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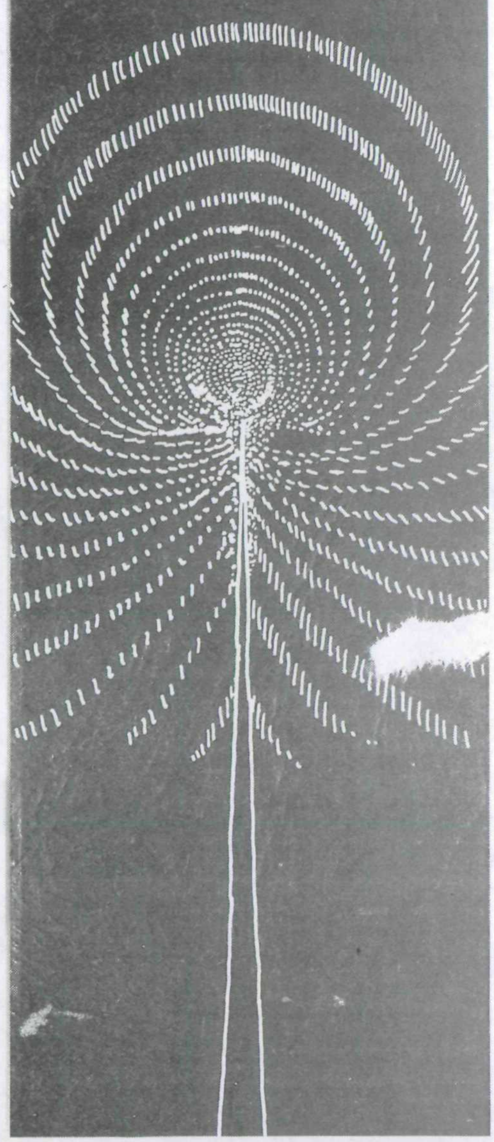

THE PROBLEM OF A FINITE STRIP COMPRESSED BETWEEN TWO ROUGH RIGID STAMPS

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ABSTRACT

A finite strip compressed between two rough rigid stamps is considered. The elastostatic problem is formulated in terms of a singular integral equation from which the proper stress singularities at the corners are determined. The singular integral equation is solved numerically to determine the stresses along the fixed ends of the strip. The effect of material properties and strip geometry on the stress intensity factor is presented graphically.

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INTRODUCTION

The problem of a finite strip compressed between two rough rigid stamps has been of considerable interest from both mechanics and mathematical points of view. In particular, the problem is very frequently encountered by the experimentalists in rock mechanics, who use the standard crushing test for various rock specimens. It is well known that rocks are nonhomogeneous and contain relatively large voids. However, the experiments [1] have shown that the failure of a compressed rock specimen initiates at the corners due to the high stress concentration at these locations. Hence, a stress analysis, specifically near the ends of the strip, is essential for a better understanding of the failure mechanism. For the sake of convenience, a homogeneous and isotropic finite strip will be analyzed in this paper.

Numerous analytical studies have been devoted to the finite strip problem but none of the methods provides a solution which can directly give the correct behavior of stresses near the corners without presenting convergence difficulties. The best solution known so far is given by Benthem and Minderhoud [2] who used the eigenfunction expansion technique to solve a semi-infinite and finite cylinder problem with remarkable success. The method is equally well applicable to the finite strip problem; however, it requires a prior knowledge of the stress singularities by alternate means.

An integral transform technique has recently been used by Gupta [3] to solve a semi-infinite strip problem. In this paper, the method used in [3] has been extended for a finite strip problem, where one needs to use the finite integral transforms. The final integral equation contains a divergent infinite series from which a singular kernel can be isolated and a singular integral equation may be obtained. This equation may then be solved numerically by using the Gauss-Jacobi integration technique.

FORMULATION OF THE PROBLEM

Consider a strip of width 2h and length 2L compressed between two rough rigid stamps. Stamps have to be rough to ensure no sliding at the ends. The shear modulus and Poisson's ratio of the strip are μ and ν respectively. The problem described above can be recovered by the superposition of a homogeneous (I) and a disturbance (II) problem as shown in Figure 1. Solution of I is simply given as

$$
\begin{aligned}
&\sigma_{xx}{}^{I}(x,y) = \sigma_{xy}{}^{I}(x,y) = 0 \\
&\sigma_{yy}{}^{I}(x,y) = -p_0 \\
&u_I(x,y) = \varepsilon_0 x \; ; \qquad \varepsilon_0 = \left(\frac{3-\kappa}{8\mu}\right) p_0 \\
&v_I(x,y) = -\varepsilon_1 y \; ; \qquad \varepsilon_1 = \left(\frac{1+\kappa}{8\mu}\right) p_0
\end{aligned}
\tag{1}
$$

where $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

The disturbance problem II must have the input function as the displacement in x-direction at $y=\pm L$ plane, equal to the negative of that in I. Since the problem is symmetrical about $x=0$ and

$y=0$ planes, it is sufficient to consider one quarter of the medium only. Hence, the boundary conditions for II are written as

$$\sigma_{xx}(h,y) = \sigma_{xy}(h,y) = 0 , \qquad |y|<L,$$

$$\left.\begin{aligned} v(x,0) &= 0 \\ \sigma_{xy}(x,0) &= 0 \end{aligned}\right\} \quad |x|<h \tag{2}$$

$$\left.\begin{aligned} u(x,L) &= -\varepsilon_0 x \\ v(x,L) &= v_0 \end{aligned}\right\} \quad |x|<h \tag{3}$$

where v_0 is an unknown constant determined from the following equilibrium condition:

$$\int_{-h}^{h} \sigma_{yy}(x,L)dx = 0 \tag{4}$$

Note that this problem is a special case of a general problem of a parallel array of rigid inclusions lying in a strip. The finite strip problem is recovered when the inclusions extend to the strip surfaces. Also, in the inclusion problem, the boundary conditions (3) would be replaced by a set of mixed boundary conditions [3].

