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CLASSICAL EIGHTH- AND LOWER-ORDER RUNGE-KUTTA-NYSTRÖM FORMULAS WITH A NEW STEPSIZE CONTROL PROCEDURE FOR SPECIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

by Erwin Fehlberg

George C. Marshall Space Flight Center
Marshall Space Flight Center, Ala. 35812

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16. Abstract

New Runge-Kutta-Nyström formulas of the eighth, seventh, sixth, and fifth order are derived for the special second-order (vector) differential equation $\ddot{x} = f(t,x)$. In contrast to Runge-Kutta-Nyström formulas of an earlier NASA report by this author, these formulas provide a stepsize control procedure based on the leading term of the local truncation error in $\dot{x}$. This new procedure is more accurate than the earlier Runge-Kutta-Nyström procedure of this author (with stepsize control based on the leading term of the local truncation error in x) when integrating close to singularities. Two central orbits are presented as examples. For these orbits, the accuracy and speed of the formulas of this report are compared with those of Runge-Kutta-Nyström and Runge-Kutta formulas of earlier NASA reports by this author.

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CLASSICAL EIGHTH- AND LOWER-ORDER RUNGE-KUTTA-NYSTRÖM FORMULAS WITH A NEW STEPSIZE CONTROL PROCEDURE FOR SPECIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

INTRODUCTION

1. In an earlier report [1] this author derived Runge-Kutta-Nyström (RKN) formulas for a special class of second-order (vector) differential equations,

$$\ddot{x} = f(t,x) \quad , \tag{1}$$

which do not contain the first derivative $\dot{x}$ on the right-hand side. In report [1], the stepsize control for the Runge-Kutta-Nyström formulas was based on the leading term of the local truncation error in x.

2. In this report we will derive Runge-Kutta-Nyström formulas for differential equations (1) with stepsize control based on the leading term of the local truncation error in $\dot{x}$. The derivation of such formulas is similar to the derivation in report [1]. However, because of the higher accuracy in $\dot{x}$ required for our new formulas, their derivation will be more involved. The new formulas will, in general, require a slightly higher number of evaluations of the differential equations per integration step compared with the formulas of report [1]: for the eighth-order formula, 13 evaluations instead of 11; for the seventh-order formula, 10 evaluations instead of 9; and for the sixth-order formula, 8 evaluations instead of 7, whereas the fifth-order formula requires 6 evaluations only as the corresponding formula of report [1].

The number of evaluations per step still compares favorably with the number of evaluations of the Runge-Kutta formulas for the first-order differential equations of report [2].

3. The formulas of this report are well suited for problems in which the derivative $\dot{x}$ might assume large values. If $\dot{x}$ becomes large, the stepsize control procedure of report [1] might lead to somewhat less accurate values for $\dot{x}$. A stepsize that is small enough for x might, for large $\dot{x}$, no longer be sufficiently small for $\dot{x}$. In Section V an

example (Orbit 2) is presented in which formulas of this report show a higher accuracy than corresponding formulas of report [1].

4. Similar to the formulas of report [1], the formulas of this report represent pairs of integration formulas for $\dot{x}$ which differ from one another by one additional evaluation of the differential equations. The orders of these formulas differ by one. Therefore, the difference of the two formulas represents an approximation for the leading term of the truncation error in $\dot{x}$ for the lower-order formula.

For x, in this report, a Runge-Kutta-Nyström formula is used, the order of which is equal to the order of the higher-order formula for $\dot{x}$.

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5. Since the equations of condition for our Runge-Kutta-Nyström formulas are derived in [1], the reader is referred to their listing in Table 1 of [1].

For an eighth-order Runge-Kutta-Nyström formula, we allow for 13 evaluations per step:

$$\left.\begin{aligned} f_0 &= f(t_0, x_0) \\ f_\kappa &= f\left(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0 \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} f_\lambda\right) \\ & \qquad (\kappa = 1, 2, \ldots, 13) \end{aligned}\right\} . \tag{2}$$

There are actually 14 evaluations. However, as in our earlier report [1], we require that the last evaluation f_{13} of our differential equations can be taken over as first evaluation f_0 for the next step. Therefore, only the very first integration step will require 14 evaluations. Our new Runge-Kutta-Nyström formulas are of the form:

$$x = x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{13} c_\kappa f_\kappa + 0(h^{10}) \tag{3}$$

$$\left.\begin{aligned}\dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{12} \dot{c}_\kappa f_\kappa + 0(h^9) \\ \hat{\dot{x}} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{13} \hat{\dot{c}}_\kappa f_\kappa + 0(h^{10})\end{aligned}\right\} . \qquad \text{(3) (con.)}$$

The quantities t_0, x_0, $\dot{x}_0$ in (2) and (3) are the initial values for the integration step under consideration, while h stands for the integration stepsize. The coefficients α_κ, $\gamma_{\kappa\lambda}$, c_κ, $\dot{c}_\kappa$, and $\hat{\dot{c}}_\kappa$ must now be determined in such a way that the first equation (3) is a ninth-order approximation of x and the last two equations (3) are eighth- and ninth-order approximations of $\dot{x}$.

Similar to report [1], we assume that

$$\left.\begin{aligned}\hat{\dot{c}}_\kappa &= \dot{c}_\kappa \qquad (\kappa = 0, 1, 2, \ldots, 11) \\ \hat{\dot{c}}_{12} &= 0 \\ \hat{\dot{c}}_{13} &= \dot{c}_{12}\end{aligned}\right\} . \qquad (4)$$

Because of our assumption for the 14th evaluation (2), the following relations must hold:

$$\left.\begin{aligned}\gamma_{13\lambda} &= c_\lambda \qquad (\lambda = 0, 1, 2, \ldots, 12) \\ \alpha_{12} &= \alpha_{13} = 1\end{aligned}\right\} . \qquad (5)$$

6. We now have to determine the coefficients α_κ, $\gamma_{\kappa\lambda}$, c_κ, $\dot{c}_\kappa$ of formulas (2) and (3). We start with the computation of the weight coefficients c_κ and $\dot{c}_\kappa$. From Table 1 of [1] we obtain, by assuming that

$$\left.\begin{aligned}c_1 &= c_2 = c_3 = c_4 = c_5 = c_6 = 0 \\ \dot{c}_1 &= \dot{c}_2 = \dot{c}_3 = \dot{c}_4 = \dot{c}_5 = \dot{c}_6 = 0\end{aligned}\right\} , \qquad (6)$$

the following equations of condition for the coefficients c_κ and $\dot{c}_\kappa$, respectively:

$$\left.\begin{aligned}
c_7\,\alpha_7 + c_8\,\alpha_8 + c_9\,\alpha_9 + c_{10}\,\alpha_{10} + c_{11}\,\alpha_{11} + c_{12} &= \frac{1}{6}\\
c_7\,\alpha_7^2 + c_8\,\alpha_8^2 + c_9\,\alpha_9^2 + c_{10}\,\alpha_{10}^2 + c_{11}\,\alpha_{11}^2 + c_{12} &= \frac{1}{12}\\
c_7\,\alpha_7^3 + c_8\,\alpha_8^3 + c_9\,\alpha_9^3 + c_{10}\,\alpha_{10}^3 + c_{11}\,\alpha_{11}^3 + c_{12} &= \frac{1}{20}\\
c_7\,\alpha_7^4 + c_8\,\alpha_8^4 + c_9\,\alpha_9^4 + c_{10}\,\alpha_{10}^4 + c_{11}\,\alpha_{11}^4 + c_{12} &= \frac{1}{30}\\
c_7\,\alpha_7^5 + c_8\,\alpha_8^5 + c_9\,\alpha_9^5 + c_{10}\,\alpha_{10}^5 + c_{11}\,\alpha_{11}^5 + c_{12} &= \frac{1}{42}\\
c_7\,\alpha_7^6 + c_8\,\alpha_8^6 + c_9\,\alpha_9^6 + c_{10}\,\alpha_{10}^6 + c_{11}\,\alpha_{11}^6 + c_{12} &= \frac{1}{56}\\
c_7\,\alpha_7^7 + c_8\,\alpha_8^7 + c_9\,\alpha_9^7 + c_{10}\,\alpha_{10}^7 + c_{11}\,\alpha_{11}^7 + c_{12} &= \frac{1}{72}
\end{aligned}\right\}\qquad(7)$$

$$\left.\begin{aligned}
\dot{c}_7\,\alpha_7 + \dot{c}_8\,\alpha_8 + \dot{c}_9\,\alpha_9 + \dot{c}_{10}\,\alpha_{10} + \dot{c}_{11}\,\alpha_{11} + \dot{c}_{12} &= \frac{1}{2}\\
\dot{c}_7\,\alpha_7^2 + \dot{c}_8\,\alpha_8^2 + \dot{c}_9\,\alpha_9^2 + \dot{c}_{10}\,\alpha_{10}^2 + \dot{c}_{11}\,\alpha_{11}^2 + \dot{c}_{12} &= \frac{1}{3}\\
\dot{c}_7\,\alpha_7^3 + \dot{c}_8\,\alpha_8^3 + \dot{c}_9\,\alpha_9^3 + \dot{c}_{10}\,\alpha_{10}^3 + \dot{c}_{11}\,\alpha_{11}^3 + \dot{c}_{12} &= \frac{1}{4}\\
\dot{c}_7\,\alpha_7^4 + \dot{c}_8\,\alpha_8^4 + \dot{c}_9\,\alpha_9^4 + \dot{c}_{10}\,\alpha_{10}^4 + \dot{c}_{11}\,\alpha_{11}^4 + \dot{c}_{12} &= \frac{1}{5}\\
\dot{c}_7\,\alpha_7^5 + \dot{c}_8\,\alpha_8^5 + \dot{c}_9\,\alpha_9^5 + \dot{c}_{10}\,\alpha_{10}^5 + \dot{c}_{11}\,\alpha_{11}^5 + \dot{c}_{12} &= \frac{1}{6}\\
\dot{c}_7\,\alpha_7^6 + \dot{c}_8\,\alpha_8^6 + \dot{c}_9\,\alpha_9^6 + \dot{c}_{10}\,\alpha_{10}^6 + \dot{c}_{11}\,\alpha_{11}^6 + \dot{c}_{12} &= \frac{1}{7}\\
\dot{c}_7\,\alpha_7^7 + \dot{c}_8\,\alpha_8^7 + \dot{c}_9\,\alpha_9^7 + \dot{c}_{10}\,\alpha_{10}^7 + \dot{c}_{11}\,\alpha_{11}^7 + \dot{c}_{12} &= \frac{1}{8}\\
\dot{c}_7\,\alpha_7^8 + \dot{c}_8\,\alpha_8^8 + \dot{c}_9\,\alpha_9^8 + \dot{c}_{10}\,\alpha_{10}^8 + \dot{c}_{11}\,\alpha_{11}^8 + \dot{c}_{12} &= \frac{1}{9}
\end{aligned}\right\}\;.\qquad(8)$$

Equations (7) and (8) must hold if the right-hand sides of (3) are to be ninth-order approximations of x or $\hat{\dot{x}}$.

