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A NONLINEAR THEORY FOR SONIC-BOOM CALCULATIONS IN A STRATIFIED ATMOSPHERE

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A NONLINEAR THEORY FOR SONIC-BOOM CALCULATIONS IN A STRATIFIED ATMOSPHERE

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SUMMARY

The exact solutions to the equations of gas dynamics are given with respect to the axis of slender lifting bodies in a stratified atmosphere. The boundary condition is satisfied by using slender-body theory. The solution predicts the magnitude of the pressure rise of the sonic boom and estimates the nonlinear effects in the vicinity of the cutoff point.

INTRODUCTION

The theories of Friedman, Kane, and Sigalla (ref. 1) and Hayes, Haefeli, and Kulsrud (ref. 2) on sonic-boom propagation in a stratified atmosphere have the advantage of being able to treat nonsteady flight maneuvers. However, they have the disadvantage of not being fully nonlinear theories. Nonlinear effects such as focusing at the caustic are now of great interest. Here a theory is presented in which the exact nonlinear equations for supersonic flow are solved for large distances from lifting bodies. This paper is an extension of the author's previous papers (refs. 3 and 4) to a stratified atmosphere in which the speed of sound changes with the altitude of flight. The pressure p and the density ρ are assumed to obey the hydrostatic law $dp = -\rho g\, dz$. The gas is considered to be nonviscous, steady, homentropic, and homoenergetic. The shock front should be attached at the nose of the slender pointed body.

The theory presented can be used to calculate the gas properties in the far field of nonaxisymmetric bodies by employing area-rule concepts.

SYMBOLS

A	cross-sectional area of body
A'	first derivative of cross-sectional area of body

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a	speed of sound
$a(r)$	speed of sound at distance r from the body
$a(z)$	speed of sound at altitude z
c_p	specific heat at constant pressure
c_v	specific heat at constant volume

$$D_r = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial r} - \frac{\partial x}{\partial r}\frac{\partial}{\partial \xi} = x_\xi \partial_r - x_r \partial_\xi$$

$$D_\xi = \frac{\partial}{\partial \xi} = \partial_\xi$$

$$D_\psi = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial}{\partial \xi} = x_\xi \partial_\psi - x_\psi \partial_\xi$$

$$E = \frac{2g}{n}$$

$$\left.\begin{aligned} \vec{e}^{\,1} &= \text{grad } x^1 \\ \vec{e}^{\,2} &= \text{grad } x^2 \\ \vec{e}^{\,3} &= \text{grad } x^3 \end{aligned}\right\} \text{Cartesian base vectors}$$

$F_m^{(j)}$	function of integration
$G_m^{(j)}(\xi)$	function of integration
g	acceleration due to gravity
$\vec{g}$	volume force due to gravity

$$\left.\begin{aligned} &H^{(1)}(A) \\ &H^{(2)}(A) \end{aligned}\right\} \text{functions dependent on the cross section (see eqs. (44))}$$

h	specific enthalpy

M	Mach number
$\vec{m}$	generatrix of Mach cone
n	number of degrees of freedom of the gas (air: $n = 5$)
$\vec{n}$	generatrix of the wave normal cone
p	pressure
R	distance from the body at which the boundary condition is satisfied
r	radial distance from the body
s	entropy
$\vec{s}, \vec{t}$	locally dependent base vectors
t	time
U	gravitational potential
w	magnitude of the velocity
$\vec{w}$	velocity vector
$\vec{w}_i$	components of the velocity vector in a locally dependent basis ($i = 1, 2, 3$)
x	abscissa of the characteristic surface
x_s	abscissa of the shock front

$$\left.\begin{aligned} x^1 &= x \\ x^2 &= r \cos \psi \\ x^3 &= r \sin \psi \end{aligned}\right\} \text{Cartesian coordinates}$$

z	altitude of flight

α angle of attack

β Prandtl factor, $\sqrt{M^2 - 1}$

γ ratio of specific heats

$\Delta = w^{(0)^2} - a^{(0)^2}_{(r)}$

$\Delta p = p - p_\infty$

η Prandtl transformation, βr

ϑ angle of inclination of the streamlines

$$\Lambda = \frac{\left[w^{(0)^2} - a^{(0)^2}_{(r)}\right]^{1/4}}{a^{(0)^3}_{(r)}} \Pi \frac{1}{2\sqrt{r}}$$

μ Mach angle

ξ characteristic variable

$\xi_i = (\xi, r, \psi)$ independent variables in locally dependent basis (i = 1, 2, 3)

$$\Pi = \frac{\left[a^{(0)^2}_{(z)}\right]^{(n+1)/4}}{\left[w^{(0)^2} - a^{(0)^2}_{(z)}\right]^{1/4}}$$

ρ density

$\Sigma = a^{(0)^2}_{(r)}$

$\tau^i = (w, \vartheta, \varphi + \psi)$ where $i = 1, 2, 3$

φ azimuth angle

ϕ potential function

ψ cylindrical variable (azimuth angle)

ω variable of integration

Subscripts:

∞ state of undisturbed flow

gr values at the ground

Harmonic (or Fourier) numbers are denoted by numbers in the subscript position or by the subscript m. The order of magnitude is indicated by numbers in parentheses in the superscript position or by the superscript (j). The Greek letters λ, σ, and ν used as subscripts and superscripts denote the Einstein summation convention. A comma preceding a subscript denotes differentiation with respect to that subscript.

ANALYSIS

Basic Equations

The gas considered here is assumed to be in equilibrium. There are no heat or mass sources. To describe the state of the gas, it is sufficient to calculate the vector of velocity, the density, and the pressure. This can be done by means of the equations of continuity, momentum, and energy. From these equations it may easily be seen that

$$\frac{ds}{dt} = 0 \tag{1}$$

As the entropy is assumed to be constant in the overall undisturbed field and, according to equation (1), it does not change along streamlines, the entropy is constant in the disturbed field as well. That is, the gas is homentropic.

Now the change of energy is to be considered. The equation of momentum is

$$(\vec{w} \cdot \text{grad})\vec{w} = -\frac{1}{\rho} \text{grad } p + \vec{g} \tag{2}$$

Combining equation (2) with the identity

$$(\vec{w} \cdot \text{grad})\vec{w} \equiv \text{grad } \frac{w^2}{2} - \vec{w} \times \text{curl } \vec{w} \tag{3}$$

gives

$$\operatorname{grad}\frac{w^2}{2} - \vec{w}\times\operatorname{curl}\vec{w} = -\frac{1}{\rho}\operatorname{grad} p + \vec{g} \tag{4}$$

where

$$w^2 = \vec{w}\cdot\vec{w}$$

As the volume forces due to gravity are irrotational,

$$\vec{g} = \operatorname{grad} U = (0,0,-g)$$

Projecting equation (4) onto the direction of the velocity vector $\vec{w}$ yields

$$d\left(\frac{w^2}{2} - U\right) = -\frac{1}{\rho}\,dp \tag{5}$$

as $\vec{w}\times\operatorname{curl}\vec{w} \equiv 0$ along the streamlines (Beltrami flow is not considered here). Equation (5) can be integrated for a given relation between the pressure p and the density ρ. According to thermodynamics,

$$\rho = \rho(p,s) = \bar{\rho}\left(\frac{p}{\bar{p}}\right)^{1/\gamma}\exp\left(\frac{\bar{s} - s}{c_p}\right) \tag{6}$$

where the bar over symbols indicates an arbitrary initial state of the gas. For homentropic flow, equation (6) is rewritten as $p = \rho^{\gamma}$. Const. Thus, the integration of equation (5) yields

$$\frac{w^2}{2} + gz + h = H \tag{7}$$

where H is the total enthalpy. Along the streamline the total enthalpy is a constant. Since the undisturbed flow field was assumed to be homoenergetic, the disturbed flow field is homoenergetic as well, according to equation (7). As long as no shocks are taken into account, the result of these considerations is that the overall flow field is homoenergetic and homentropic. Far from the body weak shocks are expected. For weak shocks the entropy increases as the third power of the pressure coefficient. Thus, so long as only weak shocks are considered, the flow can be assumed to be homentropic.

The basic equations can now be formulated. According to equation (1) the flow is irrotational. This condition is expressed by the equation of irrotational flow:

$$\operatorname{curl} \vec{w} = 0 \tag{8}$$

The equation of continuity is

$$\frac{dp}{dt} + \rho \operatorname{div} \vec{w} = 0 \tag{9}$$

By use of the relations $\rho = \rho(p)$ and $a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$, equation (9) can be rewritten

$$\frac{1}{\rho}\frac{dp}{dt} + a^2 \operatorname{div} \vec{w} = 0$$

For steady flow this equation can be expressed in terms of the speed of sound, the velocity, and a term due to gravity:

$$a^2 \operatorname{div} \vec{w} - w\vec{w} \cdot \operatorname{grad} w + \vec{w} \cdot \vec{g} = 0 \tag{10}$$

The last basic equation is derived by considering the position of the characteristic surfaces. Assume that the quantity ξ denotes a characteristic variable along the generatrix $\vec{n}$ of the wave normal cone. The unit vector $\vec{m}$ along the generatrix of the Mach cone is perpendicular to $\vec{n}$ by definition; that is, $\vec{m} \cdot \vec{n} = 0$. As $\vec{n} = \frac{\operatorname{grad} \xi}{|\operatorname{grad} \xi|}$ or, according to the definition of the speed of sound, $(\vec{w} \cdot \vec{n}) = -a$, the last basic equation is written in terms of the speed of sound, the velocity $\vec{w}$, and the characteristic variable ξ:

$$a^2 \operatorname{grad} \xi \cdot \operatorname{grad} \xi = (\vec{w} \cdot \operatorname{grad} \xi)^2 \tag{11}$$

Change to a Characteristic Coordinate System

The basic equations have been written in an invariant form. Now a coordinate system must be selected to carry out the solution. In a Cartesian frame, a point has the cylindrical coordinates $x^1 = x$, $x^2 = r \cos \psi$, and $x^3 = r \sin \psi$. However, here a locally dependent basis will be used, in which the velocity vector $\vec{w}$ has to be expressed. According to figure 1, the velocity vector $\vec{w}$ is given by

$$\vec{w} = w\left[\cos\vartheta\,\vec{e}^{\,1} + \sin\vartheta(\cos\varphi\,\vec{s} + \sin\varphi\,\vec{t})\right] \tag{12}$$

where

$$\vec{s} = \cos\psi\,\vec{e}^{\,2} + \sin\psi\,\vec{e}^{\,3}$$

$$\vec{t} = -\sin\psi\,\vec{e}^{\,2} + \cos\psi\,\vec{e}^{\,3}$$

The velocity vector is defined by its magnitude w, by the angle of inclination ϑ of the streamline, and the azimuth angle $\varphi + \psi$. If w, ϑ, and $\varphi + \psi$ are considered to be new variables of the velocity vector, new components of the velocity are defined as

$$\vec{w}_i = \frac{\partial\vec{w}}{\partial\tau^i} \qquad \tau^i = (w,\vartheta,\varphi+\psi) \qquad (i = 1, 2, 3) \tag{13}$$

Up to now the unknown functions have been expressed by the cylindrical coordinates $\vec{w} = \vec{w}(\xi,r,\psi)$, $\xi = \xi(x,r,\psi)$, and $a = a(x,r,\psi)$. Here the problem is to calculate the position of the characteristic surface rather than to determine the characteristic surface itself. So, following the Poincaré-Lighthill-Kuo (PLK) method, the dependent variable ξ is exchanged for the independent variable x. The new set of independent variables is now ξ, r, and ψ. The velocity vector $\vec{w}$, the speed of sound a, and the abscissa x are to be determined as functions of these variables. That is, defining $\xi_i = (\xi,r,\psi)$ where $i = 1, 2, 3$, determine $\vec{w} = \vec{w}(\xi_i)$, $x = x(\xi_i)$, and $a = a(\xi_i)$. The frame of reference of these coordinates must be constructed and the differential operators determined in these given directions. The final results are (see ref. 3),

$$\left.\begin{aligned} x_\xi \operatorname{grad} &= \vec{e}^{\,1}D_\xi + \vec{s}D_r + \frac{1}{r}\vec{t}D_\psi \\ \operatorname{curl}\vec{w} &= \operatorname{grad}\tau^\sigma \times \vec{w}_\sigma \\ \operatorname{div}\vec{w} &= \operatorname{grad}\tau^\sigma \cdot \vec{w}_\sigma \end{aligned}\right\} \tag{14}$$

where D_ξ, D_r, and D_ψ are abbreviations for the operators defined by the equations $D_\xi = \partial_\xi$, $D_r = x_\xi\partial_r - x_r\partial_\xi$, and $D_\psi = x_\xi\partial_\psi - x_\psi\partial_\xi$. Equations (12) to (14) have to be introduced into the basic set of differential equations (eqs. (8), (10), and (11)) to carry out the solution in the given frame of reference $\left(\vec{e}^{\,1},\vec{s},\vec{t}\right)$.

