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Approximations to the Plasma Dispersion Function

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by

A. L. Brinca

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INSTITUTE FOR PLASMA RESEARCH
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1. INTRODUCTION

Linear wave propagation in hot collisionless plasmas is described by the linearized Vlasov and Maxwell equations. In uniform media, the utilization of spatial and temporal transforms of those equations leads to the consideration of integrals of the type (Hilbert transform)

$$I(\xi) = \int_{-\infty}^{\infty} du \, \frac{g(u)}{u - \xi} \qquad (\xi = \xi_r + i\xi_i) \ , \tag{1}$$

where $g(u)$ is a functional of the equilibrium velocity distribution and ξ_r, in the initial value problem, represents a Landau or gyro resonant normalized velocity.

The prominence of Maxwellian velocity distributions has led to the tabulation[1] of the plasma dispersion function $Z(\xi)$, defined as

$$Z(\xi) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} du \, \frac{\exp(-u^2)}{u - \xi} \qquad (\xi_i > 0) \ , \tag{2}$$

and as the analytic continuation of this for $\xi_i \lesssim 0$. The essential singularities of $\exp(-u^2)$ at $u = \pm i \infty$ preclude the computation of (2) by contour integration using an infinite semicircle and Jordan's lemma. Thus a closed form for the quadrature defining $Z(\xi)$ does not exist, and its utilization is awkward.

It is our purpose here to analyze and compare two simple approximations to $Z(\xi)$ and $Z'(\xi) = dZ(\xi)/d\xi$. The first approximation is based on the utilization of resonance velocity distributions, and is derived in Section 2. The second approximation is given in Section 3 and was proposed by Fried et al.[2] Section 4 compares these two approximations. Section 5 applies them to Landau and whistler waves, and Section 6 discusses the results, commenting on the possible improvement of the Fried et al. approximation.

2. THE RESONANCE DISTRIBUTION APPROXIMATION

The Maxwellian velocity distribution

$$F_M(v) = \frac{1}{(2\pi)^{1/2}\, v_\theta} \exp\left(-\frac{v^2}{2v_\theta^2}\right),$$

$$\langle v^2\rangle_M = \int_{-\infty}^{\infty} dv\, F_M(v)\, v^2 = v_\theta^2 \quad , \int_{-\infty}^{\infty} dv\, f_M(v) = 1 , \tag{3}$$

is sometimes[3] approximated by the n^{th}-order resonance distribution

$$F_{Rn}(v) = \frac{\left[2v_\theta(2n-3)^{1/2}\right]^{2n-1}}{2\pi} \frac{[(n-1)!]^2}{(2n-2)!} \frac{1}{\left[v^2+(2n-3)v_\theta^2\right]^n} ,$$

$$\langle v^2\rangle_{Rn} = v_\theta^2 \qquad , \int_{-\infty}^{\infty} dv\, F_{Rn}(v) = 1 . \tag{4}$$

In terms of the complex v-plane, the essential singularities of $F_M(v)$ at $v = \pm i\infty$ are simulated by two n^{th}-order poles at $v = \pm i(2n-3)^{1/2} v_\theta$. These poles tend to $v = \pm i\infty$ as $n \to \infty$, and , as shown in Appendix A, the Maxwellian distribution is retrieved in the limiting process:

$$\lim_{n\to\infty} F_{Rn}(v) = F_M(v) . \tag{5}$$

This result suggests that $Z(\xi)$ might be approximated by

$$Z_{Rn}(\xi) = \frac{(2y_n)^{2n-1}}{2\pi} \frac{[(n-1)!]^2}{(2n-2)!} \int_{-\infty}^{\infty} \frac{du}{(u^2+y_n^2)^n(u-\xi)} \qquad (\xi_i > 0) \tag{6}$$

where $y_n = (n-3/2)^{1/2}$ and $n \geq 2$. The quadrature can now be performed by contour integration. Closing the path with an infinite semicircle in the lower complex u-plane, we obtain ($\xi_1 > 0$)

$$\int_{-\infty}^{\infty} \frac{du}{(u^2+y_n^2)^n(u-\xi)} = -\frac{2\pi i}{(n-1)!} \lim_{u=-iy_n} \frac{d^{n-1}}{du^{n-1}} \left[\frac{1}{(u-iy_n)^n(u-\xi)}\right] , \tag{7}$$

so that Z_{Rn} becomes

$$Z_{Rn}(\xi) = -\frac{(n-1)!}{(2n-2)!} \sum_{m=0}^{n-1} \frac{(2n-m-2)!}{(n-m-1)!} \frac{(i2y_n)^m}{(\xi+iy_n)^{m+1}} \qquad (\xi_1 > 0) . \tag{8}$$

Integration by parts readily shows that $Z'(\xi) = dZ(\xi)/d\xi$ can be obtained from (2) by substituting $d[\exp(-u^2)]/du$ for $\exp(-u^2)$. Approximating the derivative of the Maxwellian by the derivative of the resonance distribution amounts to term by term differentiation of (8). The resonance distribution approximation of $Z'(\xi)$ is then

$$Z'_{Rn}(\xi) = \frac{(n-1)!}{(2n-2)!} \sum_{m=1}^{n} \frac{(2n-m-1)!}{(n-m)!} \frac{m(i2y_n)^{m-1}}{(\xi+iy_n)^{m+1}} \qquad (\xi_i > 0) . \tag{9}$$

We note that it is sufficient to find an approximation to $Z(\xi)$ and $Z'(\xi)$ in one of the four quadrants of the ξ-plane because of the symmetry properties of $Z(\xi)$ and its derivatives,[4]

$$Z^{(n)}(-\xi^*) = (-1)^{n+1} [Z^{(n)}(\xi)]^* , \tag{10}$$

and the knowledge of their analytic continuations into the lower ξ-plane,[4]

$$Z^{(n)}(\xi) = \left[Z^{(n)}(\xi^*)\right]^* + 2i\,\pi^{1/2} (-1)^n H_n(\xi)\exp(-\xi^2) \qquad (\xi_1 < 0) , \tag{11}$$

where the Hermite polynomials $H_n(\xi)$ satisfy the recurrence relation

$$H_n(\xi) = 2\xi H_{n-1}(\xi) - H'_{n-1}(\xi) \quad , \quad H_0(\xi) = 1 .$$

3. THE TWO-POLE APPROXIMATION

Instead of approximating the Maxwellian distribution and then computing its Hilbert transform, as done in Section 2, Fried et al.[2] have approximated Z and Z' directly, obtaining for $\xi_i > 0$

$$Z_F(\xi) = \frac{1}{2\hat{a}_r}\left[\frac{1}{a(a-\xi)} - \frac{1}{a^*(a^*+\xi)}\right] ,$$

$$\frac{1}{a} = 0.55 + i\,\frac{\pi^{1/2}}{2} = \hat{a}_r + i\,\hat{a}_i , \tag{12}$$

and

$$Z_F'(\xi) = \frac{1}{2\hat{b}_r}\left[\frac{1}{b(b-\xi)^2} + \frac{1}{b^*(b^*+\xi)^2}\right]$$

$$\frac{1}{b} = 0.45 + i\,0.86 = \hat{b}_r + i\,\hat{b}_i . \tag{13}$$

In the lower half-plane, $\xi_i < 0$, Equation (11) is used to analytically continue Z_F and Z_F' .

