. In general, however, that assumption is not valid; therefore, adaptive linear filters that learn the characteristics of the channel have been considered.

The transversal or digital filter portion of the optimum receiver is easily constructed and readily lends itself to adaptive techniques. Also, as a result of the channel dispersion of the pulses, the communication efficiency has very often been limited in the rate of transmission by the intersymbol interference, rather than by the additive noise. Hence, much attention has been focused on the design of adaptive transversal filters or equalizers.

A $(2N + 1)$-dimensional digital filter will be used for the equalizer in this study. The filter coefficients are $c_{-N}, \ldots, c_0, \ldots, c_N$; the filter input is the sampled values at nT of the channel output

$x(t)$; and the behavior of the filter is described by

$$y_n = \mathbf{C}^+ \mathbf{X}_n$$

where y_n is the equalizer output at nT; $\mathbf{C}$ is the coefficient vector

$$\mathbf{C} = \begin{bmatrix} c_{-N} \\ \cdot \\ \cdot \\ \cdot \\ c_0 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{bmatrix}$$

$\mathbf{X}_n$ is the input vector

$$\mathbf{X}_n = \begin{bmatrix} x_{n+N} \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ \cdot \\ \cdot \\ \cdot \\ x_{n-N} \end{bmatrix}$$

and the symbol + is used for the vector transpose.

The filter input at nT is given by

$$x_n = a_n h_0 + \sum_{k \neq n} a_k h_{n-k} + n(nT) \tag{6}$$

where $h_m = h(mT)$. The summation portion of equation (6) is the intersymbol interference caused by the dispersion of the modulation pulse by the channel. The equalizer, with the proper values for its coefficients, will reduce this term.

A training period is used during which the equalizer coefficients converge to the optimum values according to a strategy. During this training period, the transmitter sends a sequence of identical pulses, with sufficient guard time to prevent interpulse interference. The desired equalizer response d_k is the transmitted pulse, sampled at T-second intervals. For example,

$$d_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

for no input encoding, and

$$d_k = \begin{cases} 1 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for duobinary input encoding. (See app. A for a discussion of input encoding techniques.) In either case, the filter output error at the kT sample is equal to the difference between the filter output y_k and the desired output d_k:

$$e_k = y_k - d_k \tag{9}$$

A simple and effective technique for adaptive equalization with no input encoding was developed by Lucky (ref. 4), using the tapped delay line filter for the equalizer. The equalizer parameters were chosen to minimize a peak distortion criterion specified by

$$D = \sum_{k \neq 0} \frac{|y_k|}{|y_0|} \tag{10}$$

The optimum values for the tap gains, in the sense of minimizing the peak distortion, are those that make

$$y_k = 0 \qquad \text{for } k = -N, \ldots, -1, 1, \ldots, N \tag{11}$$

with the constraint $y_0 = 1$.

The strategy used for the adaptive implementation was the steepest descent or gradient technique, using only polarity information as specified by

$$c_j = c_j - \Delta \operatorname{sgn} y_j \qquad j \neq 0 \tag{12}$$

where Δ is a small positive number. A major limitation of this technique is that convergence of the strategy to the optimum coefficients is assured only for relatively low dispersion channels. Mathematically, it is required that an initial distortion D_0, which is given by

$$D_0 = \sum_{k \neq 0} \frac{|x_k|}{x_0}$$

be less than 1. This is equivalent to requiring that the unequalized channel in the absence of noise be capable of supporting binary transmission without error. This limitation was imposed by the chosen criterion and not by the strategy.

Subsequently, Lucky and Rudin (ref. 5) proposed and implemented an adaptive equalizer for minimizing a weighted mean-square difference between an ideal channel response and the actual equalized channel response. The strategy used was again the modified steepest descent technique.

The basic approach to adaptive adjustment of a set of weights in which a mean-square-error criterion is used with a gradient search procedure was considered by Widrow and Hoff (ref. 6). They noted that no derivative computation is needed. Lucky and Rudin (ref. 5) were the first to apply the mean-square-error criterion with the gradient search procedure to the field of adaptive equalization.

This approach was applied to synchronous data transmission in the time domain by Gersho (ref. 7), Lytle (ref. 8), and Niessen (ref. 9). In the absence of noise, the mean-square output error is given by

$$\mathcal{E} = \sum_k e_k^2 = \sum_k (\mathbf{C}^+ \mathbf{X}_k - d_k)^2 \tag{13}$$

The gradient of the mean-square error for the $(k + 1)$th training pulse is used to adjust the coefficient values according to

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \frac{\alpha}{2} \nabla_{\mathbf{C}^k} \mathcal{E} \tag{14}$$

where $\mathbf{C}^k$ is the vector value after the kth iteration and the constant α is called the step size. The evaluated gradient is given by

$$\nabla_{\mathbf{C}^k} \mathcal{E} = 2 \sum_n \mathbf{X}_n e_n = 2(\mathbf{A}\mathbf{C}^k - \mathbf{g}) \tag{15}$$

where the vector $\mathbf{g} = \sum_n d_n \mathbf{X}_n$. The matrix $\mathbf{A}$ is called the channel correlation matrix and is given by

$$\mathbf{A} = \sum_n \mathbf{X}_n \mathbf{X}_n^+$$

The ijth entry is equal to

$$\{\mathbf{A}\}_{ij} = \sum_n x_n x_{n+|i-j|}$$

from which it is obvious that $\mathbf{A}$ is symmetric, and that all entries on any diagonal are equal. This special form is known as the Toeplitz form. With a nonzero input sequence x_k, it is also positive definite. If the channel were known, i.e., if the matrix $\mathbf{A}$ were specified a priori, the optimum preset equalizer that minimizes the mean-square output error would have its coefficients equal to

$$\mathbf{C}_{\text{opt}} = \mathbf{A}^{-1}\mathbf{g} \tag{16}$$

The simplicity of using a gradient search for the minimum can be seen from the ease of the gradient's implementation. No derivatives are necessary and only a digital cross-correlation of the output error e_k with the input sequence x_k is needed, as can be seen from equation (15). It should also be obvious from equation (15) that the gradient produces a system of linear equations and, hence, iterative techniques that solve systems of algebraic equations should be applicable.

The initial guess for the equalizer values normally used is 1 for the center tap and 0 elsewhere. With this choice, the equalizer output is identical to the input, thus causing no further distortion.

Substituting the evaluated gradient into the algorithm yields

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \alpha(\mathbf{A}\mathbf{C}^k - \mathbf{g}) \tag{17}$$

Then the coefficient vector error at the kth iteration, which is equal to the difference between the actual coefficient vector value and the optimum setting, is given by

$$\epsilon^k = (\mathbf{I} - \alpha\mathbf{A})\epsilon^{k-1} \tag{18}$$

where the matrix, $\mathbf{I} - \alpha\mathbf{A}$, is the governing matrix for the gradient technique. The coefficient error at the kth iteration can be expressed in terms of the initial coefficient error

$$\epsilon^k = (\mathbf{I} - \alpha\mathbf{A})^k \epsilon^0 \tag{19}$$

Using vector and matrix norm inequalities, equations (18) and (19) become

$$\|\epsilon^k\| \leqslant \|\mathbf{I} - \alpha\mathbf{A}\| \, \|\epsilon^{k-1}\| \tag{20}$$

and

$$\|\epsilon^k\| \leqslant \|(\mathbf{I} - \alpha\mathbf{A})^k\| \, \|\epsilon^0\| \tag{21}$$

The vector and matrix norms used in equations (20) and (21) are the euclidian and spectral norms, respectively. They are defined as

$$\|\mathbf{V}\| = \sqrt{\sum_i v_i^2}$$

$$\|\mathbf{A}\| = [\rho(\mathbf{A}^*\mathbf{A})]^{1/2}$$

where $\rho(\mathbf{B})$, the spectral radius, is equal to

$$\rho(\mathbf{B}) = \max_i |\lambda_i|$$

with λ_i the ith eigenvalue of $\mathbf{B}$, and $\mathbf{A}^*$ is the matrix adjoint of $\mathbf{A}$. For nonzero initial errors, the normalized coefficient mean-square error is bounded by

$$\frac{\|\epsilon^k\|}{\|\epsilon^0\|} \leqslant \|(\mathbf{I} - \alpha\mathbf{A})^k\| \tag{22}$$

Because the matrix, $\mathbf{I} - \alpha\mathbf{A}$, is hermitian, the spectral norm is equal to the spectral radius. Also

$$\|(\mathbf{I} - \alpha\mathbf{A})^k\| = \|\mathbf{I} - \alpha\mathbf{A}\|^k$$

The technique definitely converges if the constant α is chosen to be in the interval $(0, 2/\lambda_{max})$ where λ_{max} is the largest eigenvalue of the correlation matrix $\mathbf{A}$ (Widrow (ref. 10)). The step size that minimizes the upper bound, $\|\mathbf{I} - \alpha\mathbf{A}\|^k$, for the normalized mean-square coefficient error, $\|\epsilon^k\|/\|\epsilon^0\|$, was derived by Gersho (ref. 7). Its value is

$$\alpha_0 = \frac{2}{\lambda_{max} + \lambda_{min}} \tag{23}$$

The minimum reduction of the mean-square coefficient error at each iteration with this step size is given by

$$\|\mathbf{I} - \alpha_0\mathbf{A}\| = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} \tag{24}$$

Therefore, for channel correlation matrices where the spread of the eigenvalues is small, i.e., low-distortion channels with no encoding of the transmitted signal, the optimum fixed step-size gradient will converge fairly rapidly. But for channels with high dispersion, the eigenvalue spread is large and hence the convergence rate is very slow. Although the optimum fixed step size is an improvement, its rate of convergence is still too slow. Because the time spent in a training period is useless for data transmission, many different techniques have been investigated to accelerate the convergence. The best results were obtained by Schonfeld and Schwartz (ref. 11). Their algorithm is the variable step-size gradient

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \frac{\alpha_k}{2}(\nabla_{\mathbf{C}^k} \mathscr{E}) \tag{25}$$

where α_k is chosen to minimize the norm of the tap gain error after M iterations. The step-size values α_k are given by the reciprocal of the zeros of the Mth-order Chebyshev polynomial. After M iterations, the algorithm of equation (25) is repeatedly applied until the coefficients converge to the optimum value of equation (16). The minimum reduction of the coefficient error norm for M iterations is

$$2\frac{(R-1)^M}{(\sqrt{R}+1)^{2M}+(\sqrt{R}-1)^{2M}}$$

where R is the condition number of the matrix $\mathbf{A}$. For hermitian matrices, this is equal to the quotient $\lambda_{max}/\lambda_{min}$. Although the variable step-size gradient is faster than the optimum fixed step-size gradient for M iterations, the error norm does not necessarily decrease at each iteration; simulations conducted actually showed that the error norm initially increases.

In a subsequent paper, Schonfeld and Schwartz (ref. 12) investigated a second-order variable step-size gradient technique: the Chebyshev semi-iterative method. This algorithm updates the equalizer parameters with the $(k+1)$th training pulse according to

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \alpha_k \left(\frac{1}{2}\nabla_{\mathbf{C}^k}\mathscr{E}\right) + \beta_k(\mathbf{C}^k - \mathbf{C}^{k-1}) \tag{26}$$

where the coefficients α_k and β_k are chosen a priori to minimize the mean-square coefficient error at each iteration. The convergence rate for M iterations is identical to that for the first-order variable step-size gradient, but this algorithm has the property that it always decreases the mean-square error. Yet for highly dispersive channels and for partial-response encoding techniques, although it is an improvement over the fixed step-size gradient, convergence is still slow.

In all three techniques, the optimum fixed step-size gradient, the variable step-size gradient, and the second-order variable gradient, the step sizes are functions of the minimum and maximum eigenvalues of the channel correlation matrix. Because exact determination of the eigenvalues is not feasible, upper and lower bounds for these eigenvalues are used. (See app. B.) In general, these bounds are extremely poor and hence the step sizes are greatly underestimated. Because of this, the actual convergence rates are exceedingly slower than the theoretical ones.

Successive Overrelaxation Algorithm

In this study, the successive overrelaxation iterative technique is proposed as the algorithm for the adaptive equalizer. The equalizer coefficients are adjusted to minimize the output mean-square error. The algorithm determines the ith coefficient value at the $(k + 1)$ iteration according to

$$c_i^{k+1} = c_i^k - \frac{\omega}{a_{ii}} \left(\sum_{j=-N}^{i-1} a_{ij} c_j^{k+1} + \sum_{j=i}^{N} a_{ij} c_j^k - g_i \right) \tag{27}$$

where ω is the relaxation factor. After some manipulation, equation (27) can be incorporated into matrix notation. The vector behavior of the overrelaxation algorithm is described by

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \omega(\mathbf{D} - \omega\mathbf{E})^{-1}(\mathbf{A}\mathbf{C}^k - \mathbf{g}) \tag{28}$$

where $\mathbf{D}$ and $\mathbf{E}$ are strictly diagonal and lower triangular matrices such that $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{E}^+$.

The fixed step-size gradient technique changes the value of the ith coefficient by

$$c_i^{k+1} = c_i^k - \alpha \left(\sum_{j=-N}^{N} a_{ij} c_j^k - g_i \right) \tag{29}$$

The gradient technique, after computation of the new value of c_1, retains the old value and uses it in updating the other coefficient values. This is also true for the variable step-size gradient techniques. On the other hand, the relaxation algorithm is sequential in nature. First it updates c_1 and then incorporates it into the change in c_2, etc. If the ith coefficient is being updated, the new values of the first $(i - 1)$ coefficients and the old values of the remaining coefficients are used in computing the adjustment. The relaxation algorithm incorporates the most recent values in determining the change in the coefficient values. This is equivalent to conducting $(2N + 1)$ one-dimensional searches at each iteration.

The relaxation algorithm requires less storage than the gradient methods because it needs storage for only one equalizer vector whereas the fixed and first-order variable gradients require storage for $\mathbf{C}^{k+1}$ and $\mathbf{C}^k$ and the second-order gradient for $\mathbf{C}^{k+1}$, $\mathbf{C}^k$, and $\mathbf{C}^{k-1}$. The relaxation and fixed gradient methods require about the same number of calculations for each iteration, whereas the variable gradients require more. The implementation is very similar to that for the gradient techniques except for the fact that it is sequential. Figure 4 demonstrates the adaptive implementation of the transversal filter equalizer using the algorithm.

In evaluating an iterative technique, the following properties should be investigated: convergence, rate of convergence, and behavior of the algorithm in a noisy environment. The rate of convergence and the behavior in noise are discussed in later sections of this report. The convergence of the relaxation algorithm is established in the following discussion.

For the equalizer problem, the channel correlation matrix $\mathbf{A}$ is positive definite hermitian; $\mathbf{D}(\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{E}^+)$ is diagonal with positive entries and hence $\mathbf{D} - \omega\mathbf{E}$ is nonsingular because $\mathbf{E}$ is strictly lower triangular. Convergence is guaranteed for all possible channels when the relaxation parameter is in the open interval (0, 2) by Ostrowski's theorem (ref. 13):

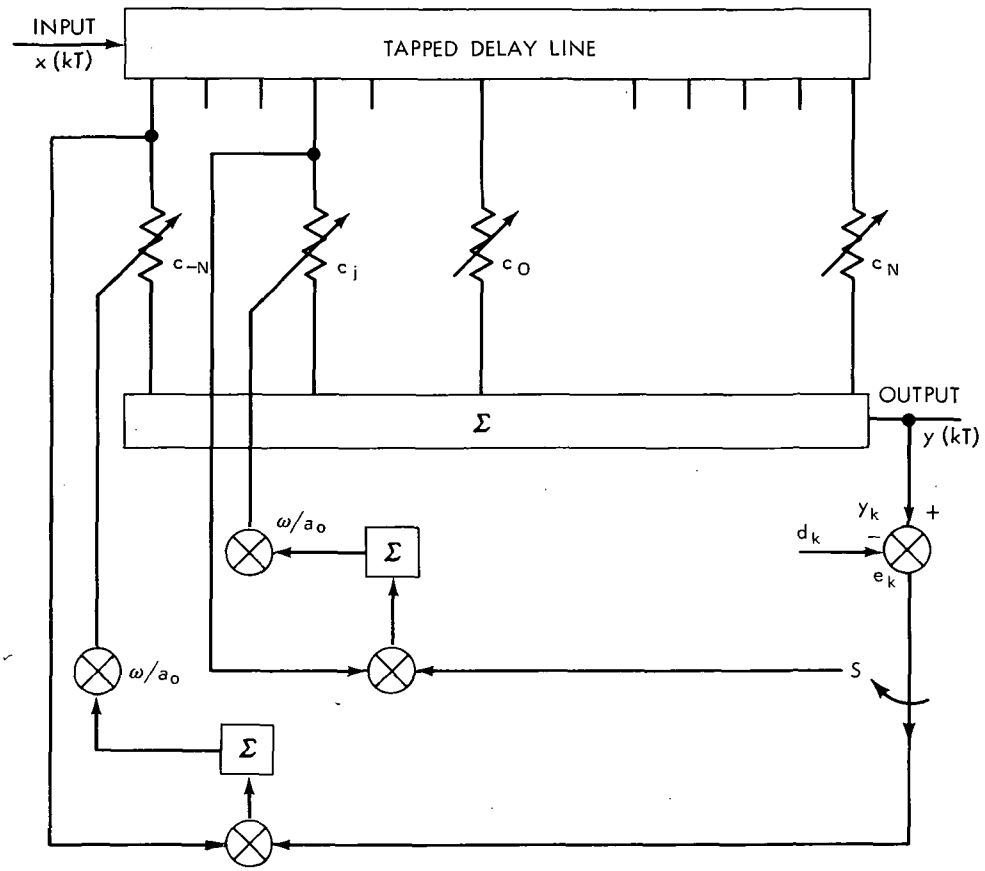

Figure 4.—Adaptive equalizer.

