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IN A TWO FLUID SOLAR WIND

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Submitted to the Physics Department on Jan 19,1973 in partial fulfillment of the requirements for the degree of Master of Science

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by

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ABSTRACT

We investigate the properties of low frequency ($\omega \ll \omega_{pc}$) transverse waves in a two-fluid solar wind having a radial magnetic field and radial streaming velocity. In order to examine what effects this streaming medium has on the waves, we decompose waves which are assumed to be linearly polarized into left and right circularly polarized waves. We compute analytic expressions valid to first order in ω/ω_{pc} for the radial amplitude and phase dependence of these constituent waves. We show that after travelling a distance Δr, these waves have different amplitudes and phases. The former result causes their superposition to become elliptical, rather than linear. The latter causes the axis of the ellipse of polarization to rotate through a well-defined angle. Analytic expressions are obtained for the eccentricity of the ellipse and for the angle of rotation. In analogy with regular Faraday rotation, in which the plane of polarization of a linear polarized wave rotates, we denote the effect as generalized Faraday rotation.

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CHAPTER 1

INTRODUCTION

Since the formulation of solar wind theory (Parker, 1958) much effort has been directed towards understanding perturbations in the steady coronal expansion. Waves, discontinuities, turbulence and shocks are observed perturbations in the steady flow of the solar wind. This thesis concerns the first of these phenomena, waves. We undertake a theoretical investigation of the properties of transverse field and plasma perturbations in the constant radial flow of a two-fluid solar wind. Observationally, the presence of such transverse waves has been firmly demonstrated (Belcher and Davis, 1971). As consequences of this study we shall obtain agreement with previously investigated amplitude dependence of transverse fluctuations for a one-fluid solar wind (Parker, 1965, Belcher, 1971), and obtain new results predicting the generalized Faraday rotation of low frequency transverse waves.

The expanding solar corona is a hydromagnetic configuration which is stable with respect to small perturbations. With the advent of direct observations of solar wind fluctuations (Bridge et al., 1964, Coleman et al., 1963, 1966, 1968, Neugebauer and Snyder, 1962, 1965, 1967, Siscoe et al., 1968), characteristic features contained in the plasma and field fluctuations emerged. It seemed plausible (Davis, 1966) that observed fluctuations could be caused by propagating Alfvén or magnetoacoustic waves. A thorough study of spectral and cross-spectral analysis of Mariner 2 plasma and field data by Coleman (1967, 1968), indicated that outwardly propagating Alfvén waves could account for many of the observed fluctuations. This statistical approach neither gave patterns of occurence nor explicit examples of wave forms, however. Unti and Neugebauer (1968) were the first to identify a specific example of a quasi-periodic Alfvén wave. Belcher, Davis, and Smith (1969), in a preliminary analysis of Mariner 5 plasma and field data identified outwardly propagating Alfvén waves as frequently occuring phenomena, although these waves were mainly non-sinusoidal and aperiodic.

A comprehensive study of Alfvén waves (Belcher et al., 1969, Belcher and Davis, 1971), suggests that the outwardly propagating waves observed primarily in high velocity streams and on their trailing edge are remnants of a broad spectrum of MHD waves generated inside the Alfvénic critical point (Hartle and Sturrock, 1968). This supports Parker's idea (1965) that one can listen at 1 A.U. to the noise generated at the Sun. Hollweg (1972) suggests that super-granulation patterns generate Alfvénic disturbances which propagate upwards through the photosphere. Parker's suggestion (1965) that waves do work on the wind led Belcher (1971) and Alazakri and Couturier (1971) to reformulate the basic solar wind problem from the point of view of a new energy source — Alfvén waves. These waves propagate in and are convected by the streaming medium and could play a principal role in the fast, hot, tenuous winds that sometimes come from the active Sun. They could accelerate the wind and heat it upon dissipation. It should be noted that the Alfvén mode is the only hydromagnetic wave which is not strongly Landau damped (Barnes, 1966, 1968), so that the longer wavelengths observed at 1 A.U. are most probably of solar origin.

Other causes of fluctuating phenomena may be due to the differing temperatures in coronal regions. Different temperatures in regions on the Sun lead to different expansion rates for coronal gases. The colder regions expand more slowly than the hotter ones, so that hot gas may eventually overtake cool gas. This leads to compression, discontinuities, and wave generation as these two streams interact (Parker, 1963, Sarahbai, 1963, Lee, 1971). The large velocity difference provides the energy to drive wave fluctuations (Jokippi and Davis, 1969), although the predominance of purely outwardly propagating Alfvén waves is not adequately explained in this manner (Belcher and Davis, 1971). Coleman (1968) suggests that the large scale shear resulting from varying wind velocity leads to turbulence in which the energy of the shear cascades down through a hierarchy of eddies to some very small scale at which dissipation converts the fluid motion into heat. The interested reader may pursue further discussion of these two differing points of view in Parker (1969). Blast waves from sudden coronal commencements generate waves (Parker, 1963) as do finite amplitude Alfvén waves when they are incident upon shock waves (Scholer and Belcher, 1971).

The present study investigates the problem of low frequency transverse wave propagation in a solar wind having two species, protons and electrons. We seek the wave amplitude dependence as a function of distance from a chosen reference level. We do not inquire about the mechanism of wave production, but rather assume that at the reference level r_o we have linearly polarized waves of a given amplitude. We assume that beyond r_o there is neither subsequent wave generation nor wave damping into thermal motion. As these waves propagate in and are convected by the two-component solar wind, we examine what effects the streaming medium has on their properties. We decompose the linearly polarized waves into left and right handed circularly polarized waves, and focus our attention on the properties of these constituent waves. We shall see that as the right and left circularly polarized waves propagate in and are convected to larger r, their respective amplitude and phases behave in different ways. Thus after a distance Δr, we no longer have linear polarization, but, rather elliptical polarization with the axis of the ellipse of polarization turned through a well-defined angle. Analytic expressions will be obtained for the amplitude and phase dependence of the right and left handed waves, and for the angle of rotation. In analogy with regular Faraday rotation, in which there is no such amplitude change, we denote the effect as generalized Faraday rotation.

CHAPTER 2

THE DIFFERENTIAL EQUATION FOR ELLIPTICALLY POLARIZED WAVES IN A TWO-FLUID PLASMA

For a two-component plasma with magnetic field B, the relevant equations of motion and the appropriate Maxwell's equations in the presence of a spherically symmetric gravitational potential Φ are

$$m_p n_p \left(\frac{\partial}{\partial t} + \vec{V}_p \cdot \vec{\nabla} \right) \vec{V}_p = \frac{1}{c} \vec{J}_p \times \vec{B} + n_p e \vec{E} - \vec{\nabla} P_p - m_p n_p \vec{\nabla} \Phi \tag{1}$$

$$\frac{\partial}{\partial t} m_p n_p + \vec{\nabla} \cdot m_p n_p \vec{V}_p = 0 \tag{2}$$

$$m_e n_e \left(\frac{\partial}{\partial t} + \vec{V}_e \cdot \vec{\nabla} \right) \vec{V}_e = \frac{1}{c} \vec{J}_e \times \vec{B} - n_e e \vec{E} - \vec{\nabla} P_e - m_e n_e \vec{\nabla} \Phi \tag{3}$$

$$\frac{\partial}{\partial t} m_e n_e + \vec{\nabla} \cdot m_e n_e \vec{V}_e = 0 \tag{4}$$

$$c \vec{\nabla} \times \vec{B} = 4\pi (\vec{J}_p + \vec{J}_e) + \frac{\partial}{\partial t} \vec{E} \tag{5}$$

$$c \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \tag{6}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{7}$$

where

$$\vec{J}_p = n_p e \vec{V}_p \qquad \vec{J}_e = -n_e e \vec{V}_e \tag{8}$$

and

$$\Phi = -\frac{G M_{SUN}}{r} \tag{9}$$

and

$$P_p = n_p K T_p \qquad P_e = n_e K T_e \tag{10}$$

In the above equations the supscript p (e) refers to protons (electrons) and n refers to the number density. We assume that there are no collisions between species and that the respective pressures are scalar quantities. In the one-fluid MHD limit, these equations reduce to the set used by Belcher (1971) in solving for the WKB wave amplitudes.

We assume that our plasma parameters $\vec{B}$, $\vec{V}$, and $\vec{E}$ are perturbed due to the presence of short wavelength transverse waves. Denoting these perturbations by $\delta\vec{B}$, $\delta\vec{V}$, and $\delta\vec{E}$ we further assume that they are superimposed upon background conditions of radial streaming and a radial magnetic field. We also assume that the propagation vector of these disturbances is parallel to the background field. It will be shown that these waves locally obey the standard two-fluid dispersion relation for transverse waves. In analogy with the one-fluid case (Belcher, 1971), we will solve for the wave amplitudes as a function of r using the WKB approximation that wave quantities vary on a scale much smaller than the scale height. We confine our analysis to the equatorial plane of a spherical coordinate system. Consistent with other workers (Yeh, 1970, Hartle and Sturrock, 1968), we assume equal number density $n_p = n_e = n$ to preserve charge neutrality of the plasma and equal radial streaming velocity $\vec{V}_p = \vec{V}_e = \vec{V}$ to preserve charge neutrality of the Sun. The radial component of $\vec{E}$ arises due to a small charge separation, insuring the absence of a net fow of charge from the Sun.

We thus look for wave solutions to (1) through (6) of the form

$$\begin{aligned}\vec{V} &= V(r)\,\hat{r} + \delta V^{\theta}(r,t)\,\hat{\theta} + \delta V^{\phi}(r,t)\,\hat{\phi}\\ \vec{B} &= B(r)\,\hat{r} + \delta B^{\theta}(r,t)\,\hat{\theta} + \delta B^{\phi}(r,t)\,\hat{\phi}\\ \vec{E} &= E(r)\,\hat{r} + \delta E^{\theta}(r,t)\,\hat{\theta} + \delta E^{\phi}(r,t)\,\hat{\phi}\end{aligned} \tag{11}$$

To solve for the first order WKB wave amplitudes, we work with the transverse components of (1), (3), (5), and (6). Since $\vec{\nabla}p$ and $\vec{\nabla}\Phi$ have only radial components, our equations then reduce to the following

$$\begin{aligned}\frac{\partial}{\partial t}\delta V_p^{\theta} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_p^{\theta} &= \frac{e}{m_p c}\left(B\,\delta V_p^{\phi} - V\,\delta B^{\phi}\right) + \frac{e}{m_p}\delta E^{\theta}\\ \frac{\partial}{\partial t}\delta V_p^{\phi} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_p^{\phi} &= \frac{e}{m_p c}\left(V\,\delta B^{\theta} - B\,\delta V_p^{\theta}\right) + \frac{e}{m_p}\delta E^{\phi}\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial}{\partial t}\delta V_e^{\theta} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_e^{\theta} &= -\frac{e}{m_e c}\left(B\,\delta V_e^{\phi} - V\,\delta B^{\phi}\right) - \frac{e}{m_e}\delta E^{\theta}\\ \frac{\partial}{\partial t}\delta V_e^{\phi} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_e^{\phi} &= -\frac{e}{m_e c}\left(V\,\delta B^{\theta} - B\,\delta V_e^{\theta}\right) - \frac{e}{m_e}\delta E^{\phi}\end{aligned} \tag{13}$$

$$\begin{aligned}-\frac{c}{r}\frac{\partial}{\partial r} r\,\delta B^{\phi} &= 4\pi n e\left(\delta V_p^{\theta} - \delta V_e^{\theta}\right) + \frac{\partial}{\partial t}\delta E^{\theta}\\ \frac{c}{r}\frac{\partial}{\partial r} r\,\delta B^{\theta} &= 4\pi n e\left(\delta V_p^{\phi} - \delta V_e^{\phi}\right) + \frac{\partial}{\partial t}\delta E^{\phi}\end{aligned} \tag{14}$$

$$\begin{aligned}-\frac{1}{r}\frac{\partial}{\partial r} r\,\delta E^{\phi} &= -\frac{1}{c}\frac{\partial}{\partial t}\delta B^{\theta}\\ \frac{1}{r}\frac{\partial}{\partial r} r\,\delta E^{\theta} &= -\frac{1}{c}\frac{\partial}{\partial t}\delta B^{\phi}\end{aligned} \tag{15}$$

It is convenient to define the following variables

$$\begin{aligned}\delta V^{\pm} &= \delta V^{\theta} \pm i\,\delta V^{\phi}\\ \delta B^{\pm} &= \delta B^{\theta} \pm i\,\delta B^{\phi}\\ \delta E^{\pm} &= \delta E^{\theta} \pm i\,\delta E^{\phi}\end{aligned} \tag{16}$$

It will be seen (Chapter 4) that this choice associates the upper (lower) sign with the left (right) handed polarized wave.

In this equivalent formulation, equations (12) through (15) become

$$\frac{\partial}{\partial t}\delta V_p^{\pm} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_p^{\pm} \pm i\omega_{pc}\delta V_p^{\pm} \mp \frac{ieV}{m_p c}\delta B^{\pm} - \frac{e}{m_p}\delta E^{\pm} = 0 \tag{17}$$

$$\frac{\partial}{\partial t}\delta V_e^{\pm} + \frac{V}{r}\frac{\partial}{\partial r} r\,\delta V_e^{\pm} \mp i\omega_{ec}\delta V_e^{\pm} \pm \frac{ieV}{m_e c}\delta B^{\pm} + \frac{e}{m_e}\delta E^{\pm} = 0 \tag{18}$$

$$\pm\frac{i}{r}\frac{\partial}{\partial r} r\,\delta E^{\pm} + \frac{1}{c}\frac{\partial}{\partial t}\delta B^{\pm} = 0 \tag{19}$$

$$\pm\frac{ic}{r}\frac{\partial}{\partial r} r\,\delta B^{\pm} - 4\pi ne\left(\delta V_p^{\pm} - \delta V_e^{\pm}\right) - \frac{\partial}{\partial t}\delta E^{\pm} = 0 \tag{20}$$

where $\omega_{pc} = \frac{eB}{m_p c}$ and $\omega_{ec} = \frac{eB}{m_e c}$ are the proton and electron cyclotron frequencies respectively.