Fried et al. have derived this form of $Z_F(\xi)$ by requiring that the two-pole approximation displayed the symmetry properties and asymptotic behavior of $Z(\xi)$. The imaginary part of $1/a$ was obtained by imposing the condition $Z(\xi=0) = Z_F(\xi=0)$, and the real part of $1/a$ was chosen, with an "eyeballing" procedure, to minimize $|Z-Z_F|$. In Section 6 we shall comment on other possible criteria to choose the value of $1/a$.

The derivation given in Appendix B, or the substitution in (1) of $g(u)$ by the $f_F(u)$ given below, shows that the form of the (velocity) distribution $f_F(u)$ implicitly utilized in the Fried et al. approximation of $Z(\xi)$ is

$$f_F(u) = \frac{4(a_r^2+a_i^2)}{(1.21+\pi)\pi^{1/2}}\ \frac{1}{[(u+a_r)^2 + a_i^2][(u-a_r)^2 + a_i^2]} . \tag{14}$$

Whereas the essential singularities of the Maxwellian at $u = \pm i\infty$ are simulated by two n^{th}-order poles at $u = \pm i(n-3/2)^{1/2}$ in the resonance distribution method, the approximation given by Fried et al. implies the use of a velocity distribution with four single poles located at $u = \pm a_r \pm i\, a_i$ (or $u = \pm b_r \pm b_i$ in the case of Z_F') .

4. COMPARISON OF THE TWO APPROXIMATIONS

The plasma dispersion function $Z(\xi)$ represents the Hilbert transform of

$$f_M(u) = \frac{1}{\pi^{1/2}} \exp(-u^2) \ . \tag{15}$$

Similarly, the approximations $Z_F(\xi)$ and $Z_{Rn}(\xi)$ are Hilbert transforms of $f_F(u)$ given in (14), and

$$f_{Rn}(u) = \frac{(2y_n)^{2n-1}}{2\pi} \frac{[(n-1)!]^2}{(2n-2)!} \frac{1}{(u^2+2y_n^2)^n} \ . \tag{16}$$

These normalized velocity distributions are depicted in Fig. 1, with $n = 2$, for $u \geq 0$. We note that f_F is a very good approximation to f_M for small values of u whereas f_{Rn} is a better asymptotic approximation to f_M. Because of the type of weighing imposed by the numerator of the integrand in (2) near $u \approx \xi$, we expect Z_F to be a better approximation to Z than Z_{RZ}.

The actual comparison between Z and Z', and their approximations Z_A and Z_A' with $A = F$, R2 and R6, is given in Figs. 2-4 by representing the complex error $\Delta(\xi) = Z_A^{(\prime)}(\xi) - Z^{(\prime)}(\xi)$ along three distinct paths in the first quadrant of the complex ξ-plane. It is not necessary to explore the other quadrants of the ξ-plane because of the symmetry properties outlined at the end of Section 2.

These figures show that, except for large $|\xi|$, when simpler asymptotic expressions may be used to simulate the plasma dispersion function and its derivative, the approximations suggested by Fried et al. are far superior to the approximations based on the resonance distribution. As the illustrations suggest, it would be necessary to go to high values of n to improve on the simulation suggested by Fried et al. However, since the resonance approximation has n terms and involves poles of up to the n^{th}-order, it is clear that its use for large n becomes awkward and, in the light of the Fried et al. approximation, unjustified.

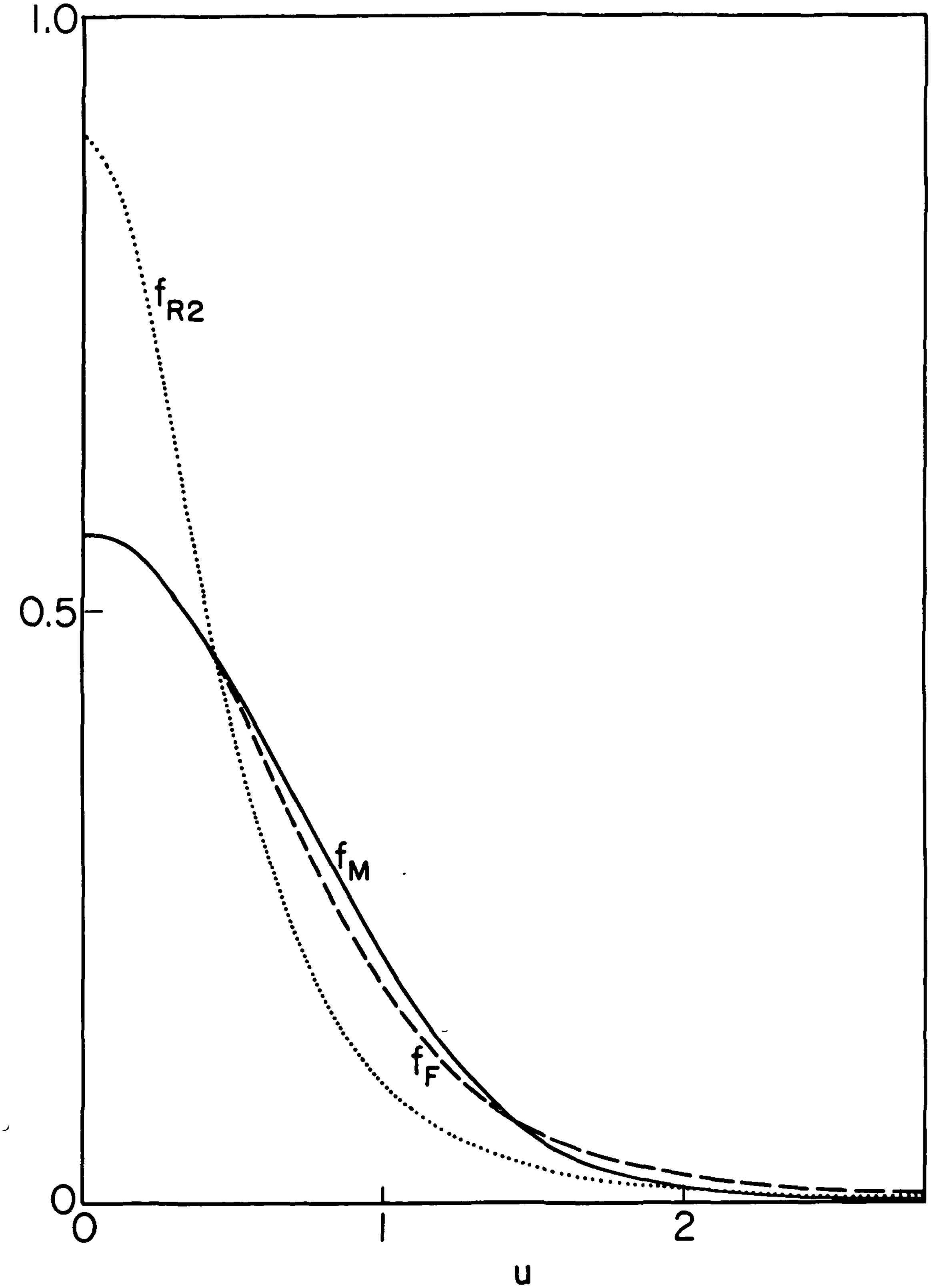

FIG. 1. Comparison of the velocity distributions defined by (3), (4) for n = 2 , and (14) with $u = v/2^{1/2}v_\theta > 0$.

