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COINCIDENCE DEGREE AND PERIODIC SOLUTIONS
OF NEUTRAL EQUATIONS

by

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## 1. Introduction

This paper is devoted to the problem of existence of periodic solutions for some nonautonomous neutral functional differential equations. It is essentially an application of a basic theorem on the Fredholm alternative for periodic solutions of some linear neutral equations recently obtained by one of the authors [2] and of a generalized Leray-Schauder theory developed by the second one [3, 4]. Although their proofs are surprisingly simple, the obtained results are nontrivial extensions to the neutral case of a number of recent existence theorems for periodic solutions of functional differential equations. In particular, section 3 generalizes some existence criteria due to one of the authors [5] and a corresponding recent extension by J. Cronin [6], the example following Theorem 4.1 improves a condition for existence given by Lopes [14] for the equation of a transmission line problem, and Theorem 5.1 generalizes a result due to R. E. Fennell [7]. Lastly, criteria analogous to Theorem 5.2 for the retarded case can be found in [8]. For partly related results concerning periodic solutions of neutral functional differential equations, see [9].

## 2. Fredholm Alternative for Linear Equations

Let $C([a,b],\mathbb{R}^n)$ be the space of continuous functions from $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a fixed $r \geq 0$, let $C = C([-r,0],\mathbb{R}^n)$ with norm $|\varphi| = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|$

for $\varphi \in C$. If $x \in C([\sigma-r,\sigma+\delta],\mathbb{R}^n)$ for some $\delta > 0$, let $x_t \in C$, $t \in [\sigma,\sigma+\delta]$ be defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$. Suppose $\omega > 0$ fixed, $A: \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, $A(t+\omega)\varphi = A(t)\varphi$ for all $(t,\varphi) \in \mathbb{R} \times C$, $A(t)\varphi$ is linear in $\varphi$ and there exists a continuous function $\gamma: [0,\infty) \to \mathbb{R}$, $\gamma(0) = 0$, such that

$$|A(t)\varphi^s| \leq \gamma(s)|\varphi^s|, \quad 0 \leq s \leq r$$

for all $t \in \mathbb{R}$ and all functions $\varphi^s \in C$ such that $\varphi^s(\theta) = 0$ for $\theta \in [-r,-s]$. Let $D: \mathbb{R} \times C \to \mathbb{R}^n$ be defined by $D(t)\varphi = \varphi(0) - A(t)\varphi$. The operator $D$ is said to be <u>stable</u> if the zero solution of the functional equation $D(t)y_t = 0$ is uniformly asymptotically stable; that is, there are constants $K$, $\alpha > 0$ such that if $y(\varphi)$ is the solution of $D(t)y_t = 0$ with $y_0 = \varphi$, then

$$|y_t(\varphi)| \leq Ke^{-\alpha t}|\varphi|, \quad t \geq 0, \quad \varphi \in C. \tag{2.1}$$

Let $\mathcal{P}_\omega = \{x \in C(\mathbb{R},\mathbb{R}^n): x(t+\omega) = x(t),\ t \in \mathbb{R}\}$, $\mathcal{H}_\omega = \{H \in C(\mathbb{R},\mathbb{R}^n): H(0) = 0$ and $H(t) = \alpha t + h(t)$ for some $\alpha \in \mathbb{R}^n$, $h \in \mathcal{P}_\omega\}$. For any $h \in \mathcal{P}_\omega$, let $|h| = \sup_{0\leq t\leq\omega} |h(t)|$ and for any $H \in \mathcal{H}_\omega$, $H(t) = \alpha t + h(t)$, $\alpha \in \mathbb{R}^n$, $h \in \mathcal{P}_\omega$, let $|H| = |\alpha| + |h|$.

<u>Proposition 2.1.</u> <u>If</u> $D$ <u>is stable, then, for any</u> $c \in \mathbb{R}^n$, <u>there is a unique solution</u> $Mc$ <u>of the equation</u> $D(t)x_t = c$ <u>in</u> $\mathcal{P}_\omega$. <u>Furthermore,</u> $M$ <u>is a continuous linear operator from</u> $\mathbb{R}^n$ <u>to</u> $\mathcal{P}_\omega$.

