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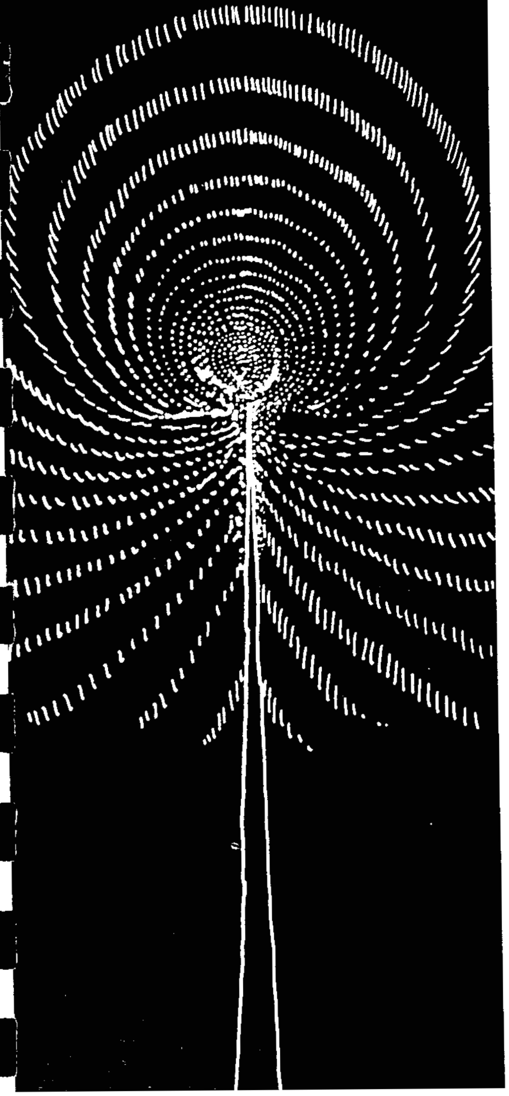

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ABSTRACT

The problem of an elastic solid containing a semi-infinite plane crack subjected to concentrated shears parallel to the edge of the crack is considered in this paper. A closed form solution using four distinct harmonic functions (none of which can be taken arbitrarily) is found to satisfy the finite displacement and inverse square root stress singularity at the edge of the crack. Explicit expressions in terms of elementary functions are given for the distribution of stress and displacement in the solid. These are obtained by employing Fourier and Kontorovich-Lebedev integral transforms and certain singular solutions of Laplace equations in three dimensions. The variations of the intensity of the local stress field along the crack border are shown graphically. The present analysis offers an example which is in contrast with the conclusion established in the literature that one of the four Papkovich-Neuber functions in three-dimensional elasticity may be arbitrarily set to zero.

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INTRODUCTION

The general notion of the classical theory of elasticity [1-3]* is that one of the four harmonic functions appearing in the Papkovich-Neuber solution of the displacement equations of equilibrium may be set to zero (or arbitrarily chosen) without loss in completeness of the solution. A counter example is provided in this paper on an unfinished problem initiated by Uflyand [4]. The problem, basically, is concerned with determining the distribution of stress in a half-space when its plane surface is subjected to mixed conditions separated by an infinite rectilinear boundary.

The solution to the problem of concentrated normal and shear forces applied to the surfaces of a semi-infinite plane crack serves as the Green's function for a number of crack problems of interest in fracture mechanics. A sketch of the crack configuration is shown in Figure (1). In terms of the coordinates indicated, the surfaces of the crack are defined by $\theta = \mp\pi$, $0<r<\infty$, $-\infty<z<\infty$. Without affecting the generality of the problem, the points of application of the loads may be taken as $r = a$, $\theta = \mp\pi$, $z = 0$. Because of skew-symmetry, it suffices to consider a half-space $y \geq 0$ with mixed conditions specified on the plane $y = 0$. It was Uflyand [4] who made the first attempt to solve this problem by using only three

*Numbers in square brackets designate references at the end.

harmonic functions in the space $y \geq 0$ and the Kontorovich-Lebedev integral transform in the variable r. The fourth function in the classical Papkovich-Neuber representation of the displacement field was arbitrarily neglected. This technique has failed to yield a solution for the case of two equal and opposite force R directed parallel to the edge of the crack [4, page 382].

The objective of this paper is two fold. First, the Green's function for the three-dimensional crack problem is constructed. Next, it is shown that the fourth Papkovich-Neuber function contributes to the solution and is not zero. Four distinct harmonic functions are required to satisfy the finite displacement and the inverse square root of r conditions as $r \to 0$. The state of displacement and stress throughout the solid is determined in closed form in terms of elementary functions. These are obtained by employing a Fourier transform in the variable z, Kontorovich-Lebedev transform in r and singular solutions of the Laplace equation in three dimensions [5]. The stress-intensity factors due to the loading R are determined explicitly and their variations along the crack border are shown graphically.

