BIFURCATION AND STABILITY FOR A NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

by

Nathaniel Chafee*

Division of Applied Mathematics
Brown University
Providence, R.I. 02912

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List of symbols

Roman letters: lower case

f	t
n	u
q	x

Roman letters: upper case

U	X
V	

Greek letters: lower case

λ lambda	ϕ phi
ξ xi	ω omega
π pi	

Mathematical symbols

∞ infinity

$|| \, ||_1$

This note is a brief report on some research conducted by the author and E.F. Infante in 1971. A complete report on this same research is scheduled to appear in a separate article [1].

Let f be a given function continuously mapping the real line R into itself. Let λ be a given non-negative real number. Let $\phi:[0,\pi] \to R$ be any C^1-smooth function such that $\phi(0) = \phi(\pi) = 0$. We shall be discussing the following problem. Find a function u continuously mapping the domain $\{(x,t)\colon 0 \le x \le \pi,\ 0 \le t < +\infty\}$ into R such that (i) the partial derivatives u_t and u_{xx} are defined and continuous on $[0,\pi] \times (0,+\infty)$; (ii) u satisfies the equations

$$u_t(x,t) = u_{xx}(x,t) + \lambda f(u(x,t)) \qquad (0 \le x \le \pi,\ 0 < t < +\infty) \qquad (1a)$$

$$u(0,t) = u(\pi,t) = 0 \qquad (0 \le t < +\infty) \qquad (1b)$$

$$u(x,0) = \phi(x) \qquad (0 \le x \le \pi)\ . \qquad (1c)$$

By a solution of (1) we mean a function u having the properties just specified.

Our primary goal in studying (1) is to determine the asymptotic behavior of solutions u of (1) as $t \to +\infty$. The investigation takes place under the following hypotheses concerning f.

(H_1) f is a C^2-smooth function mapping R into itself.

(H_2) $f(0) = 0$ and $f'(0) > 0$.

(H_3) $\limsup_{|\xi| \to +\infty} \xi^{-1} f(\xi) = 0$

(H_4) $\operatorname{sgn} f''(\xi) = -\operatorname{sgn} \xi$ for all $\xi \in R$.

In that which follows we shall let X denote the space of all C^1-smooth functions $\phi:[0,\pi] \to R$ such that $\phi(0) = \phi(\pi) = 0$. On X we impose a norm $|| \; ||_1$ by setting $||\phi||_1 = \sup\{|\phi'(x)|: 0 \leq x \leq \pi\}$ for all $\phi \varepsilon X$. X is a Banach space under $|| \; ||_1$.

It can be shown that, for any $\phi \varepsilon X$ and $\lambda \varepsilon [0,+\infty)$, Eqs. (1) have a unique solution $u(\phi,\lambda)$ defined on $[0,\pi] \times [0,+\infty)$. A non-trivial aspect of this assertion is the statement that the domain of definition for $u(\phi,\lambda)$ is all of $[0,\pi] \times [0,+\infty)$. We shall briefly return to this matter below.

For any $\phi \varepsilon X$, $\lambda \varepsilon [0,+\infty)$, $x \varepsilon [0,\pi]$, and $t \varepsilon [0,+\infty)$, we can let $u(x,t;\phi,\lambda)$ denote the value of $u(\phi,\lambda)$ at (x,t). With this in mind, we can define, for any $\lambda \varepsilon [0,+\infty)$, a nonlinear semigroup $\{U_\lambda(t)\}$ on X by setting $U_\lambda(t)\phi = u(\cdot,t;\phi,\lambda)$ for all $\phi \varepsilon X$ and $t \varepsilon [0,+\infty)$. It can be shown that $\{U_\lambda(t)\}$ is strongly continuous.

Let $\lambda \varepsilon [0,+\infty)$. By an equilibrium solution of (1) (corresponding to λ) we mean a function $u_0 \varepsilon X$ such that $U_\lambda(t)u_0 = u_0$ for all $t \varepsilon [0,+\infty)$. By virtue of (H_2), the origin $\phi_0 = 0$ in X is an equilibrium solution of (1) for every $\lambda \varepsilon [0,+\infty)$.

To discuss the existence of other equilibrium solutions for (1), we introduce a sequence of real numbers $\{\lambda_n\}_{n=1}^{+\infty}$ by setting $\lambda_n = n^2/f'(0)$ for each integer $n \geq 1$. By virtue of (H_2), we have $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$. We are now ready to state our first theorem.

<u>Theorem 1.</u> For any integer $n \geq 1$ and any number $\lambda \varepsilon [\lambda_n,+\infty)$, Eqs. (1) have two equilibrium solutions $u_n^{\pm}(\lambda)$ possessing the following three properties:

(i) $u_n^{\pm}(\lambda) = 0$ if and only if $\lambda = \lambda_n$.

(ii) The mappings $\lambda \longmapsto u_n^{\pm}(\lambda)$ from $[\lambda_n,+\infty)$ into X are each continuous. In particular, $u_n^{\pm}(\lambda) \to 0$ as $\lambda \to \lambda_n$. Also, $||u_n^{\pm}(\lambda)||_1 \to +\infty$ as $\lambda \to +\infty$.