For given coefficients α_κ, system (7) represents a system of seven equations for the six coefficients c_7, c_8, c_9, c_{10}, c_{11}, and c_{12}. Since there is one equation more than there are coefficients c_κ, there must be a restricting condition between the coefficients α_κ.

By eliminating from (7) the coefficients c_{12}, c_{11}, c_{10}, . . . one after the other, we obtain the following condition:

$$\left.\begin{aligned}
\alpha_7 &= \frac{1}{3} \cdot \frac{N_1}{D_1} \\
N_1 &= 84\, \alpha_8 \alpha_9 \alpha_{10} \alpha_{11} - 42\, (\alpha_8 \alpha_9 \alpha_{10} + \alpha_8 \alpha_9 \alpha_{11} + \alpha_8 \alpha_{10} \alpha_{11} + \alpha_9 \alpha_{10} \alpha_{11}) \\
&\quad + 24\, (\alpha_8 \alpha_9 + \alpha_8 \alpha_{10} + \alpha_8 \alpha_{11} + \alpha_9 \alpha_{10} + \alpha_9 \alpha_{11} + \alpha_{10} \alpha_{11}) \\
&\quad - 15\, (\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 10 \\
D_1 &= 70\, \alpha_8 \alpha_9 \alpha_{10} \alpha_{11} - 28\, (\alpha_8 \alpha_9 \alpha_{10} + \alpha_8 \alpha_9 \alpha_{11} + \alpha_8 \alpha_{10} \alpha_{11} + \alpha_9 \alpha_{10} \alpha_{11}) \\
&\quad + 14\, (\alpha_8 \alpha_9 + \alpha_8 \alpha_{10} + \alpha_8 \alpha_{11} + \alpha_9 \alpha_{10} + \alpha_9 \alpha_{11} + \alpha_{10} \alpha_{11}) \\
&\quad - 8\, (\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 5
\end{aligned}\right\} \quad (9)$$

Similarly, system (8) yields the following two restricting conditions:

$$\left.\begin{aligned}
\alpha_7 &= \frac{1}{2} \cdot \frac{N_2}{D_2} \\
N_2 &= 70\, \alpha_8 \alpha_9 \alpha_{10} \alpha_{11} - 42\, (\alpha_8 \alpha_9 \alpha_{10} + \alpha_8 \alpha_9 \alpha_{11} + \alpha_8 \alpha_{10} \alpha_{11} + \alpha_9 \alpha_{10} \alpha_{11}) \\
&\quad + 28\, (\alpha_8 \alpha_9 + \alpha_8 \alpha_{10} + \alpha_8 \alpha_{11} + \alpha_9 \alpha_{10} + \alpha_9 \alpha_{11} + \alpha_{10} \alpha_{11}) \\
&\quad - 20\, (\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 15
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} D_2 &= 70\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 35\,(\alpha_8\,\alpha_9\,\alpha_{10} + \alpha_8\,\alpha_9\,\alpha_{11} + \alpha_8\,\alpha_{10}\,\alpha_{11} + \alpha_9\,\alpha_{10}\,\alpha_{11}) \\ &+ 21\,(\alpha_8\,\alpha_9 + \alpha_8\,\alpha_{10} + \alpha_8\,\alpha_{11} + \alpha_9\,\alpha_{10} + \alpha_9\,\alpha_{11} + \alpha_{10}\,\alpha_{11}) \\ &- 14\,(\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 10 \end{aligned}\right\} \begin{matrix}(10)\\(\text{con.})\end{matrix}$$

and

$$\left.\begin{aligned} \alpha_7 &= \frac{1}{3} \cdot \frac{N_3}{D_3} \\ N_3 &= 126\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 84\,(\alpha_8\,\alpha_9\,\alpha_{10} + \alpha_8\,\alpha_9\,\alpha_{11} + \alpha_8\,\alpha_{10}\,\alpha_{11} + \alpha_9\,\alpha_{10}\,\alpha_{11}) \\ &+ 60\,(\alpha_8\,\alpha_9 + \alpha_8\,\alpha_{10} + \alpha_8\,\alpha_{11} + \alpha_9\,\alpha_{10} + \alpha_9\,\alpha_{11} + \alpha_{10}\,\alpha_{11}) \\ &- 45\,(\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 35 \\ D_3 &= N_2 \end{aligned}\right\} (11)$$

Assuming that

$$\alpha_8 + \alpha_9 = 1 \ , \quad \alpha_{10} + \alpha_{11} = 1 \ , \tag{12}$$

the conditions (9), (10), and (11) reduce to

$$\left.\begin{aligned} \alpha_7 &= \frac{1}{3}\,\frac{84\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 18\,(\alpha_8\,\alpha_9 + \alpha_{10}\,\alpha_{11}) + 4}{70\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 14\,(\alpha_8\,\alpha_9 + \alpha_{10}\,\alpha_{11}) + 3} \\ \alpha_7 &= \frac{1}{2} \\ \alpha_7 &= \frac{1}{3} \cdot \frac{126\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 24\,(\alpha_8\,\alpha_9 + \alpha_{10}\,\alpha_{11}) + 5}{70\,\alpha_8\,\alpha_9\,\alpha_{10}\,\alpha_{11} - 14\,(\alpha_8\,\alpha_9 + \alpha_{10}\,\alpha_{11}) + 3} \end{aligned}\right\} . \tag{13}$$

Inserting $\alpha_7 = \frac{1}{2}$ into the first and third equation (13) yields both times the following relation between $\alpha_{10}\,\alpha_{11}$ and $\alpha_8\,\alpha_9$:

$$\alpha_{10}\,\alpha_{11} = \frac{1}{6} \cdot \frac{6\,\alpha_8\alpha_9 - 1}{7\,\alpha_8\alpha_9 - 1} \quad . \tag{14}$$

We select $\alpha_8 = \frac{1}{3}$, $\alpha_9 = \frac{2}{3}$ as two values compatible with the first equation (12) and leading to relatively reasonable values for the remaining coefficients of our formula.

From (14) we then obtain $\alpha_{10}\,\alpha_{11} = \frac{1}{10}$ and because of the second equation (12), $\alpha_{10} = \frac{1}{10}\,(5 - \sqrt{15})$, $\alpha_{11} = \frac{1}{10}\,(5 + \sqrt{15})$.

Listing all the coefficients α_κ that enter equations (7) and (8), we have

$$\left.\begin{aligned} &\alpha_7 = \frac{1}{2}\,,\quad \alpha_8 = \frac{1}{3}\,,\quad \alpha_9 = \frac{2}{3}\,,\quad \alpha_{10} = \frac{1}{10}\,(5 - \sqrt{15})\,, \\ &\alpha_{11} = \frac{1}{10}\,(5 + \sqrt{15}) \quad . \end{aligned}\right\} \tag{15}$$

For these coefficients α_κ, equations (7) and (8) can be solved with respect to c_κ and $\dot{c}_\kappa$. We find that

$$\left.\begin{aligned} &c_7 = \frac{16}{315}\,,\quad c_8 = \frac{243}{1540}\,,\quad c_9 = \frac{243}{3080}\,,\quad c_{10} = \frac{25}{1386}\,(5 + \sqrt{15})\,, \\ &c_{11} = \frac{25}{1386}\,(5 - \sqrt{15})\,,\quad c_{12} = 0 \end{aligned}\right\} \tag{16}$$

and

$$\left.\begin{aligned} &\dot{c}_7 = \frac{32}{315}, \quad \dot{c}_8 = \frac{729}{3080}, \quad \dot{c}_9 = \frac{729}{3080}, \quad \dot{c}_{10} = \frac{125}{693}, \\ &\dot{c}_{11} = \frac{125}{693}, \quad \dot{c}_{12} = \frac{9}{280} . \end{aligned}\right\} \tag{17}$$

7. Next we have to find the coefficients $\gamma_{\kappa\lambda}$. Similarly as in [1], we assume for the $\gamma_{\kappa\lambda}$ the following relations:

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\,\alpha_\lambda = \frac{1}{6}\,\alpha_\kappa^3 \qquad (\kappa = 2, 3, 4, \ldots, 12) , \tag{18}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\,\alpha_\lambda^2 = \frac{1}{12}\,\alpha_\kappa^4 \qquad (\kappa = 2, 3, 4, \ldots, 12) , \tag{19}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\,\alpha_\lambda^3 = \frac{1}{20}\,\alpha_\kappa^5 \qquad (\kappa = 4, 5, 6, \ldots, 12) , \tag{20}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\,\alpha_\lambda^4 = \frac{1}{30}\,\alpha_\kappa^6 \qquad (\kappa = 5, 6, 7, \ldots, 12) . \tag{21}$$

We further assume that

$$\left.\begin{aligned} \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = \gamma_{91} = \gamma_{101} = \gamma_{111} = \gamma_{121} = 0 \\ \gamma_{72} = \gamma_{82} = \gamma_{92} = \gamma_{102} = \gamma_{112} = \gamma_{122} = 0 \\ \gamma_{73} = \gamma_{83} = \gamma_{93} = \gamma_{103} = \gamma_{113} = \gamma_{123} = 0 \end{aligned}\right\} \tag{22}$$

and

$$\left.\begin{aligned} &\sum_{\kappa=7}^{11} c_\kappa \gamma_{\kappa 4} = 0 \\ &\sum_{\kappa=7}^{11} \hat{c}_\kappa \gamma_{\kappa 4} + \hat{c}_{12} \begin{Bmatrix} \gamma_{124} \\ c_4 \end{Bmatrix} = 0 \end{aligned}\right\} \quad . \tag{23}$$

The upper line in (23) holds for the eighth-order formula and the lower line for the ninth-order formula. Because of $c_4 = 0$, we obtain from (23)

$$\gamma_{124} = 0 \quad . \tag{24}$$

As pointed out in [1], such assumptions as (18) to (21) and (23) convert the necessary and sufficient equations of condition (Table 1 of [1]) for the coefficients $\gamma_{\kappa\lambda}$ into a system of sufficient equations of condition that can be solved in a relatively easy manner.

