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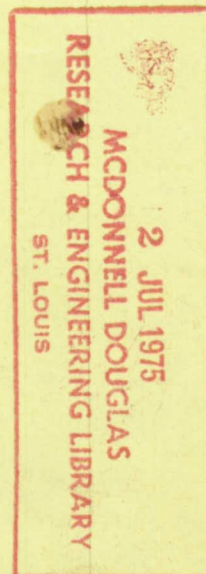
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DYNAMIC ERRORS IN A TUNED FLEXURE-MOUNTED STRAPDOWN
GYRO INTERIM SCIENTIFIC REPORT

John E. Bortz

Analytic Sciences Corporation
Reading, Massachusetts

September 1972



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TR-147-5

DYNAMIC ERRORS IN A TUNED
FLEXURE-MOUNTED STRAPDOWN GYRO

Interim Scientific Report

September 1972

Prepared Under:

Contract Number NAS 12-678

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Marshall Space Flight Center
Huntsville, Alabama

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NASA-CR-124161 DYNAMIC ERRORS IN A
TUNED FLEXURE-MOUNTED STRAPDOWN GYRO
Interim Scientific Report, Jan. - Jul.
1972 (Analytic Sciences Corp.) 62 p BC
5.25

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FOREWORD

The Analytic Sciences Corporation has conducted a series of studies in support of the Marshall Space Flight Center strapdown technology program. This document describes investigations of the performance of the tuned, flexure-mounted class of gyros and covers research performed during the period January 1972 through July 1972.

Mr. Jack H. Fagan of TASC performed the computer analysis work for this effort and Prof. Charles E. Hutchinson of the University of Massachusetts provided extensive analytical support. Both contributions are appreciatively acknowledged.

Mr. Ralph Kissel
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TABLE OF CONTENTSABSTRACT

Motion induced errors in a tuned, flexure-mounted strapdown gyro are investigated. Analytic expressions are developed for errors induced by linear vibrations, angular motion, and detuning. Sensor-level errors (gyro drift rate) and system-level errors (navigation errors) that are stimulated by an actual dynamic motion environment are computed.

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1.

INTRODUCTION

1.1 BACKGROUND

This report presents an analysis of the error torques acting upon the rotor of a tuned, flexure-mounted gyro in a strapdown application. This gyro was developed as a practical instrument in the early 1960's. Two of the first papers treating its development are by Howe (Ref. 1) and Savet (Ref. 2). A two-degree-of-freedom gyro, whose rotor is supported by a gimbal configured as shown in Fig. 1.1-1, is an example of such a gyro. The rotor is attached to an internal gimbal by a flexure mount which ideally permits rotation only about one axis, as illustrated in the figure, and provides a restoring torque proportional to the angle of rotation. The gimbal is connected to the spin motor shaft by another flexure mount, whose axis of flexure is nominally orthogonal to that of the outer one and normal to the axis of the spin motor shaft. In general, a tuned, flexure-mounted gyro can have any number of additional internal gimbals functioning in parallel, with flexure mounts between them.

In the configuration shown, the gyro can be arranged so there is no average net torque on the rotor. This occurs because the dynamic motion of the gimbal, under a condition referred to as tuning (described analytically in Chapter 2), supplies an average inertia torque to the rotor that exactly balances the average torque arising from the normal flexure of the rotor supporting members (the flexure mounts).

Three types of dynamic error torques in the tuned, flexure-mounted gyro are analyzed in this report:

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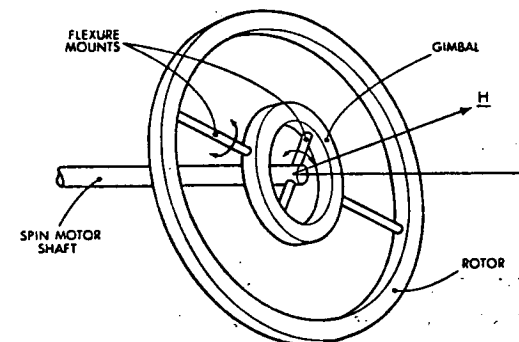


Figure 1.1-1 Flexure-Mounted Gyro

- Angular motion induced torques
- Specific force induced torques
- Torques caused by rotor speed variations

The errors generated by the rebalance loop itself, such as scale factor errors and temperature sensitivities, are not analytically described although, in the simulation work performed in this study (refer to Chapter 5), scale factor asymmetries have been included.

In order to predict gyro drift under operational conditions, a dynamic environment has been chosen and the propagation of gyro error torques into system level error in the presence of this dynamic environment has been investigated. The description of the environment that is used in this study is obtained from a report (Ref. 3) which describes the vibration spectra of the angular and linear vibrations and their cross correlations occurring in a CH-46C helicopter. In particular,

TASC's System Error Prediction Program (SEPP)* is used to compute the system level errors generated by the Teledyne SDG-1 under the specified dynamic motion conditions.

1.2 OVERVIEW OF REPORT

The basic torque equations for a flexure-mounted gyro are stated in Chapter 2. These equations are then developed into a useful form for the description of the dynamic errors in a practical tuned gyro.

The dynamic response of a tuned, flexure-mounted gyro is described in Chapter 3. The error torques induced by gyro case rotation are derived in Section 3.2. A functional description of the gyro is presented and transfer functions are derived. Consideration is given to the form and response of a general linear rebalance loop. Section 3.3 discusses the angular rate measurement process and the errors in that process.

Chapter 4 is devoted to a mathematical derivation of the gyro errors due to case angular motion, case specific force, and rotor speed variations.

The propagation of the dynamic errors into navigation errors has been studied for the case of a helicopter flight environment using the System Error Prediction Program (SEPP). A description of the SEPP and the dynamic errors computed are presented in Chapter 5.

* This program is described in Ref. 4.

2.

EQUATIONS OF MOTION

Equations are presented in this chapter describing the dynamics of the gyro rotor's motion relative to the case. The coordinate frames used in the development are introduced in Section 2.1, the basic flexure-mounted gyro equations are presented in Section 2.2, and in Section 2.3, these equations are put into practical form for analysis of a tuned, flexure-mounted gyro for strapdown applications.

2.1 COORDINATE FRAMES

Two almost coincident coordinate frames are used in analyzing the flexure-mounted gyro. One is the case-fixed triad x, y, z with the z axis along the motor shaft axis as shown in Fig. 2.1-1. The other triad, ξ, η, ζ , has its ζ axis along the rotor axis of symmetry, but is non-rotating, relative to the gyro case, about the ζ axis. Thus the ξ and η axes are in the rotor plane of symmetry but do not spin with the rotor. This coordinate frame will be called the rotor referenced frame or simply the rotor frame.

From these definitions, it can be seen that when the motor shaft axis and rotor axis of symmetry are coincident, the ξ, η, ζ triad can be made to coincide with the x, y, z triad. In general, the case fixed and the rotor frames are not coincident and differ by the rotor deflection angles θ_x and θ_y , which define the rotation of the rotor frame relative to the case frame, is shown in Fig. 2.1-1. That is, θ_x represents a rotation of the rotor relative to the case about the x axis, etc. When θ_x and θ_y are small,

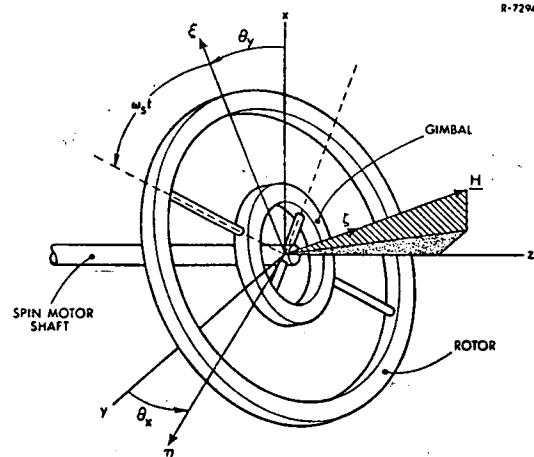


Figure 2.1-1 Gyro Coordinate Systems*

as we shall assume they are, the two frames are related by the direction cosine matrix

$$C_r^c = \begin{bmatrix} 1 & 0 & -\theta_y \\ 0 & 1 & \theta_x \\ \theta_y & -\theta_x & 1 \end{bmatrix} \quad (2.1-1)$$

which transforms a case referenced vector $\underline{x}_c$ into the rotor coordinate frame according to the rule

$$\underline{x}_r = C_r^c \underline{x}_c \quad (2.1-2)$$

* A gyro with a single gimbal is shown for purposes of illustration. The development which follows is not restricted, however, to the specific configuration illustrated.

2.2 GYRO EQUATIONS: THE STATIC CASE

A set of differential equations has been developed by Savet (Ref. 2) for the rotor deflection angles θ_x and θ_y for the static situation in which there is no motion of the gyro case. These equations are:

$$\left(A + \frac{a}{2}\right) \ddot{\theta}_x + f \dot{\theta}_x + \left[k - \left(a - \frac{c}{2}\right) \omega_s^2\right] \theta_x + (C+a) \omega_s \dot{\theta}_y + \lambda \theta_y = -q_x \cos 2\omega_s t - q_y \sin 2\omega_s t - M_x - \epsilon_x \quad (2.2-1)$$

and

$$\left(A + \frac{a}{2}\right) \ddot{\theta}_y + f \dot{\theta}_y + \left[k - \left(a - \frac{c}{2}\right) \omega_s^2\right] \theta_y - (C+a) \omega_s \dot{\theta}_x - \lambda \theta_x = q_y \cos 2\omega_s t - q_x \sin 2\omega_s t - M_y - \epsilon_y \quad (2.2-2)$$

where

$A, C(a, c)$ are the rotor (gimbal) moments of inertia about an axis in the plane normal to the axis of symmetry and about the axis of symmetry respectively.

θ_x, θ_y are the rotor deflection angles relative to the case about the case fixed x and y axes.

f is the rotor deflection angle damping coefficient.

k is the torque constant for the gimbal and rotor flexure mounts.

ω_s is the angular velocity of the rotor relative to the gyro case about the rotor axis of symmetry.

λ is a cross axis torque constant of the rotor shaft about the shaft axis of symmetry (for elaboration, see Ref. 2).

ϵ_x, ϵ_y are error torques.

M_x, M_y are torques applied externally to the rotor by the rotor torquing (rebalance) electronics.