Since $\vec{g} = -g\vec{e}^{\,2}$, the equations are written as functions of the coordinates $\xi_i = (\xi, r, \psi)$:

$$\sin \vartheta \sin \varphi \, D_r w + \frac{1}{r} \sin \vartheta \cos \varphi \, D_\psi w - w \cos \vartheta \sin \varphi \, D_r \vartheta + \frac{w}{r} \cos \vartheta \cos \varphi \, D_\psi \vartheta$$

$$- w \sin \vartheta \cos \varphi \, D_r \varphi - \frac{w}{r} \sin \vartheta \sin \varphi \, D_\psi \varphi - \frac{w}{r} \sin \vartheta \sin \varphi \, x_\xi = 0 \tag{15a}$$

$$\sin \vartheta \sin \varphi \, D_\xi w - \frac{1}{r} \cos \vartheta \, D_\psi w + w \cos \vartheta \sin \varphi \, D_\xi \vartheta + \frac{w}{r} \sin \vartheta \, D_\psi \vartheta + w \sin \vartheta \cos \varphi \, D_\xi \varphi = 0 \tag{15b}$$

$$-\sin \vartheta \cos \varphi \, D_\xi w + \cos \vartheta \, D_r w - w \cos \vartheta \cos \varphi \, D_\xi \vartheta - w \sin \vartheta \, D_r \vartheta + w \sin \vartheta \sin \varphi \, D_\xi \varphi = 0 \tag{15c}$$

$$\left(a^2 - w^2\right)\left[\cos \vartheta \, D_\xi w + \sin \vartheta \left(\cos \varphi \, D_r w + \frac{1}{r} \sin \vartheta \, D_\psi w\right)\right] + a^2 w \left[-\sin \vartheta \, D_\xi \vartheta\right.$$

$$\left. + \cos \vartheta \left(\cos \varphi \, D_r \vartheta + \frac{1}{r} \sin \vartheta \, D_\psi \vartheta\right)\right] + a^2 w \sin \vartheta \left[-\sin \varphi \, D_r \varphi + \frac{1}{r} \cos \varphi \, D_\psi \varphi\right]$$

$$- x_\xi g w \sin \vartheta \cos \psi = 0 \tag{15d}$$

$$w^2 \left(\cos \vartheta - \sin \vartheta \cos \varphi \, x_r - \sin \vartheta \sin \varphi \, \frac{1}{r} x_\psi\right)^2 = a^2 \left(1 + x_r^{\,2} + \frac{1}{r^2} x_\psi^{\,2}\right) \tag{15e}$$

$$w^2 + n a^2 + 2gz = \text{Constant} \tag{15f}$$

The last equation – the energy equation – gives the interrelation between the speed of sound and the velocity. Although this set consists of six equations for the five unknown functions w, ϑ, φ, a, and ξ, all of them must be used. The equations $\operatorname{curl} \vec{w} = 0$ are dependent on each other and have, together with the equation of continuity, the trivial solution $w = \operatorname{grad} \phi$, since $\operatorname{curl} \cdot \operatorname{grad} \phi \equiv 0$.

Solution by a Perturbation Approach

The system of nonlinear partial differential equations (15) is solved for large distances by a perturbation method. If the unknown functions are assumed to have the form of a perturbation series

$$C = C^{(0)} + C^{(1)} + C^{(2)} + \ldots \tag{16}$$

that is valid for large distances, the nonlinear differential equations can be split up into orders of magnitude. The equations for the different orders of magnitude can be easily solved in a step-by-step procedure. The series (eq. (16)) does not make any statement about the values of the orders of magnitude. It only expresses the assumption that for large distances r the zeroth-order term is larger than the first-order term, and so forth. The special properties of this series will be determined in this paper. There is some a priori knowledge about the zeroth order which is put into the calculation. For large distances r, to zeroth order

$$w^{(0)} = \text{Constant} \qquad a^{(0)} = \text{Constant} \qquad \vartheta^{(0)} = 0 \qquad x_\xi^{(0)} = \text{Constant} \qquad x_\psi^{(0)} = 0 \tag{17}$$

Contrary to the previous paper (ref. 3), $x_r^{(0)}$ is now dependent on r. This complicates the calculations greatly. Furthermore, for large distances the following assumptions are made concerning the derivatives:

$$\partial_r \approx \frac{1}{r} \qquad \partial_\xi \approx \partial_\psi \qquad \partial_r << \partial_\xi \qquad \partial_r << \partial_\psi \tag{18}$$

but

$$x_r \approx x_\psi \approx x_\xi$$

According to the linearized two-dimensional equation of characteristics $x = \xi + \beta r$, $x_\xi^{(0)}$ has been set equal to 1. Equations (16) to (18) must be introduced into equations (15) to separate the equations for the different orders of magnitude.

The final differential equations are, to zeroth order,

$$w^{(0)^2} - a^{(0)^2}\left[1 + x_r^{(0)^2}\right] = 0 \tag{19a}$$

and to the first order,

$$\left[\varphi^{(0)}\vartheta^{(1)}\right]_{,\xi} = 0 \tag{19b}$$

$$w^{(0)}\vartheta_\xi^{(1)} + x_r^{(0)}w_\xi^{(1)} = 0 \tag{19c}$$

$$x_r^{(0)} w_\psi^{(1)} + w^{(0)} \vartheta_\psi^{(1)} = 0 \tag{19d}$$

$$a^{(0)^2} x_r^{(0)} x_r^{(1)} = -\vartheta^{(1)} x_r^{(0)} w^{(0)^2} + w^{(0)} w^{(1)} - a^{(0)} a^{(1)} \left[1 + x_r^{(0)^2}\right] \tag{19e}$$

$$w^{(0)} \left[\varphi^{(1)} \vartheta^{(1)}\right]_{,\xi} = \frac{1}{r} w_\psi^{(1)} \tag{19f}$$

$$w^{(0)} \left[\vartheta_r^{(1)} + \frac{1}{r} \vartheta^{(1)}\right] - x_r^{(0)} w_r^{(1)} - \frac{g w^{(0)} \cos \psi}{a^{(0)^2}} \vartheta^{(1)} = 0 \tag{19g}$$

$$n a^{(0)} a^{(1)} + w^{(0)} w^{(1)} = 0 \tag{19h}$$

While $w^{(0)}$ is a constant in the whole field, $a^{(0)}$ is dependent on the altitude of flight z. By means of the integrated hydrostatic law,

$$n a^2(z) + 2gz = n a_{gr}^2 \tag{20}$$

the speed of sound can be determined for an arbitrary height or given distance from the body. In equation (20), z is the altitude of flight, $a(z)$ is the speed of sound at the altitude z, a_{gr} is the speed of sound at the ground, and n is the number of degrees of freedom of the gas. The local speed of sound $a(r)$ at the distance r away from the body is, according to figure 2,

$$a^2(r) = a_{gr}^2 - \frac{2g}{n} (z + r \cos \psi) \tag{21}$$

To indicate the zeroth-order form of equation (21), this equation is rewritten as

$$a^{(0)^2}(r) = a_{gr}^2 - E(z + r \cos \psi) = a^{(0)^2}(z) - E \cos \psi \cdot r \tag{22}$$

where E denotes a factor depending on gravity, $E = 2g/n$.

By developing a Fourier series in ψ the number of independent variables in equations (19) can be reduced from three to two. The proposed Fourier series is

$$C^{(j)}(\xi,r,\psi) = \sum_{\lambda=-\infty}^{\infty} C_{\lambda}^{(j)}(\xi,r)e^{i\lambda\psi} \qquad (j \geqq 1) \tag{23}$$

for the variables x, w, ϑ, and a and

$$\varphi^{(j)}(\xi,r,\psi) = \sum_{\lambda=-\infty}^{\infty} i\varphi_{\lambda}^{(j)}(\xi,r)e^{i\lambda\psi} \qquad (j \geqq 1) \tag{24}$$

for the azimuth angle φ. Naturally, the number of equations is increased by this procedure from seven equations in three variables to an infinite number of equations in two variables. However, for a lifting body of revolution the first two terms of the Fourier series will suffice.

After having introduced the Fourier series (eqs. (23) and (24)) into the set of equations (19), the differential equations are rewritten as

$$\left[\varphi^{(0)}\vartheta_m^{(1)}\right]_{,\xi} = 0 \tag{25a}$$

$$x_r^{(0)^2} = \frac{w^{(0)^2} - a^{(0)^2}}{a^{(0)^2}} \tag{25b}$$

$$w^{(0)}\vartheta_{m,\xi}^{(1)} + x_r^{(0)}w_{m,\xi}^{(1)} = 0 \tag{25c}$$

$$w^{(0)}\vartheta_{m,r}^{(1)} + \frac{w^{(0)}}{r}\vartheta_m^{(1)} - x_r^{(0)}w_{m,r}^{(1)} - \frac{gw^{(0)}\cos\psi}{a^{(0)^2}}\vartheta_m^{(1)} = 0 \tag{25d}$$

$$x_r^{(0)}w_m^{(1)} + w^{(0)}\vartheta_m^{(1)} = 0 \tag{25e}$$

$$w^{(0)}\left[\sum_{\nu}\varphi_{m-\nu}^{(1)}\vartheta_{\nu}^{(1)}\right]_{,\xi} = \frac{m}{r}w_m^{(1)} \tag{25f}$$

$$a^{(0)^2}x_r^{(0)}x_{m,r}^{(1)} = -\vartheta_m^{(1)}x_r^{(0)}w^{(0)^2} + w^{(0)}w_m^{(1)} - a^{(0)}a_m^{(1)}\left[1 + x_r^{(0)^2}\right] \tag{25g}$$

$$na^{(0)}a_m^{(1)} + w^{(0)}w_m^{(1)} = 0 \tag{25h}$$

Here the subscript m indicates Fourier numbers and the superscripts indicate the order of magnitude. The summation index ν can be restricted to $0 \leqq \nu \leqq 2$ as the body should be symmetrical with respect to the plane $\psi = 0$ and the first Fourier coefficient is assumed to be much greater than the next one, and so forth.