BASIC EQUATIONS

The equilibrium of an isotropic and homogeneous solid is governed by the Navier displacement equations which in the absence of body forces and in vector notations appear as

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \qquad (1)$$

where $\underset{\sim}{u}$ is the displacement vector, $\underset{\sim}{\nabla}$ is the gradient operator and ν designates Poisson's ratio of the material. Denoting the projections of the displacement in the directions of cylindrical coordinates by (u_r, u_θ, u_z), the Papkovich-Neuber general representation of equations (1) gives the displacement field

$$u_r = 4(1-\nu)(f_1\cos\theta + f_2\sin\theta) - \frac{\partial F}{\partial r} \qquad (2a)$$

$$u_\theta = 4(1-\nu)(f_2\cos\theta - f_1\sin\theta) - \frac{1}{r}\frac{\partial F}{\partial \theta} \qquad (2b)$$

$$u_z = 4(1-\nu)f_3 - \frac{\partial F}{\partial z} \qquad (2c)$$

in which the following abbreviation has been introduced

$$F = f_0 + (r\cos\theta)f_1 + (r\sin\theta)f_2 + zf_3 \qquad (3)$$

and $f_n(r,\theta,z)$, $n = 0,1,2,3$, are space harmonic functions. The corresponding stress field is readily obtained from equations (2) and (3) and the usual stress-displacement relations in linear elastostatics. In particular the stresses associated with the θ-plane are found:

$$\frac{\sigma_\theta}{2\mu} = -(1-2\nu)(\cos\theta\frac{\partial f_1}{\partial r} + \sin\theta\frac{\partial f_2}{\partial r}$$

$$+ \frac{2(1-\nu)}{r}(\cos\theta\frac{\partial f_2}{\partial\theta} - \sin\theta\frac{\partial f_1}{\partial\theta}) + 2\nu\frac{\partial f_3}{\partial z}$$

$$+ \frac{\partial^2 f_0}{\partial r^2} + \frac{\partial^2 f_0}{\partial z^2} - \frac{1}{r}(\cos\theta\frac{\partial^2 f_1}{\partial\theta^2} + \sin\theta\frac{\partial^2 f_2}{\partial\theta^2})$$

$$+ z(\frac{\partial^2 f_3}{\partial r^2} + \frac{\partial^2 f_3}{\partial z^2}) \tag{4a}$$

$$\frac{\tau_{r\theta}}{2\mu} = \frac{\partial G}{\partial r} + \frac{2(1-\nu)}{r}(\cos\theta\frac{\partial f_1}{\partial\theta} + \sin\theta\frac{\partial f_2}{\partial\theta}) \tag{4b}$$

$$\frac{\tau_{\theta z}}{2\mu} = \frac{\partial G}{\partial z} + \frac{2(1-\nu)}{r}\frac{\partial f_3}{\partial\theta} \tag{4c}$$

In equations (4), μ depicts the shear modulus of the solid and the function $G(r,\theta,z)$ is defined through the relation

$$G = (1-2\nu)(f_2\cos\theta - f_1\sin\theta) - \frac{1}{r}\frac{\partial f_0}{\partial\theta} - \cos\frac{\partial f_1}{\partial\theta}$$

$$- \sin\theta\frac{\partial f_2}{\partial\theta} - \frac{z}{r}\frac{\partial f_3}{\partial\theta} \tag{5}$$

The remaining stress components σ_z, σ_r and τ_{rz} are not needed for the mere purpose of completing the analysis and will not be mentioned.

THE PLANE CRACK PROBLEM

Consider the case of a pair of equal and opposite concentrated shear forces R applied to the surfaces of a plane as shown in Figure 1 (with P = Q = 0). For this problem, the displacements and stresses are skew-symmetric with respect to the variable z. Hence, u_r, u_θ, σ_θ and $\tau_{r\theta}$ are odd in z while u_z and $\tau_{\theta z}$ are even in the same variable. In addition, the deformed solid also exhibits skew-symmetry in the coordinate θ, namely u_r, u_z, σ_θ being odd in θ and u_θ, $\tau_{r\theta}$, $\tau_{\theta z}$ even in θ. The latter condition suggests that the problem can be formulated for the upper half-space $y \geq 0$ with appropriate conditions prescribed on the planes $\theta = 0$ and $\theta = \pi$. In observing these conditions, the continuity of the solid across the plane $\theta = 0$ implies

$$u_r(r,o,z) = 0 \tag{6a}$$

$$u_z(r,o,z) = 0 \tag{6b}$$

$$\sigma_\theta(r,o,z) = 0 \tag{6c}$$

On the upper surface of the crack $\theta = \pi$, the loading is described by

$$\tau_{\theta z}(r,\pi,z) = \tau_o(r,z) \tag{7a}$$

$$\tau_{r\theta}(r,\pi,z) = 0 \tag{7b}$$

$$\sigma_\theta(r,\pi,z) = 0 \tag{7c}$$

in which $\tau_0(r,z)$ is a specified function describing the applied shear loading. Aside from the requirements given by equations (6) and (7), the regularity conditions at infinity must also be satisfied, i.e., the displacements and stresses must behave as L^{-1} and L^{-2} when

$$L = (r^2+z^2)^{1/2} \to \infty$$

Furthermore, near the crack boundary ($r \to 0$), the displacements must be finite and the stresses are expected to have the usual square root singularity, $r^{-1/2}$.