The variables q_x and q_y are functions of the rotor deflection angles and are given by

$$q_x = \frac{a}{2} \ddot{\theta}_x + a \omega_s \dot{\theta}_y - \left(a - \frac{c}{2}\right) \omega_s^2 \theta_x \quad (2.2-3)$$

$$q_y = \frac{a}{2} \ddot{\theta}_y - a \omega_s \dot{\theta}_x - \left(a - \frac{c}{2}\right) \omega_s^2 \theta_y \quad (2.2-4)$$

As can be seen from Eqs. (2.2-1) and (2.2-2), the flexure-mounted gyro exhibits the response characteristics of a coupled two-degree-of-freedom oscillator. The inertia coefficient $A + a/2$ is composed of a rotor contribution (A) and a gimbal contribution ($a/2$). A composite moment of inertia $\tilde{A}$ can be defined as

$$\tilde{A} = A + a/2 \quad (2.2-5)$$

The damping coefficient f is primarily attributable to unintentional phenomena such as viscous air damping. Damping may, of course, be intentionally introduced via the feedback torques M_x and M_y .

The effective restraint torque coefficient K

$$K = k - \left(a - \frac{c}{2}\right) \omega_s^2 \quad (2.2-6)$$

is of considerable interest in the flexure-mounted family of gyros. Neglecting, for the present, the drive shaft torque coefficient λ , it can be seen from Eqs. (2.2-1) and (2.2-2) that if $K = 0$, then the gyro has the

response characteristics of a free gyro, i.e., a gyro with no elastic restraint torques.

For a given set of gimbal moments of inertia, a and c , and for a given flexure mount torque coefficient k , Eq. (2.2-6) shows that K is zero when

$$\omega_s = \omega_t \triangleq \sqrt{\frac{k}{a - c/2}} \quad (2.2-7)$$

Equation (2.2-7) is known as the tuning condition and a gyro for which Eq. (2.2-7) holds is known as a tuned gyro. As can be seen from the dependency of the tuning condition upon spin angular velocity, this is a dynamic tuning. It is made possible by the forced motion of the gimbal, which exchanges energy with the rotor in an oscillatory manner. This energy exchange is phased so as to counteract, when averaged over one complete rotor spin cycle, the torque exerted on the rotor by the flexure mounts.

2.3 A PRACTICAL TUNED GYRO

An obvious simplification can be made in Eqs. (2.2-1) and (2.2-2) for a tuned gyro, i.e., a gyro for which Eq. (2.2-7) holds. Also, for the purposes of analyzing the dominant dynamic error effects, it is reasonable to ignore friction effects in the flexure mounts and set $f = 0$. It is also reasonable to ignore as small the rotor shaft torques $\lambda \theta_x$ and $\lambda \theta_y$. With these simplifications, Eqs. (2.2-1) and (2.2-2) reduce to

$$\tilde{A} \ddot{\theta}_x + \tilde{C} \omega_s \dot{\theta}_y = -q_x \cos 2\omega_s t - q_y \sin 2\omega_s t - M_x - \epsilon_x \quad (2.3-1)$$

$$\tilde{A} \ddot{\theta}_y - \tilde{C} \omega_s \dot{\theta}_x = q_y \cos 2\omega_s t - q_x \sin 2\omega_s t - M_y - \epsilon_y \quad (2.3-2)$$

where

$\tilde{C} = \bar{C} + a$ is the effective rotor moment of inertia about the axis of symmetry

ω_t is the rotor angular velocity when the tuning condition is satisfied

For a single gimbal configuration such as the one shown in Fig. 2.1-1, there can be an inherent stability problem. This arises if the rotor inertia moments satisfy the relationship $C \approx 2A$. Then to the same degree of approximation,

$$\tilde{C} = 2\tilde{A} \quad (2.3-3)$$

The homogeneous equation set related to Eqs. (2.3-1) and (2.3-2), is therefore

$$\tilde{A}\ddot{\theta}_x + 2\tilde{A}\omega_s\dot{\theta}_y = 0 \quad (2.3-4)$$

$$\tilde{A}\ddot{\theta}_y - 2\tilde{A}\omega_s\dot{\theta}_x = 0 \quad (2.3-5)$$

The characteristic equation of this equation set is

$$\tilde{A}^2 s^2 [s^2 + (2\omega_s)^2] = 0 \quad (2.3-6)$$

where we have used Laplace operator notation. Equation (2.3-6) shows that the undamped natural frequency ω_n of the system is $2\omega_s$. As can be seen from Eqs. (2.3-1) and (2.3-2), the forcing functions on the right hand side are sinusoidal at a frequency $2\omega_s$. This can cause unbounded growth in θ_x and θ_y .

The stability problem can be averted, as indicated by Craig (Refs. 5, 6), by providing more than one gimbal and arranging the gimbals in parallel. In the n gimbal configuration, there are n sets of oscillatory forcing terms. For $n \geq 2$ and for gimbal flexure mounts spaced $2\pi/n$ radians apart the forcing terms can be caused to sum to zero. Craig points out (Ref. 6) that "from a practical standpoint, at least three gimbals are required for complete elimination" of the q_x and q_y terms.

It is assumed that a multigimbal configuration is used to achieve stability and that the forcing terms that oscillate at $2\omega_s$ vanish. We therefore take for the basic equation set

$$\tilde{A}\ddot{\theta}_x + \tilde{C}\ddot{\theta}_y = -M_x - \epsilon_x \quad (2.3-7)$$

$$\tilde{A}\ddot{\theta}_y - \tilde{C}\ddot{\theta}_x = -M_y - \epsilon_y \quad (2.3-8)$$

This set describes the dynamic performance of a tuned, flexure-mounted gyro whose case is not subjected to angular motion. Savet (Ref. 2) directs attention to the fact that, as indicated by the absence of spring restraint terms on the left hand sides of Eqs. (2.3-7) and (2.3-8), the tuned, flexure-mounted gyro exhibits the response characteristics of a free gyro.

3.

GYRO DYNAMIC RESPONSE

The dynamic response of a tuned, flexure-mounted gyro is described in Chapter 2 for the situation in which the gyro case is not subjected to angular motion. In this chapter, we consider the general circumstance of an arbitrary angular motion environment.

3.1 ANGULAR MOTION DYNAMICS

In Appendix A, it is shown that the components of the rotor inertial reaction torque vector $\underline{L}$ are

$$\begin{aligned} L_x = & A(\dot{\Omega}_x + \ddot{\theta}_x) + H(\dot{\Omega}_y + \dot{\theta}_y) + (C - A)\Omega_y\Omega_z \\ & + \theta_x \left[(C - A)(\Omega_z^2 - \Omega_y^2) + H\Omega_z \right] \\ & + \theta_y \left[(C - A)\Omega_x\Omega_y + (C - A)\dot{\Omega}_z \right] \end{aligned} \quad (3.1-1)$$

$$\begin{aligned} L_y = & A(\dot{\Omega}_y + \ddot{\theta}_y) - H(\dot{\Omega}_x + \dot{\theta}_x) - (C - A)\Omega_x\Omega_z \\ & + \theta_x \left[(C - A)\Omega_x\Omega_y - (C - A)\dot{\Omega}_z \right] \\ & - \theta_y \left[(C - A)(\Omega_z^2 - \Omega_x^2) - H\Omega_z \right] \end{aligned} \quad (3.1-2)$$

where

$\Omega_x, \Omega_y, \Omega_z$ are the components of the angular velocity of the gyro case with respect to inertial space

θ_x, θ_y are the deflection angles of the rotor relative to the gyro case, as shown in Fig. 2.1-1

A, C are the rotor* moments of inertia about axes perpendicular and parallel respectively to the rotor axis of symmetry

$H = C\omega_s$ is the magnitude of the rotor angular momentum

As stated at the close of the preceding chapter, the tuned, flexure-mounted gyro is essentially a free gyro. Therefore, the only external torques** applied to the gyro are those provided by the torque generator, M_x and M_y . Hence, to the degree of approximation for which Eqs. (3.1-1) and (3.1-2) are valid (see Appendix A), we can write

$$M_x = -L_x \quad (3.1-3)$$

$$M_y = -L_y \quad (3.1-4)$$

since the applied torques are ideally equal and opposite to the inertia reaction torques.

* The gimbal moments of inertia have been assumed to be zero. This assumption is based on the observation that although the gimbal inertias are of direct importance in tuning the gyro, they are of much lesser consequence in the dynamic motion error model, since $A \approx A + a/2$ and $C \approx C - a$.

** This assertion is relaxed in Section 4.3 where the additional torques that exist when the gyro tuning condition of Eq. (2.2-7) is not met are considered.

Using Eqs. (3.1-3) and (3.1-4) in Eqs. (3.1-1) and (3.1-2) and rearranging slightly yields

$$A\delta_x + H\delta_y = -A\dot{\Omega}_x - H\dot{\Omega}_y - M_x - \epsilon_x \quad (3.1-5)$$

$$A\delta_y - H\delta_x = -A\dot{\Omega}_y + H\dot{\Omega}_x - M_y - \epsilon_y \quad (3.1-6)$$

where the error torques ϵ_x and ϵ_y are identified by comparison with Eqs. (3.1-1) and (3.1-2) as

$$\epsilon_{x_{ang}} = A\dot{\Omega}_x + (C-A)\Omega_y\Omega_z + \theta_x \left[(C-A)(\Omega_z^2 - \Omega_y^2) + H\Omega_z \right] + \theta_y \left[(C-A)\Omega_x\Omega_z + (C-A)\dot{\Omega}_z \right] \quad (3.1-7)$$

$$\epsilon_{y_{ang}} = A\dot{\Omega}_y - (C-A)\Omega_x\Omega_z - \theta_y \left[(C-A)(\Omega_z^2 - \Omega_x^2) - H\Omega_z \right] + \theta_x \left[(C-A)\Omega_x\Omega_y - (C-A)\dot{\Omega}_z \right] \quad (3.1-8)$$

The subscript "ang" has been appended to the components ϵ_x and ϵ_y to indicate that these are the error torques arising from gyro case angular motion. In Chapter 4 the error torques due to specific force and rotor speed variations will be discussed. Before proceeding with an examination of the gyro error components, however, we first develop a functional description of the gyro and its rebalance loop.