The solution of equations (25a) and (25b) can be written as

$$\varphi^{(0)} = \text{Constant} \tag{26}$$

and

$$x^{(0)} = \frac{1}{E\cos\psi}\left[a^{(0)}(z)\sqrt{w^{(0)^2} - a^{(0)^2}(z)} - a^{(0)}(r)\sqrt{w^{(0)^2} - a^{(0)^2}(r)}\right]$$
$$+ \frac{w^{(0)^2}}{E\cos\psi}\left[\tan^{-1}\sqrt{\frac{a^{(0)^2}(z)}{w^{(0)^2} - a^{(0)^2}(z)}} - \tan^{-1}\sqrt{\frac{a^{(0)^2}(r)}{w^{(0)^2} - a^{(0)^2}(r)}}\right] + \xi \tag{27a}$$

For $E = 0$ or $\cos\psi = 0$, l'Hospital's rule gives the limit:

$$x^{(0)} = r\sqrt{\frac{w^{(0)^2} - a^{(0)^2}(r)}{a^{(0)^2}(r)}} + \xi \tag{27b}$$

The integration of equation (25c) with respect to ξ gives

$$x_r^{(0)}w_m^{(1)} + w^{(0)}\vartheta_m^{(1)} = w^{(0)}F_m^{(1)}(r) \tag{28}$$

If the integration function $F_m^{(1)}(r)$ is set equal to zero, equation (28) becomes the same as equation (25e).

Differentiating this equation with respect to r yields

$$\dot{x}_r^{(0)}w_m^{(1)} + x_r^{(0)}w_{m,r}^{(1)} + w^{(0)}\vartheta_{m,r}^{(1)} = 0 \tag{29}$$

Adding equations (29) and (25d) eliminates the quantity $w^{(1)}_{m,r}$:

$$2\vartheta^{(1)}_{m,r} + \vartheta^{(1)}_{m}\left[\frac{1}{r} - \frac{g \cos \psi}{a^{(0)2}_{(r)}} - \frac{\dot{x}^{(0)}_{r}}{x^{(0)}_{r}}\right] = 0 \tag{30}$$

The integration gives

$$\vartheta^{(1)}_{m} = G^{(1)}_{m}(\xi) r^{-1/2} \Pi \frac{\Delta^{1/4}}{\Sigma^{(n+1)/4}} \tag{31}$$

Here $G^{(1)}_{m}(\xi)$ is a function of integration which is to be determined by the boundary condition. For brevity the abbreviations Π, Δ, and Σ are used henceforth:

$$\Pi = \frac{\left[a^{(0)2}_{(z)}\right]^{(n+1)/4}}{\left[w^{(0)2} - a^{(0)2}_{(z)}\right]^{1/4}}$$

$$\Delta = w^{(0)2} - a^{(0)2}_{(r)}$$

$$\Sigma = a^{(0)2}_{(r)}$$

From equation (28) the first-order velocity component is calculated as

$$w^{(1)}_{m} = w^{(0)} \frac{G^{(1)}_{m}(\xi)}{\sqrt{r}} \frac{\Pi}{\Delta^{1/4}\Sigma^{(n-1)/4}} \tag{32}$$

Equation (25g) can be rewritten, by using equations (25h) and (28), as

$$x^{(1)}_{m,r} = -\frac{n+1}{n} \frac{\left[1 + x^{(0)2}_{r}\right]^{2}}{x^{(0)2}_{r}} \vartheta^{(1)}_{m}$$

or

$$x_{m,r}^{(1)} = -\frac{n+1}{n} \frac{G_m^{(1)}(\xi)}{\sqrt{r}} \frac{{w^{(0)}}^4 \Pi}{\Delta^{3/4} \Sigma^{(n+5)/4}} \tag{33}$$

This equation is solved by developing the terms $\Sigma^{-(n+5)/4}$ and $\Delta^{-3/4}$ in a power series.

After the development in a series in terms of r, the integration can easily be carried out:

$$x_m^{(1)} = -\frac{n+1}{n} G_m^{(1)}(\xi) \frac{{w^{(0)}}^4}{a^{(0)2}_{(z)}\left[{w^{(0)}}^2 - a^{(0)2}_{(z)}\right]} \left(2\sqrt{r} - \frac{E \cos \psi}{6} r^{3/2}\left[\frac{3}{{w^{(0)}}^2 - a^{(0)2}_{(z)}} + \frac{n+5}{a^{(0)2}_{(z)}}\right] + \frac{(E \cos \psi)^2}{80} r^{5/2}\left\{\frac{21}{\left[{w^{(0)}}^2 - a^{(0)2}_{(z)}\right]^2} + \frac{n^2 + 14n + 45}{\left[a^{(0)2}_{(z)}\right]^2} + \frac{6(n+5)}{a^{(0)2}_{(z)}\left[{w^{(0)}}^2 - a^{(0)2}_{(z)}\right]}\right\} + \ldots\right) \tag{34}$$

Here the function of integration has been chosen to be zero. The calculations still contain the quantity n, the number of degrees of freedom of the gas. For air, which obeys the model of a molecule with two atoms, $n = 5$.

By a development analogous to that of the first-order set of equations, a set of second-order equations is written as

$$x_r^{(0)} w_\xi^{(2)} + w^{(0)} \vartheta_\xi^{(2)} = w_r^{(1)} - \frac{w_\xi^{(1)} w^{(1)}}{x_r^{(0)} w^{(0)}} \left\{1 + \frac{1}{n}\left[1 + {x_r^{(0)}}^2\right]^2\right\} \tag{35a}$$

$$-x_r^{(0)}w_r^{(2)} + w^{(0)}\vartheta_r^{(2)} + \frac{1}{r}w^{(0)}\vartheta^{(2)} - \frac{g\cos\psi\, w^{(0)}}{a^{(0)^2}}\vartheta^{(2)} = +\frac{n+1}{n}\frac{w^{(1)}w_r^{(1)}}{w^{(0)}x_r^{(0)}}\left[1 - x_r^{(0)^4}\right] + w^{(1)}\vartheta_r^{(1)}\left\{x_r^{(0)^2} + \frac{2}{n}\left[1 + x_r^{(0)^2}\right]\right\}$$

$$+\frac{2}{r}w^{(1)}\vartheta^{(1)}\frac{\left[1 + x_r^{(0)^2}\right]}{n} - \frac{w^{(0)}}{r}\left[\varphi^{(1)}\vartheta^{(1)}\right]_{,\psi} \tag{35b}$$

$$x_r^{(0)}w_\psi^{(2)} + w^{(0)}\vartheta_\psi^{(2)} = w_r^{(0)}\left[\varphi^{(1)}\vartheta^{(1)}\right]_{,r} + w^{(0)}\varphi^{(1)}\vartheta^{(1)} - \vartheta^{(1)}\vartheta_\psi^{(1)}\frac{w^{(0)}}{x_r^{(0)}}\left\{\frac{1}{x_r^{(0)^2}} + \frac{1}{n}\left[\frac{1 + x_r^{(0)^2}}{x_r^{(0)}}\right]^2\right\} \tag{35c}$$

$$2x_r^{(0)}x_r^{(2)}a^{(0)^2} = 2w^{(0)}w^{(2)} - 2\vartheta^{(2)}x_r^{(0)}w^{(0)^2} - 2\left[1 + x_r^{(0)^2}\right]a^{(0)}a^{(2)} + w^{(1)^2}\left\{\left(\frac{n+1}{n}\right)^2\left[\frac{1 + x_r^{(0)^2}}{x_r^{(0)}}\right]^2\left[2x_r^{(0)^2} - 1\right] + x_r^{(0)^2}\right\} \tag{35d}$$

$$w^{(1)^2} + 2w^{(0)}w^{(2)} + na^{(1)^2} + 2na^{(0)}a^{(2)} = 0 \tag{35e}$$

The formal solution for this set of equations is obtained in a manner similar to that for the first-order equations. By investigating the solution for the second-order angle of inclination $\vartheta^{(2)}$, it can be shown that this quantity is really much smaller than the first-order angle of inclination. The solutions will be provided to enable an investigation of the behavior of the second-order quantities at the caustic, that is, the condition $w^2 - a^2(r) << 1$. (However, the second-order calculations have not been included in the sample calculations.) The differential equation for the second-order angle of inclination is

$$2\vartheta_{m,r}^{(2)} + \vartheta_m^{(2)}\left[\frac{1}{r} - \frac{g\cos\psi}{a^{(0)^2}(r)} - \frac{w^{(0)^2}E\cos\psi}{2\Delta\Sigma}\right] = \Pi^2 G_{m-\nu}^{(1)}G_\nu^{(1)}\left\{\frac{\Delta}{r^2\Sigma^{7/2}} + \frac{E\cos\psi}{r}\left[\frac{w^{(0)^2}}{2\Delta^2\Sigma^{5/2}} - \frac{5\Delta}{2\Sigma^{9/2}}\right]\right\}$$

$$+ \Pi^2 G_{m-\nu}^{(1)}G_\nu^{(1)}\frac{E\cos\psi}{nr}\left[\frac{-11}{2}\frac{w^{(0)^2}}{\Sigma^{9/2}} + \frac{w^{(0)^2}}{2\Delta^2\Sigma^{5/2}}\right]$$

$$+ \Pi\int_0^\xi G_m^{(1)}\,d\xi\left\{\left(-\frac{3}{4} + m^2\right)\frac{1}{r^{5/2}\Delta^{1/4}\Sigma} + \frac{E\cos\psi}{r^{3/2}}\left(\frac{3}{4\Delta^{1/4}\Sigma^2} - \frac{1}{2\Delta^{5/4}\Sigma}\right)\right.$$

$$\left. + \frac{(E\cos\psi)^2}{r^{1/2}}\left[\frac{-3}{2\Delta^{1/4}\Sigma^3} + \frac{7w^{(0)^2}}{8\Delta^{9/4}\Sigma^2} - \frac{21}{16\Delta^{9/4}\Sigma}\right]\right\} + \dot{F}_m^{(2)}(r) - F_m^{(2)}(r)\frac{\dot{x}_r^{(0)}}{x_r^{(0)}} \tag{36}$$

The left-hand side of this differential equation has the same structure as the analogous equation in the first-order calculation (eq. (30)). Hence, the second-order angle of inclination $\vartheta^{(2)}$ can be determined as

$$\vartheta_m^{(2)} = \Pi G_{m-\nu}^{(1)} G_\nu^{(1)} \int_\infty^r \frac{\Delta^{3/4}}{r^{3/2}\Sigma^2} dr + 2G_m^{(2)}\Lambda + \left(m^2 - \frac{3}{4}\right)\Lambda \int_0^\xi G_m^{(1)}(\xi)d\xi \int_\infty^r \frac{\Sigma^{1/2}}{\Delta^{1/2}r^2} dr$$

$$+ \Pi\Lambda G_{m-\nu}^{(1)} G_\nu^{(1)} E \cos\psi \int_\infty^r \left(\frac{w^{(0)^2}}{2r^{1/2}\Delta^{9/4}\Sigma} - \frac{5\Delta^{3/4}}{2r^{1/2}\Sigma^3}\right) dr$$

$$+ \Pi\Lambda G_{m-\nu}^{(1)} G_\nu^{(1)} \frac{E\cos\psi\, w^{(0)^2}}{n} \int_\infty^r \left(\frac{-11}{2r^{1/2}\Delta^{1/4}\Sigma^3} + \frac{1}{2r^{1/2}\Delta^{9/4}\Sigma}\right) dr$$

$$+ \Lambda E \cos\psi \int_0^\xi G_m^{(1)}(\xi)d\xi \int_\infty^r \left(\frac{3}{4r\Delta^{1/2}\Sigma^{1/2}} - \frac{\Sigma^{1/2}}{2\Delta^{3/2}r}\right) dr$$

$$+ \Lambda(E\cos\psi)^2 \int_0^\xi G_m^{(1)}(\xi)d\xi \int_\infty^r \left(\frac{-3}{2\Delta^{1/2}\Sigma^{3/2}} + \frac{7w^{(0)^2}}{8\Delta^{5/2}\Sigma^{1/2}} - \frac{21\Sigma^{1/2}}{16\Delta^{5/2}}\right) dr \qquad (37)$$

where

$$\Lambda = \frac{\Delta^{1/4}}{2\Sigma^{3/2}r^{1/2}}\Pi$$

and $G^{(2)}$ is the second-order integration function. In calculating the integration functions $G^{(1)}$ and $G^{(2)}$, it will be shown that $G^{(2)}$ is really much smaller than $G^{(1)}$. The integration functions are assumed to obey the relation (eq. (23)):

$$G^{(j)} = G^{(j)}(\xi,\psi) = \sum_\lambda G_\lambda^{(j)}(\xi)e^{i\lambda\psi}$$