<u>Proof</u>: Following the proof of Lemma 3.4 in [10], there are constants $b > 0$, $a > 0$ and an appropriate equivalent norm in $C$ such that the solution $x(\varphi, c)$ of $D(t)x_t = c$, $x_0 = \varphi$, satisfies $|x_t(\varphi, c)| \leq \leq |c|b + |\varphi| \exp(-at)$, $t \geq 0$, $\varphi \in C$, $c \in \mathbb{R}^n$. If $T\varphi = x_\omega(\varphi, c)$, then $T$ is a contraction mapping. Thus, if $d > 0$ is sufficiently large that $|c|b + d \exp(-a\omega) < d$, then $T$ has a unique fixed point such that $|\varphi| < d$. Consequently, there is a solution of the equation in $\mathcal{P}_\omega$. The fact that $D$ is stable implies the uniqueness, linearity and continuous dependence on $c$.

Let us rephrase Proposition 2.1 in a different way. Let $L: \mathcal{P}_\omega \to \mathcal{A}_\omega$ be the continuous linear mapping defined by

$$Lx(t) = D(t)x_t - D(0)x_0, \quad t \in \mathbb{R}.$$

Proposition 2.1 implies that

$$\ker L = \{x \in \mathcal{P}_\omega : \exists\, c \in \mathbb{R}^n \text{ with } x = Mc\}$$

is an n-dimensional subspace of $\mathcal{P}_\omega$. Let $P: \mathcal{P}_\omega \to \mathcal{P}_\omega$ be a continuous projection onto $\ker L$.

For the statement of the next proposition, let $Q: \mathcal{A}_\omega \to \mathcal{A}_\omega$ be the continuous projection defined by

$$QH(t) = \omega^{-1} H(\omega) t, \quad t \in \mathbb{R}.$$

Proposition 2.2. If $D$ is stable, then $\operatorname{Im} L = \ker Q$ and there is a continuous linear operator $K\colon \operatorname{Im} L \to \ker P$ such that $K$ is a right inverse of $L$. Thus, $L$ is a Fredholm operator with index 0.

Proof: The second proof given in [2] of the Fredholm alternative holds equally well for the equation $D(t)x_t = H(t)$. Thus, from [2], $\dim \ker L = \operatorname{codim} \operatorname{Im} L$. Proposition 2.1 implies $\dim \ker L = n$. For the equation $Lx = H$ to have a solution, it is clearly necessary that $H \in \ker Q$. Since $\operatorname{codim} \ker Q = n$, it follows that $\operatorname{Im} L = \ker Q$. The existence of the bounded right inverse follows from [2] or one may apply the closed graph theorem to $L|(I-P)\mathcal{P}_\omega$.

For the applications, it is necessary to be able to compute $\ker L$. In some simple cases, this is easily accomplished. For example, if $a(t) = a(t+\omega)$, $t \in \mathbb{R}$, is an $n \times n$ matrix with $|a(t)| \le k < 1$ for $t \in \mathbb{R}$, then the unique solution $Mc$ in $\mathcal{P}_\omega$ of

$$x(t) - a(t)x(t-r) = c \tag{2.2}$$

is given by

$$(Mc)(t) = \left[ I + \sum_{k=0}^{\infty} \prod_{j=0}^{k} a(t-jr) \right] c. \tag{2.3}$$

Another case particularly interesting in the applications is when $D(t)\varphi$ is independent of $t$. Then $\ker L = \{$constant functions in $\mathcal{P}_\omega\}$.