Inserting relations (6) into equations (2a), (2c) and (4a) and making use of (3), the functions f_n ($n = 0,1,2,3$) can be determined from the following system of equations in the region $\theta = 0$

$$(3-4\nu)f_1 - \frac{\partial f_0}{\partial r} - r\frac{\partial f_1}{\partial r} - z\frac{\partial f_3}{\partial r} = 0 \tag{8a}$$

$$(3-4\nu)f_3 - \frac{\partial f_0}{\partial z} - r\frac{\partial f_1}{\partial z} - z\frac{\partial f_3}{\partial z} = 0 \tag{8b}$$

$$(1-2\nu)\frac{\partial f_1}{\partial r} + \frac{2(1-\nu)}{r}\frac{\partial f_2}{\partial \theta} + 2\nu\frac{\partial f_3}{\partial z} + \frac{\partial^2 f_0}{\partial r^2} + \frac{\partial^2 f_0}{\partial z^2}$$

$$- \frac{1}{r}\frac{\partial^2 f_1}{\partial \theta^2} + z\left(\frac{\partial^2 f_3}{\partial r^2} + \frac{\partial^2 f_3}{\partial z^2}\right) = 0 \tag{8c}$$

Upon recognizing the identity

$$- \frac{1}{r} \frac{\partial^2 f_1}{\partial \theta^2} = \frac{\partial f_1}{\partial r} + r\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) f_1, \quad \theta = 0 \tag{9}$$

equations (8) are satisfied by requiring

$$f_0(r,o,z) = 0 \tag{10a}$$

$$f_1(r,o,z) = 0 \tag{10b}$$

$$f_3(r,o,z) = 0 \tag{10c}$$

$$\frac{\partial f_2}{\partial \theta} (r,o,z) = 0 \tag{10d}$$

In a similar manner, conditions (7) when used in conjunction with (4), give rise to the following relations for $\theta = \pi$, i.e.,

$$\frac{\partial G}{\partial z} + \frac{2(1-\nu)}{r} \frac{\partial f_3}{\partial \theta} = \frac{\tau_0(r,z)}{2\mu} \tag{11a}$$

$$\frac{\partial G}{\partial r} - \frac{2(1-\nu)}{r} \frac{\partial f_1}{\partial \theta} = 0 \tag{11b}$$

$$(1-2\nu) \frac{\partial f_1}{\partial r} - \frac{2(1-\nu)}{r} \frac{\partial f_2}{\partial \theta} + 2\nu \frac{\partial f_3}{\partial z} + \frac{\partial^2 f_0}{\partial r^2} + \frac{\partial^2 f_0}{\partial z^2}$$

$$+ \frac{1}{r} \frac{\partial^2 f_1}{\partial \theta^2} + z\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) f_3 = 0 \tag{11c}$$

where G can be found from

$$G = -(1-2\nu)f_2 - \frac{1}{r}\frac{\partial f_0}{\partial \theta} + \frac{\partial f_1}{\partial \theta} - \frac{z}{r}\frac{\partial f_3}{\partial \theta}, \quad \theta = \pi \tag{12}$$

Further simplification may be achieved by setting

$$G = 0, \quad \theta = \pi \tag{13}$$

On account of equation (13), it follows that

$$\frac{\partial G}{\partial z} = \frac{\partial G}{\partial r} = 0, \quad \theta = \pi \tag{14}$$

and as an immediate consequence, equations (11a) and (11b) yield the crack-surface conditions

$$\frac{\partial f_3}{\partial \theta} = \frac{r\tau_0(r,z)}{4\mu(1-\nu)}, \quad \theta = \pi \tag{15}$$

$$\frac{\partial f_1}{\partial \theta} = 0, \quad \theta = \pi \tag{16}$$

Equations (15) and (16) together with (10b) and (10c) provide the necessary relations for the evaluation of f_1 and f_3. The remaining functions f_0 and f_2 are obtained from equations (13), (15), (16) and the fact that the loading is applied at the point $r = a$, $z = a$. This yields

$$(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} = 0, \quad \theta = \pi \tag{17}$$

From equations (10a) and (10d), it is not difficult to verify that

$$\frac{\partial}{\partial y}\left[(1-2\nu)f_2 - \frac{\partial f_0}{\partial y}\right] = 0, \ \theta = 0 \tag{18}$$

The mixed conditions (17) and (18) provide a relation between the potentials f_2 and $\frac{\partial f_0}{\partial y}$ which is given in equation (30). The next step in the analysis is to derive a second relation between the same potentials. This may be accomplished by using equations (11c) and (10). Utilizing the identities

$$\begin{aligned} &\frac{\partial}{\partial r} = -\frac{\partial}{\partial x}, \quad -\frac{1}{r}\frac{\partial}{\partial \theta} = \frac{\partial}{\partial y} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \theta = \pi \\ &\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} = -\frac{\partial^2}{\partial y^2} \end{aligned} \tag{19}$$

equation (11c) can be transformed into

$$\begin{aligned} 2(1-\nu)\frac{\partial f_2}{\partial y} - (1-2\nu)\frac{\partial f_1}{\partial x} + 2\nu\frac{\partial f_3}{\partial z} - \frac{\partial^2 f_0}{\partial y^2} - \frac{\partial^2 f_1}{\partial y \partial \theta} \\ - z\frac{\partial^2 f_3}{\partial y^2} = 0, \ \theta = \pi \end{aligned} \tag{20}$$

At this point, it is expedient to add the term $(y\frac{\partial^2 f_3}{\partial y \partial z})$ (which vanishes* in the region $\theta = \pi$) to both sides of equation (20). The resulting relation takes the form

*The second derivative of f_3 is assumed to be regular for $\theta = \pi$. This condition, incidently, is easily confirmed once the potential f_3 is known. Refer to equation (26).

$$\frac{\partial}{\partial y}\Big[2(1-\nu)f_2 - \frac{\partial f_0}{\partial y} - \frac{\partial f_1}{\partial \theta} + y\,\frac{\partial f_3}{\partial z} - z\,\frac{\partial f_3}{\partial y}$$

$$- (1-2\nu)\int_{\infty}^{y}\Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_3}{\partial z}\Big)dy\Big] = 0,\ \theta = \pi \qquad (21)$$

The above limits of integration have been introduced to ensure boundedness at the infinitely distant points.