Theorem: Let $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{E}^+$ be an $n \times n$ hermitian matrix where $\mathbf{D}$ is hermitian and positive definite and $\mathbf{D} - \omega\mathbf{E}$ is nonsingular for $0 \leqslant \omega \leqslant 2$. Then $\rho(\mathcal{L}_\omega) < 1$; i.e., the algorithm converges, if and only if $\mathbf{A}$ is positive definite and $0 < \omega < 2$, where $\mathcal{L}_\omega = \mathbf{I} - \omega(\mathbf{D} - \omega\mathbf{E})^{-1}\mathbf{A}$ is the matrix associated with the relaxation method.

Two-Dimensional Example

In this section, a heuristic argument for the use of the successive overrelaxation method is presented, and the behaviors of both the fixed step-size gradient and relaxation methods are analyzed for a two-dimensional equalizer.

In a noise-free environment for a two-dimensional equalizer, the equal mean-square-error surfaces (fig. 5) are ellipses with the major axis proportional to $a_0 + |a_1|$ and the minor axis to $a_0 - |a_1|$. The channel correlation matrix is given by

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} \tag{30}$$

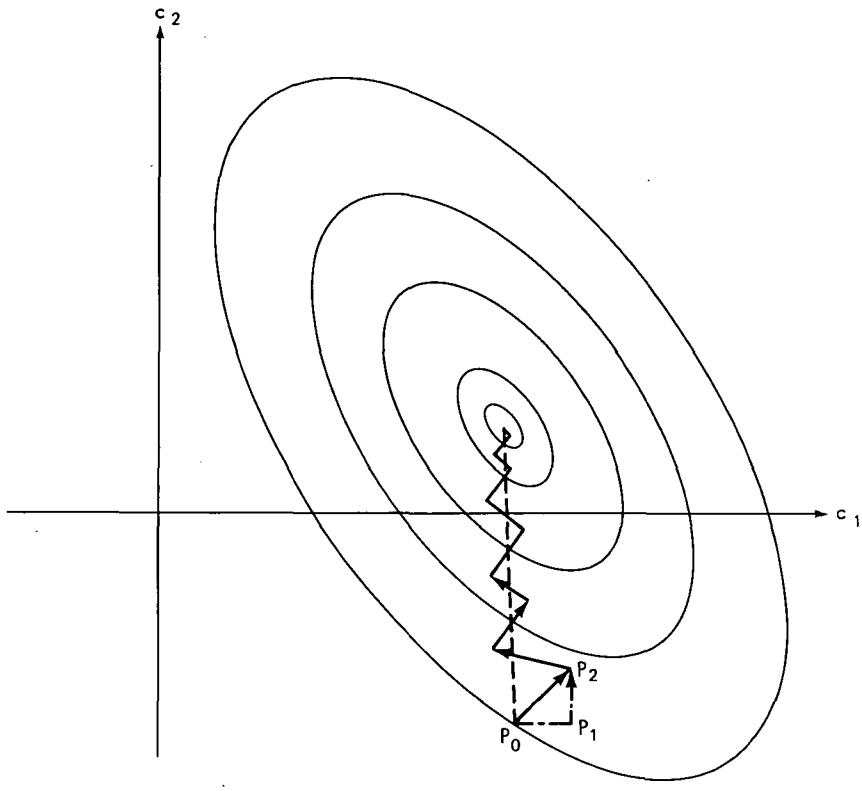

Figure 5.—Mean-square-error surfaces for two-dimensional equalizer.

with a_0 positive. The eigenvalues of this matrix are

$$\lambda_{1,2} = a_0 \pm a_1$$

If the error surface were known, the optimum way of proceeding from the initial guess P_0 to the minimum is represented by the dashed line of figure 5. The optimum "step size" with this direction is equal to the length of the dashed line. This step size should not be confused with the constant α as in the gradient step size (eq. (14)); it is the total correction to the coefficient. For example, in the gradient, this step size would be equal to $-(\alpha/2)\nabla_{C^k}\xi$. Unfortunately, the surface is always unknown and, therefore, local exploration at the point P_0 must be used to determine a suitable path and the step-size magnitude. The steepest descent path or gradient direction is chosen, because infinitesimally it is the path of most rapid descent. The gradient is also an indication of the step-size magnitude: The gradient is large when the present position is far from the minimum and decreases as the minimum is approached.

Figure 5 describes the progress of the gradient algorithm (eq. (14)) toward the minimum value. It is seen that the gradient oscillates around the optimum path (ref. 14). These oscillations significantly impede the progress.

The gradient can be viewed as the sum of one-dimensional searches along the respective coefficient axes. Decomposing the gradient into components along the c_1 and c_2 axes, respectively, the motion

along the c_1 axis from P_0 to P_1 is feasibly the best one can do under the circumstances because the change is due to the gradient projection onto the c_1 axis evaluated at P_0. The motion from P_1 to P_2 along the c_2 axis is still probably in the correct direction but the step magnitude is not the best because it is derived from the gradient evaluated at P_0. Because the gradient is a function of position in all cases excluding the circular error surfaces, using the gradient evaluated at P_0 is no longer the best strategy for motion at P_1. A better strategy would be to use the gradient projection evaluated at P_1 along the c_2 axis. This strategy makes more efficient use of the available information than the gradient algorithm. The successive overrelaxation iterative technique uses this strategy. Therefore if the constants α of equation (29) and ω/a_{ii} of equation (27) are equal, one would expect the relaxation method to be faster than the gradient method.

The case in which the error surfaces are circular corresponds to a channel that causes no dispersion but only value scaling of the transmitted signal. The relaxation and gradient techniques both converge in one iteration.

Because the eigenvalues of a 2 × 2 matrix can be found analytically, it is possible to determine the optimum step size (eq. (23)) for the fixed step-size gradient and hence compare the two techniques analytically. The optimum step size α_0 has a value of $1/a_0$. The spectral norm or the minimum reduction in the error norm (eq. (24)) at each iteration is given by

$$\|\mathbf{G}\| = \frac{|a_1|}{a_0} \equiv |a| \leqslant \frac{1}{2} \tag{31}$$

where the matrix $\mathbf{G}$, which is equal to $\mathbf{I} - \alpha_0 \mathbf{A}$, is the matrix governing the behavior of the gradient technique. For $\omega = 1$, the step size for the relaxation iterative technique is equal to the optimum gradient step size. For the relaxation parameter greater than or less than 1, the step size is larger or smaller, respectively, than the gradient. The relaxation matrix $\mathcal{L}_1$ is given by

$$\mathcal{L}_1 = (\mathbf{D} - \mathbf{E})^{-1}\mathbf{E}^{+} \tag{32}$$

where $\mathbf{D}$ and $\mathbf{E}$ are diagonal and strictly lower triangular matrices, such that the channel correlation matrix can be uniquely decomposed into

$$\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{E}^{+} \tag{33}$$

In this case

$$\mathbf{D} = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} 0 & 0 \\ -a_1 & 0 \end{bmatrix}$$

The relaxation matrix is not symmetric, and hence the norm of the kth power is not equal to the kth power of the norm. The spectral norm of the kth power of the relaxation matrix can be found analytically and is equal to

$$\|\mathcal{L}_1^k\| = |a|^{2k-1}(1 + a^2)^{1/2} \tag{34}$$

It is now possible to compare the bounds (eq. (22)) on the normalized coefficient mean-square error for both techniques. For all values of k greater than 1, the spectral norm for the kth power of the relaxation technique is smaller than that for the gradient. Therefore, the average reduction for k iterations ($k > 1$) is larger for the relaxation method; hence it should converge faster.

For large numbers of iterations, the relaxation norm behaves according to

$$\|\mathcal{L}_1^k\|^{1/k} \to |a|^2 \tag{35}$$

This is equal to the square of the fixed gradient norm (eq. (31)).

The minimum average reduction at each iteration for the variable step-size gradient techniques asymptotically is given by

$$\frac{|a|}{1 + \sqrt{1 - |a|^2}}$$

Because $|a| \leqslant 1/2$, the minimum reduction at each iteration for relaxation is larger than that for the variable gradients. Hence, the relaxation technique can be asymptotically twice as fast as the fixed gradient and at least as fast as the variable step-size gradients.

For the two-dimensional equalizer with $\omega = 1$ and the optimum α, examination of the equations governing the adjustment of the equalizer coefficient (eq. (27) for the relaxation method and eq. (29) for the fixed gradient) yields an amazing fact. For any starting point, the values of c_2 at the kth

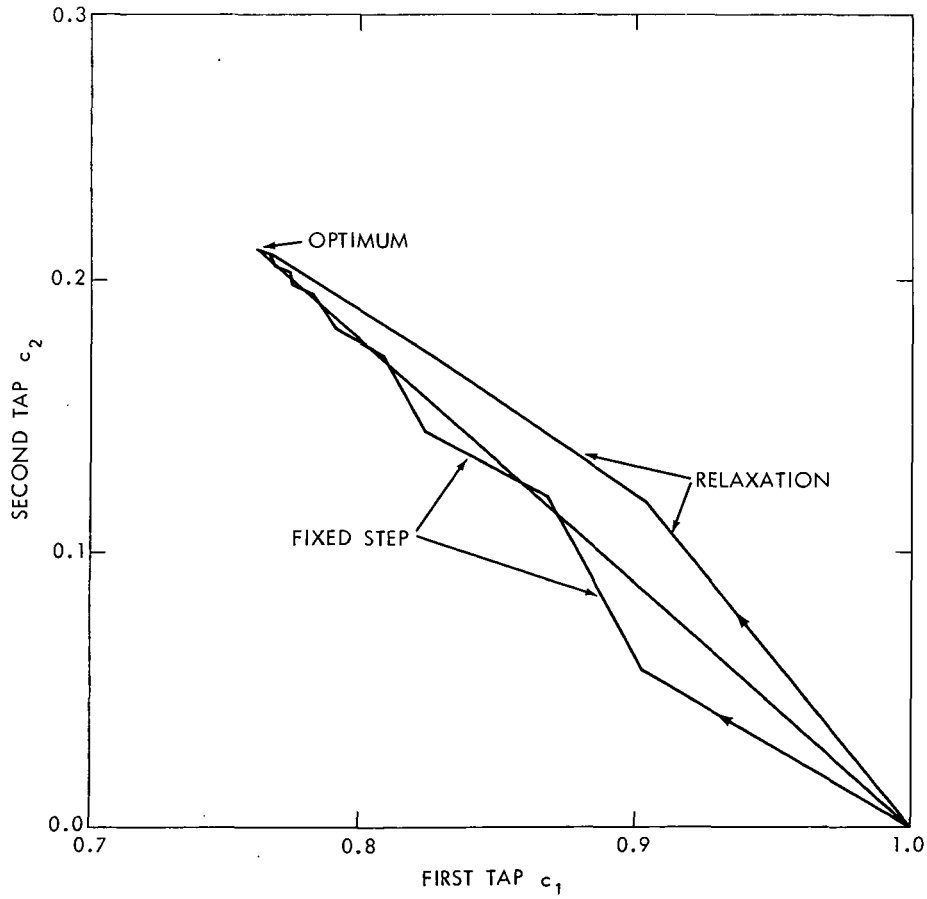

Figure 6.—Convergence of relaxation and gradient algorithms for two-dimensional case; duobinary encoding was used with condition number $R = 3.28$.

relaxation iteration and the $2k$ gradient iteration are identical; also, the values of c_1 at the $(k + 1)$ relaxation and the $(2k + 1)$ gradient iterations are identical. This implies that if convergence occurs in M iterations for the relaxation method, the gradient technique requires $(2M - 1)$ iterations to converge. Asymptotically, if M is large, the relaxation method is twice as fast as the gradient. This supports the asymptotic result obtained earlier from the spectral norms.

Figure 6 shows the results of a simulation for both techniques for the two-dimensional equalizer. Not surprisingly, the simulation results support the theoretical ones. Convergence for the relaxation method was obtained in 18 iterations, whereas 35 iterations were required for the gradient. The simulation agrees with the behavior of the gradient algorithm as portrayed in figure 5. The relaxation algorithm, on the other hand, did not oscillate; and after the first iteration its direction was essentially the same. The relaxation factor $\omega = 1$ does not yield the best asymptotic results, as will be demonstrated later. This implies that the technique has the potential of at least halving the time spent in a training mode over the fixed step-size gradient.

In the following section, the convergence properties of the successive relaxation algorithm are investigated and, where possible, compared with those for the gradient in a noise-free environment. The effects of the channel additive noise and of finite precision are determined separately in a later section. The simulation results of the implementation of this algorithm for the adaptive equalizer are then presented and the improvements that are possible over both the fixed and variable step-size gradients are demonstrated. Improvements at least of the order observed in the two-dimensional equalizer are shown to be possible for a wide range of channel dispersions.

CONVERGENCE PROPERTIES

The successive overrelaxation iterative technique is proposed as the algorithm for the iterative adjustment of the equalizer coefficients to minimize the mean-square error. As mentioned before, the method in the $(k + 1)$ iteration is characterized by the use of the latest estimate of the coefficient values c_i^{k+1} in all subsequent computations and corrections. The technique is specified by equation (27):

$$c_i^{k+1} = c_i^k - \left(\sum_{j=-N}^{i-1} a_{ij} c_j^{k+1} + \sum_{j=i}^{N} a_{ij} c_j^k - g_i \right) \frac{\omega}{a_{ii}} \tag{27}$$

where a_{ij} is the ij entry of the channel correlation matrix **A**. In matrix notation this is

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \omega(\mathbf{D} - \omega\mathbf{E})^{-1}(\mathbf{A}\mathbf{C}^k - \mathbf{g}) \tag{28}$$

where **D** is a diagonal matrix formed with the diagonal entries of **A**, and **E** is a strictly lower triangular matrix with entries equal to the negative entries of **A** below the main diagonal. Note that the matrix $\omega(\mathbf{D} - \omega\mathbf{E})^{-1}$ is similar to the step size in the gradient technique.

The coefficient vector error at the kth iteration, i.e., the difference between the actual coefficient value and the optimum setting, is given by

$$\epsilon^k = \mathcal{L}_\omega \epsilon^{k-1} = \mathcal{L}_\omega^k \epsilon^0 \tag{36}$$

where $\mathcal{L}_\omega = \mathbf{I} - \omega(\mathbf{D} - \omega\mathbf{E})^{-1}\mathbf{A}$ is the relaxation matrix. For a matrix iterative technique to converge for all initial values, it is necessary that the successive powers of the matrix associated with the method approach the zero matrix ($\mathcal{L}_\omega^k \to 0$). Convergence is guaranteed if and only if the spectral radius of the associated matrix is strictly less than 1. Because the matrix $\mathbf{A}$ is positive definite Toeplitz for all possible distortions, the successive overrelaxation method converges for all values of the relaxation parameter ω in the open interval (0, 2). (See Ostrowski (ref. 13).)

Using the matrix spectral norm and vector euclidian norm, equation (36) becomes

$$\|\epsilon^k\| \leqslant \|\mathcal{L}_\omega^k\| \, \|\epsilon^0\| \qquad k \geqslant 0 \tag{37}$$

For nonzero initial errors, $\|\mathcal{L}_\omega^k\|$ gives an upper-bound estimate for the ratio $\|\epsilon^k\|/\|\epsilon^0\|$, and serves as a basis for comparison of different iterative methods. With $\|\mathcal{L}_\omega^k\| < 1$, $\|\mathcal{L}_\omega^k\|$ is the minimum reduction in the normalized coefficient mean-square error for k iterations. An average rate of convergence (ref. 15) for M iterations is defined as

$$R(\mathbf{F}^m) = -\ln \frac{\|\mathbf{F}^m\|}{m} \qquad \text{for all } m \geqslant 1 \text{ such that } \|\mathbf{F}^m\| < 1 \tag{38}$$

where the matrix $\mathbf{F}$ is the governing matrix for the technique. If $\mathbf{F}$ were symmetric, then

$$\|\mathbf{F}^k\| = \|\mathbf{F}\|^k = [\rho(\mathbf{F})]^k \tag{39}$$

where $\rho(\mathbf{F})$ is the spectral radius or the largest eigenvalue in magnitude. The previously defined average rate of convergence is then equal to a single value, which is identical to the asymptotic rate of convergence given by

$$R_\infty(\mathbf{F}) = -\ln \rho(\mathbf{F}) \tag{40}$$

On the other hand, if $\mathbf{F}$ is not symmetric, the equality of equation (39) will most likely not hold. Therefore, the average rate of convergence, which is defined for all $k \geqslant 1$, will possess an infinite set of values that need not be related. The average rate of convergence, as k increases, converges to the asymptotic rate of convergence.