For completeness, the second-order differential equation for the abscissa of the Mach cone will be given as well. According to equation (35d),

$$x_{m,r}^{(2)} = -\vartheta_m^{(2)} \frac{w^{(0)^2}}{\Delta} - \Pi^2 G_{m-\nu}^{(1)} G_\nu^{(1)} \frac{w^{(0)^2}}{2r\Delta^2\Sigma^{3/2}} \left[1 + \frac{w^{(0)^4}}{n\Sigma^2}\right]$$

$$+ \Pi w^{(0)^2} \int_0^\xi G_m^{(1)}(\xi)d\xi \left[\frac{1}{4r^{3/2}\Delta^{5/4}\Sigma} + \frac{E\cos\psi}{r^{1/2}}\left(\frac{1}{4\Delta^{9/4}\Sigma} - \frac{1}{\Delta^{5/4}\Sigma^2}\right)\right]$$

$$+ \Pi^2 w^{(0)^2} G_{m-\nu}^{(1)} G_\nu^{(1)} \frac{1}{2r\Delta^2\Sigma^{9/2}} \left\{\frac{1}{n^2}\left[3w^{(0)^4}\Delta - w^{(0)^4}\Sigma\right]\right.$$

$$\left. + \frac{1}{n}\left[4w^{(0)^4}\Delta - 2w^{(0)^4}\Sigma + w^{(0)^2}\Sigma\Delta\right] + \left[2w^{(0)^4}\Delta - w^{(0)^4}\Sigma + \Delta^2\Sigma\right]\right\} \tag{38}$$

If the influence due to gravity is set equal to zero, all the previous equations can be reduced to the corresponding equations in reference 3.

As this theory is based on the hyperbolic differential equation of wave propagation, the solution to the problem at the cutoff point $w = a(r)$, which is a parabolic problem, cannot be obtained. Here, the Mach cone and the wave normal cone degenerate to a plane and a straight line, respectively, when the Mach angle $\mu = 90^o$ (fig. 3).

Coordinates of the Shock Front

The differential equation for the coordinates of the shock front in a stratified atmosphere does not differ essentially from equation (28) of the previous paper (ref. 3). For a stratified atmosphere, this equation is written as

$$\left[1 + x_r^{(0)} x_{s,r}\right]^2 \left(1 + x_r^{\,2} + \frac{1}{r^2} x_\psi^{\,2}\right) = \left(1 + x_r x_{s,r} + \frac{1}{r^2} x_\psi x_{s,\psi}\right)^2 \left[1 + x_r^{(0)^2}\right] \tag{39}$$

This equation must be developed in orders of magnitude and in Fourier series in ψ in the same manner as the set of equations (15). The coordinate x_s of the shock front is calculated to be

$$x_s = x^{(0)} + \frac{1}{2} x_m^{(1)} \tag{40}$$

In this paper the bow shock is the only shock that is considered and calculated.

Satisfying the Boundary Condition

The integration of the system of equations (25) and (35) introduced the functions of integration $G_m^{(j)}(\xi)$, which have to be determined from the boundary condition. Although these equations have been integrated for large distances r, the boundary condition can be formulated only for small r – that is, at the body itself. However, here the boundary condition will be matched with the nonlinear solution at some distance R. If linear slender-body theory is used, the boundary condition means another approximation. But slender-body theory has here the advantage of short, compact solutions. To handle the problem in such a way means that the disturbances sent out by the body are assumed to run along straight characteristics (zeroth-order theory). At the distance R, the straight characteristics are matched with the characteristics resulting from the exact theory that are valid for large distances r. At first glance it seems to be a problem to find an appropriate matching distance R. But numerical testing showed that this parameter can be chosen in a broad range of values without essentially changing the numerical results (ref. 3).

It is convenient to calculate the functions of integration by considering the angle of inclination ϑ of the streamline. The inclination can be expressed by

$$\tan \vartheta = \frac{v}{u_\infty + u} \tag{41}$$

where v is a vertical velocity component and $u_\infty + u$ is a horizontal velocity component. For small disturbances this equation can be approximated by

$$\vartheta = \frac{v}{u_\infty} = \beta \frac{\partial \phi}{\beta\, \partial r} = \beta \frac{\partial \phi}{\partial \eta} \tag{42}$$

where $\beta = \sqrt{M^2 - 1}$ and $\eta = \beta r$.

If equation (42) is developed in perturbation and Fourier series, it can be rewritten as

$$\vartheta_0^{(1)} + \vartheta_0^{(2)} + \left[\vartheta_1^{(1)} + \vartheta_1^{(2)}\right] 2 \cos \psi + \left[\vartheta_2^{(1)} + \vartheta_2^{(2)}\right] 2 \cos 2\psi = \left\{\phi_{\eta_0}^{(1)} + \phi_{\eta_0}^{(2)} + \left[\phi_{\eta_1}^{(1)} + \phi_{\eta_1}^{(2)}\right] 2 \cos \psi + \left[\phi_{\eta_2}^{(1)} + \phi_{\eta_2}^{(2)}\right] 2 \cos 2\psi\right\} \beta \tag{43}$$

By comparing equation (43) with equations (31) and (37), the functions of integration can be expressed in terms of ϕ_η, which itself will be determined by slender-body theory. This will now be demonstrated for a lifting body of revolution.

Lifting Body of Revolution

The quantities $\phi^{(j)}_{\eta_m}$ can be taken from reference (3):

$$\left.\begin{aligned}
\phi^{(1)}_{\eta_0} &= \frac{1}{2\pi\eta}\,\frac{\sqrt{\xi}(\xi+\eta)}{\sqrt{\xi+2\eta}}\,H^{(1)}(A) \\
\phi^{(2)}_{\eta_0} &= \frac{1}{2\pi\eta}\,\frac{\sqrt{\xi}(\xi+3\eta)}{\sqrt{(\xi+2\eta)^3}}\,H^{(2)}(A) \\
\phi^{(1)}_{\eta_1} &= \frac{\alpha\beta}{2\pi\eta^2}\,\frac{\sqrt{\xi}(\xi+\eta)^2}{\sqrt{\xi+2\eta}}\,H^{(1)}(A) \\
\phi^{(2)}_{\eta_1} &= \frac{\alpha\beta}{2\pi\eta^2}\,\frac{\sqrt{\xi}(3\xi^2+10\eta\xi+7\eta^2)}{2\sqrt{(\xi+2\eta)^3}}\,H^{(2)}(A)
\end{aligned}\right\} \qquad (44)$$

where

$$H^{(1)}(A) = \frac{A'}{\xi} + \frac{1}{2\xi^2}(A'\xi - A) + \frac{3}{8\xi^3}\left[A'\xi^2 - 2A\xi + 2\int_0^\xi A(\omega)d\omega\right]$$

$$+ \frac{5}{16\xi^4}\left[A'\xi^3 - 3A\xi^2 + 6\int_0^\xi A(\omega)\omega\, d\omega\right] + \ldots$$

$$H^{(2)}(A) = -A' + \frac{1}{2\xi}(A'\xi - A) + \frac{1}{8\xi^2}\left[A'\xi^2 - 2A\xi + 2\int_0^\xi A(\omega)d\omega\right]$$

$$+ \frac{1}{16\xi^3}\left[A'\xi^3 - 3A\xi^2 + 6\int_0^\xi A(\omega)\omega\, d\omega\right] + \ldots$$

Here $A' = \frac{dA}{d\xi}$; α denotes the angle of attack, which is measured counterclockwise; and $\eta = \beta R$, where R is the distance from the body where the boundary condition will be satisfied.

Thus the functions of integration are determined as

$$G_0^{(1)} = \frac{\sqrt{\xi}(\xi + \eta)}{2\pi\sqrt{R}\sqrt{\xi + 2\eta}} H^{(1)}(A) \tag{45a}$$

$$G_1^{(1)} = \frac{\alpha\sqrt{\xi}(\xi + \eta)^2}{2\pi R^{3/2}\sqrt{\xi + 2\eta}} H^{(1)}(A) \tag{45b}$$

$$\begin{aligned} G_0^{(2)} = {} & \frac{\sqrt{\xi}(\xi + 3\eta)}{2\pi\sqrt{R}\sqrt{(\xi + 2\eta)^3}} H^{(2)}(A) - \Pi\left[G_0^{(1)^2} + 2G_1^{(1)^2}\right]\int_\infty^R \frac{\Delta^{3/4}}{2r^{3/2}\Sigma^2} dr \\ & + \frac{3}{8}\int_0^\xi G_0^{(1)}(\xi)d\xi\int_\infty^R \frac{\Sigma^{1/2}}{\Delta^{1/2}r^2} dr - \Pi\left[G_0^{(1)^2} + 2G_1^{(1)^2}\right]\frac{E \cos \psi}{4n}\int_\infty^R \left[\frac{-11w^{(0)^2}}{r^{1/2}\Delta^{1/4}\Sigma^3} \right. \\ & \left. + \frac{w^{(0)^2}}{r^{1/2}\Delta^{9/4}\Sigma}\right]dr - \int_0^\xi G_0^{(1)}(\xi)d\xi \, \frac{E \cos \psi}{4}\int_\infty^R \left(\frac{3}{2r\Delta^{1/2}\Sigma^{1/2}} - \frac{\Sigma^{1/2}}{\Delta^{3/2}r}\right)dr \\ & - \int_0^\xi G_0^{(1)}(\xi)d\xi\left(\frac{E \cos \psi}{2}\right)^2\int_\infty^R \left[\frac{-3}{\Delta^{1/2}\Sigma^{3/2}} + \frac{7w^{(0)^2}}{4\Delta^{5/2}\Sigma^{1/2}} - \frac{21\Sigma^2}{8\Delta^{5/2}}\right]dr \end{aligned} \tag{45c}$$

$$G_1^{(2)} = \frac{\alpha\sqrt{\xi}(3\xi^2 + 10\eta\xi + 7\eta^2)}{4\pi R^{3/2}\sqrt{(\xi + 2\eta)^3}} H^{(2)}(A) - \Pi G_0^{(1)} G_1^{(1)} \int_\infty^R \frac{\Delta^{3/4}}{r^{3/2}\Sigma^2} dr$$

$$- \frac{1}{8}\int_0^\xi G_1^{(1)}(\xi)d\xi \int_\infty^R \frac{\Sigma^{1/2}}{\Delta^{1/2} r^2} dr - \Pi G_0^{(1)} G_1^{(1)} \frac{E \cos\psi}{2} \int_\infty^R \left[\frac{w^{(0)^2}}{r^{1/2}\Delta^{9/4}\Sigma} - \frac{5\Delta^{3/4}}{r^{1/2}\Sigma^3}\right] dr$$

$$- \Pi G_0^{(1)} G_1^{(1)} \frac{E \cos\psi}{2n} \int_\infty^R \left[\frac{-11 w^{(0)^2}}{r^{1/2}\Delta^{1/4}\Sigma^3} + \frac{w^{(0)^2}}{r^{1/2}\Delta^{9/4}\Sigma}\right] dr$$

$$- \int_0^\xi G_1^{(1)}(\xi)d\xi \frac{E \cos\psi}{4} \int_\infty^R \left(\frac{3}{2r\Delta^{1/2}\Sigma^{1/2}} - \frac{\Sigma^{1/2}}{\Delta^{3/2} r}\right) dr$$

$$- \int_0^\xi G_1^{(1)}(\xi)d\xi \left(\frac{E \cos\psi}{2}\right)^2 \int_\infty^R \left[\frac{-3}{\Delta^{1/2}\Sigma^{3/2}} + \frac{7 w^{(0)^2}}{4\Delta^{5/2}\Sigma^{1/2}} - \frac{21\Sigma^{1/2}}{8\Delta^{5/2}}\right] dr \tag{45d}$$

Within the limits of a small error, equations (45) can be integrated for $a(z) = a(R)$.

