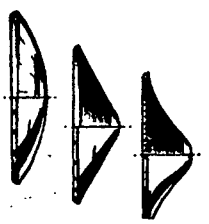
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FEASIBILITY STUDY OF THE NUMERICAL INTEGRATION OF SHELL EQUATIONS USING THE FIELD METHOD

By Gerald A. Cohen



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FEASIBILITY STUDY OF THE NUMERICAL INTEGRATION OF SHELL EQUATIONS USING THE FIELD METHOD

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SUMMARY

The "field method" for the numerical solution of even order linear boundary value problems in ordinary differential equations is formulated. In essence, the field method converts the boundary value problem into two successive initial value problems, each defined over the same one-dimensional domain as the boundary value problem. In the first of these problems the dependent variables are the "field functions", which appear in linear algebraic relations (the "field relations") satisfied by the dependent variables of the original boundary value problem. In general, for open branch domains the number of (scalar) field relations is equal to one-half the order of the boundary value problem. The field functions are obtained over the domain of the independent variable by a standard forward integration technique. In the second initial value problem the dependent variables are one-half of the original dependent variables, the remaining variables being given in terms of these by the field relations. The field functions appear as known terms in the differential equations of this problem, which is more properly termed a final value problem since it is solved by a backward integration over the domain of the independent variable.

In this report the field method is developed for arbitrary open branch domains subjected to general linear boundary conditions. Although closed branches are within the scope of the method, they are not treated here. The numerical feasibility of the method has been demonstrated by implementing it in a computer program for the linear static analysis of open branch shells of revolution under asymmetric loads. For such problems the field method eliminates the well-known numerical problem of "long subintervals" associated with the rapid growth of extraneous solutions. Also, the method appears to execute significantly faster than other numerical integration methods.

INTRODUCTION

Most problems in the response of axisymmetric shell structures may be reduced to the solution of equivalent linear statics problems. The earliest known application of numerical integration techniques to the solution of such problems was presented by Goldberg, et al. (ref. 1). In its most rudimentary form the mathematical problem is formulated as a two-point linear boundary value problem in ordinary differential equations. A set of linearly independent complementary solutions and a particular solution of the differential equations are obtained by a forward integration scheme (such as Runge-Kutta). Linear algebraic equations for superposition constants are then solved so that superposition of these auxiliary solutions gives the solution satisfying the boundary conditions. This method is, of course, nothing more than an attempt to use a well-known analytical approach for such problems as the basis of a numerical analysis.

Although this approach works well when the auxiliary solutions are obtained in closed form, it is numerically ill-conditioned in the sense that for sufficiently long intervals of integration the superposition of the auxiliary solutions involves the small difference of large numbers at points remote from the initial point.* This problem of "long subintervals" was subsequently circumvented by Cohen (ref. 2) and Kalnins (ref. 3), who divided the range of integration into suitably small subintervals. By using fresh initial values for the auxiliary solutions at the initial point of each subinterval, the growth of these functions is held to reasonable limits. Also, by allowing rings and other discontinuities to exist at the end points of each subinterval (i.e., the "boundaries"), the problem was simultaneously generalized to a multi-point boundary value problem. One difficulty with this approach is that the limit on subinterval length depends on factors in addition to the properties of the structure itself, e.g. the circumferential wave number and, for eigenvalue problems, the eigenvalue shift. Also some problems may not be treatable simply because more subintervals are required than allowed by computer storage limitations.

Later, Zarghamee and Robinson (ref. 4) further improved the method by proposing the use of initial values for the auxiliary solutions which imply satisfaction of the boundary conditions. Although this idea reduces the number of complementary solutions required by a factor of two, it does not eliminate the long subinterval problem. Their method was subsequently generalized to an arbitrary open branch domain [known as a "tree" in the terminology of geometric graph theory (ref. 5)] and general linear boundary conditions by Anderson, et al. (ref. 6). Cohen (ref. 7) further refined this method by improving the treatment of general boundary

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*If, instead, a numerical reintegration for the desired solution is attempted, the rapid growth of extraneous solutions associated with round-off errors will have the same effect.

conditions and generalizing it to domains which include a single closed branch.

About the same time as the Zarghamee method was being developed, Jordan and Shelly (ref. 8) demonstrated a numerical integration technique for two-point boundary value problems which eliminates the long subinterval problem. This method, which they termed the "field method", does not use complementary and particular solutions at all, but replaces the boundary value problem by two initial value problems to be solved in succession. The field method was later formulated by Miller (ref. 9) for two-point boundary value problems governed by a general ordinary linear differential equation of even order.

The purpose of the present study is twofold: 1) to formulate the field method for a general even order multi-point boundary value problem defined on an arbitrary tree, and 2) to demonstrate the numerical feasibility of the method by implementing it in a computer program for static response of open branch shells of revolution subjected to arbitrary boundary conditions.

SYMBOLS

a,b,c,d	$p \times p$ matrix coefficients of linear differential equations [eqs. (1)]
B,D	$p \times p$ matrix coefficients of linear boundary conditions [eq. (2)]
$C_1^{(0)}, C_1^{(2)}$	meridional stretching and bending stiffnesses
e	$p \times p$ diagonal scaling matrix [eq. (14b)]
f,g	inhomogeneous $p \times 1$ matrices of linear differential equations [eqs. (1)]
I	$p \times p$ identity matrix
L	inhomogeneous $p \times 1$ matrix of linear boundary conditions [eq. (2)]
ℓ	cylindrical length
M_1	meridional stress couple
n	circumferential harmonic number
P,Q,S	shell forces per unit of circumferential length in axial, radial, and circumferential directions

p	half-order of boundary value problem
R	spherical radius of curvature
R_2	circumferential radius of curvature
r	small circle radius
SC_1, SC_2	scale factors used as elements of e matrix
s	arc distance
s, ϕ, z	meridional, circumferential, and normal coordinates of shell reference surface
t	effective wall thickness
u, w	$p \times p$ and $p \times 1$ field function matrices [eq. (6)]
x, y	axial and radial coordinates
y, z	generalized force and displacement $p \times 1$ response matrices
Δ	net change across a vertex [eqs. (3) and (7)]
κ	$\underline{B}^{-1}\underline{D}$
ξ, η, v	shell displacements in axial, radial, and circumferential directions
χ	meridional rotation

Subscripts:

0	value at initial shell edge

Superscripts:

T	matrix transpose
$( )'$	$d( )/ds$
$( )^+$	value at vertex on exiting arc
$( )^-$	value at vertex on entering arc
$(\tilde{})$	modified variable for singular arcs

ANALYTICAL FORMULATION

In order to formulate the field method in a general context, it will be convenient to introduce some elementary concepts from the theory of geometric graphs. Figure 1 shows the reference meridian of a hypothetical shell of revolution. The heavy dotted points depict boundaries of the shell, which are defined as one of the following types of points:

(1) branch points
(2) branch extremities
(3) ring or ring load points
(4) shell property or load discontinuity points

The boundaries thus divide the meridian into a number of subintervals. In contrast to other numerical integration methods, no additional (artificial) boundaries are required simply to reduce subinterval length.

A geometric graph (ref. 5) is defined as a set of points, called vertices, and a set of non-self-intersecting curves, called arcs, satisfying the following requirements:

(1) Each closed arc contains precisely one vertex.
(2) Each open arc contains precisely two vertices, viz. its end points.
(3) The arcs have no common points, except for the vertices.

It is clear from figure 1 that if we identify the boundaries as vertices and the subintervals as arcs, the reference meridian of a shell of revolution is nothing more than a geometric graph.

The following graph terminology will be used:

(1) Chain - a continuous sequence of arcs from an initial vertex to a terminal vertex. (A non-self-intersecting chain is said to be simple.)
(2) Circuit - a chain whose initial and terminal vertices coincide.
(3) Connected graph - a graph for which every pair of vertices is joined by at least one chain.
(4) Tree - a connected graph which contains no circuits.