The eikonal approximation (Weinberg, 1962) consists in assuming that the spatial dependence of the quantities $\delta V_p^{\pm}$, $\delta V_e^{\pm}$, $\delta E^{\pm}$, and $\delta B^{\pm}$ is contained in a common factor exp(i $S^{\pm}(r)$). We thus set

$$\left.\begin{array}{l} \delta V^{\pm} = \delta V_1^{\pm} \\ \delta E^{\pm} = \delta E_1^{\pm} \\ \delta B^{\pm} = \delta B_1^{\pm} \end{array}\right. \quad e^{i\left(S^{\pm}(r) - \omega t\right)} \tag{21}$$

with $\delta V_1^{\pm}$, $\delta E_1^{\pm}$, and $\delta B_1^{\pm}$ to be determined. Defining

$$K^{\pm} = \frac{d}{dr} S^{\pm}(r) \tag{22}$$

and inserting (21) into (17) through (20) we obtain

$$\left(-i\omega + Vik^{\pm} \pm i\omega_{pc} + \frac{V}{r}\right)\delta V_{p1}^{\pm} \mp \frac{ieV}{m_p c}\delta B_1^{\pm} - \frac{e}{m_p}\delta E_1^{\pm} = 0 \tag{23}$$

$$\left(-i\omega + Vik^{\pm} \mp i\omega_{ec} + \frac{V}{r}\right)\delta V_{e1}^{\pm} \pm \frac{ieV}{m_e c}\delta B_1^{\pm} + \frac{e}{m_e}\delta E_1^{\pm} = 0 \tag{24}$$

$$\mp ck^{\pm}\delta E_1^{\pm} \pm \frac{ic}{r}\delta E_1^{\pm} - i\omega\delta B_1^{\pm} = 0 \tag{25}$$

$$\mp ck^{\pm}\delta B_1^{\pm} \pm \frac{ic}{r}\delta B_1^{\pm} - 4\pi ne\,\delta V_{p1}^{\pm} + 4\pi ne\,\delta V_{e1}^{\pm} + i\omega\delta E_1^{\pm} = 0 \tag{26}$$

Let the scale length over which the unperturbed quantities vary be h, which is of the order r. Then comparing the terms $V\delta V_1^{\pm}/r$ and $Vik^{\pm}\delta V_1^{\pm}$ in (23) and (24) we note that the ratio is $\lambda^{\pm}/h$ where $\lambda^{\pm} = 2\pi/k^{\pm}$. We assume that $h \gg \lambda^{\pm}$ so that this ratio is much less than 1. Hence we may drop $V\delta V_1^{\pm}/r$ with respect to $Vik^{\pm}\delta V_1^{\pm}$ in order to obtain an equation correct to zeroth order in $\lambda^{\pm}/h$. Similar reasoning in (25) and (26) suggests that we drop $\pm ic\delta E_1^{\pm}/r$ and $\pm ic\delta B_1^{\pm}/r$ respectively. We thus obtain the following set of equations correct to zeroth order in $\lambda^{\pm}/h$

$$\left(-i\omega + Vik^{\pm} \pm i\omega_{pc}\right)\delta V_{p1}^{\pm} \mp \frac{ieV}{m_p c}\delta B_1^{\pm} - \frac{e}{m_p}\delta E_1^{\pm} = 0 \tag{27}$$

$$\left(-i\omega + Vik^{\pm} \mp i\omega_{ec}\right)\delta V_{e1}^{\pm} \pm \frac{ieV}{m_e c}\delta B_1^{\pm} + \frac{e}{m_e}\delta E_1^{\pm} = 0 \tag{28}$$

$$\mp ck^{\pm}\delta E_1^{\pm} - i\omega\delta B_1^{\pm} = 0 \tag{29}$$

$$\mp ck^{\pm}\delta B_1^{\pm} - 4\pi ne\,\delta V_{p1}^{\pm} + 4\pi ne\,\delta V_{e1}^{\pm} + i\omega\delta E_1^{\pm} = 0 \tag{30}$$

B

For a two-component plasma, the Alfvén velocity is

$$\vec{V}_A = \frac{\vec{B}}{\sqrt{4\pi n(m_p+m_e)}} = \frac{\vec{B}}{\sqrt{4\pi\rho}} \tag{31}$$

and the plasma frequency is

$$\omega_p^2 = 4\pi n e^2 \left(\frac{1}{m_e} + \frac{1}{m_p} \right) \tag{32}$$

In order to obtain a dispersion relation for $K^{\pm}$ in a two-fluid plasma, we set the determinant of (27) through (30) equal to zero. Using (32) we obtain

$$(\omega - K^{\pm}V \mp \omega_{pc})(\omega - K^{\pm}V \pm \omega_{ec})(\omega^2 - c^2(K^{\pm})^2) = \omega_p^2 (\omega - K^{\pm}V)^2 \tag{33}$$

From the definition of the index of refraction we obtain

$$n^2(\omega) = \frac{c^2(K^{\pm})^2}{\omega^2} = 1 - \frac{\omega_p^2 (\omega - K^{\pm}V)^2}{\omega^2(\omega - K^{\pm}V \mp \omega_{pc})(\omega - K^{\pm}V \pm \omega_{ec})} \tag{34}$$

We note that there are two resonances occuring at $\omega - K^{+}V = \omega_{pc}$ and at $\omega - K^{-}V = \omega_{ec}$. This can be easily understood if we transform to the frame of reference moving at the non-relativistic wind velocity V. The frequency measured in this frame is the Doppler shifted frequency $\omega - K^{\pm}V \equiv \omega_{*}$. This immediately follows from the Lorentz transformation in the non-relativistic limit. The resonances occur when the electric field vector of the right (left) polarized wave rotates with the same velocity as the protons (electrons) in their cyclotron rotation about $\vec{B}$.

We can further exploit the elegance of the Lorentz transformation (Olbert, private communication) to obtain agreement between (33) and conventional two-fluid dispersion relations (Van Kampen and Felderhoff, 1967) by the substitution $(\omega - K^{\pm}V) \to \omega_{*}$.

Furthermore in the frame co-moving with the wind velocity V it is a straightforward matter to obtain $\omega_*^2 = (K \pm V)^2$ from (33) under the assumption that the phase velocity is much smaller than the speed of light c. This anticipates the results (58) and (66) in Chapter 3 and, for continuity, will not be given here.

Having solved for the dispersion relation correct to zeroth order in $\frac{\lambda^\pm}{h}$ we now seek to solve for our unknown amplitudes $\delta V_1^\pm$, $\delta E_1^\pm$, and $\delta B_1^\pm$. To accomplish this we follow the method of Weinberg (1962) which involves retaining terms to first order in $\frac{\lambda^\pm}{h}$ in equations (17) through (20). We write

$$\left.\begin{aligned} \delta V^\pm &= (\delta V_1^\pm + \delta V_2^\pm) \\ \delta B^\pm &= (\delta B_1^\pm + \delta B_2^\pm) \\ \delta E^\pm &= (\delta E_1^\pm + \delta E_2^\pm) \end{aligned}\right\} e^{i(S^\pm(r) - \omega t)} \tag{35}$$

where $\delta V_1^\pm$, $\delta B_1^\pm$, and $\delta E_1^\pm$ satisfy (27) through (30), and substitute these expressions into (17) through (20). This process yields

$$(-i\omega + Vik^\pm \pm i\omega_{pc})\delta V_{p2}^\pm \mp \frac{ieV}{m_p c}\delta B_2^\pm - \frac{e}{m_p}\delta E_2^\pm = -V\frac{\partial}{\partial r}(\delta V_{p1}^\pm + \delta V_{p2}^\pm) - \frac{V}{r}(\delta V_{p1}^\pm + \delta V_{p2}^\pm) \tag{36}$$

$$(-i\omega + Vik^\pm \mp i\omega_{ec})\delta V_{e2}^\pm \pm \frac{ieV}{m_e c}\delta B_2^\pm + \frac{e}{m_e}\delta E_2^\pm = -V\frac{\partial}{\partial r}(\delta V_{e1}^\pm + \delta V_{e2}^\pm) - \frac{V}{r}(\delta V_{e1}^\pm + \delta V_{e2}^\pm) \tag{37}$$

$$\mp ck^\pm \delta E_2^\pm - i\omega\,\delta B_2^\pm = \mp ic\frac{\partial}{\partial r}(\delta E_1^\pm + \delta E_2^\pm) \mp \frac{ic}{r}(\delta E_1^\pm + \delta E_2^\pm) \tag{38}$$

$$\mp ck^{\pm}\delta B_2^{\pm} - 4\pi ne\,\delta V_{p2}^{\pm} + 4\pi ne\,\delta V_{e2}^{\pm} + i\omega\,\delta E_2^{\pm} = \mp ic\frac{\partial}{\partial r}\left(\delta B_1^{\pm} + \delta B_2^{\pm}\right) \mp \frac{ic}{r}\left(\delta B_1^{\pm} + \delta B_2^{\pm}\right) \tag{39}$$

where we have used (27) through (30) in the above. The first conclusion to be drawn is that $\delta V_2^{\pm}$, $\delta E_2^{\pm}$, and $\delta B_2^{\pm}$ are smaller than $\delta V_1^{\pm}$, $\delta E_1^{\pm}$, and $\delta B_1^{\pm}$, respectively, by a factor $k^{\pm}h$, where h is the scale length within which the quantities vary. For example, from (36) we group the $\delta V_{p2}^{\pm}$ terms and compare them with corresponding terms in $\delta V_{p1}^{\pm}$ under the assumption $h \sim r$. Doing this we obtain $k^{\pm}h\,\delta V_{p2} \sim \delta V_{p1}$. Thus if $k^{\pm}h \gg 1$, that is $\lambda^{\pm}/h \ll 1$, we will be justified in dropping the $\delta V_2^{\pm}$, $\delta E_2^{\pm}$, and $\delta B_2^{\pm}$ to first order in $\lambda^{\pm}/h$ on the right hand sides of (36) through (39). This is just the WKB appriximation. We now have

$$(-i\omega + ViK^{\pm} \pm i\omega_{pc})\delta V_{p2}^{\pm} \qquad 0 \qquad \mp\frac{ieV}{m_p c}\delta B_2^{\pm} - \frac{e}{m_p}\delta E_2^{\pm} = -V\frac{\partial}{\partial r}\delta V_{p1}^{\pm} - \frac{V}{r}\delta V_{p1}^{\pm} \tag{40}$$

$$0 \qquad (-i\omega + ViK^{\pm} \mp i\omega_{ec})\delta V_{e2}^{\pm} \qquad \pm\frac{ieV}{m_e c}\delta B_2^{\pm} + \frac{e}{m_e}\delta E_2^{\pm} = -V\frac{\partial}{\partial r}\delta V_{e1}^{\pm} - \frac{V}{r}\delta V_{e1}^{\pm} \tag{41}$$

$$0 \qquad 0 \qquad -i\omega\,\delta B_2^{\pm} \qquad \mp cK^{\pm}\delta E_2^{\pm} = \mp ic\frac{\partial}{\partial r}\delta E_1^{\pm} \mp \frac{ic\,\delta E_1^{\pm}}{r} \tag{42}$$

$$-4\pi ne\,\delta V_{p2}^{\pm} \qquad 4\pi ne\,\delta V_{e2}^{\pm} \qquad \mp cK^{\pm}\delta B_2^{\pm} \qquad + i\omega\,\delta E_2^{\pm} = \mp ic\frac{\partial}{\partial r}\delta B_1^{\pm} \mp \frac{ic\,\delta B^{\pm}}{r} \tag{43}$$

where we have written the left hand sides of the equations in matrix-like form in anticipation of the next step. In order to obtain a single differential equation involving only $\delta V_1^{\pm}$, $\delta E_1^{\pm}$, and $\delta B_1^{\pm}$ we eliminate the quantities $\delta V_2^{\pm}$, $\delta E_2^{\pm}$, and $\delta B_2^{\pm}$ from

(40) through (43) by using the technique of matrix symmetrization of Weinberg (1962). Let us view (40) through (43) as a matrix equation of the form

$$A\,\underline{X}_2 = B\,\underline{X}_1 \tag{44}$$

where

$$\underline{X}_2 = \begin{pmatrix} \delta V_{p2}^{\pm} \\ \delta V_{e2}^{\pm} \\ \delta B_2^{\pm} \\ \delta E_2^{\pm} \end{pmatrix} \qquad \underline{X}_1 = \begin{pmatrix} \delta V_{p1}^{\pm} \\ \delta V_{e1}^{\pm} \\ \delta B_1^{\pm} \\ \delta E_1^{\pm} \end{pmatrix}$$

$$A = \begin{pmatrix} -i\omega + Vik^{\pm} \pm i\omega_{pc} & 0 & \mp ieV/m_p c & -e/m_p \\ 0 & -i\omega + Vik^{\pm} \mp i\omega_{ec} & \pm ieV/m_e c & e/m_e \\ 0 & 0 & -i\omega & \mp ck^{\pm} \\ -4\pi ne & 4\pi ne & \mp ck^{\pm} & i\omega \end{pmatrix}$$

$$B = \begin{pmatrix} -V\frac{\partial}{\partial r} - \frac{V}{r} & 0 & 0 & 0 \\ 0 & -V\frac{\partial}{\partial r} - \frac{V}{r} & 0 & 0 \\ 0 & 0 & 0 & \mp ic\frac{\partial}{\partial r} \mp \frac{ic}{r} \\ 0 & 0 & \mp ic\frac{\partial}{\partial r} \mp \frac{ic}{r} & 0 \end{pmatrix} \tag{45}$$

Since A is not symmetric, we premultiply both sides of our equation by another matrix C, so that the resultant matrix C A is symmetric. Trial and error has shown that we should choose C in the following manner:

$$C = \begin{pmatrix} 4\pi n m_p & 0 & 0 & 0 \\ 0 & 4\pi n m_e & 0 & 0 \\ 0 & 0 & 1 & \pm iV/c \\ 0 & 0 & \mp iV/c & 1 \end{pmatrix} \tag{46}$$