The displacement and stress fields for the strip can be expressed as a superposition of two transform solutions. One is the solution for a finite strip ($|x|<h$, $|y|<L$) with symmetry about $x=0$ and $y=0$ planes, and the other is the infinite strip ($|x|<\infty$, $0<y<L$) with $x=0$ as the plane of symmetry. Expressing the solution as

$$u(x,y) = -\sum_{n=0}^{\infty} \{\frac{1}{\alpha_n}[f_n - \frac{\kappa-1}{2} g_n]\sinh(\alpha_n x) + xg_n\cosh(\alpha_n x)\}\cos\alpha_n y$$

$$+ \frac{2}{\pi}\int_0^{\infty} \frac{\phi(\xi)}{\xi}[\{\kappa - \xi L\coth(\xi L)\}\cosh(\xi y) + \xi y\sinh(\xi y)]\sin\xi x d\xi$$

$$v(x,y) = \sum_{n=0}^{\infty} \{\frac{1}{\alpha_n}[f_n + \frac{\kappa+1}{2} g_n]\cosh(\alpha_n x) + xg_n\sinh(\alpha_n x)\}\sin\alpha_n y$$

$$+ \frac{2}{\pi}\int_0^{\infty} \frac{\phi(\xi)}{\xi}[\xi L\coth(\xi L)\sinh(\xi y) - \xi y\cosh(\xi y)]\cos\xi x d\xi$$

$$\frac{\sigma_{xx}(x,y)}{2\mu} = -\sum_{n=0}^{\infty} [f_n\cosh(\alpha_n x) + \alpha_n xg_n\sinh(\alpha_n x)]\cos\alpha_n y$$

$$+ \frac{2}{\pi}\int_0^{\infty}\phi(\xi)[\{\frac{\kappa+3}{2} - \xi L\coth(\xi L)\}\cosh(\xi y) + \xi y\sinh(\xi y)]\cos\xi x d\xi \tag{5}$$

$$\frac{\sigma_{yy}(x,y)}{2\mu} = \sum_{n=0}^{\infty} [(f_n + 2g_n)\cosh(\alpha_n x) + \alpha_n xg_n\sinh(\alpha_n x)]\cos\alpha_n y$$

$$- \frac{2}{\pi}\int_0^{\infty}\phi(\xi)[\{\frac{\kappa-1}{2} - \xi L\coth(\xi L)\}\cosh(\xi y) + \xi y\sinh(\xi y)]\cos\xi x d\xi$$

$$\frac{\sigma_{xy}(x,y)}{2\mu} = \sum_{n=1}^{\infty} [(f_n + g_n)\sinh(\alpha_n x) + \alpha_n xg_n\cosh(\alpha_n x)]\sin\alpha_n y$$

$$- \frac{2}{\pi}\int_0^{\infty}\phi(\xi)[\{\frac{\kappa+1}{2} - \xi L\coth(\xi L)\}\sinh(\xi y) + \xi y\cosh(\xi y)]\sin\xi x d\xi$$

and $\qquad \alpha_n = \frac{n\pi}{L}$

it may be seen that this solution identically satisfies the conditions $v(x,0) = 0$, and $\sigma_{xy}(x,0) = 0$, of (2). The unknowns $\phi(\xi)$, f_n and g_n must be determined by the first two conditions of (2) and the conditions (3). The first two conditions of (2) may be written as

$$f_n\cosh(\alpha_n h) + \alpha_n hg_n\sinh(\alpha_n h) = D_n$$

$$f_n\sinh(\alpha_n h) + g_n[\sinh(\alpha_n h) + \alpha_n h\cosh(\alpha_n h)] = E_n \tag{6}$$

where

$$D_n = \frac{2}{\pi}\int_0^\infty \phi(\xi)[\{\frac{\kappa+3}{2} - \xi L\coth(\xi L)\}I_{1n}(\xi) + I_{2n}(\xi)]\cos\xi h d\xi$$

$$E_n = \frac{2}{\pi}\int_0^\infty \phi(\xi)[\{\frac{\kappa+1}{2} - \xi L\coth(\xi L)\}I_{3n}(\xi) + I_{4n}(\xi)]\sin\xi h d\xi \tag{7}$$

and

$$I_{1n}(\xi) = \begin{cases} \frac{\sinh(\xi L)}{\xi L}, & n=0 \\ (-1)^n \frac{2\xi\sinh(\xi L)}{(\alpha_n{}^2 + \xi^2)L}, & n\geq 1 \end{cases}$$

$$I_{3n}(\xi) = (-1)^n \frac{2\alpha_n\sinh(\xi L)}{(\alpha_n{}^2 + \xi^2)L}, \qquad n\geq 0 \tag{8}$$

$$I_{2n}(\xi) = \xi \frac{d}{d\xi} I_{1n}(\xi) \; ; \qquad I_{4n}(\xi) = \xi \frac{d}{d\xi} I_{3n}(\xi)$$

It may be noted that in order to obtain the displacement in x-direction at $y=\pm L$, certain shear stresses must be applied at those planes. Let this shear stress be the unknown function in the problem which has to be determined so that (3) is satisfied. Hence, from (5)

$$\frac{\sigma_{xy}(x,L)}{2\mu} = -\frac{\kappa+1}{\pi}\int_0^\infty \phi(\xi)\sin\xi x \sinh(\xi L)d\xi \tag{9}$$

$$= \frac{G(x)}{2\mu}, \qquad |x|<h$$

The unknown function $\phi(\xi)$ can be written in terms of the new unknown $G(x)$ by inverting the integral in (9) to give

$$\phi(\xi) = -\frac{1}{\mu(\kappa+1)}\int_0^h G(t)\frac{\sin\xi t}{\sinh(\xi L)}dt \tag{10}$$

The first condition of (3) can now be expressed as

$$\frac{\partial u}{\partial x}(x,L) = \frac{2}{\pi} \lim_{y\to L} \int_0^\infty \phi(\xi)[\kappa\cosh(\xi y) - \xi L\coth(\xi L)\cosh(\xi y)$$

$$+ \xi y\sinh(\xi y)]\cos\xi x d\xi \tag{11}$$

$$- \sum_{n=0}^{\infty}(-1)^n\{[f_n - \frac{\kappa-3}{2} g_n]\cosh(\alpha_n x) + x\alpha_n g_n\sinh(\alpha_n x)\}$$

$$= -\varepsilon_o \ , \qquad |x|<h$$

Note that displacement derivative is used in (11) instead of displacement in order to maintain a dimensional consistency in (10) and (11). Equations (6) are now solved simultaneously to obtain

$$f_o = D_o$$

$$\frac{1}{2} f_n = \frac{D_n[\sinh(\alpha_n h) + \alpha_n h\cosh(\alpha_n h)] - E_n\alpha_n h\sinh(\alpha_n h)}{\sinh(2\alpha_n h) + 2\alpha_n h}$$

$$n\geq 1 \tag{12}$$

$$\frac{1}{2} g_n = \frac{- D_n\sinh(\alpha_n h) + E_n\cosh(\alpha_n h)}{\sinh(2\alpha_n h) + 2\alpha_n h}$$

where substituting (8) into (7), D_n and E_n can be expressed as

$$D_n = \frac{2}{\pi}\int_0^\infty \phi(\xi)[(-1)^n \frac{m\xi\sinh(\xi L)}{(\alpha_n^2 + \xi^2)L} \{\frac{\kappa+1}{2} + \frac{2\alpha_n^2}{\alpha_n^2 + \xi^2}\}]\cos\xi h d\xi$$