3.2 GYRO FUNCTIONAL DESCRIPTION

Equations (3.1-5) and (3.1-6) can be written using Laplace operator notation as

$$\begin{bmatrix} As^2 & Hs \\ -Hs & As^2 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} -As & -H \\ H & -As \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} - \begin{bmatrix} M_x \\ M_y \end{bmatrix} - \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.2-1)$$

where we have employed a vector-matrix format for convenience. Using the relation $H \approx 2A\omega_s^*$ in Eq. (3.2-1) and inverting, we obtain

$$\begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} \frac{G(s)}{2\omega_s} & -\frac{G(s)}{s} \\ \frac{G(s)}{s} & \frac{G(s)}{2\omega_s} \end{bmatrix} \left\{ \begin{bmatrix} -\frac{Hs}{2\omega_s} & -H \\ H & -\frac{Hs}{2\omega_s} \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} - \begin{bmatrix} M_x \\ M_y \end{bmatrix} - \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \right\} \quad (3.2-2)$$

where the function $G(s)$ is

$$G(s) = \frac{1}{H \left(\left(\frac{s}{2\omega_s} \right)^2 + 1 \right)} \quad (3.2-3)$$

Equation (3.2-2) is diagrammed in Fig. 3.2-1.

From the figure we see that if $\underline{\epsilon} = \underline{M} = \underline{0}$, then, dropping the s subscript from ω_s for convenience,

$$\theta_x = -H \left(\frac{G(s)}{s} \right) \Omega_x - \left(\frac{Hs}{2\omega} \right) \left(\frac{G(s)}{2\omega} \right) \Omega_x + \left(\frac{Hs}{2\omega} \right) \left(\frac{G(s)}{s} \right) \Omega_y - H \left(\frac{G(s)}{2\omega} \right) \Omega_y$$

Using Eq. (3.2-3), this becomes

$$\theta_x = -\frac{1}{s} \Omega_x \quad (3.2-4)$$

* This is a particularly good approximation for the Teledyne SDG-1 gyro treated in Chapter 5. It results from the fact that, for that instrument, $C \approx 2A$. We will make the latter approximation in several places in this report without repeating the justification.

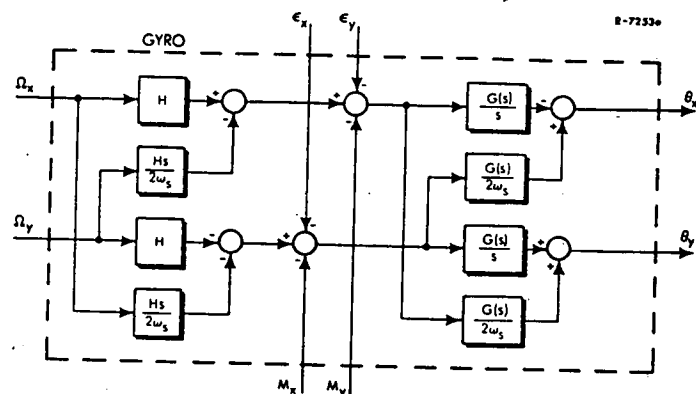


Figure 3.2-1 Tuned, Flexure-Mounted Gyro

Likewise, under these conditions

$$\theta_y = -\frac{1}{s} \Omega_y \quad (3.2-5)$$

We conclude that in the presence of a steady input angular rate, a rebalance torque is required that is proportional to that rate.

Let us now consider a linear torquing scheme of the form

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} F_c(s) & F_d(s) \\ -F_d(s) & F_c(s) \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (3.2-6)$$

where $F_d(s)$ is the direct feedback path transfer function and $F_c(s)$ is the cross axis transfer function. Both are realizable linear networks. The

gyro and rebalance electronics then are configured as shown in Fig. 3.2-2. Inserting Eq. (3.2-6) into Eq. (3.2-2) gives

$$\begin{bmatrix} 1 + \frac{F_c(s)G(s)}{2\omega} + \frac{F_d(s)G(s)}{s} & -\frac{F_c(s)G(s)}{s} + \frac{F_d(s)G(s)}{2\omega} \\ \frac{F_c(s)G(s)}{s} - \frac{F_d(s)G(s)}{2\omega} & 1 + \frac{F_c(s)G(s)}{2\omega} + \frac{F_d(s)G(s)}{s} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = -\frac{1}{s} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} - \begin{bmatrix} \frac{G(s)}{2\omega} & -\frac{G(s)}{s} \\ \frac{G(s)}{s} & \frac{G(s)}{2\omega} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.2-7)$$

which can be inverted to obtain

$$\begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = -\frac{1}{\Delta(s)} \begin{bmatrix} s + F_d(s)G(s) + \frac{s}{2\omega} F_c(s)G(s) & F_c(s)G(s) - \frac{s}{2\omega} F_d(s)G(s) \\ -F_c(s)G(s) + \frac{s}{2\omega} F_d(s)G(s) & s + F_d(s)G(s) + \frac{s}{2\omega} F_c(s)G(s) \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} + \begin{bmatrix} G(s) \left[\frac{s^2}{2\omega} + \frac{1}{H} F_c(s) \right] & -G(s) \left[s + \frac{1}{H} F_d(s) \right] \\ G(s) \left[s + \frac{1}{H} F_d(s) \right] & G(s) \left[\frac{s^2}{2\omega} + \frac{1}{H} F_c(s) \right] \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.2-8)$$

where Eq. (3.2-3) has been used for reduction of some of the expressions arising in the inversion and where

$$\Delta(s) \triangleq s^2 + G(s) \left[\frac{F_c^2(s)}{H} + \frac{s^2}{\omega} F_c(s) + 2sF_d(s) + \frac{F_d^2(s)}{H} \right] \quad (3.2-9)$$

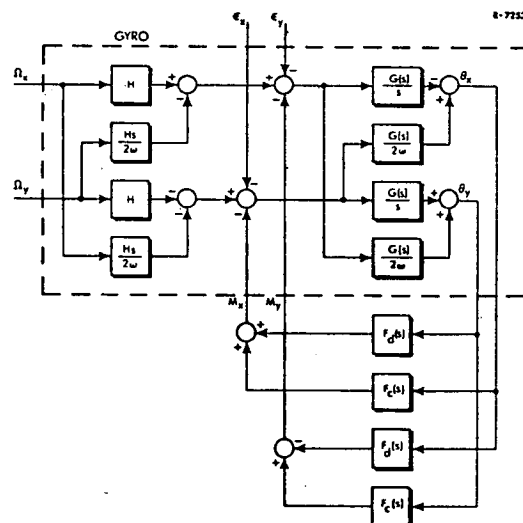


Figure 3.2-2 Tuned, Flexure-Mounted Gyro and Rebalance Loops

For the linear torquing scheme as represented by Eq. (3.2-6), when Eq. (3.2-8) is used, the torque $\underline{M}$ is found to be

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} sF_c(s) + \frac{s}{2\omega} G(s) (F_c^2(s) + F_d^2(s)) & sF_d(s) + G(s) (F_c^2(s) + F_d^2(s)) \\ -sF_d(s) - G(s) (F_c^2(s) + F_d^2(s)) & sF_c(s) + \frac{s}{2\omega} G(s) (F_c^2(s) + F_d^2(s)) \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} - \begin{bmatrix} sG(s) \left(\frac{s}{2\omega} F_c(s) + F_d(s) \right) + \frac{G(s)}{H} (F_c^2(s) + F_d^2(s)) & -sG(s) (F_c(s) - \frac{s}{2\omega} F_d(s)) \\ sG(s) (F_c(s) - \frac{s}{2\omega} F_d(s)) & sG(s) \left(\frac{s}{2\omega} F_c(s) + F_d(s) \right) + \frac{G(s)}{H} (F_c^2(s) + F_d^2(s)) \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.2-10)$$

It is of interest to evaluate the steady-state torques generated by a constant angular velocity input vector $\underline{\Omega}$. The final value theorem of Laplace transform theory states (Ref. 7) that if $F(s)$ is the Laplace transform of $f(t)$, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (3.2-11)$$

To apply this theorem to Eq. (3.2-10) we need to make some assumptions about the transfer functions $F_d(s)$ and $F_c(s)$. Assume that

$$F_d(0) = k_d \quad F_c(0) = k_c \quad (3.2-12)$$

From Eq. (3.2-3), note that

$$G(0) = \frac{1}{H} \quad (3.2-13)$$

Using Eqs. (3.2-12) and (3.2-13) in Eq. (3.2-9),

$$\Delta(0) = \frac{k_d^2 + k_c^2}{H^2} \quad (3.2-14)$$

The Laplace transforms of the constant input rate vectors $\underline{\Omega}$ and $\underline{\epsilon}$ are

$$\underline{\Omega}(s) = \frac{1}{s} \begin{bmatrix} \Omega_{xss} \\ \Omega_{yss} \end{bmatrix} \quad \underline{\epsilon}(s) = \frac{1}{s} \begin{bmatrix} \epsilon_{xss} \\ \epsilon_{yss} \end{bmatrix} \quad (3.2-15)$$

Then using Eqs. (3.2-12) through (3.2-15) in Eq. (3.2-10) and applying (3.2-11), we obtain

$$\lim_{t \rightarrow \infty} \begin{bmatrix} M_x(t) \\ M_y(t) \end{bmatrix} = \begin{bmatrix} M_{xss} \\ M_{yss} \end{bmatrix} = \begin{bmatrix} 0 & -H \\ H & 0 \end{bmatrix} \begin{bmatrix} \Omega_{xss} \\ \Omega_{yss} \end{bmatrix} - \begin{bmatrix} \epsilon_{xss} \\ \epsilon_{yss} \end{bmatrix} \quad (3.2-16)$$

Thus in the case of a constant angular velocity about the x axis, there is a steady-state torque applied about the y axis to balance the input moment. As is expected, the x axis balance torque is zero when there is no input rate about the y axis nor error torque about the x axis.

3.3 ANGULAR RATE MEASUREMENTS

In a strapdown gyro, the calibrated feedback torques are used as a measure of motion about the case input axes. For a linear rebalance loop, the measurement $\underline{M}$ is taken to be linearly related to the input angular rate vector.