If we proceed in a way quite similar to that of [1], the application of (6) and (18) to (23) reduces Table 1 of [1] to

$$\left.\begin{aligned} &c_0 + c_7 + c_8 + c_9 + c_{10} + c_{11} = \frac{1}{2} \\ &\hat{c}_0 + \hat{c}_7 + \hat{c}_8 + \hat{c}_9 + \hat{c}_{10} + \hat{c}_{11} + c_{12} = 1 \end{aligned}\right\} \quad , \tag{25}$$

to (7), (8), and to the following four equations:

$$\left.\begin{aligned} &c_7 P_{75} + c_8 P_{85} + c_9 P_{95} + c_{10} P_{105} + c_{11} P_{115} = \frac{1}{3024} \\ &\hat{c}_7 P_{75} + \hat{c}_8 P_{85} + \hat{c}_9 P_{95} + \hat{c}_{10} P_{105} + \hat{c}_{11} P_{115} + \hat{c}_{12} \begin{Bmatrix} P_{125} \\ \frac{1}{42} \end{Bmatrix} = \frac{1}{336} \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}&\dot{c}_7\,\alpha_7\,P_{75}+\dot{c}_8\,\alpha_8\,P_{85}+\dot{c}_9\,\alpha_9\,P_{95}+\dot{c}_{10}\,\alpha_{10}\,P_{105}\\&+\dot{c}_{11}\,\alpha_{11}\,P_{115}+\dot{c}_{12}\cdot\frac{1}{42}=\frac{1}{378}\\&\dot{c}_7\,P_{76}+\dot{c}_8\,P_{86}+\dot{c}_9\,P_{96}+\dot{c}_{10}\,P_{106}+\dot{c}_{11}\,P_{116}+\dot{c}_{12}\,\frac{1}{56}=\frac{1}{504}\end{aligned}\right\}\quad.\qquad\begin{matrix}(26)\\(\text{con.})\end{matrix}$$

In (26) we have used the abbreviations

$$\gamma_{\mu 1}\,\alpha_1^{\nu}+\gamma_{\mu 2}\,\alpha_2^{\nu}+\ldots+\gamma_{\mu\mu-1}\,\alpha_{\mu-1}^{\nu}=P_{\mu\nu}\quad. \tag{27}$$

Equations (26) correspond to equations (IX,27), (VIII,27)$^{\cdot}$, (IX,43)$^{\cdot}$, and (IX,53)$^{\cdot}$ of Table 1 in [1].

From the second equation (26), we obtain

$$P_{125}=\frac{1}{42}\quad. \tag{28}$$

The coefficients $\gamma_{\kappa\lambda}$ can now be determined from equations (18), (19), (20), (21), (23), and (26).

8. There are restrictive conditions for the α_κ's that we derive in the following. From equations $(18)_{\kappa=2}$ and $(19)_{\kappa=2}$, we obtain

$$\alpha_1=\frac{1}{2}\,\alpha_2\quad. \tag{29}$$

From equations $(18)_{\kappa=5}$, $(19)_{\kappa=5}$, $(20)_{\kappa=5}$, and $(21)_{\kappa=5}$, the following condition results:

$$\alpha_2 = \alpha_5 \frac{5\,\alpha_3\,\alpha_4 - 3\,(\alpha_3 + \alpha_4)\,\alpha_5 + 2\,\alpha_5^2}{10\,\alpha_3\,\alpha_4 - 5\,(\alpha_3 + \alpha_4)\,\alpha_5 + 3\,\alpha_5^2} \tag{30}$$

and from equations $(18)_{\kappa=7}$, $(19)_{\kappa=7}$, $(20)_{\kappa=7}$, and $(21)_{\kappa=7}$,

$$\alpha_4 = \alpha_7 \cdot \frac{5\,\alpha_5\,\alpha_6 - 3(\alpha_5 + \alpha_6)\,\alpha_7 + 2\,\alpha_7^2}{10\,\alpha_5\,\alpha_6 - 5(\alpha_5 + \alpha_6)\,\alpha_7 + 3\,\alpha_7^2} \quad . \tag{31}$$

Having selected α_7 by equation (15), the coefficients α_3, α_5, and α_6 can be chosen arbitrarily, while the coefficients α_4, α_2, and α_1 are determined by equations (31), (30), and (29).

9. Since the expressions for the coefficients $\gamma_{\kappa\lambda}$ can be derived in a straightforward manner, we only sketch their derivation:

from $(18)_{\kappa=2}$, we obtain γ_{21};

from $(18)_{\kappa=3}$, $(19)_{\kappa=3}$, we obtain γ_{31}, γ_{32};

from $(18)_{\kappa=4}$, $(19)_{\kappa=4}$, $(20)_{\kappa=4}$, we obtain γ_{41}, γ_{42}, γ_{43};

from $(18)_{\kappa=5}$, $(19)_{\kappa=5}$, $(20)_{\kappa=5}$, we obtain γ_{52}, γ_{53}, γ_{54};

from $(18)_{\kappa=6}$, $(19)_{\kappa=6}$, $(20)_{\kappa=6}$, $(21)_{\kappa=6}$, we obtain γ_{62}, γ_{63}, γ_{64}, γ_{65};

from $(18)_{\kappa=7}$, $(19)_{\kappa=7}$, $(20)_{\kappa=7}$, we obtain γ_{74}, γ_{75}, γ_{76};

and

from $(18)_{\kappa=8}$, $(19)_{\kappa=8}$, $(20)_{\kappa=8}$, $(21)_{\kappa=8}$, we obtain γ_{84}, γ_{85}, γ_{86}, γ_{87}.

With

$$\gamma_{94} = 0 \quad , \tag{32}$$

we find from

$(18)_{\kappa=9}$, $(19)_{\kappa=9}$, $(20)_{\kappa=9}$, $(21)_{\kappa=9}$ the coefficients γ_{95}, γ_{96}, γ_{97}, γ_{98} .

Now we can consider (23) as a system of two equations for the two coefficients γ_{104} and γ_{114}.

The first three equations (26) can serve to express P_{105} and P_{115} by P_{75}, P_{85}, P_{95} and the weight factors c_κ and $\dot{c}_\kappa$. Using the values (16) and (17) for the weight factors, we find that these first three equations (26) are not independent: The second equation is equal to the sum of the first and the third equation. Therefore, we may omit one of these three equations and obtain P_{105} and P_{115} from the remaining two equations.

Since γ_{104} and P_{105} are now known, equations $(18)_{\kappa=10}$, $(19)_{\kappa=10}$, $(20)_{\kappa=10}$, $(21)_{\kappa=10}$, and the equation (27) defining P_{105} yield γ_{105}, γ_{106}, γ_{107}, γ_{108}, and γ_{109}.

From the fourth equation (26), we obtain P_{116} as a function of P_{76}, P_{86}, P_{96}, P_{106} and the weight factors $\dot{c}_\kappa$. The values for γ_{114}, P_{115}, and P_{116} being known, we find γ_{115}, γ_{116}, γ_{117}, γ_{118}, γ_{119}, and γ_{1110} from equations $(18)_{\kappa=11}$, $(19)_{\kappa=11}$, $(20)_{\kappa=11}$, $(21)_{\kappa=11}$ and equation (27) for P_{115} and P_{116}.

Putting

$$\gamma_{125} = 0 \tag{33}$$

and using (28), we have five equations: $(18)_{\kappa=12}$, $(19)_{\kappa=12}$, $(20)_{\kappa=12}$, $(21)_{\kappa=12}$, and the definition (27) for P_{125} to express γ_{127}, γ_{128}, γ_{129}, γ_{1210}, and γ_{1211} by γ_{126}.

10. The coefficient γ_{126} remains undetermined. Collecting from Table 1 of [1] all terms that contribute to the leading term of the truncation error for the second equation (3), we find that for given coefficients α_κ this leading term is proportional to γ_{126}. Therefore, by varying γ_{126} we can

influence the local truncation error in $\dot{x}$. However, we must not choose $\gamma_{126} = 0$, because our second equation (3) would then become a ninth-order formula and our stepsize control procedure would break down.

11. The coefficients $\gamma_{\kappa 0}$ are determined from the well-known relation,

$$\sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} = \frac{1}{2}\alpha_{\kappa}^{2} \quad , \tag{34}$$

and the weight factors c_0 and $\dot{c}_0$ from (25).

12. This terminates the computation of the coefficients of our eighth-order Runge-Kutta-Nyström formula. We shall label our formula RKN 8(9)-13-$\dot{x}$ to indicate that we have a pair of formulas of eighth- and ninth-order (in $\dot{x}$) with 13 evaluations per step and with stepsize control based on the leading term of the local truncation error in $\dot{x}$.

In Table 1 the pattern of such a formula RKN 8(9)-13-$\dot{x}$ is presented. All coefficients different from 0 and from 1 are indicated by asterisks (*). Comparing Table 1 with Tables 5, 8, and 11 of the lower-order formulas makes it easy to recognize the structure of our formulas.

In Table 2 are listed the coefficients of a formula RKN 8(9)-13-$\dot{x}$ in fraction form. The parameters α_3, α_5, and α_6 were chosen as $\frac{1}{2}$, 1, and $\frac{1}{9}$, respectively, since this choice leads to coefficients that are still relatively reasonable.