$$E_n = \frac{2}{\pi}\int_0^\infty \phi(\xi)[(-1)^n \frac{m\alpha_n\sinh(\xi L)}{(\alpha_n^2 + \xi^2)L} \{\frac{\kappa+1}{2} - \frac{2\xi^2}{\alpha_n^2 + \xi^2}\}]\sin\xi h d\xi \tag{13}$$

where

$$m = \begin{cases} 1 \ , & n=0 \\ 2 \ , & n\geq 1 \end{cases}$$

In order to reduce (11) to an integral equation in $G(x)$, f_n and g_n must be substituted from (12) and (13), and equation (10) must be used to eliminate $\phi(\xi)$. Note that in this symmetric problem the shear stress $G(x)$ is an odd function, i.e., $G(x) = -G(-x)$. Using this property, equation (10) and the relations given in [4], the first integral in (11) yields a Cauchy kernel as follows:

$$\int_{-h}^{h} G(t)dt \lim_{y\to L} \int_0^\infty [\kappa\cosh(\xi y) - \xi L\coth(\xi L)\cosh(\xi y) + \xi y\sinh(\xi y)]\frac{\sin\xi(t-x)}{\sinh(\xi L)}d\xi$$

$$= \kappa\int_{-h}^{h} G(t)\frac{\pi}{2L}\coth\left(\frac{\pi[t-x]}{2L}\right)dt - \int_{-h}^{h} G(t)dt\int_0^\infty \frac{\xi L}{\sinh^2(\xi L)}\sin\xi(t-x)d\xi \tag{14}$$

$$= \kappa\int_{-h}^{h}\frac{G(t)}{t-x}dt + \int_{-h}^{h} G(t)K_{1F}(t,x)dt$$

where

$$K_{1F}(t,x) = \frac{\kappa\pi}{2L}\coth\left(\frac{\pi}{2L}[t-x]\right) - \frac{\kappa}{t-x} - \int_0^\infty \frac{\xi L}{\sinh^2(\xi L)}\sin\xi(t-x)d\xi \tag{15}$$

Note that $K_{1F}(t,x)$ is a Fredholm kernel and is bounded in $-h \le (t,x) \le h$. Now substituting $\phi(\xi)$ from (10) into (13), the functions D_n and E_n can be written as

$$\pi\mu(\kappa+1)D_0 = \frac{\kappa+1}{2}\int_{-h}^{h} G(t)dt\int_0^\infty \frac{\sin\xi(h-t)}{\xi L}d\xi$$

$$\pi\mu(\kappa+1)D_n = \int_{-h}^{h} G(t)dt\int_0^\infty (-1)^n\frac{2\xi}{(\alpha_n^2+\xi^2)L}\left\{\frac{\kappa+1}{2} + \frac{2\alpha_n^2}{\alpha_n^2+\xi^2}\right\}\sin\xi(h-t)d\xi$$

$$n \ge 1 \tag{16}$$

$$\pi\mu(\kappa+1)E_n = -\int_{-h}^{h} G(t)dt\int_0^\infty (-1)^n\frac{2\alpha_n}{(\alpha_n^2+\xi^2)L}\left\{\frac{\kappa+1}{2} - \frac{2\xi^2}{\alpha_n^2+\xi^2}\right\}\cos\xi(h-t)d\xi$$

Using the tables of Fourier transforms in [4], the expressions in (16) can be reduced as

$$2\mu(\kappa+1)D_0 = \frac{\kappa+1}{2}\int_{-h}^{h} G(t)dt = 0$$

$$2\mu(\kappa+1)D_n = (-1)^n \frac{2}{L} \int_{-h}^{h} G(t)[\frac{\kappa+1}{2} + \alpha_n(h-t)]e^{-\alpha_n(h-t)} dt$$

$$n \geq 1 \qquad (17)$$

$$2\mu(\kappa+1)E_n = -(-1)^n \frac{2}{L} \int_{-h}^{h} G(t)[\frac{\kappa-1}{2} + \alpha_n(h-t)]e^{-\alpha_n(h-t)} dt$$

Using (12), (14) and (17), equation (11) can be expressed in terms of the unknown G(t) and a constant g_0 which cannot be evaluated by the prescribed boundary conditions. This constant must be determined by using the equilibrium condition (4). The final singular integral equation may be written as

$$\int_{-h}^{h} G(t)[\frac{\kappa}{t-x} + K_{1F}(t,x) + K(t,x)]dt = \frac{(\kappa+1)(3-\kappa)}{8} \pi p_0(1-\lambda) \qquad (18)$$

$$|x|<h$$

where

$$\lambda = \frac{4\mu}{p_0} g_0 \qquad (19)$$

g_0 being an unknown constant (see (11)). From (5) it may be shown that the constants g_0 and $v(x,L) = v_0$ are related by

$$v_0 = \frac{\kappa+1}{2} L g_0 \qquad (20)$$

$$K(t,x) = \frac{\pi}{L} \sum_{n=1}^{\infty} e^{-\alpha_n(h-t)} \; k(t,x,\alpha_n)$$

$$\begin{aligned} k(t,x,\alpha_n) = \frac{2}{\sinh(2\alpha_n h) + 2\alpha_n h} \Big[&\cosh(\alpha_n x)\{[\frac{\kappa+1}{2} + \alpha_n(h-t)][\alpha_n h\cosh(\alpha_n h) \\ &+ \frac{\kappa-1}{2}\sinh(\alpha_n h)] + [\frac{\kappa-1}{2} + \alpha_n(h-t)][\alpha_n h\sinh(\alpha_n h) + \frac{\kappa-3}{2}\cosh(\alpha_n h)]\} \\ &- x\alpha_n \sinh(\alpha_n x)\{[\frac{\kappa+1}{2} + \alpha_n(h-t)]\sinh(\alpha_n h) \\ &+ [\frac{\kappa-1}{2} + \alpha_n(h-t)]\cosh(\alpha_n h)\}\Big] \end{aligned} \qquad (21)$$