We thus obtain the following matrix equation

$$\begin{pmatrix} 4\pi n m_p(-i\omega + Vik^{\pm} \pm i\omega_{pc}) & 0 & \mp i4\pi ne\frac{V}{c} & -4\pi ne \\ 0 & 4\pi n m_e(-i\omega + Vik^{\pm} \mp i\omega_{ec}) & \pm i4\pi ne\frac{V}{c} & 4\pi ne \\ \mp i4\pi ne\frac{V}{c} & \pm i4\pi ne\frac{V}{c} & -i\omega - iVk^{\pm} & \mp ck^{\pm} \mp \frac{V\omega}{c} \\ -4\pi ne & 4\pi ne & \mp ck^{\pm} \mp \frac{V\omega}{c} & i\omega + iVk^{\pm} \end{pmatrix} \begin{pmatrix} \delta V_{p2}^{\pm} \\ \delta V_{e2}^{\pm} \\ \delta B_2^{\pm} \\ \delta E_2^{\pm} \end{pmatrix}$$

$$= \begin{pmatrix} -4\pi n m_p\left(V\frac{\partial}{\partial r} + \frac{V}{r}\right) & 0 & 0 & 0 \\ 0 & -4\pi n m_e\left(V\frac{\partial}{\partial r} + \frac{V}{r}\right) & 0 & 0 \\ 0 & 0 & V\frac{\partial}{\partial r} + \frac{V}{r} & \mp ic\frac{\partial}{\partial r} \mp \frac{ic}{r} \\ 0 & 0 & \mp ic\frac{\partial}{\partial r} \mp \frac{ic}{r} & -V\frac{\partial}{\partial r} - \frac{V}{r} \end{pmatrix} \begin{pmatrix} \delta V_{p1}^{\pm} \\ \delta V_{e1}^{\pm} \\ \delta B_1^{\pm} \\ \delta E_1^{\pm} \end{pmatrix} \tag{47}$$

We now have (40) through (43) in terms of symmetric matrices. Denoting C A = A' and C B = B' we have

$$A' X_2 = B' X_1 \qquad \text{with } A'_{ij} = A'_{ji} \text{ and } B'_{ij} = B'_{ji} \tag{48}$$

Premultiplying (48) by the transposed vector

$$\underline{X}_1^T = \left(\delta V_{p1}^{\pm} \quad \delta V_{e1}^{\pm} \quad \delta B_1^{\pm} \quad \delta E_1^{\pm} \right) \tag{49}$$

yields

$$\underline{X}_1^T A' \underline{X}_2 = \underline{X}_1^T B' \underline{X}_1 \tag{50}$$

Taking the transpose of both sides gives

$$\underline{X}_2^T A' \underline{X}_1 = \underline{X}_1^T B' \underline{X}_1 \tag{51}$$

But the left hand side of (51) is zero by virtue of equations (27) through (30), leaving us with a single differential equation in the unknown amplitudes $\delta V_{p1}^{\pm}$, $\delta V_{e1}^{\pm}$, $\delta E_1^{\pm}$, and $\delta B_1^{\pm}$. For simplicity of notation we now drop the subscript 1 on our unknown amplitudes. Performing the indicated matrix multiplications on the right hand side of (51), using (47) and (49), yields

$$\begin{aligned} 0 = & -4\pi n m_p V \delta V_p^{\pm} \frac{\partial}{\partial r} \delta V_p^{\pm} - 4\pi n m_p \frac{V}{r} (\delta V_p^{\pm})^2 - 4\pi n m_e V \delta V_e^{\pm} \frac{\partial}{\partial r} \delta V_e^{\pm} \\ & -4\pi n m_e \frac{V}{r} (\delta V_e^{\pm})^2 + V \delta B^{\pm} \frac{\partial}{\partial r} \delta B^{\pm} + \frac{V}{r} (\delta B^{\pm})^2 \mp ic \delta E^{\pm} \frac{\partial}{\partial r} \delta B^{\pm} \\ & \mp \frac{ic}{r} \delta E^{\pm} \delta B^{\pm} \mp ic \delta B^{\pm} \frac{\partial}{\partial r} \delta E^{\pm} \mp ic \delta B^{\pm} \delta E^{\pm} \\ & - V \delta E^{\pm} \frac{\partial}{\partial r} \delta E^{\pm} - \frac{V}{r} (\delta E^{\pm})^2 \end{aligned} \tag{52}$$

We thus obtain a single differential equation involving the unknown amplitudes $\delta V_p^{\pm}$, $\delta V_e^{\pm}$, $\delta E^{\pm}$, and $\delta B^{\pm}$ and their derivatives. In order to solve it, we must relate the amplitudes to one another. This is accomplished by choosing $\delta B^{\pm}$ as the independent amplitude in (27) through (30) and expressing the other amplitudes in terms of it. This process yields the following relationships between the amplitudes

$$\delta V_p^{\pm} = \frac{-\frac{\delta B^{\pm}}{B}\left(\frac{\omega}{K^{\pm}} - V\right)}{\left(1 \mp \frac{\omega - K^{\pm}V}{\omega_{pc}}\right)} \tag{53}$$

$$\delta V_e^{\pm} = \frac{-\frac{\delta B^{\pm}}{B}\left(\frac{\omega}{K^{\pm}} - V\right)}{\left(1 \pm \frac{\omega - K^{\pm}V}{\omega_{ec}}\right)} \tag{54}$$

$$\delta E^{\pm} = \pm \frac{\omega}{icK^{\pm}} \delta B^{\pm} \tag{55}$$

By substituting (53) through (55) into (52) and using the dispersion relation (33) to obtain an expression for the phase velocity $\frac{\omega}{K^{\pm}}$, we can solve (52) analytically to first order in $\frac{\lambda^{\pm}}{h}$ for short wavelength oscillations of arbitrary frequency ω.

CHAPTER 3

SOLUTIONS FOR THE WKB WAVE AMPLITUDES TO FIRST ORDER IN ω/ω_{pc}

In the previous chapter we obtained the dispersion relation (34) for $K^{\pm}$ in a two-fluid plasma correct to zeroth order in $\lambda^{\pm}/h$. From this dispersion relation we may derive the phase speed of the wave to terms of zeroth order in ω/ω_{pc}. Recalling (31) and (32) we may easily establish the identity

$$\omega_{pc}\,\omega_{ec} = \omega_p^2 V_A^2 / C^2 \tag{56}$$

From (33) we have

$$\left(\frac{\omega - K^{\pm}V}{\omega_{pc}} \mp 1\right)\left(\frac{\omega - K^{\pm}V}{\omega_{ec}} \pm 1\right)\left(\frac{\omega^2/(K^{\pm})^2}{C^2} - 1\right)\omega_{pc}\,\omega_{ec}\,C^2 (K^{\pm})^2 = \omega_p^2 (\omega - K^{\pm}V)^2 \tag{57}$$

Using (56) and neglecting terms of order $\dfrac{\omega - K^{\pm}V}{\omega_{pc}}$ and $\left(\dfrac{V^{\pm}_{phase}}{C}\right)^2$ where $V^{\pm}_{phase} = \dfrac{\omega}{K^{\pm}}$ we obtain

$$\omega = K^{\pm}(V + V_A) \tag{58}$$

This expression correct to zero order in ω/ω_{pc}, agrees with that of Belcher (1971) for the outward propagating wave. In order to solve for the unknown amplitudes $\delta V_p^{\pm}$, $\delta V_e^{\pm}$, $\delta E^{\pm}$, and $\delta B^{\pm}$

we substitute this phase relation into (53) through (55). Since we desire, at first, only the relations correct to zero order in $\frac{\omega}{\omega_{pc}}$ we neglect terms of order $\frac{\omega - k^{\pm}V}{\omega_{pc}}$ and smaller. This process yields the following relationships between the amplitudes:

$$\delta V_p^{\pm} = \delta V_e^{\pm} = \frac{\delta B^{\pm}}{\sqrt{4\pi n(m_p+m_e)}} = \frac{\delta B^{\pm}}{\sqrt{4\pi\rho}} \tag{59}$$

and

$$\delta E^{\pm} = \pm\frac{\omega}{ick^{\pm}}\delta B^{\pm} = \mp\frac{i}{c}(V+V_A)\,\delta B^{\pm} \tag{60}$$

Substituting (59) and (60) into (52) and using conservation of mass (equations (2) and (4) and the divergence free property of the magnetic field we can obtain a solution correct to zero order in $\frac{\omega}{\omega_{pc}}$ for the unknown amplitude $\delta B^{\pm}$. After some algebra we obtain

$$0 = \frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial\rho}{\partial r}(3V+V_A) - 2(V+V_A)\,\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} \tag{61}$$

and since $\frac{V_A}{V} = \frac{V_A^0}{V_0}\left(\frac{\rho}{\rho_0}\right)^{1/2}$ this easily integrates to

$$\delta B^{\pm}(r) = \delta B_0\left(\frac{\rho}{\rho_0}\right)^{3/4}\left[\frac{1+V_A^0/V_0}{1+\frac{V_A^0}{V_0}\left(\frac{\rho}{\rho_0}\right)^{1/2}}\right] \tag{62}$$

an expression first obtained for a one-fluid MHD plasma by Parker (1965). Notice that this case is the most general because no assumptions about the behavior of V, a stipulation we will encounter later, were made. If we do specialize to the case $V = V_0$ = constant we obtain

$$\delta B^{\pm}(r) = \delta B_0 \left(\frac{r_0}{r}\right)^{3/2} \left[\frac{1+\frac{V_A^0}{V_0}}{1+\frac{V_A^0}{V_0}\frac{r_0}{r}}\right] \tag{63}$$

as our expression correct to zero order in ω/ω_{pc}. Notice that this latter assumption implies (by conservation of mass) that $\rho \sim 1/r^2$.

Having obtained this zero order expression, we now seek to obtain an expression correct to first order in $\frac{\omega}{\omega_{pc}}$ for the case that $V = V_0$ = constant. To do this, we must obtain a first order expression for the phase velocity $\frac{\omega}{K^{\pm}}$. In order to obtain the phase velocity to first order in $\frac{\omega}{\omega_{pc}}$ we substitute (58) into (33). This process yields

$$\frac{\omega}{K^{\pm}} = V + V_A\left[1 \mp \frac{1}{2}\frac{V_A}{V+V_A}\left(\frac{\omega}{\omega_{pc}} - \frac{\omega}{\omega_{ec}}\right)\right] \tag{64}$$

where terms to second order in $\frac{\omega}{\omega_{pc}}$ have been neglected. Since $m_e/m_p = 1/1837 \ll 1$ we obtain

$$\frac{\omega}{K^{\pm}} = V + V_A\left[1 \mp \frac{1}{2}\frac{V_A}{V+V_A}\frac{\omega}{\omega_{pc}}\right] \tag{65}$$

Notice that in the absence of streaming, the phase velocity is modified to first order in $\frac{\omega}{\omega_{pc}}$ to

$$\frac{\omega}{K^{\pm}} = V_A\left[1 \mp \frac{1}{2}\left(\frac{\omega}{\omega_{pc}} - \frac{\omega}{\omega_{ec}}\right)\right] \tag{66}$$

which is the standard expression to this order in the two-fluid model with no streaming.

Since we now have a first order solution for the phase velocity, we may compute the first order relations between the unknown amplitudes $\delta V_{p1}^{\pm}$, $\delta V_{e1}^{\pm}$, $\delta E_1^{\pm}$ and $\delta B_1^{\pm}$ which we originally saw to zero order in $\frac{\omega}{\omega_{pc}}$ in (53) through (55). Denoting

$$\epsilon(r) = \epsilon \equiv \frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V+V_A} \tag{67}$$

and using (65) we obtain

$$\delta V_p^{\pm} = \frac{-\frac{\delta B^{\pm}}{B} V_A(1\mp\epsilon)}{\left[1 \mp \frac{K^{\pm}V_A(1\mp\epsilon)}{\omega_{pc}}\right]} = \frac{-\delta B^{\pm}}{\sqrt{4\pi\rho}}(1\pm\epsilon) \tag{68}$$

$$\delta V_e^{\pm} = \frac{-\frac{\delta B^{\pm}}{B} V_A(1\mp\epsilon)}{\left[1 \pm \frac{m_e}{m_p}\frac{K^{\pm}V_A(1\mp\epsilon)}{\omega_{pc}}\right]} = \frac{-\delta B^{\pm}}{\sqrt{4\pi\rho}}(1\mp\epsilon) \tag{69}$$

$$\delta E^{\pm} = \pm\frac{\omega}{icK^{\pm}}\delta B^{\pm} = \mp\frac{i}{c}\left[V+V_A(1\mp\epsilon)\right]\delta B^{\pm} \tag{70}$$

where we now restrict ourselves to frequencies $\omega \ll \omega_{pc}$ such that $\epsilon \ll 1$. In (68) we neglect terms of order ϵ^2. In addition, in (69) we neglect terms of order $(m_e/m_p)\epsilon$. Upon substitution of (68) through (70) into (52) we have (Appendix 1) to first order in ϵ

$$0=\frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial\rho}{\partial r}(3V+V_A)-2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}\pm\frac{\partial\epsilon}{\partial r}\left[(V_A-V)(\delta B^{\pm})^2\right]$$

$$+2\epsilon\left[\pm(V_A-V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}\pm\left(V-\frac{V_A}{2}\right)(\delta B^{\pm})^2\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\mp\frac{V(\delta B^{\pm})^2}{r}\right] \tag{71}$$

where we have neglected terms of order m_e/m_p. Now upon neglect of terms of order V_a^2/V^2 when they are multiplied by ω/ω_{pc} we get

$$\epsilon=\frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V+V_A}=\frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V}\left(1+\frac{V_A}{V}\right)^{-1}\simeq\frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V}=\frac{1}{2}\frac{\omega}{\omega_{pc}^{\circ}}\frac{V_A^{\circ}}{V_o}\frac{r}{r_o} \tag{72}$$

so that

$$\frac{\partial\epsilon}{\partial r}=\frac{\epsilon}{r} \tag{73}$$

Then after some algebra our differential equation becomes (Appendix 2)

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\mp\frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V}\frac{1}{r}=\frac{d}{dr}\ln\delta B^{\pm}(r) \tag{74}$$

where, again, terms of order V_a^2/V^2 have been neglected if they are multiplied by ω/ω_{pc}. This may be easily integrated using (72), giving

$$\ln\frac{\delta B^{\pm}}{\delta B^{\circ}}=\ln\left\{\left(\frac{r_o}{r}\right)^{3/2}\left[\frac{1+V_A^{\circ}/V_o}{1+\frac{V_A^{\circ}}{V_o}\left(\frac{\rho}{\rho_o}\right)^{1/2}}\right]\right\}\mp\frac{1}{2}\frac{\omega}{\omega_{pc}^{\circ}}\frac{V_A^{\circ}}{V_o}\frac{r-r_o}{r_o} \tag{75}$$