In Table 3 are listed the coefficients of Table 2 in decimal form (32 decimal digits). In Table 3 we set $\gamma_{126} = \frac{5}{2}$, since this value yielded the most accurate results for the examples that we ran on the computer. Table 4 represents a flow chart for programming our formula RKN 8(9)-13-$\dot{x}$.

SECTION II. SEVENTH-ORDER FORMULA RKN 7(8)

13. Since seventh-order Runge-Kutta-Nyström formulas are derived in a very similar way as the eighth-order formula of Section I, we restrict ourselves to briefly outlining their derivation without presenting all details.

We allow 10 evaluations per step for the seventh-order formula. Assuming

$$\left.\begin{aligned} \alpha_9 &= \alpha_{10} = 1 \\ c_1 &= c_2 = c_3 = c_4 = c_5 = 0 \\ \dot{c}_1 &= \dot{c}_2 = \dot{c}_3 = \dot{c}_4 = \dot{c}_5 = 0 \end{aligned}\right\} \quad , \tag{35}$$

we obtain for the weight factors c_κ and $\dot{c}_\kappa$ the following equations of condition:

$$\left.\begin{aligned} c_6\,\alpha_6 + c_7\,\alpha_7 + c_8\,\alpha_8 + c_9 &= \frac{1}{6} \\ c_6\,\alpha_6^2 + c_7\,\alpha_7^2 + c_8\,\alpha_8^2 + c_9 &= \frac{1}{12} \\ c_6\,\alpha_6^3 + c_7\,\alpha_7^3 + c_8\,\alpha_8^3 + c_9 &= \frac{1}{20} \\ c_6\,\alpha_6^4 + c_7\,\alpha_7^4 + c_8\,\alpha_8^4 + c_9 &= \frac{1}{30} \\ c_6\,\alpha_6^5 + c_7\,\alpha_7^5 + c_8\,\alpha_8^5 + c_9 &= \frac{1}{42} \\ c_6\,\alpha_6^6 + c_7\,\alpha_7^6 + c_8\,\alpha_8^6 + c_9 &= \frac{1}{56} \end{aligned}\right\} \tag{36}$$

and

$$\left.\begin{aligned}
\dot{c}_6\,\alpha_6 + \dot{c}_7\,\alpha_7 + \dot{c}_8\,\alpha_8 + \dot{c}_9 &= \frac{1}{2}\\
\dot{c}_6\,\alpha_6^2 + \dot{c}_7\,\alpha_7^2 + \dot{c}_8\,\alpha_8^2 + \dot{c}_9 &= \frac{1}{3}\\
\dot{c}_6\,\alpha_6^3 + \dot{c}_7\,\alpha_7^3 + \dot{c}_8\,\alpha_8^3 + \dot{c}_9 &= \frac{1}{4}\\
\dot{c}_6\,\alpha_6^4 + \dot{c}_7\,\alpha_7^4 + \dot{c}_8\,\alpha_8^4 + \dot{c}_9 &= \frac{1}{5}\\
\dot{c}_6\,\alpha_6^5 + \dot{c}_7\,\alpha_7^5 + \dot{c}_8\,\alpha_8^5 + \dot{c}_9 &= \frac{1}{6}\\
\dot{c}_6\,\alpha_6^6 + \dot{c}_7\,\alpha_7^6 + \dot{c}_8\,\alpha_8^6 + \dot{c}_9 &= \frac{1}{7}\\
\dot{c}_6\,\alpha_6^7 + \dot{c}_7\,\alpha_7^7 + \dot{c}_8\,\alpha_8^7 + \dot{c}_9 &= \frac{1}{8}
\end{aligned}\right\} \quad . \tag{37}$$

Equations (36) lead to the following restrictive conditions for the α_κ's:

$$\left.\begin{aligned}
\alpha_6 &= \frac{1}{7}\cdot\frac{14\,\alpha_7\,\alpha_8 - 7\,(\alpha_7+\alpha_8) + 4}{5\,\alpha_7\,\alpha_8 - 2\,(\alpha_7+\alpha_8) + 1}\\
\alpha_6 &= \frac{1}{2}\cdot\frac{14\,\alpha_7\,\alpha_8 - 8\,(\alpha_7+\alpha_8) + 5}{14\,\alpha_7\,\alpha_8 - 7\,(\alpha_7+\alpha_8) + 4}
\end{aligned}\right\} \quad , \tag{38}$$

whereas from (37) the conditions

$$\left.\begin{aligned}
\alpha_6 &= \frac{5\,\alpha_7\,\alpha_8 - 3\,(\alpha_7+\alpha_8) + 2}{10\,\alpha_7\,\alpha_8 - 5\,(\alpha_7+\alpha_8) + 3}\\
\alpha_6 &= \frac{1}{7}\cdot\frac{21\,\alpha_7\,\alpha_8 - 14\,(\alpha_7+\alpha_8) + 10}{5\,\alpha_7\,\alpha_8 - 3\,(\alpha_7+\alpha_8) + 2}
\end{aligned}\right\} \tag{39}$$

$$\alpha_6 = \frac{1}{2} \cdot \frac{28\,\alpha_7\,\alpha_8 - 20\,(\alpha_7 + \alpha_8) + 15}{21\,\alpha_7\,\alpha_8 - 14\,(\alpha_7 + \alpha_8) + 10} \quad \Bigg\} \qquad (39)\ (\text{con.})$$

result.

Assuming that

$$\alpha_7 + \alpha_8 = 1 \quad , \qquad (40)$$

equations (38) and (39) reduce to

$$\alpha_7\,\alpha_8 = \frac{1}{7} \qquad (41)$$

and

$$\alpha_6 = \frac{1}{2} \quad . \qquad (42)$$

From (40) and (42) we find that

$$\alpha_7 = \frac{1}{14}\,(7 - \sqrt{21}\,) \quad , \quad \alpha_8 = \frac{1}{14}\,(7 + \sqrt{21}\,) \quad . \qquad (43)$$

The α_κ-values (42) and (43) lead to the following values for the weight factors

$$\left. \begin{aligned} & c_6 = \frac{8}{45}\,, \quad c_7 = \frac{7}{360}\,(7 + \sqrt{21})\,, \quad c_8 = \frac{7}{360}\,(7 - \sqrt{21})\;, \\ & c_9 = 0 \end{aligned} \right\} \qquad (44)$$

$$\dot{c}_6 = \frac{16}{45}\;, \quad \dot{c}_7 = \frac{49}{180}\;, \quad \dot{c}_8 = \frac{49}{180}\;, \quad \dot{c}_9 = \frac{1}{20}\;. \qquad (45)$$

14. Similar to our assumptions in Section I, we now assume that the following relations for the coefficients $\gamma_{\kappa\lambda}$ hold:

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_{\lambda} = \frac{1}{6} \alpha_{\kappa}^{3} \qquad (\kappa = 2, 3, 4, \ldots, 9) \quad , \tag{46}$$

$$\sum_{\kappa=1}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_{\lambda}^{2} = \frac{1}{12} \alpha_{\kappa}^{4} \qquad (\kappa = 2, 3, 4, \ldots, 9) \quad , \tag{47}$$

$$\sum_{\kappa=1}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_{\lambda}^{3} = \frac{1}{20} \alpha_{\kappa}^{5} \qquad (\kappa = 4, 5, 6, \ldots, 9) \quad . \tag{48}$$

We further assume that

$$\left.\begin{aligned} \gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = \gamma_{91} &= 0 \\ \gamma_{62} = \gamma_{72} = \gamma_{82} = \gamma_{92} &= 0 \end{aligned}\right\} \tag{49}$$

and

$$\dot{c}_6 \gamma_{63} + \dot{c}_7 \gamma_{73} + \dot{c}_8 \gamma_{83} = 0 \quad . \tag{50}$$

These assumptions reduce the equations of condition for $\gamma_{\kappa\lambda}$ listed in Table 1 of [1] to the following four equations:

$$\left.\begin{aligned} c_6 P_{64} + c_7 P_{74} + c_8 P_{84} &= \frac{1}{1680} \\ \dot{c}_6 P_{64} + \dot{c}_7 P_{74} + \dot{c}_8 P_{84} + \dot{c}_9 \left\{\begin{matrix} P_{94} \\ \frac{1}{90} \end{matrix}\right\} &= \frac{1}{210} \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned}\dot{c}_6\,\alpha_6\,P_{64} + \dot{c}_7\,\alpha_7\,P_{74} + \dot{c}_8\,\alpha_8\,P_{84} + \dot{c}_9 \cdot \frac{1}{30} &= \frac{1}{240}\\ \dot{c}_6\,P_{65} + \dot{c}_7\,P_{75} + \dot{c}_8\,P_{85} + \dot{c}_9 \cdot \frac{1}{42} &= \frac{1}{336}\end{aligned}\right\} \quad . \qquad \begin{matrix}(51)\\(\text{con.})\end{matrix}$$

These equations correspond to equations (VIII, 15), (VII, 15)$^{\bullet}$, (VIII, 21)$^{\bullet}$, and (VIII, 27)$^{\bullet}$ of Table 1 in [1].

From the second equation (51), it follows that

$$P_{94} = \frac{1}{90} \quad . \tag{52}$$

15. Again, there are restrictive conditions for the α_κ's. From equations $(46)_{\kappa=2}$ and $(47)_{\kappa=2}$, we obtain

$$\alpha_1 = \frac{1}{2}\,\alpha_2 \quad , \tag{53}$$

and from equations $(46)_{\kappa=4}$, $(47)_{\kappa=4}$, and $(48)_{\kappa=4}$,

$$\alpha_2 = \frac{1}{5}\,\alpha_4\,\frac{5\alpha_3 - 3\alpha_4}{2\alpha_3 - \alpha_4} \quad . \tag{54}$$

We also assume that

$$\alpha_5 = \alpha_8 \quad . \tag{55}$$

The coefficients α_3 and α_4 can be chosen arbitrarily while the coefficients α_2, α_1, and α_5 are determined by (54), (53), and (55).