It should be noted that the infinite series appearing in the kernel $K(t,x)$ is a divergent series and becomes infinite for $t\to h$, $x\to\pm h$. This divergent series can be reduced to a convergent infinite sum by separating the singular part of the series. This singular part of the series is obtained by taking the asymptotic value of the function $k_n(t,x,\alpha_n)$ as $n\to\infty$. Let

$$K_s(t,x) = \frac{\pi}{L}\sum_{n=0}^{\infty} e^{-\alpha_n(h-t)} k_\infty(t,x,\alpha_n) - \frac{\pi}{L} k_\infty(t,x,0) \tag{22}$$

where $K_s(t,x)$ is the singular part of the kernel $K(t,x)$. From (21) it follows that

$$k_\infty(t,x,\alpha_n) = e^{-\alpha_n h}\Big[\cosh(\alpha_n x)\{4\alpha_n^2 h(h-t) + 2\alpha_n[h\kappa + (h-t)(\kappa-2)] + (\kappa-1)^2\} - 2x\alpha_n \sinh(\alpha_n x)\{\kappa + 2\alpha_n(h-t)\}\Big] \tag{23}$$

and

$$k_\infty(t,x,0) = (\kappa-1)^2$$

Using the following result [5]

$$\sum_{n=0}^{\infty} \alpha_n^m e^{-\alpha_n(2h-t)}\begin{Bmatrix}\cosh(\alpha_n x)\\ \sinh(\alpha_n x)\end{Bmatrix} = \frac{d^m}{dt^m}\sum_{n=0}^{\infty} e^{-\alpha_n(2h-t)}\begin{Bmatrix}\cosh(\alpha_n x)\\ \sinh(\alpha_n x)\end{Bmatrix}$$

$$= \frac{1}{2}\frac{d^m}{dt^m}\left[\frac{1}{1-e^{-\frac{\pi}{L}(2h-t-x)}} \pm \frac{1}{1-e^{-\frac{\pi}{L}(2h-t+x)}}\right] \tag{24}$$

the singular kernel $K_s(t,x)$ now becomes

$$K_s(t,x) = \frac{\pi}{2L}\Bigg[\{(\kappa-1)^2 + 2[h\kappa + (h-t)(\kappa-2)]\frac{d}{dt} + 4h(h-t)\frac{d^2}{dt^2}\} \cdot \{\frac{1}{1-e^{-\frac{\pi}{L}(2h-t-x)}} + \frac{1}{1-e^{-\frac{\pi}{L}(2h-t+x)}}\}$$

$$- 2x\{\kappa\frac{d}{dt} + 2(h-t)\frac{d^2}{dt^2}\}\{\frac{1}{1-e^{-\frac{\pi}{L}(2h-t-x)}} - \frac{1}{1-e^{-\frac{\pi}{L}(2h-t+x)}}\}\Bigg] \tag{25}$$

Note that the second term in (22) does not have any contribution in (25) since G(t) is an odd function of t and

$$\int_{-h}^{h} G(t)\ k_{\infty}(t,x,0)dt = (\kappa-1)^2 \int_{-h}^{h} G(t)dt = 0 \qquad (26)$$

In order to analyze the behavior of the unknown function G(t) near the end points, the dominant part of the equation consisting of the Cauchy kernel and the singular kernel $K_s(t,x)$ must be considered. For the purpose of analysis, it is convenient to express the kernel $K_s(t,x)$ in terms of a generalized Cauchy kernel and a Fredholm kernel, i.e., expressing

$$\frac{1}{1 - e^{-\frac{\pi}{L}(2h-t-x)}} = \frac{L}{\pi(2h-t-x)} + O(2h-t-x)$$

and $\qquad (27)$

$$\frac{1}{1 - e^{-\frac{\pi}{L}(2h-t+x)}} = \frac{L}{\pi(2h-t+x)} + O(2h-t+x)$$

$K_s(t,x)$ can be written as

$$K_s(t,x) = K_{1s}(t,x) + K_{3F}(t,x)$$

where

$$K_{1s}(t,x) = \frac{1}{2}\ \{\kappa^2 - 3 + 12(h-x)\frac{d}{dx} - 4(h-x)^2\ \frac{d^2}{dx^2}\}\ \frac{1}{2h-t-x}$$

$$+ \frac{1}{2}\ \{\kappa^2 - 3 - 12(h+x)\frac{d}{dx} - 4(h+x)^2\ \frac{d^2}{dx^2}\}\ \frac{1}{2h-t+x} \qquad (28)$$

The dominant part of the singular integral equation (18) is now written as

$$\frac{1}{\pi} \int_{-h}^{h} G(t)[\frac{\kappa}{t-x} + K_{1s}(t,x)]dt = \frac{(\kappa+1)(3-\kappa)}{8}\ p_o(1-\lambda) - A(x) \qquad (29)$$

$$|x|<h$$

where $A(x)$ is a bounded function containing all the terms coming from the Fredholm kernels, i.e.,

$$A(x) = \int_{-h}^{h} G(t)[K_{1F}(t,x) + K(t,x) - K_s(t,x) + K_{3F}(t,x)]dt \qquad (30)$$

The unknown function $G(t)$ is assumed to have integrable singularities at $t=\pm h$ and, following [6], may be expressed as

$$G(t) = \frac{H(t)}{(h^2-t^2)^{\alpha}} = \frac{H(t)e^{\pi i\alpha}}{(t-h)^{\alpha}(t+h)^{\alpha}}, \qquad |t|<h \qquad (31)$$

where $Re(\alpha) < 1$ and $H(t)$ satisfies a Hölder condition in the closed interval $|t|\leq h$. The general procedure for determining α from the dominant part of the singular integral equation (29) has been treated in detail in [7]. Also, the left hand side of (29) is identical to that obtained in [3] where the corresponding equation is analyzed to determine the power of the stress singularity α. Hence α is the first root of the following transcendental equation [3]:

$$2\kappa\cos\pi\alpha - (\kappa^2+1) + 4(\alpha-1)^2 = 0 \qquad (32)$$

Note that this equation depends only on the Poisson's ratio of the finite strip and yields a real value of α for any material $0\leq\nu\leq 0.5$.

SOLUTION OF THE INTEGRAL EQUATION

Without any loss of generality, the integral equation (29) can be normalized with respect to h by using the transformation:

$$\tau = \frac{t}{h}, \qquad y = \frac{x}{h}, \qquad G(t) = G(h\tau) = \phi(\tau) \qquad (33)$$

Hence (26) can be expressed as

$$\int_{-1}^{1} \phi(\tau)\left[\frac{\kappa}{\tau-y} + hK_s(h\tau,hy) + hK_{1F}(h\tau,hy) + hK_F(h\tau,hy)\right]d\tau$$

$$= \frac{(3-\kappa)(\kappa+1)}{8}\,\pi p_o(1-\lambda)\ , \qquad |y|<1 \tag{34}$$

where

$$K_F(h\tau,hy) = \frac{\pi}{L}\sum_{n=1}^{\infty} e^{-\alpha_n h(1-\tau)}\,[k(h\tau,hy,\alpha_n) - k_\infty(h\tau,hy,\alpha_n)]$$

and (31) becomes

$$\phi(\tau) = \frac{\psi(\tau)}{(1-\tau^2)^{\alpha}} \tag{35}$$

where α is given by (32). Equation (34) can now be solved by using Gauss-Jacobi Integration formula. This technique has been described in [7] and has been used in [3]. Since the unknown $\phi(\tau)$ represents the shear stress at the end of the strip, it must be unbounded and should have integrable singularity, i.e., $0<\mathrm{Re}(\alpha)<1$. Hence the index of the singular integral equation (34) is +1 and it must be solved subject to the condition (26). A set of N×N simultaneous algebraic equations are obtained.