Let us define $\Delta r = r-r_o$ so that our full expression for $\delta B^{\pm}(r)$ is

$$\delta B^{\pm}=\delta B_o\left(\frac{r_o}{r}\right)^{3/2}\left[\frac{1+V_A^{\circ}/V_o}{1+\frac{V_A^{\circ}}{V_o}\left(\frac{\rho}{\rho_o}\right)^{1/2}}\right]e^{\mp\frac{1}{2}\frac{\omega}{\omega_{pc}^{\circ}}\frac{V_A^{\circ}}{V_o}\frac{\Delta r}{r_o}} \tag{76}$$

Labelling (63) by $\delta B_{0^{th}\,\text{order}}$ we see that

$$\delta B^{\pm}_{1^{st}\,\text{ORDER}} = \delta B_{0^{th}\,\text{ORDER}}\; e^{\mp\frac{1}{2}\frac{\omega}{\omega^{\circ}_{pc}}\frac{V^{\circ}_{A}}{V_{o}}\frac{\Delta r}{r_{o}}} \tag{77}$$

It is convenient (Olbert, private communication) to introduce a length $\mathcal{K}$ defined by

$$\frac{1}{\mathcal{K}} = \frac{1}{2}\,\frac{\omega}{\omega^{\circ}_{pc}}\,\frac{V^{\circ}_{A}}{V_{o}}\,\frac{1}{r_{o}} \tag{78}$$

which, from (72), is just ϵ/r . Then (77) becomes

$$\delta B^{\pm}_{1^{st}\,\text{ORDER}} = \delta B_{0^{th}\,\text{ORDER}}\; e^{\mp\Delta r/\mathcal{K}} \tag{79}$$

Hence after propagating a distance $\mathcal{K}$ the left (right) handed polarized waves will decay (grow) by a factor e . We shall see that for the case of the solar wind, $\mathcal{K}$ is extremely large. Hence the relative amplitudes of the left and right handed waves are nearly identical for all Δr of interest. However for the cases of astrophysical interest when Δr is extremely large, the length $\mathcal{K}$ and the variation in amplitude will be important.

In summary, this first order solution for the WKB wave amplitudes has been derived under the following main assumptions:

(1) $V = V_o = \text{constant} \ll c$

(2) Only terms to first order in $\frac{\omega}{\omega_{pc}}$ are kept

(3) Terms of order V_a^2/V^2 neglected if multiplied by $\frac{\omega}{\omega_{pc}}$

It thus represents a reasonable solution for outwardly propagating low frequency transverse waves in the solar wind. Under the first assumption, it is valid more than halfway back to the sun. The form of the solution predicts a new observable effect in the solar wind and is the basis of the next chapter.

CHAPTER 4

GENERALIZED FARADAY ROTATION OF LOW FREQUENCY TRANSVERSE WAVES IN THE SOLAR WIND

We have just seen that, to first order in ω/ω_{pc}, the wave amplitudes of circularly polarized waves either grow or decay after travelling distance Δr . We have also seen that circularly polarized waves of right and left handedness propagate at different phase velocities in the streaming two-fluid medium. We will show that the former result causes linearly polarized waves to become elliptical (with the eccentricity decreasing as Δr increases), while the latter results in a rotation of the plane of polarization. These two results are collectively referred to by the term "generalized Faraday rotation".

For clarity, the conventions used for denoting the handedness of circularly polarized waves will be discussed. For a wave with electric field $\vec{E}$ and magnetic field $\vec{B}$ travelling towards an observer, the term "left handed" is applied if the electric field vector $\vec{E}$ rotates in a counterclockwise direction when viewed by the observer. For clockwise rotation of $\vec{E}$, the observer calls the wave "right handed". This definition is the one used in classical optics (Jackson, 1962, Born and Wolf, 1964, Stone, 1963) and is the one used in this thesis. The opposite convention is commonly employed in plasma physics (Spitzer, 1962, Boyd and Sanderson, 1969, Stix, 1962). Having chosen a spherical coordinate system with origin at the sun, we must choose a combination of variables of the form $\hat{\theta} \pm i\hat{\phi}$ to denote an outwardly propagating left (right) handed wave (Figure 1). This justifies the choice (16) for the amplitudes. With all conventions explained, we next describe ordinary Faraday rotation.

In ordinary Faraday rotation of electromagnetic waves, the difference in phase velocity between left and right circularly polarized waves results in the pure rotation of the plane of polarization. We define

$$r^{\pm} = r^{\theta} \pm i r^{\phi} \tag{80}$$

so that

$$r^{\theta} = \frac{1}{2}\left(r^{+} + r^{-}\right) \tag{81}$$

and

$$r^{\phi} = \frac{1}{2i}\left(r^{+} - r^{-}\right) \tag{82}$$

We note that the upper (lower) sign in (80) corresponds to the left (right) handed wave under our chosen sign convention. In accordance with classical definitions where one focuses attention upon the electric vector $\vec{E}$, we make the correspondence

$$r^{\pm} \longrightarrow E^{\pm} \tag{83}$$

Ordinary Faraday rotation assumes that at some reference level r_0 there exists a linearly polarized wave of the form

$$E^{\theta} = E\cos\omega t \qquad E^{\phi} = 0 \tag{84}$$

which can be decomposed into a left and right circularly polarized wave of the form

$$\vec{E}^{\pm} = E e^{i(S^{\pm} - \omega t)}\left(\hat{\theta} \pm i\hat{\phi}\right) \tag{85}$$

where E is a real number representing the amplitude of the circularly polarized wave and

$$S^{\pm} = \int_{r_0}^{r} K^{\pm}(r)\,dr \tag{86}$$

where $K^{\pm}(r) = \frac{2\pi}{\lambda^{\pm}(r)}$. In (85) and the following, the appearance of the exponential implies that we take the real part of the expression when we talk about real oscillations. Notice that at $r=r_0$ we have

$$E^{\theta} = E\cos\omega t \qquad E^{\phi} = 0 \tag{87}$$

as required by (84). After travelling a distance $\Delta r=r-r_0$, (85) becomes

$$\vec{E}^{\pm} = E e^{i\left[\frac{1}{2}(S^{+}+S^{-}) \pm \frac{1}{2}(S^{+}-S^{-}) - \omega t\right]} (\hat{\theta} \pm i\hat{\phi}) \tag{88}$$

where we have decomposed $S^{\pm}$ into terms symmetric and anti-symmetric in S^{+} and S^{-} of the form

$$S^{\pm} = \frac{1}{2}(S^{+}+S^{-}) \pm \frac{1}{2}(S^{+}-S^{-}) \tag{89}$$

Then denoting

$$\varphi = \frac{1}{2}(S^{+}+S^{-}) - \omega t \tag{90}$$

and

$$\psi = \frac{1}{2}(S^{+}-S^{-}) \tag{91}$$

we have at r

$$\vec{E}^{+} = E e^{i\varphi} e^{i\psi} (\hat{\theta} + i\hat{\phi}) \qquad \vec{E}^{-} = E e^{i\varphi} e^{-i\psi} (\hat{\theta} - i\hat{\phi}) \tag{92}$$

from which it follows from (81) and (82) that

$$E^{\theta} = E e^{i\varphi} \cos\psi \qquad E^{\phi} = E e^{i\varphi} \sin\psi \tag{93}$$

Since E^{θ} and E^{ϕ} oscillate in phase at r, they combine into a linear oscillation which is rotated by an angle ψ with respect to the incident orientation. The phase of common oscillation has been changed from its original value at $r = r_0$ in the amount φ (Sommerfeld, 1964). Hence in the ordinary Faraday effect, when the left and the right handed waves arrive at $r > r_0$ they have equal amplitude but unequal phase. This results in a pure rotation of the plane of polarization.

Let us investigate the consequences of allowing the left and right handed waves to have variable amplitudes when they arrive at r as well as variable phases. Instead of (83) we will make the correspondence

$$r^{\pm} \rightarrow B^{\pm} \tag{94}$$

This choice is made because we have chosen $\delta B^{\pm}$ as the independent amplitude (Chapter 2). Let B_1 be the amplitude and S^{-} be the phase for the right-handed wave. Let B_2 be the amplitude and S^{+} be the phase for the left-handed wave. Hence the analytic representations at r are the following

$$\begin{aligned} \vec{B} &= B_1\cos(S^{-}-\omega t)\,\hat{\theta} + B_1\sin(S^{-}-\omega t)\,\hat{\phi} && \text{right} \\ \vec{B} &= B_2\cos(S^{+}-\omega t)\,\hat{\theta} - B_2\sin(S^{+}-\omega t)\,\hat{\phi} && \text{left} \end{aligned} \tag{95}$$

Notice that if we set the phases S^{-} and S^{+} equal to each other at r, the superposition of the right and the left polarized wave results in an elliptically polarized wave of semi-major axis $B_1 + B_2$ and semi-minor axis $|B_1 - B_2|$. This is easily seen in Figure 2 since the $\hat{\theta}$ components are always in phase and the

$\hat{\phi}$ components are always 180° cut of phase. Taking the case that S^- and S^+ are unequal, we rewrite the left handed wave from (95) as

$$B = B_2 \cos(S^- - \omega t + 2\chi')\,\hat{\theta} - B_2 \sin(S^- - \omega t + 2\chi')\,\hat{\phi} \tag{96}$$

where

$$\chi' = \frac{S^+ - S^-}{2} \tag{97}$$

This process emphasizes the explicit phase difference $S^+ - S^-$ between the two waves.

Combining the right and left handed waves at r yields

$$\begin{aligned} B^{\theta}_{TOTAL} &= B_1 \cos(S^- - \omega t) + B_2 \cos(S^- - \omega t + 2\chi') \\ &= (B_1 + B_2 \cos 2\chi')\cos(S^- - \omega t) \\ &\qquad + (-B_2 \sin 2\chi')\sin(S^- - \omega t) \end{aligned} \tag{98}$$

and

$$\begin{aligned} B^{\phi}_{TOTAL} &= B_1 \sin(S^- - \omega t) - B_2 \sin(S^- - \omega t + 2\chi') \\ &= (B_1 - B_2 \cos 2\chi')\sin(S^- - \omega t) \\ &\qquad + (-B_2 \sin 2\chi')\cos(S^- - \omega t) \end{aligned} \tag{99}$$

Thus

$$\begin{aligned} B^{\theta}_{TOTAL} &= (A + iB)\,e^{-i(S^- - \omega t)} \\ B^{\phi}_{TOTAL} &= (B + iC)\,e^{-i(S^- - \omega t)} \end{aligned} \tag{100}$$

with

$$\begin{aligned} A &= B_1 + B_2 \cos 2\chi' \\ B &= -B_2 \sin 2\chi' \\ C &= B_1 - B_2 \cos 2\chi' \end{aligned} \tag{101}$$

Hence

$$\vec{B}_{TOTAL} = \left[(A\hat{\theta}+B\hat{\phi})+i(B\hat{\theta}+C\hat{\phi})\right]e^{-i(S^{-}-\omega t)} \tag{102}$$

We therefore define

$$\begin{aligned}\vec{D} &= A\hat{\theta}+B\hat{\phi} \\ &= (B_1+B_2\cos 2\chi')\hat{\theta}+(-B_2\sin 2\chi')\hat{\phi}\end{aligned} \tag{103}$$

and

$$\begin{aligned}\vec{E} &= B\hat{\theta}+C\hat{\phi} \\ &= (-B_2\sin 2\chi')\hat{\theta}+(B_1-B_2\cos 2\chi')\hat{\phi}\end{aligned} \tag{104}$$

so that

$$\vec{B}_{TOTAL} = (\vec{D}+i\vec{E})e^{-i(S^{-}-\omega t)} \tag{105}$$

As before, the resultant oscillation is the real part of (105) and is expressed as

$$\vec{B}_{TOTAL} = \vec{D}\cos(S^{-}-\omega t)+\vec{E}\sin(S^{-}-\omega t) \tag{106}$$

Determining the position of $\vec{B}_{TOTAL}$ at $S^{-}-\omega t = 0$ and $S^{-}-\omega t = \pi/2$ gives us two conjugate radii of the ellipse of oscillation traced out by (104) as a function of time (Stone, 1963). By computing $\frac{d}{dt}\vec{B}_{TOTAL}$ at these same two times, we deduce that the direction of motion is from $\vec{D}$ to $\vec{E}$, and proceeds along the smaller of the two arcs. Since the right and left handed waves are out of phase at r, we should expect that their superposition leads to a wave rotated by an angle ϕ just as in ordinary Faraday rotation. The fact that they are of different amplitude implies that the resultant wave is elliptical, rather than linear (Figure 3).