## 3. Existence Theorems for Nonlinear Equations

With the above notations, let us consider the neutral functional differential equation

$$\frac{d}{dt} D(t)x_t = f(t,x_t) \tag{3.1}$$

where $D$ is stable and $f: \mathbb{R} \times C \to \mathbb{R}^n$ is $\omega$-periodic with respect to $t$, continuous and takes bounded sets into bounded sets. If we define $N: \mathcal{P}_\omega \to \mathcal{A}_\omega$ by

$$Nx(t) = \int_0^t f(s,x_s)ds, \quad t \in \mathbb{R},$$

it is clear that finding $\omega$-periodic solutions of (3.1) is equivalent to solving the operator equation $Lx = Nx$ in $\mathcal{P}_\omega$ with $L$ defined in (2.7). To apply coincidence degree theory to this problem still requires that $N$ should be compact, i.e. continuous and taking bounded sets of $\mathcal{P}_\omega$ into relatively compact sets of $\mathcal{P}_\omega$.

<u>Proposition 3.1</u>. <u>Under the conditions listed above,</u> $N$ <u>is compact</u>.

<u>Proof</u>: The continuity is obvious. If $S > 0$ and $x \in \mathcal{P}_\omega$ is such that $|x| \leq S$, then $|x_t| \leq S$ for every $t \in S$ and thus $|f(s,x_s)| \leq T$ for some $T > 0$ and every $s \in \mathbb{R}$. It then follows easily that

$$|Nx(t)| \leq T(1+2\omega),\ t \in \mathbb{R}$$

and

$$|Nx(t_1) - Nx(t_2)| \leq T|t_1-t_2|,\quad t_1,\ t_2 \in \mathbb{R}$$

and Proposition 3.1 is a consequence of the Arzela-Ascoli theorem.

A direct application of Propositions 2.2, 3.1 above and of Theorem 5.1(i) of [3] gives the following

Theorem 3.1. If there exists an open bounded set $\Omega \subset \mathcal{P}_\omega$ whose boundary $\partial\Omega$ contains no $\omega$-periodic solution of (3.1) and if the $\mathcal{L}_+$-coincidence degree $d[(L,N),\Omega]$ is not zero, then equation (3.1) has at least one $\omega$-periodic solution in $\Omega$.

This result is quite general but requires the solution of two difficult problems, namely, finding $\Omega$ (it is an a priori bound problem) and estimating $d[(L,N),\Omega]$. Theorem 7.2 of [3] reduces this last question to the study of Brouwer degree of some well-defined finite-dimensional mapping if the a priori estimate condition is slightly strengthened. Let $g: \mathbb{R} \times C \times [0,1] \to \mathbb{R}^n$, $(t,\varphi,\lambda) \to g(t,\varphi,\lambda)$ be $\omega$-periodic with respect to $t$, continuous, taking bounded sets into bounded sets and such that

$$g(t,\varphi,1) \equiv f(t,\varphi),\ (t,\varphi) \in \mathbb{R} \times C. \qquad (3.2)$$

Let $M: \mathbb{R}^n \to \mathcal{P}_\omega$ be the mapping defined in Proposition 2.1 and define $\mathcal{G}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathcal{G}(a) = \omega^{-1}\int_0^\omega g(t,(Ma)_t,0)\,dt.$$

If $D(t)\varphi$ is independent of $t$, one can put $M = I$, the identity in this definition. Theorem 7.2 of [3] implies the following

Theorem 3.2. *Suppose there exists an open bounded set* $\Omega \subset \mathcal{P}_\omega$ *for which the following conditions are satisfied.*

1. *For each* $\lambda \in (0,1)$, *the equation*

$$\frac{d}{dt} D(t)x_t = \lambda g(t,x_t,\lambda)$$

*has no* $\omega$-*periodic solution on* $\partial\Omega$.

2. $\mathcal{G}(a) \neq 0$ *for every* $a \in \mathbb{R}^n$ *such that* $Ma$ *belongs to* $\partial\Omega$.

3. *The Brouwer degree* $d_B[\mathcal{G},\tilde{\Omega},0]$ *is not zero, where* $\tilde{\Omega} = \{a \in \mathbb{R}^n: Ma$ *belongs to* $\Omega\}$.