In the present formulation of the field method, we shall confine our attention to boundary value problems defined over one-dimensional domains representable as (i.e., isomorphic to) a tree (fig. 2). It is important to recognize that every pair of distinct vertices of a tree is joined by precisely one chain, since connectivity implies the existence of at least one chain, whereas the absence of circuits implies the existence of at most one chain. It therefore follows that the cutting of any arc of a tree will disconnect the tree into two separate parts, a fact which is used in setting up the one-dimensional coordinate of the tree.

For each arc, the arc distance s will be used as the independent variable. Let us assume that the arcs have been ordered and oriented (with respect to the direction of increasing s), but defer for the time being the manner in which this was done.

Definition of Boundary Value Problem

A system of ordinary differential equations of order 2p may always be written as a system of 2p first-order equations. If we group one-half of the dependent variables in the p × 1 matrix y and the other half in the p × 1 matrix z, the system of first-order equations may be written, for linear systems, as two matrix differential equations, viz.

$$y' + ay + bz = f \tag{1a}$$

$$z' + cy + dz = g \tag{1b}$$

where prime denotes differentiation with respect to s; a, b, c, and d are p × p matrix functions of s; and f and g are p × 1 matrix functions of s.

Equations (1) are defined at every interior point of each arc of the tree. They are supplemented by linear boundary conditions, defined at the vertices of the tree, of the form

$$B\Delta y + Dz = L \tag{2}$$

where B and D are p × p matrices, L is a p × 1 matrix, and

$$\Delta y = \sum y^{+} - \sum y^{-} \tag{3}$$

Here, y^{+} and y^{-} represent the values of y at the vertex on exiting (s increasing away from the vertex) and entering (s increasing towards the vertex) arcs respectively. As implied by the form of eq. (2), it is assumed that z is continuous at vertices.* A vertex and its boundary condition are said to be singular if the matrix B is singular; otherwise they are called regular.†

The majority of boundary value problems in mechanics are self-adjoint. This property is synonymous with being derivable from a

*It is assumed on physical grounds that one-half of the dependent variables are continuous at vertices, and these shall be grouped in the vector z. The vector y may be viewed as a generalized "force" vector corresponding to the generalized "displacement" vector z, and -Δy represents the net external force entering the vertex.

†A singular boundary condition implies a relationship between the components of the displacement vector z, i.e., kinematic constraint.

variational principle. Physically, this corresponds to systems which do not involve energy losses. For the system defined by eqs. (1) and (2), the conditions of self-adjointness may be shown to be

$$b = b^T \tag{4a}$$

$$c = c^T \tag{4b}$$

$$d = -a^T \tag{4c}$$

$$\kappa = \kappa^T \tag{4d}$$

where $$\kappa = B^{-1}D \tag{5}$$

and the superscript T denotes matrix transpose. Although condition (4d) strictly applies only to regular vertices, it may be used also for singular vertices if a singular vertex is viewed as a limiting case of a sequence of regular vertices. Thus, for self-adjoint problems a singular vertex is the limit of a sequence of regular vertices, for each of which eq. (4d) holds true.*

The Field Method

Since a cut at an interior point of any arc of a tree disconnects the tree into two separate parts, it is clear that the value of z at the cut can serve as a boundary condition which, along with the differential equations (1) and boundary conditions (2) over one of the parts, determines y and z over that part independently of the corresponding data over the remaining part.† In particular, the value of z determines y at the cut, which in view of the linearity of the problem is expressed by

$$y = uz + w \tag{6}$$

where u is a $p \times p$ matrix and w is a $p \times 1$ matrix. Equation (6) is called a field relation since it is satisfied by the "field" of all possible solutions y,z (depending on the unused data on the remaining part).

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*Physically speaking, kinematic constraint may be approximated as close as one pleases by a sufficiently stiff spring.

†It is assumed that giving z at the cut determines a unique solution, i.e., no nontrivial solution of the homogeneous forms of eqs. (1) and (2) satisfying the cut condition z = 0 exists. This is certainly the case if $y \equiv 0$ for all solutions of the homogeneous forms of eqs. (1) and (2) subject to an arbitrary homogeneous cut condition. Physically, this corresponds to a situation where the only possible response under no load is a rigid body displacement.

Correspondingly, u and w are called field functions. It follows that if we order and orient the arcs of the tree so that for every such cut all of one part is described by s before any of the other part, then the determination of u and w will be an initial value problem. Such an arc ordering and orientation will be achieved if at each interior vertex (i.e., one incident with two or more arcs):

(1) there is precisely one exiting arc,
(2) all entering arcs are ordered (i.e., described by s) before the exiting arc, and
(3) the exiting arc is ordered immediately after the last entering arc.

Implicit in these conditions is the requirement that one of the vertices of the first (last) arc is incident with no other arc, i.e., represents a branch extremity, and this arc is oriented away from (towards) this vertex. Assuming that the s-coordinate is set up in this manner, it is clear that, for a tree, eq. (3) reduces to

$$\Delta y = y^{+} - \sum y^{-} \tag{7}$$

since there will be at most one exiting arc for each vertex.

Although u and w exist at interior points of each arc, this is not true at the initial vertex of an arc if the vertex is singular. (Such an arc itself will be called singular; otherwise an arc is regular.) The condition $|B| = 0$ is equivalent to the specification of a linear combination of the components of z at the vertex. This relationship plus eq. (6) would constitute initially on a singular arc $p + 1$ equations in the p unknown components of z. Compatibility of these equations implies a linear relationship between the components of y^{+}. Since, however, y^{+} depends on as yet unused data on the remaining part of the tree, it follows that eq. (1), i.e., u and w, cannot exist initially on singular arcs. From the foregoing, it is clear that singular arcs require special treatment. The discussion of singular arcs is postponed until after the next section, in which the basic method is presented.

Trees without singular arcs.- Let us assume for the moment that all vertices are regular, except possibly one vertex corresponding to a branch extremity. In this case, the arcs can be ordered so that the sole arc incident with the singular vertex (if it exists) is the final arc, which is then oriented towards the vertex. Since the initial vertex of every arc is then regular, no singular arcs exist. This leads to the simplest form of the field method.

In order to derive the differential equations for u and w, differentiate eq. (6) with respect to s, use eqs. (1) to eliminate y' and z', and eq. (6) to eliminate y to obtain

$$(u' - ucu + au - ud + b)z + w' - ucw + aw + ug - f = 0 \tag{8}$$

Since eq. (8) must be satisfied identically for all z (which depends on data at points of greater s, whereas u and w do not), it follows that

$$u' - ucu + au - ud + b = 0 \tag{9a}$$

$$w' - ucw + aw + ug - f = 0 \tag{9b}$$

Equations (9) are the differential equations for u and w.

The corresponding initial conditions for regular arcs are derived by substituting eq. (6) into eq. (2) to obtain

$$(\Delta u + B^{-1}D)z + \Delta w - B^{-1}L = 0 \tag{10}$$

where the symbol Δ has the same meaning as in eq. (7), viz. $\Delta u = u^+ - \sum u^-$ and $\Delta w = w^+ - \sum w^-$. Since eq. (10) is also an identity with respect to z, one obtains the initial values

$$u^+ = -B^{-1}D + \sum u^- \tag{11a}$$

$$w^+ = B^{-1}L + \sum w^- \tag{11b}$$

After integrating the initial value problem (9), (11) over the whole tree and storing the field functions u and w, eqs. (2) and (6) at the final vertex are solved simultaneously to give z there, viz.

$$z = (D - Bu)^{-1}(L + Bw) \tag{12}$$

With this value as an initial condition, a backward integration of eqs. (1b) and (6), i.e.

$$z' + (cu + d)z = g - cw \tag{13}$$

is performed over the whole tree, using z-continuity at interior vertices. From the values of z so obtained, y is calculated from eq. (6), completing the solution.

