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CONTINUOUS DEPENDENCE OF FIXED POINTS OF CONDENSING MAPS

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Many problems in analysis are concerned with the dependence upon parameters of fixed points of maps. For contraction mappings, criteria are relatively easy to obtain and have been known for some time. In the study of solutions of functional differential equations, more general results were needed by Hale and Cruz [3] and Melvin [8]. Kasriel and Nashed [4] and Cain and Nashed [1] have also obtained some theorems in this direction under the name of stability of fixed points of nonlinear mappings. It is the purpose of this paper to give a rather general fixed-point theorem for condensing maps depending on a parameter, to prove continuous dependence and to indicate how many of the previous results are special cases.

An interesting theorem on continuous dependence of a fixed point of a map obtained by means of an asymptotic fixed-point theorem is contained in the paper of Lopes [6].

We begin with a few definitions and known results. If A is a bounded set of a Banach space X, define the measure $\alpha(B)$ of noncompactness of a set B to be $\alpha(B) = \inf\{d>0:$ B has a finite covering of diameter less than $d\}$. A bounded set B has $\alpha(B) = 0$ if and only if the closure $\overline{B}$ of B is compact. This concept was introduced by Kuratowski [5] and later Darbo [2] showed that $\alpha(B)$ satisfies the following properties:

i) $\alpha(\overline{co}\, B) = \alpha(B)$, where $\overline{co}\, B$ is the closed convex hull of B,

ii) $\alpha(B+C) \leq \alpha(B) + \alpha(C)$,

iii) $\alpha(B \cup C) = \max(\alpha(B),\alpha(C))$.

Suppose Γ is a subset of X and $T: \Gamma \to X$ is a continuous mapping. The map T is said to be condensing if for any bounded set $B \subset \Gamma$, $\alpha(B) > 0$, it follows that $\alpha(TB) < \alpha(B)$. The map T is said to be an α-contraction if there is a $k \in [0,1)$ such that for any bounded set $B \subset \Gamma$, $\alpha(TB) \leq k\alpha(B)$. An α-contraction is necessarily condensing, but the converse may not be true. For linear operators, the concepts are equivalent. Any mapping which is the sum of a contraction operator and a completely continuous operator is an α-contraction. A fundamental result which will be used throughout is the following generalization of Schauder's fixed-point theorem (see Sadovskii [9] and Martelli [7]).

Lemma 1. If Γ is a closed bounded convex subset of X and $T: \Gamma \to \Gamma$ is condensing, then T has a fixed point.

Suppose $X, \mathcal{L}$ are Banach spaces, Λ is a subset of $\mathcal{L}$, Γ is a closed bounded subset of X and $T: \Gamma \times \Lambda \to X$ is a given mapping satisfying

h_1) $T(\cdot,\lambda)$ is continuous for each $\lambda \in \Lambda$ and there exists a $\lambda_0 \in \Lambda$ such that $T(x,\lambda)$ is continuous at x,λ_0 for each $x \in \Gamma$.

h_2) The equation

$$x = T(x,\lambda) \tag{1}$$

for $\lambda = \lambda_0$ has a unique solution $x(\lambda_0)$ in Γ.

h_3) For every $\Gamma' \subset \Gamma$, $\alpha(\Gamma') > 0$, there is an open neighborhood $B = B(\Gamma')$ of λ_0 such that for any precompact set $\Lambda' \subset \Lambda \cap B$, we have

$$\alpha(T(\Gamma',\Lambda')) < \alpha(\Gamma').$$

Theorem 1. For each $\lambda \in \Lambda \cap B(\Gamma)$, suppose (1) has a solution $x(\lambda) \in \Gamma$. Then $x(\lambda) \to x(\lambda_0)$ as $\lambda \to \lambda_0$.