The integration functions $G^{(1)}$ and $G^{(2)}$ are compared in figure 4 for a fighter airplane. The cross-sectional area distribution of that airplane is given in figure 5 for $M = 1.6$ Mach cuts. The negative areas are due to the fact that the engine stream tube exit areas are smaller than the intake. It can be seen from figure 4 that the integration functions satisfy the assumption of equation (16), since $G^{(1)} \gg G^{(2)}$ is the fundamental condition for the gas properties of first order, such as $\vartheta^{(1)}$ and $w^{(1)}$, to be greater than those of second order.

Pressure Signature

The pressure coefficient is approximated by

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^{\ 2}} = -2\frac{\partial\phi}{\partial x} = -2\left[\frac{w}{w^{(0)}}\cos\vartheta - 1\right] \tag{46}$$

The pressure Δp at the altitude $z + r \cos \psi$ has been nondimensionalized with respect to the dynamic pressure at the ground. By using the equation for the speed of sound at the the ground, $a_{gr}^2 = \frac{n+2}{n} \frac{p_{gr}}{\rho_{gr}}$, equation (46) can be rewritten as

$$\frac{\Delta p}{p_{gr}} = -\frac{n+2}{n} \frac{w^{(0)^2}}{a_{gr}^2} \left[\frac{w}{w^{(0)}} \cos \vartheta - 1 \right] \tag{47}$$

To use this equation, the solutions for w and ϑ have to be introduced.

DISCUSSION

Some pressure signatures will be given for a fighter airplane. The point of intersection of the Mach line of the undisturbed flow through the nose of the body and a line parallel to the abscissa x at the distance r away from the body gives the origin of the plots of the pressure signatures, as shown in figure 6. The points of multiple values on the pressure signature indicate the occurrance of shocks. In figure 7 the pressure signature is shown for the Mach number $M = 1.5$, the altitude of flight $z = 10\ 000$ m, and the speed of sound at the ground $a_{gr} = 300$ m/sec at the distance $r = 100$ m away from the body. Here the factor $\alpha \cos \psi$ has been varied. Naturally the pressure rise is greater at the bottom of the body ($\cos \psi = -1$) than at the top ($\cos \psi = 1$) for an angle of attack $\alpha = -0.1$. In figure 8 the decreasing pressure with increasing distance from the craft is demonstrated. In both figures only the bow shock has been calculated. A comparison with the theory given in reference 2 is shown in figure 9. For this calculation a fairly good agreement has been obtained.

At the caustic the speed of sound is of the same order of magnitude as the velocity; that is, $w^2 - a^2(r) << 1$. For the cutoff point, this can be written as $w^2 = a^2(r) = a_{gr}^2 - E(z + r \cos \psi)$. The locus of the cutoff can be calculated for given values of ψ, z, a, and $M(z)$:

$$r_{cutoff} = \frac{1}{\cos \psi} \left[\frac{a_{gr}^2 - M^2(z) a^2(z)}{E} - z \right]$$

Because of the parabolic character of the flow, the pressure rise cannot be investigated at the cutoff point. At this point the assumed series, equation (16), breaks down. As can be seen from equations (31) and (32), the first-order angle of inclination $\vartheta^{(1)}$ decreases continuously and becomes zero at the cutoff point, whereas the first-order velocity component decreases at first and then increases, reaching a high value near the cutoff point.

Thus, at the cutoff point the first-order quantity $w^{(1)}$ is inconsistent with the assumption of equation (26), whereas the result for $\vartheta^{(1)}$ is compatible with this assumption. However, in the second-order calculation even the angle of inclination $\vartheta^{(2)}$, equation (37), contains singularities at the cutoff. However, so long as the Mach number is slightly supersonic, the pressure can be calculated. The pressure rise is shown in figure 10. First, a pressure decrease with increasing distance from the body can be observed. Then, ahead of the cutoff point, the pressure increases because of the focusing and then augments very rapidly near the cutoff point. It should be pointed out that the most rapid increase of the pressure occurs within a distance of a few centimeters. For completeness the cutoff distance, at which the calculation breaks down, has been included in the figure. For this point, geometric acoustics give an infinite pressure rise. It is the author's opinion that so long as the initial equations do not contain any damping mechanism, such as viscosity or entropy increase, the pressure rise at the cutoff cannot be expected to be finite.

There are some uncertainties about how the pressure wave is reflected at the ground and at the cutoff point. Figure 11 (from ref. 5) shows the reflection of the pressure wave at the ground as measured from various tower microphones. It seems to indicate that the pressure wave is reflected in the same way as a wave at a solid wall; that is, the compression wave is reflected as a compression wave and the rarefaction wave as a rarefaction wave. The pressure signatures of the wave traveling toward the ground and the reflected wave can be superimposed. The abscissa of the reflected wave is $x_{refl} = -x$, whereas the pressure signatures of the reflected wave are given by equation (47). However, in this case no obstacles at the ground are taken into account; the ground is considered to be plane and smooth. Figure 12 indicates the complexity of the boundary condition in the mixed-flow region of the caustic. It is not clear whether the atmosphere can be considered to behave as a "soft" solid wall at the cutoff point. Figure 13 (from ref. 5) shows some measurements at the cutoff point. Here the phenomenon of the U-shaped wave can be observed. The incident N-wave is changed at the focus with a component phase shift of $\pi/2$ radians, whereas the reflected wave, after passage through the focus, again has an N-waveform. In first-order calculations the U-shaped waveform has not been obtained by the present theory. It is hoped to get an improvement by including the second order in the calculations.

Although some progress has been made by Seebass (ref. 6) in calculating the sonic line at the caustic by using Tricomi's equation, this problem needs more investigation for the understanding of the complete behavior at the caustic and the cutoff point.

CONCLUSIONS

The exact nonlinear system of partial differential equations for supersonic flow is solved for large distances in a nonhomogeneous atmosphere by using a perturbation method. The unknown functions were expanded in a perturbation and a Fourier series. The system of differential equations was derived and solved for each order of magnitude and Fourier order. The integration introduced a function of integration which could be used to satisfy the boundary condition. To calculate the far field, the boundary condition was not satisfied at the body itself, but at some distance R away from the body. The matching point does not influence the numerical solutions in a broad interval of values of R. The example calculations give reasonable sonic-boom signatures. They are in good agreement with those obtained by previous theories.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., February 14, 1973.

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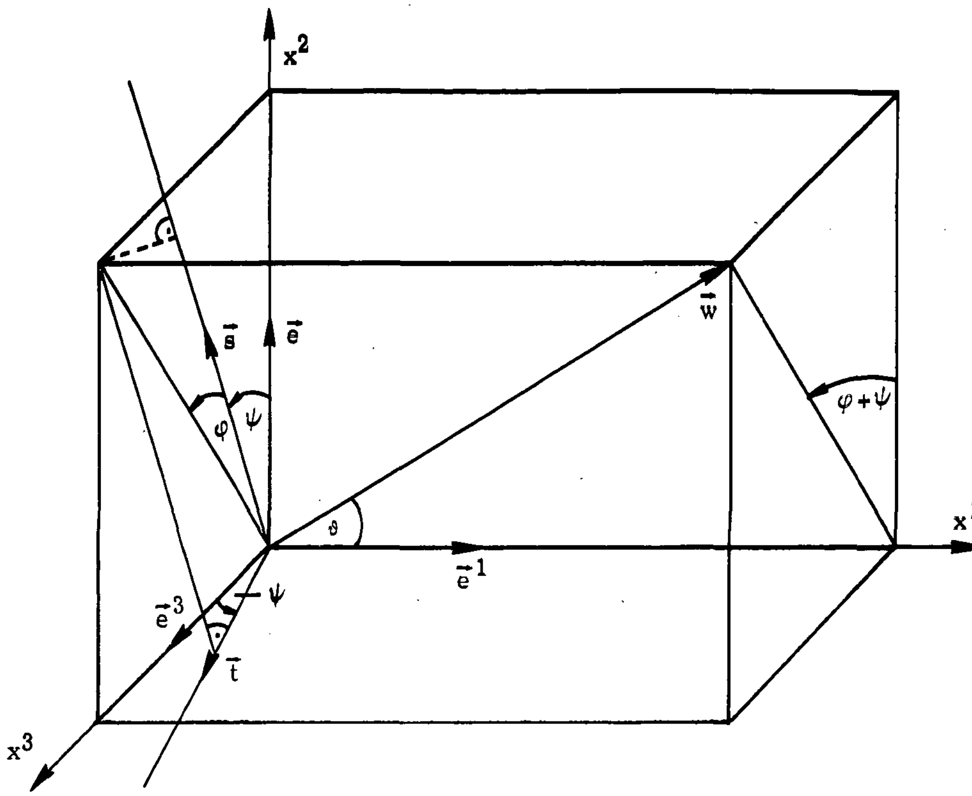


Figure 1.- The velocity vector.

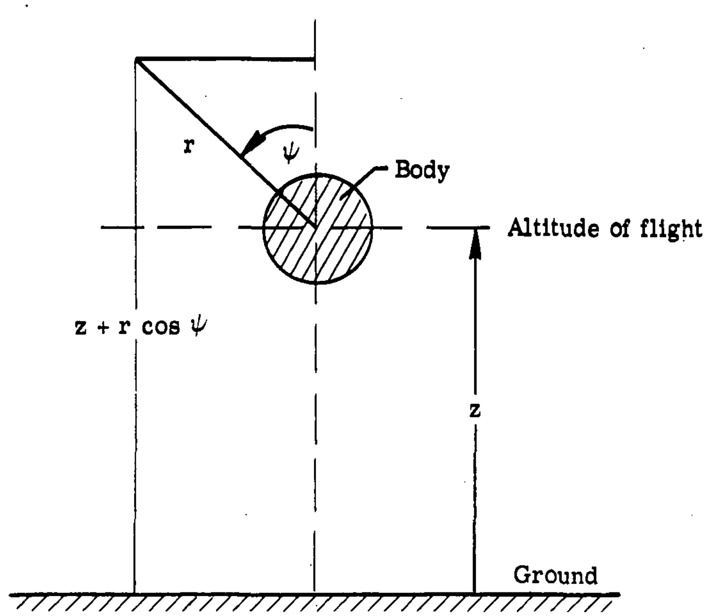


Figure 2.- Location of the body relative to the ground.