In order to find the angle ψ which the semi-major axis of the ellipse makes with the $\hat{\Theta}$ axis, we seek to find $\vec{D}_o$ and $\vec{E}_o$, the principal radii of the ellipse. Principal radii have the property of lying on the symmetry axes and obey the equation

$$\vec{D}_o \cdot \vec{E}_o = 0 \tag{107}$$

Let us choose a time t_o such that $\vec{D}$ will coincide with $\vec{D}_o$ and $\vec{E}$ will coincide with $\vec{E}_o$.

$$\begin{aligned} B_{TOTAL}(t_o) &= \vec{D}_o = \vec{D}\cos(\delta - \omega t_o) + \vec{E}\sin(\delta - \omega t_o) \\ B_{TOTAL}(t_o + \tfrac{\pi}{2\omega}) &= \vec{E}_o = -\vec{D}\sin(\delta - \omega t_o) + \vec{E}\cos(\delta - \omega t_o) \end{aligned} \tag{108}$$

Using (107) we get

$$\begin{aligned} 0 = (E^2 - D^2)&\sin(\delta - \omega t_o)\cos(\delta - \omega t_o) \\ &+ (\vec{E}\cdot\vec{D})\left[\cos^2(\delta - \omega t_o) - \sin^2(\delta - \omega t_o)\right] \end{aligned}$$

Thus

$$\tan 2(\delta - \omega t_o) = \frac{2\vec{D}\cdot\vec{E}}{D^2 - E^2} \tag{110}$$

from which we may determine $\delta - \omega t_o$ and hence solve for $\vec{D}_o$ and $\vec{E}_o$ by using (108). Then

$$\vec{D}_o \cdot \hat{\Theta} = D_o \cos\psi \tag{111}$$

determines ψ, the angular orientation. Using (101) and (105) it is easy to show that (110) reduces to

$$\tan 2(\delta - \omega t_o) = \tan(-2\chi') \tag{112}$$

and thus that

$$\vec{D}_\circ = \vec{D}\cos(-\chi') + \vec{E}\sin(-\chi')$$
$$\vec{E}_\circ = -\vec{D}\sin(-\chi') + \vec{E}\cos(-\chi') \tag{113}$$

So

$$\begin{aligned}\vec{D}_\circ \cdot \hat{\theta} &= A\cos(-\chi') + B\sin(-\chi') \\ &= (B_1 + B_2\cos 2\chi')\cos\chi' + B_2 \sin 2\chi' \sin\chi' \\ &= (B_1 + B_2)\cos\chi'\end{aligned} \tag{114}$$

But since $D_\circ = \sqrt{(B_1 + B_2)^2}$ we obtain

$$\cos\psi = \cos\chi' \tag{115}$$

Thus we obtain the usual Faraday rotation $\psi = \chi' = \dfrac{\delta^+ - \delta^-}{2}$ which we derived earlier. A shorter derivation of (115) (Lazarus, private communication) is given in Appendix 3. The eccentricity at r is given by

$$\xi = \sqrt{1 - E_\circ^2 / D_\circ^2} \tag{116}$$

which reduces to (Appendix 3)

$$\xi = \frac{2\sqrt{B_1 B_2}}{B_1 + B_2} \tag{117}$$

and simplifies using (78) and (79) to

$$\xi = \operatorname{sech}\frac{\Delta r}{K} \tag{118}$$

Using (65) and (33) to compute δ^+ and δ^- we form the phase difference to get the following:

$$\frac{S^+ - S^-}{2} = \frac{1}{2}\int_{r_0}^{r} \frac{\omega^2}{\omega_{pc}} \frac{V_A^2\, dr}{(V+V_A)^3} = \frac{1}{2}\frac{\omega^2}{\omega_{pc}^0} V_A^{o2} \int_{r_0}^{r} \frac{dr}{\left(V_0 + V_A^0 \frac{r_0}{r}\right)^3} \tag{119}$$

neglecting terms of order ϵ^2. This integrates to

$$\frac{1}{2}\frac{\omega^2}{\omega_{pc}^0}\frac{V_A^{o2}}{V_0^3}\left[r + \frac{2V_A^0 r}{(V_0+V_A)} + \frac{V_0 V_A^0 r_0}{2(V_0+V_A)^2} - \frac{3V_A^0}{V_0} r_0 \log(V_0+V_A) - \frac{3V_A^0}{V_0} r_0 \log r\right]_{r_0}^{r} \tag{120}$$

the leading term of which is

$$\frac{1}{2}\frac{\omega^2}{\omega_{pc}^0}\frac{V_A^{o2}}{V_0^3}\Delta r \tag{121}$$

where $\Delta r = r - r_o$ is the distance from the reference level. Then from (115), (120), and (118) we can predict the angular orientation and the eccentricity of the resultant elliptically polarized wave as a function of r.

We have seen that when a linearly polarized wave propagates in and is convected by a two-fluid solar wind, its constituent right and left handed circularly polarized components undergo amplitude and phase changes. Upon recombination of the right and left handed wave, we find an elliptically polarized wave whose semi-major axis is oriented at an angle ψ with respect to the original orientation of the linearly polarized wave. We shall now consider examples of this phenomenon for cases of interest in the study of the solar wind and in astrophysical contexts.

Chapter 5

RESULTS AND CONCLUSIONS

Ordinary Faraday rotation concerns the pure rotation of the plane of polarization of high frequency electromagnetic waves. We have shown that for low frequency waves ($\omega/\omega_{pc} \ll 1$) in a two-fluid solar wind, there exists both a rotation of the plane of polarization and a gradual transition from linear to elliptical polarization. Examples will be given to illustrate these conclusions.

From (118) we may easily obtain Figure 4 showing the variation of the eccentricity as a function of the dimensionless ratio $\Delta r/\mathcal{K}$. In Figure 5 we show representative ellipses for the cases $\xi = 1$, 0.9, and 1/e occuring at $\Delta r/\mathcal{K} = 0$, 0.47, and 1.653 respectively. It must be remembered that for the variation of ξ to be physically meaningful at large r we must not violate the assumptions under which (118) was derived. In particular we must stay well away in frequency from the local proton cyclotron frequency over the entire path length Δr. We have already seen (Chapter 2) that a resonance occurs at this frequency when one transforms to a frame of reference co-moving at the wind velocity V. Furthermore, examination of the index of refraction (34) shows that the right handed wave solution does not yield real values for n beyond ω_{pc} (Van Kampen and Felderhoff, 1967). Since we assume $B \sim 1/r^2$ and $\omega_{pc} \sim B$, we are forced to examine waves of smaller and smaller frequency as we seek to obtain results applicable at larger and larger r. We must also be careful not to violate the WKB approximation that $\frac{\lambda^{\pm}}{h} \ll 1$. We have seen in (58) that the zero order phase velocity is $\frac{\omega}{k^{\pm}} = V + V_A$ from which it follows that $\lambda^{\pm} = \frac{2\pi}{k^{\pm}} = 2\pi\left(\frac{V+V_A}{\omega}\right)$. Under the assumption that $V = V_o$ = constant it is easy to show that $V_a \sim 1/r$. So if we are at large r, we must choose ω to be very small in order not to violate $\frac{\omega}{\omega_{pc}} \ll 1$ locally. But this means $\lambda^{\pm} \gg 1$. In order not to violate the WKB approximation the scale length h must be enormous.

With these thought in mind, let us examine the behavior of the eccentricity for observed solar wind parameters. We choose the reference level r_o at 100 solar radii (0.46 A.U.) so that the condition $V = V_o$ = constant is met. Consistent with observed average solar wind velocities, we take this constant equal to 400 km/sec (Lazarus, 1969). We take a field at the sun of 2 gauss, giving a reference level field of 20 gamma (1 gamma= 10^{-5} gauss). We set V_a^o = 100 km/sec, since the observed Alfvén velocity at 1 A.U. is approximately 50 km/sec (Belcher and Davis, 1971). For a wave having an inertial period T of 1 minute, we obtain from (78) that $\mathcal{K} \sim 146 r_o \sim 68$ A.U. Computing the distance Δr at which ξ decreases to 0.9ξ yields $\Delta r \sim 32$A.U. Hence for a reasonable solar wind case, the wave remains linearly polarized over a very large range of r (the orbit of Pluto is 39 A.U.).

We can also examine the behavior of the angle of rotation through which an initially linearly polarized wave turns as it propagates in and is convected by a two-fluid solar wind. From (121) we expect (from the ω^2 dependence) that waves of longer inertial periods rotate less in the distance Δr than waves of shorter periods. In Figure 6 we show the angle of rotation ψ (determined from (115) and (120)) as a function of distance for waves of inertial periods of 3, 4, 5, 6, 7, and 8 minutes. The same reference level parameters of V_o = 400 km/sec, V_a^o = 100 km/sec B_{sun} = 2 gauss were chosen. A striking feature in Figure 6 is the linearity of the rotation with distance, as in ordinary Faraday rotation in an infinite homogeneous medium. This can easily be explained in physical terms upon realization of the relations between plasma density and magnetic field in our assumed model. The radial background magnetic field varies as $1/r^2$, so that $\omega_{pc} \propto 1/r^2$ and $\frac{\omega}{\omega_{pc}} \propto r^2$. Thus one might expect the amount of rotation to increase more rapidly than r, since ω comes closer and closer to ω_{pc} as r increases. But the assumption that V = V_o = constant constrains the density to fall off as $1/r^2$, so that the Alfvén velocity falls off as 1/r. Fortuitously, this decrease in V_a with r exactly balances the increase in ω/ω_{pc} with r. This immediately follows by rewriting (121) in terms of the local Alfvén velocity and local cyclotron frequency. When expressed in these parameters, the leading term in the expression for the rotation angle becomes $\frac{1}{2} \frac{\omega^2}{V_o^3} \frac{V_A^2}{\omega_{pc}} \Delta r$. The cyclotron frequency varies as B, i.e. $\omega_{pc} \propto 1/r^2$. Assuming V = V_o = constant constrains the density to vary as $1/r^2$. Thus the square of the Alfvén velocity also varies as $1/r^2$. Hence the ratio V_a^2/ω_{pc} is constant and the rotation is predominantly linear with distance, as in ordinary Faraday rotation. Figure 7 shows the dependence of rotation angle (measured at 1 A.U.) upon the square of the frequency measured in an inertial frame for the solar wind case in Figure 6.

In summary these calculations predict that linearly polarized waves will gradually become elliptical as they propagate in and are convected by a two-fluid solar wind. In addition they undergo Faraday rotation so that the semi-major axis of the ellipse of polarization is turned through a well-defined angle. In principle this effect can be observed if at some distance r_0 there exists linearly polarized waves which do not thereafter damp into thermal motions as they travel in the two-fluid medium.

Appendix 1

Equation (52) consists of 12 terms with the upper (lower) sign representing left (right) handed polarized waves. Using (68) and substituting into the first term of (52) yields

$$-4\pi n m_p V \delta V_p^{\pm} \frac{\partial}{\partial r} \delta V_p^{\pm} = -4\pi n m_p V \frac{\delta B^{\pm}}{\sqrt{4\pi\rho}}(1 \pm \epsilon) \frac{\partial}{\partial r} \frac{\delta B^{\pm}}{\sqrt{4\pi\rho}}(1 \pm \epsilon) \tag{A1.1}$$

which becomes

$$-4\pi n m_p V \left(\frac{\delta B^{\pm}}{\sqrt{4\pi\rho}}(1 \pm \epsilon)^2 \frac{\partial}{\partial r} \frac{\delta B^{\pm}}{\sqrt{4\pi\rho}} \pm \frac{(\delta B^{\pm})^2}{4\pi\rho}(1 \pm \epsilon) \frac{\partial \epsilon}{\partial r} \right) \tag{A1.2}$$

Neglecting terms of order ϵ^2 and $\epsilon \frac{\partial \epsilon}{\partial r}$ yields

$$-4\pi n m_p V \left(\left\{ \frac{\delta B^{\pm}}{4\pi\rho} \frac{\partial}{\partial r} \delta B^{\pm} - \frac{(\delta B^{\pm})^2}{4\pi\rho} \frac{1}{2\rho} \frac{\partial \rho}{\partial r} \right\} + 2\epsilon \left[\pm \frac{\delta B^{\pm}}{4\pi\rho} \frac{\partial}{\partial r} \delta B^{\pm} \right.\right.$$

$$\left.\left. \mp \frac{(\delta B^{\pm})^2}{4\pi\rho} \frac{1}{2\rho} \frac{\partial \rho}{\partial r} \right] + \frac{\partial \epsilon}{\partial r} \left[\pm \frac{(\delta B^{\pm})^2}{4\pi\rho} \right] \right) \tag{A1.3}$$

Using (68) in the second term of (52) yields

$$-4\pi n m_p \frac{V}{r} (\delta V_p^{\pm})^2 = -4\pi n m_p \frac{V}{r} \left(\left\{ \frac{V_A^2 (\delta B^{\pm})^2}{B^2} \right\} + 2\epsilon \left[\pm \frac{V_A^2 (\delta B^{\pm})^2}{B^2} \right] \right) \tag{A1.4}$$

where terms of order ϵ^2 have been neglected.

Using (69) we notice that the substitution $\epsilon \to -\epsilon$ implies $\delta V_e^{\pm} \to \delta V_p^{\pm}$. In addition, the third term of (52) may bo obtained from the first by the substitution $m_e \to m_p$. Hence the third term of (52) may be obtained from (A1.3) using these substitutions, The result is

$$-4\pi n m_e V\left(\left\{\frac{\delta B^{\pm}}{4\pi\rho}\frac{\partial}{\partial r}\delta B^{\pm}-\frac{(\delta B^{\pm})^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right\}+2\epsilon\left[\mp\frac{\delta B^{\pm}}{4\pi\rho}\frac{\partial}{\partial r}\delta B^{\pm}\right.\right. \tag{A1.5}$$

$$\left.\left.\pm\frac{(\delta B^{\pm})^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right]+\frac{\partial\epsilon}{\partial r}\left[\mp\frac{(\delta B^{\pm})^2}{4\pi\rho}\right]\right)$$

Similar considerations apply to the fourth term of (52) which may be obtained from (A1.4). The result is

$$-4\pi n m_e\frac{V}{r}(\delta V_e^{\pm})^2=-4\pi n m_e\frac{V}{r}(\delta V_e^{\pm})^2\left(\left\{\frac{V_A^2(\delta B^{\pm})^2}{B^2}\right\}+2\epsilon\left[\mp\frac{V_A^2(\delta B^{\pm})^2}{B^2}\right]\right) \tag{A1.6}$$

The fifth and sixth terms are straightforward. The seventh term follows easily from (70). The eighth and tenth terms add to give

$$\mp\frac{2ic}{r}\delta E^{\pm}\delta B^{\pm}=\left\{\frac{-2(V+V_A)(\delta B^{\pm})^2}{r}\right\}+2\epsilon\left[\frac{-V_A(\delta B^{\pm})^2}{r}\right] \tag{A1.7}$$

using (70). The ninth term is

$$\mp ic\,\delta B^{\pm}\frac{\partial}{\partial r}\delta E^{\pm}=\mp ic\,\delta B^{\pm}\frac{\partial}{\partial r}\left(\pm\frac{\omega}{ick^{\pm}}\delta B^{\pm}\right)$$

$$=-(\delta B^{\pm})^2\frac{\partial}{\partial r}\frac{\omega}{k^{\pm}}-\frac{\omega}{k^{\pm}}\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} \tag{A1.8}$$

where (70) has been used. Using (65) for the phase velocity and taking $V = V_0$ = constant yields

$$-(\delta B^{\pm})^2(1\mp\epsilon)\frac{\partial V_A}{\partial r}\pm V_A(\delta B^{\pm})^2\frac{\partial\epsilon}{\partial r}-\delta B^{\pm}(V+V_A)\frac{\partial}{\partial r}\delta B^{\pm}\pm\epsilon V_A\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} \tag{A1.9}$$