With reference to the region $\theta = 0$ and making use of some elementary transformations analogous to those in equations (19), equations (10) confirm that the relation (21) is also valid in the region $\theta = 0$. This means that equation (21) holds across the entire plane $y = 0$ of the half space $y \geq 0$. The expression inside the bracket in equation (21) is recognized as a harmonic function. It follows from Green's formula [6] that

$$2(1-\nu)f_2 - \frac{\partial f_0}{\partial y} = \frac{\partial f_1}{\partial \theta} + z\,\frac{\partial f_3}{\partial y} - y\,\frac{\partial f_3}{\partial z}$$

$$+ (1-2\nu)\int_{\infty}^{y}\Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_3}{\partial z}\Big)dy,\ y \geq 0 \qquad (22)$$

exists throughout the entire semi-infinite solid $y \geq 0$. Equation (22) provides the second equation for finding f_2 and f_0.

METHOD OF SOLUTION

In this section the potentials appearing in the basic displacement representation are determined by employing methods of integral transforms and singular (primitive) solutions of Laplace's equation [5].

The function $f_3(r,\theta,z)$ is governed by the mixed conditions (10c) and (15). This calls for the application of a Fourier cosine transform with respect to the variable z and a Kontorovich-Lebedev transform* in r [4]. Seeking $f_3(r,\theta,z)$ by the expression

$$f_3 = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \int_0^\infty \phi(s,t) \frac{\sinh(\theta t)}{t\cosh(\pi t)} K_{it}(rs)\cos(sz)dsdt \qquad (23)$$

where K_{it} is the Macdonald function [7] and $\phi(s,t)$ is an arbitrary parameter such that equation (10c) is satisfied. By introducing the appropriate inversions and accounting for the concentrated load at ($r = a$, $z = 0$), $\phi(s,t)$ is determined from equation (15) as

$$\phi(s,t) = -\frac{Rt\sinh(\pi t)K_{it}(rs)}{(2\pi)^{1/2}\pi^2 2\mu(1-\nu)} \qquad (24)$$

The result of putting equation (24) into (23) and employing the integral representation [7] is

$$K_{it}(rs)K_{it}(as) = \int_0^\infty K_0(s\xi)\cos(t\eta)d\eta$$
$$\xi = (r^2+2ar\cosh\eta+a^2)^{1/2} \qquad (25)$$

*See Appendix for brief outline of this transform.

The integrals in equations (23) may now be evaluated to render

$$f_3(r,\theta,z) = - \frac{R}{4\mu\pi^2(1-\nu)\rho} \tan^{-1} \left(\frac{\sqrt{2a(r-x)}}{\rho}\right) \tag{26}$$

where ρ represents the distance of any point in $y \geq 0$ from the point of application of R, i.e.,

$$\rho = [(x+a)^2 + y^2 + z^2]^{1/2} \tag{27}$$

The evaluation of $f_1(r,\theta,z)$ is dictated by the "homogeneous" conditions (10b) and (16) governed by the Laplace equation in three dimensions. However, the choice $f_1 \equiv 0$ is inappropriate as it does not lead to the correct solution (see [4], pages 381-383). This, in fact, is the crucial step in the present analysis and contradicts the general notion in the open literature that one of the harmonic functions contained in the Papkovich-Neuber solution can be dropped. In order to arrive at the proper expression for f_1, a singular solution of Laplace equation which satisfies conditions (10b) and (16) and gives rise to well-behaved displacements at infinity is proposed [5]

$$f_1(r,\theta,z) = c_1 \frac{\sin \frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g(\zeta)] \tag{28}$$

where Im designates the imaginary part of an analytic function of the variable ζ defined by

$$\zeta = r + a + iz, \quad i = \sqrt{-1} \tag{29}$$

and c_1 is a constant introduced for convenience of writing the ensuing development. Note that as $r \to 0$, the expression (28) is undefined. On physical ground, however, the displacements must be finite in that region. This condition will be improved later on.

To find the other potentials f_0 and f_2, equations (17) and (18) suggest the following relation involving the special solution introduced in (28), i.e.,

$$(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} = \frac{A}{2} \frac{\cos\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g(\zeta)] \tag{30}$$