*Then equation* (3.1) *has at least one* $\omega$-*periodic solution in* $\Omega$.

Another useful special case of Theorem 3.1 follows at once from Theorem 7.3 of [3]. Suppose that the mapping $g$ defined above verifies (3.2) and the supplementary condition

$$g(t,-\varphi,0) = -g(t,\varphi,0), \quad (t,\varphi) \in \mathbb{R} \times C. \tag{3.3}$$

Theorem 3.3. Suppose there exists an open bounded set $\Omega \subset \mathcal{P}_\omega$ symmetric with respect to the origin, containing it and such that $\partial\Omega$ contains no $\omega$-periodic solution of each equation

$$\frac{d}{dt} D(t)x_t = g(t,x_t,\lambda), \quad \lambda \in [0,1]$$

with $g$ verifying (3.2) and (3.3). Then equation (3.1) has at least one $\omega$-periodic solution in $\Omega$.

Let us note that (3.3) will always be satisfied if $g(t,\varphi,0)$ is linear with respect to $\varphi$. Also, Theorems 3.1, 3.2 and 3.3 are respective generalizations of Theorems 2, 3 and 4 of [5] which all correspond to the case of retarded functional differential equations, i.e. $D\varphi = \varphi(0)$, and $\Omega$ an open ball. Also, an extension to the neutral case of Theorem 1 of [6] is easily obtained by a suitable choice of $\Omega$ and the properties of coincidence degree.

## 4. An Application

Let us consider the neutral equation

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} A_k x(t-\tau_k)\right] = \operatorname{grad} V[x(t)] + e(t) \tag{4.1}$$

where, $V: \mathbb{R}^n \to \mathbb{R}$ is of class $\mathscr{C}^1$, $e \in \mathscr{P}_\omega$, $\tau_k \in [-r,0)$ $(k = 1,2,\dots,N)$ and the $n \times n$ constant matrices $A_k$ are such that

$$\sum_{k=1}^{N} |A_k| = 1 - \alpha, \quad \alpha > 0. \tag{4.2}$$

Let $\bar{e}$ be the mean value of $e$.

<u>Theorem 4.1. If there exists</u> $R > 0$ <u>such that</u> $\bar{e} + \omega^{-1}\int_0^\omega \operatorname{grad} V[x(t)]dt \neq 0$ <u>for every</u> $x \in \mathscr{P}_\omega$ <u>satisfying</u> $\inf_{t\in R} |x(t)| \geq R$ <u>and if the Brouwer degree</u> $d_B[\bar{e} + \operatorname{grad} V, B(0,R),0]$ <u>is not zero, then equation</u> (4.1) <u>has at least one</u> $\omega$-<u>periodic solution.</u>

<u>Proof</u>: Let $|\cdot|$ and $\langle\cdot,\cdot\rangle$ respectively denote the Euclidean norm and the inner product in $\mathbb{R}^n$. It is well known [10] that condition (4.2) implies that the operator $D: \varphi \to \varphi(0) - \sum_{k=1}^{N} A_k\varphi(-\tau_k)$ is stable and the right hand side of (4.1) clearly takes bounded sets into bounded sets. Let us consider the family of equations

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} A_k x(t-\tau_k)\right] = \lambda \operatorname{grad} V[x(t)] + \lambda e(t), \qquad \lambda \in (0,1). \tag{4.3}$$

If $x$ is any $\omega$-periodic solution of (4.3) for some $\lambda \in (0,1)$ then