To determine any of these rates of convergence, it is necessary to find the eigenvalues of matrices. Eigenvalues are the solutions of the Mth-order associated polynomial where M is the equalizer dimension. With the exception of a few cases, it is impossible to find a workable analytic solution for the roots of a general polynomial. Some of the exceptions are quadratic polynomials and those arising from tridiagonal matrices. Finding the average rate of convergence implies solving for the eigenvalues of $\mathbf{F}^{k+}\mathbf{F}^k$ for all values of k. Because this is almost impossible, much of the work in this study has been concentrated in determining or bounding the spectral radius and norm of the matrix $\mathcal{L}_\omega$.

To compare two techniques analytically is difficult; and even in cases for which the eigenvalues can be found analytically, a comparison may not be possible. For a comparison to be made, it is necessary that the eigenvalues of the two associated matrices have a functional relationship, or be bounded by each other, or have bounds which themselves are bounded by the other set of eigenvalues.

In forming the relaxation matrix, the correlation matrix has been altered in a nonlinear fashion and, in general, a relationship does not exist between the eigenvalues of the relaxation matrix and the

channel correlation matrix **A**. Hence, no relationship is expected between the eigenvalues of the relaxation and gradient ($\mathbf{I} - \alpha\mathbf{A}$) matrices.

For the past decade and a half, mathematicians have been extremely interested in the convergence properties of the relaxation method for the solution of systems of linear equations (refs. 13 and 15 to 23). They (refs. 18 and 21) were able to determine or bound the asymptotic rates of convergence for matrices that were p-cyclic or had associated Jacobi matrices ($\mathbf{B} = \mathbf{D}^{-1}(\mathbf{E} + \mathbf{E}^{+})$) that were nonnegative and convergent ($\rho(\mathbf{B}) < 1$). Of the p-cyclic class of matrices, the only subclass that is compatible with the equalization problem is the 2-cyclic class (tridiagonal matrices). In this study, the results for nonnegative Jacobi matrices will be extended to include all Jacobi matrices with entries having the same sign.

Further analytical results were not obtained, hence numerical evaluation of the spectral radius was conducted for several channels. These simulations suggested an upper bound for the relaxation spectral radius that is valid for a large portion of the parameter range. This upper bound indicates that the type of improvement obtained for the 2-cyclic case is possible for more general equalizer problems.

A bound for the spectral norm of the relaxation matrix will be developed that demonstrates that the technique is coefficient mean-square-error reducing at each iteration for certain parameter values for channels with light or moderate intersymbol interference or channels that give rise to tridiagonal correlation matrices. Perturbation theory will be used to analytically demonstrate that the technique is norm decreasing for a small parameter range for all possible distortions. Numerical evaluations of the spectral norm for several channels have supported and extended the theoretical range.

The upper bound $\|\mathbf{F}^k\|$ for the normalized coefficient mean-square error, as suggested by equation (37), was numerically evaluated as a function of the iteration number k for the relaxation, the optimum, and estimated fixed step-size gradient methods.

2-Cyclic Matrices

To give rise to tridiagonal matrices, the output of the channel can have only two nonzero samples: the transmitted sample and its echo. The channel matrix for a $(2N + 1)$-dimensional digital filter as the equalizer has the form

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_1 & a_0 & a_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & a_1 & a_0 & a_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & a_0 \end{bmatrix} \tag{41}$$

Surprisingly, the eigenvalues of this matrix can be found analytically and are the roots of a $(2N + 1)$-dimensional Chebyshev polynomial of the second kind. For the fixed step-size gradient technique, the

optimum step size α_0 can be determined and is $1/a_0$. The spectral radius (also the norm, because the gradient matrix is real symmetric) for this step size is given by

$$\|\mathbf{G}\| = \rho(\mathbf{G}) = 2|a| \cos \frac{\pi}{2\mathbf{N} + 2} \tag{42}$$

where $a = a_1/a_0$ and $\mathbf{G} = \mathbf{I} - \alpha_0 \mathbf{A}$. For 2-cyclic matrices, the optimum fixed step-size gradient technique is equivalent to the Jacobi iterative method which is used for the solution of systems of linear equations. Hence, the comparison theorems between the relaxation and Jacobi methods are directly applicable. For the 2-cyclic class of matrices, Young (ref. 21) discovered that a functional relationship exists between the eigenvalues of the relaxation matrix and the Jacobi matrix. It is

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2 \tag{43}$$

where λ and μ are nonzero eigenvalues of the relaxation and Jacobi matrices, respectively, and ω is the relaxation parameter.

That such a functional relationship actually exists is itself interesting, but the importance of this result lies in the fact that a direct comparison can be made between the two techniques. Also, it is the basis for the determination of the values of ω yielding the best asymptotic results.

The spectral radius can now be determined analytically as a function of ω. Solving the functional relationship for the nonzero eigenvalues of the relaxation matrix, the following is obtained:

$$\lambda = \frac{\omega^2\mu^2 + 2(1 - \omega) \pm \omega\mu\sqrt{\omega^2\mu^2 + 4(1 - \omega)}}{2} \tag{44}$$

For each nonzero eigenvalue of the Jacobi matrix, there corresponds two eigenvalues for the relaxation method; in the interval $(0, \omega_0)$ where ω_0 is the largest value of ω that insures real roots, the larger root is obtained by using the plus sign for the square root. The value of ω_0 is given by

$$\omega_0 = \frac{2}{1 + \sqrt{1 - \mu^2}}$$

The larger eigenvalue is real, positive, and an increasing function of the Jacobi eigenvalues. Therefore, the maximum eigenvalue (spectral radius of $\mathcal{L}_\omega$) is obtained when the spectral radius of the Jacobi matrix corresponding to $\mathbf{A}$ is used as the eigenvalue μ. In the interval $(0, \omega_b)$ where

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(\mathbf{B})}} \tag{45}$$

and is the largest value of ω that insures real eigenvalues, the spectral radius is given by

$$\rho(\mathcal{L}_\omega) = \frac{\omega^2\rho^2(\mathbf{B}) + 2(1 - \omega) + \omega\rho(\mathbf{B})\sqrt{\omega^2\rho^2(\mathbf{B}) + 4(1 - \omega)}}{2} \tag{46}$$

For $\omega \geq \omega_b$, the eigenvalues of equation (44) are complex, but all have the same magnitude. Hence in this range, the spectral radius is equal to

$$\rho(\mathcal{L}_\omega) = \omega - 1 \tag{47}$$

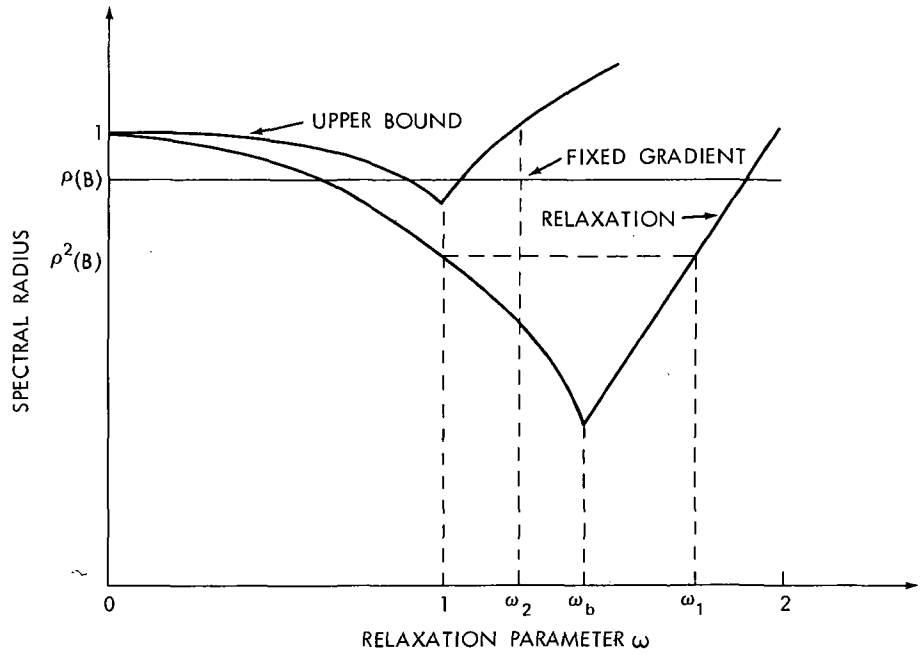

Figure 7.—Spectral radii for relaxation and fixed gradient methods; and relaxation norm upper bound for 2-cyclic matrices.

The spectral radius for the relaxation method is sketched as a function of ω in figure 7 for 2-cyclic matrices. It is monotonically decreasing for ω in the interval $(0, \omega_b)$ and monotonically increasing for $\omega > \omega_b$. It has a nondifferential minimum at ω_b. The spectral radius of the fixed gradient (Jacobi) technique is superimposed on the same graph. For $\omega = 1$, the spectral radius of the relaxation method is equal to the square of the spectral radius of the Jacobi method, $\rho^2(\mathbf{B})$. This implies that the relaxation method is asymptotically twice as fast for this parameter value.

From figure 7, it is evident that there exists a large region for ω such that the relaxation method is asymptotically faster than the gradient. Furthermore, in the region $[1, \omega_1]$ with $\omega_1 = 1 + \rho^2(\mathbf{B})$, the relaxation method is at least twice as fast as the gradient and has the fastest convergence rate for a relaxation factor of ω_b.

An asymptotic comparison can also be made for the variable step-size gradient techniques and the relaxation iterative method. Both the first- and second-order variable gradient methods have the same rate of convergence for M iterations. The minimum reduction in the normalized coefficient mean-square error for M iterations by using these techniques is given by (ref. 11)

$$\frac{\|\epsilon^m\|}{\|\epsilon^0\|} \leqslant \frac{1}{T_M(\gamma)} \tag{48}$$

where $\gamma = 1/\rho(\mathbf{B})$ and $T_M(\gamma)$ is the classical Chebyshev polynomial

$$T_M(\gamma) = \frac{\exp\left(M \cosh^{-1} \gamma\right) + \exp\left(-M \cosh^{-1} \gamma\right)}{2}$$

The average reduction per iteration is simply

$$\left(\frac{\|\epsilon^M\|}{\|\epsilon^0\|}\right)^{1/M} \leqslant \left(\frac{1}{T_M(\gamma)}\right)^{1/M} \tag{49}$$

The asymptotic result (eq. (50)) is obtained by letting M approach infinity in equation (49) and using

$$\cosh^{-1}\gamma = \ln\left(\gamma + \sqrt{\gamma^2 + 1}\right)$$

$$\lim_{M\to\infty}\left(\frac{\|\epsilon^M\|}{\|\epsilon^0\|}\right)^{1/M} \leqslant \frac{\rho(\mathbf{B})}{1 + \sqrt{1 - \rho^2(\mathbf{B})}} \tag{50}$$

The asymptotic result for the relaxation method with the optimum relaxation factor ω_b is

$$\lim_{M\to\infty}\left(\frac{\|\epsilon^M\|}{\|\epsilon^0\|}\right)^{1/M} \leqslant \left(\frac{\rho}{1 + \sqrt{1 - \rho^2}}\right)^2 \tag{51}$$

The upper bound of equation (51) is equal to the square of the right-hand side of equation (50). This implies that asymptotically the variable gradient techniques need twice as many iterations as the optimum relaxation method to have the same minimum reduction.

The second-order variable step-size gradient method uses the semi-iterative Chebyshev acceleration method to improve the convergence of the fixed step-size gradient. If the acceleration method is applied to improve the optimum relaxation, the Chebyshev acceleration method is identical to the relaxation technique applied M times where M is the order of the semi-iterative method (ref. 19).

The optimum relaxation method yields asymptotic convergence which is twice as fast as the variable step-size gradient techniques and at least twice as fast as the optimum fixed step-size gradient (ref. 19).

Nonnegative Jacobi Matrices

In the previous section, the properties of the successive overrelaxation method were investigated under the assumption that the correlation matrix **A** is 2 cyclic. It should be clear that the basic assumption that **A** is 2 cyclic allowed the functional relationship between the eigenvalues to be deduced and hence was the steppingstone for the analysis. Mathematicians have been able to extend somewhat the results to Jacobi matrices that are symmetric, nonnegative, and convergent (refs. 15 and 18). For this class of matrices, the Jacobi and fixed step-size gradient techniques are not necessarily the same; and the results, therefore, are not immediately applicable. The following observations, however, are germane.

To use any gradient technique successfully, it is necessary to estimate the range for the step-size values, and to choose a suitable step size. Convergence is guaranteed for all step sizes in the range $(0, 2/\lambda_{max})$, when λ_{max} is the largest eigenvalue of **A**. If the eigenvalues are known exactly, the optimum fixed step size in the sense of convergence is

$$\alpha_0 = 2/(\lambda_{max} + \lambda_{min}) \tag{52}$$

Exact determination of the eigenvalues is not feasible. Hence estimates are used in determining this step size. Commonly used bounds (app. B) for the minimum and maximum eigenvalues are given by

$$|\lambda_{\min}| \geqslant \min_k \left(|a_{kk}| - \sum_{j \neq k} |a_{kj}| \right) \tag{53}$$

$$|\lambda_{\max}| \leqslant \max_k \sum_j |a_{kj}| \tag{54}$$

For the lower bound to be useful, i.e., positive, the correlation matrix must be diagonally dominant. If these bounds are used in determining the step size according to equation (52), the gradient technique reduces to the Jacobi method. Therefore, if the correlation matrix **A** is diagonally dominant and has a Jacobi matrix that is nonnegative, the comparative results are applicable. This class of matrices has the 2-cyclic matrices as a subclass.

Let ρ be the spectral radius of the fixed step-size gradient (Jacobi) technique. The spectral radius of the relaxation method for ω in the interval (0, 1] obeys the following inequalities (ref. 18):

$$\frac{2(1-\omega) + \omega^2\rho^2 + \omega\rho\sqrt{\omega^2\rho^2 - 4(\omega - 1)}}{2} \leqslant \rho(\mathcal{L}_\omega) < \frac{2(1-\omega) + \omega\rho}{2 - \omega\rho} \tag{55}$$

Equality occurs if and only if **A** is 2 cyclic. It has also been shown that for this case the actual relaxation spectral radius in this interval is a monotonically decreasing function of the relaxation parameter (ref. 15). Hence, the fastest convergence in the interval (0, 1] is obtained with $\omega = 1$. The inequalities of equation (55) reduce to

$$\rho^2 \leqslant \rho(\mathcal{L}_1) < \frac{\rho}{2 - \rho} < \rho \tag{56}$$

For a relaxation factor of ω_b, the optimum choice in the 2-cyclic case, the spectral radius is bounded by

$$\omega_b - 1 \leqslant \rho(\mathcal{L}_{\omega_b}) < \sqrt{\omega_b - 1} \tag{57}$$

again, with equality if and only if **A** is 2 cyclic. Although a precise determination of the relaxation factor that minimizes the convergence has not been obtained, equation (57) indicates that asymptotic rates similar to those obtained for the 2-cyclic case are possible. Improvements in convergence are not guaranteed by using factors larger than 1.

The upper bounds of equations (56) and (57) are both smaller than the spectral norm for the fixed gradient. Hence, the relaxation method is better asymptotically than the gradient for these values of the relaxation factor.

The upper bound of equation (57) is equal to

$$\sqrt{\omega_b - 1} = \frac{\rho}{1 + \sqrt{1 - \rho^2}} \tag{58}$$

If the upper and lower bounds (eqs. (53) and (54)) for the eigenvalues are also used for the variable step-size gradient techniques, the asymptotic results of equation (50) hold and are equal to equation (58). This implies that for a relaxation factor of ω_b, the relaxation method is at least as fast as the variable gradient techniques.

Nonpositive Diagonally Dominant Jacobi Matrices

In the following discussion, the preceding results will be extended to include correlation matrices that are nonnegative and diagonally dominant. The Jacobi matrix **B** now has all its entries nonpositive ($b_{ij} \leqslant 0$). Because $\mathbf{B} = -|\mathbf{B}|$ where **B** is the absolute value of the Jacobi matrix, the spectral radii of both are identical:

$$\rho(\mathbf{B}) = \rho(|\mathbf{B}|)$$

$|\mathcal{L}_\omega|$ and $|\mathbf{B}|$ are nonpositive and therefore are members of the class of matrices just discussed. Hence, they must obey the inequalities of equations (55) to (57):

$$\rho^2(|\mathbf{B}|) \leqslant \rho(|\mathcal{L}_1|) < \frac{\rho(|\mathbf{B}|)}{2 - \rho(|\mathbf{B}|)} = \frac{\rho(\mathbf{B})}{2 - \rho(\mathbf{B})} \tag{59}$$

$$\omega_b - 1 \leqslant \rho(|\mathcal{L}_{\omega_b}|) < \sqrt{\omega_b - 1} \tag{60}$$

Furthermore, because for any $n \times n$ matrix **M**,

$$\rho(\mathbf{M}) \leqslant \rho(|\mathbf{M}|)$$

the spectral radius of the relaxation method is bounded by the right-hand side of those inequalities. For $\omega = 1$ and $\omega = \omega_b$,

$$\rho(\mathcal{L}_1) \leqslant \rho(|\mathcal{L}_1|) < \frac{\rho}{2 - \rho} \tag{61}$$

$$\rho(\mathcal{L}_{\omega_b}) \leqslant \rho(|\mathcal{L}_{\omega_b}|) < \sqrt{\omega_b - 1} \tag{62}$$

Unfortunately, it is not necessarily true that the spectral radius is a monotonically decreasing function of ω in the interval (0, 1]; but it is bounded by a monotonically decreasing function. Two-cyclic matrices are also a subclass, but the asymptotic convergence rate of non-2-cyclic matrices can be better than that for the 2-cyclic matrices. Again the relaxation method is faster than the fixed gradient for both $\omega = 1$ and ω_b and faster than the Chebyshev gradient for $\omega = \omega_b$.