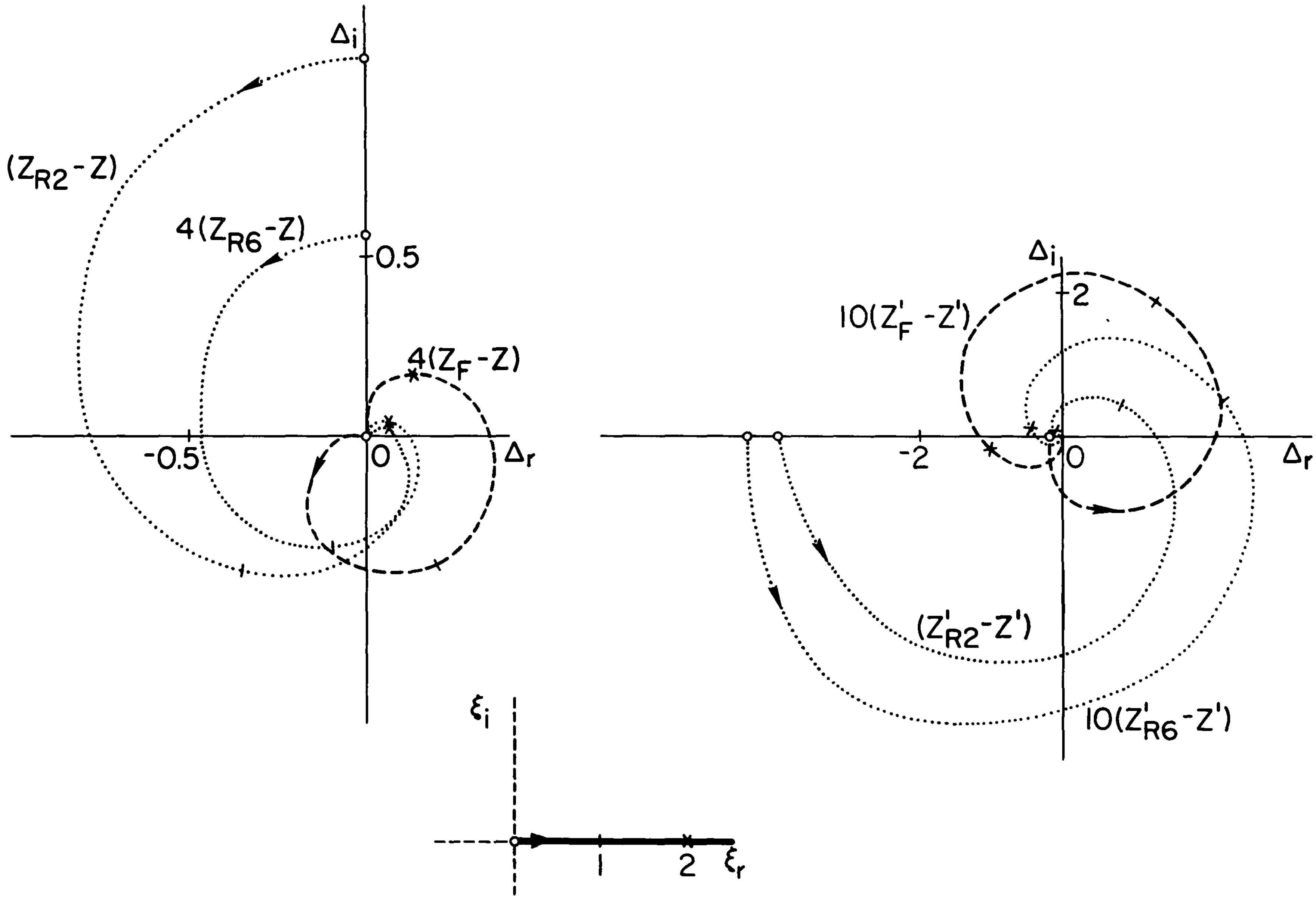

FIG. 2. Errors $\Delta(\xi) = Z_A - Z$ and $Z'_A - Z'$ for A = F , R2 and R6 along $\xi_i = 0$.

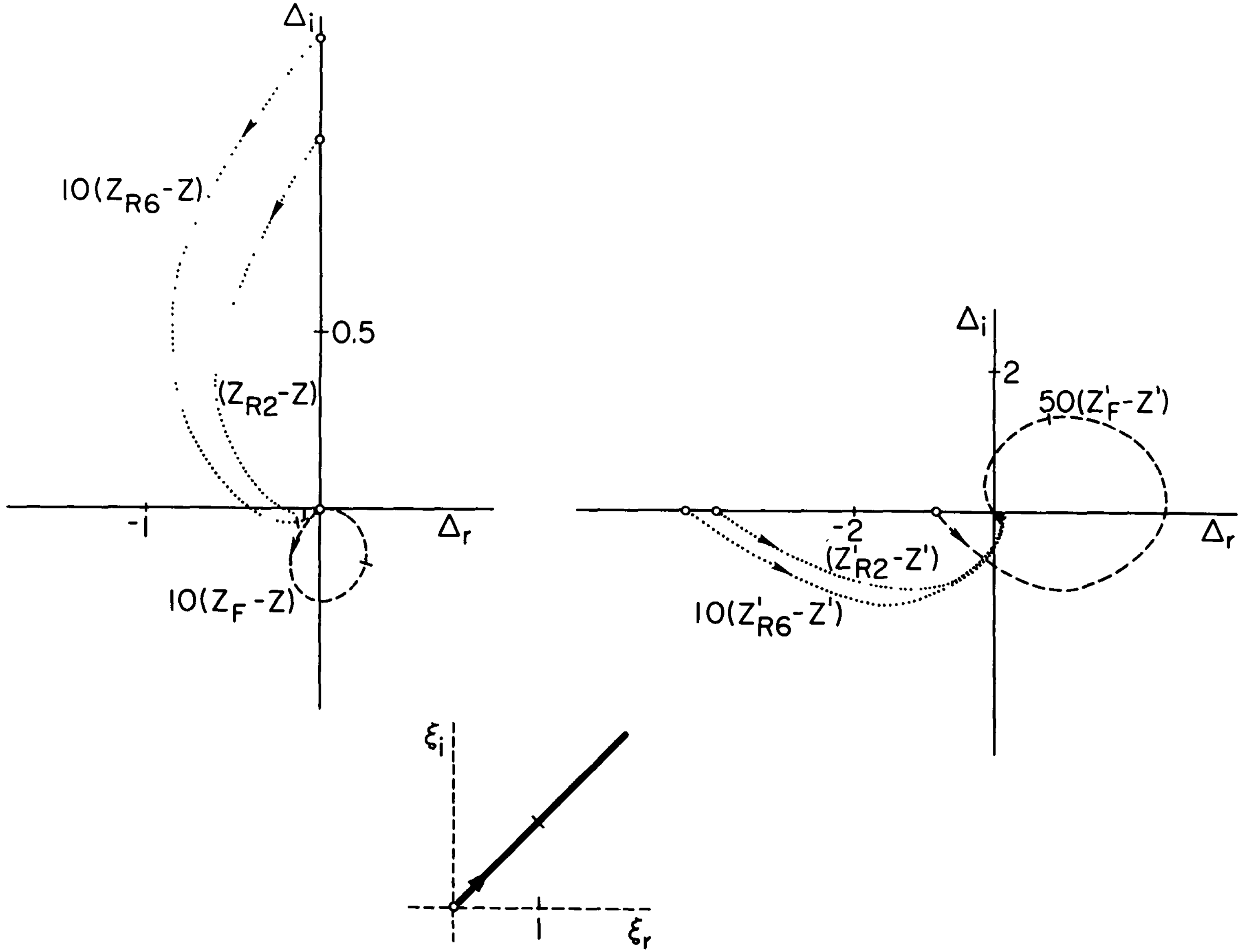

FIG. 3. Errors $\Delta(\xi) = Z_A - Z$ and $Z'_A - Z'$ for $A = F$, R2 and R6 along $\xi_r = \xi_i$.