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \hat{\Omega}_x \\ \hat{\Omega}_y \end{bmatrix} \quad (3.3-1)$$

where $\hat{\Omega}$ is a vector of the estimated angular rates about the two sensitive axes and the a_{ij} of the geometry matrix have yet to be specified. If we assume that the error torques are zero and that a steady-state condition exists, then, using Eq. (3.2-1), we see that

$$\begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} 0 & -H \\ H & 0 \end{bmatrix} \begin{bmatrix} \hat{\Omega}_x \\ \hat{\Omega}_y \end{bmatrix} \quad (3.3-2)$$

or, solving for $\hat{\Omega}_x$ and $\hat{\Omega}_y$,

$$\begin{aligned} \hat{\Omega}_x &= \frac{1}{H} M_y \\ \hat{\Omega}_y &= -\frac{1}{H} M_x \end{aligned} \quad (3.3-3)$$

To investigate the errors in this measurement scheme, we must consider the general case where the error torques are not zero and a steady state has not been reached. Substituting Eq. (3.3-2) into Eq. (3.2-1) yields

$$\begin{bmatrix} As^2 & Hs \\ -Hs & As^2 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} -As & -H \\ H & -As \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} - \begin{bmatrix} 0 & -H \\ H & 0 \end{bmatrix} \begin{bmatrix} \hat{\Omega}_x \\ \hat{\Omega}_y \end{bmatrix} - \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.3-4)$$

If we define the rate measurement error vector $\delta\hat{\Omega}$ to be

$$\delta\hat{\Omega} = \hat{\Omega} - \underline{\Omega} \quad (3.3-5)$$

then Eq. (3.3-4) becomes

$$\begin{bmatrix} 0 & H \\ -H & 0 \end{bmatrix} \begin{bmatrix} \delta\hat{\Omega}_x \\ \delta\hat{\Omega}_y \end{bmatrix} = \begin{bmatrix} As^2 & Hs \\ -Hs & As^2 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} + \begin{bmatrix} As & 0 \\ 0 & As \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} + \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} \quad (3.3-6)$$

or, solving for $\delta\hat{\Omega}$,

$$\begin{bmatrix} \delta\hat{\Omega}_x \\ \delta\hat{\Omega}_y \end{bmatrix} = \begin{bmatrix} s & -\frac{As^2}{H} \\ \frac{As^2}{H} & s \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} + \begin{bmatrix} 0 & -\frac{As}{H} \\ \frac{As}{H} & 0 \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} + \frac{1}{H} \begin{bmatrix} -\epsilon_y \\ \epsilon_x \end{bmatrix} \quad (3.3-7)$$

Equation (3.3-7) is the equation for the dynamic rate measurement errors in a two-degree-of-freedom strapdown gyro.

Summary — The operation of the flexure-mounted strapdown gyro has been discussed in this chapter. The gyro dynamics equation was restructured to reflect the effects of gyro case angular velocity.

4. DYNAMIC ERROR MODEL

A principal objective of this report is the development of a model for the dynamic errors of a tuned, flexure-mounted strapdown gyro. In this chapter, models are presented for dynamic errors due to angular motion, specific force, and rotor speed variations.

4.1 ANGULAR MOTION ERROR TORQUES

The dynamic response of the tuned, flexure-mounted gyro, when subjected to arbitrary angular motion, is treated in Chapter 3 from an instrument dynamics point of view. In this section the angular motion error torques, as presented in Eqs. (3.1-7) and (3.1-8), are discussed term by term. These equations are repeated here for convenience.

$$\epsilon_{x_{ang}} = (C-A) \Omega_y \Omega_z + \theta_x \left[(C-A) (\Omega_z^2 - \Omega_y^2) + H \Omega_z \right] + \theta_y \left[(C-A) \Omega_x \Omega_y + (C-A) \dot{\Omega}_z \right] \quad (4.1-1)$$

$$\epsilon_{y_{ang}} = -(C-A) \Omega_x \Omega_z - \theta_y \left[(C-A) (\Omega_z^2 - \Omega_x^2) - H \Omega_z \right] + \theta_x \left[(C-A) \Omega_x \Omega_y - (C-A) \dot{\Omega}_z \right] \quad (4.1-2)$$

4.1.1 Anisoinertia

The terms $(C-A)\Omega_y\Omega_z$ in Eq. (4.1-1) and $(C-A)\Omega_x\Omega_z$ in Eq. (4.1-2) are referred to as the anisoinertia error torques. They are critical terms because they can generate constant error torques (bias drift rates) when case angular rates are constant or when properly phased oscillatory angular motion is experienced. Note that the primary gyroscopic input for the x axis is $H\Omega_x = C\omega_s\Omega_x$. The x-axis anisoinertia term is $(C-A)\Omega_z\Omega_y \approx C\Omega_z\Omega_y/2$. Therefore, the anisoinertia error torques may be written

$$\begin{bmatrix} \epsilon_{x \text{ anisoinertia}} \\ \epsilon_{y \text{ anisoinertia}} \end{bmatrix} = \frac{C}{2} \begin{bmatrix} \Omega_y\Omega_z \\ -\Omega_x\Omega_z \end{bmatrix} \quad (4.1-3)$$

We see, by comparing the terms in Eq. (4.1-3) with the input terms $-C\omega_s\Omega_y$ and $C\omega_s\Omega_x$, that the ratio of the coefficient of the anisoinertia term to that of the input term is $\Omega_z/2\omega_s$. For a constant non-negligible case angular velocity about the z axis, this could result in a significant error.* In a single degree-of-freedom gyro, an attempt is made to balance the moments of inertia in order to suppress the anisoinertia term. This balancing is facilitated by the fact that in the single-degree-of-freedom gyro, the inertia moments C and A are moments for the entire suspended element which consists of the rotor and its relatively substantial supporting gimbal. Such a balance of inertias would also be desirable in a flexure-mounted gyro. In fact, as can be seen from Eqs. (4.1-1) and (4.1-2), this would eliminate many of the dynamic error torques due to angular motion as well as the anisoinertia error torque.

* Quantitative estimates of this error are given in Chapter 5.

However, such a balance is more difficult to achieve since it must be accomplished by variation of the rotor parameters alone.* For practical reasons, such a balance is not attained.

In the event of oscillatory motion about the z axis (the spin axis), a phenomenon known as decoupling comes into play. This occurs when the rotor speed control loop stiffness tends to degrade. As oscillation frequency about the spin axis begins to exceed the rotor speed control loop bandwidth, sinusoidal variations begin to occur in the speed of the rotor relative to the gyro case; the rotor spins at a constant rate relative to inertial space but does not follow the case oscillations about the spin axis. Consequently the inertia of the rotor about its axis of symmetry, C, is eliminated from the effective anisoinertia. For high frequency angular rates about the gyro spin axis Eq. (4.1-3) becomes

$$\begin{aligned} \begin{bmatrix} \epsilon_{x \text{ anisoinertia}} \\ \epsilon_{y \text{ anisoinertia}} \end{bmatrix} &= \left(\frac{C}{2} - C\right) \begin{bmatrix} \Omega_y\Omega_z \\ -\Omega_x\Omega_z \end{bmatrix} \\ &= -\frac{C}{2} \begin{bmatrix} \Omega_z\Omega_z \\ -\Omega_x\Omega_z \end{bmatrix} \end{aligned} \quad (4.1-4)$$

That is, for the case where $C \approx 2A$, the sign of the anisoinertia coefficient is reversed when Ω_z occurs at frequencies above the rotor speed control bandwidth.

* To remove the $(C-A)$ terms in Eqs. (4.1-1) and (4.1-2) would require a rotor shape approaching that of a sphere.

4.1.2 Cross Coupling

The terms $\theta_x [(C-A)(\Omega_z^2 - \Omega_y^2) + H\Omega_z]$ in Eq. (4.1-1) and $-\theta_y [(C-A)(\Omega_z^2 - \Omega_x^2) - H\Omega_z]$ in Eq. (4.1-2) are referred to as the cross-coupling error torques. Now

$$H = C\omega_s; \quad C-A \cong C/2 \quad (4.1-5)$$

and since $\omega_s \gg \Omega_i$, $i = x, y, z$, the terms involving H predominate. We can write the dominant cross coupling terms as

$$\begin{bmatrix} \epsilon_{x_{cc}} \\ \epsilon_{y_{cc}} \end{bmatrix} = \begin{bmatrix} H\Omega_z & 0 \\ 0 & H\Omega_z \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (4.1-6)$$

Using Eq. (3.2-8) and neglecting error torques we can relate the rotor deflection angles to the input rates in the form

$$\begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = - \begin{bmatrix} Q_1(s) & Q_2(s) \\ -Q_2(s) & Q_1(s) \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} \quad (4.1-7)$$

where

$$Q_1(s) = \frac{s + F_d(s)G(s) + \frac{s}{2\omega} F_c(s)G(s)}{\Delta(s)} \quad (4.1-8)$$

$$Q_2(s) = \frac{F_c(s)G(s) - \frac{s}{2\omega} F_d(s)G(s)}{\Delta(s)} \quad (4.1-9)$$

Combining Eq. (4.1-7) with Eq. (4.1-6), we obtain for the cross coupling torques

$$\begin{bmatrix} \epsilon_{x_{cc}} \\ \epsilon_{y_{cc}} \end{bmatrix} = - \begin{bmatrix} Q_1(s) & Q_2(s) \\ -Q_2(s) & Q_1(s) \end{bmatrix} \begin{bmatrix} H\Omega_x \Omega_z \\ H\Omega_y \Omega_z \end{bmatrix} \quad (4.1-10)$$

In order to estimate the significance of these terms, suppose that Ω_x , Ω_y , and Ω_z are oscillatory at a frequency below the first break frequency in $Q_1(s)$ and $Q_2(s)$. In that case we can replace $Q_1(s)$ and $Q_2(s)$ by $Q_1(0)$ and $Q_2(0)$ respectively. With the aid of Eqs. (3.2-13) and (3.2-14), we see from Eqs. (4.1-8) and (4.1-9) that,

$$Q_1(0) = \frac{Hk_d}{k_d^2 + k_c^2} \quad (4.1-11)$$

$$Q_2(0) = \frac{Hk_c}{k_d^2 + k_c^2} \quad (4.1-12)$$

In a typical linear rebalance loop, at low frequencies the direct rebalance torque is much larger than the cross axis rebalance torque, i.e., $k_d \gg k_c$. Hence Eqs. (4.1-11) and (4.1-12) can be approximated by

$$Q_1(0) = \frac{H}{k_d} \triangleq k_q \quad (4.1-13)$$

and

$$Q_2(0) = 0 \quad (4.1-14)$$

Thus to a good approximation, the cross coupling torques of Eq. (4.1-10) are

$$\begin{bmatrix} \epsilon_{x_{cc}} \\ \epsilon_{y_{cc}} \end{bmatrix} = k_q H \begin{bmatrix} -\Omega_x \Omega_z \\ -\Omega_y \Omega_z \end{bmatrix} \quad (4.1-15)$$

These terms are potentially serious because of the rectification that occurs when Ω_x or Ω_y are sinusoids that are in phase with Ω_z . The cross coupling error mechanism is also excited by constant input rate components Ω_x and Ω_z or Ω_y and Ω_z .

4.1.3 Rotor-to-Case Misalignment

The terms $\theta_y [(C-A)\Omega_y\Omega_x + (C-A)\dot{\Omega}_z]$ in Eq. (4.1-1) and $\theta_x [(C-A)\Omega_y\Omega_x - (C-A)\dot{\Omega}_z]$ in Eq. (4.1-2) arise from rotor-to-case dynamic misalignments. In a single-degree-of-freedom gyro (Ref. 8), such terms arise because of suspension system compliance or because of misalignments generated during the gyro manufacturing process. In a two-degree-of-freedom gyro, the rotor misalignments θ_y and θ_x arise as a consequence of the normal rotor deflections that occur when there is case rotation.