For diagonally dominant correlation matrices whose associated Jacobi matrices have entries all of the same size, it can be concluded that the relaxation method is asymptotically faster than the estimated gradient techniques. Furthermore, the asymptotic improvements are similar to those obtained for 2-cyclic matrices.

Spectral Radius for Several Examples

The spectral radius for the optimum and estimated fixed step-size gradient and the relaxation method were evaluated numerically by a computer as a function of the relaxation factor for two

channels with and without duobinary encoding of the transmitted signal. The step size for the estimated fixed step-size gradient was determined by using the bounds given by equations (B-2) and (B-3) of appendix B for the minimum and maximum eigenvalues.

The condition number R is equal to the quotient of the maximum eigenvalue by the minimum eigenvalue of the channel correlation matrix **A**. The condition number without partial-response encoding is a measure of the distortion of the transmitted pulse by the channel. For a channel with no distortion, the correlation matrix is a diagonal matrix and $R = 1$. Even with a poor initial guess for the vector coefficients, both techniques will converge in one iteration.

With partial-response encoding and a distortionless channel, the correlation matrix is not diagonal. In particular for duobinary encoding, the matrix **A** is tridiagonal with a non-one-condition number. If an incorrect initial guess is made, neither technique will converge in one iteration even though the channel is distortionless. Because **A** is 2 cyclic, the relaxation method with a factor in the interval $[1, 1 + \rho^2]$ will be asymptotically at least twice as fast as the optimum fixed gradient and, for $\omega = \omega_b$, twice as fast as the Chebyshev gradient. Partial-response encoding introduces more correlation between the equalizer coefficient and hence greatly slows down the convergence rate. With a distortion channel, the duobinary encoding technique increases the condition number R. With channels having condition numbers of 3.28 and 17.81, the duobinary encoding technique increased the condition numbers to 150.4 and 173.5. Convergence is dependent only upon the matrix associated with the technique and not upon the vector g of equation (28). Because g contains the partial-response encoding information, the iterative techniques cannot distinguish between a moderate distortion channel with partial-response encoding and an extremely high distortion channel without encoding. Therefore, we can view the examples used as different channels without encoding and with channel distortions varying from moderate to extreme.

The spectral radii for the relaxation, the optimum, and the estimated fixed step-size gradients are plotted as functions of ω in figures 8 to 11 for the channels with condition numbers 3.28, 17.81, 150.4, and 173.5. The spectral radius for relaxation is observed to be a continuous function of ω having one minimum. Its functional behavior is reminiscent of that for channels that give rise to 2-cyclic matrices. (See fig. 7.) Also plotted is the 2-cyclic-type functional relationship between the spectral radii for relaxation and for the optimum fixed step-size gradient.

For $R = 3.28$, the relaxation radius is minimum at $\omega = 1.08$, and the asymptotic rate of convergence is more than twice as fast as the optimum and 28 times faster than the estimated gradient. For ω in the interval $[0.95, 1.25]$, the optimum gradient is at least twice as slow as the relaxation. For $\omega = 1$, relaxation is 2 and 26 times as fast as the optimum and estimated gradients, respectively. For a relaxation factor of 1.5, the optimum gradient is slightly faster but the relaxation method is 10 times better than the estimated. The spectral radius is bounded by the radii relationship for values of ω up to 1.05. The relationship has a minimum at $\omega = 1.083$.

Figure 9 gives the results for $R = 17.81$. The radius is minimum at $\omega \simeq 1.3$ and yields asymptotic results that are 6 and 40 times better than the optimum and estimated fixed gradients. The relaxation method is at least twice as fast as the optimum for ω in $[0.95, 1.7]$ and the estimated gradient for ω in $[0.25, 1.9]$. For relaxation factors of 1 and 1.5, the relaxation method is 2 and 4 times as fast as

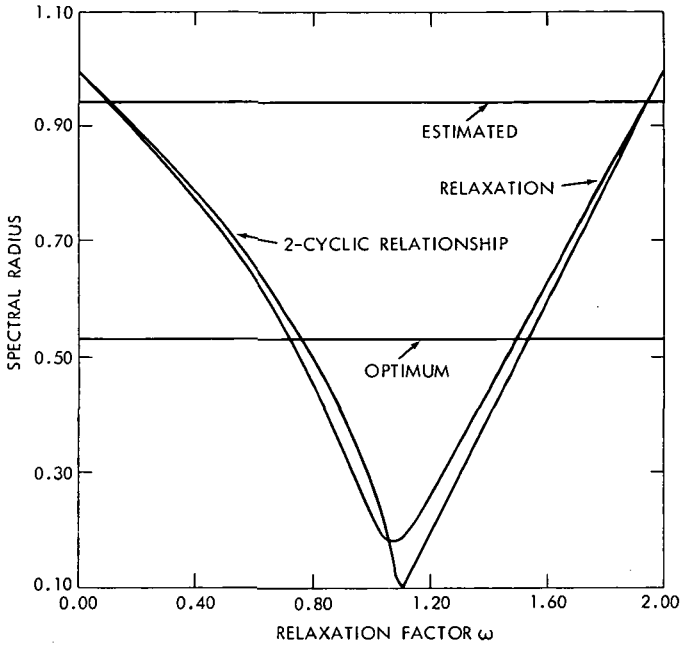

Figure 8.—Spectral radii for fixed step-size gradient and relaxation methods with R = 3.28 and 17 taps.

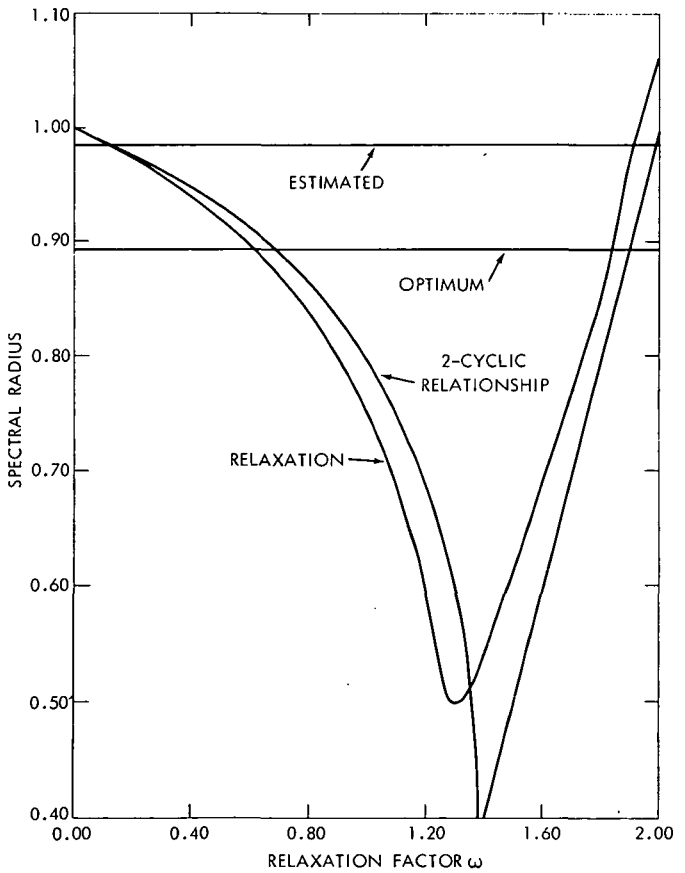

Figure 9.—Spectral radii for fixed step-size gradient and relaxation methods with R = 17.81 and 17 taps.

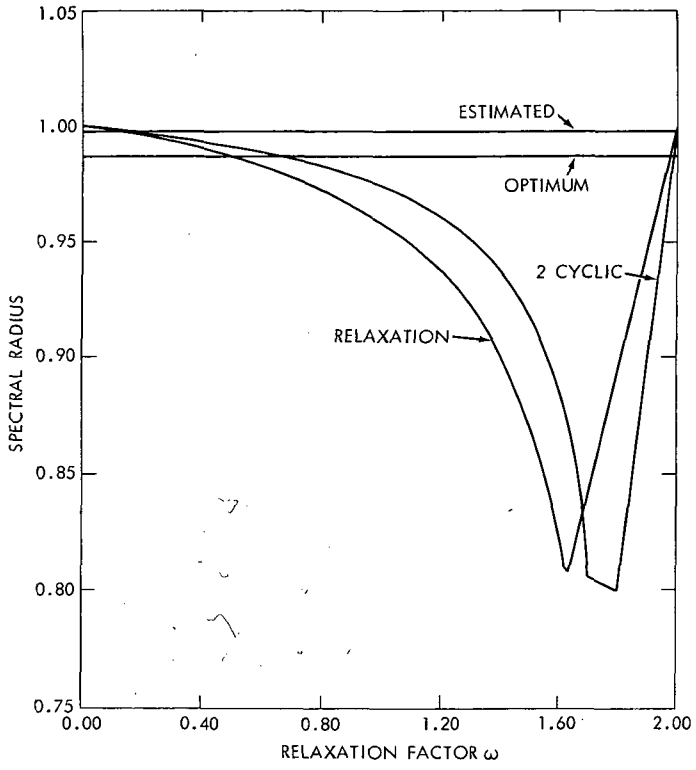

Figure 10.—Spectral radii for fixed step-size gradient and relaxation methods with duobinary encoding, R = 150.4, and 17 taps.

the optimum and 19 and 32 times faster than the estimated. The spectral radius is smaller than the relationship for values of ω up to 1.35. The relationship minimum is at 1.38.

For duobinary encoding with R = 150.4 (fig. 10), the minimum is at 1.64 and here relaxation is about 16 and 86 times faster. It is asymptotically at least 3 times faster than the optimum with a relaxation factor in [1.2, 1.9] and than the estimated in [0.3, 1.9]. For ω equal to 1 and 1.5, it is 3 and 10 times faster than the optimum. The spectral radius is smaller than the relationship which has its minimum at $\omega = 1.72$ for $\omega \leqslant 1.68$.

The results are plotted in figure 11 for duobinary encoding with R = 173.5. The relaxation method has its best asymptotic result at 1.32 where it is 8 and 32 times faster than the optimum and estimated gradients. It is at least 3 times faster than the optimum for ω in [0.9, 1.7] and than the estimated in [0.3, 1.9]. For relaxation factors of 1 and 1.5, the optimum is 4 to 6 times slower. The spectral radius is bounded by the 2-cyclic relationship for values of ω less than 1.4. The relationship has its minimum at 1.74.

For the examples considered, the relaxation spectral radius is bounded by the 2-cyclic-type relationship for values of ω up to the value yielding the spectral radius minimum; therefore, asymptotic improvements similar to those obtained for the 2-cyclic case are possible for channels with general characteristics. Although no clear pattern emerges for the minimum, values of ω around 1 seem to yield better asymptotic results for small and moderate distortion channels, and values around 1.5 for large and enormous distortions. The amount of distortion present can be determined from the estimated condition number (keeping in mind that the estimate is poor).

The minima for the evaluated spectral radii occurred for values of ω larger than 1. This suggests that values of $\omega < 1$ need not be considered because better asymptotic results occur for $\omega > 1$. This was analytically proven for the special cases considered.

Norm-Decreasing Property

A bound for the relaxation spectral norm will be developed for general correlation matrices in the following sections. This bound, although not in a suitable form for comparisons, proves that the relaxation method is norm decreasing at each iteration for certain intervals of the relaxation factor values.

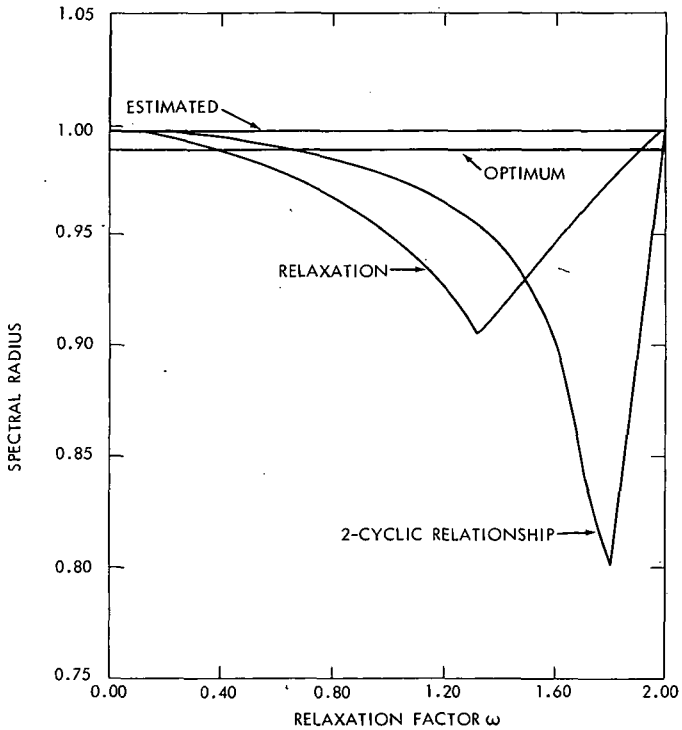

Figure 11.—Spectral radii for fixed step-size gradient and relaxation methods with duobinary encoding, $R = 173.5$, and 17 taps.

From an alternate representation of the relaxation matrix, an upper bound for the spectral norm can be found. The relaxation matrix $\mathcal{L}_\omega$ can be expressed as

$$\mathcal{L}_\omega = (\mathbf{D} - \omega\mathbf{E})^{-1}[(1-\omega)\mathbf{D} + \omega\mathbf{E}^+] \tag{63}$$

The matrix $(\mathbf{D} - \omega\mathbf{E})^{-1}$ can be expanded in a power series in $\omega\mathbf{D}^{-1}\mathbf{E}$. Because $\mathbf{E}$ is a strictly lower triangular matrix, $\mathbf{E}^m$ is identically zero for all $m \geqslant 2N + 1$. The relaxation matrix becomes

$$\mathcal{L}_\omega = \left(\mathbf{I} + \frac{\omega}{a_0}\mathbf{E} + \frac{\omega^2}{a_0^2}\mathbf{E}^2 + \cdots + \frac{\omega^{2N}}{a_0^{2N}}\mathbf{E}^{2N}\right)\left[(1-\omega)\mathbf{I} + \frac{\omega\mathbf{E}^+}{a_0}\right]$$

Using norm sum and product inequalities, the spectral norm of the relaxation matrix is bounded by

$$\|\mathcal{L}_\omega\| \leqslant \left(1 + \frac{\omega}{a_0}\|\mathbf{E}\| + \cdots + \frac{\omega^{2N}}{a_0^{2N}}\|\mathbf{E}\|^{2N}\right)\left(|1-\omega| + \frac{\omega}{a_0}\|\mathbf{E}\|\right) \tag{64}$$

The finite series can be expressed as a quotient of two polynomials. A more workable form of equation (64) is

$$\|\mathcal{L}_\omega\| \leqslant \frac{1 - (\omega^{2N}/a_0^{2N})\|\mathbf{E}\|^{2N}}{1 - (\omega/a_0)\|\mathbf{E}\|}\left(|1-\omega| + \frac{\omega}{a_0}\|\mathbf{E}\|\right) \tag{65}$$

For a zero relaxation factor, the upper bound $N_u(\omega)$ reduces to 1, which is the actual value of $\mathcal{L}_0$. Differentiating the bound with respect to ω, one obtains

$$\frac{\partial N_u(\omega)}{\partial\omega} = \left(\frac{\|\mathbf{E}\|}{a_0} + 2\omega\frac{\|\mathbf{E}\|^2}{a_0^2} + \cdots + 2N\omega^{2N-1}\frac{\|\mathbf{E}\|^{2N}}{a_0^{2N}}\right)\left(|1-\omega| + \frac{\omega}{a_0}\|\mathbf{E}\|\right)$$

$$+ \left(1 + \frac{\omega}{a_0}\|\mathbf{E}\| + \cdots + \frac{\omega^{2N}}{a_0^{2N}}\|\mathbf{E}\|^{2N}\right)\left[\operatorname{sgn}(\omega - 1) + \frac{\|\mathbf{E}\|}{a_0}\right] \tag{66}$$

For $\omega > 1$, the derivative is positive, which implies that the bound is monotonically increasing. For a relaxation factor less than 1, after some manipulation the derivative becomes

$$\frac{\partial N_u(\omega)}{\partial \omega} = \left(2\frac{\|\mathbf{E}\|}{a_0} - 1\right)\left(1 + 2\omega\frac{\|\mathbf{E}\|}{a_0} + \cdots + 2N\omega^{2N-1}\frac{\|\mathbf{E}\|^{2N-1}}{a_0^{2N-1}}\right) + \left(\frac{\|\mathbf{E}\|}{a_0} - 1\right)(2N+1)\omega^{2N}\frac{\|\mathbf{E}\|^{2N}}{a_0^{2N}} \tag{67}$$

It is definitely negative for $\|\mathbf{E}\| \leq a_0/2$ and therefore the spectral radius is a monotonically decreasing function of ω having a minimum value of $1 - \|\mathbf{E}\|^{2N}/(1 - \|\mathbf{E}\|)\|\mathbf{E}\|$ at $\omega = 1$. The spectral norm of $\mathbf{E}$ is another measure of the intersymbol interference present in the channel, in that as the intersymbol interference increases, $\|\mathbf{E}\|$ increases. The restriction $\|\mathbf{E}\| \leq a_0/2$ is equivalent to limitation to channels with light or moderate intersymbol interference or that give rise to 2-cyclic correlation matrices. For these channels, the bound indicates that the average rate of convergence is smallest at $\omega = 1$ and that the method is at each iteration monotonically decreasing the mean-square coefficient error for a range of ω. This bound for the 2-cyclic case is sketched in figure 7 and is smaller than the fixed gradient norm for some combinations of equalizer dimension and channel distortion. For channels that have $\|\mathbf{E}\| > a_0$, the bound is an increasing function of ω and hence not very useful.