16. We now compute the coefficients $\gamma_{\kappa\lambda}$. The coefficients γ_{21}, γ_{31}, γ_{32}, γ_{42}, γ_{43}, γ_{52}, γ_{53}, γ_{54}, γ_{63}, γ_{64}, and γ_{65} are computed from equations (46), (47), and (48) in the same way as they were computed in Section I from (18), (19), (20), and (21). The only difference is that γ_{41} and γ_{62} are now zero.

Inserting (44) and (45) into the first three equations (51), we find that these three equations are not independent of each other: The second equation (51) is equal to the sum of the first and the third equation (51). Therefore, we can omit one of the three equations (51) and use the other two equations (51) to express P_{74} and P_{84} by P_{64}.

From $(46)_{\kappa=7}$, $(47)_{\kappa=7}$, $(48)_{\kappa=7}$, P_{74}, we obtain γ_{73}, γ_{74}, γ_{75}, γ_{76}.

The coefficient γ_{83} is determined by (50).

From $(46)_{\kappa=8}$, $(47)_{\kappa=8}$, $(48)_{\kappa=8}$, P_{84}, we obtain γ_{84}, γ_{85}, γ_{86}, γ_{87}.

By the special choice (55) of α_5, the fourth equation (51) is also satisfied.

Setting

$$\gamma_{95} = 0 \quad , \tag{56}$$

equations

$(46)_{\kappa=9}$, $(47)_{\kappa=9}$, $(48)_{\kappa=9}$, P_{94}, yield γ_{94}, γ_{96}, γ_{97}, γ_{98} as functions of γ_{93}.

The leading term of the local truncation error in our seventh-order formula for $\dot{x}$ is proportional to γ_{93}. Therefore, by varying γ_{93}, we can again influence the local truncation error in $\dot{x}$.

17. Tables 5, 6, and 7 show the pattern for a formula RKN 7(8)-10-$\dot{x}$ and the coefficients for such a formula in fraction and in decimal form. For the free parameter α_3 and α_4, we chose $\frac{1}{7}$ and $\frac{5}{7}$ since we then obtain relatively simple values for the remaining coefficients of our formula. By setting $\gamma_{93} = \frac{1}{10}$, we got the best accuracy in the examples that we ran on the computer.

For the seventh- and lower-order formulas we do not include flow charts because they would be very similar to the flow chart for the eighth-order formula (Table 4).

SECTION III. SIXTH-ORDER FORMULA RKN 6(7)

18. Again, we only briefly outline the derivation of our sixth-order formula, which requires eight evaluations per step. Assuming that

$$\left.\begin{aligned} \alpha_7 &= \alpha_8 = 1 \\ c_1 &= c_2 = c_3 = 0 \\ \dot{c}_1 &= \dot{c}_2 = \dot{c}_3 = 0 \end{aligned}\right\} , \qquad (57)$$

the weight factors c_κ and $\dot{c}_\kappa$ are obtained from

$$\left.\begin{aligned} c_4 \alpha_4 + c_5 \alpha_5 + c_6 \alpha_6 + c_7 &= \frac{1}{6} \\ c_4 \alpha_4^2 + c_5 \alpha_5^2 + c_6 \alpha_6^2 + c_7 &= \frac{1}{12} \\ c_4 \alpha_4^3 + c_5 \alpha_5^3 + c_6 \alpha_6^3 + c_7 &= \frac{1}{20} \\ c_4 \alpha_4^4 + c_5 \alpha_5^4 + c_6 \alpha_6^4 + c_7 &= \frac{1}{30} \\ c_4 \alpha_4^5 + c_5 \alpha_5^5 + c_6 \alpha_6^5 + c_7 &= \frac{1}{42} \end{aligned}\right\} \qquad (58)$$

and

$$\left.\begin{aligned} \dot{c}_4 \alpha_4 + \dot{c}_5 \alpha_5 + \dot{c}_6 \alpha_6 + \dot{c}_7 &= \frac{1}{2} \\ \dot{c}_4 \alpha_4^2 + \dot{c}_5 \alpha_5^2 + \dot{c}_6 \alpha_6^2 + \dot{c}_7 &= \frac{1}{3} \end{aligned}\right\} \qquad (59)$$

$$\left.\begin{aligned}
\dot{c}_4\ \alpha_4^3 + \dot{c}_5\ \alpha_5^3 + \dot{c}_6\ \alpha_6^3 + \dot{c}_7 &= \frac{1}{4} \\
\dot{c}_4\ \alpha_4^4 + \dot{c}_5\ \alpha_5^4 + \dot{c}_6\ \alpha_6^4 + \dot{c}_7 &= \frac{1}{5} \\
\dot{c}_4\ \alpha_4^5 + \dot{c}_5\ \alpha_5^5 + \dot{c}_6\ \alpha_6^5 + \dot{c}_7 &= \frac{1}{6} \\
\dot{c}_4\ \alpha_4^6 + \dot{c}_5\ \alpha_5^6 + \dot{c}_6\ \alpha_6^6 + \dot{c}_7 &= \frac{1}{7}
\end{aligned}\right\} \quad . \qquad \begin{matrix}(59)\\(\text{con.})\end{matrix}$$

Equations (58) lead to the restrictive condition,

$$\alpha_4 = \frac{1}{7}\ \frac{14\ \alpha_5\ \alpha_6 - 7\ (\alpha_5 + \alpha_6) + 4}{5\ \alpha_5\ \alpha_6 - 2\ (\alpha_5 + \alpha_6) + 1} \quad , \qquad (60)$$

and equations (59) to the two restrictive conditions,

$$\left.\begin{aligned}
\alpha_4 &= \frac{5\ \alpha_5\ \alpha_6 - 3\ (\alpha_5 + \alpha_6) + 2}{10\ \alpha_5\ \alpha_6 - 5\ (\alpha_5 + \alpha_6) + 3} \\
\alpha_4 &= \frac{1}{7}\ \frac{21\ \alpha_5\ \alpha_6 - 14\ (\alpha_5 + \alpha_6) + 10}{5\ \alpha_5\ \alpha_6 - 3\ (\alpha_5 + \alpha_6) + 2}
\end{aligned}\right\} \quad . \qquad (61)$$

Assuming that

$$\alpha_5 + \alpha_6 = 1 \quad , \qquad (62)$$

equations (60) and (61) simplify to

$$\left.\begin{aligned} \alpha_4 &= \frac{1}{7} \cdot \frac{14\,\alpha_5\,\alpha_6 - 3}{5\,\alpha_5\,\alpha_6 - 1} \\ \alpha_4 &= \frac{1}{2} \\ \alpha_4 &= \frac{1}{7} \cdot \frac{21\,\alpha_5\,\alpha_6 - 4}{5\,\alpha_5\,\alpha_6 - 1} \end{aligned}\right\} \tag{63}$$

From (63), we obtain

$$\alpha_5\,\alpha_6 = \frac{1}{7} \quad . \tag{64}$$

Equations (62), (63), and (64) yield

$$\alpha_4 = \frac{1}{2}, \quad \alpha_5 = \frac{1}{14}\,(7 - \sqrt{21}\,), \; \alpha_6 = \frac{1}{14}\,(7 + \sqrt{21}\,) \quad . \tag{65}$$

Equations (58) and (59) then lead to

$$c_4 = \frac{8}{45}, \quad c_5 = \frac{7}{360}\,(7 + \sqrt{21}\,), \quad c_6 = \frac{7}{360}\,(7 - \sqrt{21}\,), \quad c_7 = 0 \; . \tag{66}$$

$$\dot{c}_4 = \frac{16}{45}, \quad \dot{c}_5 = \frac{49}{180}, \quad \dot{c}_6 = \frac{49}{180}, \quad \dot{c}_7 = \frac{1}{20} \quad . \tag{67}$$

19. Again, we assume that certain relations hold for the coefficients $\gamma_{\kappa\lambda}$:

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\,\alpha_\lambda = \frac{1}{6}\,\alpha_\kappa^3 \qquad (\kappa = 2, 3, \ldots, 7), \tag{68}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda} \, \alpha_\lambda^2 = \frac{1}{12} \alpha_\kappa^4 \qquad (\kappa = 2, 3, \ldots, 7) \quad , \tag{69}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda} \, \alpha_\lambda^3 = \frac{1}{20} \alpha_\kappa^5 \qquad (\kappa = 4, 5, 6, 7) \quad . \tag{70}$$

We further assume that

$$\gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = 0 \quad . \tag{71}$$

If assumptions (68) to (71) hold, Table 1 of [1] reduces to one equation for $\gamma_{\kappa\lambda}$:

$$\dot{c}_4 \, P_{44} + \dot{c}_5 \, P_{54} + \dot{c}_6 \, P_{64} + \dot{c}_7 \cdot \frac{1}{30} = \frac{1}{210} \quad , \tag{72}$$

corresponding to equation (VII, 15)* of this table.

20. From $(68)_{\kappa=2}$ and $(69)_{\kappa=2}$, we obtain the restrictive condition

$$\alpha_1 = \frac{1}{2} \alpha_2 \tag{73}$$

and from $(68)_{\kappa=4}$, $(69)_{\kappa=4}$, $(70)_{\kappa=4}$,

$$\alpha_2 = \frac{1}{5} \alpha_4 \frac{5 \, \alpha_3 - 3 \, \alpha_4}{2 \, \alpha_3 - \alpha_4} \quad , \tag{74}$$

while the coefficient α_3 can be chosen arbitrarily.

21. The computation of the coefficients $\gamma_{\kappa\lambda}$ follows the same pattern as in Sections I and II. The values for γ_{21}, γ_{31}, γ_{32}, γ_{42}, γ_{43}, γ_{52}, γ_{53}, and γ_{54} are computed from (68), (69), and (70). For the computation of γ_{62}, γ_{63}, γ_{64}, γ_{65} we also use P_{64}, as computed from (72).

Putting

$$\gamma_{72} = 0 \quad , \tag{75}$$

we can express γ_{74}, γ_{75}, γ_{76} by γ_{73} using equations (68), (69), and (70) with κ=7.

The leading term of the local truncation error in our sixth-order formula for $\dot{x}$ is proportional to γ_{73}.