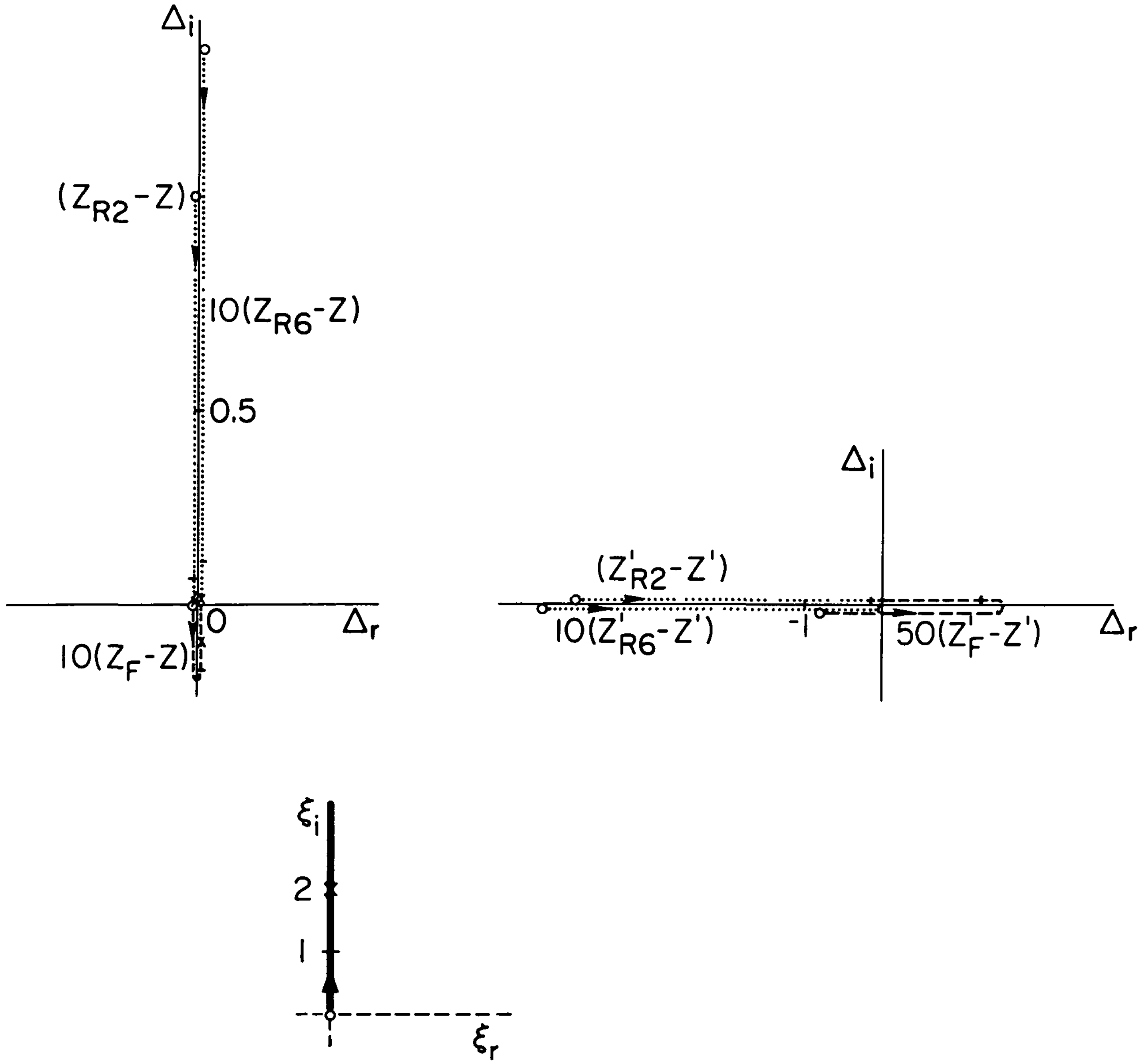

FIG. 4. Errors $\Delta(\xi) = Z_A - Z$ and $Z'_A - Z'$ for $A = F$, R2, R6 along $\xi_r = 0$.

5. APPLICATION TO LANDAU AND WHISTLER WAVES

The dispersion relation of electron Landau waves propagating as $\exp i(\omega t-kz)$ in a Maxwellian plasma is[4]

$$K^2 = Z'(\xi) \qquad \left(\xi = \frac{W}{K}\right) \ , \tag{17}$$

where we have

$$K = \frac{kv_t}{\omega_p} \ , \quad W = \frac{\omega}{\omega_p} \ , \quad v_t^2 = \frac{2\kappa T}{m_e} \ ;$$

ω_p is the electron plasma frequency; κ is Boltzmann's constant, and T and m_e are the electron temperature and mass.

In Fig. 5 we plot the solutions of this dispersion relation for real frequencies, $K = K(W = W_r)$, corresponding to the lowest order root,[5] and compare them with the curves obtained by substituting Z_F' and Z_{R2}' for Z'. We use the expressions given in (9) for $n = 2$, and (13) when $\xi_1 > 0$, and

$$Z_A'(\xi) = \left[Z_A'(\xi^*)\right]^* - i4\pi^{1/2}\,\xi\,\exp(-\xi^2) \qquad (A = RZ, F) \tag{18}$$

for $\xi_1 < 0$. (In Fig. 5 we have $\xi_1 < 0$.) Due to the difficulties associated with $|\xi| \to \infty$ as $W \to 1$, the curves obtained with Z_A' for $W < 1.08$ are extrapolations. For reference, the solution of the Bohm and Gross dispersion relation,

$$W^2 = 1 + \frac{3}{2}K^2 \ , \tag{19}$$

is also depicted.

The curves show that the two approximations yield roughly equivalent results: Z_F' is more adequate to compute K_1, whereas Z_{R2}' gives better agreement with K_r. We note that the resonance approximation was used for rather large values of $|\xi|$ thus explaining its relative success (see Section 4). For applications involving small $|\xi|$, the advantages of the Fried et al. approximation become clearer, as we shall see for the whistler case.