The theory of perturbation can be used to demonstrate that the relaxation method is norm decreasing for channels that cause large intersymbol interference. Let the relaxation factor be equal to ϵ, which is positive but small. The relaxation matrix is equal to

$$\mathcal{L}_\omega = \left(\mathbf{I} + \epsilon\frac{\mathbf{E}}{a_0} + \epsilon^2\frac{\mathbf{E}^2}{a_0^2} + \cdots + \epsilon^{2N}\frac{\mathbf{E}^{2N}}{a_0^{2N}}\right)\left(\mathbf{I} - \epsilon\frac{\mathbf{D} + \mathbf{E}^+}{a_0}\right) \tag{68}$$

Expansion yields

$$\mathcal{L}_\omega = \mathbf{I} - \frac{\epsilon}{a_0}\mathbf{A} - \frac{\epsilon^2}{a_0^2}\mathbf{E}\mathbf{A} + 0(\epsilon^3) \tag{69}$$

Let ϵ be sufficiently small so that the third term is negligible in comparison to the second term. Notice that the relaxation method has reduced to the fixed gradient technique and hence its spectral norm is less than 1. Because the norm is a continuous function of ω, there exists an interval for the relaxation factor such that the relaxation method is a coefficient-mean-square-error-decreasing iterative technique.

Numerical Evaluation of Spectral Norm

The spectral norm was numerically evaluated for the four channels for which the spectral radii were determined earlier. This was done for the purpose of collaborating the theoretical bound and of demonstrating that the technique is norm decreasing for a much larger parameter range than indicated by the theory of perturbation. Figures 12 through 15 contain the results for the various techniques. The spectral norms for the estimated and optimum fixed step-size gradient are constant with respect

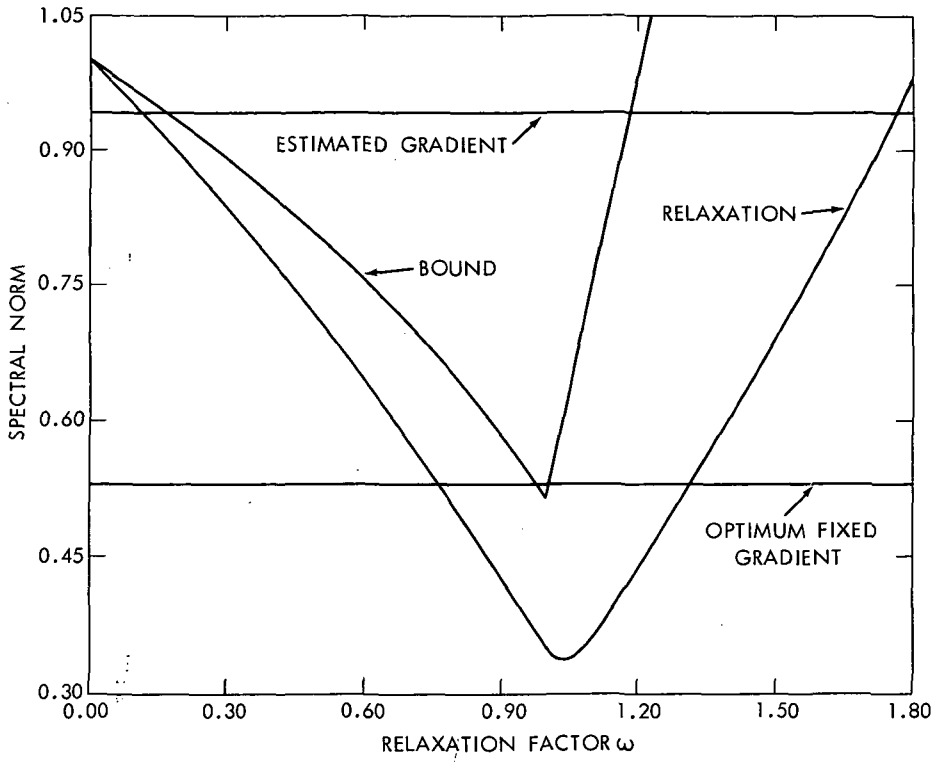

Figure 12.—Spectral norms for fixed gradient and relaxation methods with $R \approx 3.28$ and 17 taps.

to the relaxation factor. The relaxation spectral norm is observed to be a continuous concave function of ω, with the value of 1 at $\omega = 0$.

Figure 12 has the results for the channel having a condition number of 3.28. This channel will cause moderate distortion of the transmitted signal and has a norm upper bound (eq. (65)) that is not a monotonically increasing function of ω. For all other channels considered, $\|\mathbf{E}\|$ is not less than $a_0/2$ and hence the bound was an increasing function of ω. At $\omega = 1.05$, the spectral norm for the relaxation method has a minimum that is smaller than the optimum fixed gradient and is equal to the 17th power of the norm for the estimated gradient. The average rate of convergence for the relaxation method for all iterations is larger than that for the optimum and estimated gradient. Relaxation is almost twice as fast as the optimum gradient, and is 17 times faster than the estimated gradient. For ω in the interval [0.75, 1.3], the relaxation method is better than the optimum gradient and in [0.2, 1.7] faster than the estimated gradient. For a relaxation factor of 1.5, the optimum fixed step-size gradient is a little better than the relaxation, but the relaxation is six times faster than the estimated. The relaxation method will definitely decrease the mean-square coefficient error for all values of ω up to 1.9.

The results for the channel with a condition number of 17.81 are plotted in figure 13. The relaxation method has a minimum at $\omega \cong 1.125$. At this value of the relaxation factor, the relaxation method has an average rate of convergence that is two times as fast as the optimum and 18 times

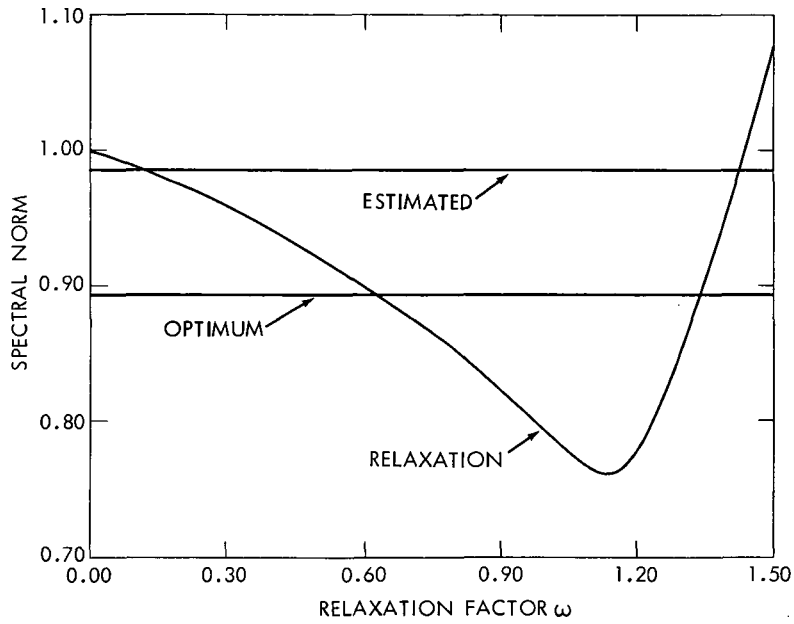

Figure 13.–Spectral norms for fixed gradient and relaxation methods with R = 17.81 and 17 taps.

faster than the estimated fixed step-size gradient. For ω in the interval [0.65, 1.35], the relaxation method is faster than the optimum; and in the interval [0.2, 1.4], better than the estimated. For values of ω up to 1.45, the relaxation norm is strictly less than 1. At $\omega = 1.5$, a comparison cannot be made because the relaxation norm is greater than 1.

The results for duobinary encoding with channels that had condition numbers of 150.4 and 173.5 are in figures 14 and 15, respectively. The relaxation norm has a minimum at ω = 1.075 for R = 150.4, and at ω = 0.825 for R = 173.5. For R = 150.4, relaxation is almost 3 times faster than the optimum, and 15 times faster than the estimated; for R = 173.5, it is better than 2 and 10 times as fast as the optimum and estimated fixed step-size gradients, respectively. For R = 150.4, in the factor intervals of [0,5, 1.2] and [0.2, 1.2], the relaxation norm is smaller than those for the optimum and estimated gradients. It is also norm decreasing for values up to 1.3. With R = 173.5 and intervals [0.2, 0.9] and [0.4, 0.9], the estimated and optimum are slower. For ω up to and including 0.925, the spectral norm is less than 1.

The numerical evaluations of the spectral norm for the different channels indicate that great improvements over the fixed step-size gradients are possible with the use of the relaxation method. They also suggest that the norm-decreasing property for small ω is valid for a large range, and that relaxation factors around 1 yield nearly the best average rates of convergence.

Numerical Evaluation of $\|\mathcal{L}_\omega^k\|$

The upper-bound estimate for the normalized mean-square coefficient error, $\|\mathcal{L}_\omega^k\|$ for relaxation, has been plotted in figures 16 and 17 for the various techniques. The relaxation factor used is 1.5.

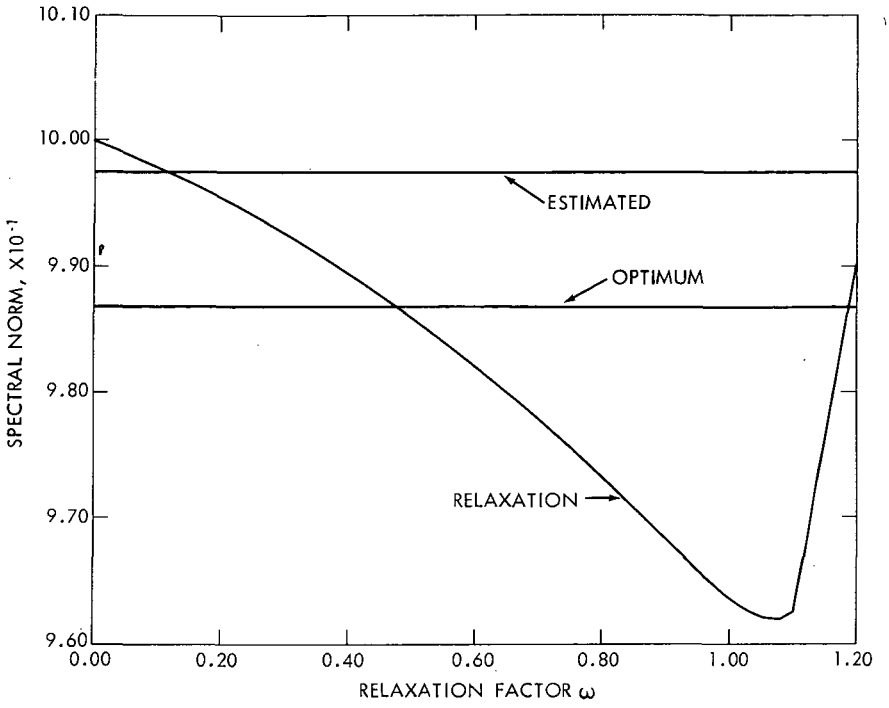

Figure 14.—Spectral norms for fixed gradient and relaxation methods with duobinary encoding, R = 150.4, and 17 taps.

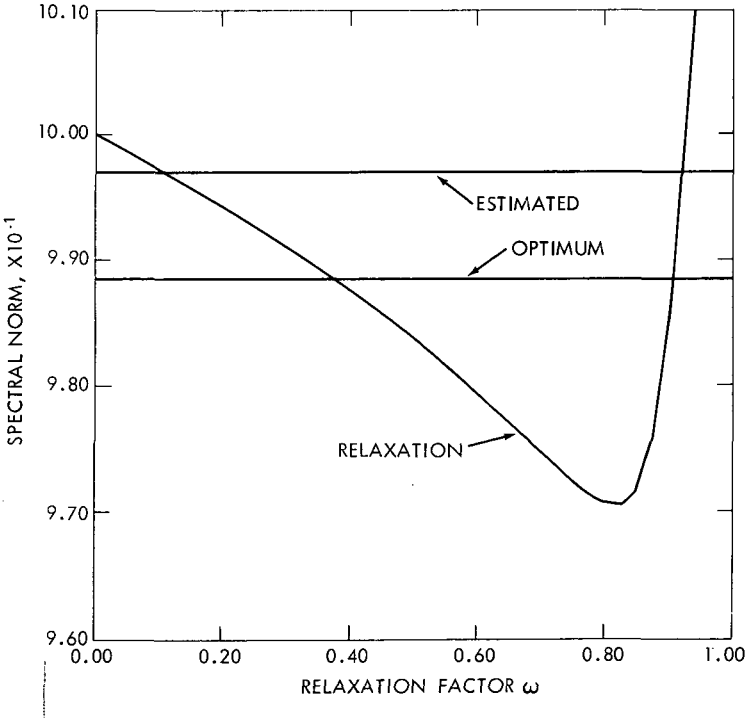

Figure 15.—Spectral norms for fixed gradient and relaxation methods with duobinary encoding, R = 173.5, and 17 taps.

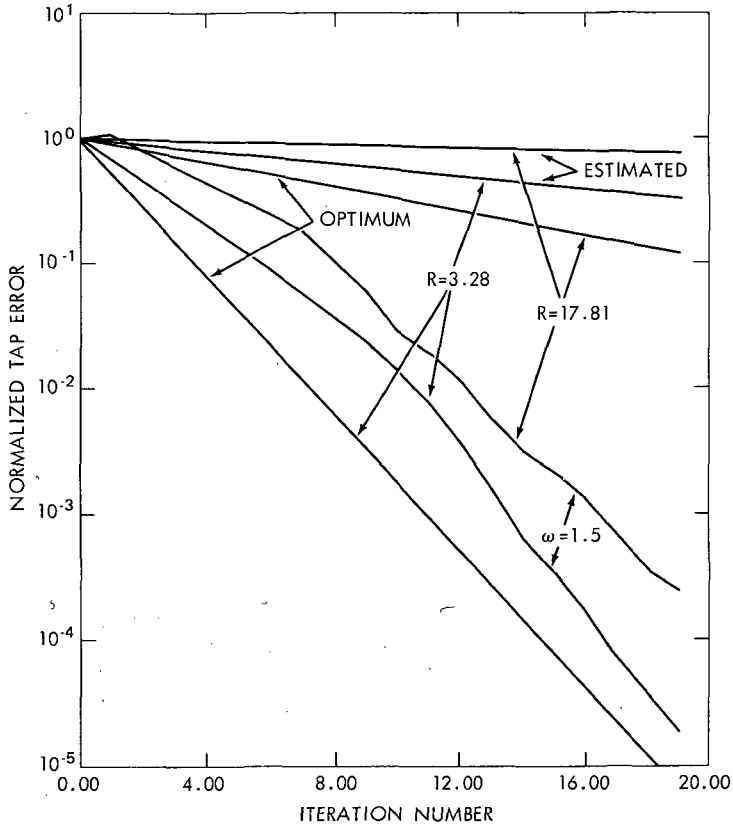

Figure 16.—Normalized tap error norms versus iteration number.

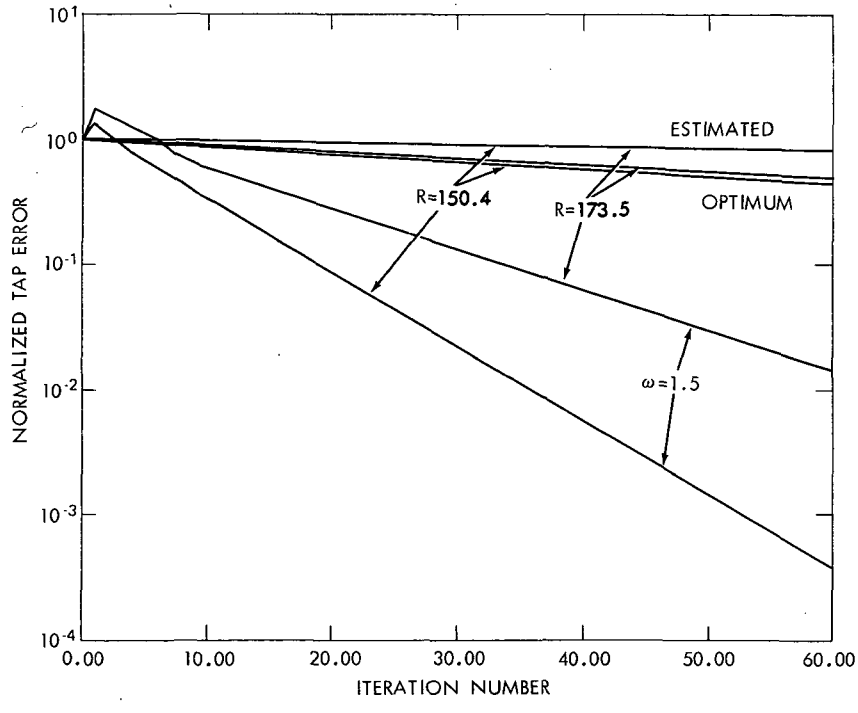

Figure 17.—Normalized tap error norms versus iteration number, duobinary system.