In order to estimate the relative contribution of the rotor-to-case misalignment torques ϵ_{mis} , consider the first term of each torque component.

$$\begin{bmatrix} \epsilon_{x_{mis}} \\ \epsilon_{y_{mis}} \end{bmatrix} = \begin{bmatrix} 0 & (C-A)\Omega_y\Omega_x \\ (C-A)\Omega_y\Omega_x & 0 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (4.1-16)$$

For comparison purposes, assume $C \approx 2A$ and rewrite the coefficient expression as follows:

$$\begin{aligned} (C-A)\Omega_y\Omega_x &= \frac{C\omega_s}{2\omega_s} \Omega_y\Omega_x \\ &= H\Omega_y \left(\frac{\Omega_x}{2\omega_s} \right) = H\Omega_x \left(\frac{\Omega_y}{2\omega_s} \right) \end{aligned} \quad (4.1-17)$$

By inserting Eqs. (4.1-13) and (4.1-14) into Eq. (4.1-7), at frequencies below the first break frequency, the rotor deflection angles can be approximated by

$$\begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = -k_q \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} \quad (4.1-18)$$

Equations (4.1-16) through (4.1-18) can be combined to obtain

$$\begin{bmatrix} \epsilon_{x_{mis}} \\ \epsilon_{y_{mis}} \end{bmatrix} = -k_q H \begin{bmatrix} \Omega_x \Omega_y \left(\frac{\Omega_y}{2\omega_s} \right) \\ \Omega_y \Omega_x \left(\frac{\Omega_x}{2\omega_s} \right) \end{bmatrix} \quad (4.1-19)$$

By comparing Eq. (4.1-19) with Eq. (4.1-15), it is seen that the misalignment torque is similar in form to the cross coupling torque except that the misalignment torque components include the multiplicative factors $\Omega_y/2\omega_s$ and $\Omega_x/2\omega_s$ for the x and y components respectively. Since $\Omega_y, \Omega_x \ll 2\omega_s$, it is reasoned that the misalignment torques are correspondingly less significant than the cross coupling torques. This conclusion is borne out by the results of the error propagation study described in Chapter 5.

4.1.4 Angular Acceleration Sensitivity

The terms $-A\dot{\Omega}_x$ in Eq. (3.1-5) and $-A\dot{\Omega}_y$ in Eq. (3.1-6), while not included in the definition of the angular motion error torques, can generate sizeable vibration induced errors at the system level. Equation (3.3-7) identifies the same terms as "dynamic" rate measurement errors (the second term on the right-hand side). In a vibration environment oscillatory angular rates imply oscillatory angular accelerations, and, from Eq. (3.3-7), oscillatory errors in the angular rate measured by the gyro. These can combine with the true rates to rectify in the strapdown attitude computations.

Suppose an angular oscillation takes place about the x-axis of a two-degree-of-freedom strapdown gyro. The frequency is assumed to be well within the gyro bandwidth. The indicated outputs of the gyro will be $\hat{\Omega}_x = \Omega_x + \delta\Omega_x$ and $\hat{\Omega}_y = \delta\Omega_y$. From Eq. (3.3-7) one component of $\delta\Omega_y$ will be $A\dot{\Omega}_x/H$. In particular, if

$$\Omega_x = W \sin \nu t$$

then

$$\delta\Omega_y = -\frac{AW\nu}{H} \cos \nu t + \dots$$

The angular error in coordinate frame orientation caused by $\delta\Omega_y$ will be

$$\begin{aligned} e_y &= \int \delta\Omega_y dt \\ &= \frac{AW}{H} \sin \nu t + \dots \end{aligned}$$

The indicated angular velocity about the x-axis of both the gyro and the system will be misresolved because of e_y , giving an apparent drift rate about a third (approximately orthogonal) axis, z, which, to first order, has a value of

$$\begin{aligned} \epsilon_z &\approx \Omega_x e_y \\ &= \frac{AW^2}{H} \sin^2 \nu t + \dots \\ &= \frac{AW^2}{2H} + \dots \end{aligned} \quad (4.1-20)$$

Equation (4.1-20) identifies a constant drift of the system coordinate frame which results from a properly phased sinusoidal gyro error and a sinusoidal true motion about another system axis. This general type of error is called pseudo-coning. Its cause here is the angular acceleration sensitivity of the gyro, as indicated in Eq. (3.3-7). Angular acceleration sensitivity can also contribute to cross coupling type errors which rectify at the sensor level.

4.2 SPECIFIC FORCE ERROR TORQUES

The equations for the torques generated by specific force are derived in Appendix B. Equations (B-8) show these torques to be

$$\epsilon_{x_{sf}} = -m\delta_{zy}f_x + m^2 \left[K_{zx}f_yf_x - K_{yx}f_zf_x + K_{zy}f_y^2 - K_{yz}f_z^2 + (K_{zz} - K_{yy})f_yf_z \right] \quad (4.2-1)$$

and

$$\epsilon_{y_{sf}} = m \delta_z^2 f_x^2 + m^2 \left[K_{xy} f_z f_y - K_{zy} f_x f_y + K_{xz} f_z^2 - K_{zx} f_x^2 + (K_{xx} - K_{zz}) f_z f_x \right] \quad (4.2-2)$$

where

m is the rotor mass

δ_z is the z component of the displacement of the rotor center of mass from the center of support*

f_x, f_y, f_z are the components of specific force

K_{ij} are the elements of the rotor compliance matrix:**

4.2.1 Mass Unbalance

The mass unbalance error torques are

$$\epsilon_{x_{unbal}} = -m \delta_z f_y \quad (4.2-3)$$

$$\epsilon_{y_{unbal}} = m \delta_z f_x \quad (4.2-4)$$

In a vibratory environment, these terms average to zero. When the specific force field has a non-zero average component, the average mass unbalance torques are also non-zero.

* The center of support can be defined with reference to Fig. 2.1-1 as the point of intersection of the rotor-to-gimbal and gimbal-to-shaft flexure mount axes. (For a multi-gimbal gyro with the gimbals arranged in parallel, all flexure mounts ideally intersect at a point.)

** These compliance matrix elements are defined here in exactly the same way as they are for a single-degree-of-freedom gyro (see Ref. 8).

4.2.2 Compliance

The compliance torques are given by

$$\epsilon_{x_{comp}} = m^2 \left[K_{zx} f_y f_x - K_{yz} f_z f_x + K_{zy} f_z^2 - K_{yz} f_z^2 + (K_{zz} - K_{yy}) f_y f_z \right] \quad (4.2-5)$$

$$\epsilon_{y_{comp}} = m^2 \left[K_{xy} f_z f_y - K_{zx} f_x f_y + K_{xz} f_z^2 - K_{zx} f_x^2 + (K_{xx} - K_{zz}) f_z f_x \right] \quad (4.2-6)$$

The compliance terms (also referred to as the "g squared" terms) produce non-zero average torques in a zero average vibration field because of the rectification that occurs. For example, assume that all cross compliances K_{ij} , $i \neq j$, are zero and that

$$f_x = f_y = f_z = f \sin \nu t \quad (4.2-7)$$

where ν is the linear vibration frequency. Then

$$\begin{bmatrix} \epsilon_{x_{comp}} \\ \epsilon_{y_{comp}} \end{bmatrix} = (mf)^2 \begin{bmatrix} K_{zz} - K_{yy} \\ K_{xx} - K_{zz} \end{bmatrix} \sin^2 \nu t \quad (4.2-8)$$

This torque has an average value

$$\begin{bmatrix} \overline{\epsilon_{x_{comp}}} \\ \overline{\epsilon_{y_{comp}}} \end{bmatrix} = \frac{1}{2} (mf)^2 \begin{bmatrix} K_{zz} - K_{yy} \\ K_{xx} - K_{zz} \end{bmatrix} \quad (4.2-9)$$

which is non-zero even though the specific force components of Eq. (4.2-7) each have an average value of zero.

4.3 DETUNING TORQUES

Equations (2.2-1) and (2.2-2) show two terms that exist in the general error model that vanish under the condition referred to as tuning. They are

$$\epsilon_x = \left[k - \left(a - \frac{c}{2} \right) \omega_s^2 \right] \theta_x \quad (4.3-1)$$

$$\epsilon_y = \left[k - \left(a - \frac{c}{2} \right) \omega_s^2 \right] \theta_y \quad (4.3-2)$$

where

k is the flexure-support torsion constant

a, c are gimbal moments of inertia

ω_s is rotor spin angular velocity.

The tuning condition, for which these terms vanish, is

$$\omega_s = \omega_t = \sqrt{\frac{k}{a - \frac{c}{2}}} \quad (4.3-3)$$

When $\omega_s = \omega_t$, the gyro is said to be tuned.

Suppose that

$$\omega_s = \omega_t + \delta\omega \quad (4.3-4)$$

Then

$$\omega_s^2 = \omega_t^2 + 2\delta\omega\omega_t + \delta\omega^2 \quad (4.3-5)$$

If $\delta\omega$ is small compared to ω_t , as will be assumed here, the last term can be ignored. Combining Eqs. (4.3-1) through (4.3-5) gives the detuning error torque components

$$\epsilon_{x \text{ detune}} = -2\delta\omega \left[\sqrt{k \left(a - \frac{c}{2} \right)} \right] \theta_x \quad (4.3-6)$$

$$\epsilon_{y \text{ detune}} = -2\delta\omega \left[\sqrt{k \left(a - \frac{c}{2} \right)} \right] \theta_y \quad (4.3-7)$$

Since rotor speed can be controlled very precisely, the principal causes of detuning are likely to be errors in measuring the rotor inertias, a and c , and the flexure spring constant, k , or a shift in the value of the latter. Consequently, detuning is typically a quasi-static phenomenon. Using this point of view, Eqs. (4.3-6) and (4.3-7) are rewritten as

$$\begin{bmatrix} \epsilon_{x \text{ detune}} \\ \epsilon_{y \text{ detune}} \end{bmatrix} = -k_{\text{detune}} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (4.3-8)$$

where k_{detune} is the quasi-static detuning factor

$$k_{\text{detune}} = 2\delta\omega \sqrt{k \left(a - \frac{c}{2} \right)} \quad (4.3-9)$$

At constant or low frequency input rates, Eq. (4.1-18) can be used in Eq. (4.3-8) to obtain

$$\begin{bmatrix} \epsilon_{x \text{ detune}} \\ \epsilon_{y \text{ detune}} \end{bmatrix} = k_{\text{detune}} k_q \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} \quad (4.3-10)$$

For a tight rebalance loop and accurate tuning k_q and k_{detune} will both be small. Consequently the detuning torques will likewise be small. It is noted that when the flexure mounted gyro is not used as a strapdown instrument, θ_x and θ_y may not necessarily be small and the detuning error torque may have correspondingly greater significance.