$x(t)$ must have a continuous first derivative (see [14]) and

$$\omega^{-1} \int_0^{\omega} \langle \dot{x}(t) - \sum_{k=1}^{N} a_k \dot{x}(t-\tau_k), \dot{x}(t) \rangle dt =$$

$$= \lambda\omega^{-1} \int_0^{\omega} \langle \text{grad } V[x(t)], \dot{x}(t) \rangle dt + \lambda\omega^{-1} \int_0^{\omega} \langle e(t), \dot{x}(t) \rangle dt,$$

which implies, using Schwarz inequality and (4.2),

$$\left( \omega^{-1} \int_0^{\omega} |\dot{x}(t)|^2 dt \right)^{1/2} \leq \alpha^{-1}\eta$$

with $\eta^2 = \omega^{-1} \int_0^{\omega} |e(t)|^2 dt$. Then, for every $t, t' \in [0,\omega]$, we have

$$|x(t) - x(t')| \leq \omega\alpha^{-1}\eta. \tag{4.4}$$

On the other hand, every $\omega$-periodic solution of (4.3) verifies the equation

$$\bar{e} + \omega^{-1} \int_0^{\omega} \text{grad } V[x(t)] dt = 0$$

and hence there must exist some $\sigma \in [0,\omega]$ for which $|x(\sigma)| < R$. Taking $t' = \sigma$ in (4.4) we obtain

$$|x| < R + \omega\alpha^{-1}\eta = S$$

for every $\omega$-periodic solution of (4.3). The result then follows from Theorem 3.2 by taking for $\Omega$ the open ball of center 0 and radius S in $\mathscr{P}_\omega$.

As an application of Theorem 4.1, let us consider the special case of a scalar equation with one delay,

$$\frac{d}{dt}[x(t) + ax(t-r)] = p(x) + e(t)$$

where $|a| < 1$, $e \in \mathscr{P}_\omega$ and $p(x)$ is a given function of $x$. This equation arises in a transmission line problem with a shunt across the line (see [13], [14]). Then, if $p$ is any continuous function such that $|p(x)| \to \infty$ if $|x| \to \infty$ and $p(x)p(-x) < 0$ for all $x$ with $|x|$ sufficiently large, there will exist one $\omega$-periodic solution. Using Liapunov functions, Lopes [14] has obtained the existence of an $\omega$-periodic solution of this special equation for $|a| < 1/2$ and $xp(x) \to +\infty$ as $|x| \to \infty$.

## 5. Neutral Functional Differential Equations with Quasibounded Nonlinearities.

We shall consider in this section $\omega$-periodic equations of the form

$$\frac{d}{dt} D(t)x_t = b(t,x_t) + f(t,x_t) \tag{5.1}$$

where D satisfies the conditions in section 2, $b\colon \mathbb{R} \times C \to \mathbb{R}^n$, $(t,\varphi) \to b(t,\varphi)$ is linear with respect to $\varphi$ and continuous,

$f: \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, takes bounded sets into bounded sets and is such that

$$\limsup_{|\varphi| \to \infty} (|\varphi|^{-1}|f(t,\varphi)|) = \inf_{0<\rho<\infty} \left( \sup_{|\varphi|\geq\rho} |\varphi|^{-1}|f(t,\varphi)|\right) = 0 \quad (5.2)$$

uniformly in $t \in \mathbb{R}$.

Let us recall that a mapping $F: X \to Y$ between normed spaces is <u>quasi-bounded</u> if the number $|\!|\!| F |\!|\!| = \limsup_{|x| \to \infty} |x|^{-1}|Fx|$ is finite, in which case it is called the <u>quasinorm</u> of $F$ [12]. We shall use in this section a mapping theorem of Granas for compact quasibounded perturbations of the identity [12] and a special case of its generalization in the frame of coincidence degree theory [4].

<u>Proposition 5.1. If $f$ satisfies the conditions above, then the mapping $N: \mathscr{P}_\omega \to \mathscr{A}_\omega$ defined by $Nx(t) = \int_0^t f(s,x_s)ds$, $t \in \mathbb{R}$, is compact, quasibounded and</u> $|\!|\!| N |\!|\!| = 0$.