In equations (28) and (30) the function $g(\zeta)$ as well as the constants c_1 and A are yet to be found. Solving equations (22) and (30) simultaneously and utilizing equations (26) and (28) yield

$$f_2 = \left\{\frac{R\sqrt{a}}{4\mu\pi^2(1-\nu)} + \frac{1}{2}[(3-4\nu)c_1 - A]\right\} \frac{\cos\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g(\zeta)]$$

$$+ (1-2\nu)\left\{\psi(x,y,z) - c_1 \int_\infty^y \frac{\sin\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g'(\zeta)]dy\right\} \tag{31a}$$

and

$$\frac{\partial f_0}{\partial y} = \left\{\frac{(1-2\nu)R\sqrt{a}}{4\mu\pi^2(1-\nu)} + \frac{1}{2}[(1-2\nu)(3-4\nu)c_1 - 2(1-\nu)A]\right\} \times$$

$$\times \frac{\cos\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g(\zeta)] + (1-2\nu)^2\left\{\psi(x,y,z) - c_1 \int^y \frac{\sin\frac{\theta}{2}}{\sqrt{r}} \times\right.$$

$$\left. \times \operatorname{Im}[g'(\zeta)]dy\right\} \tag{31b}$$

in which $g'(\zeta) = \frac{d}{d\zeta} g(\zeta)$ and the following notations have been adopted:

$$\psi(x,y,z) = \frac{R|\zeta_0|^{-2}}{4\mu\pi^2(1-\nu)} \{\frac{yz}{\rho} \tan^{-1} (\frac{\sqrt{2a(r-x)}}{\rho})$$

$$- (2a)^{1/2} \operatorname{Im}[\zeta_0(2x-\zeta_0)^{-1/2} \ln(\frac{\sqrt{r+x}+\sqrt{2x-\zeta_0}}{\zeta})]\} \quad (32)$$

$$\zeta_0 = [\zeta]_{\theta=0} = x + a + iz$$

Integrating equation (31b) with respect to y between the limits ∞ and y and applying the results

$$\int_\infty^y \frac{\cos\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g'(\zeta)]dy = \frac{1}{\sqrt{2}} \operatorname{Im} \int_\infty^y \frac{g(t)dt}{\sqrt{t-\zeta_0}}$$

$$\int_\infty^y \psi(x,y,z)dy = y\psi + \frac{R}{4\mu\pi^2(1-\nu)} \{\frac{z}{\rho} \tan^{-1} \frac{\sqrt{2a(r-x)}}{\rho}$$

$$- \sqrt{2a} \operatorname{Im}[(\zeta_0)^{-1/2}\tan^{-1}(\frac{\zeta_0}{\zeta-\zeta_0})^{1/2}]\} \quad (33)$$

$$\int_\infty^y dy \int_\infty^y \frac{\sin\frac{\theta}{2}}{\sqrt{r}} \operatorname{Im}[g'(\zeta)]dy = \frac{1}{2\sqrt{2}} \operatorname{Im}[\int_\infty^y \frac{g(t)dt}{\sqrt{t-\zeta_0}}$$

$$- 2y \int_\infty^y g(t)\frac{d}{dt}(t-\zeta_0+2x)^{-1/2}dt]$$

it is found that

$$f_0 = -\frac{(1-2\nu)R\sqrt{a}}{\sqrt{2}\mu\pi^2}\,\mathrm{Im}[\zeta_0^{-1/2}\tan^{-1}\left(\frac{\zeta_0}{\zeta-\zeta_0}\right)^{1/2}]$$

$$+\frac{1-\nu}{\sqrt{2}}[(1-2\nu)c_1 - A]\,\mathrm{Im}\int_\infty^y \frac{g(t)dt}{t-\zeta_0}$$

$$+(1-2\nu)^2[y\psi + \frac{Rz}{4\mu\pi^2(1-\nu)\rho}\tan^{-1}\left(\frac{\sqrt{2a(r-x)}}{\rho}\right)$$

$$+\frac{c_1 y}{\sqrt{2}}\,\mathrm{Im}\int_\infty^y g(t)\frac{d}{dt}(t-\zeta_0+2x)^{-1/2}dt] \tag{34}$$

The remaining boundary condition to be satisfied is equation (10a). Imposing this condition on equation (34) results in

$$[(1-2\nu)c_1 - A]\int_{\zeta_0}^\infty \frac{g(t)dt}{\sqrt{t-\zeta_0}} = -\frac{(1-2\nu)R\sqrt{a}}{2\mu(1-\nu)\pi}\zeta_0^{-1/2} \tag{35}$$

which is a standard integral equation of the Abel type [8] and thus the function $g(\zeta)$ is readily determined as

$$g(\zeta) = \frac{1}{\zeta} \tag{36}$$

provided that

$$(1-2\nu)c_1 - A = -\frac{(1-2\nu)R\sqrt{a}}{2\mu(1-\nu)\pi^2} \tag{37}$$

The expressions for the potentials f_0 and f_2 can be simplified further by defining a constant c_2 via the relation

$$c_2 = \frac{\nu R\sqrt{a}}{2\mu\pi^2(1-\nu)} + (1-\nu)c_1 \tag{38}$$

Upon substituting equations (36-38) into (31a) and (34) the following expressions are derived

$$f_0(r,\theta,z) = (1-2\nu)^2 \left[\frac{Rz}{4\mu\pi^2(1-\nu)\rho} \tan^{-1} \times \right.$$

$$\times \left(\frac{2(ar)^{1/2}\sin\frac{\theta}{2}}{\rho}\right) + y\psi] + \frac{c_1}{\sqrt{2}} \mathrm{Im}\left[\frac{\sqrt{r-x}}{(\zeta_0-2x)}\right.$$

$$\left. + (2x-\zeta_0)^{-3/2} \ln\left(\frac{\sqrt{r+x}+\sqrt{2x-\zeta_0}}{\zeta}\right)\right] \qquad (39)$$

and

$$f_2(r,\theta,z) = c_2 \frac{\cos\frac{\theta}{2}}{\sqrt{r}} \mathrm{Im}\left(\frac{1}{\zeta}\right) + (1-2\nu)\{\psi(x,y,z)$$