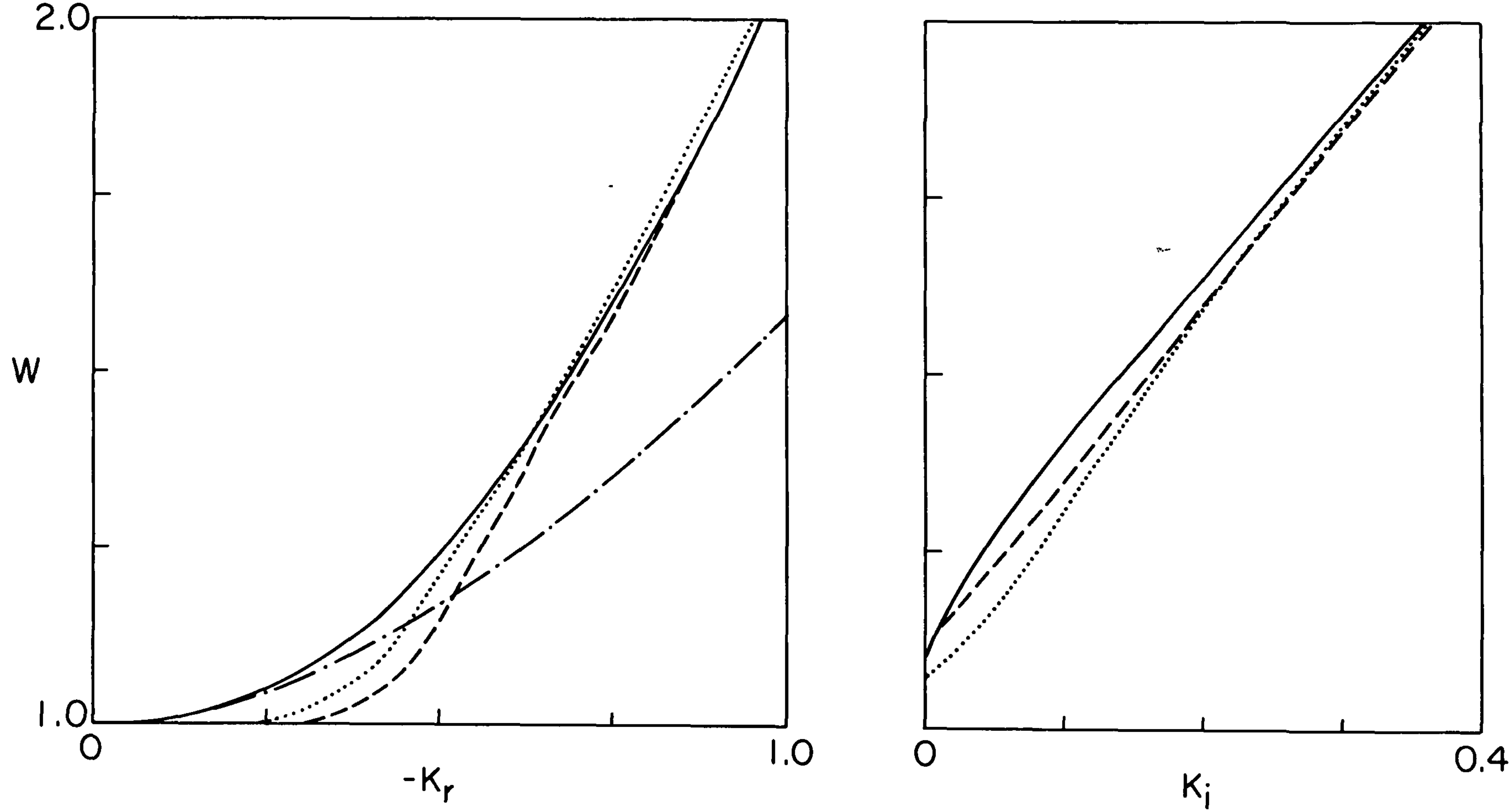

FIG. 5. Brillouin diagram of Landau waves using (—) the derivative of the plasma dispersion function; (--) the Fried *et al*. approximation (13); (···) the resonance approximation (9) with n = 2 . The solution of the Bohm and Gross dispersion relation is denoted by (— ·) .

The dispersion relation of electron whistlers propagating as $\exp i(\omega t-kz)$ along the static magnetic field in an isotropic Maxwellian plasma is[6]

$$\left(\frac{K}{W}\right)^2 = 1 + \frac{W_p^2}{WK\beta} Z(\xi) \qquad \left(\xi = \frac{W-1}{K\beta}\right) , \tag{20}$$

where we have

$$K = \frac{kc}{\omega} \quad , \quad W = \frac{\omega}{\Omega} \quad , \quad W_p = \frac{\omega_p}{\Omega} \quad , \quad \beta = \frac{v_t}{c} \quad ;$$

Ω is the electron cyclotron frequency defined by the static magnetic field; c is the speed of light in vacuum, and ω_p and v_t were defined in relation to (17).

After choosing W_p and β, we solve the dispersion relation for real frequencies, $K = K(W = W_r)$, and plot the results corresponding to the least damped root of (20) together with the solutions obtained by substituting Z_F and Z_{R2} for Z. In Figs. 6-8 we have used $W_p = 5$ and values of β corresponding to electron temperatures of 1, 10 and 100 eV. The approximations used for Z when $\xi_i > 0$ are Z_F, given by (12), and Z_{R2}, defined by (8) for $n = 2$. When $\xi_i < 0$ we have adopted the following analytic continuations

$$Z_A(\xi) = \left[Z_A(\xi^*)\right]^* + i2\pi^{1/2} \exp(-\xi^2) \qquad (A = R2,F) . \tag{21}$$

Here we find $\xi_i > 0$ for $W < 1$ and $\xi_i < 0$ for $W > 1$. For comparison, these figures also show the solution of the cold whistler dispersion relation,

$$\left(\frac{K}{W}\right)^2 = 1 + \frac{W_p^2}{W(1-W)} \quad . \tag{22}$$

Because in the region of cyclotron resonance $(W \sim 1)$ we have $|\xi| \sim 0$, we expect the results obtained with the resonance method to be poor approximations of the exact solutions in that domain (see Section 4). Indeed, the curves show that whereas Z_F yields a good