4.4 SUMMARY

Error torques excited by angular motion, specific force, and rotor speed variations have been discussed in this chapter. The most significant term for each torque mechanism is tabulated in Table 4.4-1.

TABLE 4.4-1
ERROR TORQUE SUMMARY

Name	x Component	y Component	Approximation Valid for	Full Expression Given in
Anisoinertia (Low Frequency)	$\frac{C}{2} \Omega_y \Omega_z$	$-\frac{C}{2} \Omega_x \Omega_z$	Low frequencies	Section 4.1.1
Anisoinertia (High Frequency)	$-\frac{C}{2} \Omega_y \Omega_z$	$\frac{C}{2} \Omega_x \Omega_z$	High frequencies	Section 4.1.1
Cross Coupling	$-k_q \Omega_x \Omega_z$	$-k_q \Omega_y \Omega_z$	Low frequencies	Section 4.1.2
Angular Acceleration Sensitivity	$-\frac{A \Omega_y}{H}$	$\frac{A \Omega_x}{H}$	All frequencies	Section 4.1.4
Misalignment	$-k_q H \Omega_x \Omega_y \frac{\Omega_z}{2 \omega_0}$	$-k_q H \Omega_y \Omega_x \frac{\Omega_z}{2 \omega_0}$	Low frequencies	Section 4.1.3
Mass Unbalance	$-m \delta_z f_y$	$m \delta_z f_x$	All frequencies	Section 4.2.1
Compliance	$m^2 (K_{zx} - K_{yy}) f_y f_z$	$m^2 (K_{yx} - K_{zz}) f_x f_z$	All frequencies	Section 4.2.2
Detuning	$k_{\text{detune}} k_q \Omega_x$	$k_{\text{detune}} k_q \Omega_y$	Low frequencies	Section 4.1.1

5. SYSTEM ERROR GENERATION

In this chapter, the characteristics of the Teledyne SDG-1 flexure-mounted gyro (Ref. 9), used as a strapdown sensor, are inserted into the TASC System Error Prediction Program (SEPP) to study motion-induced errors in that instrument. The SEPP is briefly described in Section 5.1. In Section 5.2 we discuss the results of the SEPP study, namely the dynamic errors of the gyro when subjected to a realistic dynamic motion environment. A brief comparison of the performance of the SDG-1 gyro with that of another gyro in the same environment is also presented.

5.1 SYSTEM ERROR PREDICTION PROGRAM

The System Error Prediction Program is used to compute system level attitude drift rates and acceleration measurement errors resulting from the interaction of strapdown IMU gyro error mechanisms with angular and linear vibrations. The information flow in the SEPP program is illustrated in Fig. 5.1-1. The motion environment can be described in detail by density spectra which relate the magnitude of the in-phase and quadrature components of all possible pairs of angular and linear vibrational motions in three dimensional space. Thirty six unique spectral density functions exist. (In the usual situation where such a mass of data is not available, a reduced number of spectra can be used.) The gyro and accelerometer error parameters (moments of inertia, rms scale factor errors, etc.) and the sensor rebalance loop transfer functions are included in the error computations. The program does not treat nonlinear rebalance

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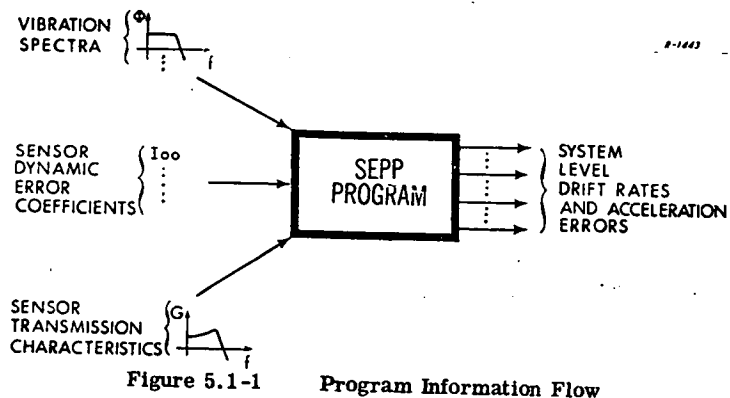
For a tight rebalance loop and accurate tuning k_q and k_{detune} will both be small. Consequently the detuning torques will likewise be small. It is noted that when the flexure mounted gyro is not used as a strapdown instrument, θ_x and θ_y may not necessarily be small and the detuning error torque may have correspondingly greater significance.

4.4 SUMMARY

Error torques excited by angular motion, specific force, and rotor speed variations have been discussed in this chapter. The most significant term for each torque mechanism is tabulated in Table 4.4-1.

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Cross Coupling	$-k_q \Omega_x \Omega_z$	$-k_q \Omega_y \Omega_z$	Low frequencies	Section 4.1.2
Angular Acceleration Sensitivity	$-\frac{A \Omega_y}{H}$	$\frac{A \Omega_x}{H}$	All frequencies	Section 4.1.4
Misalignment	$-k_q H \Omega_x \Omega_y \frac{\Omega_y}{2\omega_g}$	$-k_q H \Omega_y \Omega_x \frac{\Omega_x}{2\omega_g}$	Low frequencies	Section 4.1.3
Mass Unbalance	$-m \delta_z f_y$	$m \delta_z f_x$	All frequencies	Section 4.2.1
Compliance	$m^2 (K_{zz} - K_{yy}) f_y f_z$	$m^2 (K_{xx} - K_{zz}) f_x f_z$	All frequencies	Section 4.2.2
Detuning	$k_{\text{detune}} k_q \Omega_x$	$k_{\text{detune}} k_q \Omega_y$	Low frequencies	Section 4.1.1



loops in detail, but several pulse torqued loops can be approximated by equivalent linear continuous mechanizations which do meet the description required by the program (see Ref. 4). The SEPP program gives a detailed source-by-source breakdown of sensor-level and system-level attitude drift rate and errors in measuring specific force. It allows the major causes of vibration-induced errors to be quickly identified and provides an indication of how well specific systems can do in various dynamic environments.

5.2 STRAPDOWN SYSTEM ERRORS

The vibration power spectra used to represent the dynamic environment for the SEPP run is the helicopter environment described in Ref. 3. The Teledyne SDG-1 gyro rebalance loop characteristics are described in Ref. 9. Additional gyro parameters used in the runs are listed in Table 5.2-1. The accelerometers were assumed to be perfect

TABLE 5.2-1

SDG-1 PERFORMANCE PARAMETERS

Symbol	Value	Units	Name
m	2.50×10^2	gm	Rotor mass*
ω_s	6.28×10^2	rad/sec	Rotor angular velocity
C-A	7.04×10^2	gm cm ²	Rotor inertia difference
H	8.64×10^5	gm cm ² /sec	Rotor angular momentum
$K_{xx} - K_{zz}$	1.42×10^{-12}	cm/dyne	Major compliance*
ΔSF	50×10^{-6}		Scale factor asymmetry

in this analysis. ** Coordinate frames were employed as shown in Fig. 5.2-1. The SEPP runs were performed using two SDG-1 gyros with "Gyro 1" instrumenting the x and y axes and "Gyro 2" instrumenting the z axis of the helicopter coordinate frame. Both sensor-level errors (gyro drift rate) and system-level errors (drift in the computed coordinate frame) and errors in the processed specific force[†] arising from the sensitivity of the SDG-1 gyro to the CH-46C helicopter dynamic environment^{††} were studied.

* Obtained in a telephone conversation on December 27, 1972 with R. J. G. Craig of Teledyne Systems Company. Corresponds to 0.02 deg/hr/g².

** Acceleration errors to which imperfections in accelerometer contribute, have been found to be generally small in similar analyses that assume the same environment (see Ref. 4).

[†] Processed specific force refers to specific force that has been transformed into the stabilized coordinate frame and compensated for angular motion (coriolis) effects.

^{††} The helicopter environment was chosen since it is one of the few representative environments for which detailed angular and linear vibration spectral data are available. Also, it has been used to analyze errors in other gyros, thereby providing a basis for comparison.

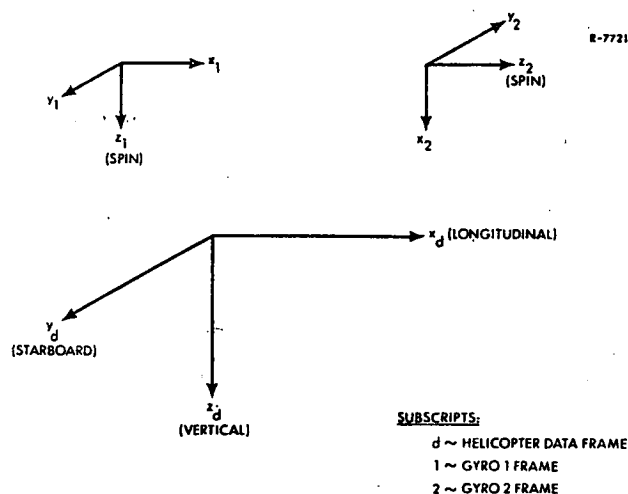


Figure 5.2-1 Coordinate Frames Employed in SEPP

At the sensor level, four error mechanisms were found to give errors in excess of 0.001 degree per hour. These mechanisms are:

- Anisoinertia
- Cross coupling
- Angular acceleration sensitivity
- Rebalance loop asymmetry

Discussion of each of the mechanisms, with the exception of the rebalance loop asymmetry, are presented in Chapter 4.

The cross coupling error is analogous to that found in a single-degree-of-freedom gyro, with one exception: It can be generated by analog rates about either gyro input axis. Because of the normal deflection of the rotor spin axis in the operation of the gyro, the actual input axes are not

aligned with the reference input axes, which are fixed in the gyro case. Hence the gyro can also sense a component of angular velocity about the nominal spin axis. This sensitivity, which is treated analytically in Section 4.1.2 is referred to as the cross coupling error mechanism.