$$- \frac{c_1}{\sqrt{2}} \mathrm{Im}\left[\frac{\sqrt{r+x}}{\zeta(2x-\zeta_0)} - (2x-\zeta_0)^{-3/2} \times\right.$$

$$\left.\times \ln\left(\frac{\sqrt{r+x}+\sqrt{x-\zeta_0}}{\zeta}\right)\right]\} \qquad (40)$$

where ψ is defined in equation (32). The final step in the analysis is to obtain another expression relating c_1 and c_2. This may be done by imposing the regularity condition on the displacements as $r \to 0$. Inserting equations (26), (28), (39) and (40) in (2) and (3) and carrying out the expansion for small r, making use of (38) and retaining the lowest-order terms, it is found that the displacements near the crack edge become

$$u_r = -\frac{c_1+c_2}{2\sqrt{r}}\frac{z\sin\frac{\theta}{2}}{a^2+z^2}[3-4\nu+(7-8\nu)\cos\theta]+O(r^0) \tag{41a}$$

$$u_\theta = \frac{c_1+c_2}{2\sqrt{r}}\frac{z\cos\frac{\theta}{2}}{a^2+z^2}[3-4\nu-(5-8\nu)\cos\theta]+O(r^0) \tag{41b}$$

$$u_z = O(r^0) \tag{41c}$$

The finite displacements requirement at the crack edge gives

$$c_1 + c_2 = 0 \tag{42}$$

It follows from equation (38) that

$$c_1 = -c_2 = -\frac{\nu R\sqrt{a}}{2\mu\pi^2(1-\nu)(2-\nu)} \tag{43}$$

and the solution is complete. The physical quantities of interest may be readily computed from equations (2) - (5) when the appropriate expressions for the harmonic functions in equations (26), (28), (39), (40) and (43) are used.

STRESSES NEAR CRACK EDGE

The shear stresses across the surface $\theta = 0$ are computed from equations (4b) and (4c) as

$$\tau_{r\theta}(r,o,z) = -\frac{4\nu R\sqrt{a}}{\pi^2(2-\nu)}\frac{z(x+a)}{(x)^{1/2}[(x+a)^2+z^2]^2} \tag{44a}$$

$$\tau_{\theta z}(r,o,z) = -\frac{R\sqrt{a}}{\pi^2(2-\nu)}\frac{(2-3\nu)(x+a)^2+(2+\nu)z^2}{(x)^{1/2}[(x+a)^2+z^2]^2} \tag{44b}$$

Equations (44) may be expressed in the standard form as

$$\tau_{r\theta}(r,o,z) = \frac{k_2}{\sqrt{2x}} + 0(x^0) \tag{45a}$$

$$\tau_{\theta z}(r,o,z) = \frac{k_3}{\sqrt{2x}} + 0(x^0) \tag{45b}$$

The stress-intensity factors for the edge-sliding and tearing modes of crack extension are expressed in terms of the non-dimensional parameter $z^* = \frac{z}{a}$ through the relations

$$k_2 = - \frac{4\sqrt{2}R}{\pi a^{3/2}} \left(\frac{\nu}{2-\nu}\right) \frac{z^*}{(1+z^{*2})^2} \tag{46a}$$

$$k_3 = - \frac{\sqrt{2}R}{\pi^2 a^{3/2}(1+z^{*2})} \left[1 - \frac{2\nu}{2-\nu} \frac{1-z^{*2}}{1+z^{*2}}\right] \tag{46b}$$

The variations of these equations with z^* are shown in Figures 2 and 3 for various values of ν.

CLOSURE

The problem of determining stresses and displacements in a three-dimensional elastic solid with a semi-infinite plane crack is reduced to the finding of four distinct harmonic functions. The surfaces of the crack are deformed by the application of concentrated shears (R) parallel to the rectilinear boundary as shown in Figure 1. The analysis reveals that none of the Papkovich-Neuber potentials appearing in the general solution of the equations of equilibrium can be neglected.

Indeed, all of the *four* functions are needed to satisfy the appropriate boundary and regularity conditions in the problem.

Methods of integral transforms involving the Macdonald function (the so-called Kontorovich-Lebedev transforms) as well as singular solutions of Laplace's equation in three dimensions are employed. The results presented here coupled with the solutions of Uflyand [4] complete the construction of the Green's functions for the semi-infinite crack problem and pave the way for immediate application in a number of problems of interest in fracture mechanics.

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APPENDIX

The Kontorovich-Lebedev transform of a function $\phi(x)$ is defined by the formula

$$\hat{\phi}(t) = \int_0^\infty \frac{\phi(x)}{x} K_{it}(x)dx \tag{47}$$

where $\phi(x)$ is assumed to be of class L^2 in the interval $[0,\infty]$ and $K_{it}(x)$ is the Macdonald function which is a solution of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - (1 - \frac{t^2}{x^2})y = 0 \tag{48}$$

Assuming that $\frac{\phi(x)}{x}$ is continuously differentiable and $x\phi(x)$ and $x \frac{d}{dx} [\frac{\phi(x)}{x}]$ are absolutely integrable in $[0,\infty]$ then the inverse of (47) is given by

$$\phi(x) = \frac{2}{\pi^2} \int_0^\infty \hat{\phi}(t)K_{it}(x)t\sinh(\pi t)dt \tag{49}$$

Further details may be found in [4].