<u>Proof</u>: The compactness follows from Proposition 3.1. Now, if $\varepsilon > 0$, it follows from (5.1) and the fact that $f$ takes bounded sets into bounded sets that there exist $\gamma(\varepsilon) > 0$ such that, for every $(t,\varphi) \in \mathbb{R} \times C$,

$$|f(t,\varphi)| \leq \varepsilon|\varphi| + \gamma.$$

Hence, for every $x \in \mathcal{P}_\omega$,

$$|Nx| = |\omega^{-1} \int_0^\omega f(s,x_s)\,ds| + \sup_{t\in[0,\omega]} |\int_0^t [f(t',x_{t'}) - \omega^{-1} \int_0^\omega f(s,x_s)\,ds]\,dt'|$$

$$\leq (1+2\omega)[\varepsilon|x| + \gamma(\varepsilon)]$$

which clearly implies $|N| = 0$.

Now we can prove the following

<u>Theorem 5.1</u>. <u>With</u> D, b <u>and</u> f <u>as above, suppose the linear equation</u>

$$\frac{d}{dt} D(t)x_t = b(t,x_t) \tag{5.3}$$

<u>has no nontrivial</u> $\omega$-<u>periodic solution</u>. <u>Then equation</u> (5.1) <u>has at least one</u> $\omega$-<u>periodic solution</u>.

<u>Proof</u>. The result is equivalent to solving the equation $Lx - Bx = Nx$ in $\mathcal{P}_\omega$ with $B: \mathcal{P}_\omega \to \mathcal{A}_\omega$ defined by $Bx(t) = \int_0^t b(s,x_s)\,ds$, $t \in \mathbb{R}$, and L,N as above. From Proposition 3.1 we know that B is a compact mapping and L being a continuous Fredholm mapping of index zero, the same is true for L - B [11]. As L - B is one-to-one by our assumption on (5.3) it will necessarily be onto and such that $(L-B)^{-1}: \mathcal{A}_\omega \to \mathcal{P}_\omega$ is continuous. The proof of Theorem 5.1 is then equivalent to the fixed point problem $x = (L-B)^{-1}Nx$ in $\mathcal{P}_\omega$ with

$(L-B)^{-1}N$ clearly compact, quasibounded and of quasinorm zero. The result then follows from Granas' theorem.

An interesting problem is now to try to drop the assumption about the nonexistence of nontrivial $\omega$-periodic solutions for (5.3). It is clear from the Fredholm alternative that conditions upon $f$ will then be needed. We consider here the simplest case, i.e. $b(t,\varphi) \equiv 0$. Let us define $\mathscr{F}: \mathbb{R}^n \to \mathbb{R}^n$ by $\mathscr{F}(a) = \omega^{-1} \int_0^\omega f(t,(Ma)_t)\,dt$. If $D(t)\varphi$ is independent of $t$, take $M = I$, the identity, in the definition of $\mathscr{F}$.

<u>Theorem 5.2</u>. <u>Let</u> $D$ <u>and</u> $f$ <u>be as above and suppose there exists</u> $\mu > 0$ <u>such that</u> $|(Mc)(t)| \geq \mu|c|$ <u>for every</u> $t \in \mathbb{R}$ <u>and every</u> $c \in \mathbb{R}^n$. <u>If there exists</u> $R_1 > 0$ <u>such that</u> $\int_0^\omega f(s,x_s)\,ds \neq 0$ <u>for every</u> $x \in \mathscr{P}_\omega$ <u>verifying</u> $\inf_{t\in\mathbb{R}} |x(t)| \geq R_1$ <u>and if</u> $d_B[\mathscr{F},\Omega_R,0]$ <u>is not zero, where</u> $\Omega_R = \{a \in \mathbb{R}^n: Ma \in B(0,R)\}$ <u>and</u> $R = \mu^{-1}|M|R_1$, <u>then the equation</u>

$$\frac{d}{dt} D(t)x_t = f(t,x_t) \tag{5.4}$$

<u>has at least one</u> $\omega$-<u>periodic solution</u>.