If the boundary value problem is self-adjoint, the numerical work is considerably reduced. In this case, the matrix u is symmetric so that only $p(p + 1)/2$ of its component functions are independent.* This fact follows from eqs. (9a) and (11a) in view of eqs. (4). Thus in this common case, the total number of independent scalar differential equations contained in eqs. (9) and (13) is $p(p + 5)/2$, which compares with $2p(p + 1)$ scalar first-order equations which must be integrated in the Zarghamee method.

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*In this case, u may be viewed as a "stiffness" matrix for that part of the tree, produced by a cut at s, which is fully described by smaller s-values.

Singular arcs.- Initially on singular arcs, the field relation (6) does not exist. Roughly speaking, one may say that the field functions u,w are infinite at such points. As discussed previously, the essential reason for this behavior is that a singular boundary condition implies a relationship between the components of z at the corresponding vertex. One is therefore motivated to make a transformation of variables to a modified z-vector, whose components are independent at the vertex. Consequently, on singular arcs new variables $\tilde{y}$ and $\tilde{z}$ are defined by*

$$\tilde{y} = y \tag{14a}$$

$$\tilde{z} = z + ey \tag{14b}$$

where e is a constant diagonal p × p matrix required for dimensional homogeneity of eq. (14b). Substitution of eqs. (14) into eq. (6) shows that $\tilde{y},\tilde{z}$ satisfy the modified field relation

$$\tilde{y} = \tilde{u}\tilde{z} + \tilde{w} \tag{15}$$

where

$$\tilde{u} = (I + ue)^{-1}u \tag{16a}$$

$$\tilde{w} = (I + ue)^{-1}w \tag{16b}$$

Because of the symmetry of the transformation (14), eqs. (16) are inverted simply by replacing e by -e, i.e.

$$u = (I - \tilde{u}e)^{-1}\tilde{u} \tag{17a}$$

$$w = (I - \tilde{u}e)^{-1}\tilde{w} \tag{17b}$$

In terms of these modified variables, eqs. (1) become

$$\tilde{y}' + \tilde{a}\tilde{y} + \tilde{b}\tilde{z} = \tilde{f} \tag{18a}$$

$$\tilde{z}' + \tilde{c}\tilde{y} + \tilde{d}\tilde{z} = \tilde{g} \tag{18b}$$

where

$$\begin{aligned} \tilde{a} &= a - be \\ \tilde{b} &= b \\ \tilde{c} &= c + e\tilde{a} - de \\ \tilde{d} &= d + eb \\ \tilde{f} &= f \\ \tilde{g} &= g + ef \end{aligned} \tag{19}$$

*An alternate transformation for singular arcs is given in Appendix A.

Equations (15) and (18) can be used to derive differential equations for $\tilde{u}$ and $\tilde{w}$ in exactly the same way as eqs. (1) and (6) were used to derive eqs. (9). Since eqs. (18) and (15) are identical in form to eqs. (1) and (6), respectively, the differential equations for $\tilde{u}$ and $\tilde{w}$ are in the same form as eqs. (9) with all variables replaced by corresponding tilde variables.

The initial values of $\tilde{u}$ and $\tilde{w}$ on singular arcs may be obtained from eqs. (11) and (16). Considering the singular vertex as the limit of a sequence of nonsingular vertices, one has from eqs. (11) and (16)

$$\tilde{u}^+ = \lim_{|B|\to 0} [I + (\sum u^- - B^{-1}D)e]^{-1}(\sum u^- - B^{-1}D)$$

$$= -\tilde{B}^{-1}\tilde{D} \tag{20a}$$

$$\tilde{w}^+ = \lim_{|B|\to 0} [I + (\sum u^- - B^{-1}D)e]^{-1}(\sum w^- + B^{-1}L)$$

$$= \tilde{B}^{-1}\tilde{L} \tag{20b}$$

where

$$\tilde{B} = B - \tilde{D}e \tag{21a}$$

$$\tilde{D} = D - B \sum u^- \tag{21b}$$

$$\tilde{L} = L + B \sum w^- \tag{21c}$$

The modified field functions $\tilde{u}$ and $\tilde{w}$ are calculated (and stored) on singular arcs by forward integration of eqs. (9) written for tilde variables, starting with the initial values (20). At the terminal vertex of singular arcs, it is convenient to replace $\tilde{u},\tilde{w}$ by u,w according to the reversion formulas (17). This is done so that the initial values of the field functions for all arcs [eqs. (11) for regular arcs and eqs. (20) for singular arcs], as well as the terminal value of z [eq. (12)], may be computed the same way, regardless of whether preceding arcs are regular or singular.

The backward integration on singular arcs is also done in terms of tilde variables, i.e., eq. (13) written for tilde variables is integrated. For this purpose, the value of z at the terminal vertex of a singular arc is replaced by $\tilde{z}$ by first computing y there from eq. (6) (recall that u and w have been stored there) and then $\tilde{z}$ from eq. (14b). After so doing, u and w at the terminal vertex are changed back to $\tilde{u}$ and $\tilde{w}$ [by using eqs. (16)], as required for the integration of the tilde form of eq. (13). From the values of $\tilde{z}$ obtained by the backward integration, $y = \tilde{y}$ is computed from eq. (15) and z is computed from eq. (14b).

For self-adjoint problems, it may be seen from eqs. (19) that the transformation (14) preserves the self-adjoint property of the differential

equations (1), i.e., eqs. (4a-c) are satisfied in terms of tilde variables. Also, it is easily shown from eq. (16a) that the transformation from u to $\tilde{u}$ (and vice versa) preserves matrix symmetry. For eq. (16a) may be written, in the case of nonsingular u, as

$$\tilde{u} = (u^{-1} + e)^{-1} \tag{22}$$

Since the inverse of a symmetric matrix is also symmetric, from eq. (22) symmetric u implies symmetric $\tilde{u}$. This result is valid even if u is singular since symmetric singular u may be approximated as closely as one pleases by symmetric nonsingular matrices. In particular, since the value of $\tilde{u}^+$ [eq. (20a)] was derived as the limit of a sequence of symmetric matrices, $\tilde{u}^+$ is itself symmetric. As for u on regular arcs, integration of the differential equations for $\tilde{u}$ preserves the symmetry of $\tilde{u}^+$. Hence, the simplifying conclusions drawn for self-adjoint problems on page 9 hold as well in the presence of singular arcs.

A common type of boundary value problem originates as the minimization of a certain positive definite functional over a tree. An example of this type of problem is one-dimensional static response of an elastic structure. Such problems are self-adjoint, and the field relation (6) necessarily exists at all interior points of each arc, since the uniqueness condition mentioned in the footnote on page 7 is satisfied. To insure that the modified field relation (15) exists at all points of a singular arc (i.e., I + ue should be nonsingular), it is necessary to choose the diagonal matrix e [see eq. (14b)] positive definite, i.e., each of its nonzero elements should be positive. For such e, specification of $\tilde{z}$ at a generic point $s = \hat{s}$ of a singular arc will uniquely determine $\tilde{y} = y$ there, since this corresponds to cutting the tree at $\hat{s}$ and the attachment there of p stable elastic springs (to that part fully described by $s < \hat{s}$). If e is not positive definite, at least one of the springs is unstable, which could lead to an instability (i.e., infinite $\tilde{u}$) for some $\hat{s}$.

SHELLS OF REVOLUTION

A pilot computer program employing the field method to obtain the linear elastic response of open branch ring-stiffened shells of revolution subject to general harmonic mechanical and thermal loads has been written. For this class of problems, the tree over which the boundary value problem is defined represents the reference meridian of the shell. The differential equations (1) are eighth order so that the response matrices y and z are 4-element column vectors. The equations are ordered so that y and z are the force and displacement vectors (fig. 3)

$$y^T = r(P,Q,S,M_1) \tag{23a}$$

$$z^T = (\xi,\eta,v,\chi) \tag{23b}$$

where P,Q,S are forces per unit of circumferential length referred to fixed axial, radial, and circumferential coordinate directions x,y,ϕ, ξ,η,v are the corresponding displacement components, M_1 is the meridional bending moment per unit of circumferential length, and χ is the corresponding rotation.