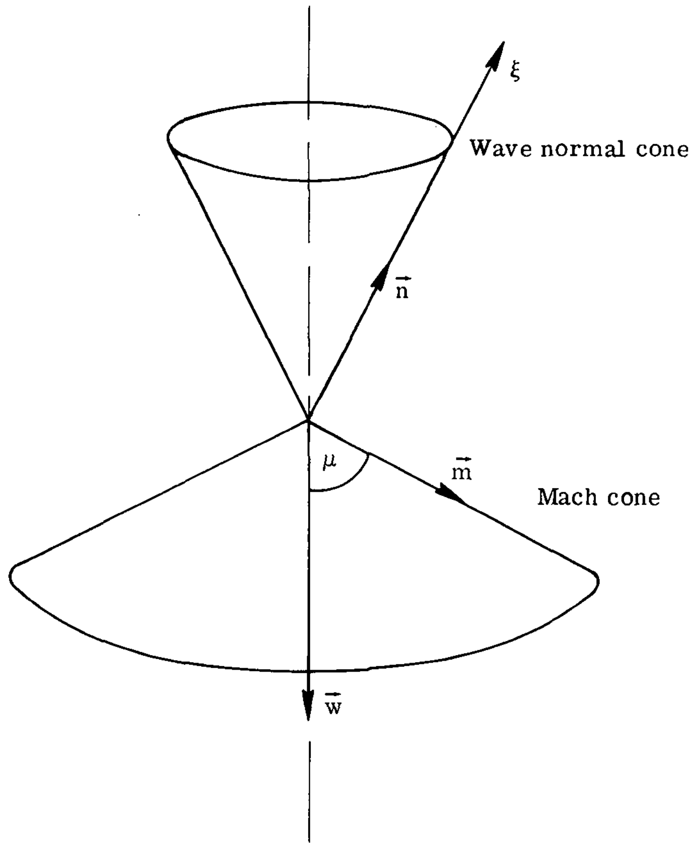


Figure 3.- Mach cone and wave normal cone.

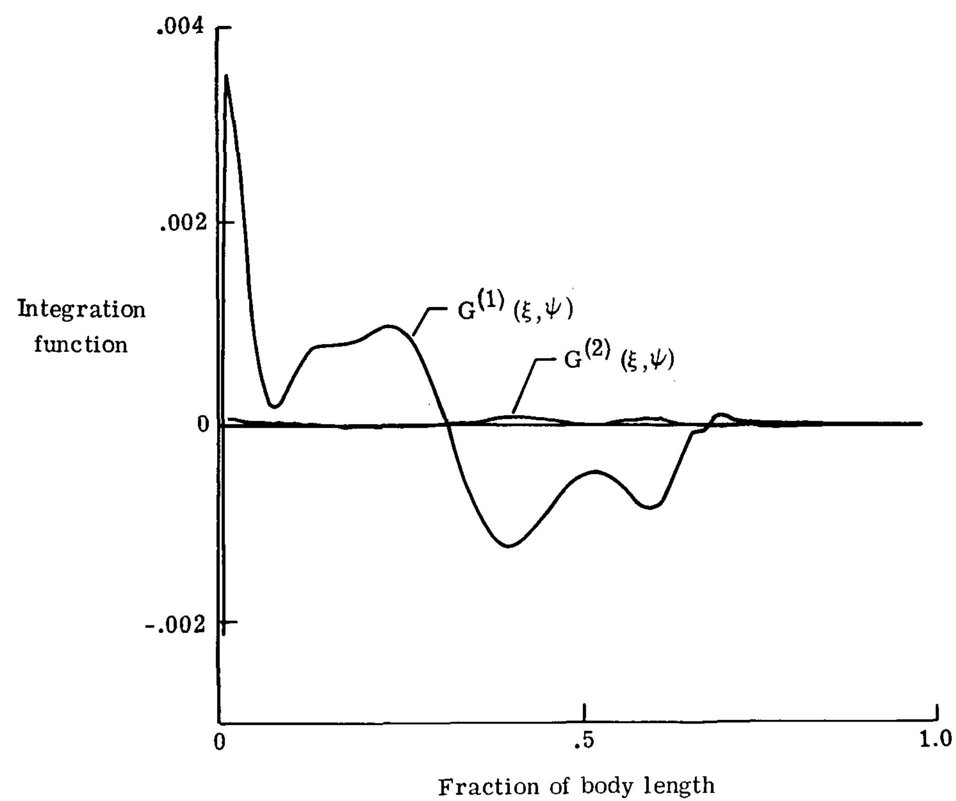


Figure 4.- Comparison of functions of integration $G^{(1)}(\xi,\psi)$ and $G^{(2)}(\xi,\psi)$. $M = 2$; $R = 20$ m; $\alpha \cos \psi = 0.1$.

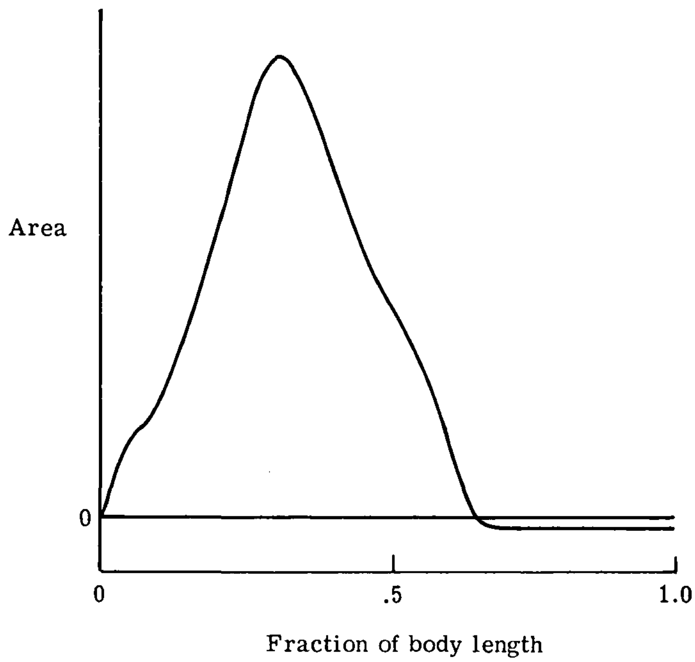


Figure 5.- Cross-sectional area distribution of a fighter airplane.

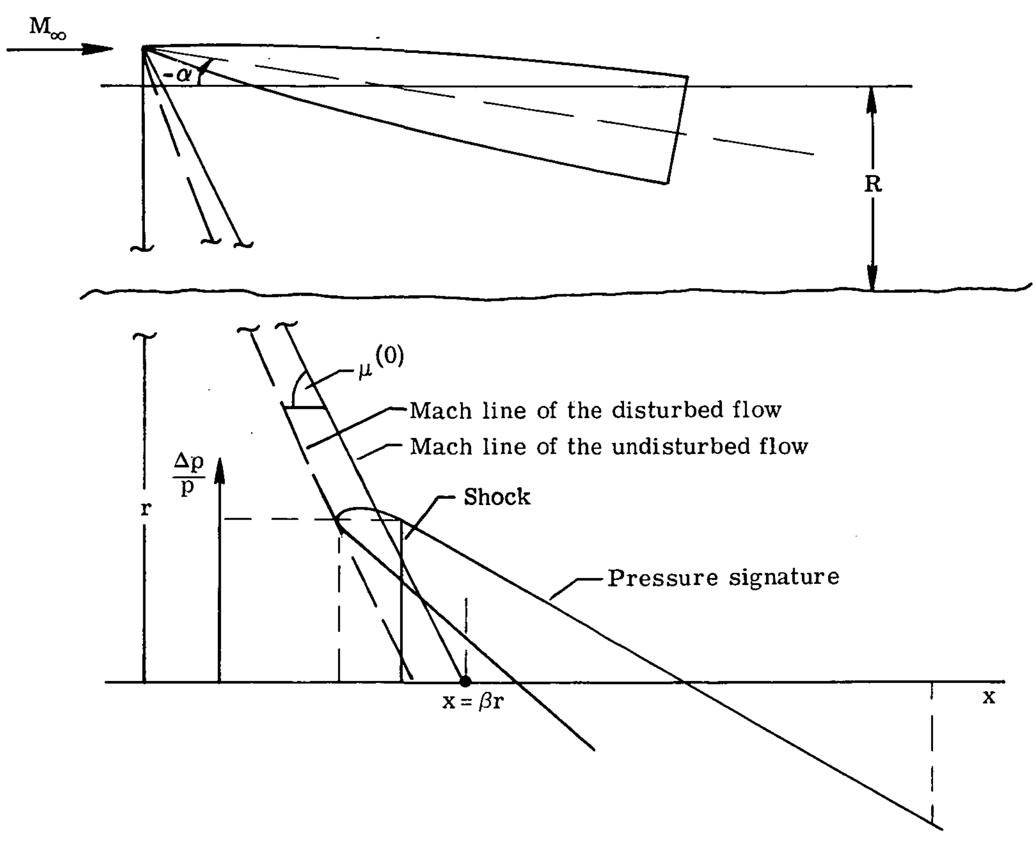


Figure 6.- Description of construction of the figures for the pressure signature.

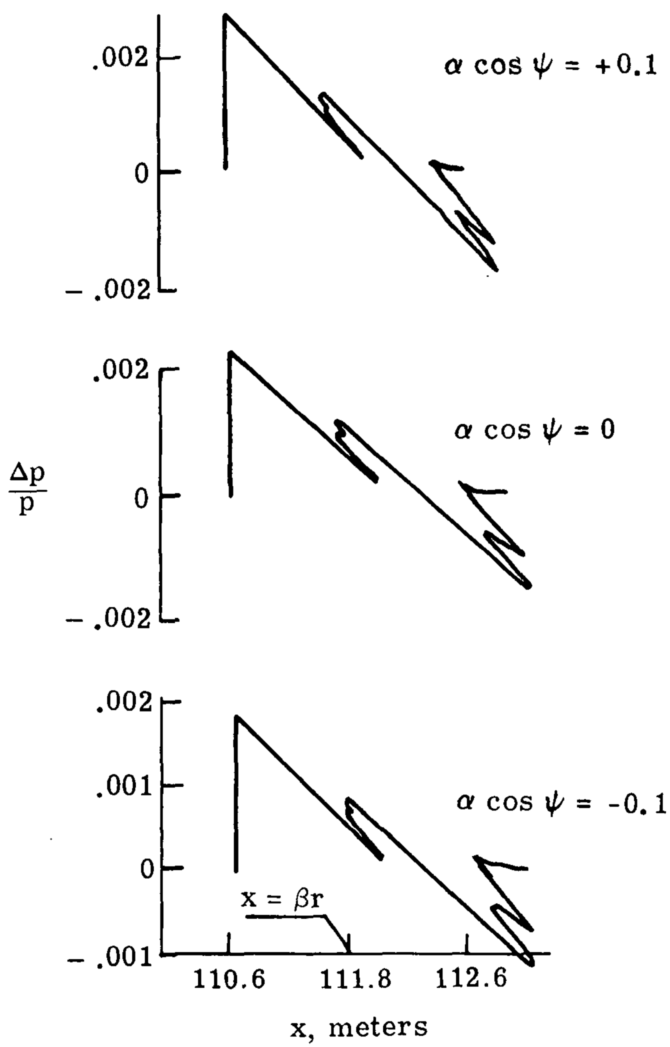


Figure 7.- Influence of $\alpha \cos \psi$ on the pressure signature. M = 1.5; z = 10 000 m; R = 20 m; r = 100 m.