22. In Tables 8, 9, and 10, we present the pattern and the numerical values for the coefficients of a sixth-order formula RKN 6(7)-8-$\dot{x}$. For our formula, we chose $\alpha_3 = \frac{1}{6}$ and $\gamma_{73} = \frac{1}{100}$.

SECTION IV. FIFTH-ORDER FORMULA RKN 5(6)

23. We allow for six evaluations per step. Assuming that

$$\left.\begin{aligned} \alpha_7 &= \alpha_8 = 1 \\ c_1 &= c_2 = 0 \\ \dot{c}_1 &= \dot{c}_2 = 0 \end{aligned}\right\} \quad , \tag{76}$$

we find the following equations of condition for the weight factors c_κ and $\dot{c}_\kappa$:

$$\left.\begin{aligned} c_3\,\alpha_3 + c_4\,\alpha_4 + c_5 &= \frac{1}{6} \\ c_3\,\alpha_3^2 + c_4\,\alpha_4^2 + c_5 &= \frac{1}{12} \\ c_3\,\alpha_3^3 + c_4\,\alpha_4^3 + c_5 &= \frac{1}{20} \\ c_3\,\alpha_3^4 + c_4\,\alpha_4^4 + c_5 &= \frac{1}{30} \end{aligned}\right\} \tag{77}$$

and

$$\left.\begin{aligned} \dot{c}_3\,\alpha_3 + \dot{c}_4\,\alpha_4 + \dot{c}_5 &= \frac{1}{2} \\ \dot{c}_3\,\alpha_3^2 + \dot{c}_4\,\alpha_4^2 + \dot{c}_5 &= \frac{1}{3} \\ \dot{c}_3\,\alpha_3^3 + \dot{c}_4\,\alpha_4^3 + \dot{c}_5 &= \frac{1}{4} \\ \dot{c}_3\,\alpha_3^4 + \dot{c}_4\,\alpha_4^4 + \dot{c}_5 &= \frac{1}{5} \\ \dot{c}_3\,\alpha_3^5 + \dot{c}_4\,\alpha_4^5 + \dot{c}_5 &= \frac{1}{6} \end{aligned}\right\} . \tag{78}$$

Equations (77) lead to the restrictive condition

$$\alpha_3 = \frac{2\,\alpha_4 - 1}{5\,\alpha_4 - 2} \tag{79}$$

and equations (78) to the conditions

$$\left.\begin{aligned} \alpha_3 &= \frac{1}{5} \cdot \frac{5\,\alpha_4 - 3}{2\,\alpha_4 - 1} \\ \alpha_3 &= \frac{3\,\alpha_4 - 2}{5\,\alpha_4 - 3} \end{aligned}\right\} . \tag{80}$$

Equations (79) and (80) can be written in the form,

$$\left.\begin{aligned} 5\,\alpha_3\,\alpha_4 - 2\,(\alpha_3 + \alpha_4) + 1 &= 0 \\ 10\,\alpha_3\,\alpha_4 - 5\,(\alpha_3 + \alpha_4) + 3 &= 0 \\ 5\,\alpha_3\,\alpha_4 - 3\,(\alpha_3 + \alpha_4) + 2 &= 0 \end{aligned}\right\} \quad . \tag{81}$$

Assuming that

$$\alpha_3 + \alpha_4 = 1 \ , \tag{82}$$

equations (81) reduce to

$$\alpha_3\,\alpha_4 = \frac{1}{5} \quad . \tag{83}$$

Equations (82) and (83) result in

$$\alpha_3 = \frac{1}{10}\,(5 - \sqrt{5}\,)\ ,\ \alpha_4 = \frac{1}{10}\,(5 + \sqrt{5}\,) \quad . \tag{84}$$

For the weight factors c_κ and $\dot{c}_\kappa$, we find from (77) and (78) that

$$c_3 = \frac{1}{24}\,(5 + \sqrt{5}\,),\ \ c_4 = \frac{1}{24}\,(5 - \sqrt{5}\,),\ \ c_5 = 0 \quad . \tag{85}$$

$$\dot{c}_3 = \frac{5}{12},\ \ \dot{c}_4 = \frac{5}{12},\ \ \dot{c}_5 = \frac{1}{12} \ . \tag{86}$$

24. The following relations are assumed to hold for the coefficients $\gamma_{\kappa\lambda}$:

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_{\lambda} = \frac{1}{6} \alpha_{\kappa}^3 \qquad (\kappa = 2, 3, 4, 5) . \tag{87}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_{\lambda}^2 = \frac{1}{12} \alpha_{\kappa}^4 \qquad (\kappa = 3, 4, 5) . \tag{88}$$

$$\dot{c}_3 \gamma_{31} + \dot{c}_4 \gamma_{41} = 0 . \tag{89}$$

Table 1 of [1] reduces again to one equation for $\gamma_{\kappa\lambda}$:

$$\dot{c}_3 P_{33} + \dot{c}_4 P_{43} + \dot{c}_5 \cdot \frac{1}{20} = \frac{1}{120} , \tag{90}$$

corresponding to equation (VI, 8)$^{\cdot}$ of this table.

From $(87)_{\kappa=2}$, we obtain γ_{21} as a function of α_1 and α_2. Similarly $(87)_{\kappa=3}$ and $(88)_{\kappa=3}$ yield γ_{31} and γ_{32} as functions of α_1, α_2, and α_3. The four equations $(87)_{\kappa=4}$, $(88)_{\kappa=4}$, (89), and (90) for the three coefficients γ_{41}, γ_{42}, γ_{43} lead to a restrictive condition for the α_{κ}'s:

$$\alpha_2 = \frac{\alpha_3 \, (\alpha_3^4 + \alpha_4^4) - \frac{3}{25}}{2 \, \alpha_3 \, \alpha_4^3 + \alpha_3^4 - \alpha_4^4} . \tag{91}$$

Inserting (84) into (91) leads to

$$\alpha_2 = \frac{1}{10} (5 + \sqrt{5}) = \alpha_4 . \tag{92}$$

Putting

$$\gamma_{52} = 0 , \tag{93}$$

we use equations $(87)_{\kappa=5}$ and $(88)_{\kappa=5}$ to express γ_{53} and γ_{54} by α_1, α_3, α_4, and γ_{51}.

Since the leading term of the local truncation error in $\dot{x}$ is proportional to γ_{51}, we can achieve a small truncation error by a proper choice of γ_{51}.

25. Tables 11, 12, and 13 show the pattern and the numerical values for the coefficients of a fifth-order formula RKN 5(6)-6-$\dot{x}$. For our formula, we chose $\alpha_1 = 1$ and $\gamma_{51} = 1/300$.

SECTION V. AN APPLICATION: NUMERICAL INTEGRATION OF CENTRAL ORBITS

26. Let us consider a particle moving in the (x, y)-plane under the influence of an attracting mass, this mass being located in the origin of the (x, y)-coordinate system. With proper units, the motion of the particle in the (x, y)-coordinate system is described by the differential equations

$$\ddot{x} = -\frac{x}{r^3} \quad , \quad \ddot{y} = -\frac{y}{r^3} \quad , \quad \left(r = \sqrt{x^2 + y^2}\right) \quad . \tag{94}$$

If we consider an elliptical orbit, equations (94) are solved by

$$\left.\begin{aligned} x &= e + \cos\psi \\ y &= b \cdot \sin\psi \\ t &= \psi + e \cdot \sin\psi \end{aligned}\right\} \quad , \tag{95}$$

the last equation (95) being Kepler's equation.

In (95) b stands for the minor semi-axis of the elliptical orbit, e for its eccentricity $\left(e = \sqrt{1 - b^2}\right.$, the major semi-axis a being equal to 1$\left.\right)$, and ψ for the eccentric anomaly.

From (95) we can obtain the coordinates x, y for the particle and the time t as functions of ψ. Differentiating (95) with respect to t, we can also express the velocities $\dot{x}$ and $\dot{y}$ by ψ:

$$\dot{x} = -\frac{\sin \psi}{1 + e \cos \psi} \quad , \quad \dot{y} = \frac{b \cos \psi}{1 + e \cos \psi} \quad . \tag{96}$$

For ψ=0 and ψ=2π (one complete orbit), we find from (95) and (96):

$$\left.\begin{array}{l} t = 0 \\ \\ t = 2\pi \end{array}\right\} x = 1 + e \quad , \quad y = 0 \quad , \quad \dot{x} = 0 \quad , \quad \dot{y} = \frac{b}{1 + e} \quad . \tag{97}$$

27. Integrating (94) for a complete orbit, we can check the accuracy of the numerical integration by comparing the results of the integration with (97). We applied our RKN formulas to two elliptical orbits, one of them (Orbit No. 2) being highly elliptical. The orbits are pictured in Figure 1 (page 50).

For comparison, the same orbits were also computed using the Runge-Kutta-Nyström formulas of report [1] and the Runge-Kutta formulas of report [2].

For Orbit No. 1 we chose $e = \frac{24}{25}$, leading to $b = \frac{7}{25}$ and the initial values

$$x_0 = \frac{49}{25} = 1.96 \quad ; \quad y_0 = 0 \quad , \quad \dot{x}_0 = 0 \quad , \quad \dot{y}_0 = \frac{1}{7} \quad . \tag{98}$$

For Orbit No. 2 we chose $e = \frac{924}{925}$, leading to $b = \frac{43}{925}$ and

$$x_0 = \frac{1849}{925} = 1.99\overline{891}\ldots, \quad y_0 = 0, \quad \dot{x}_0 = 0, \quad \dot{y}_0 = \frac{43}{1849} \quad . \tag{99}$$

In Tables 14 and 15 the results of the various integration formulas are listed for Orbits Nos. 1 and 2. The tables show for a complete revolution ($t=2\pi$) the number of integration steps, the errors, and the execution time of the various methods. All computations were performed on an IBM-7094 computer in double precision (16 decimal digits); all computer programs were written in IBM machine language.

For all formulas, the stepsize control was applied as described in No. 26 of report [1], using a tolerance of $0.1 \cdot 10^{-16}$.