The matrices a,b,c,f and g [eqs. (1)], as well as $\kappa = B^{-1}D$ and $B^{-1}L$ [eq. (2)] for ring boundaries, are given in Appendix B. Since this problem is self-adjoint, b,c and κ are symmetric and $d = -a^T$ [cf. eqs. (4)]. Corresponding shell and ring equations have been given previously (although not in precisely the same form) in reference 7.

Solutions of several problems were obtained by the field method and compared to solutions obtained by the Zarghamee method. The purpose of the numerical calculations was to:

(1) uncover any practical problems in the implementation of the field method,
(2) show that the "long subinterval" problem does not exist in the field method, and
(3) compare the execution time of the field method with that of the Zarghamee method.

Four basic shell configurations were studied:

(1) clamped spherical cap ($R/t = 91.4$; $n = 0$)
(2) branched conical shell ($r_0/t = 100$; $n = 1$)
(3) 140° sandwich cone ($r_0/t = 25.3$; $n = 0$ and $n = 1$)
(4) clamped-free cylinder ($\ell/r = 1$, $r/t = 10$; $n = 0$)

Diagrams of the first three configurations are shown in figure 4. Here, n is the harmonic number considered, R is the spherical radius of curvature, r is a small circle radius, r_0 is the initial value of r, ℓ is the cylindrical length, and t is the effective wall thickness (i.e., $\sqrt{3}$ times the core depth for case 3, and the monocoque thickness for the other cases).

Each of these cases involve singular subintervals. In the first and fourth cases, this is due to the kinematic constraint at the initial edge. The second and third cases are free structures without rigid body constraint, but artificial rigid body constraint has been provided, in each case, at interior boundaries. The third case was previously used as the sample problem for the corresponding Zarghamee program (SRA 100) in reference 10, in which the variable pressure loading used is given. In the fourth case, the loading is uniform lateral pressure.

During the course of the numerical evaluation of the field method, two practical problems were encountered. These are the choice of: 1) the elements of the scaling matrix e, and 2) the storage locations for the field functions u and w.

Choice of the Scaling Matrix e

In the general discussion of singular arcs, a scaling matrix e was introduced [eq. (14b)] without explicit definition, other than to say that it is a constant diagonal p × p (in this case, 4 × 4) matrix. In addition, it was deduced that for positive definite self-adjoint problems, to insure the existence of the modified field function $\tilde{u}$, all of the diagonal elements of e should be positive. Within this guideline there are an infinite variety of possible choices for e. Although the precise values of the diagonal elements of e have no effect on the solution, their order of magnitude determine the severity of the boundary layer in $\tilde{u}$ following a singular boundary, and hence the rate at which the numerical integration will proceed.

The first values tried for the diagonal elements e_{ii} were based on the scale factors used in the supplemental initial conditions at singular boundaries in the Zarghamee method (ref. 7). These are $e_{11} = e_{22} = e_{33} = SC_1 \approx t/rC_1^{(0)}$ and $e_{44} = SC_2 \approx t/rC_1^{(2)}$, where $C_1^{(0)}$ and $C_1^{(2)}$ are meridional stretching and bending stiffnesses, respectively. However, these values caused in sample problem 2 a severe boundary layer for $\tilde{u}$, and consequent slow forward integration, immediately following the edge at point 7 (see fig. 4) when (artificial) rigid body constraints were imposed at that edge. After several empirical changes from these values, the values $SC_1 \approx 10/C_1^{(0)}$ and $SC_2 \approx 3t/rC_1^{(2)}$ were chosen as optimal in the sense that for the sample problems studied they resulted in the least number of integration steps. The table below shows the number of derivative evaluations (four per Runge-Kutta step, whether accepted or not, plus one extra for each subinterval) during integration for three different sets of scale factors SC_1 and SC_2. In each case, the upper value represents the forward integration and the lower value the backward integration.

Scale factors	Case 1	Case 2[a]	Case 2[b]	Case 3	Case 4
$SC_1 = t/rC_1^{(0)}$	--	1441	--	--	274
$SC_2 = t/rC_1^{(2)}$	--	--	--	--	234
$100t/rC_1^{(0)}$	376	425	369	421	242
$t/rC_1^{(2)}$	312	249	217	373	194
$10/C_1^{(0)}$	344	425	361	405	234
$3t/rC_1^{(2)}$	312	249	193	373	194

[a]Singular edge (point 7 of fig. 4)

[b]Singular branch point (point 25 of fig. 4)

The corresponding numbers for case 2 when the kinematic constraints are placed at the final edge (so that there are no singular subintervals and the numerical process therefore does not involve the matrix e) are 377

and 177. Thus, it is seen that singular subintervals cause a small penalty when optimal scale factors are used.

Not only is the numerical work reduced by the proper choice of scale factors for singular subintervals but at the same time an improvement in accuracy is generally realized. This is illustrated in the following table, in which the values of the dimensionless transverse shear stress resultant Q/pt and meridional stress couple M_1/pt^2 at the clamped edge of the clamped-free cylinder (case 4) are shown for the three field solutions of the preceding table, as well as for the Zarghamee solution. Here p represents the applied pressure load.

Method	Q/pt	M_1/pt^2
Field #1	2.5181 (3.16%)	-2.8536 (4.22%)
Field #2	2.4417 (0.03%)	-2.9738 (0.18%)
Field #3	2.4413 (0.02%)	-2.9803 (0.04%)
Zarghamee	2.4409	-2.9792

In this table the numbers in parentheses are the percent differences of the field solutions with the Zarghamee solution. Taking the Zarghamee solution as a standard,* the percent error in the field method is seen to be directly related to the number of integration steps required for the different scale factors.

Although the third set of scale factors worked well in these test cases, no claim of universality is made for them. As programmed they are constant over the whole meridian, whereas it is only necessary that they be constant over each subinterval. One can anticipate that, for shells with large property variations along the meridian, it would be better to calculate different values of these scale factors for each singular subinterval. Also, for highly orthotropic shells for which the shear modulus is orders of magnitude different from the normal moduli, it may be desirable to introduce a third scale factor based on the shell wall shear stiffness. However, these refinements are not considered essential to demonstrating feasibility of the field method and are beyond the scope of the present study.

Storage Locations for Field Functions

When preparing the data deck for the computer program, the user must specify storage points for the response vectors y and z. The location of these points is arbitrary within the limitation that they are equally spaced within each subinterval. The backward integration for the z-vector is constrained to land on each of these specified points.

*The Zarghamee solution is considered the most accurate of the four solutions since it does not require interpolation of calculated functions as does the field method. (In this regard, see next section.)

In the field method, it is necessary to store the field functions u and w at the time they are calculated during the forward integration. Interpolation of these stored values is then made during the backward integration of eq. (13). In the initial version of the computer program, it was convenient to store these functions at the predesignated y,z storage points. Thus, in this version, the forward integration of eqs. (9) for u and w is also constrained to land on these points. However, because the variation of the field functions is of a totally different character than that of the response functions, this is generally a poor choice of storage locations for them. In general, the field functions have a narrow zone of rapid variation immediately following a boundary, but are otherwise slowly varying. In order to be able to interpolate accurately for intermediate values of u and w with the minimum number of storage points, the spacing of these points should vary with the rate of variation of u and w, i.e., the more rapid the variation of u and w, the closer together their storage points should be. Therefore the natural place to store them is at the end of each integration step, the size of which is automatically adjusted during execution according to their rate of variation. At the same time this would allow the forward integration to proceed at its own pace without being restricted by predesignated data points.