Angular acceleration sensitivity, which is discussed in Section 4.1.4, is similar to output axis sensitivity in single-degree-of-freedom gyros. It is based on the fact that angular acceleration of the gyro case will cause output angles to grow. This growth is indistinguishable from that caused by the angular rates the gyro seeks to measure. In particular, an angular acceleration about one input axis has the same effect as an angular rate about the other input axis. Furthermore the error produced is often of just the right phase relative to true angular motion to produce a rectification in the system attitude calculations.

Rebalance loop asymmetry refers to the difference between the rebalance scale factor when torquing in a positive sense and the scale factor when torquing in a negative sense.

The gyro drift rate generated by the interaction of the helicopter dynamic environment with each of the error mechanisms listed above is shown graphically in Fig. 5.2-2. As can be seen from the difference in error components, the propagation characteristics are sensitive to gyro orientation.

In addition to the constant gyro drift rates induced by the dynamic motion environment, there is also induced a class of errors referred to as "system-level" errors. There are system-level drifts known as undetected coning and pseudo-coning and system-level acceleration errors known as undetected sculling and pseudo sculling. Basically, these errors arise from the failure of the gyros (and accelerometers also in the general

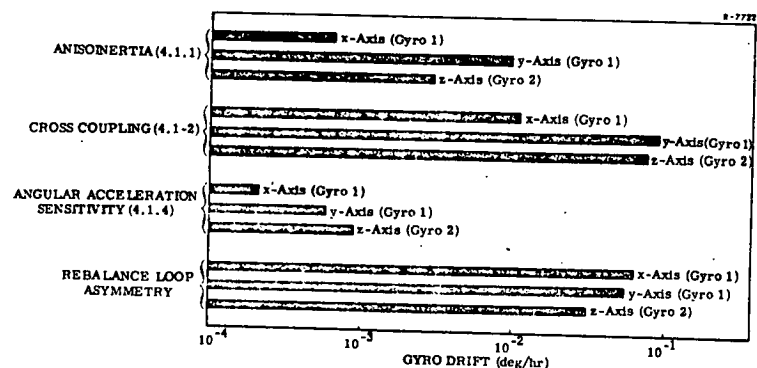


Figure 5.2-2 Vibration Induced Gyro Errors

case where perfect accelerometers have not been postulated) to detect high frequency oscillations or because oscillatory sensor errors are rectified in the system-level calculations. (For a treatment of the system level error phenomena, the reader is directed to Ref. 4.)

The system-level errors induced by the helicopter environment are shown graphically in Figs. 5.2-3 and 5.2-4. The only error mechanism that contributes in any meaningful way to either pseudo-coning or pseudo-sculling error is the cross axis sensitivity of the instrument to angular acceleration (analogous to the output axis sensitivity in a single-degree-of-freedom gyro). Undetected coning and sculling errors arise because the dynamic motion spectral bandwidth for the helicopter exceeds the measurement bandwidth of the gyros and hence there are high frequency angular oscillations not recognized by the gyro.

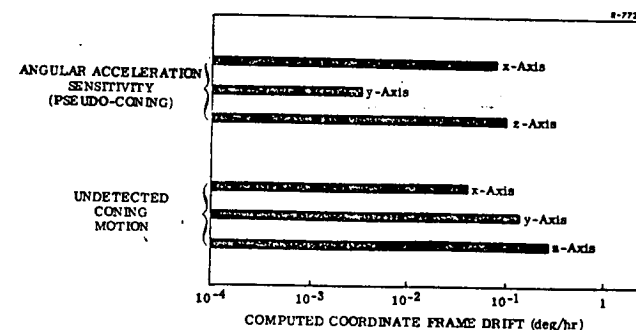


Figure 5.2-3 Vibration Induced System Drift Rates

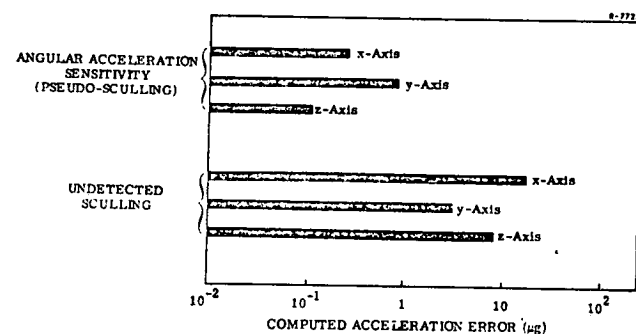


Figure 5.2-4 Vibration Induced Pseudo-Sculling Errors

Table 5.2-2 summarizes the sensor-level and system-level errors induced by the helicopter environment. This table has been prepared in the same format as the error summary for a system based on the Norden 1139 single-degree-of-freedom gyro (Table 4.2-3, Ref. 4) in order to facilitate a better understanding of the relative capabilities of the two types of gyros. The rebalance loop asymmetry error for the SDG-1 gyro is about half the error obtained in Ref. 4 for the Norden 113

TABLE 5.2-2

VIBRATION INDUCED ERRORS: SDG-1 GYRO

Vehicle:	CH-46C Twin Rotor Helicopter		
Gyro:	Teledyne SDG-1 Gyro		
Accelerometer:	Perfect Accelerometer Assumed		
Principal Sources of Vibration-Induced Error			
<u>Gyro Drift Rates (deg/hr)</u>	Axes		
	<u>X</u>	<u>Y</u>	<u>Z</u>
• Anisoinertia	-0.001	0.009	-0.003
• Cross Coupling	-0.011	0.091	0.081
• Angular Acceleration Sensitivity	—	0.001	0.001
• Rebalance Loop Asymmetry	0.065	0.061	0.035
<u>System Drift Rates (deg/hr)</u>			
• Angular Acceleration Sensitivity	0.075	0.003	0.201
• Undetected Coning Motion	0.042	-0.115	0.310
<u>System Acceleration Errors (μg)</u>			
• Angular Acceleration Sensitivity	0.28	0.40	-0.11
• Undetected Sculling Motion	-15.5	2.8	8.1

gyro. This is a direct consequence of the fact that the asymmetry parameter used for the SDG-1 was half that used for the Norden 1139. The most significant system-level error mechanism is the angular acceleration sensitivity exhibited by the SDG-1. This produces pseudo-coning errors at the system level that are of the same order of magnitude as those produced by a similar effect in the Norden 1139.

The undetected coning motion error is somewhat larger for the SDG-1 than for the Norden 1139. Recall that undetected coning motion is coning motion that occurs at a frequency above the detection bandwidth of

the gyro. The SDG-1 rebalance loop bandwidth used in computing the errors listed in Table 5.2-2 was 25 Hz while the bandwidth for the Norden 1139 was 70 Hz. This explains the difference in the undetected coning motion errors.

The SEPP program was also exercised for a system containing SDG-1 gyros in which the bandwidth of the gyros had been increased to about 50 Hz, through appropriate changes in rebalance loop gains and compensation. For purposes of comparison the major errors generated in that system are shown in Table 5.2-3. Tables 5.2-2 and 5.2-3 reveal that the gyro loop with a wider bandwidth has significantly smaller errors due to cross coupling and undetected coning and sculling motion. The cross coupling errors are reduced because the gyro has a stiffer rebalance loop and the undetected coning and sculling are less because the gyro correctly measures more of the motion spectrum. Comparing the two tables also shows that rebalance loop asymmetry errors go up, also because more of the motion spectrum is being passed by the gyro. Pseudo-coning errors caused by angular acceleration sensitivity are lower in Table 5.2-3, evidently because of the tighter rebalance loop. Of course, certain errors not related to bandwidth, such as anisoinertia, are unchanged. The total of the major deterministic errors (i.e., excluding those with random error coefficients such as rebalance loop asymmetry and rotor bearing compliance) is somewhat smaller for the gyro with the 50 Hz bandwidth, mainly because of the reduction in z-axis undetected coning. It is not clear that the larger bandwidth is generally superior for all motion environments.

TABLE 5.2-3
VIBRATION INDUCED ERRORS:
SDG-1 GYRO WITH WIDE BANDWIDTH

Vehicle:	CH-46C Twin Rotor Helicopter		
Gyro:	Teledyne SDG-1 Gyro (Modified)		
Accelerometer:	Perfect Accelerometer Assumed		
Principal Sources of Vibration-Induced Error			
<u>Gyro Drift Rates (deg/hr)</u>		Axes	
	$\bar{x}$	$\bar{y}$	$\bar{z}$
• Anisoinertia	-0.001	0.009	-0.003
• Cross Coupling	-0.001	—	-0.001
• Angular Acceleration Sensitivity	—	—	—
• Rebalance Loop Asymmetry	0.074	0.069	0.013
<u>System Drift Rates (deg/hr)</u>			
• Angular Acceleration Sensitivity	-0.067	0.002	0.168
• Undetected Coning Motion	—	—	—
<u>System Acceleration Errors (μg)</u>			
• Angular Acceleration Sensitivity	0.33	0.24	-0.11
• Undetected Sculling Motion	-2.3	-3.1	2.4

5.3 SUMMARY

The error propagation characteristics of a navigation system comprised of two Teledyne SDG-1 gyros and three perfect accelerometers has been studied using the TASC System Error Prediction Program. The dynamic motion environment of the CH-46C helicopter was used as the excitation function. The major error mechanisms and the magnitudes of the errors produced by these mechanisms are summarized in Tables 5.2-2 and 5.2-3 for two different sets of rebalance loop gains and compensations.

6.

SUMMARY AND CONCLUSIONS

This report has described the operation of a tuned, flexure-mounted gyro as a strapdown sensor. Transfer functions describing the gyro dynamic performance are derived and discussed for the case of a linear rebalance loop.

A dynamic error model is derived for the gyro, describing the instrument's sensitivity to

- Angular motion
- Specific force
- Rotor speed variations

In order to obtain an understanding of the performance capability of a strapdown navigation system utilizing a pair of tuned, flexure-mounted gyros, a sample case was investigated using the TASC System Error Propagation Program. The Teledyne SDG-1 gyro was chosen as the system gyro and the dynamic motion environment was assumed to be that of a CH-46 helicopter. Leading contributors to individual gyro drift were found to be:

- Cross coupling
- Rebalance loop scale factor asymmetry

The dominant contributors to system-level errors such as coordinate frame drift were:

- Sensitivity to angular accelerations
- Bandwidth limitations on the gyro

In general a system containing the SDG-1 gyro was found to have dynamic errors comparable in magnitude to a system (investigated in Ref. 4) which employs the Norden 1139 gyro.

A second set of calculations were made for an SDG-1 gyro with an increased bandwidth. The system errors calculated were smaller, but not significantly so.

As stated without change, it implies that all or many systems were compared with SDG-1 system and the (perhaps only) one that was similar to the SDG-1 system was the one with the Norden 1139.