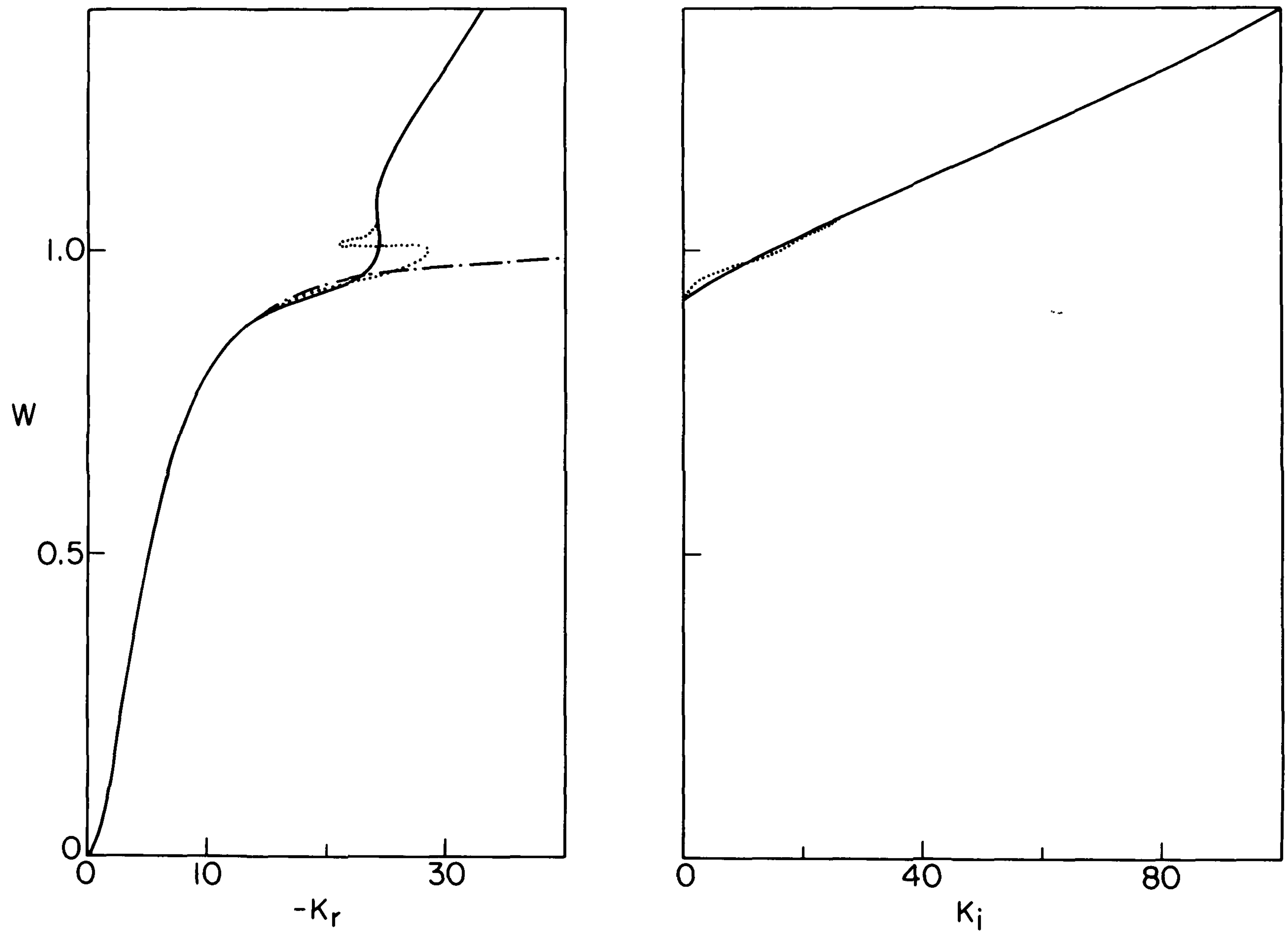

FIG. 6. Brillouin diagram of whistler waves using (—) the plasma dispersion function; (--) the Fried et al. approximation (12); (···) the resonance approximation (8) with n = 2 . The solution of the cold plasma dispersion relation is denoted by (— ·). The electron plasma temperature is 1 eV .

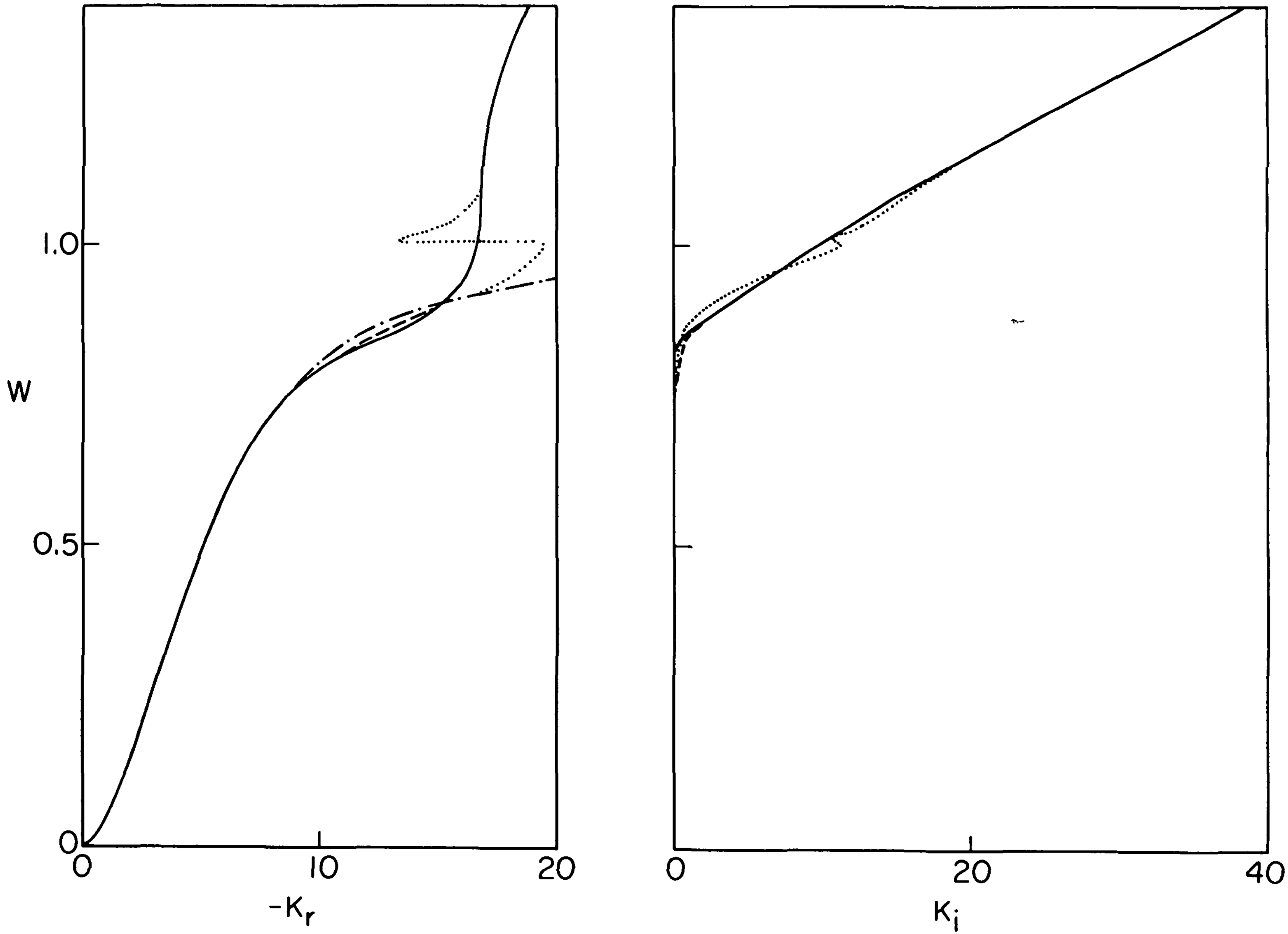

FIG. 7. Brillouin diagram of whistler waves using (—) the plasma dispersion function; (--) the Fried <u>et al</u>. approximation (12); (···) the resonance approximation (8) with n = 2 . The solution of the cold plasma dispersion relation is denoted by (— ·). The electron plasma temperature is 10 eV .