In all of the cases studied, excellent agreement with the Zarghamee method was obtained for forces and displacements at the terminal shell edge. On the other hand, in some cases small errors of the order of one percent crept into the field method solution at other points due to the inadequacy of the specified y,z data points for the description of the field functions u and w, resulting in interpolation errors for u and w during the backward integration. This is illustrated by the following table giving the dimensionless components of the z-vector obtained by each method at the initial (point 1 of fig. 4) and final (point 27 of fig. 4) edges of the branched shell (case 2, singular edge).*

Edge	Method	$\xi/t \times 10^2$	$\eta/t \times 10^2$	$v/t \times 10^3$	$\chi \times 10^3$
Initial	Field	-0.8352	-1.4904	0.3847	-2.7215
	Zarghamee	-0.8055	-1.4595	0.3833	-2.6505
Final	Field	-2.2100	-2.8237	4.2476	3.1863
	Zarghamee	-2.2096	-2.8227	4.2460	3.1824

This case showed the greatest loss of accuracy due to inadequate storage of the field functions of all cases studied. For example, the corresponding results for the n = 1 harmonic of the 140° sandwich cone (case 3), which has almost twice as many storage points (as well as fewer real boundaries) as case 2, are given in the following table.

*This and all following comparisons are based on the same relative error control for the Runge-Kutta integration routine used in both methods.

Edge	Method	$\xi/t \times 10^2$	$\eta/t \times 10^2$	$\nu/t \times 10^2$	$\chi \times 10^3$
Initial	Field	-0.9268	0.7079	-0.0784	4.5021
	Zarghamee	-0.9353	0.7110	-0.0786	4.5386
Final	Field	5.7869	-1.5461	1.4774	0.5932
	Zarghamee	5.7874	-1.5463	1.4776	0.5933

Long Subintervals and Execution Time

In order to demonstrate that subinterval length has no effect on the numerical solution obtained by the field method, the clamped-free cylindrical shell configuration (case 4) was used. However, in order to put it into the range where several subintervals would be required in the Zarghamee method (for which the axial length of each subinterval should be less than approximately $5\sqrt{rt}$), the shell dimensions were changed so that $\ell/r = 2$ and $r/t = 100$. Also, for this case a comparison of the Zarghamee and field method integration times for the response due to uniform fifth harmonic lateral pressure loading was made. For the Zarghamee setup, in order to avoid the long subinterval problem, the meridian was divided into four subintervals, each subinterval having four interior data points. For the field method only one interior boundary was used; this was specified at one-eighth the total length from the initial edge in order to change the spacing of data points qualitatively in accordance with the change in variation of the field functions. Four interior data points were specified in the first subinterval and six interior data points in the second subinterval. In order to avoid interaction with the choice of the scaling matrix e, singular arcs were avoided in this comparison by choosing the clamped edge to be the final edge, the initial edge then being free. Integration times (CPU times in seconds) for the CDC 6400 computer are shown in the table below.

Integration	Field	Zarghamee
Forward	1.753 (290)	5.156 (516)
Backward	0.541 (170)	--
Total	2.294 sec	5.156 sec

The numbers given in parentheses are the number of derivative evaluations (four per Runge-Kutta step plus one extra for each subinterval) made during each integration.

In spite of the rather meagre number of data points used to store the field functions, the accuracy of the field solution is good. The table below compares the dimensionless displacement components at the free edge obtained by each of the methods.

Method	$\xi/t \times 10$	$\eta/t \times 10^{-1}$	v/t	$\chi \times 10$
Field	4.4166	2.2023	-4.3587	1.9081
Zarghamee	4.4222	2.2033	-4.3595	1.9232

CONCLUDING REMARKS

The field method of solution of general even order linear boundary value problems defined on an arbitrary open branch one-dimensional domain has been formulated. The method has been implemented in a computer program for the static elastic response of open branch ring-stiffened shells of revolution subjected to general asymmetric (harmonic) loads. By studying specific sample problems the numerical feasibility of the method for axisymmetric shell problems has been demonstrated. For such problems the method eliminates the long subinterval problem of other numerical integration methods. In addition, it executes considerably faster than the Zarghamee method, which is the fastest numerical integration method currently in use. Numerical problems associated with kinematic constraints have been shown to be resolvable by the use of an "optimum" transformation of variables. It has also been shown that it is inadvisable to store the field functions at the same locations used to store the physical response functions. Rather, to obtain the greatest speed and accuracy for the least storage, the field functions should be stored at the end of each integration step, the size of which is automatically adjusted during execution according to the rate at which they vary.

Based on the demonstrated advantages of the field method over other numerical integration methods, it is recommended that the method be further developed with the object of obtaining a practical set of axisymmetric shell programs based on it. Such a set of programs should have the following features:

(1) They should be applicable to shells with many closed branches.
(2) They should be essentially free of numerical ill-conditioning problems (requiring, for example, double precision calculations). In particular, the user should not have to concern himself with questions of ill-conditioning when setting up a data deck.
(3) Data deck preparation should be simple, at least no more difficult than that for other programs capable of treating the same problem.
(4) They should execute faster and require less storage than existing programs for similar problems.

APPENDIX A

ALTERNATE VARIABLES FOR SINGULAR ARCS

In place of eqs. (14) defining $\tilde{y},\tilde{z}$ on singular arcs, one could use the transformation

$$\bar{y} = y + ez \tag{A-1a}$$

$$\bar{z} = z \tag{A-1b}$$

where e is a constant diagonal p × p matrix required for dimensional homogeneity of eq. (A-1a). In a similar fashion to the development of eqs. (15)-(21) for the tilde variables, the following analogous relations for the bar variables may be derived. The field relation is

$$\bar{z} = \bar{\bar{u}}\bar{y} + \bar{w} \tag{A-2}$$

where

$$\bar{u} = (u + e)^{-1} \tag{A-3a}$$

$$\bar{w} = -\bar{u}w \tag{A-3b}$$

The inverse of eqs. (A-3) is

$$u = \bar{u}^{-1} - e \tag{A-4a}$$

$$w = -\bar{u}^{-1}\bar{w} \tag{A-4b}$$

The differential equations for $\bar{y}$ and $\bar{z}$ are

$$\bar{y}' + \bar{a}\bar{y} + \bar{b}\bar{z} = \bar{f} \tag{A-5a}$$

$$\bar{z}' + \bar{c}\bar{y} + \bar{d}\bar{z} = \bar{g} \tag{A-5b}$$

where

$$\begin{aligned} \bar{a} &= a + ec \\ \bar{b} &= b - ae + e\bar{d} \\ \bar{c} &= c \\ \bar{d} &= d - ce \\ \bar{f} &= f + eg \\ \bar{g} &= g \end{aligned} \tag{A-6}$$

The differential equations for the modified field functions $\bar{u}$ and $\bar{w}$ are

$$\bar{u}' - \bar{\bar{u}}\bar{\bar{b}}\bar{u} + \bar{\bar{d}}\bar{u} - \bar{\bar{u}}\bar{a} + \bar{c} = 0 \qquad \text{(A-7a)}$$

$$\bar{w}' - \bar{\bar{u}}\bar{\bar{b}}\bar{w} + \bar{\bar{d}}\bar{w} + \bar{\bar{u}}\bar{f} - \bar{g} = 0 \qquad \text{(A-7b)}$$

The initial values of $\bar{u}$ and $\bar{w}$ on singular arcs are

$$\bar{u}^{+} = -\bar{D}^{-1}\bar{B} \qquad \text{(A-8a)}$$

$$\bar{w}^{+} = \bar{D}^{-1}\bar{L} \qquad \text{(A-8b)}$$

where

$$\bar{B} = B \qquad \text{(A-9a)}$$

$$\bar{D} = D - B(\sum u^{-} + e) \qquad \text{(A-9b)}$$

$$\bar{L} = L + B \sum w^{-} \qquad \text{(A-9c)}$$

With this modification for singular arcs, the calculation procedure is similar to that of the tilde modification (see pp. 10-12), except that in this case the backward integration on singular arcs is for $\bar{y}$. The differential equation for $\bar{y}$ is obtained from eq. (A-5a), which in view of the field relation (A-2) may be written as

$$\bar{y}' + (\bar{a} + \bar{\bar{b}}\bar{u})\bar{y} = \bar{f} - \bar{\bar{b}}\bar{w} \qquad \text{(A-10)}$$

The integration of eq. (A-10) is started with the value of $\bar{y}$, computed from eq. (A-1a), at the terminal vertex of the singular arc.