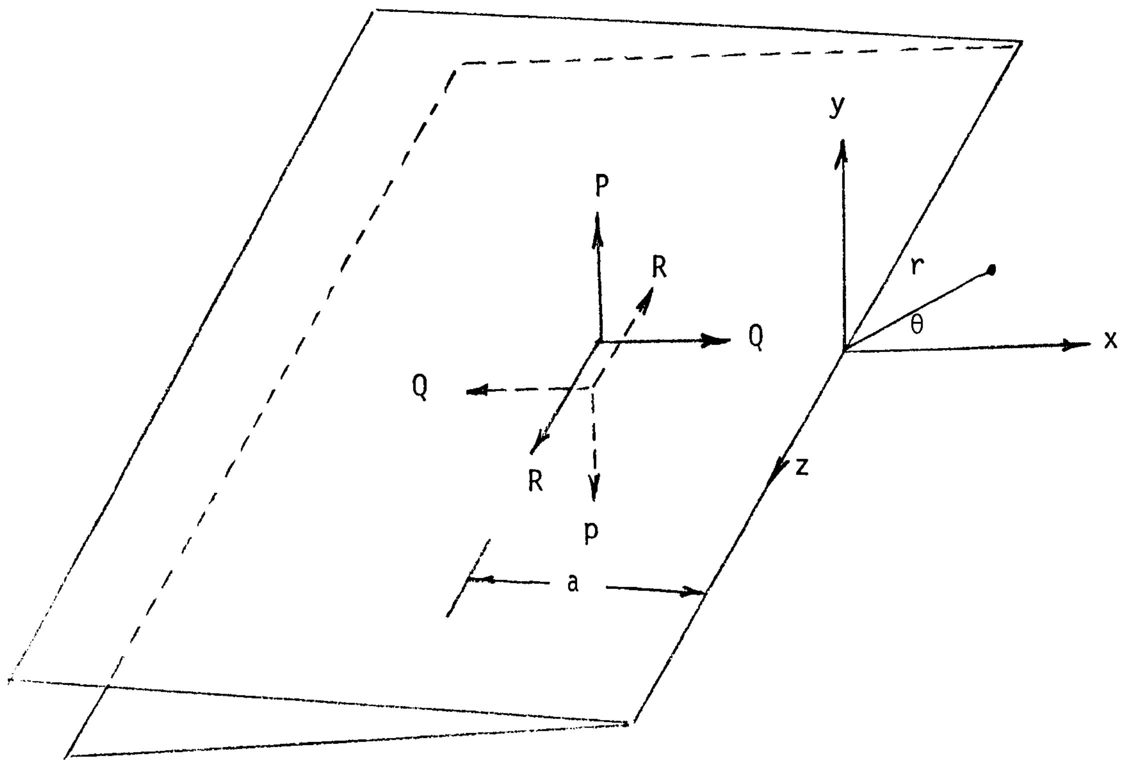

Figure 1. Semi-infinite plane crack in elastic solid.