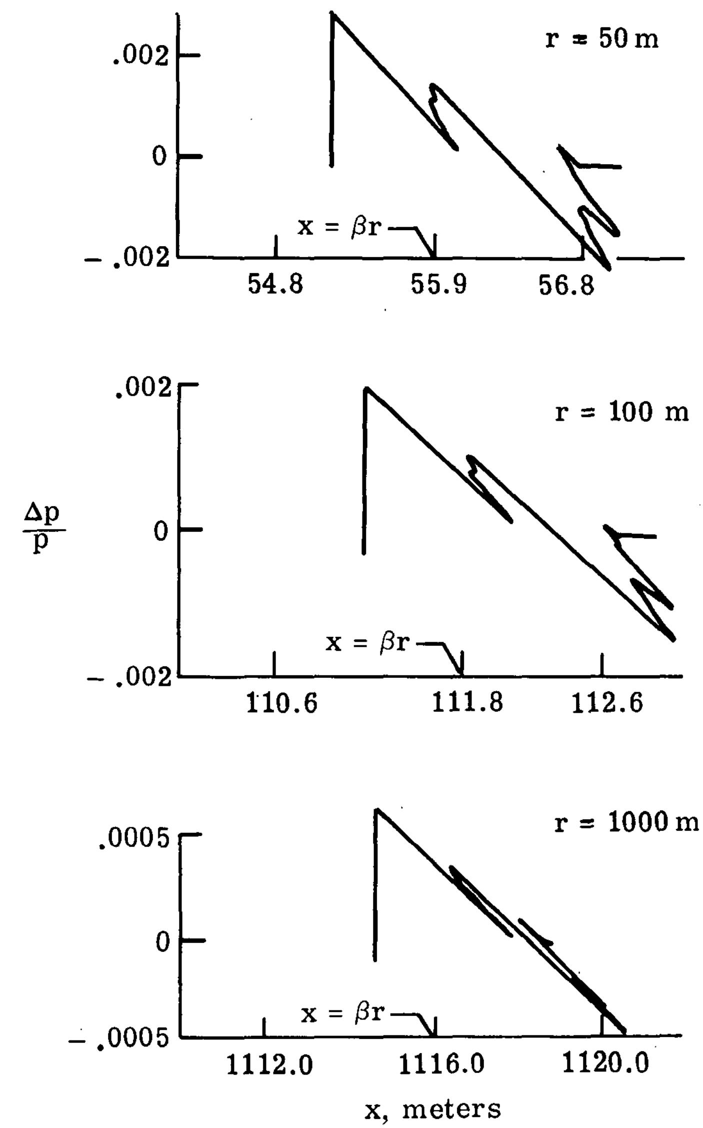
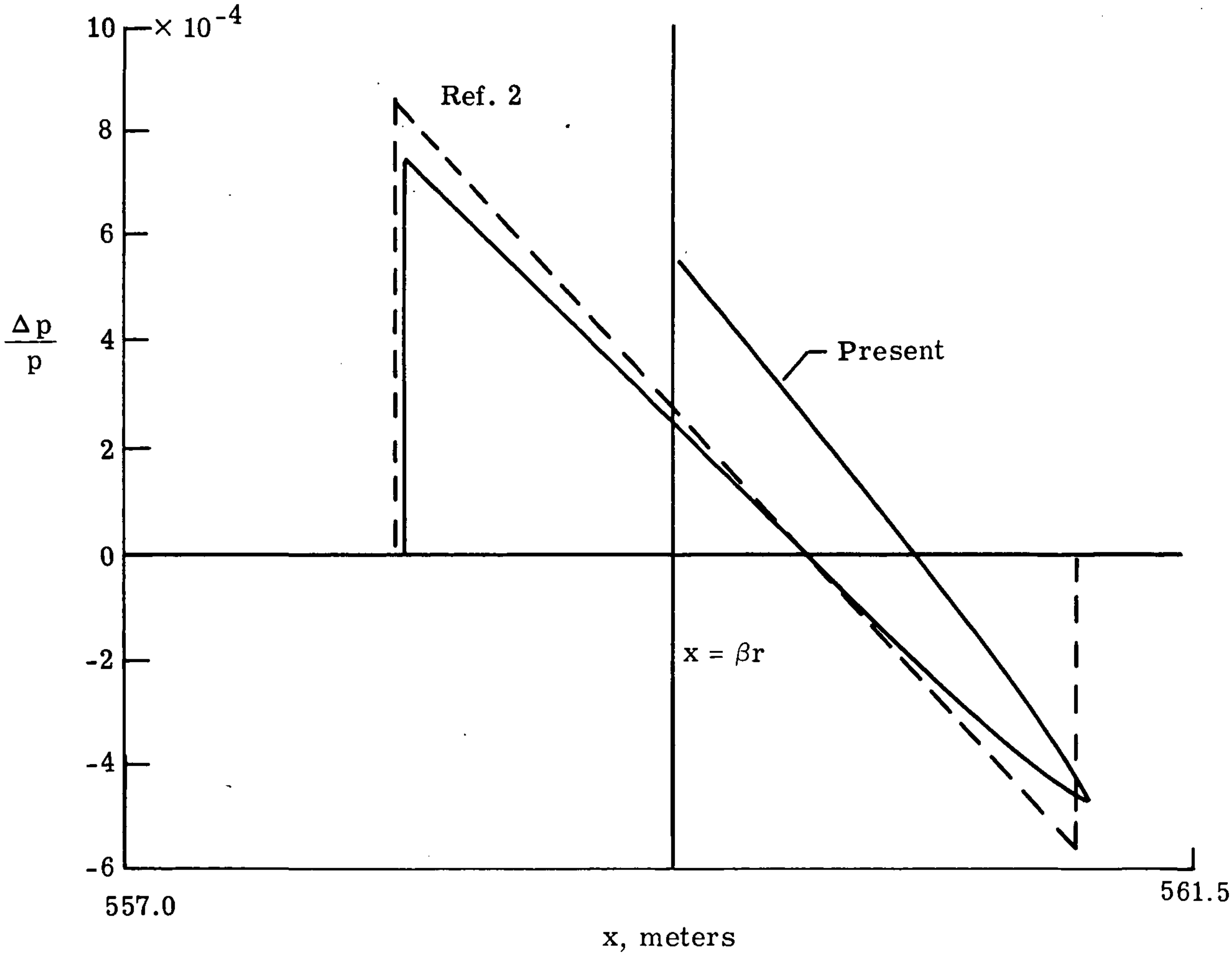
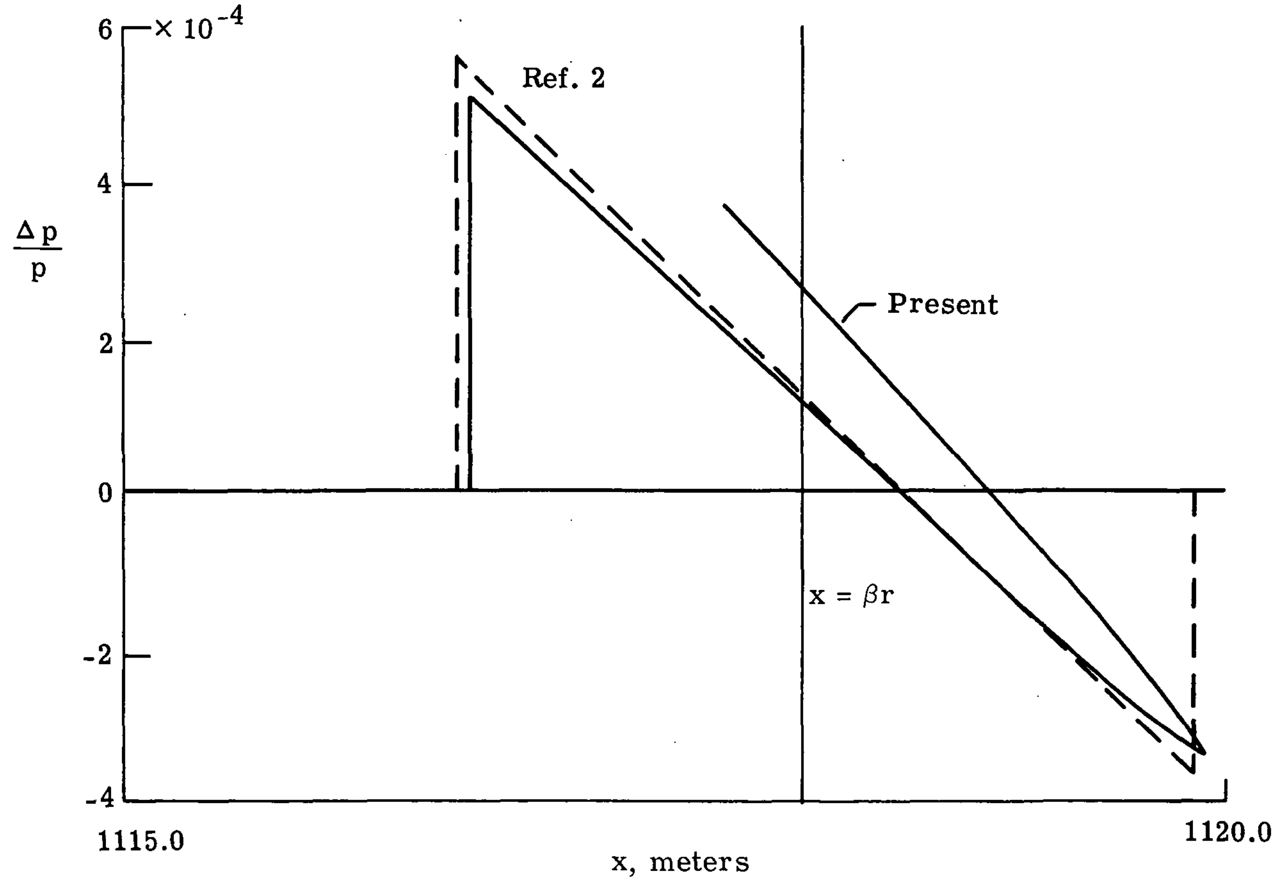


Figure 8.- The pressure signature at various distances r. $\alpha \cos \psi = 0$; M = 1.5; z = 1000 m; R = 20 m.



(a) r = 500 m.

Figure 9.- Comparison of present theory with theory of reference 2. Parabolic body shape, $A = 0.04\pi(x^2 - 2x^3 + x^4)$; $\alpha \cos \psi = 0$; M = 1.5; R = 20 m.



(b) r = 1000 m.

Figure 9.- Concluded.

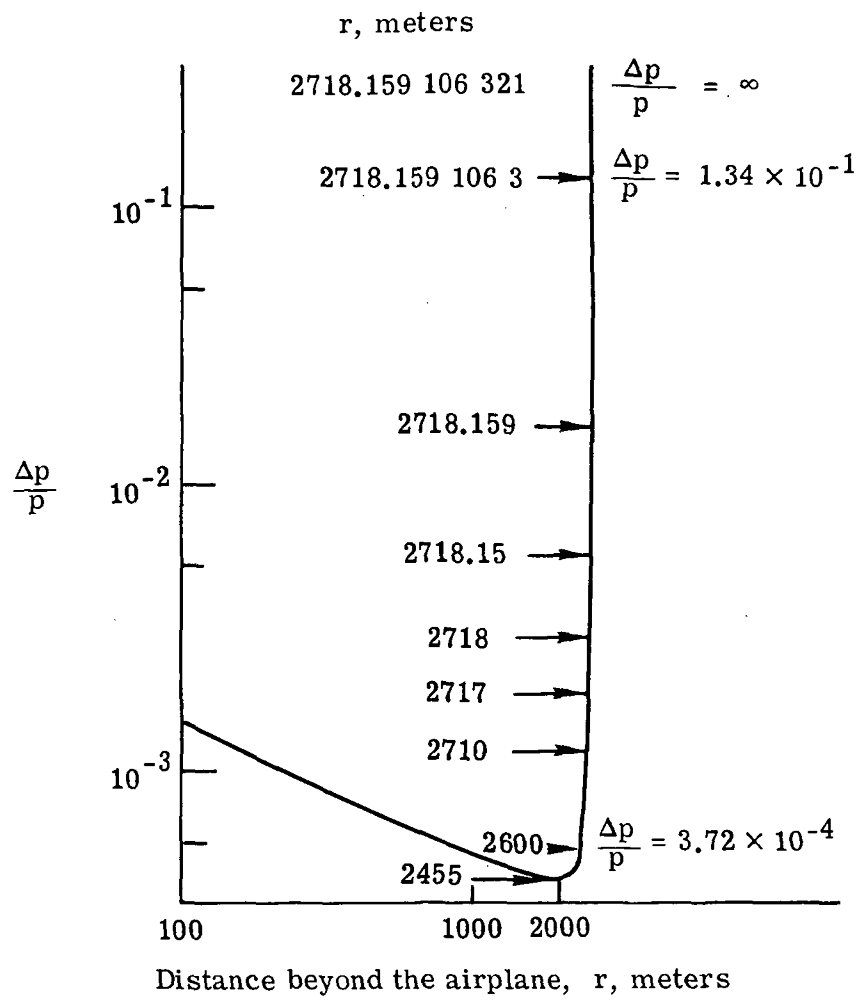


Figure 10.- The pressure rise at the caustic. $M = 1.1$; $z = 10\,000$ m; $a_{gr} = 300$ m/sec.