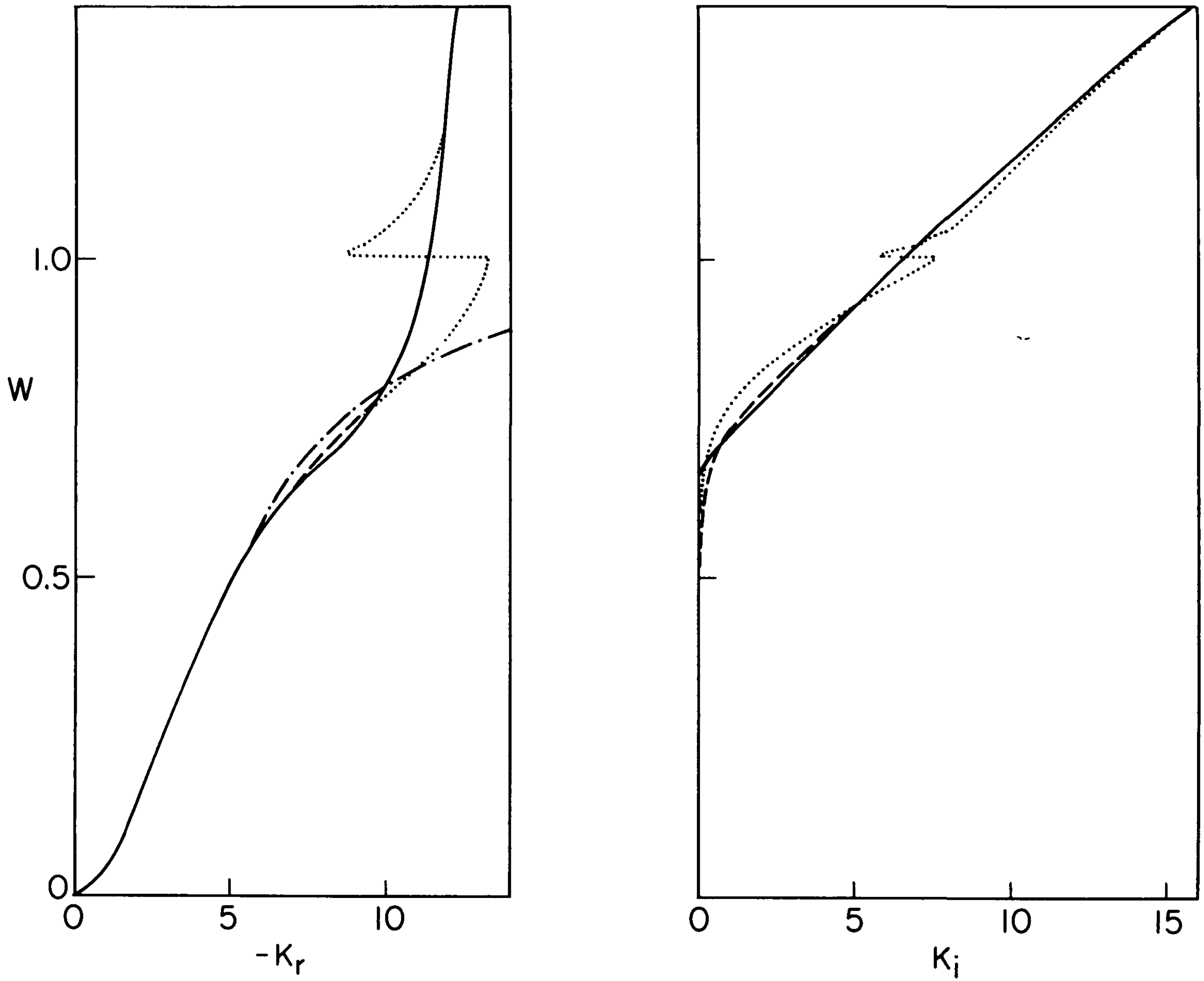

FIG. 8. Brillouin diagram of whistler waves using (—) the plasma dispersion function; (--) the Fried <u>et al</u>. approximation (12); (···) the resonance approximation (8) with n = 2. The solution of the cold plasma dispersion relation is denoted by (— ·). The electron plasma temperature is 100 eV .

approximation, the solutions based on the resonance method yield poor results that worsen with an increase in the electron temperature. In particular, we find a discontinuity in the results for $\lim_{\delta \to 0} (W = 1 \pm \delta)$ which is brought about by the combination of the error $Z_{R2}(\xi) - Z(\xi)$ at $\xi = 0$, and the adopted analytic continuation (21). This discontinuity is removed if we utilize (8) both for $\xi_1 > 0$ and $\xi_1 < 0$, i.e. if we analytically continue the algebraic expression (8) disregarding the characteristics of the function that (8) is trying to simulate. However, this procedure, although removing the discontinuity, shall bring about larger errors than the adopted (21) as $W(> 1)$ increases.

6. DISCUSSION

The results presented in Section 4, showing the superiority of the two-pole approximation of the plasma dispersion function with respect to the resonance approximation, are confirmed by the applications given in Section 5. It should be noted, however, that the apparent simplicity of both approximations, equations (8) and (12), is deceiving. When the problem under consideration requires the utilization of the plasma dispersion function both in the upper and lower half planes of its argument ($\xi_i > 0$ and $\xi_i < 0$, as e.g. in the whistler problem for $W < 1$ and $W > 1$) it is necessary to utilize the analytic continuation given by (11). The algebraic simplicity of the approximations is then lost.

The criterion adopted by Fried et al. to obtain the two-pole approximation was described in Section 3. Accepting the general form (12) of the approximation, which satisfies the symmetry properties and asymptotic behavior of $Z(\xi)$, we are left with the choice of the complex parameter $1/a = \hat{a}$. Imposition of the condition $Z(\xi = 0) = Z_F(\xi = 0)$ determines $\hat{a}_i = \pi^{1/2}/2$. But, if instead of minimizing $|Z_F - Z|$ by "eyeballing", $\hat{a}_r$ is chosen to minimize the square relative error $\int |1 - Z_F/Z|^2 \, d\xi$ over the most critical path[2], i.e. the real axis (here taken between $\xi_r = 0$ and 4), we find that the optimum value of $\hat{a}_r$ is ≈ 0.69 and not 0.55 as proposed by Fried et al. Also, the condition $Z(\xi = 0) = Z_F(\xi = 0)$ might be discarded when Z_F is not utilized for small arguments. Then, minimizing the square relative error between $\xi_r = 0$ and 4 for $\hat{a}_r = 0.69$, we find that the optimum value of $\hat{a}_i$ is ≈ 0.85. Figure 9 depicts the relative errors $\Delta = (\hat{Z} - Z)/|Z|$ for $\hat{Z} = Z_F$ given by (12), and $\hat{Z} = Z_{FO}$, where Z_{FO} is obtained from (12) by putting $\hat{a} = 1/a = 0.69 + i\,0.85$. We find that Z_{FO} yields smaller errors than Z_F for $\xi_r > 0.4$.