$$\sum_{j=1}^{N} A_j\psi(\tau_j)\left[\frac{\kappa}{\tau_j-y_i} + h\{K_s(h\tau_j,hy_i) + K_{1F}(h\tau_j,hy_i) + hK_F(h\tau_j,hy_i)\}\right]$$

$$= \frac{(\kappa+1)(3-\kappa)}{8}\,\pi p_o(1-\lambda) \tag{36}$$

$$\sum_{j=1}^{N} A_j\psi(\tau_j) = 0$$

where from [7] τ_j and y_i are given as the roots of the following equations:

$$P_N^{(-\alpha,-\alpha)}(\tau_j) = 0\ , \qquad (j=1,\ldots,N)$$

$$P_{N-1}^{(1-\alpha,1-\alpha)}(y_i) = 0\ , \qquad (i=1,\ldots,N-1)$$

and A_j's are the corresponding weighting constants [7]. $\psi(\tau_j)$ are computed by solving (36) numerically and the shear stress $\sigma_{xy}(x,L)$ can then be expressed as

$$\sigma_{xy}(x,L) = G(x) = \frac{h^{2\alpha}\psi(x/h)}{(h^2-x^2)^{\alpha}} , \qquad |x|<h \tag{37}$$

Since λ is an unknown constant in equation (36) (see (19) and (20)), the numerical solution of this equation yields $\frac{G(x)}{p_o(1-\lambda)}$. λ in turn is determined from the equilibrium condition (4).

NORMAL STRESS AND STRESS INTENSITY FACTOR

After having solved for the shear stress in the disturbance problem, the remaining stress and strain fields can be computed from the corresponding equation in (5). An important quantity of interest is the normal stress at the ends of the strip. Also, this normal stress will be used to determine the unknown constant λ. Starting from the fourth equation in (5) and using (12), (10), (16) and (14), the normal stress $\sigma_{yy}(x,L)$ can be expressed as

$$\sigma_{yy}(x,L) - \lambda p_o = \frac{2}{(\kappa+1)\pi} \int_{-h}^{h} G(t)[\frac{\kappa-1}{2(t-x)} + K_{3F}(t,x) + K_{2s}(t,x) + K_{4F}(t,x)]dt , \qquad |x|<h \tag{38}$$

where

$$K_{4F}(t,x) = \frac{\pi}{L} \sum_{n=1}^{\infty} e^{-\alpha_n(h-t)} [k_1(t,x,\alpha_n) - k_{1\infty}(t,s,\alpha_n)]$$

$$k_1(t,x,\alpha_n) = \frac{2}{\sinh(2\alpha_n h) + 2\alpha_n h}\Big[\cosh(\alpha_n x)\{[\tfrac{\kappa+1}{2} + \alpha_n(h-t)][\alpha_n h\cosh(\alpha_n h)$$

$$- \sinh(\alpha_n h)] + [\tfrac{\kappa-1}{2} + \alpha_n(h-t)][\alpha_n h\sinh(\alpha_n h) - 2\cosh(\alpha_n h)]\}$$

$$- x\alpha_n\sinh(\alpha_n x)\{\sinh(\alpha_n h)[\tfrac{\kappa+1}{2} + \alpha_n(h-t)]$$

$$+ \cosh(\alpha_n h)[\tfrac{\kappa-1}{2} + \alpha_n(h-t)]\}\Big] \cdot$$

$$k_{1\infty}(t,x,\alpha_n) = e^{-\alpha_n h}\Big[\cosh(\alpha_n x)\{1 - [\kappa + 2\alpha_n(h-t)](3 - 2\alpha_n h)\}$$

$$- 2\alpha_n x\sinh(\alpha_n x)\{\kappa + 2\alpha_n(h-t)\}\Big] \qquad (39)$$

$$K_{2s}(t,x) = \frac{\pi}{2L}\Bigg[\{(1-3\kappa) + 2[\kappa h - 3(h-t)]\frac{d}{dt} + 4h(h-t)\frac{d^2}{dt^2}\}\{\frac{1}{1 - e^{-\frac{\pi}{L}(2h-t-x)}}$$

$$+ \frac{1}{1 - e^{-\frac{\pi}{L}(2h-t+x)}}\} - 2x\{\kappa\frac{d}{dt} + 2(h-t)\frac{d^2}{dt^2}\}\{\frac{1}{1 - e^{-\frac{\pi}{L}(2h-t-x)}}$$

$$- \frac{1}{1 - e^{-\frac{\pi}{L}(2h-t+x)}}\}\Bigg]$$

$$K_{3F}(t,x) = \frac{\kappa-1}{2}\{\frac{\pi}{2L}\coth(\frac{\pi}{2L}[t-x]) - \frac{1}{t-x}\} - \int_0^\infty \frac{\xi L}{\sinh^2(\xi L)}\sin\xi(t-x)d\xi$$

Using the solution of (36) in (38), the unknown constant λ can now be computed from (4). This enables the determination of the shear and normal stresses at the fixed end from the corresponding equations.

The behavior of $\sigma_{yy}(x,L)$ near the corner points $t \to \pm h$ can be determined by considering the dominant part of the equation (38), which can be written as

$$(\kappa+1)\sigma_{yy}(x,L) = \frac{1}{\pi}\int_{-h}^{h} G(t)\Big[\frac{\kappa-1}{t-x} + \{-(3\kappa+5) + 2(\kappa+7)(h-x)\frac{d}{dx}$$

$$- 4(h-x)^2 \frac{d^2}{dx^2}\} \frac{1}{2h-t-x} + \{-(3\kappa+5) - 2(\kappa+7)(h+x)\frac{d}{dx}$$

$$- 4(h+x)^2 \frac{d^2}{dx^2}\} \frac{1}{2h-t+x}\Big]dt \qquad (41)$$

Using (31) and relations from [6] and [3], the dominant part of the normal stress becomes

$$(\kappa+1)\sigma_{yy}(x,L) = \frac{1}{(2h)^{\alpha}\sin\pi\alpha}\,[(\kappa-1)(\cos\pi\alpha + 1) - 2(\kappa+1)(\alpha-1)$$

$$+ 4(\alpha-1)^2]\Big[\frac{H(-h)}{(h+x)^{\alpha}} - \frac{H(h)}{(h-x)^{\alpha}}\Big], \qquad x\to\pm h \qquad (42)$$