APPENDIX B

MATRICES FOR AXISYMMETRIC SHELLS AND RINGS

Symbols

A	ring section area
E	ring elastic modulus
e_x, e_y	ring centroidal eccentricities
F_x, F_y, F_ϕ	harmonic amplitudes of ring force loads per unit of circumferential length
GJ	ring torsional stiffness
I_x, I_y, I_{xy}	ring section moments of inertia
k	ring stiffness matrix
L_1, L_2	harmonic amplitudes of shell moment loads per unit of surface area in meridional and circumferential directions
ℓ_e, ℓ_f, ℓ_t	ring load vectors
N_x, N_y, N_ϕ	harmonic amplitudes of ring moment loads per unit of circumferential length
X_1, X_2, X_3	harmonic amplitudes of shell force loads per unit of surface area in meridional, circumferential, and normal directions
ε	ring eccentricity matrix
$\Theta_1(0), \Theta_2(0)$	harmonic amplitudes of meridional and circumferential thermal force loads
$\Theta_1(1), \Theta_2(1)$	harmonic amplitudes of meridional and circumferential thermal moment loads
θ	harmonic amplitude of ring free thermal strain
λ_{ij}	orthotropic shell wall normal stiffness coefficients

μ_{ij} orthotropic shell wall shear stiffness coefficients

ρ ring centroidal radius

Shell

For elastic orthotropic shells of revolution subjected to symmetric n'th harmonic loads, the coefficient and load matrices of eqs. (1) are given below. For the definitions of the stiffness (λ_{ij},μ_{ij}) and load (X_i,L_i,$\Theta_i^{(m)}$) variables in these matrices, see reference 7.

$$a = \frac{1}{r}\begin{bmatrix} n^2\lambda_{23}r'/R_2 & n^2\lambda_{23}r'^2/r & n(r/R_2-\mu_{12}/r) & n^2\lambda_{24}r'/r \\ -(\lambda_{13}+n^2\lambda_{23}/R_2)(r/R_2) & -(\lambda_{13}+n^2\lambda_{23}/R_2)r' & nr' & -(\lambda_{14}+n^2\lambda_{24}/R_2) \\ -n(\lambda_{13}+\lambda_{23}/R_2)(r/R_2) & -n(\lambda_{13}+\lambda_{23}/R_2)r' & r' & -n(\lambda_{14}+\lambda_{24}/R_2) \\ (1-\lambda_{23}/R_2)rr' & -r^2/R_2-\lambda_{23}r'^2 & n\mu_{12} & -\lambda_{24}r' \end{bmatrix}$$

$$b = -\frac{1}{r}\begin{bmatrix} n^2(n^2\lambda_{22}r'^2+\mu_{11})/r^2 & -n^2(\lambda_{12}+n^2\lambda_{22}/R_2)r'/r & -n^3(\lambda_{12}+\lambda_{22}/R_2)r'/r & -n^2(\lambda_{22}r'^2+\mu_{11})/r \\ & \lambda_{11}+2n^2\lambda_{12}/R_2+n^4\lambda_{22}/R_2^2 & n[\lambda_{11}+(n^2+1)\lambda_{12}/R_2+n^2\lambda_{22}/R_2^2] & (\lambda_{12}+n^2\lambda_{22}/R_2)r' \\ & & n^2(\lambda_{11}+2\lambda_{12}/R_2+\lambda_{22}/R_2^2) & n(\lambda_{12}+\lambda_{22}/R_2)r' \\ \text{Symmetric} & & & \lambda_{22}r'^2+n^2\mu_{11} \end{bmatrix}$$

$$c = -\frac{1}{r}\begin{bmatrix} \lambda_{33}(r/R_2)^2 & \lambda_{33}r'(r/R_2) & 0 & \lambda_{34}(r/R_2) \\ & \lambda_{33}r'^2 & 0 & \lambda_{34}r' \\ & & \mu_{22} & 0 \\ \text{Symmetric} & & & \lambda_{44} \end{bmatrix}$$

$$f = \begin{Bmatrix} r[-(r/R_2)X_1+r'(X_3-nL_1/r)]+n^2(r'/r)[\Theta_2(1)-\lambda_{23}\Theta_1(0)-\lambda_{24}\Theta_1(1)] \\ -r[r'X_1+(r/R_2)(X_3-nL_1/r)]+[\lambda_{13}+n^2\lambda_{23}/R_2]\Theta_1(0)+[\lambda_{14}+n^2\lambda_{24}/R_2]\Theta_1(1)-\Theta_2(0)-n^2\Theta_2(1)/R_2 \\ -rX_2+(r/R_2)L_1+n[(\lambda_{13}+\lambda_{23}/R_2)\Theta_1(0)+(\lambda_{14}+\lambda_{24}/R_2)\Theta_1(1)-\Theta_2(0)-\Theta_2(1)/R_2] \\ -rL_2+r'[\lambda_{23}\Theta_1(0)+\lambda_{24}\Theta_1(1)-\Theta_2(1)] \end{Bmatrix}$$

$$g = \begin{Bmatrix} (r/R_2)[\lambda_{33}\Theta_1(0)+\lambda_{34}\Theta_1(1)] \\ r'[\lambda_{33}\Theta_1(0)+\lambda_{34}\Theta_1(1)] \\ 0 \\ \lambda_{34}\Theta_1(0)+\lambda_{44}\Theta_1(1) \end{Bmatrix}$$

Rings

For elastic isotropic ring boundaries, the coefficient and load matrices $\kappa = B^{-1}D$ and $B^{-1}L$ of eq. (2) are (ref. 7)

$$\kappa = -\varepsilon^T k \varepsilon$$

$$B^{-1}L = -\varepsilon^T(\ell_f + \ell_t - k\ell_e)$$

where

$$k = \frac{1}{\rho}\begin{bmatrix} n^2(n^2EI_y+GJ)/\rho^2 & n^4EI_{xy}/\rho^2 & n^3EI_{xy}/\rho^2 & -n^2(EI_y+GJ)/\rho \\ & EA+n^4EI_x/\rho^2 & n(EA+n^2EI_x/\rho^2) & -n^2EI_{xy}/\rho \\ & & n^2(EA+EI_x/\rho^2) & -nEI_{xy}/\rho \\ \text{Symmetric} & & & EI_y+n^2GJ \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & e_y \\ 0 & 1 & 0 & -e_x \\ ne_x/r & ne_y/r & \rho/r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\ell_f^T = (\rho F_x + nN_y,\ \rho F_y - nN_x,\ \rho F_\phi - N_x,\ \rho N_\phi)$$

$$\ell_t^T = EA\theta(0,1,n,0)$$

$$\ell_e^T = \theta(e_x,e_y,0,0)$$

REFERENCES

1. Goldberg, J. E.; Bogdanoff, J. L.; and Marcus, L.: On the Calculation of the Axisymmetric Modes and Frequencies of Conical Shells. J. Acoust. Soc. Am., vol. 32, 1960, pp.738-742.

2. Cohen, G. A.: Computer Analysis of Asymmetrical Deformation of Orthotropic Shells of Revolution. AIAA J., vol. 2, no. 5, May 1964, pp. 932-934.