<u>Proof</u>. We will use Propositions 3.1 and 5.1 above and Theorem 4.1 of [4]. The proof will be complete if we show the existence of $\alpha \geq 0$ and $R > 0$ such that every $\omega$-periodic solution $x$ of (5.4)

satisfies the inequality

$$|Px| < \alpha|(I-P)x| + R. \tag{5.5}$$

If $x$ is any $\omega$-periodic solution of (5.4), then $\int_0^\omega f(s,x_s)ds = 0$ and hence there will exist $\sigma \in [0,\omega]$ such that $|x(\sigma)| < R_1$. Therefore, if $c \in \mathbb{R}^n$ is such that $Px = Mc$, we have

$$\mu|c| \le |(Mc)(\sigma)| < R_1 + |(I-P)x(\sigma)| \le R_1 + |(I-P)x|,$$

which implies

$$|Px| \le |M|\ |c| < \mu^{-1}|M|R_1 + \mu^{-1}|M|\ |(I-P)x|$$

and (5.5) holds with $\alpha = \mu^{-1}|M|$ and $R = \mu^{-1}|M|R_1$.

Let us remark that if $D(t)\varphi$ is independent of $t$, ker $L$ is the subspace of $\mathscr{P}_\omega$ of constant functions and the positive number $\mu$ involved above always exists and can be taken equal to one. Hence, a simple example for Theorem 5.2 is given by the scalar equation

$$\frac{d}{dt}[x(t) - ax(t-r)] = g(x_t) + e(t) \tag{5.6}$$

where $a \in (-1,1)$, $e \in \mathscr{P}_\omega$ has mean value zero, $g\colon C \to \mathbb{R}^n$ is continuous, quasibounded with quasinorm zero, takes bounded sets into bounded sets and is such that, for some $R > 0$, either $g(\varphi)\varphi(\theta) > 0$ or $g(\varphi)\varphi(\theta) < 0$, for every $\theta \in [-r,0]$ and every $\varphi \in C$ such that $\inf_{[-r,0]} |\varphi(\theta)| \geq R$. It is the case, for example, for the equation

$$\frac{d}{dt}[x(t) + ax(t-r)] = b\,\frac{x(t-r)}{|x(t-r)|^{1/2}} + e(t)$$

if $|a| < 1$, $b \neq 0$, $e \in \mathscr{P}_\omega$ has mean value zero and $y/|y|^{1/2}$ is extended by 0 at $y = 0$.

To apply Theorem 5.2 to a scalar equation of the form

$$\frac{d}{dt}[x(t) - a(t)x(t-r)] = g(x_t) + e(t)$$

with $g$ and $e$ as above and $a \in \mathscr{P}_\omega$, the crucial point is to prove the existence of $\mu > 0$ such that $|(Mc)(t)| \geq \mu|c|$ for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$. The following propositions give answers to this problem. For the sake of brevity, we shall say that the operator $M$ associated with the scalar equation $x(t) - a(t)x(t-r) = c$ has <u>property</u> $\mu$ if there exists $\mu > 0$ such that $|(Mc)(t)| \geq \mu|c|$ for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$.

<u>Proposition 5.2</u>. <u>If</u> $|a(t)| \leq k$ <u>for all</u> $t \in \mathbb{R}$ <u>and</u> $k \in [0,1/2)$, <u>then</u> $M$ <u>has property</u> $\mu$.

Proof. From the relation

$$(Mc)(t) - a(t)(Mc)(t-r) = c$$

one obtains easily $|Mc| \leq (1-k)^{-1}|c|$ and hence

$$|(Mc)(t)| \geq |c| - k|(Mc)(t-r)| \geq |c|(1-2k)(1-k)^{-1}$$

for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$.