28. Tables 14 and 15 show clearly that, for these examples, the Runge-Kutta-Nyström formulas of this report and those of report [1] are considerably faster than our Runge-Kutta formulas of report [2]. The fifth-order formula of this report is also faster than the fifth-order formula of [1]. For higher orders both types of Runge-Kutta-Nyström formulas required, in these examples, approximately the same time on the computer. However, for Orbit No. 2, the seventh- and eighth-order formulas of this report yield more accurate results than the corresponding formulas of [1]. It is believed that this increase in accuracy is gained by the new procedure of stepsize control that we have applied in this report. In the vicinity of singularities it seems to be more reasonable to base the stepsize control on the $\dot{x}$-values than on the x-values. Orbit No. 2 comes rather close to the attracting mass ($x = y = 0$), and the particle moves fast in the vicinity of the mass. Therefore, in this case, a stepsize control based on the $\dot{x}$-values seems to be more appropriate to the problem.

Computation Laboratory
George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Marshall Space Flight Center, Alabama 35812, February 25, 1973
014-00-00-00-62

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[1] FEHLBERG, E.: Classical Eighth- and Lower-Order Runge-Kutta-Nyström Formulas with Stepsize Control for Special Second-Order Differential Equations. NASA TR R-381, March 1972.

[2] FEHLBERG, E.: Classical Fifth-, Sixth-, Seventh-, and Eighth-Order Runge-Kutta Formulas with Stepsize Control. NASA TR R-287, October 1968.

TABLE 1. PATTERN FOR RKN 8(9)-13-$\dot{x}$

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$: 0	1	2	3	4	5	6	7	8	9	10	11	12	c_κ	$\dot{c}_\kappa$
0	0	0													*	*
1	*	*													0	0
2	*	*	*												0	0
3	*	*	*	*											0	0
4	*	*	*	*	*										0	0
5	1	*	0	*	*	*									0	0
6	*	*	0	*	*	*	*								0	0
7	*	*	0	0	0	*	*	*							*	*
8	*	*	0	0	0	*	*	*	*						*	*
9	*	*	0	0	0	0	*	*	*	*					*	*
10	*	*	0	0	0	*	*	*	*	*	*				*	*
11	*	*	0	0	0	*	*	*	*	*	*	*			*	*
12	1	*	0	0	0	0	0	*	*	*	*	*	*		0	*
13	1	*	0	0	0	0	0	0	*	*	*	*	*	0		

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TABLE 2. COEFFICIENTS (IN FRACTION FORM) FOR RKN 8(9)-13-$\dot{x}$

$$\alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{2}{3}, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = \frac{1}{3}, \quad \alpha_5 = 1, \quad \alpha_6 = \frac{1}{9}, \quad \alpha_7 = \frac{1}{2}, \quad \alpha_8 = \frac{1}{3}, \quad \alpha_9 = \frac{2}{3},$$

$$\alpha_{10} = \frac{1}{10}\left(5 - \sqrt{15}\right), \quad \alpha_{11} = \frac{1}{10}\left(5 + \sqrt{15}\right), \quad \alpha_{12} = 1, \quad \alpha_{13} = 1$$

$$\gamma_{10} = \frac{1}{18}$$

$$\gamma_{20} = \frac{2}{27}, \quad \gamma_{21} = \frac{4}{27}$$

$$\gamma_{30} = \frac{7}{128}, \quad \gamma_{31} = \frac{5}{64}, \quad \gamma_{32} = -\frac{1}{128}$$

$$\gamma_{40} = \frac{89}{3240}, \quad \gamma_{41} = \frac{31}{540}, \quad \gamma_{42} = \frac{11}{1080}, \quad \gamma_{43} = -\frac{16}{405}$$

$$\gamma_{50} = \frac{11}{120}, \quad \gamma_{51} = 0, \quad \gamma_{52} = \frac{9}{40}, \quad \gamma_{53} = -\frac{4}{15}, \quad \gamma_{54} = \frac{9}{20}$$

$$\gamma_{60} = \frac{33\ 259}{7\ 085\ 880}, \quad \gamma_{61} = 0, \quad \gamma_{62} = \frac{343}{157\ 464}, \quad \gamma_{63} = -\frac{4708}{885\ 735}, \quad \gamma_{64} = \frac{1879}{393\ 660}, \quad \gamma_{65} = -\frac{139}{885\ 735}$$

$$\gamma_{70} = \frac{29}{1920}, \quad \gamma_{71} = 0, \quad \gamma_{72} = 0, \quad \gamma_{73} = 0, \quad \gamma_{74} = \frac{99}{2560}, \quad \gamma_{75} = \frac{1}{30\ 720}, \quad \gamma_{76} = \frac{729}{10\ 240}$$

$$\gamma_{80} = \frac{13}{1215}, \quad \gamma_{81} = 0, \quad \gamma_{82} = 0, \quad \gamma_{83} = 0, \quad \gamma_{84} = \frac{1}{144}, \quad \gamma_{85} = \frac{1}{77\ 760}, \quad \gamma_{86} = \frac{87}{2240}, \quad \gamma_{87} = -\frac{8}{8505}$$

$$\gamma_{90} = \frac{22}{1215}, \quad \gamma_{91} = 0, \quad \gamma_{92} = 0, \quad \gamma_{93} = 0, \quad \gamma_{94} = 0, \quad \gamma_{95} = \frac{1}{4860}, \quad \gamma_{96} = \frac{3}{28}, \quad \gamma_{97} = \frac{256}{8505}, \quad \gamma_{98} = \frac{1}{15}$$

$$\gamma_{100} = \frac{1}{420\ 000}\left(7561 - 1454 \cdot \sqrt{15}\right), \quad \gamma_{101} = 0, \quad \gamma_{102} = 0, \quad \gamma_{103} = 0, \quad \gamma_{104} = -\frac{9}{800\ 000}\left(1373 + 45 \cdot \sqrt{15}\right)$$

$$\gamma_{105} = \frac{1}{1\ 344\ 000}\left(379 - 145\sqrt{15}\right), \quad \gamma_{106} = \frac{729}{78\ 400\ 000}\left(6997 - 1791\sqrt{15}\right), \quad \gamma_{107} = \frac{1}{183\ 750}\left(999 - 473\sqrt{15}\right)$$

$$\gamma_{108} = \frac{27}{5\ 600\ 000}\left(19\ 407 - 3865\sqrt{15}\right), \quad \gamma_{109} = \frac{297}{700\ 000}\left(78 - 19 \cdot \sqrt{15}\right)$$

$$\gamma_{110} = \frac{1}{840\ 000}\left(12\ 647 + 2413\sqrt{15}\right), \quad \gamma_{111} = 0, \quad \gamma_{112} = 0, \quad \gamma_{113} = 0, \quad \gamma_{114} = -\frac{9}{800\ 000}\left(1373 - 45 \cdot \sqrt{15}\right)$$

$$\gamma_{115} = -\frac{1}{336\ 000}\left(29 - 61\sqrt{15}\right), \quad \gamma_{116} = \frac{729}{19\ 600\ 000}\left(14\ 743 + 3789\sqrt{15}\right), \quad \gamma_{117} = \frac{1}{183\ 750}\left(999 + 143\sqrt{15}\right)$$

$$\gamma_{118} = \frac{27}{5\ 600\ 000}\left(20\ 157 + 4315 \cdot \sqrt{15}\right), \quad \gamma_{119} = \frac{27}{1\ 400\ 000}\left(1641 + 463\sqrt{15}\right), \quad \gamma_{1110} = -\frac{1}{56}\left(27 + 7\sqrt{15}\right)$$

$$\gamma_{120} = \frac{9}{280} - \frac{35}{6561} \cdot \gamma_{126}, \quad \gamma_{121} = 0, \quad \gamma_{122} = 0, \quad \gamma_{123} = 0, \quad \gamma_{124} = 0, \quad \gamma_{125} = 0, \quad \gamma_{126} = \gamma_{126}$$

$$\gamma_{127} = \frac{16}{315} - \frac{160}{19\ 683} \cdot \gamma_{126}, \quad \gamma_{128} = \frac{243}{1540} + \frac{35}{2673} \cdot \gamma_{126}, \quad \gamma_{129} = \frac{243}{3080} + \frac{7}{2673} \cdot \gamma_{126}$$

$$\gamma_{1210} = \frac{25}{1386}\left(5 + \sqrt{15}\right) - \frac{3500}{216\ 513}\left(31 + 8\sqrt{15}\right) \cdot \gamma_{126}, \quad \gamma_{1211} = \frac{25}{1386}\left(5 - \sqrt{15}\right) - \frac{3500}{216\ 513}\left(31 - 8\sqrt{15}\right) \cdot \gamma_{126}$$

$$\gamma_{130} = c_0 = \frac{9}{280}, \quad \gamma_{131} = c_1 = 0, \quad \gamma_{132} = c_2 = 0, \quad \gamma_{133} = c_3 = 0, \quad \gamma_{134} = c_4 = 0, \quad \gamma_{135} = c_5 = 0$$

$$\gamma_{136} = c_6 = 0, \quad \gamma_{137} = c_7 = \frac{16}{315}, \quad \gamma_{138} = c_8 = \frac{243}{1540}, \quad \gamma_{139} = c_9 = \frac{243}{3080}, \quad \gamma_{1310} = c_{10} = \frac{25}{1386}\left(5 + \sqrt{15}\right)$$

$$\gamma_{1311} = c_{11} = \frac{25}{1386}\left(5 - \sqrt{15}\right), \quad \gamma_{1312} = c_{12} = 0$$

$$\dot{c}_0 = \frac{9}{280}, \quad \dot{c}_1 = 0, \quad \dot{c}_2 = 0, \quad \dot{c}_3 = 0, \quad \dot{c}_4 = 0, \quad \dot{c}_5 = 0, \quad \dot{c}_6 = 0, \quad \dot{c}_7 = \frac{32}{315}, \quad \dot{c}_8 = \frac{729}{3080},$$

$$\dot{c}_9 = \frac{729}{3080}, \quad \dot{c}_{10} = \frac{125}{693}, \quad \dot{c}_{11} = \frac{125}{693}, \quad \dot{c}_{12} = \frac{9}{280}$$

Truncation Error in $\dot{x}$: $TE_{\dot{x}} = \frac{9}{280}\left(f_{12} - f_{13}\right) h$