Defining the stress intensity factors as [3]

$$K_1 = \lim_{x\to h} \sqrt{2}\,(h-x)^{\alpha}\,\sigma_{yy}(x,L)$$

$$K_2 = \lim_{x\to h} \sqrt{2}\,(h+x)^{\alpha}\,\sigma_{xy}(x,L) \qquad (43)$$

and using (31) and (41), K_1 and K_2 are expressed in terms of the unknown function $H(x)$ as:

$$K_1 = -\frac{\sqrt{2}\,H(h)}{(\kappa+1)(2h)^{\alpha}\sin\pi\alpha}\,[(\kappa-1)(\cos\pi\alpha+1) - 2(\kappa+1)(\alpha-1) + 4(\alpha-1)^2]$$

$$K_2 = \sqrt{2}\,\frac{H(h)}{(2h)^{\alpha}} \qquad (44)$$

NUMERICAL RESULTS AND DISCUSSION

The total solution of the finite strip problem shown in Figure 1 is now obtained by summing the two problems I and II. Hence,

$$\sigma_{yy}^T(x,L) = \sigma_{yy}^I(x,L) + \sigma_{yy}(x,L) = -p_0 + \sigma_{yy}(x,L)$$
$$\sigma_{xy}^T(x,L) = \sigma_{xy}^I(x,L) + \sigma_{xy}(x,L) = \sigma_{xy}(x,L) \tag{45}$$

Note that equations (18) and (38) depend only on the Poisson's ratio of the strip. If the Poisson's ratio of the strip is zero, the disturbance problem ceases to exist and the total problem as shown in Figure 1 becomes identical to the homogeneous problem I. The same conclusion can also be arrived at by putting $\kappa=3$ in equation (17). Thus, for $\nu=0$

$$\sigma_{xy}^T(x,y) = \sigma_{xy}^I(x,y) = 0$$
$$\sigma_{yy}^T(x,y) = \sigma_{yy}^I(x,y) = -p_0 \tag{46}$$

The effect of the disturbance problem increases as the Poisson's ratio of the strip increases (to a maximum value 0.5). These effects are presented in detail in [3] for a semi-infinite strip problem, hence, will not be repeated here. In this study, since the effect of the strip size on the disturbance problem is of primary importance, the results are presented only for one value of the Poisson's ratio $\nu = 0.3$. Figures 2 and 3 show the variations of shear and normal stresses, respectively, along the fixed end of the strip for various values of strip length to width ratios. For a value $\frac{L}{h} = 10$, the results are identical to those obtained for a semi-infinite strip [3]. Figures 2 and 3 show that the effect of the disturbance problem decreases with a decrease in the length of the strip. This implies that the effect of the decrease in strip length on the edge stress field is similar to that of reducing the Poisson's ratio. The two effects are not identical, however, since the power α of stress singularity decreases when Poisson's ratio is decreased and it remains unchanged when the strip length is reduced.

The variation of the stress intensity factor K_2 (as in (44)) with respect to the strip length is given in Figure 4. As expected, the stress intensity factor decreases with the reduction in the strip length. In limit, it would go to zero for $\frac{L}{h} \to 0$ and would tend to that of a semi-infinite strip for $\frac{L}{h} >> 1$. A similar trend is predicted by Benthem and Minderhoud [2] for a finite cylinder problem. As is seen from (44), the stress intensity factor K_1 depends on K_2, and their ratio K_2/K_1 is only a function of the Poisson's ratio of the strip. From (44)

$$\frac{K_2}{K_1} = - \frac{(\kappa+1)\sin\pi\alpha}{[(\kappa-1)(\cos\pi\alpha + 1) - 2(\kappa+1)(\alpha-1) + 4(\alpha-1)^2]} \tag{47}$$

The power α of stress singularity in (47) is related to the Poisson's ratio via equation (32).

This ratio K_2/K_1 is not affected by the size of the strip. Figure 5 shows a variation of K_2/K_1 with respect to ν. The result is quite significant for this finite strip compression problem. If the rigid stamps are rough enough so that the coefficient of friction, f, between the stamps and strip surfaces, is greater than K_2/K_1, the contact condition may be assumed to be that of perfect adhesion, i.e., no sliding would occur and the solution given in this paper would be valid. If $f < K_2/K_1$, the problem becomes that of a finite strip compressed between two rigid stamps with friction. Figure 5 shows that in a crushing test, to determine whether the end condition is that of perfect adhesion or sliding, as a first approximation one may assume that $K_2/K_1 = \nu$. The above comments and the results shown by Figure 5 were included in [3]; however, they must be repeated here since they are even more relevant to the physical problem considered in this paper.

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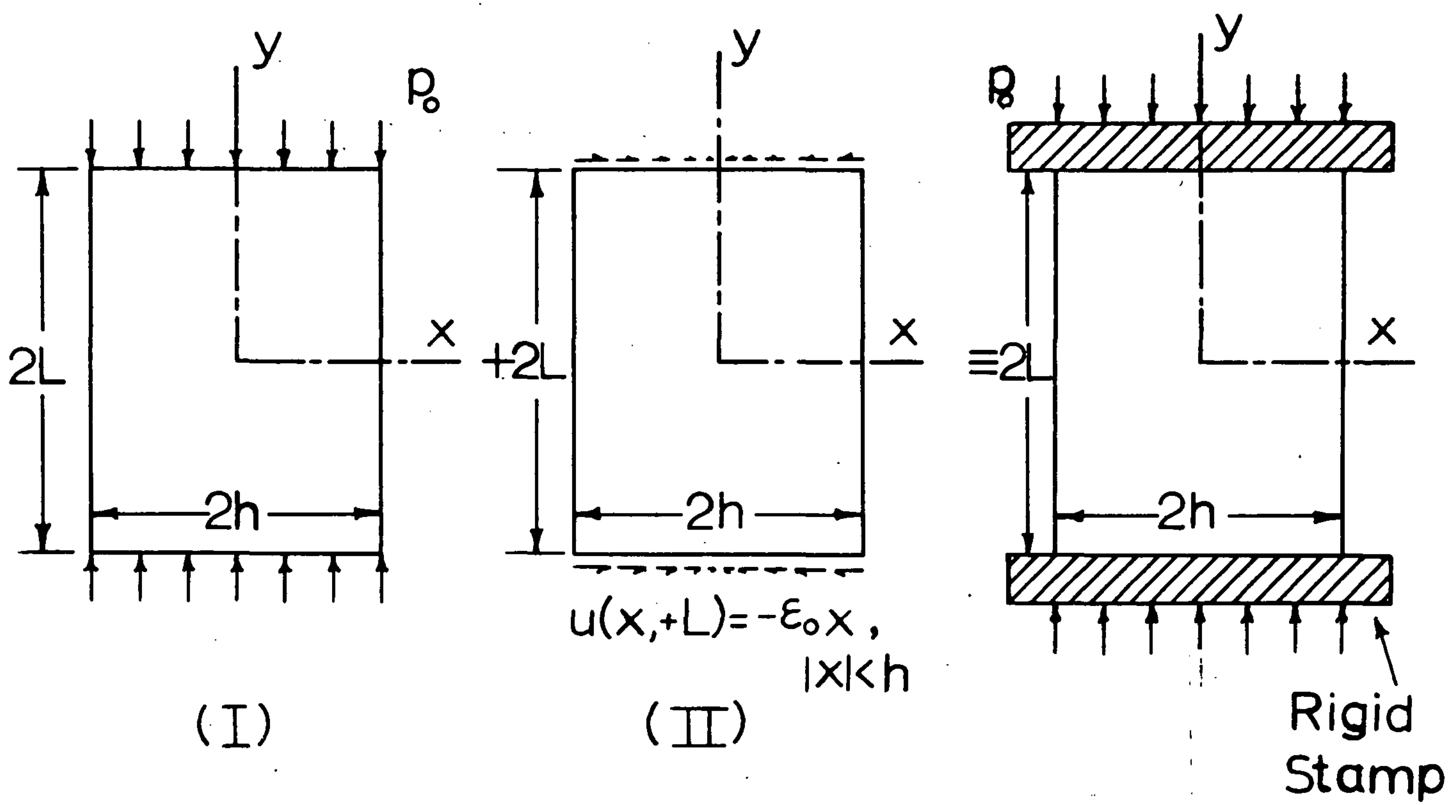