The following example will show that Proposition 5.2 is the best possible without supplementary assumptions on the oscillatory character of $a(t)$. Let $\omega = p$, $p$ a positive integer, $r = 1$ and $a(t)$ be a p-periodic continuous function such that $|a(t)| \leq k < 1$, $t \in \mathbb{R}$, $a(0) = -k$, $a(m) = k$ $(m = 1,2,\ldots,p-1)$. Then, if $x(t)$ is the solution of $x(t) - a(t)x(t-1) = 1$, property $\mu$ clearly will not hold if we exhibit one $t \in [0,p]$ such that $x(t) = 0$. Using formula (2.3) and the form of $a(t)$ we have

$$\begin{aligned} x(0) &= 1 - k(1+k+k^2+\cdots+k^{p-1} - k^p(1+k+\cdots+k^{p-1} - \cdots \\ &= 1 - k(\frac{1-k^p}{1-k} - k^p(\frac{1-k^p}{1-k} - \cdots \\ &= 1 - k(\frac{1-k^p}{1-k})(1-k^p+k^{2p}-\cdots) = 1 - k(1-k^p)(1-k)^{-1}(1+k^p)^{-1} \\ &= (1+k^p)^{-1}(1-k)^{-1}(1-2k+k^p) = \gamma(k). \end{aligned}$$

It is easy to show that $\gamma(k)$ is strictly positive in $[0,1/2)$ and is strictly negative in a neighborhood of 1. Thus, $x(0) = 0$ for some $k \in [\frac{1}{2},1)$ and this zero is arbitrary close to $\frac{1}{2}$ if we take $p$ sufficiently large, as follows at once from the form of $\gamma(k)$.

It is, however, possible to improve the condition upon $k$ when $a(t)$ has a constant sign as follows from

<u>Proposition 5.3.</u> *If* $|a(t)| \leq k < 1$ *and* $a(t)$ *has constant sign, then* $M$ *has property* $\mu$.

<u>Proof.</u> Let us first consider the case where $0 \leq a(t) \leq k$ for every $t \in \mathbb{R}$. Then $M$ has property $\mu$ because

$$|(Mc)(t)| = |1+a(t)+a(t)a(t-r)+\cdots|\ |c| \geq |c|.$$

Now suppose that $-k \leq a(t) \leq 0$ for every $t \in \mathbb{R}$. It is clear that the unique $\omega$-periodic solution $x$ of $x(t) - a(t)x(t-r) = c$ is the limit of the sequence $\{x^m(t)\}$ of $\omega$-periodic functions defined by

$$x^0(t) = c, \quad x^{m+1}(t) = c + a(t)x^m(t-r), \quad m = 0,1,2,\ldots .$$

If $c > 0$, then $x^1(t) = [1+a(t)]c \geq (1-k)c > 0$, $x^2(t) = c + a(t)x^1(t-r)$ $\geq [1-k(1-k)]c = (1-k+k^2)c > 0$, and if $x^m(t) \geq [1-k+k^2+\ldots+(-1)^m k^m]c > 0$, then $x^{m+1}(t) = c + a(t)x^m(t-r) \geq c\{1-k[1-k+\cdots+(-1)^m k^m]\} =$ $= [1-k+k^2+\cdots+(-1)^{m+1}k^{m+1}]c > 0$. Hence, by induction and passing to the limit, we have $|(Mc)(t)| \geq (1+k)^{-1}|c|$. Finally, suppose that $c < 0$. Then,

$$c \leq x^1(t) = c + a(t)c \leq (1-k)c < 0$$

and hence

$$c \leq x^2(t) = c + a(t)x^1(t-r) \leq (1-k)c < 0.$$

If we suppose that $c \leq x^{m-1}(t) \leq (1-k)c < 0$, then $0 \leq a(t)x^{m-1}(t-r) \leq -ck$ and hence

$$0 > (1-k)c \geq x^m(t) = c + a(t)x^{m-1}(t) \geq c.$$

By induction and passing to the limit we have $0 > (1-k)c \geq (Mc)(t) \geq c$ and hence $|(Mc)(t)| \geq (1-k)|c|$, which achieves the proof.

Corollary 5.1. *If* $a$ *is a constant verifying* $0 < |a| < 1$ *then, for every* $b \in \mathcal{P}_\omega$ *such that* $|b(t)| < \min(|a|, |1-a|)$, $t \in \mathbb{R}$, *the mapping* $M$ *associated with* $x(t) - [a+b(t)]x(t-r) = c$ *has property* $\mu$.

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