3. Kalnins, A.: Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical Loads. J. Appl. Mech., vol. 31, 1964, pp. 467-476.

4. Zarghamee, M S.; and Robinson, A. R.: A Numerical Method for Anaysis of Free Vibrations of Spherical Shells. AIAA J., vol. 5, no. 7, July 1967, pp. 1256-1261.

5. Busacker, R. G.; and Saaty, T. L.: Finite Graphs and Networks. McGraw-Hill Book Co., Inc., 1965, Chpt. 1.

6. Anderson, M. S.; Fulton, R. E.; Heard, W. L.; and Walz, J. E.: Stress, Buckling, and Vibration Analysis of Shells of Revolution. Computers & Structures (Pergamon Press), vol. 1, 1971, pp. 157-192, Appendix A.

7. Cohen, G. A.: Computer Analysis of Ring-Stiffened Shells of Revolution. NASA CR-2085, 1973.

8. Jordan, P. F.; and Shelly, P. E.: Stabilization of Unstable Two-Point Boundary Value Problems. AIAA J., vol. 4, no. 5, May 1966, pp. 923-924.

9. Miller, R. E.: Use of the Field Method for Numerical Integration of Two-Point Boundary Value Problems. AIAA J., vol. 5, no. 4, April 1967, pp. 811-813.

10. Cohen, G. A.: User Document for Computer Programs for Ring-Stiffened Elastic Shells of Revolution. NASA CR-2086, 1973.