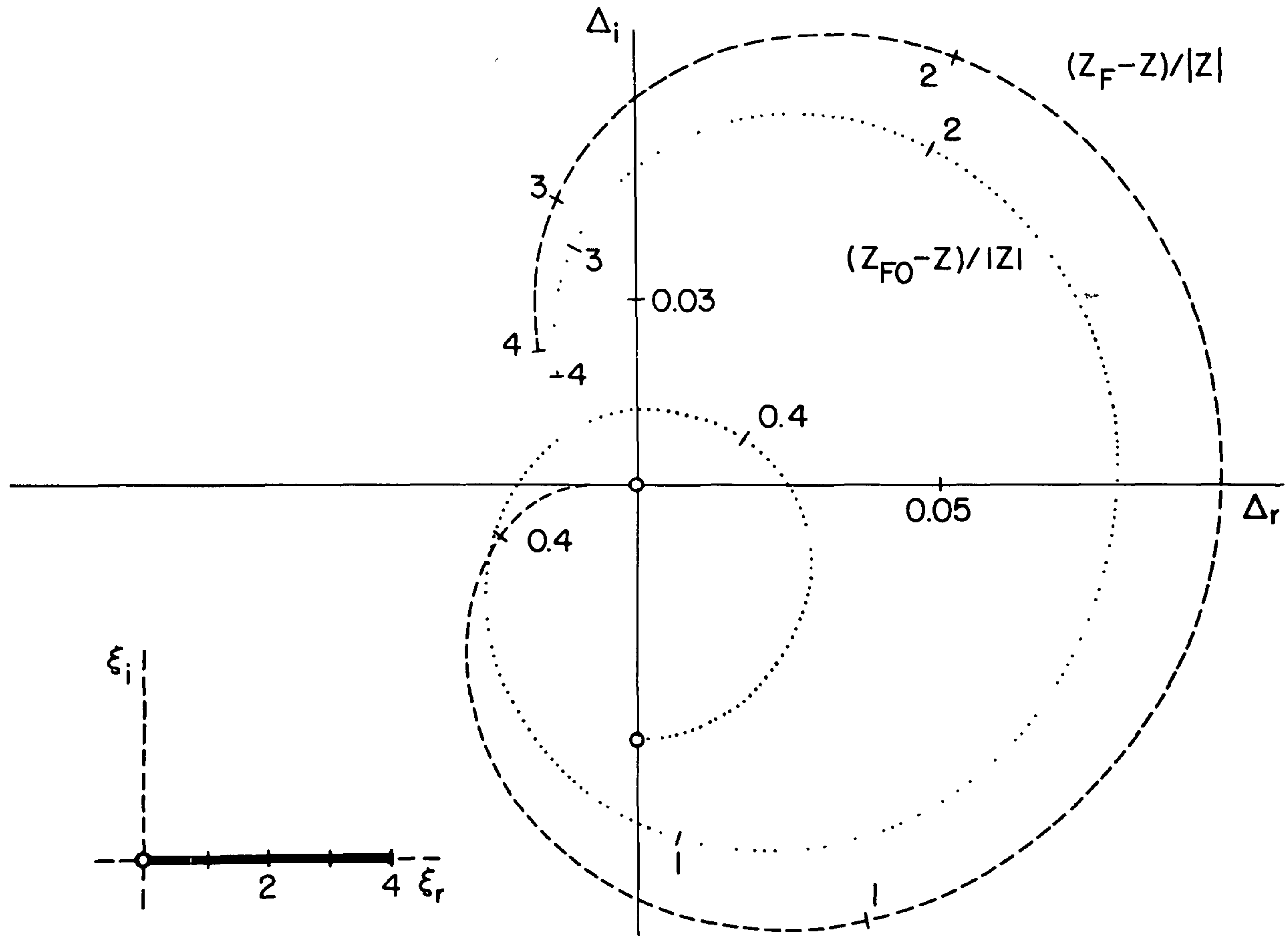

FIG. 9. Relative errors $\Delta(\xi) = (Z_A - Z)/|Z|$ for $A = F$, i.e. (12) with $1/a = 0.55 + i\,\pi^{1/2}/2$, and $A = FO$, i.e. (12) with $1/a = 0.69 + i\,0.85$, along $\xi_i = 0$.

ACKNOWLEDGMENTS

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APPENDIX A

THE MAXWELLIAN DISTRIBUTION AS A LIMIT

To demonstrate (5), we use Stirling's formula to write

$$\lim_{n \to \infty} \frac{[(n-1)!]^2}{(2n-2)!} \approx \lim_{n \to \infty} 2^{-2(n-1)} (\pi n)^{1/2} \quad , \qquad (A.1)$$

and note that

$$\lim_{n \to \infty} \left\{ \frac{[2v_\theta (2n-3)^{1/2}]^{2n-1}}{2\pi [v^2 + (2n-3) v_\theta^2]^n} \right\} = \lim_{n \to \infty} \left[\frac{2^{2(n-1)}}{\pi v_\theta (2n)^{1/2}} \left(1 + \frac{v^2}{2v_\theta^2} \frac{1}{n} \right)^{-n} \right] . \qquad (A.2)$$

Substitution in (4) yields

$$\lim_{n \to \infty} F_{Rn}(v) = \lim_{n \to \infty} \left[\frac{1}{(2\pi)^{1/2} v_\theta} \left(1 + \frac{v^2}{2v_\theta^2} \frac{1}{n} \right) \right]^{-n} = F_M(v) \quad . \qquad (A.3)$$

APPENDIX B

DERIVATION OF $f_F(u)$

To obtain $f_F(u)$ we solve the integral equation

$$Z_F(\xi) = \int_{-\infty}^{\infty} du \, \frac{f_F(u)}{u-\xi} \qquad \left(u = \frac{v}{2^{1/2} v_\theta} \ , \ \xi_1 > 0\right). \qquad (B.1)$$

Denoting the positive frequency part of $f_F(u)$ by $f_F^+(u)$, that is[7]

$$f_F^+(u) = \frac{1}{2\pi}\int_0^{\infty} ds \, \hat{f}_F(s) \exp isu \qquad (u_1 \leq 0) \ ,$$

$$\hat{f}_F(s) = \int_{-\infty}^{\infty} du \, f_F(u) \exp(-isu) \ , \qquad (B.2)$$

we can write

$$Z_F(\xi) = \int_{-\infty}^{\infty} du \, \frac{f_F(u)}{u-\xi} = i \, 2\pi \, f_F^+(\xi) \qquad (\xi_1 > 0) \ . \qquad (B.3)$$

Because $\hat{f}_F(s)$ is the Fourier transform of $f_F(u)$ and we expect this velocity distribution to be real and even (it simulates the Maxwellian distribution), it follows the $\hat{f}_F(s)$ will also be real and even.

Combining (12), (B.2) and (B.3) we find

$$2\pi f_F^+(\xi) = \frac{i}{1.1}\left[\frac{1}{a(\xi-a)} + \frac{1}{a^*(\xi+a^*)}\right] = \int_0^{\infty} ds \, \hat{f}_F(s) \exp is\xi \qquad (\xi_1 > 0) \ , \qquad (B.4)$$

so that introducing $p = -i\xi$ we obtain

$$\frac{1}{1.1a(p+ia)} + \frac{1}{1.1a^*(p-ia^*)} = \int_0^\infty ds\ \hat{f}_F(s)\ \exp(-ps) \qquad (p_r = \xi_i > 0)\ . \tag{B.5}$$

We retrieve $\hat{f}_F(s)$, for $s > 0$, by inverting this Laplace transform:

$$\hat{f}_F(s) = \frac{\exp(-ias)}{1.1\ a} + \frac{\exp\ ia^*s}{1.1\ a^*}$$

$$= \cos\ (a_r s)\ \exp\ a_i s + \frac{\pi^{1/2}}{1.1}\ \sin\ (a_r s)\ \exp\ a_i s \qquad (s > 0)\ . \tag{B.6}$$

The desired velocity distribution is then obtained by inverting the Fourier transform, and recalling that $\hat{f}_F(s)$ is a real and even function of s :

$$f_F(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds\ \hat{f}_F(s)\ \exp\ isu = \frac{1}{\pi}\int_0^\infty ds\ \hat{f}_F(s)\ \cos\ (su)$$

$$= \frac{|a_i|}{2\pi}\left[\frac{1}{(u+a_r)^2+a_i^2} + \frac{1}{(u-a_r)^2+a_i^2}\right] \tag{B.7}$$

$$+ \frac{1}{2.2\pi^{1/2}}\left[\frac{u + a_r}{(u+a_r)^2+ a_i^2} - \frac{u - a_r}{(u-a_r)^2+ a_i^2}\right].$$

Noting that $1.1\ |a_i| = \pi^{1/2}\ a_r$, we finally have

$$f_F(u) = \frac{4(a_r^2 + a_i^2)}{(1.21+\pi)\pi^{1/2}}\ \frac{1}{\left[(u+a_r)^2 + a_i^2\right]\left[(u-a_r)^2 + a_i^2\right]}$$

with (B.8)

$$a_r = \frac{2.2}{1.21+\pi} \quad , \qquad a_i = -\frac{2\pi^{1/2}}{1.21+\pi}$$

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