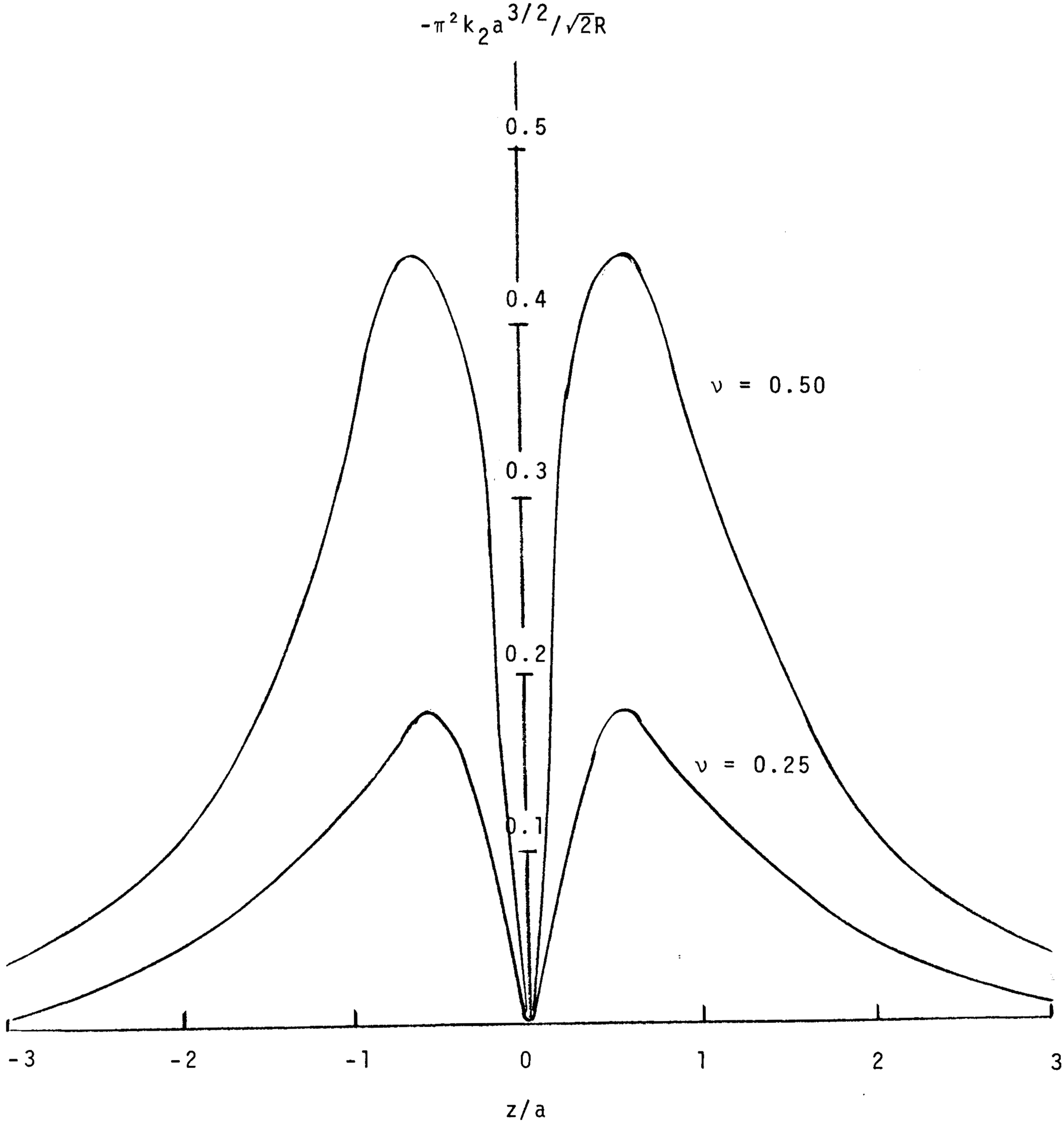

Figure 2. Variations of k_2 along crack border.

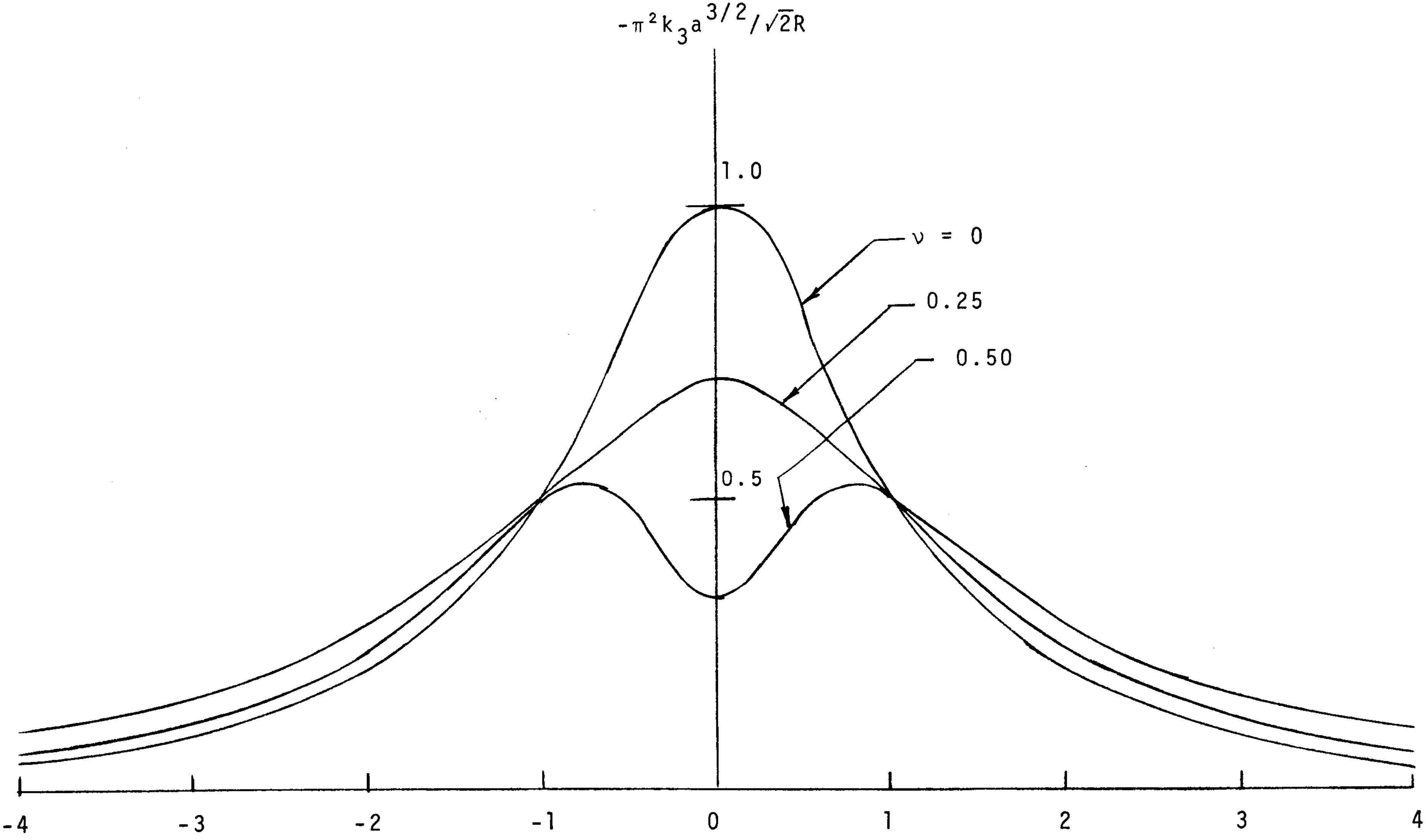

Figure 3. Variations of k_3 along crack border.