For $R = 3.28$ the optimum fixed step-size gradient is, not surprisingly, faster, reaching an error of 10^{-3} in 11 steps, whereas with relaxation, it required 14 iterations, but relaxation is at least seven times faster than the estimated gradient. For $R = 17.81$, relaxation is faster (two times better than the optimum and seven times better than estimated) even though initially it is not norm decreasing. For the duobinary cases, relaxation is again much faster, but initially not norm decreasing. The plots are almost linear on the log scale after a few iterations, indicating that mean-square error is decreasing proportionally to some factor raised to the kth power, where k is the iteration number. The factor is very close to the values of the relaxation spectral radius with $\omega = 1.5$. Relaxation is seven and six times better than the optimum gradient, and nine times better than the estimated gradient for $R = 150.4$ and 173.5, respectively.

Conclusions

Analytically, the relaxation algorithm has been demonstrated to be asymptotically faster than the gradient techniques for the following cases:

(1) Channels that give rise to 2-cyclic correlation matrices

(2) Channels that have diagonally dominant Jacobi matrices with all entries of the same sign and with equations (53) and (54) used as the estimates for the eigenvalues

Although these results have not been analytically extended to channels with general characteristics, numerical evaluations of the spectral radius for the considered channels strongly suggest that similar improvements to those for the 2-cyclic case can be obtained for general channels.

EFFECTS OF NOISE

In the preceding section, the behavior of the relaxation algorithm was investigated in a noise-free environment and with infinite precision. Physical systems, on the other hand, are usually corrupted by noise and are limited to finite precision. In this section, the effects of additive channel noise and limited precision will be investigated separately.

When the incoming signal is corrupted by additive noise, the filter coefficients become random variables. Their final mean is the value about which the coefficients will oscillate after convergence, and the variance is a measure of the peak-to-peak oscillations. When the system is limited by finite precision, this limitation can be viewed as the actual value plus an additive noise that is uniformly distributed with zero mean and variance proportional to the difference in permissible coefficient levels. Here again, the coefficients are random variables, but with zero means. The variance bounds for both cases are similar.

Additive Channel Noise

In the presence of additive channel noise, the input to the equalizer is z_k, where $z_k = x_k + n_k$. The noise samples n_k are assumed to be independent random variables that are identically distributed, gaussian with zero mean and variance σ^2. The equalizer dimension is chosen to contain the dispersed

pulse. The measured gradient becomes

$$\frac{1}{2}\nabla_{C^k}\mathcal{E} = \sum_l Z_l(Z_l^+ C^k - d_l) \tag{70}$$

which in matrix notation is

$$\frac{1}{2}\nabla_{C^k}\mathcal{E} = \mathbf{H}_k C^k - \mathbf{g} - \mathbf{g}_k$$

with Z_l the input vector

$$Z_l = \begin{bmatrix} x_{l-N} + n_{l-N+k(2N+1)} \\ \vdots \\ x_{l+} + n_{l+k(2N+1)} \\ \vdots \\ x_{l+N} + n_{l+N+k(2N+1)} \end{bmatrix} \tag{71}$$

and $\mathbf{H}_k$, the new channel correlation matrix given by

$$\mathbf{H}_k = \sum_m Z_m Z_m^+$$

$$\mathbf{g} = \sum_m d_m \mathbf{X}_m$$

$$\mathbf{g}_k = \sum_m d_m \mathbf{n}_{m+k(2N+1)}$$

The relaxation strategy is

$$C^{k+1} = C^k - \omega(\mathbf{D}_k - \omega\mathbf{E}_k)^{-1}(\mathbf{H}_k C^k - \mathbf{g} - \mathbf{g}_k) \tag{72}$$

where $\mathbf{H}_k = \mathbf{D}_k - \mathbf{E}_k - \mathbf{E}_k^+$ with $\mathbf{D}_k$ and $\mathbf{E}_k$ the previously defined diagonal and strictly lower triangular portions of $\mathbf{H}_k$. The filter coefficients are now random variables that will converge in the mean to the solution $\langle C \rangle$ of

$$\mathbf{E}\left\{(\mathbf{D}_k - \omega\mathbf{E}_k)^{-1}(\mathbf{H}_k\langle C\rangle - \mathbf{g} - \mathbf{g}_k)\right\} = 0 \tag{73}$$

if the spectral radius of the expected value of the relaxation matrix $\langle \mathbf{M}_k \rangle$ is less than 1.

The correlation matrix is positive definite hermitian with probability 1. Therefore $(\mathbf{D}_k - \omega\mathbf{E}_k)$ is nonsingular for all values of the relaxation parameter in the interval (0, 2). Then, by Ostrowski's theorem the spectral radius of the relaxation matrix and, hence, of its expected value is strictly less than 1 with probability 1. Therefore, convergence is guaranteed for all possible channel distortions, as long as the relaxation factor is in the interval (0, 2).

The filter coefficients will oscillate around the mean value $\langle C \rangle$. Notice that coefficients do not converge to the noise-free optimum setting and the mean final square error is larger. This bias cannot

be eliminated with the present receiver, but if the receiver were altered to contain an adaptive matched filter, the bias could be reduced.

The coefficient error, the difference between the coefficient values and the optimum values, is given by

$$\epsilon^k = \mathbf{M}_k \epsilon^{k-1} - \omega \mathbf{h}_k \tag{74}$$

where

$$\mathbf{M}_k = (\mathbf{D}_k - \omega \mathbf{E}_k)^{-1} [(1 - \omega)\mathbf{D}_k + \omega \mathbf{E}_k^+]$$

$$\mathbf{h}_k = (\mathbf{D}_k - \omega \mathbf{E}_k)^{-1} (\mathbf{H}_k \langle \mathbf{C} \rangle - \mathbf{g} - \mathbf{g}_k)$$

The expected value of this error tends toward zero as the number of iterations increase because the technique converges. In the following discussion, the variance will be shown to be bounded.

The mean-square error for a vector $\mathbf{V}$ is equal to $[\mathbf{E}(\mathbf{V}^+\mathbf{V})]^{1/2}$. Let the norm of a random vector be the mean-square error because it satisfies all the properties associated with norms.

$$\|\mathbf{V}\|^2 = \mathbf{E}(\mathbf{V}^+\mathbf{V}) = \langle \mathbf{V}^+\mathbf{V} \rangle \tag{75}$$

For deterministic vectors, this reduces to the euclidean norm used earlier.

Premultiplying equation (74) by ϵ^{k+} and taking expected values of both sides, the following is obtained:

$$\|\epsilon^k\|^2 = \mathbf{E}\left\{\epsilon^{k-1+}\mathbf{M}_k^+\mathbf{M}_k\epsilon^{k-1}\right\} - 2\omega\mathbf{E}\left\{\epsilon^{k-1+}\mathbf{M}_k^+\mathbf{h}_k\right\} + \omega^2\|\mathbf{h}_k\|^2 \tag{76}$$

The relaxation matrix at the kth iteration is independent of the coefficient error at the $(k - 1)$ iteration. Using the Schwarz inequality, the first term of the right-hand side of equation (76) becomes

$$\mathbf{E}\left\{\epsilon^{k-1+}\mathbf{M}_k^+\mathbf{M}_k\epsilon^{k-1}\right\} \leqslant \|\epsilon^{k-1}\| \, \|\langle \mathbf{M}_k^+\mathbf{M}_k \rangle \epsilon^{k-1}\| \leqslant \mu\|\epsilon^{k-1}\|^2 \tag{77}$$

where $\mu = \|\langle \mathbf{M}_k^+\mathbf{M}_k \rangle\|$ and the matrix norm is the usual spectral norm. The middle term is equal to

$$\omega^2 \langle \epsilon^{k-1+}\mathbf{H}_k(\mathbf{D}_k - \omega\mathbf{E}_k^+)^{-1}\mathbf{h}_k \rangle \tag{78}$$

Again using the Schwarz inequality, expression (78) is bounded by

$$\omega^2 \|\langle \epsilon^{k-1} \rangle\| \, \|\langle \mathbf{H}_k^+(\mathbf{D}_k - \omega\mathbf{E}_k^+)^{-1}\mathbf{h}_k \rangle\| \tag{79}$$

Due to convergence, $\|\langle \epsilon^k \rangle\| \to 0$; hence, the bound is zero and expression (78) is also zero for large k.

For large k, equation (76) becomes

$$\|\epsilon^k\|^2 \leqslant \mu\|\epsilon^{k-1}\|^2 + \omega^2\|\mathbf{h}_k\|^2 \tag{80}$$

Define the sequence of numbers as described by the first-order driven difference equation

$$q^k = \mu q^{k-1} + \omega^2\|\mathbf{h}_k\|^2 \tag{81}$$

The norm of the coefficient error is bounded by this sequence

$$\|\epsilon^k\|^2 \leqslant q^k \tag{82}$$

Solving the first-order difference equation in terms of the initial value,

$$q^k = \mu^k q^0 + \omega^2 \|\mathbf{h}\|^2 \sum_{i=0}^{k-1} \mu^i \tag{83}$$

Asymptotically, if $\mu < 1$, the solution becomes in the limit

$$q^k \to \frac{\omega^2 \|\mathbf{h}\|^2}{1 - \mu} \tag{84}$$

Therefore, the variance of the coefficient error in steady state is bounded by

$$\|\epsilon^\infty\|^2 \leqslant \frac{\omega^2 \|\mathbf{h}\|^2}{1 - \mu} \tag{85}$$

For the fixed step-size gradient technique, a similar bound is obtained

$$\|\epsilon^\infty\|_{\text{gradient}} \leqslant \frac{\alpha^2 \|\mathbf{h}_1\|^2}{1 - \rho^2} \tag{86}$$

where

$$\mathbf{h}_1 = \mathbf{H}_k \langle \mathbf{C} \rangle - \mathbf{g} - \mathbf{g}_k \qquad \text{and} \qquad \rho = \|\langle \mathbf{I} - \alpha \mathbf{H}_k \rangle\|$$

The bounds for the variance of both techniques are similar in nature. Initially, it would seem that the dominant factor in decreasing the variance would be the spectral norms ρ or μ, but it turns out that making the step size smaller than the optimum more than compensates for the increase in the spectral norm. Because the eigenvalue sum $\lambda_{max} + \lambda_{min}$ is overestimated, the step size is decreased, and hence the resultant variance is reduced for the fixed gradient technique. There appears to be a tradeoff between the speed of convergence and the variance value. Consider the fixed step-size gradient and the Robbins-Monroe technique as examples. The Robbins-Monroe method is a variable step-size gradient that forces the variance to be zero at convergence. Convergence is extremely slow and may even require an infinite amount of steps. The fixed step-size gradient converges much more quickly, but has a finite nonzero variance. This tradeoff is also evident with the fixed step-size gradient for different step sizes.

Quantization Noise

The effects of finite coefficient precision on the equalizer will be investigated in the following sections. The assumption made thus far is that the variable parameters are continuous; i.e., they can assume any value. In an actual realization of a digital filter, all the coefficients are discrete because the number of binary bits or the word length is finite. Therefore, the initial coefficient vector choice and all subsequent values must belong to a finite set of numbers. The correction algorithm can be modified to satisfy this constraint. Consider the relaxation algorithm, equation (28):

$$\mathbf{C}^{k+1} = \mathbf{C}^k - [\omega(\mathbf{D} - \omega\mathbf{E})^{-1}(\mathbf{A}\mathbf{C}^k - \mathbf{g})] \tag{28}$$

The constraint is satisfied by forcing the correction, the bracketed term, to be a member of the finite set. The quantization effect (finite precision) is analogous to the effect of noise superimposed on the original analog value. The relaxation algorithm becomes

$$\mathbf{C}^{k+1} = \mathbf{C}^k - \omega(\mathbf{D} - \omega\mathbf{E})^{-1}(\mathbf{A}\mathbf{C}^k - \mathbf{g}) - \mathbf{e}^k \tag{87}$$

where the noise $\mathbf{e}^k$ is uniformly distributed in the interval $[-E_0/2, E_0/2]$ with E_0 the separation between different quantization levels. The different noise samples are assumed to be uncorrelated and have zero mean and variance equal to $E_0^2/12$. Taking the expected value of equation (87), it is seen that the filter coefficients converge in the mean to the optimum value (infinite precision):

$$\langle \mathbf{C}^k \rangle \to \mathbf{A}^{-1}\mathbf{g} \tag{88}$$

Defining the vector error as the difference between the actual and the average filter coefficient values

$$\mathbf{Q}^k = \mathbf{C}^k - \langle \mathbf{C}^k \rangle \tag{89}$$

equation (87) becomes

$$\mathbf{Q}^{k+1} = \mathcal{L}_\omega \mathbf{Q}^k - \mathbf{e}^k \tag{90}$$

where $\mathcal{L}_\omega$ is the relaxation matrix. The solution of this first-order difference equation is

$$\mathbf{Q}^{k+1} = \mathcal{L}_\omega^{k+1}\mathbf{Q}^0 - \sum_{m=0}^{k} \mathcal{L}_\omega^m \mathbf{e}^{k-m}$$

Taking norms of both sides and using the norm sum inequality,

$$\|\mathbf{Q}^{k+1}\| \leqslant \|\mathcal{L}_\omega^{k+1}\| \, \|\mathbf{Q}^0\| + \left\| \sum_{m=0}^{k} \mathcal{L}_\omega^m \mathbf{e}^{k-m} \right\|$$

Because for large $k \|\mathcal{L}_\omega^k\| \to 0$ (due to convergence), equation (91) is obtained:

$$\|\mathbf{Q}^{k+1}\| \leqslant \left\| \sum_{m=0}^{k} \mathcal{L}_\omega^m \mathbf{e}^{k-m} \right\| \tag{91}$$

Because the quantization noise is uncorrelated and $\|\mathbf{e}^k\|^2 = (2N+1)E_0^2/12$, the following relation is obtained:

$$\|\mathbf{Q}^{k+1}\| \leqslant E_0 \sum_{m=0}^{k} \|\mathcal{L}_\omega^m\| \sqrt{\frac{2N+1}{12}} \tag{92}$$

A closed-form solution for the summation can be obtained if the relaxation factor lies in the region where the spectral norm is less than unity:

$$\|\mathbf{Q}^{k+1}\| \leqslant \frac{E_0}{1-\beta}(1-\beta^k)\sqrt{\frac{2N+1}{12}} \tag{93}$$

with

$$\beta = \|\mathcal{L}_\omega\|$$

Similarly for the fixed step-size gradient technique, the coefficients converge to the optimum value (88) and the coefficient standard deviation is bounded by

$$\|\mathbf{Q}^{k+1}\| \leq \frac{E_0}{1-\rho}(1-\rho^k)\sqrt{\frac{2N+1}{12}} \tag{94}$$

The variance bounds (93) and (94) are monotonically increasing functions of both the respective spectral norm and the equalizer dimensions. The relaxation bound is definitely smaller than the fixed gradient bound in those regions where the relaxation norm is smaller than the gradient norm. (See the section entitled "Convergence Properties.")

As the distortion caused by the channel increases, the condition number and, hence, both of the spectral norms increase. This in turn causes an increase in the variance bound. If the equality of the bound holds, quantization can cause large oscillations about the final value in poor conditioned matrices, i.e., large condition numbers.

DIGITAL SIMULATIONS

The data transmission system of figure 1 was simulated on a computer. Intersymbol interference was generated by sending a pulse with a raised cosine transformed through a channel with parabolic delay and amplitude ripple $1 + a_r \cos(2\omega T)$. The channel amplitude ripple a_r was varied to simulate different intersymbol interference conditions; a_r values of 0.3 and 0.65 resulted in correlation matrix condition numbers R of 3.28 and 17.81, respectively, where $R = \lambda_{max}/\lambda_{min}$. Figure 18 shows the resulting pulse samples. The pulse yielding a condition number of 3.28 has a peak distortion D_0 of 3.98, and the pulse with condition number 17.81 has a peak distortion of 5.78, where

$$D_0 = \sum_{i \neq 0} \frac{|x_i|}{x_0}$$

Both pulses would lead to a divergent Lucky iteration. Because the dispersion is for 17 samples at most, a 17-dimensional digital filter was used as the equalizer receiver. The duobinary encoding technique was also simulated using the same channels. The performance of the successive overrelaxation and gradient methods as the algorithms for adaptive equalization were investigated for these channels for both normal and duobinary transmission.

Figure 19 shows the output mean-square error $\|\mathbf{e}^k\|$ versus the iteration number k when the fixed step-size gradient, first-order variable step-size gradient, and the successive overrelaxation algorithms are used for the adjustment of the equalizer coefficients for the first pulse of figure 18 ($R = 3.28$). The upper bound λ_u, derived from the trace of the correlation matrix, and the lower bound λ_l, estimated from the pulse spectrum, are used to estimate the minimum and the maximum eigenvalues. (These bounds are determined by eqs. (B-3) and (B-2), respectively.) The relaxation method uses a factor of 1.5 and converges in 13 steps. This is at least twice as fast as the Chebyshev gradient and about six times faster than the fixed step-size gradient. A relaxation factor of 1.1 further improved the convergence by a factor of 2.