But since

$$\frac{\partial V_A}{\partial r} = -\frac{V_A}{2\rho}\frac{\partial \rho}{\partial r} - \frac{2V_A}{r} \qquad \text{(A1.10)}$$

we obtain

$$(\delta B^{\pm})^2\left(\frac{V_A}{2\rho}\frac{\partial\rho}{\partial r}+\frac{2V_A}{r}\right)-\delta B^{\pm}(V+V_A)\frac{\partial}{\partial r}\delta B^{\pm}+\epsilon\left[\pm V_A\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}\right.$$

$$\left.+(\delta B^{\pm})^2\left(\mp\frac{V_A}{2\rho}\frac{\partial\rho}{\partial r}\mp\frac{2V_A}{r}\right)\right]+\frac{\partial\epsilon}{\partial r}\left[\pm V_A(\delta B^{\pm})^2\right] \qquad \text{(A1.11)}$$

for the ninth term of (52). The eleventh term is easily shown to be

$$-V\delta E^{\pm}\frac{\partial}{\partial r}\delta E^{\pm} = \frac{V}{c^2}\frac{\omega}{k^{\pm}}\delta B^{\pm}\frac{\partial}{\partial r}\left(\frac{\omega}{k^{\pm}}\delta B^{\pm}\right) \qquad \text{(A1.12)}$$

while the twelfth is

$$-\frac{V}{r}(\delta E^{\pm})^2 = \frac{V}{c^2}\frac{1}{r}\left(\frac{\omega}{k^{\pm}}\right)^2(\delta B^{\pm})^2 \qquad \text{(A1.13)}$$

Both of these terms may be neglected since $\frac{VV_{PHASE}}{c^2} \ll 1$. Adding up all these terms and grouping terms of like order yields

$$0=\left\{-4\pi nm_pV\frac{\delta B^{\pm}}{4\pi\rho}\frac{\partial}{\partial r}\delta B^{\pm}+4\pi nm_pV\frac{(\delta B^{\pm})^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right\}+2\epsilon\left[\mp\frac{4\pi nm_pV\delta B^{\pm}}{4\pi\rho}\frac{\partial}{\partial r}\delta B^{\pm}\right.$$

$$\left.\pm\frac{4\pi nm_pV(\delta B^{\pm})^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right]+\frac{\partial\epsilon}{\partial r}\left[\mp\frac{4\pi nm_pV(\delta B^{\pm})^2}{4\pi\rho}\right]+\left\{-\frac{4\pi nm_pV}{r}\frac{V_A^2(\delta B^{\pm})^2}{B^2}\right\}$$

$$+2\epsilon\left[\mp \frac{4\pi n m_p V}{r}\frac{V_A^2(\delta B^\pm)^2}{B^2}\right]+\left\{-\frac{4\pi n m_e V\delta B^\pm}{4\pi\rho}\frac{\partial}{\partial r}\delta B^\pm+\frac{4\pi n m_e V}{4\pi\rho}\frac{(\delta B^\pm)^2}{2\rho}\frac{\partial\rho}{\partial r}\right\}$$

$$+2\epsilon\left[\pm\frac{4\pi n m_e V\delta B^\pm}{4\pi\rho}\frac{\partial}{\partial r}\delta B^\pm \mp \frac{4\pi n m_e V(\delta B^\pm)^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right]$$

$$+\frac{\partial\epsilon}{\partial r}\left[\pm\frac{4\pi n m_e V(\delta B^\pm)^2}{4\pi\rho}\right]+\left\{-\frac{4\pi n m_e V}{r}\frac{V_A^2(\delta B^\pm)^2}{B^2}\right\}+2\epsilon\left[\pm\frac{4\pi n m_e V}{r}\frac{V_A^2(\delta B^\pm)^2}{B^2}\right]$$

$$+\left\{\frac{1}{2\rho}\frac{\partial\rho}{\partial r}V_A(\delta B^\pm)^2+\frac{2V_A(\delta B^\pm)^2}{r}-(V+V_A)\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm\right\}+\epsilon\left[\mp\frac{V_A(\delta B^\pm)^2}{2\rho}\frac{\partial\rho}{\partial r}\right.$$

$$\left.\mp\frac{2V_A(\delta B^\pm)^2}{r}\pm V_A\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm\right]+\frac{\partial\epsilon}{\partial r}\left[\pm V_A(\delta B^\pm)^2\right]+\left\{-\frac{2(V+V_A)}{r}(\delta B^\pm)^2\right\}$$

$$+2\epsilon\left[\pm\frac{V_A(\delta B^\pm)^2}{r}\right]+\left\{V\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm+\frac{V}{r}(\delta B^\pm)^2\right\}$$

$$+\left\{-(V+V_A)\delta B^\mp\frac{\partial}{\partial r}\delta B^\mp\right\}+\epsilon\left[\pm V_A\delta B^\mp\frac{\partial}{\partial r}\delta B^\mp\right] \qquad (A1:14)$$

as may easily be verified. The terms in curly brackets are of zero order in $\frac{\omega}{\omega_{pc}}$ while the others are of first order. Combining the terms in curly brackets in (A1.14) we obtain

$$\left\{-4\pi n(m_p+m_e)\frac{V\delta B^\pm}{4\pi\rho}\frac{\partial}{\partial r}\delta B^\pm+4\pi n(m_p+m_e)\frac{V(\delta B^\pm)^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}-\frac{4\pi\rho V}{r}\frac{V_A^2(\delta B^\pm)^2}{B^2}\right.$$

$$\left.+\frac{1}{2\rho}\frac{\partial\rho}{\partial r}V_A(\delta B^\pm)^2-V_A\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm-\frac{V}{r}(\delta B^\pm)^2-V\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm-V_A\delta B^\pm\frac{\partial}{\partial r}\delta B^\pm\right\} \qquad (A1.15)$$

Combining the terms in square brackets in (A1.14) we obtain

$$2\epsilon\left[\mp 4\pi n(m_p-m_e)\frac{V\delta B^\pm}{4\pi\rho}\frac{\partial}{\partial r}\delta B^\pm \pm 4\pi n(m_p-m_e)\frac{V(\delta B^\pm)^2}{4\pi\rho}\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\right.$$

$$\mp 4\pi n(m_p - m_e)\frac{V}{r}\frac{V_A^2(\delta B^{\pm})^2}{B^2} \pm \frac{V_A}{2}\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} \mp \frac{V_A(\delta B^{\pm})^2}{2}\frac{1}{2\rho}\frac{\partial \rho}{\partial r}$$

$$\mp \frac{V_A(\delta B^{\pm})^2}{r} \pm \frac{V_A(\delta B^{\pm})^2}{r} \pm \frac{V_A}{2}\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}\Bigg] + \frac{\partial \epsilon}{\partial r}\Bigg[\pm V_A(\delta B^{\pm})^2 \tag{A1.16}$$

$$\mp 4\pi n(m_p - m_e)\frac{V(\delta B^{\pm})^2}{4\pi\rho}\Bigg]$$

Simplifying (A1.15) yields

$$\left\{-2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + (V+V_A)\frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial \rho}{\partial r} - 2V\frac{(\delta B^{\pm})^2}{r}\right\} \tag{A1.17}$$

Neglecting terms of order m_e/m_p in (A1.16) yields

$$2\epsilon\left[\pm(V_A - V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} \pm \left(V - \frac{V_A}{2}\right)\frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial \rho}{\partial r} \mp \frac{V(\delta B^{\pm})^2}{r}\right]$$

$$+\frac{\partial \epsilon}{\partial r}\left[\pm(V_A - V)(\delta B^{\pm})^2\right] \tag{A1.18}$$

which is the first order term in (71). From conservation of mass (the sum of (2) and (4)) we have

$$\rho V r^2 = \text{CONSTANT} \tag{A1.19}$$

But since $V = V_\circ$ = constant we get

$$\frac{1}{2\rho}\frac{\partial \rho}{\partial r} = -\frac{1}{r} \tag{A1.20}$$

using (A1.20) in (A1.17) we get

$$\left\{-2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + (3V+V_A)(\delta B^{\pm})^2\frac{1}{2\rho}\frac{\partial \rho}{\partial r}\right\} \tag{A1.21}$$

which is the zero order term in (71)

Appendix 2

Upon substitution of (73) into (71) we get

$$0 = \frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial \rho}{\partial r}(3V+V_A) - 2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + 2\epsilon\Big[\pm(V_A-V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}$$

$$\pm\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\left(V-\frac{V_A}{2}\right)(\delta B^{\pm})^2 \mp \frac{V(\delta B^{\pm})^2}{r}\Big] \pm \frac{\epsilon}{r}(\delta B^{\pm})^2(V_A-V) \tag{A2.1}$$

which simplifies to

$$0 = \frac{(\delta B^{\pm})^2}{2\rho}\frac{\partial \rho}{\partial r}(3V+V_A) - 2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + 2\epsilon\Big[\pm(V_A-V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}$$

$$\pm\frac{1}{2\rho}\frac{\partial\rho}{\partial r}\left(V-\frac{V_A}{2}\right)(\delta B^{\pm})^2 \mp \frac{3}{2}\frac{V(\delta B^{\pm})^2}{r} \pm \frac{1}{2}\frac{V_A(\delta B^{\pm})^2}{r} \tag{A2.2}$$

Using (A1.20) in the above yields

$$0 = -\frac{(\delta B^{\pm})^2}{r}(3V+V_A) - 2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + 2\epsilon\Big[\pm(V_A-V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}$$

$$+\left(V-\frac{V_A}{2}\right)\frac{(\delta B^{\pm})^2}{r} \mp \frac{3}{2}\frac{V(\delta B^{\pm})^2}{r} \pm \frac{1}{2}\frac{V_A(\delta B^{\pm})^2}{r}\Big] \tag{A2.3}$$

Thus

$$0 = -(3V+V_A)\frac{(\delta B^{\pm})^2}{r} - 2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm} + 2\epsilon\Big[\pm(V_A-V)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}$$

$$\mp\frac{5}{2}\frac{V(\delta B^{\pm})^2}{r} \pm \frac{V_A(\delta B^{\pm})^2}{r}\Big] \tag{A2.4}$$

Hence

$$(3V+V_A)\frac{(\delta B^{\pm})^2}{r}\left[1 \pm \epsilon\frac{5V-2V_A}{3V+V_A}\right] = -2(V+V_A)\delta B^{\pm}\frac{\partial}{\partial r}\delta B^{\pm}\left[1 \mp \epsilon\frac{V_A-V}{V_A+V}\right] \tag{A2.5}$$

So

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\left[\frac{1\pm\epsilon\,\frac{5V-2V_A}{3V+V_A}}{1\mp\epsilon\,\frac{V_A-V}{V_A+V}}\right]=\frac{\partial}{\partial r}\ln\delta B^{\pm} \tag{A2.6}$$

Neglecting terms of order ϵ^2 yields

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\left[1\pm\epsilon\,\frac{5V-2V_A}{3V+V_A}\pm\epsilon\,\frac{V_A-V}{V_A+V}\right]=\frac{\partial}{\partial r}\ln\delta B^{\pm} \tag{A2.7}$$

which simplifies to

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\left[1\pm\epsilon\,\frac{2V^2+5VV_A-V_A^2}{3V^2+4VV_A+V_A^2}\right]=\frac{\partial}{\partial r}\ln\delta B^{\pm} \tag{A2.8}$$

Using (72) we get

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\left[1\pm\frac{1}{3}\frac{\omega}{\omega_{pc}}\frac{V_A}{V}\left(1+\frac{5}{2}\frac{V_A}{V}-\frac{1}{2}\frac{V_A^2}{V^2}\right)\left(1+\frac{4}{3}\frac{V_A}{V}+\frac{V_A^2}{V^2}\right)^{-1}\right] \tag{A2.9}$$

and neglecting terms of order $V_a^2\,/\,V^2$ if they are multiplied by ω/ω_{pc} we obtain

$$-\frac{(3V+V_A)}{2(V+V_A)}\frac{1}{r}\mp\frac{1}{2}\frac{\omega}{\omega_{pc}}\frac{V_A}{V}\frac{1}{r}=\frac{\partial}{\partial r}\ln\delta B^{\pm} \tag{A2.10}$$

which is (74).

Appendix 3

Equation (115) may be derived with a minimum of algebra by using the following method suggested by Dr. A.J.Lazarus: Let us combine a left circularly polarized wave of the form $r = B_1 \exp(i\omega t)$ with a right circularly polarized wave of the form $r = B_2 \exp(-i\omega t)$. For outwardly propagating waves, the sence of rotation agrees with the convention chosen in this thesis. Let the right circularly polarized wave differ in phase from the left by $2\chi'$. The resultant oscillation is expressed as

$$r = B_1 e^{i\omega t} + B_2 e^{-i\omega t + 2i\chi'} \tag{A3.1}$$

Thus

$$r = e^{i\chi'}\left(B_1 e^{i(\omega t - \chi')} + B_2 e^{-i(\omega t - \chi')}\right) \tag{A3.2}$$

Defining $\gamma \equiv \omega t - \chi'$ we obtain

$$\begin{aligned} r &= e^{i\chi'}\left(B_1 e^{i\gamma} + B_2 e^{-i\gamma}\right) \\ &= e^{i\chi'}\left(\left[\frac{B_1+B_2}{2}\right](e^{i\gamma}+e^{-i\gamma}) + \left[\frac{B_1-B_2}{2}\right](e^{i\gamma}-e^{-i\gamma})\right) \\ &= e^{i\chi'}\left((B_1+B_2)\cos\gamma + i(B_1-B_2)\sin\gamma\right) \end{aligned} \tag{A3.3}$$

Now suppose that the phase difference $2\chi'$ is zero. Then the resultant oscillation has the form

$$r = (B_1+B_2)\cos\omega t + i(B_1-B_2)\sin\omega t \tag{A3.4}$$

which represents an elliptically polarized wave of semi-major axis $B_1 + B_2$ and semi-minor axis $|B_1 - B_2|$. The same result was obtained in (95) and illustrated in Figure 2. When $2\chi'$ is different from zero, (A3.3) represents an ellipse rotated by an angle χ' with respect to the $\hat{\Theta}$ axis. The same result was obtained more laboriously in (98) through (115) and illustrated in Figure 3.