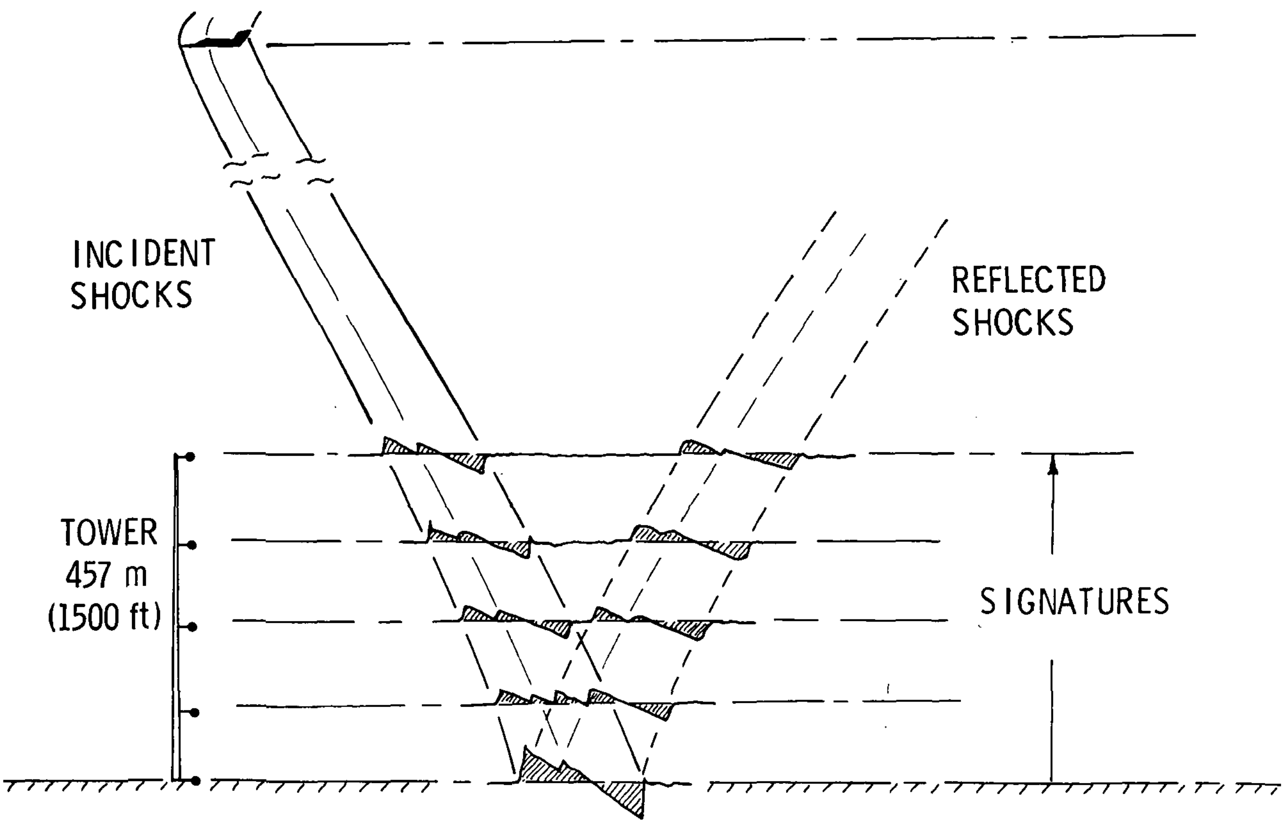


Figure 11.- Measured sonic-boom signatures at various heights above the ground for an F-104 aircraft in steady, level flight at a Mach number above cutoff (M = 1.3) and an altitude of 10.26 km. (From ref. 5.)

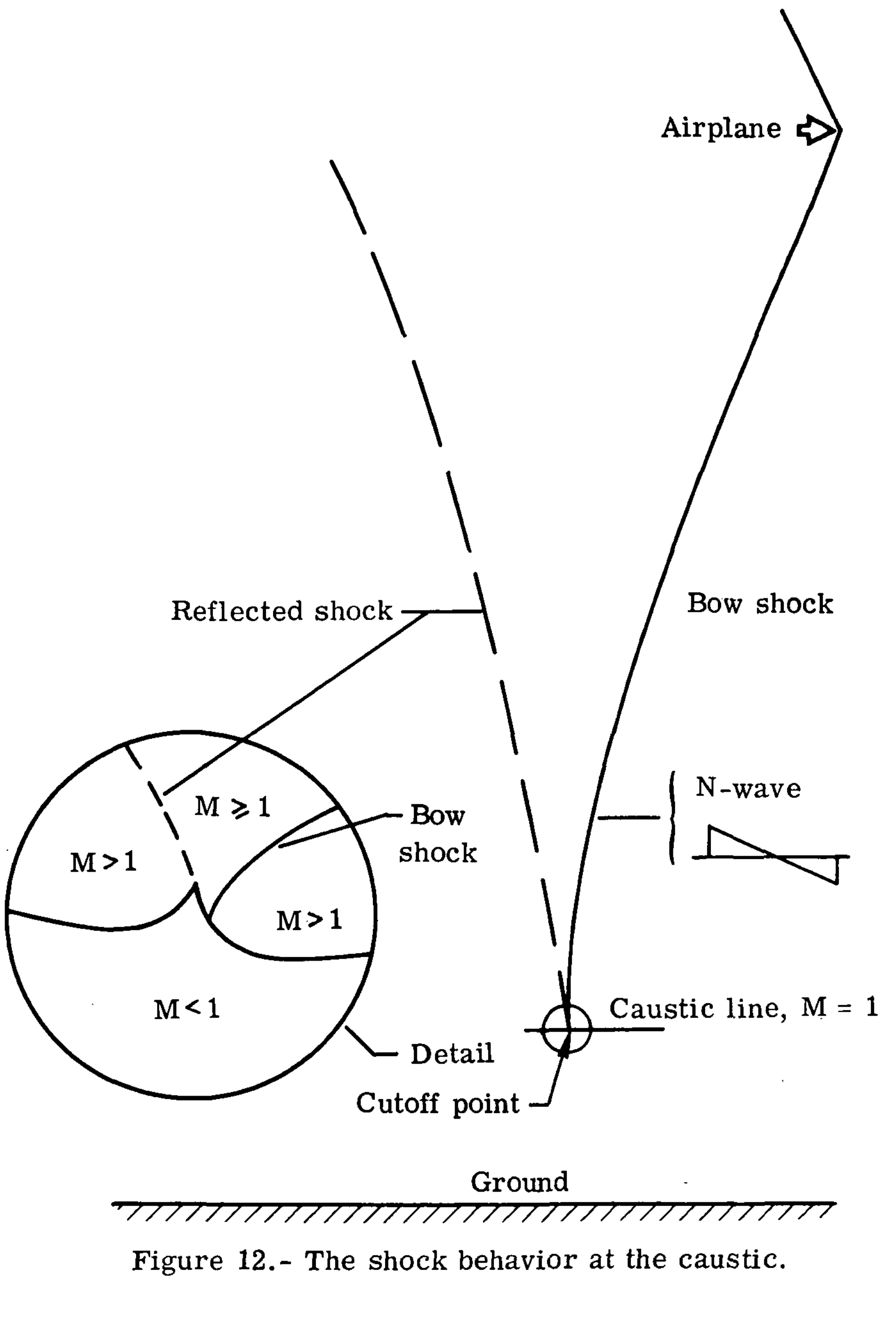


Figure 12.- The shock behavior at the caustic.

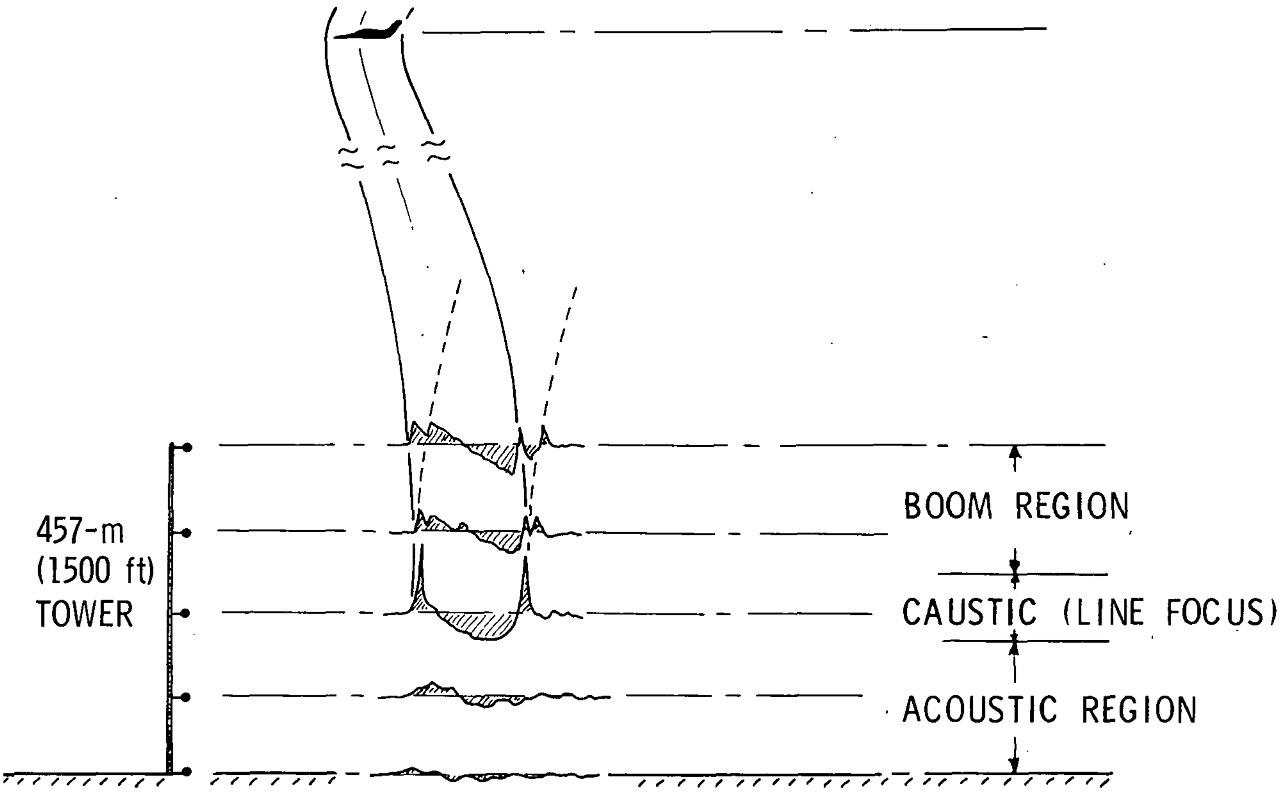


Figure 13.- Measured sonic-boom signatures at various heights above the ground for an F-104 aircraft in steady, level flight at cutoff Mach number (M = 1.095) and an altitude of 10.26 km. (From ref. 5.)

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