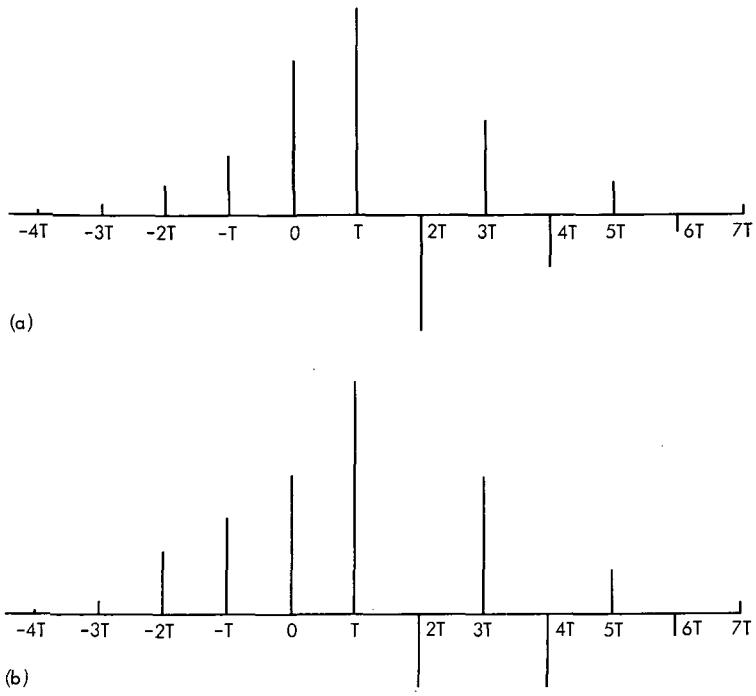

Figure 18.—Received pulses. (*a*) $R = 3.28$, $D_0 = 3.98$. (*b*) $R = 17.81$, $D_0 = 5.78$.

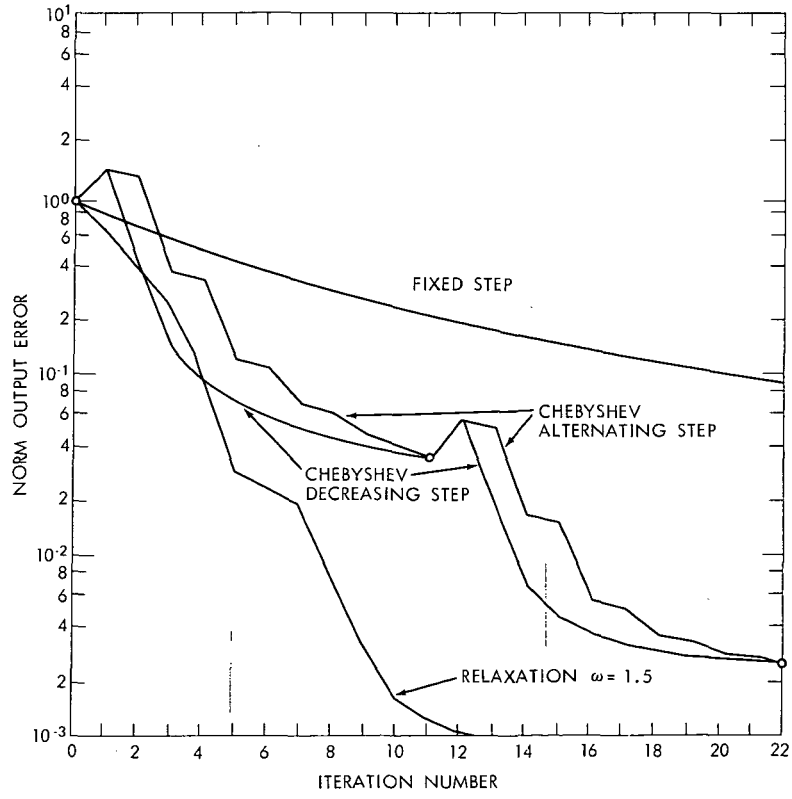

Figure 19.—Comparison of 11th-degree Chebyshev, relaxation, and fixed step-size gradient algorithms with $R = 3.28$, $\lambda_u/\lambda_l = 35.1$, and 17 taps.

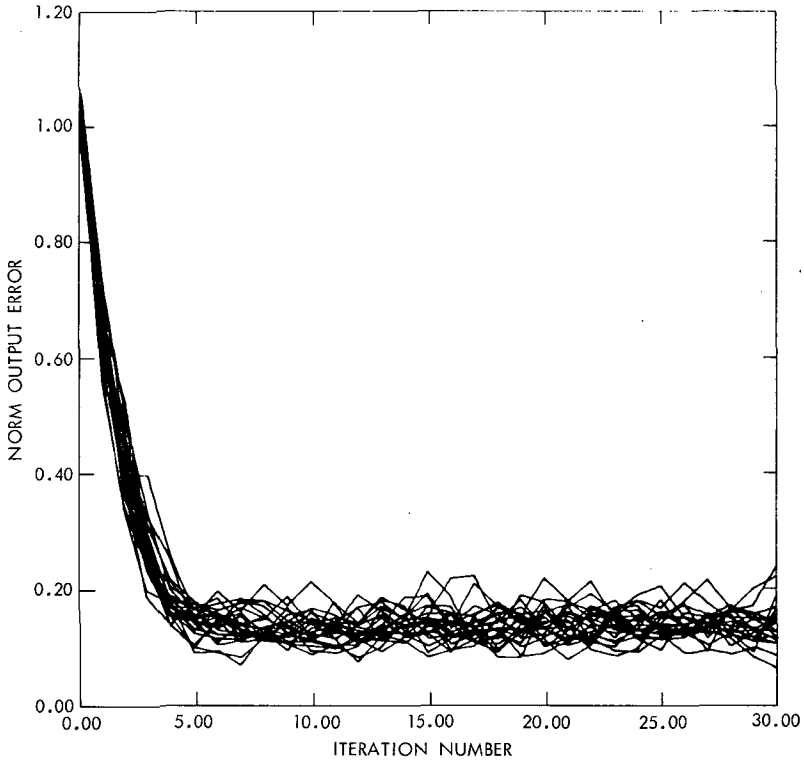

Figure 20.—Convergence of relaxation algorithm for 30 sample runs with S/N = 30 dB, R = 3.2, and 17 taps.

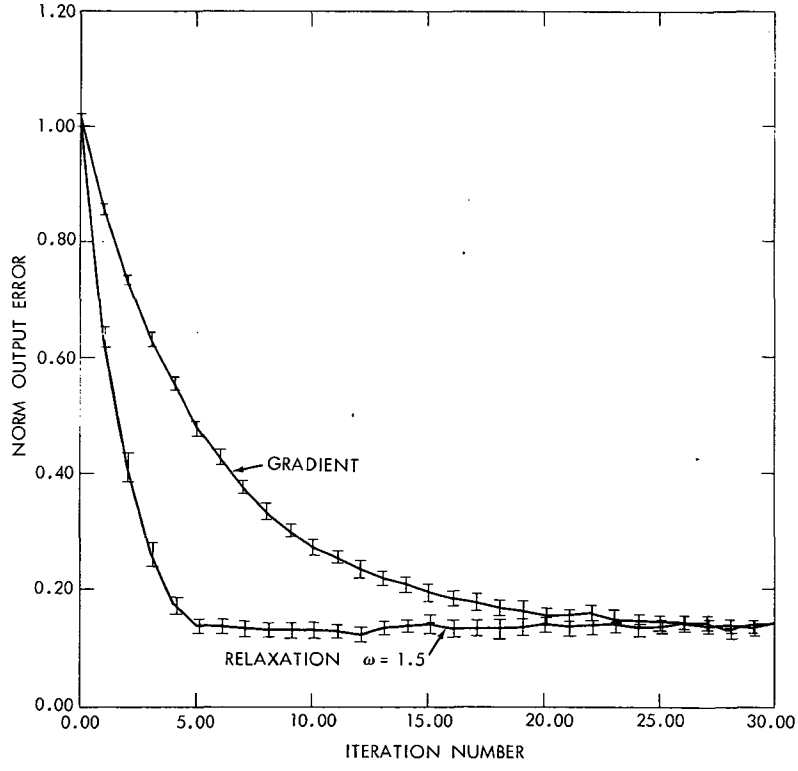

Figure 21.—Comparison of relaxation and fixed step-size gradient algorithms with S/N = 30 dB, R = 3.2, and 17 taps.

Next, gaussian noise was added to the input pulse samples. Figures 20 and 21 contain the results for a 30-dB input-signal-to-noise ratio (S/N). Thirty sample runs were conducted with independent noise samples. Figure 20 shows the relaxation algorithm behavior for all 30 independent runs. The computed average error norm is plotted in figure 21, with the standard deviation marked by vertical lines for the relaxation and fixed step-size gradient techniques. The eigenvalues are again estimated (from eqs. (B-3) and (B-2)) and the relaxation factor used is 1.5. The fixed step-size gradient algorithm requires at least five times as many iterations as the relaxation algorithm for convergence in the mean. The resultant standard deviation for relaxation is slightly larger than that for the gradient. The first-order variable gradient required nine iterations (ref. 11); the relaxation method required five.

For the channel yielding the second pulse (R = 17.81) of figure 18, the equalization results with both gradient techniques and the relaxation method are plotted as a function of the iteration number in figure 22. The relaxation algorithm with $\omega = 1.5$ converged in 12 iterations. This is four times faster than the Chebyshev algorithm and about seven times faster than the fixed gradient.

Figure 23 shows the output mean-square error and the standard deviation for 30 independent noise runs with S/N = 30 dB for the fixed step gradient and the relaxation method. The relaxation algorithm ($\omega = 1.5$) converged in seven iterations. This is at least seven times faster than the estimated

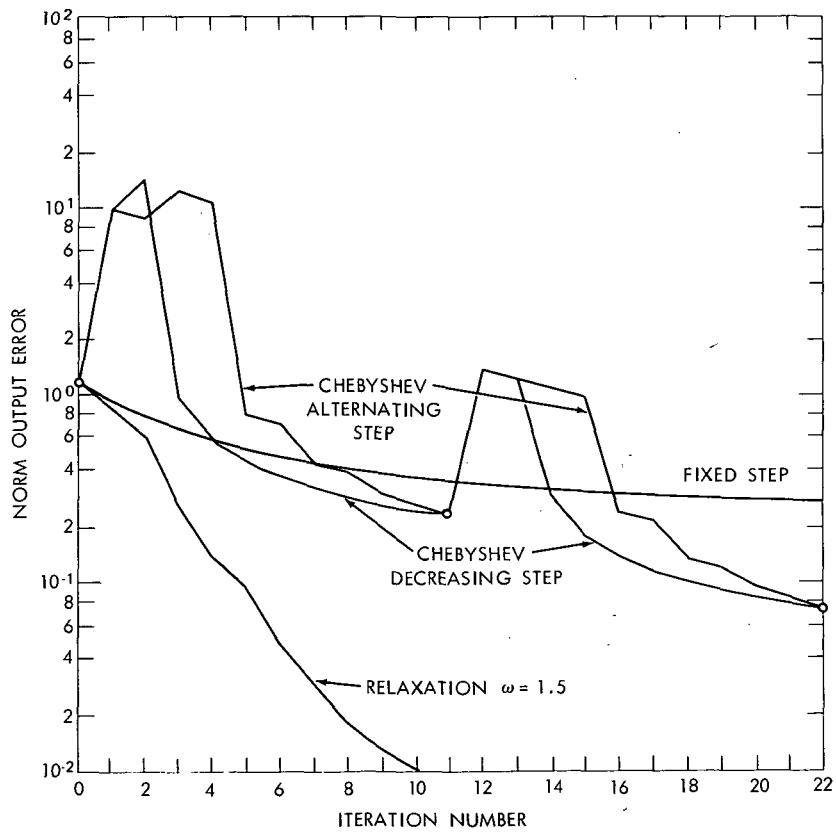

Figure 22.—Comparison of 11th-degree Chebyshev, relaxation, and fixed step-size gradient algorithms with R = 17.81, λ_u/λ_l = 164, and 17 taps.

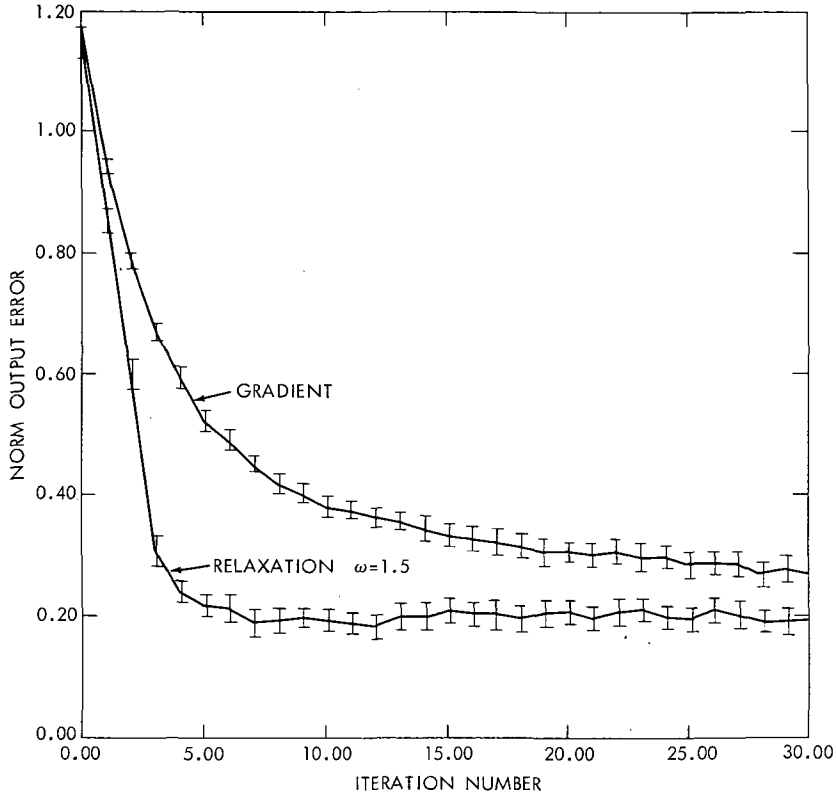

Figure 23.—Comparison of relaxation and fixed step-size gradient algorithms with S/N = 30 dB, R = 15.8, and 17 taps.

gradient with a slightly higher standard deviation. The Chebyshev gradient required 20 iterations (ref. 11).

Figures 24 and 25 contain the equalization results obtained when the duobinary encoding technique was applied to the transmitted signal for the channels with condition numbers R = 3.28 and 17.81, respectively. The condition numbers for the channel correlation matrix increased enormously to 150.4 and 173.5, respectively. The fixed step-size gradient with the optimum step size α_0 and the successive overrelaxation method with ω = 1.5 were used for the iterative adjustment of the equalizer parameters in the noiseless case. Convergence for the relaxation method occurred in 40 iterations for R = 150.4, and in about 70 iterations for R = 173.5. The gradient required on the order of six times more iterations for either channel. This implies that the gradient method requires at least 200 more iterations. For this simulation, the eigenvalues of **A** were determined exactly to find the optimum fixed step size; estimates would have to be used in real-time operation and hence the gradient technique would require even more iterations.

Figures 26 and 27 show the average value and standard deviation of the output error norm for 30 independent noise runs with S/N = 30 dB. Equations (B-2) and (B-3) were used to estimate the step size. The relaxation technique is again substantially faster (at least seven times), but with a higher standard deviation for both channels.

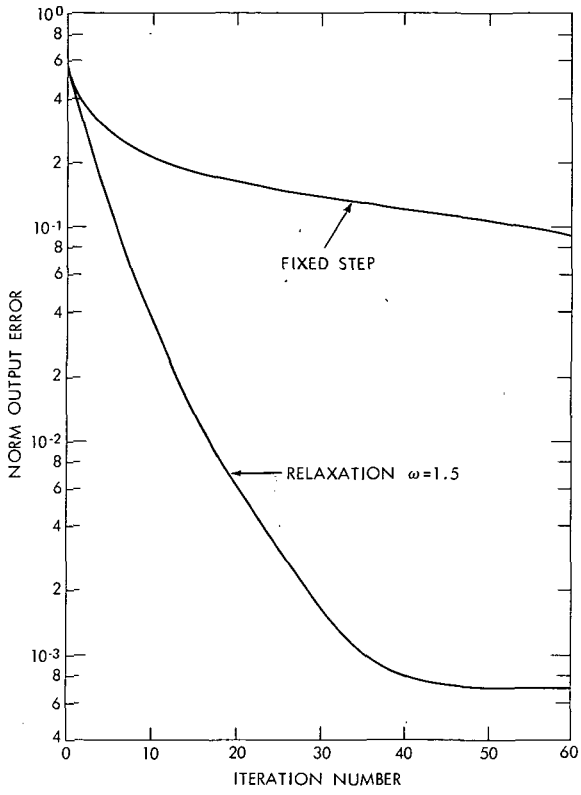

Figure 24.—Comparison of relaxation and fixed step-size gradient algorithms for duobinary encoding with R = 150.4 and 17 taps.