Figure 1. Crushing of a Finite Strip Between Two Rough Rigid Stamps

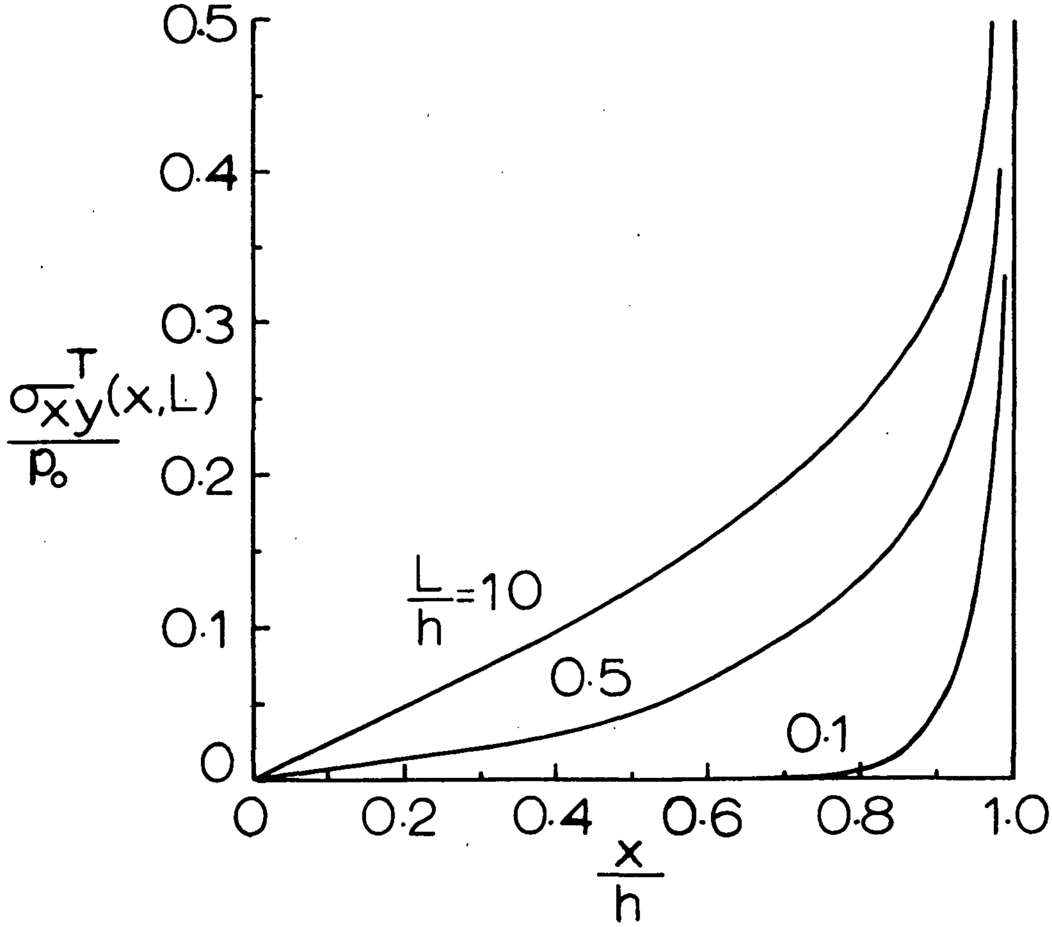

Figure 2. Shear Stress vs. the Strip Length for the Finite Strip. $\nu = 0.3$

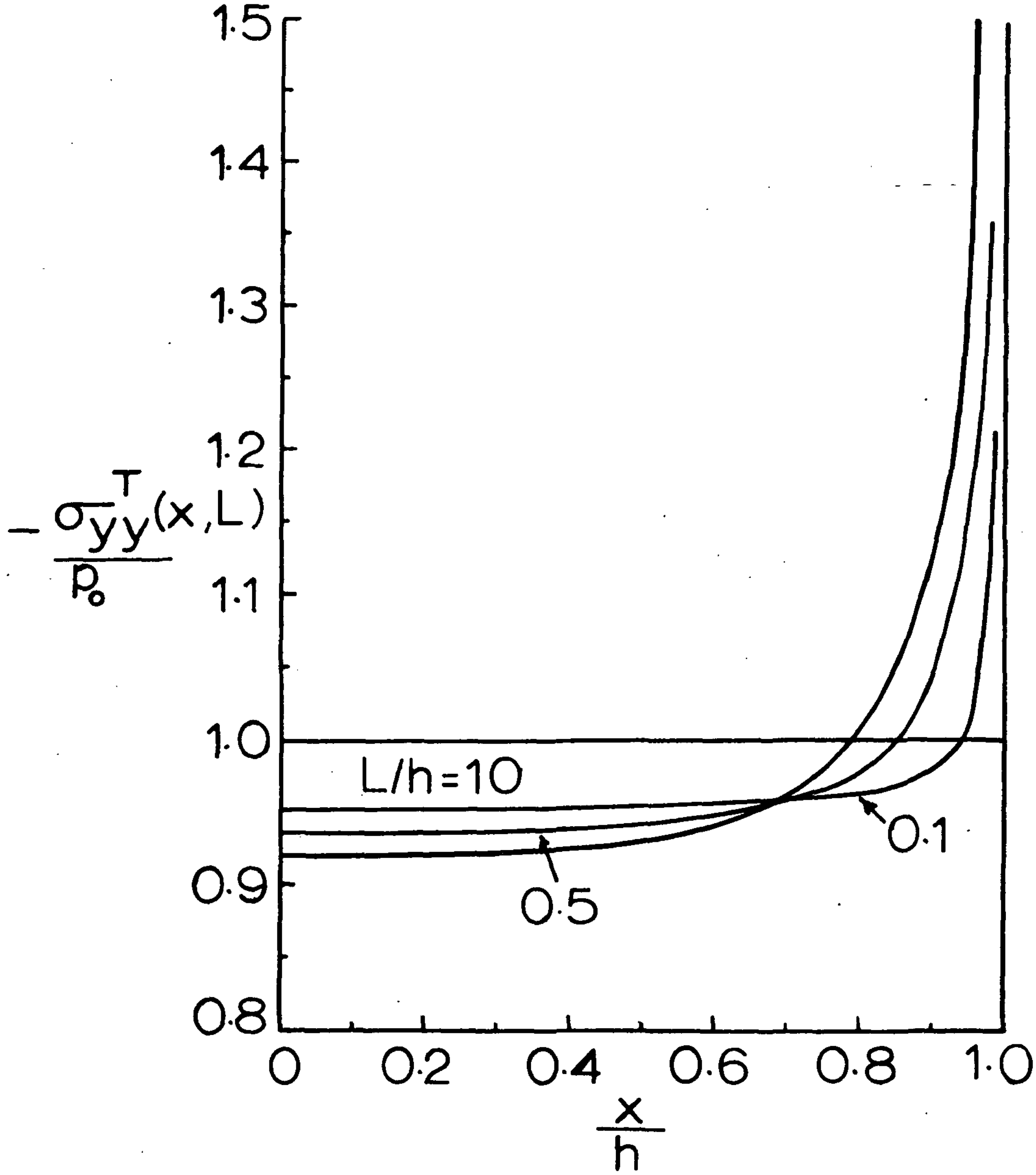