Proof. Suppose $\{\lambda_k\} \subset \Lambda \cap B(\Gamma)$ is an arbitrary sequence converging to λ_0 as

$k \to \infty$, and let $x(\lambda_k) \varepsilon \Gamma$ be a solution of (1) for $\lambda = \lambda_k$. If $\Gamma' = \{x(\lambda_k)\}$, choose k so large that $\Lambda' = \{\lambda_k\} \subset \Lambda \cap B(\Gamma')$. Since Λ' is precompact and (h_3) is satisfied,

$$\alpha(\Gamma') = \alpha(\{T(x_{\lambda_k}, \lambda_k)\} \leq \alpha(T(\Gamma', \Lambda')) < \alpha(\Gamma')$$

if $\alpha(\Gamma') > 0$. Since this is impossible, $\alpha(\Gamma') = 0$ and Γ' is precompact. Since Γ is closed, there is a subsequence $\{\nu_k\}$ of $\{\lambda_k\}$ and $z \varepsilon \Gamma$ such that $x(\nu_k) \to z$ as $k \to \infty$. Condition (h_1) implies $z = T(z, \lambda_0)$ and (h_2) implies $z = x(\lambda_0)$. Since every convergent subsequence of the $\{x(\lambda_k)\}$ must converge to the same limit, it follows that $x(\lambda_k) \to x(\lambda_0)$ as $k \to \infty$. Since the sequence $\{\lambda_k\}$ was an arbitrary sequence converging to λ_0, the proof of the theorem is complete.

Corollary 1. Suppose T satisfies (h_2) and

> h_4) $T(\cdot, \lambda)$ is condensing for each $\lambda \varepsilon \Lambda$, and $T(x, \lambda)$ is continuous at λ_0 uniformly for $x \varepsilon \Gamma$.

Then T satisfies (h_1) - (h_3) and the conclusions of Theorem 1 are valid.

Proof. Since a condensing map is continuous, (h_4) implies (h_1). Thus we need only show that (h_3) is satisfied. Let $B_\beta = \{\lambda : |\lambda - \lambda_0| < \beta\}$. Uniform continuity at λ_0 implies for any $\varepsilon > 0$ there is a $\beta(\varepsilon) > 0$ such that $|T(x, \lambda) - T(x, \lambda_0)| \leq \varepsilon$ if $\lambda \varepsilon B_{\beta(\varepsilon)}$, $x \varepsilon \Gamma$. If $\Gamma' \subset \Gamma$ and $\alpha(\Gamma') > 0$, then $T(\cdot, \lambda_0)$ condensing implies there is a $k = k(\Gamma', \lambda_0) \varepsilon [0,1)$ such that $\alpha(T(\Gamma', \lambda_0)) \leq k\alpha(\Gamma')$. Let $2\varepsilon = \delta\alpha(\Gamma')$ and choose $\varepsilon > 0$ so small that $k + \delta < 1$. Since

$$T(x, \lambda) = T(x, \lambda_0) + [T(x, \lambda) - T(x, \lambda_0)],$$

we have, for any $\Lambda' \subset \Lambda \cap B_{\beta(\varepsilon)}$,

$$\alpha(T(\Gamma',\Lambda')) \leq \alpha(T(\Gamma',\lambda_0)) + 2\varepsilon \leq (k+\delta)\alpha(\Gamma') < \alpha(\Gamma') .$$

Thus, (h_3) is satisfied and the corollary is proved.

Corollary 2. If (h_1) - (h_3) are satisfied, Γ is also convex and $T:\Gamma \times \Lambda \to \Gamma$, then there is a solution $x(\lambda)$ of (1) for each $\lambda \varepsilon \Lambda \cap B(\Gamma)$ and $x(\lambda) \to x(\lambda_0)$ as $\lambda \to \lambda_0$.

Proof. Use Lemma 1 and Theorem 1.

We now give some specific mappings which satisfy the conditions of Theorem 1 and, in particular, condition (h_3). Suppose X, Y, $\mathcal{L}$ are Banach spaces, Γ is a closed bounded set of X and Λ is a subset of $\mathcal{L}$. Suppose

$$Q: \Gamma \times \Lambda \to Y \tag{2}$$

$$G: \Gamma \times \overline{Q(\Gamma,\Lambda)} \times \Lambda \to X \tag{3}$$

$$T: \Gamma \times \Lambda \to X,\ T(x,\lambda) \overset{\text{def}}{=} G(x,Q(x,\lambda),\lambda) \tag{4}$$

Lemma 2. If Q, G, T are defined as above and for each precompact $\Lambda' \subset \Lambda$ and each precompact $Y' \subset \overline{Q(\Gamma,\Lambda)}$ we have

h_5) $G(\cdot,y,\lambda):\Gamma \to X$ is a contraction uniformly with respect to $(y,\lambda) \varepsilon Y' \times \Lambda'$

h_6) $Q(\Gamma,\Lambda')$ is precompact ,

then T satisfies condition (h_3).