We now focus our attention on the eccentricity of the elliptically polarized wave. Equation (116) may be put in the form

$$\xi = \frac{\sqrt{D_0^2 - E_0^2}}{D_0} \tag{A3.5}$$

From (113) we have

$$\begin{aligned} \vec{D}_0 &= \vec{D}\cos\chi' - \vec{E}\sin\chi' \\ E_0 &= D\sin\chi' + E\cos\chi' \end{aligned} \tag{A3.6}$$

from which it is easy to show that

$$D_0^2 - E_0^2 = (D^2 - E^2)\cos 2\chi' - 2\vec{D}\cdot\vec{E}\sin 2\chi' \tag{A3.7}$$

and hence that

$$\xi^2 = \frac{(D^2 - E^2)\cos 2\chi' - 2\vec{D}\cdot\vec{E}\sin 2\chi'}{D^2\cos^2\chi' + E^2\sin^2\chi' - \vec{D}\cdot\vec{E}\sin 2\chi'} \tag{A3.8}$$

Now from (105) and (101) we have the following

$$\frac{4B_1B_2\cos^2 2\chi' + 4B_1B_2\sin^2 2\chi'}{\left[(B_1^2+B_2^2+2B_1B_2\cos 2\chi')\cos^2\chi' + (B_1^2+B_2^2-2B_1B_2\cos 2\chi')\sin^2\chi' + 2B_1B_2\sin^2 2\chi'\right]} \tag{A3.9}$$

which simplifies to

$$\xi = \frac{2\sqrt{B_1 B_2}}{B_1 + B_2} \tag{A3.10}$$

Notice that if $B_1 = B_2$ then $\xi = 1$, which means that the ellipse degenerates into a straight line. In other words, we recover the case of regular Faraday rotation of a linear polarized wave rotated by an angle χ' with respect to the incident orientation.

From (79) we have

$$\delta B^{\pm}_{1^{ST}\,\mathrm{ORDER}} = \delta B_{0^{TH}\,\mathrm{ORDER}}{}' \, e^{\mp\frac{\Delta r}{\kappa}} \tag{A3.11}$$

The upper (lower) sign corresponds to the left (right) hand mode. According to (94) and (95) we take $B_2 = B_L$ and $B_1 = B_R$. Then

$$\begin{aligned} B_1 + B_2 &= \delta B_{0^{TH}\,\mathrm{ORDER}} \left(e^{\frac{\Delta r}{\kappa}} + e^{-\frac{\Delta r}{\kappa}} \right) \\ &= 2\,\delta B_{0^{TH}\,\mathrm{ORDER}} \cosh \frac{\Delta r}{\kappa} \end{aligned} \tag{A3.12}$$

And

$$B_1 B_2 = \delta B^2_{0^{TH}\,\mathrm{ORDER}} \tag{A3.13}$$

Thus from (A3.10)

$$\xi = \frac{1}{\cosh \frac{\Delta r}{\kappa}} = \operatorname{sech} \frac{\Delta r}{\kappa} \tag{A3.14}$$

which is (118) in the text.

FIGURE CAPTIONS

Figure

1 Spherical coordinate system with origin at the Sun. An outwardly propagating wave travelling towards an observer is represented by the wave vector $\vec{k}$. The wave is "left-handed" since $\vec{E}$ rotates counterclockwise.

2 Representation of a right circularly polarized wave (RCP) of magnitude B_1 and a left circularly polarized wave of magnitude B_2 (with $B_2 < B_1$). The waves are taken to be in phase and are propagating out of the page. The ellipse shown is the superposition of the two above waves.

3 Superposition of right and left circularly polarized waves when they differ in phase. The rotation angle ψ is shown relative to the $\hat{\Theta}$ axis. Conjugate radii D and E are shown along with principle radii D_o and E_o. The wave is propagating out of the page.

4 The eccentricity ξ of the elliptically polarized wave as a function of $\Delta r/\kappa$.

5 Ellipses with varying magnitude of eccentricity ξ. $\xi = 1$ corresponds to the reference level where a linearly polarized wave exists. $\xi = 0.9$ corresponds to $\Delta r/\kappa = 0.47$. $\xi = 1/e$ corresponds to $\Delta r/\kappa = 1.653$.

6 Rotation angle ψ as a function of distance for waves having inertial periods of 3,4,5,6,7,and 8 minutes. The reference level is indicated at 0.46 A.U.

7 Rotation angle ψ observed at 1 A.U. as a function of ω^2 for the waves of Figure 6.

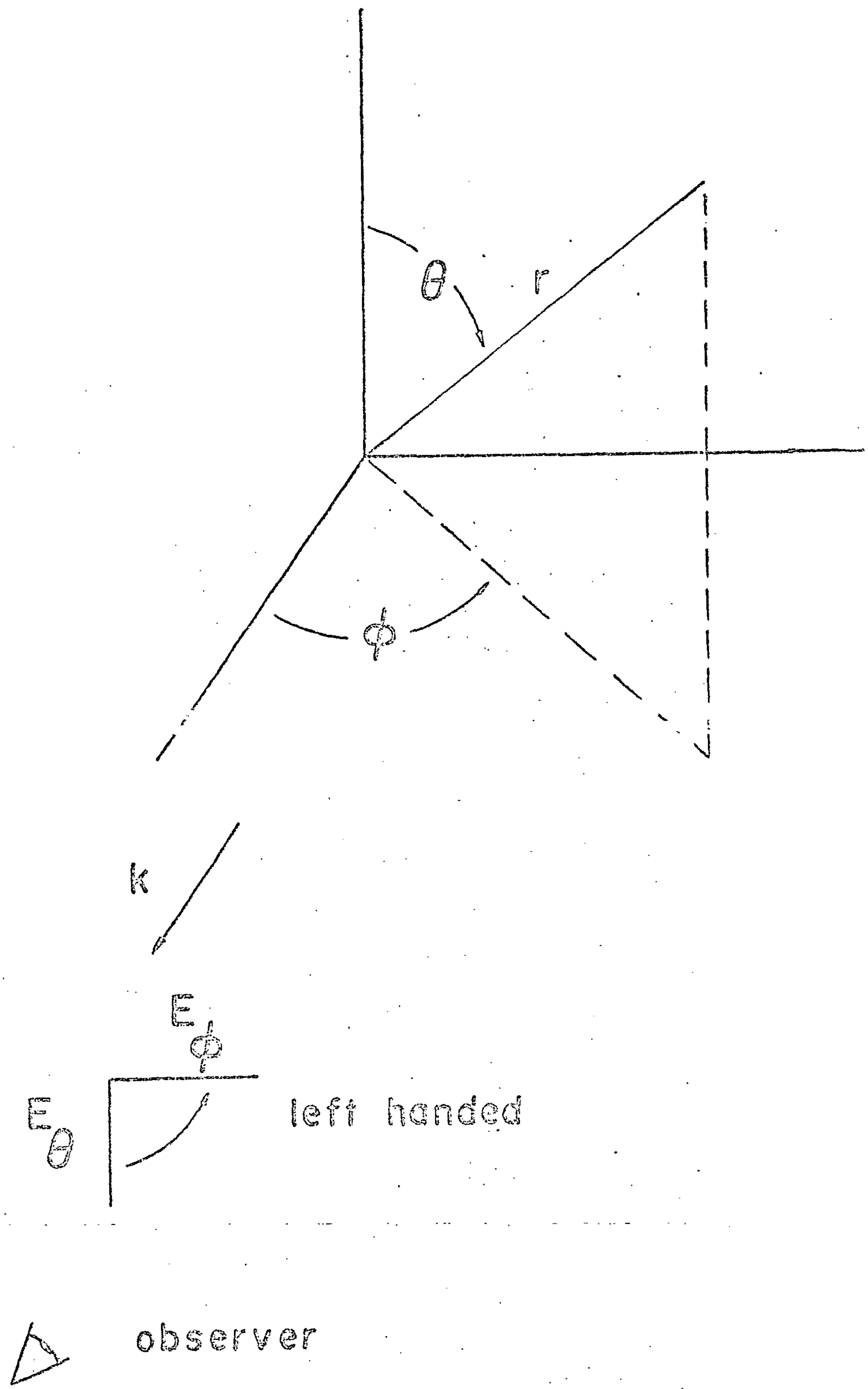
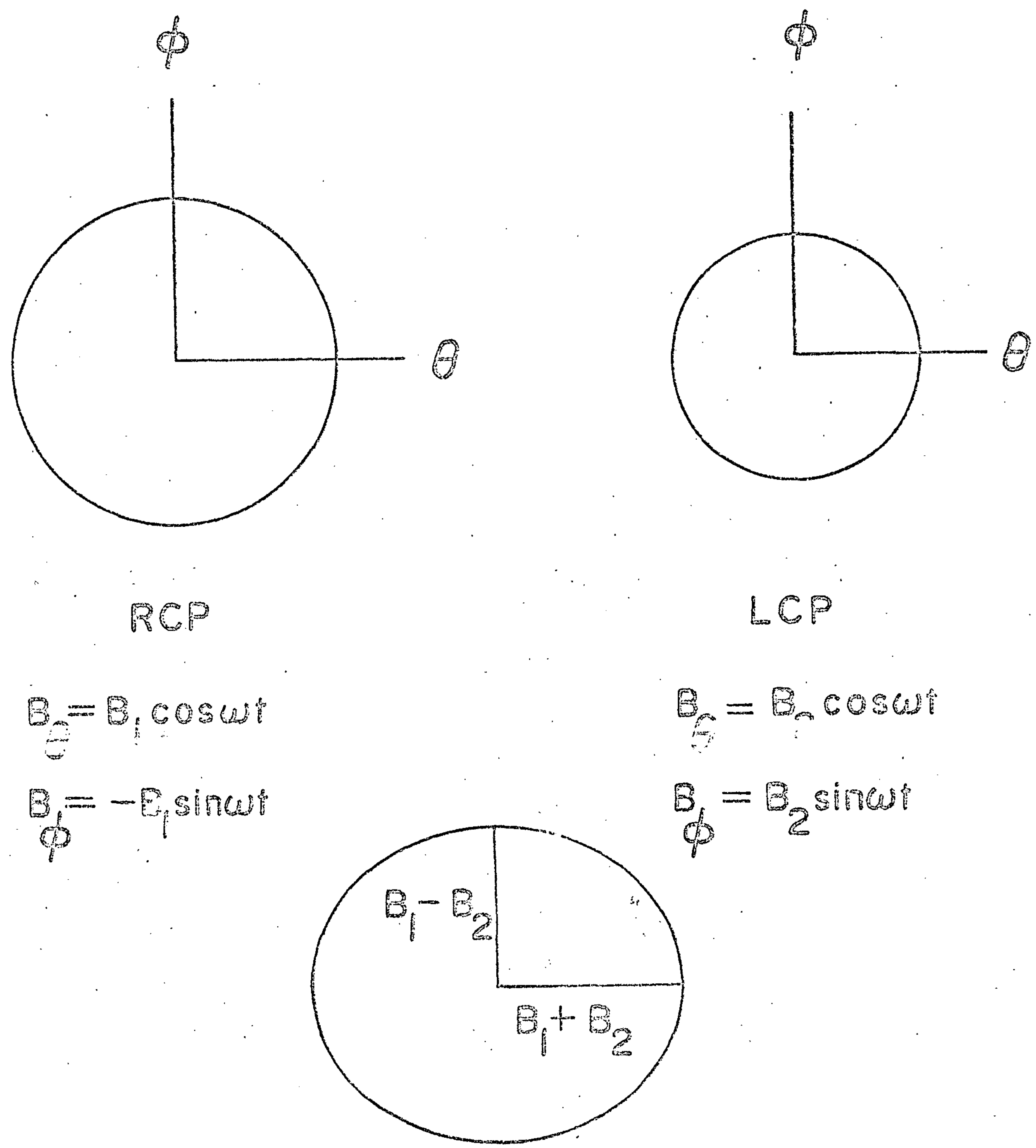


FIGURE I



ϕ
θ
ϕ
θ
RCP
LCP
$B_\theta = B_1 \cos\omega t$
$B_\phi = -B_1 \sin\omega t$
$B_\theta = B_2 \cos\omega t$
$B_\phi = B_2 \sin\omega t$
$B_1 - B_2$
$B_1 + B_2$