APPENDIX A

ANGULAR MOTION TORQUES

In this appendix we compute the rotor's reaction torque due to gyro case angular velocity. The reaction torque vector $\underline{L}$ is given by

$$\underline{L} = [\dot{\underline{H}}]_i \quad (A-1)$$

where $\underline{H}$ is the rotor angular momentum vector and $[\]_i$ indicates that the derivative is taken with respect to an inertial coordinate frame. Applying the Law of Coriolis to this relationship yields the torque in terms of derivatives relative to the rotor-fixed coordinate frame

$$\underline{L} = [\dot{\underline{H}}]_r + \underline{\omega} \times \underline{H} \quad (A-2)$$

where $\underline{\omega}$ is the angular velocity of the rotor relative to an inertial frame and $[\]_r$ indicates that the derivative is taken with respect to a coordinate frame related to the rotor principal axes. In particular the coordinate frame we will use has two axes in the plane normal to the rotor axis of symmetry and one axis along the rotor spin (symmetry) axis. The first two axes are not fixed in the rotor, however. They rotate with respect to the rotor at a speed equal to and opposite in direction from rotor spin. The $\underline{\omega}$ vector is the sum of the gyro case angular velocity vector $\underline{\Omega}$ and the gimbal angle derivative vector $\underline{\dot{\theta}}$.

$$\underline{\omega} = \underline{\Omega} + \underline{\dot{\theta}} \quad (A-3)$$

In gyro case coordinates,

$$(\underline{\omega})_c = \begin{bmatrix} \Omega_x + \dot{\theta}_x \\ \Omega_y + \dot{\theta}_y \\ \Omega_z \end{bmatrix} \quad (A-4)$$

The coordinate frame most natural for Eq. (A-2) is the rotor referenced coordinate frame* since the axes are along principal inertia axes of the rotor.

To convert from case to rotor coordinates, we apply the coordinate transformation, which for small rotor deflection angles θ is given by Eq. (2.1-1):

$$C_r^c = \begin{bmatrix} 1 & 0 & -\theta_y \\ 0 & 1 & \theta_x \\ \theta_y & -\theta_x & 1 \end{bmatrix} \quad (A-5)$$

Applying this to Eq. (A-4) and neglecting second-order products of rotor deflection angles and derivatives, we obtain

$$(\underline{\omega})_r = \begin{bmatrix} \Omega_x + \dot{\theta}_x - \theta_y \Omega_z \\ \Omega_y + \dot{\theta}_y + \theta_x \Omega_z \\ \Omega_z + \theta_y \Omega_x - \theta_x \Omega_y \end{bmatrix} \quad (A-6)$$

* As explained in Chapter 2, the third (ζ) axis of the rotor referenced frame is parallel to the rotor axis of symmetry although this frame is non-rotating relative to the gyro case about this axis. For convenience, the rotor referenced frame is referred to as the rotor frame.

The angular momentum vector $\underline{H}$ is

$$\underline{H} = J(\underline{\omega} + \underline{\omega}_s) \quad (A-7)$$

where for rotor principal axes, the inertia tensor J is

$$J = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \quad (A-8)$$

and the rotor spin angular velocity $\underline{\omega}_s$ is, in rotor axes,

$$(\underline{\omega}_s)_r = \begin{bmatrix} 0 \\ 0 \\ \omega_s \end{bmatrix} \quad (A-9)$$

Combining the four previous equations, the angular momentum, resolved in the rotor coordinate frame, is

$$(\underline{H})_r = \begin{bmatrix} A(\Omega_x + \dot{\theta}_x - \theta_y \Omega_z) \\ A(\Omega_y + \dot{\theta}_y + \theta_x \Omega_z) \\ C(\Omega_z + \theta_y \Omega_x - \theta_x \Omega_y + \omega_s) \end{bmatrix} \quad (A-10)$$

By inserting Eqs. (A-6) and (A-10) into Eq. (A-2), we obtain in the rotor ξ, η, ζ coordinates shown in Fig. 2.1-1,

$$\begin{aligned} L_\xi &= A(\dot{\Omega}_x + \dot{\theta}_x) + H(\dot{\Omega}_y + \dot{\theta}_y) + (C-A)\Omega_y\Omega_z \\ &\quad + \theta_x[(C-A)(\Omega_z^2 - \Omega_y^2) + H\Omega_z] \\ &\quad + \theta_y[(C-A)\Omega_y\Omega_x - A\dot{\Omega}_z] \end{aligned} \quad (A-11)$$

$$\begin{aligned}
L_{\eta} = & A(\dot{\Omega}_y + \ddot{\theta}_y) - H(\dot{\Omega}_x + \ddot{\theta}_x) - (C-A)\Omega_x\Omega_z \\
& - \dot{\theta}_y \left[(C-A)(\Omega_z^2 - \Omega_x^2) - H\Omega_z \right] \\
& + \dot{\theta}_x \left[(C-A)\Omega_y\Omega_x + A\dot{\Omega}_z \right]
\end{aligned} \quad (A-12)$$

$$L_{\zeta} = C\dot{\Omega}_z \quad (A-13)$$

where the torque vector given above is

$$(\underline{L})_r = \begin{bmatrix} L_{\xi} \\ L_{\eta} \\ L_{\zeta} \end{bmatrix} \quad (A-14)$$

and is coordinatized in the rotor frame and where we have defined

$$H \triangleq C\omega_s \approx 2A\omega_s \quad (A-15)$$

In deriving Eqs. (A-11) through (A-13), it is assumed that second-order products involving gimbal angles or their derivatives such as $\theta_i\theta_j$, $\theta_i\dot{\theta}_j$, $\dot{\theta}_i\dot{\theta}_j$, $\theta_i\dot{\Omega}_j$, or $\dot{\theta}_i\Omega_j$ are negligible. Only the most significant term has been retained in Eq. (A-13).

The torque can be expressed in gyro case coordinates, using the rotor-to-case transformation

$$\underline{L}_c = (C_r^c)^T \underline{L}_r \quad (A-16)$$

Then, using Eqs. (A-14) and (A-5), we get

$$L_x = L_{\xi} + \theta_y L_{\zeta} \quad (A-17)$$

$$L_y = L_{\eta} - \theta_x L_{\zeta} \quad (A-18)$$

The z component of the torque vector is of no interest since it is about the rotor shaft axis and not a θ_x or θ_y rotor deflection pick-off axis. Using Eqs. (A-11) through (A-13) in Eqs. (A-17) and (A-18), we have

$$\begin{aligned}
L_x = & A(\dot{\Omega}_x + \ddot{\theta}_x) + H(\dot{\Omega}_y + \ddot{\theta}_y) + (C-A)\Omega_y\Omega_z \\
& + \dot{\theta}_x \left[(C-A)(\Omega_z^2 - \Omega_y^2) + H\Omega_z \right] \\
& + \dot{\theta}_y \left[(C-A)\Omega_y\Omega_x + (C-A)\dot{\Omega}_z \right]
\end{aligned} \quad (A-19)$$

and

$$\begin{aligned}
L_y = & A(\dot{\Omega}_y + \ddot{\theta}_y) - H(\dot{\Omega}_x + \ddot{\theta}_x) - (C-A)\Omega_x\Omega_z \\
& - \dot{\theta}_y \left[(C-A)(\Omega_z^2 - \Omega_x^2) - H\Omega_z \right] \\
& + \dot{\theta}_x \left[(C-A)\Omega_y\Omega_x - (C-A)\dot{\Omega}_z \right]
\end{aligned} \quad (A-20)$$

We may identify the terms $H\Omega_y$ in L_x and $-H\Omega_x$ in L_y as the "ideal" torque components, i.e., the components which in an ideal gyro would be attributed to the case angular motion. The other terms contribute to gyro errors as is shown in Chapters 3 and 4.

APPENDIX B

SPECIFIC FORCE TORQUES

Two types of error torques are considered in this Appendix to arise from the specific force experienced by the gyro case. They are the mass unbalance torques, given by

$$\underline{\epsilon}_1 = m (\underline{\delta} \times \underline{f}) \quad (\text{B-1})$$

where

m is the rotor mass

$\underline{\delta}$ is the displacement of the center of mass from the center of support

$\underline{f}$ is specific force

and the compliance torque, given by

$$\underline{\epsilon}_2 = m^2 (\underline{f} \times K \underline{f}) \quad (\text{B-2})$$

where K is the rotor compliance matrix.

For the mass unbalance torque, we will take $\underline{\delta}$ to be, in case coordinates,

$$\underline{\delta} = \begin{bmatrix} 0 \\ 0 \\ \delta_z \end{bmatrix} \quad (\text{B-3})$$

B-1

since the rotor is spinning at a virtually constant angular velocity and the z component alone has a non-zero average value. Likewise in case coordinates,

$$\underline{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \quad (\text{B-4})$$

Using Eqs. (B-3) and (B-4) in Eq. (B-1) yields for the x and y components

$$\begin{bmatrix} \epsilon_{1x} \\ \epsilon_{1y} \end{bmatrix} = m \delta_z \begin{bmatrix} -f_y \\ f_x \end{bmatrix} \quad (\text{B-5})$$

For the compliance torque, the most general form of the compliance matrix K is

$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \quad (\text{B-6})$$

Using this and Eq. (B-4) in Eq. (B-2) yields for the x and y components

$$\begin{bmatrix} \epsilon_{2x} \\ \epsilon_{2y} \end{bmatrix} = m^2 \begin{bmatrix} K_{zx} f_x f_x + K_{zy} f_y f_y + K_{zz} f_z f_z - K_{yx} f_z f_x - K_{yy} f_z f_y - K_{yz} f_z f_z \\ K_{xx} f_z f_x + K_{xy} f_z f_y + K_{xz} f_z f_z - K_{zx} f_x f_x - K_{zy} f_x f_y - K_{zz} f_x f_z \end{bmatrix} \quad (\text{B-7})$$

B-2

The total specific force induced torques are

$$\epsilon_{lin} = \epsilon_1 + \epsilon_2$$

which upon use of Eqs. (B-5) and (B-7) is

$$\begin{bmatrix} \epsilon_{xsf} \\ \epsilon_{ysf} \end{bmatrix} = m \begin{bmatrix} -\delta f \\ z y \end{bmatrix} + m^2 \begin{bmatrix} K_{zx} f f x + K_{zy} f^2 + K_{zz} f f z - K_{yx} f f x - K_{yy} f f y - K_{yz} f^2 \\ K_{xx} f f x + K_{xy} f f y + K_{xz} f^2 - K_{zx} f^2 - K_{zy} f f y - K_{zz} f f z \end{bmatrix}$$

(B-8)

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