Proof. From (h_6), the set $\overline{Q(\Gamma,\Lambda')} \overset{\text{def}}{=} Y'$ is compact. Suppose Λ' is a precompact set of Λ. Then hypothesis (h_5) implies there is a $k\epsilon[0,1)$ such that

$$|G(x,y,\lambda)-G(\bar{x},y,\lambda)| \leq k|x-\bar{x}| \quad \text{for all } (y,\lambda)\epsilon Y' \times \Lambda'.$$

Therefore, for any $\Gamma' \subset \Gamma$

$$\alpha(G(\Gamma',Y',\Lambda')) \leq k\alpha(\Gamma') .$$

Furthermore,

$$\alpha(T(\Gamma',\Lambda')) = \alpha(\bigcup_{(x,\lambda)\epsilon\Gamma'\times\Lambda'} G(x,Q(x,\lambda),\lambda))$$

$$\leq \alpha(\bigcup_{(x,y,\lambda)\epsilon\Gamma\times Y'\times\Lambda'} G(x,y,\lambda))$$

$$= \alpha(G(\Gamma,Y',\Lambda')) \leq k\alpha(\Gamma') .$$

This proves (h_3) and the lemma.

Now let us make a few more hypotheses on Q, G; namely,

h_7) $Q(x,\lambda)$ is continuous in x for each $\lambda\epsilon\Lambda$ and
$Q(x,\lambda)$ is continuous in (x,λ) at $(x,\lambda_0)\epsilon\Gamma \times \Lambda$.

h_8) $G(x,y,\lambda)$ is continuous in y for each $(x,\lambda)\epsilon\Gamma \times \Lambda$ and $G(x,y,\lambda)$ is continuous in (y,λ) at (x,y,λ_0) for each $(x,y,\lambda)\epsilon\Gamma \times \overline{Q(\Gamma,\Lambda)} \times \Lambda$.

If conditions (h_4) - (h_8) are satisfied, then the map $T(\cdot,\lambda):\Gamma \to X$ defined by (4) satisfies (h_1). In fact, for any precompact $\Lambda' \subset \Lambda$, there is a $k\epsilon[0,1)$ such that, for any $(x,\lambda)\epsilon\Gamma \times \Lambda'$, $(\bar{x},\bar{\lambda})\epsilon\Gamma \times \Lambda'$, we have

$$|T(x,\lambda)-T(\bar{x},\bar{\lambda})| \leq |G(x,Q(x,\lambda),\lambda) - G(\bar{x},Q(x,\lambda),\lambda)|$$

$$+ |G(\bar{x},Q(x,\lambda),\lambda) - G(\bar{x},Q(\bar{x},\bar{\lambda}),\bar{\lambda})|$$

$$\leq k|x-\bar{x}| + |G(\bar{x},Q(x,\lambda),\lambda) - G(\bar{x},Q(\bar{x},\bar{\lambda}),\bar{\lambda})| .$$

Now one obtains the result by observing that the set $\Lambda' = \{\lambda_k\}$ is precompact if $\lambda_k \to \lambda_0$ and $\Lambda' = \{\lambda\}$ is compact for any given $\lambda\varepsilon\Lambda$.

We thus obtain the following

Theorem 2 [8]. Suppose $T: \Gamma \times \Lambda \to X$ is defined by (4) and conditions (h_5) - (h_8) are satisfied. Then T satisfies (h_1) - (h_3) and the conclusions of Theorem 1 are valid.

For the special case where $G(x,y,\lambda) = S(x,\lambda) + y$, we have the following:

Corollary 3 [8]. Suppose $T = S + U$, $S: \Gamma \times \Lambda \to X$, $U: \Gamma \times \Lambda \to X$, T satisfies (h_1), (h_2) and for each compact set $\Lambda' \subset \Lambda$

h_9) $S(\cdot,\lambda)$ is a contraction uniformly with respect to $\lambda\varepsilon\Lambda'$,

h_{10}) $U(\Gamma,\Lambda')$ is precompact.

Then T satisfies (h_1) - (h_3) and the conclusions of Theorem 1 are valid.