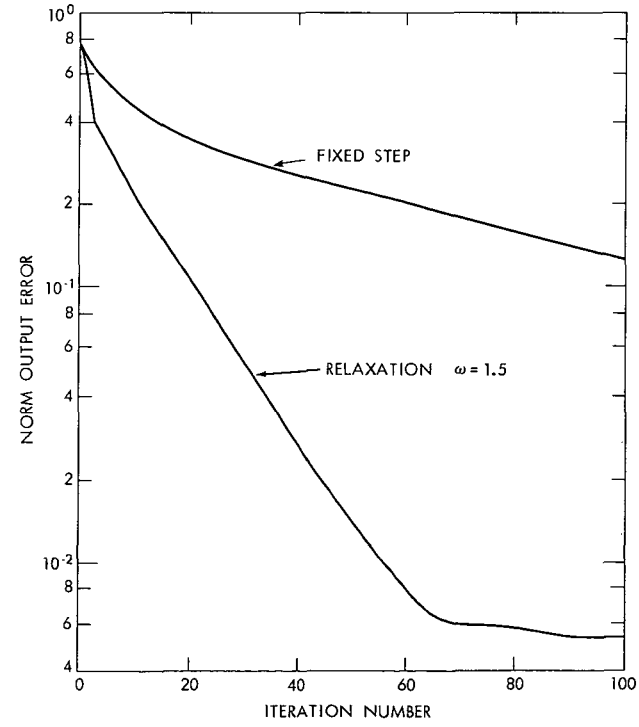

Figure 25.—Comparison of relaxation and fixed step-size gradient algorithms for duobinary encoding with R = 173.5 and 17 taps.

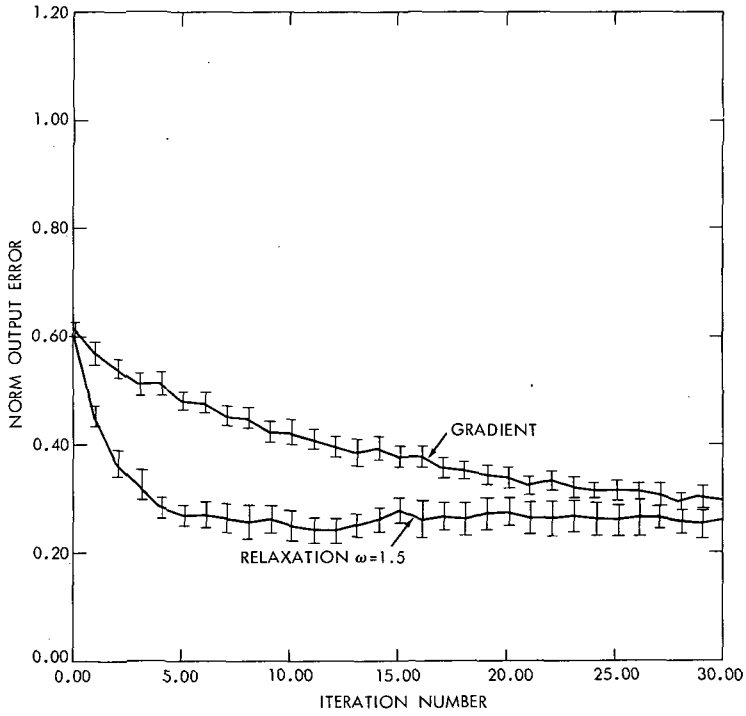

Figure 26.—Comparison of relaxation and fixed step-size gradient algorithms for duobinary encoding with $S/N = 30$ dB, $R = 83.8$, and 17 taps.

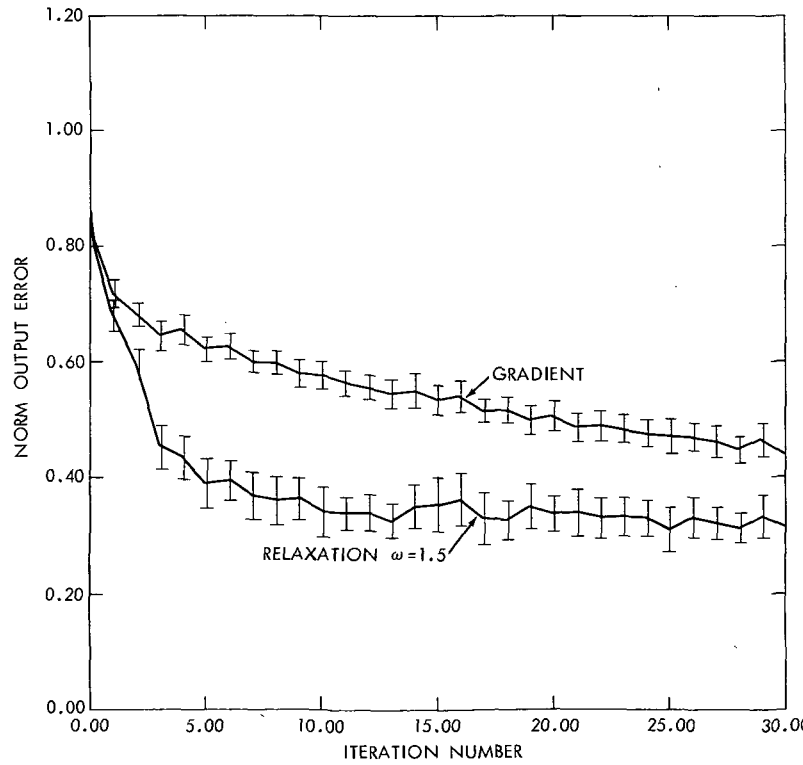

Figure 27.—Comparison of relaxation and fixed step-size gradient algorithms for duobinary encoding with $S/N = 30$ dB, $R = 103.6$, and 17 taps.

Amazingly, for all simulations done, one relaxation factor ($\omega = 1.5$) was used and it yielded convergence at least five times faster than the fixed step-size gradients and at least twice as fast as the Chebyshev gradient ($R = 3.28$ and 17.81 only). The factor $\omega = 1.5$ does not yield the minimum value for the spectral radii for any of the channels; hence asymptotically better results are possible for other parameter values. (See figs. 8 to 11.) In the numerical evaluations developed earlier, the minima for the spectral norms for these channels occurred for values closer to 1. As a matter of fact, the spectral norm was greater than 1 for $\omega = 1.5$ except for $R = 3.28$ and no encoding. Again, there were better choices for ω and yet substantial improvements were obtained.

Overall, the noiseless simulations supported the numerical results. They demonstrated that the relaxation method with $\omega = 1.5$ was consistently better than the estimated and/or the optimum fixed step-size gradient in convergence. For $R = 3.28$, further improvement was obtained by using the relaxation factor of 1.1. The minimum in the spectral radius (fig. 13) for this channel occurred for a relaxation factor of 1.08; that is, the best asymptotic results occur for this factor.

With $\omega = 1.5$ the resultant standard deviation obtained for the relaxation method was only slightly larger than that for the fixed gradient, but the improvement in the convergence rate more than compensates for this. Better variances may be obtained by using smaller values of the relaxation factor. This will in turn slow down the convergence rate because there appears to be some tradeoff between speed of convergence and standard deviation values.

Close examination of the simulations conducted with noise strongly suggests the conjecture that in a noisy environment, the intersymbol interference caused by the channel distortion is initially the dominant noise. The equalizer reduces this in a manner similar to the noiseless case with about the same convergence rates, until the additive noise becomes dominant. Because the equalizer does not have the capability of handling the additive noise, the noise in essence introduces a barrier beyond which the equalizer cannot reduce the mean-square error. When the noise seriously limits the equalizer performance, it may be possible to improve the reception by using a matched filter. The noise barrier level is not only dependent upon the noise variance but also on the sum of the squares of the coefficient values. With the same signal-to-noise ratio for all simulations, the noise variance increased with increasing condition number. This probably accounts for the different final mean-square errors.

The effect of the duobinary encoding of the transmitted signal is to vastly increase the condition number. Hence the rate of convergence is slowed down; this effect is more noticeable for the noiseless case. With duobinary encoding, convergence is about five times slower for $R = 3.28$ and seven for $R = 17.81$, with no noise. The convergence with noise is impeded by a factor of 2. Also the final mean-square error increases, although the initial error is smaller for duobinary encoding than for no encoding.

CONCLUSIONS

The successive overrelaxation iterative technique has been proposed and demonstrated to be feasible as the algorithm for the iterative adjustment of the equalization coefficients.

The commonly used fixed gradient technique has been shown to be identical to the Jacobi iterative method for 2-cyclic and diagonally dominant nonnegative Jacobi matrices. This allowed the

author to use known comparative theorems that state that substantial improvements are possible with the relaxation technique. In this study, the results were also analytically extended to diagonally dominant nonpositive Jacobi matrices.

An analytical bound was found that proves that the relaxation algorithm is definitely mean-square-error decreasing at each iteration for light or moderate channel dispersions. Perturbation theory was used to show that the mean-square-error-decreasing property is valid for general channel characteristics and small ω. Numerical examples indicate that this property is valid for a much larger parameter range.

Numerical evaluation of the spectral radius indicates that improvements similar to the analytic ones are possible for channels with general characteristics. An area open to further investigation is the analytical proof that the relaxation method is indeed faster for all possible channels.

In a noisy channel, i.e., one with additive channel noise, the relaxation algorithm was shown to converge in the mean and the variance of the equalizer coefficients was bounded. If the coefficient values are limited to finite precession, the relaxation algorithm was altered to be feasible for this problem.

Computer simulations, using pulses as shown in figure 18 with and without duobinary encoding of the transmitted signal, support this conjecture. The Chebyshev gradient technique required at least twice as many iterations (no duobinary encoding), and the fixed step-size gradient required at least five times as many.

Furthermore, convergence for relaxation is not critically connected with the estimation of the correlation eigenvalues. Both the gradient techniques suffer substantial decreases in the convergence rates because of the eigenvalue bounds.

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Appendix A

DUOBINARY SIGNALING

Even with an ideal channel, i.e., perfectly distortionless, the transmitter must have ideal low-pass characteristics, to insure no intersymbol interference with binary transmission. This system is unrealizable. Therefore much attention has been focused on duobinary and related data transmission schemes that utilize a controlled amount of intersymbol interference. The duobinary scheme arises from the use of a cosine filter as the signal-shaping characteristic. Intersymbol interference is expected and for samples at $nT - T/2$, the impulse response $g(t)$ is

$$g_n = g\left(nT - \frac{T}{2}\right) = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{A-1}$$

The transmitted signal has the form

$$s(t) = \sum_n a_n g(t - nT) \tag{1}$$

and with an ideal channel, the received signal sampled at $kT - T/2$ is

$$y_k = a_k + a_{k-1} \tag{A-2}$$

Note that the intersymbol interference comes only from the preceding sample. If the possible value of a_k is $\pm d$, the received signal has three possible values: $\pm 2d$ and 0. To prevent error propagation at the receiver, the input sequence $\{a_k\}$ is precoded. The input sequence is converted to another binary sequence $\{b_k\}$ before transmission according to

$$b_k = a_k \oplus b_{k-1} \tag{A-3}$$

where the symbol $\oplus$ represents modulo 2 addition. The sequence $\{b_k\}$ is then transmitted using $\pm d$ for 1 and 0, respectively.

The decoder is a modulo 2 adder with the following decision rule:

$$\left.\begin{aligned} y_k = \pm 2d &\rightarrow a_k = 0 \\ y_k = 0 &\rightarrow a_k = 1 \end{aligned}\right\} \tag{A-4}$$

With more realistic channels, the received signal (equalizer output) is

$$y_k = a_k x_0 + a_{k-1} x_1 + \sum_{n \neq k,\, k-1} a_n x_{k-n} \qquad \text{(A-5)}$$

The last term is the additional intersymbol interference but the values a_k and a_{k-1} have been scaled down.

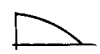
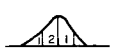
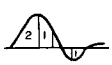
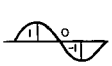
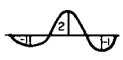
In a training period, only one pulse is transmitted, thereby making the a_j all equal. The desired equalizer output is the sampled transmitted signal of equation (A-1):

$$d_k = \begin{cases} 1 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

The advantages of duobinary encoding are the use of realizable filters that do not further add to the intersymbol interference and the sampling rate insensitivity. However, the duobinary signal has three levels that must be distinguished and this requires a higher signal-to-noise ratio for equal performance with binary transmission.

The duobinary data transmission scheme has been generalized to partial-response encoding. Kretzmer (ref. 24) has tabulated and classified a number of these partial-response systems, which appear to have useful properties. The responses, their frequency characteristics, their signal-to-noise ratio degradations over ideal binary, and their speed tolerances before peak eye closure is unity are shown in table A-1. Notice particularly the last two frequency characteristics. These functions go to zero at zero frequency (in addition to the Nyquist frequency) and thus become attractive for the many occasions in which frequencies near dc are prohibited or are severely attenuated. The desired response after equalization is the sampled values of the impulse response $x(t)$. The first example is duobinary.

Table A-1.—Partial-Response Systems
[From Kretzmer (ref. 24)]

Impulse response $x(t)$	Frequency characteristic $x(\omega)$	$x(\omega)$	Number received levels	Speed tolerance, percent	S/N degradation over ideal binary, dB
		$2T \cos \frac{\omega T}{2}$	3	43	2.1
		$4T \cos^2 \frac{\omega T}{2}$	5	40	6.0
		$T(2 + \cos \omega T - \cos 2\omega T)$ $+ iT(\sin \omega T - \sin 2\omega T)$	5	38	[a]1.2
		$2T \sin \omega T$	3	15	2.1
		$4T \sin \omega T$	5	8	6.0

[a]With precoding it is 7.2 dB.

Appendix B

ESTIMATION OF EIGENVALUE BOUNDS

To use the gradient algorithm properly, it is necessary to bound the eigenvalues of the correlation matrix **A** so as to estimate the optimum step size.

One set of bounds on the eigenvalues of the signal-plus-noise correlation matrix was derived by Gersho (ref. 7) (see also Grenander and Szego (ref. 25)):

$$\lambda_{\max} \leqslant \lambda_u = \max_{|\omega| \leqslant \pi/T} [|X^*(\omega)|^2 + (2N+1)S(\omega)] \tag{B-1}$$

and

$$\lambda_{\min} \geqslant \lambda_l = \min_{|\omega| \leqslant \pi/T} [|X^*(\omega)|^2 + (2N+1)S(\omega)] \tag{B-2}$$

where $X^*(\omega)$ and $S(\omega)$ are the sampled signal Fourier transform and input noise spectral density, respectively. The spectral bounds require finding the minima and maxima of a function. Because this cannot be done a priori, an algorithm must be used. In general, the implementation at the receiver may be extremely difficult and time consuming. It is therefore required that bounds easily implemented be used; i.e., bounds based on input signal measurements.

The upper bound λ_u is fairly easy to obtain from input signal measurements. Consider the positive definite matrix **A** with positive eigenvalues:

$$\text{trace } \mathbf{A} = \sum_i \lambda_i > \lambda_{\max}$$

Then a simple bound is

$$\lambda_u = \text{trace } \mathbf{A} = (2N+1)\left[\sum_i x_i^2 + (2N+1)\sigma^2\right] \tag{B-3}$$

where $\sigma^2 = E(n_i^2)$ is the variance of input noise samples.

The use of this bound forces the step size α to be smaller than the optimum. This will definitely slow down the convergence, but will most likely reduce the error variance. A much tighter bound is obtained by using the theorem of Frobenius or from the Gershgorin disks:

$$\lambda_{\max} \leqslant \lambda_u = (2N+1)\sigma^2 + \max_k \sum_l \left| \sum_n X_n X_{n+|l+k|} \right| \tag{B-4}$$

where it is understood that $x_j = 0$ for all j not in the interval $[-N, N]$.

A lower bound on the minimum eigenvalue is also obtained from the Gershgorin disks or by Frobenius' theorem:

$$\lambda_{\min} \geqslant \lambda_l = (2N+1)\sigma^2 + \sum_n x_n^2 - \max_k \sum_{l \neq k} \left| \sum_n x_n x_{n+|l+k|} \right| \tag{B-5}$$

The lower bound given by equation (B-5) is useful only if it is positive. Although it appears as if the maximum value must again be determined, this can be avoided if equation (B-5) yields a positive lower bound. This is done by noting that the sum $\lambda_{\max} + \lambda_{\min}$ must be estimated, and the estimate of the sum is given by

$$\lambda_l + \lambda_u = 2 \sum_n (x_n^2 + \sigma^2) \tag{B-6}$$

For the noiseless case ($\sigma = 0$), equation (B-5) will be positive only if the channel correlation matrix **A** is diagonally dominant. In this case, the gradient technique becomes identical to the Jacobi iterative method.

On the other hand, if equation (B-5) becomes negative, zero can be used as the lower bound. Then it is necessary to determine the maxima in equation (B-4). This is easier than determining that of equation (B-1), because in equation (B-4), $(N + 1)$ sums need only be computed and compared. In this case, the gradient will still converge and it may turn out that the estimated step size will be closer to the optimum step size. If the error variance is more important, then the upper bound of (B-3) should be used because its estimated step size will be much smaller than that of (B-4).

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