FIGURE 2

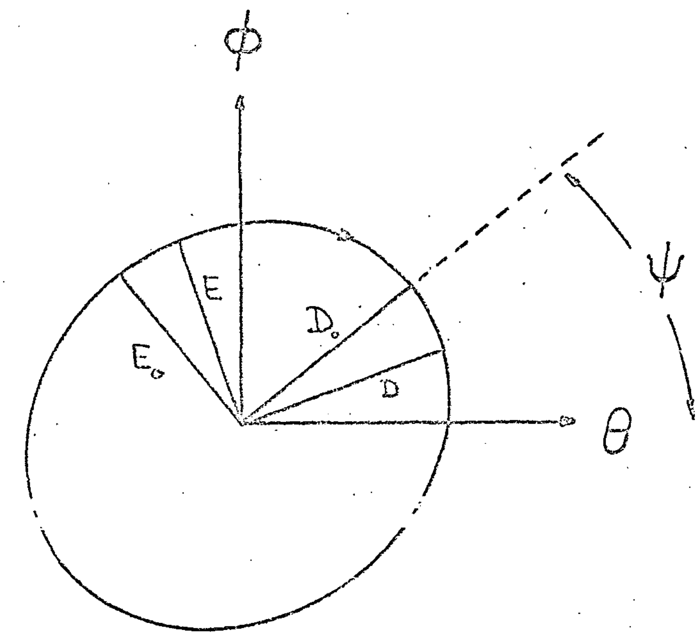


FIGURE 3

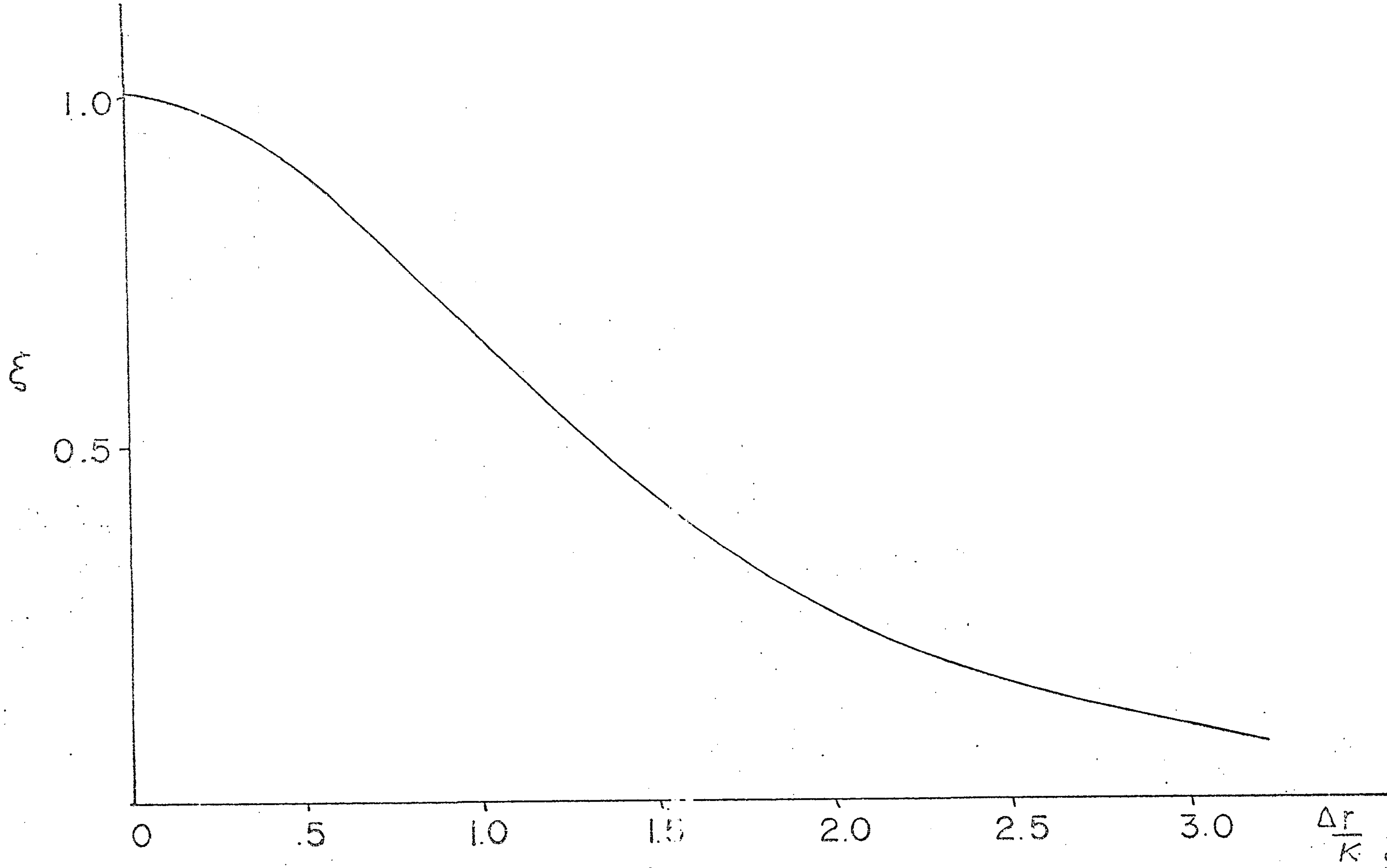


FIGURE 4

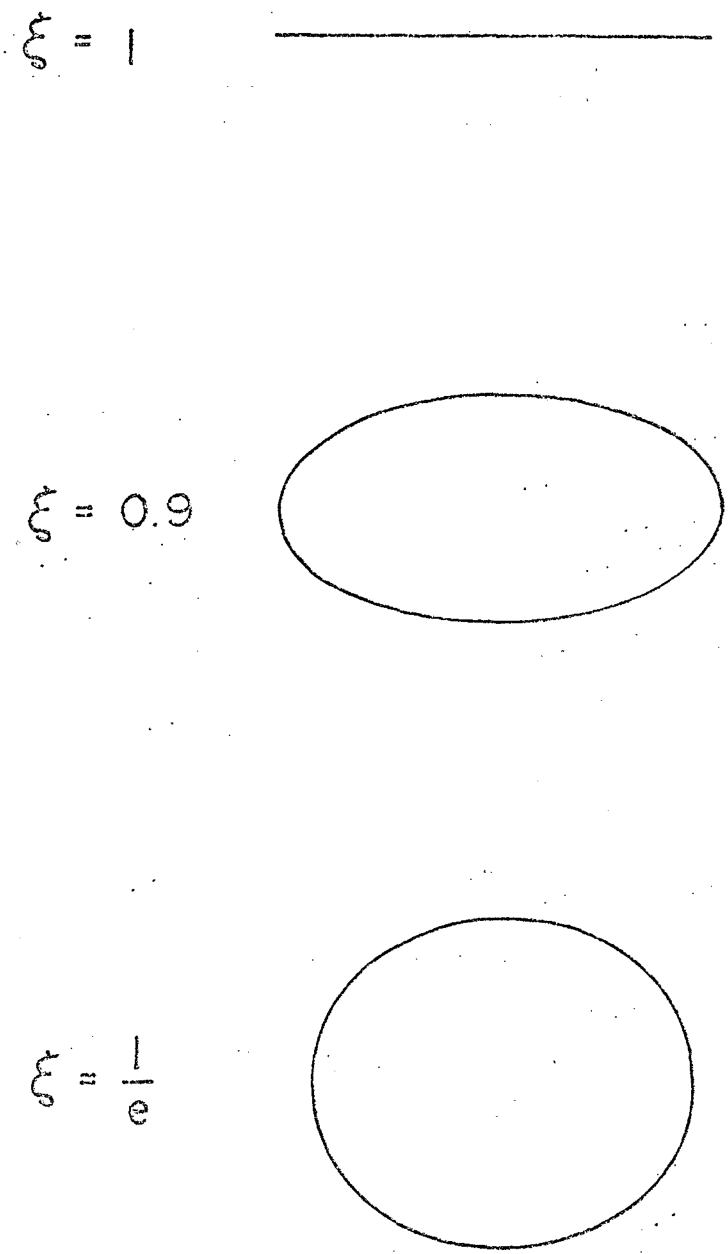


FIGURE 5

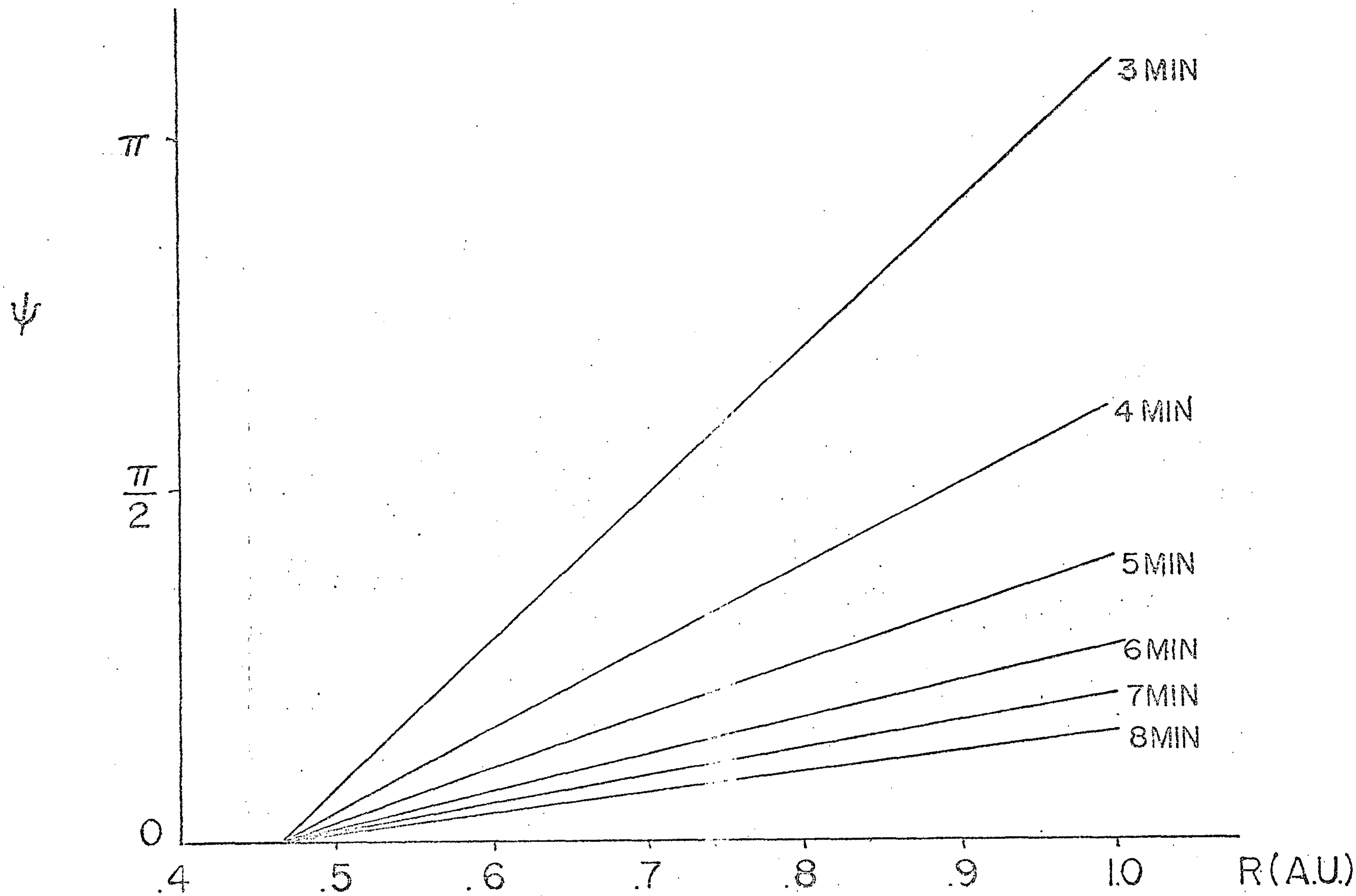


FIGURE 6

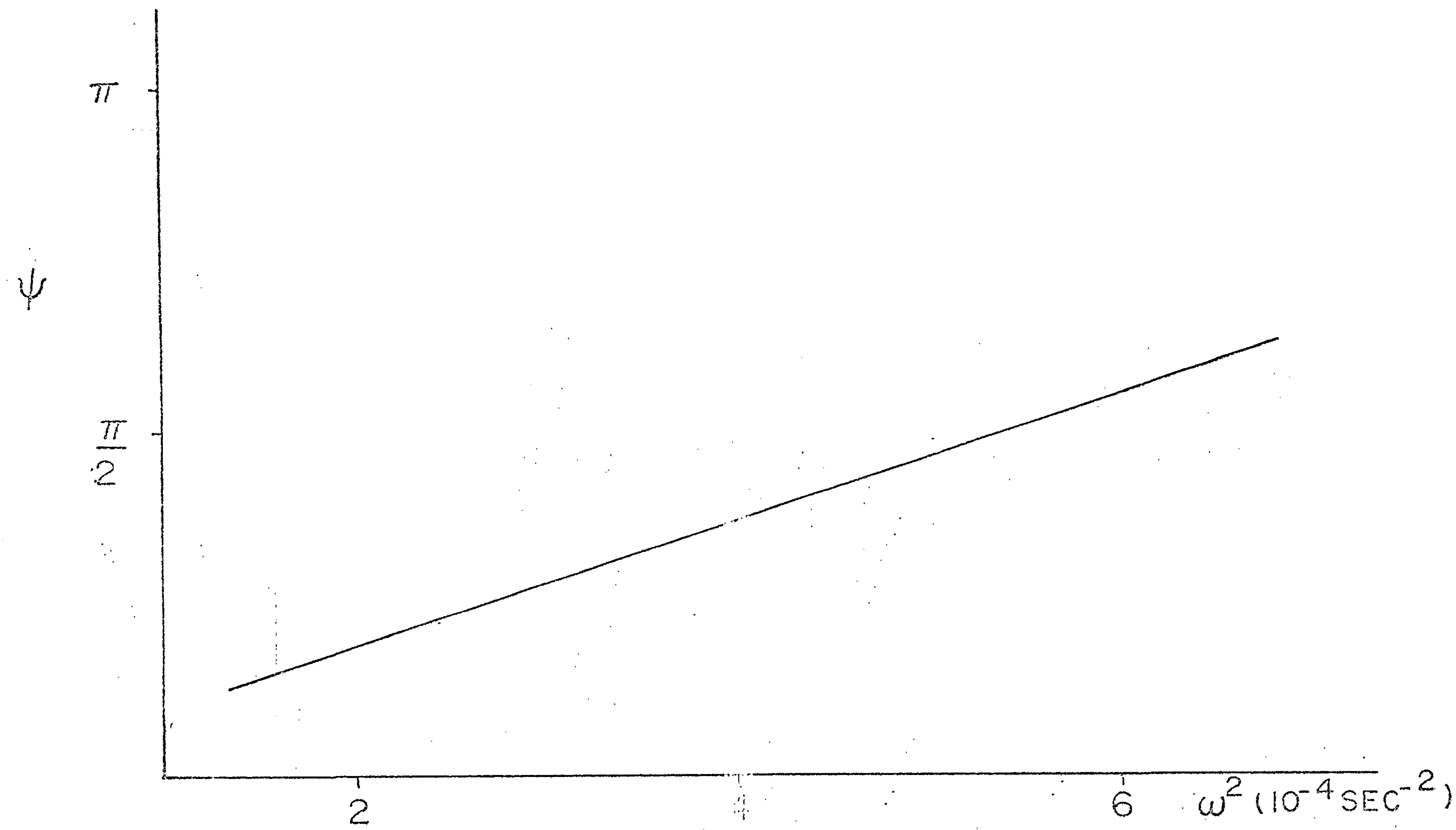


FIGURE 7

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