Figure 3. Normal Stress vs. the Strip Length for the Finite Strip. $\nu = 0.3$

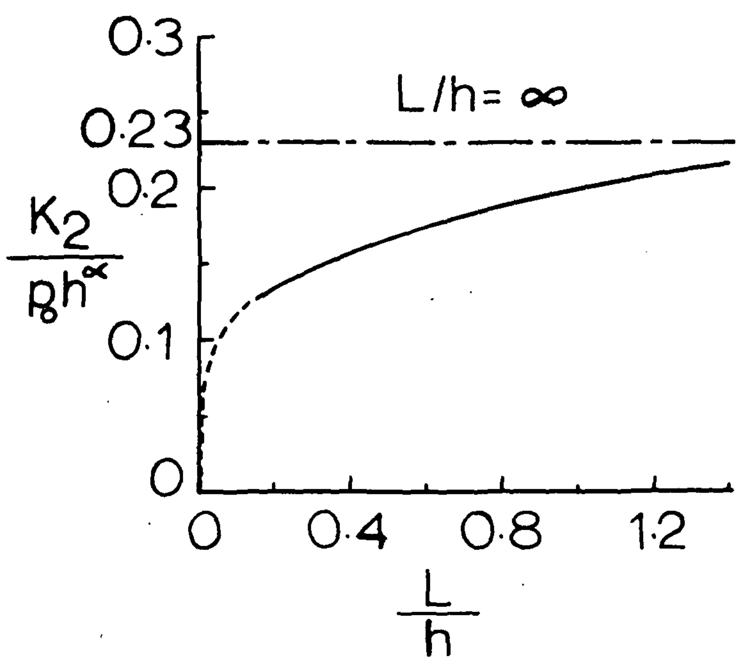

Figure 4. Stress Intensity Factor vs. the Strip Length. $\nu = 0.3$

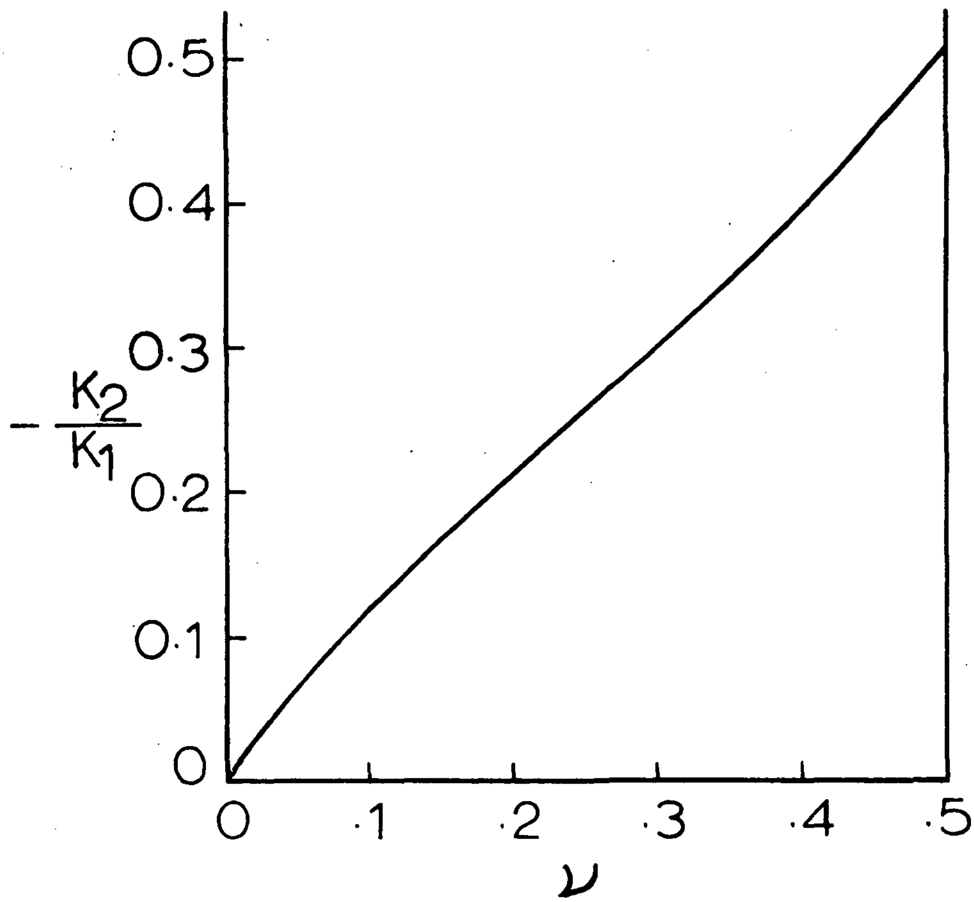

Figure 5. K_2/K_1 vs. Strip Poisson's Ratio ν.