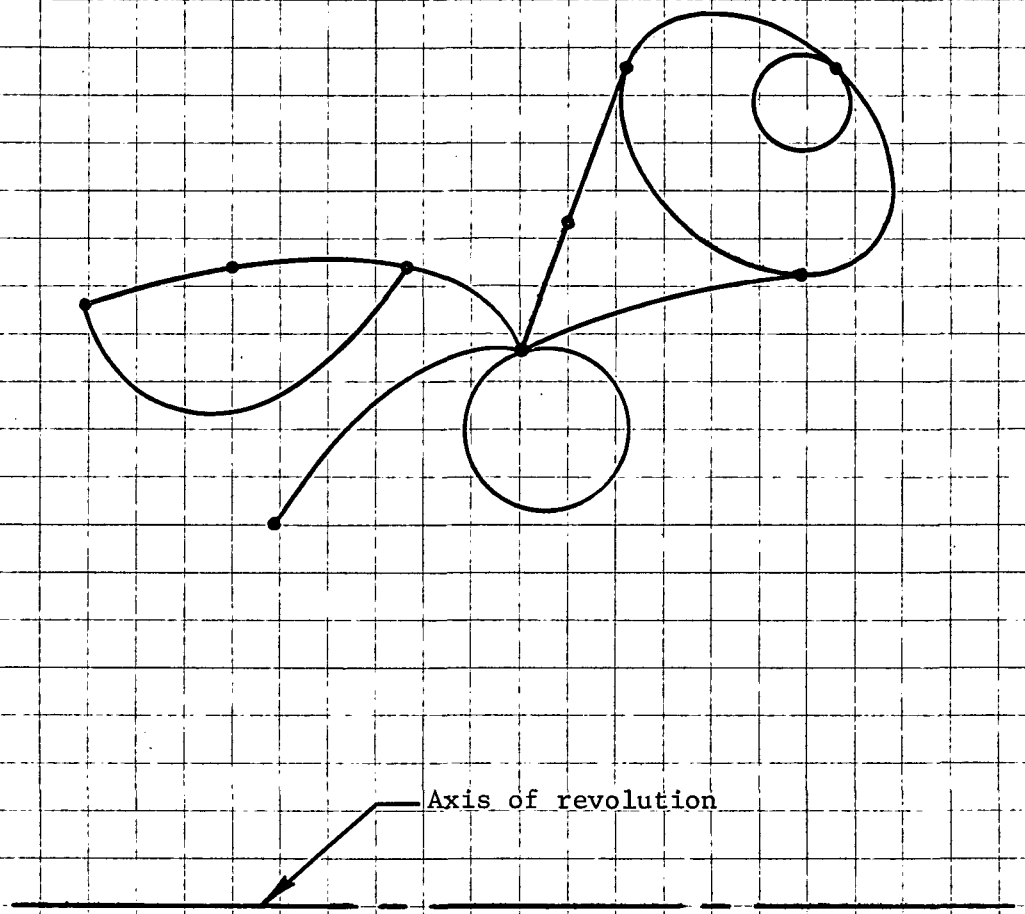


FIGURE 1. REFERENCE MERIDIAN OF A HYPOTHETICAL SHELL OF REVOLUTION

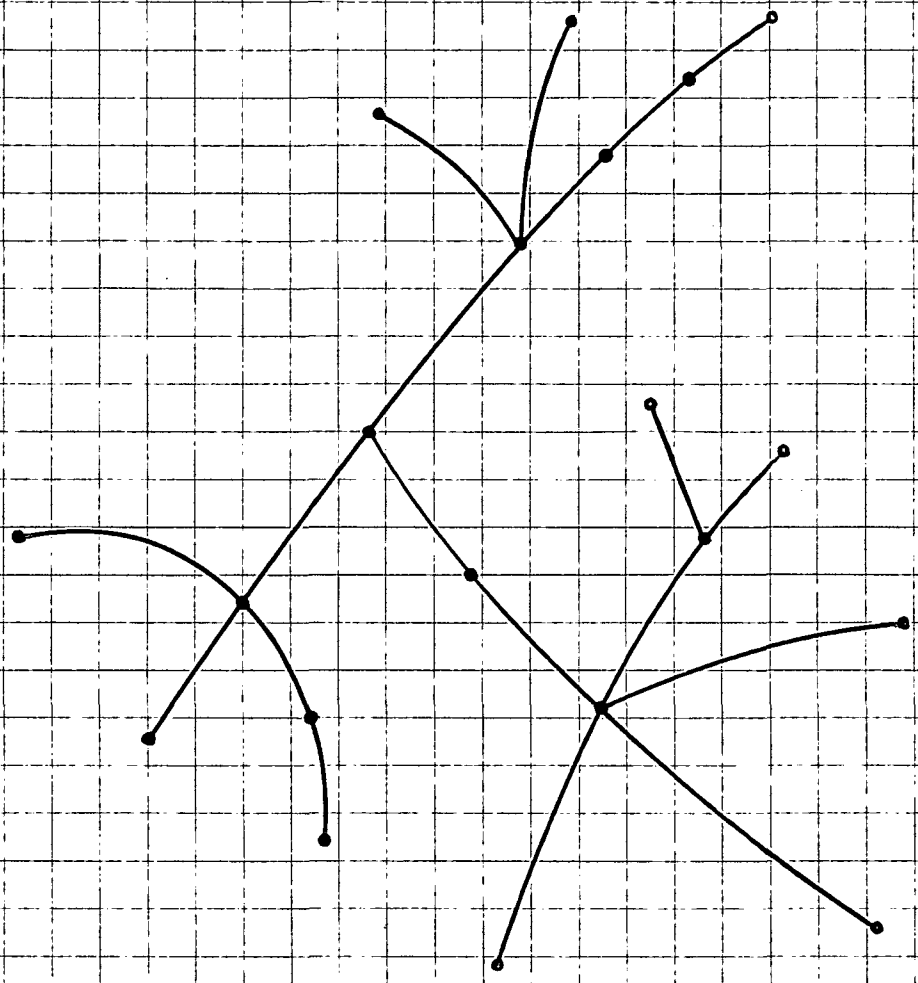


FIGURE 2. A HYPOTHETICAL TREE

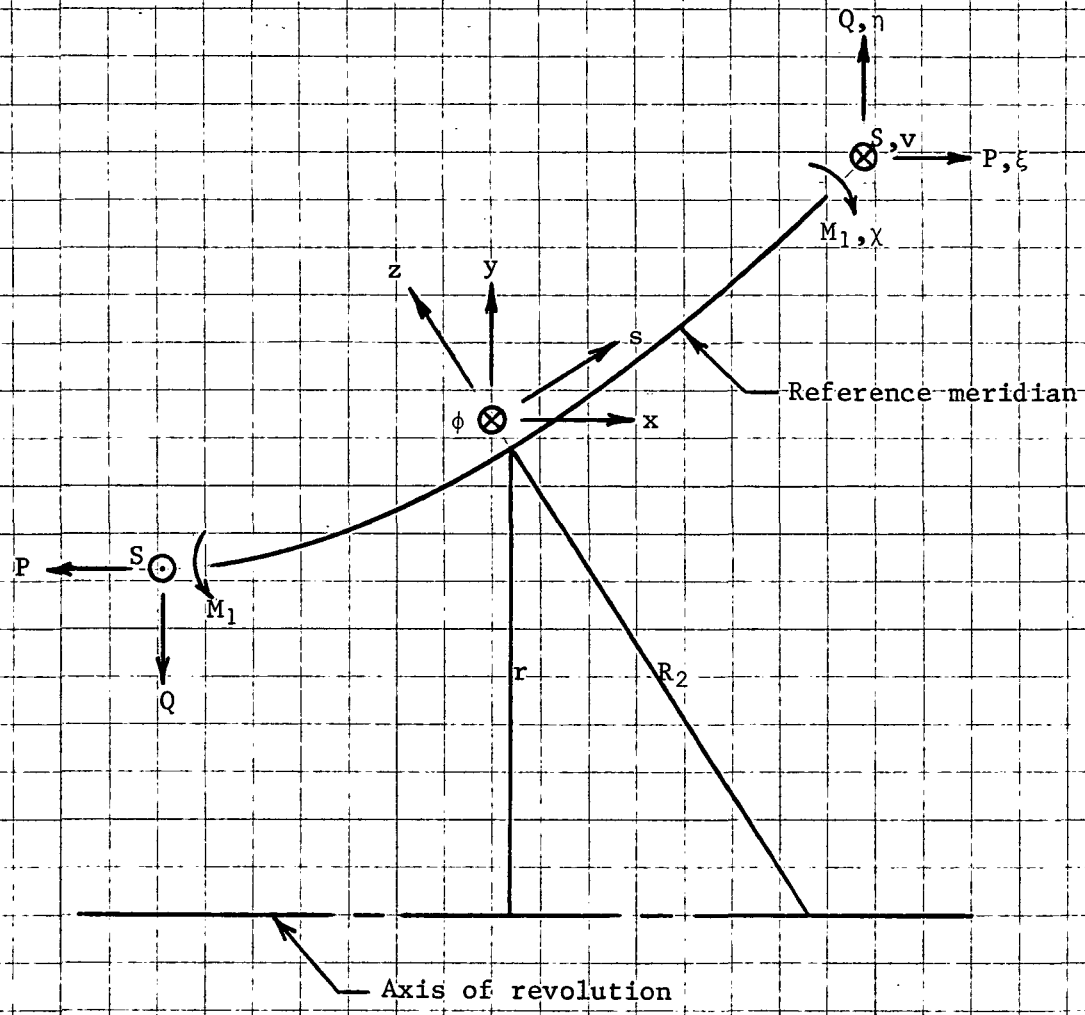
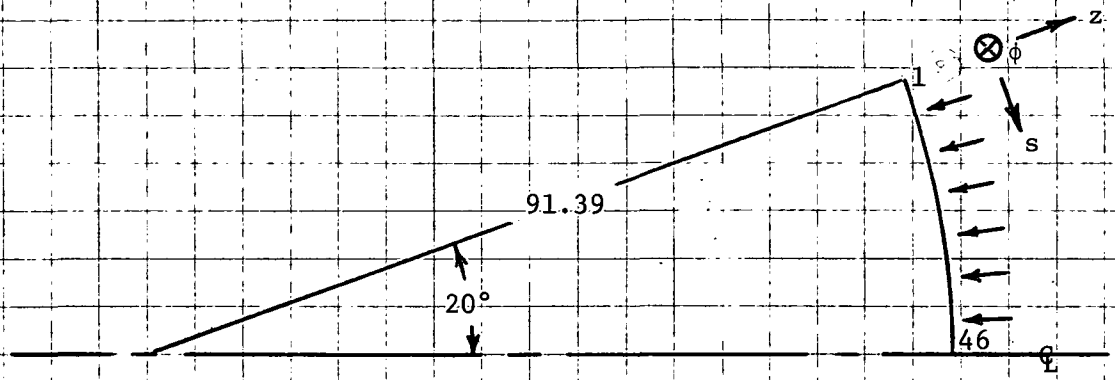
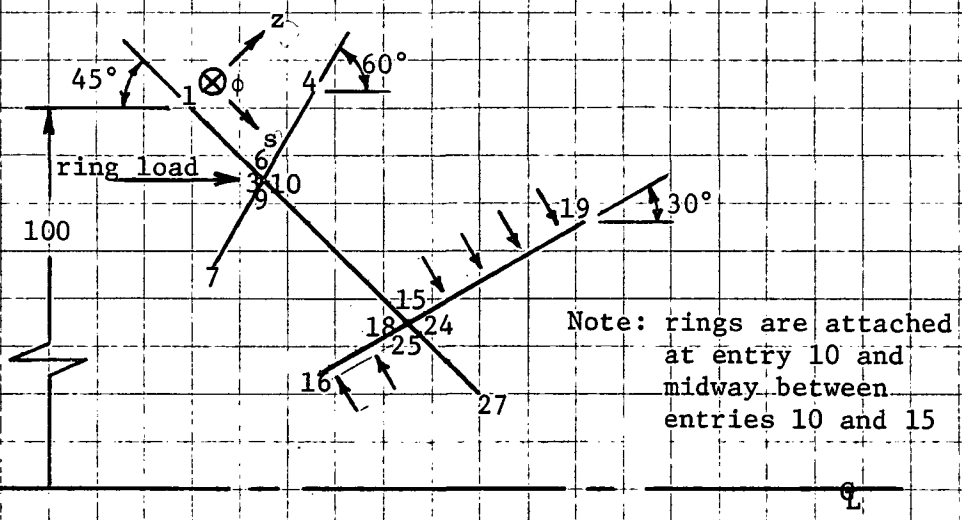


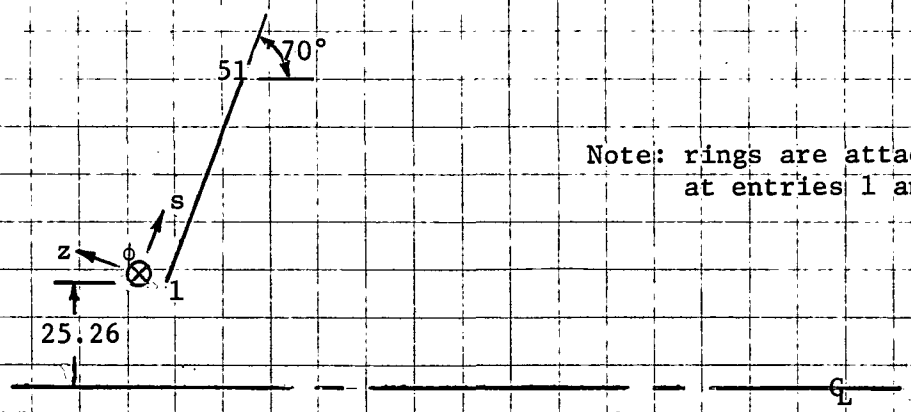
FIGURE 3. SHELL COORDINATES AND RESPONSE VARIABLES



(a) Clamped shperical cap



(b) Branched conical shell



(c) 140° sandwich cone

FIGURE 4. SAMPLE SHELL CONFIGURATIONS (Dimensions given in units of effective wall thickness)