Corollary 4 [3]. Suppose T is defined as in Corollary 3, satisfies condition (h_2) of Theorem 1; Q satisfies (h_{10}) and also

h_{11}) $S(\cdot,\lambda)$ is a contraction for each $\lambda\varepsilon\Lambda$, $U(\cdot,\lambda)$ is continuous for each $\lambda\varepsilon\Lambda$.

h_{12}) $S(x,\lambda)$, $U(x,\lambda)$ are continuous at λ_0 uniformly for $x\varepsilon\Gamma$.

Then T satisfies $(h_1) = (h_3)$ and the conclusions of Theorem 1 are valid.

Proof. This follows from Corollary 1.

In a similar manner, one can generalize Theorem 1.2A of [8].

As another application, let us consider a problem of Cain and Nashed [1] concerning stable solutions of nonlinear equations. In [1], generally locally convex spaces are considered, but we can see how their results for Banach spaces can be obtained from Theorem 1. Suppose we consider all continuous maps A of some open ball U into X. We may then define $||A_1-A_2||_U = \sup_{x\epsilon\overline{U}}||A_1x-A_2x||$. Let $S(x,r) = \{y\epsilon X:||y-x||<r\}$.

Definition. Suppose $x_0 \epsilon U$, $A_0: U \to X$ and $A_0x_0 = y_0$. The solution x_0 of $A_0u = y_0$ is said to be stable if for any $r > 0$, $S(x,r) \subset U$, there is a $d > 0$, $a > 0$ such that for any $y\epsilon S(y_0,d)$, $A_1: U \to X$, $||A_1-A_0||_U < a$, there is an $x\epsilon S(x_0,r)$ such that $A_1x = y$.

Without loss in generality, one may assume $y_0 = 0$. Thus, if $A_0 = I - T_0$, then x_0 is a fixed point of T_0. If, for a fixed $y\epsilon X$, $A_1: U \to X$, we define

$$x - T_0x + (A_1x - A_0x) - y \overset{\text{def}}{=} x - T(x,A_1,y)$$

then showing that a solution x_0 of $A_0x_0 = 0$ is stable is equivalent to showing that each of the operators $T(x,A_1,y)$ has a fixed point $x\epsilon S(x_0,r)$ for every $||y|| < d$, $||A_1-A_0||_U < a$. In particular, x_0 will be stable if we can show the fixed points are continuous at $y = 0$, $A_1 = A_0$. The parameter λ in Theorem 1 can be taken as $\lambda = (y,A_1)$.

Following Cain and Nashed [1], let $A: X \to X$.

$$R(x_0,A,r) = \frac{1}{r}\sup\{||Ax-Ax_0||:||x-x_0||=r\}$$

$$\eta(x_0,A) = \inf\{r: R(x_0,A,r)<1\}.$$

If $\eta(x_0,T_0) = 0$, $x_0 = T_0x_0$, then x_0 is an isolated fixed point of T_0. Furthermore, there is an r which may be taken as small as desired so that

$$||T_0x-x_0|| \leq R(x_0,T_0,r)r < r \text{ if } ||x-x_0|| = r .$$

Consequently, $T_0: \partial S(x_0,r) \to S(x_0,r)$. One can now choose $r > 0$ so small and $a > 0$, $d > 0$ such that $\overline{co}\ T(S(x_0,r),A_1,y) \subset S(x_0,1)$ for $||y|| < d$, $||A_1-A_0|| < a$. Let p be a retract of $S(x_0,1)$ onto $S(x_0,r)$ and define $F(\cdot,A_1,y)$: $S(0,1) \to S(0,1)$ by $F(x,A_1,y) = T(p(x),A_1,y)$. Then the fixed points of $F(\cdot,A_1,y)$ must belong to $S(x_0,r)$. Furthermore, $F(\cdot,A_1,y)$ is a condensing map if $T(\cdot,A_1,y)$ is condensing and we have the following consequence of Theorem 1 and Lemma 1.

<u>Theorem 3</u>. If $B_0x_0 = 0$, $\eta(x_0,I-B_0) = 0$, then x_0 is a stable solution of $B_0x_0 = 0$ relative to the class of operators B such that I-B is condensing.

When the operators B are completely continuous, this is Theorem 4.1 of [1]. Theorem 4.2 of [1] in Banach spaces is also a special case of Theorem 1 and Lemma 1.

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