(iii) For any $\lambda\varepsilon(\lambda_n,+\infty)$, $u_n^{\pm}(\lambda)$ has exactly n+1 zeros $x_0^{\pm}(\lambda)$, $x_1^{\pm}(\lambda)$, ..., $x_n^{\pm}(\lambda)$ in $[0,\pi]$ with $0 = x_0^{\pm}(\lambda) < x_1^{\pm}(\lambda) < \ldots < x_n^{\pm}(\lambda) = \pi$. Moreover, for each integer q = 0,1,...,n-1, we have $(-1)^q u_n^+(x;\lambda) > 0$ if $x_q^+(\lambda) < x < x_{q+1}^+(\lambda)$ and we have $(-1)^q u_n^-(x;\lambda) < 0$ if $x_q^-(\lambda) < x < x_{q+1}^-(\lambda)$.

In addition to the preceding assertions, we have that for any $\lambda\varepsilon[0,+\infty)$ Eqs. (1) have no equilibrium solutions other than the zero solution $u_0 = 0$ and those elements $u_n^{\pm}(\lambda)$, $n \geq 1$, such that $\lambda_n \leq \lambda$.

On the basis of Assertion (ii) in Theorem 1, we may state that, for any integer $n \geq 1$, the two equilibrium solutions $u_n^{\pm}(\lambda)$ bifurcate from the zero solution as λ increases from λ_n.

Now we come to our second theorem.

Theorem 2. For any $\phi\varepsilon X$ and any $\lambda\varepsilon[0,+\infty)$, there exists an equilibrium solution $u_0(\phi,\lambda)$ of (1) such that $U_\lambda(t)\phi \to u_0(\phi,\lambda)$ as $t \to +\infty$.

The question arises, given $\phi\varepsilon X$ and $\lambda\varepsilon[0,+\infty)$, to which of the equilibrium solutions described in Theorem 1 is $u_0(\phi,\lambda)$ equal? A partial answer to this query is given in the following theorem.

Theorem 3. For any $\lambda\varepsilon[0,\lambda_1]$, the zero solution $u_0 = 0$ of (1) is globally asymptotically stable in the sense of Liapunov. In particular, for each $\phi\varepsilon X$ and $\lambda\varepsilon[0,\lambda_1]$, we have $||U_\lambda(t)\phi||_1 \to 0$ as $t \to +\infty$. For any $\lambda\varepsilon(\lambda_1,+\infty)$, the zero solution $u_0 = 0$ of (1) is unstable. For any $\lambda\varepsilon[\lambda_1,+\infty)$, the solutions $u_1^{\pm}(\lambda)$ are each asymptotically stable in the sense of Liapunov. Finally, for any integer $n \geq 2$ and any $\lambda\varepsilon[\lambda_n,+\infty)$, the solutions $u_n^{\pm}(\lambda)$ are each unstable.

Theorems 1-3 are proved in the article [1] already mentioned. We shall not repeat the proofs here but shall rather confine ourselves to making the following remarks.

Our approach to studying Eqs. (1) is to interpret (1) as a dynamical system on X and then to apply certain methods associated with the Liapunov theory of stability. The methods we have in mind are set forth in [2], [3] and [4] and are often referred to as the invariance principle in stability theory.

An essential tool in our use of the invariance principle is the following Liapunov functional:

$$V_\lambda(\phi) = \int_0^\pi \{\frac{1}{2}\phi'(x)^2 - \lambda \int_0^{\phi(x)} f(\xi)d\xi\}\, dx \qquad (\phi\varepsilon X, \lambda\varepsilon[0,+\infty)) . \qquad (2)$$

For each $\lambda\varepsilon[0,+\infty)$, Eq. (2) defines a functional V_λ mapping X into R. For any $\phi\varepsilon X$ and $\lambda\varepsilon[0,+\infty)$, it can be shown that

$$\dot{V}_\lambda(U_\lambda(t)\phi) = -\int_0^\pi |u_t(x,t;\phi,\lambda)|^2\, dx \qquad (t > 0) . \qquad (3)$$

Consider any $\phi\varepsilon X$ and $\lambda\varepsilon[0,+\infty)$. Using V_λ one can show that the solution $u(\phi,\lambda)$ is defined everywhere on $[0,\pi] \times [0,+\infty)$. This is a matter which we have mentioned earlier in this note. Of more immediate interest is the fact that, using V_λ, one can show that $u(\phi,\lambda)$ has a nonempty compact connected invariant ω-limit set $\omega(\phi,\lambda) \subset X$. Here, one also uses the invariance principle referred to two paragraphs above. That same principle together with Eq. (3) tells us that any element in $\omega(\phi,\lambda)$ must be an equilibrium solution of (1).

Therefore, one now seeks the equilibrium solutions of Eqs. (1). This means that one studies the two-point boundary-value problem

$$\begin{aligned} &u''(x) + \lambda f(u(x)) = 0 \\ &u(0) = u(\pi) = 0 \end{aligned} \qquad (0 \leq x \leq \pi,\ 0 \leq \lambda < +\infty) . \tag{4}$$

The results of our investigation are stated in Theorem 1. In particular, we see that, for any $\lambda \varepsilon [0,+\infty)$, each equilibrium solution of (1) is isolated in X. Hence, for any $\phi \varepsilon X$ and $\lambda \varepsilon [0,+\infty)$, the set $\omega(\phi,\lambda)$ consists of exactly one equilibrium solution of (1). From this there follows Theorem 2.

Theorem 3 is established using arguments from the classical theory of calculus of variations. We shall not attempt to describe these arguments here.

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