TABLE 3. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 8(9)-13-$\dot{x}$

α_1	= 0.3333 3333 3333 3333 3333 3333 3333 3333	
α_2	= 0.6666 6666 6666 6666 6666 6666 6666 6667	
α_3	= 0.5	
α_4	= 0.3333 3333 3333 3333 3333 3333 3333 3333	
α_5	= 1	
α_6	= 0.1111 1111 1111 1111 1111 1111 1111 1111	
α_7	= 0.5	
α_8	= 0.3333 3333 3333 3333 3333 3333 3333 3333	
α_9	= 0.6666 6666 6666 6666 6666 6666 6666 6667	
α_{10}	= 0.1127 0166 5379 2583 1148 2073 4600 2176	
α_{11}	= 0.8872 9833 4620 7416 8851 7926 5399 7824	
α_{12}	= 1	
α_{13}	= 1	
γ_{10}	= 0.5555 5555 5555 5555 5555 5555 5555 5556	$\cdot 10^{-1}$
γ_{20}	= 0.7407 4074 0740 7407 4074 0740 7407 4074	$\cdot 10^{-1}$
γ_{21}	= 0.1481 4814 8148 1481 4814 8148 1481 4815	
γ_{30}	= 0.5468 75	$\cdot 10^{-1}$
γ_{31}	= 0.7812 5	$\cdot 10^{-1}$
γ_{32}	=-0.7812 5	$\cdot 10^{-2}$
γ_{40}	= 0.2746 9135 8024 6913 5802 4691 3580 2469	$\cdot 10^{-1}$
γ_{41}	= 0.5740 7407 4074 0740 7407 4074 0740 7407	$\cdot 10^{-1}$
γ_{42}	= 0.1018 5185 1851 8518 5185 1851 8518 5185	$\cdot 10^{-1}$
γ_{43}	=-0.3950 6172 8395 0617 2839 5061 7283 9506	$\cdot 10^{-1}$
γ_{50}	= 0.9166 6666 6666 6666 6666 6666 6666 6667	$\cdot 10^{-1}$
γ_{51}	= 0	
γ_{52}	= 0.225	
γ_{53}	=-0.2666 6666 6666 6666 6666 6666 6666 6667	
γ_{54}	= 0.45	
γ_{60}	= 0.4693 7007 1183 8190 8810 1971 8087 2383	$\cdot 10^{-2}$
γ_{61}	= 0	
γ_{62}	= 0.2178 2756 6935 9345 6282 0708 2253 7215	$\cdot 10^{-2}$
γ_{63}	=-0.5315 3595 6013 9319 3223 7068 6492 0095	$\cdot 10^{-2}$
γ_{64}	= 0.4773 1544 9880 6076 3095 0566 4786 8719	$\cdot 10^{-2}$
γ_{65}	=-0.1569 3181 3691 4539 9018 8939 1296 4939	$\cdot 10^{-3}$
γ_{70}	= 0.1510 4166 6666 6666 6666 6666 6666 6667	$\cdot 10^{-1}$
γ_{71}	= 0	
γ_{72}	= 0	
γ_{73}	= 0	
γ_{74}	= 0.3867 1875	$\cdot 10^{-1}$
γ_{75}	= 0.3255 2083 3333 3333 3333 3333 3333 3333	$\cdot 10^{-4}$

TABLE 3. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 8(9)-13-$\dot{x}$
(Continued)

γ_{76}	= 0.7119 1406 25	$\cdot\ 10^{-1}$
γ_{80}	= 0.1069 9588 4773 6625 5144 0329 2181 0700	$\cdot\ 10^{-1}$
γ_{81}	= 0	
γ_{82}	= 0	
γ_{83}	= 0	
γ_{84}	= 0.6944 4444 4444 4444 4444 4444 4444 4444	$\cdot\ 10^{-2}$
γ_{85}	= 0.1286 0082 3045 2674 8971 1934 1563 7860	$\cdot\ 10^{-4}$
γ_{86}	= 0.3883 9285 7142 8571 4285 7142 8571 4286	$\cdot\ 10^{-1}$
γ_{87}	=-0.9406 2316 2845 3850 6760 7289 8295 1205	$\cdot\ 10^{-3}$
γ_{90}	= 0.1810 6995 8847 7366 2551 4403 2921 8107	$\cdot\ 10^{-1}$
γ_{91}	= 0	
γ_{92}	= 0	
γ_{93}	= 0	
γ_{94}	= 0	
γ_{95}	= 0.2057 6131 6872 4279 8353 9094 6502 0576	$\cdot\ 10^{-3}$
γ_{96}	= 0.1071 4285 7142 8571 4285 7142 8571 4286	
γ_{97}	= 0.3009 9941 2110 5232 2163 4332 7454 4386	$\cdot\ 10^{-1}$
γ_{98}	= 0.6666 6666 6666 6666 6666 6666 6666 6667	$\cdot\ 10^{-1}$
γ_{100}	= 0.4594 4814 6336 7656 7832 1273 3592 1819	$\cdot\ 10^{-2}$
γ_{101}	= 0	
γ_{102}	= 0	
γ_{103}	= 0	
γ_{104}	=-0.1740 6947 8190 1750 4798 1220 0310 8640	$\cdot\ 10^{-1}$
γ_{105}	=-0.1358 5013 7797 6751 8478 4965 3891 1343	$\cdot\ 10^{-3}$
γ_{106}	= 0.5624 3490 8687 4288 9606 4146 7180 2943	$\cdot\ 10^{-3}$
γ_{107}	=-0.4532 9040 6942 0996 9343 6621 7872 6371	$\cdot\ 10^{-2}$
γ_{108}	= 0.2139 7111 2333 0803 7669 1281 7128 6734	$\cdot\ 10^{-1}$
γ_{109}	= 0.1872 5071 1050 2209 3099 0486 4784 3256	$\cdot\ 10^{-2}$
γ_{110}	= 0.2618 1558 1123 7916 3028 4971 0405 9137	$\cdot\ 10^{-1}$
γ_{111}	= 0	
γ_{112}	= 0	
γ_{113}	= 0	
γ_{114}	=-0.1348 5552 1809 8249 5201 8779 9689 1360	$\cdot\ 10^{-1}$
γ_{115}	= 0.6168 2138 1305 5131 8451 1711 8731 7478	$\cdot\ 10^{-3}$
γ_{116}	= 0.1094 1595 9245 9721 8867 0006 8800 0631	$\cdot\ 10^{+1}$
γ_{117}	= 0.8450 8115 2929 3391 0997 5855 7562 8239	$\cdot\ 10^{-2}$
γ_{118}	= 0.1777 6087 9419 6241 2575 1394 6991 7887	
γ_{119}	= 0.6623 0832 0078 1351 3201 1614 2625 9057	$\cdot\ 10^{-1}$
γ_{1110}	=-0.9662 6577 5418 7842 5350 4551 0321 1566	
γ_{120}	= 0.1880 6475 4937 1829 2072 2015 3722 2113	$\cdot\ 10^{-1}$

TABLE 3. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 8(9)-13-$\dot{x}$
(Concluded)

γ_{121}	= 0									
γ_{122}	= 0									
γ_{123}	= 0									
γ_{124}	= 0									
γ_{125}	= 0									
γ_{126}	= 0.25									$\cdot 10^{+1}$
γ_{127}	= 0.3047	1545	4235	3444	9597	5497	3472	3946		$\cdot 10^{-1}$
γ_{128}	= 0.1905	2696	2749	1849	7140	7193	6294	1585		
γ_{129}	= 0.8544	3054	8874	9933	1943	7763	8822	0833		$\cdot 10^{-1}$
γ_{1210}	= -0.2344	9246	2505	6950	7305	3300	7835	2450		$\cdot 10^{+1}$
γ_{1211}	= 0.1967	6586	5030	1368	5512	2865	4566	2269		$\cdot 10^{-1}$
$\gamma_{130} = c_0$	= 0.3214	2857	1428	5714	2857	1428	5714	2857		$\cdot 10^{-1}$
$\gamma_{131} = c_1$	= 0									
$\gamma_{132} = c_2$	= 0									
$\gamma_{133} = c_3$	= 0									
$\gamma_{134} = c_4$	= 0									
$\gamma_{135} = c_5$	= 0									
$\gamma_{136} = c_6$	= 0									
$\gamma_{137} = c_7$	= 0.5079	3650	7936	5079	3650	7936	5079	3651		$\cdot 10^{-1}$
$\gamma_{138} = c_8$	= 0.1577	9220	7792	2077	9220	7792	2077	9221		
$\gamma_{139} = c_9$	= 0.7889	6103	8961	0389	6103	8961	0389	6104		$\cdot 10^{-1}$
$\gamma_{1310} = c_{10}$	= 0.1600	4659	7153	8134	3587	9856	8795	0545		
$\gamma_{1311} = c_{11}$	= 0.2032	8583	2213	6693	9300	5183	0086	9726		$\cdot 10^{-1}$
$\gamma_{1312} = c_{12}$	= 0									
$\dot{c}_0$	= 0.3214	2857	1428	5714	2857	1428	5714	2857		$\cdot 10^{-1}$
$\dot{c}_1$	= 0									
$\dot{c}_2$	= 0									
$\dot{c}_3$	= 0									
$\dot{c}_4$	= 0									
$\dot{c}_5$	= 0									
$\dot{c}_6$	= 0									
$\dot{c}_7$	= 0.1015	8730	1587	3015	8730	1587	3015	8730		
$\dot{c}_8$	= 0.2366	8831	1688	3116	8831	1688	3116	8831		
$\dot{c}_9$	= 0.2366	8831	1688	3116	8831	1688	3116	8831		
$\dot{c}_{10}$	= 0.1803	7518	0375	1803	7518	0375	1803	7518		
$\dot{c}_{11}$	= 0.1803	7518	0375	1803	7518	0375	1803	7518		
$\dot{c}_{12}$	= 0.3214	2857	1428	5714	2857	1428	5714	2857		$\cdot 10^{-1}$

TABLE 4. FLOW CHART FOR RKN 8(9)-13-$\dot{x}$

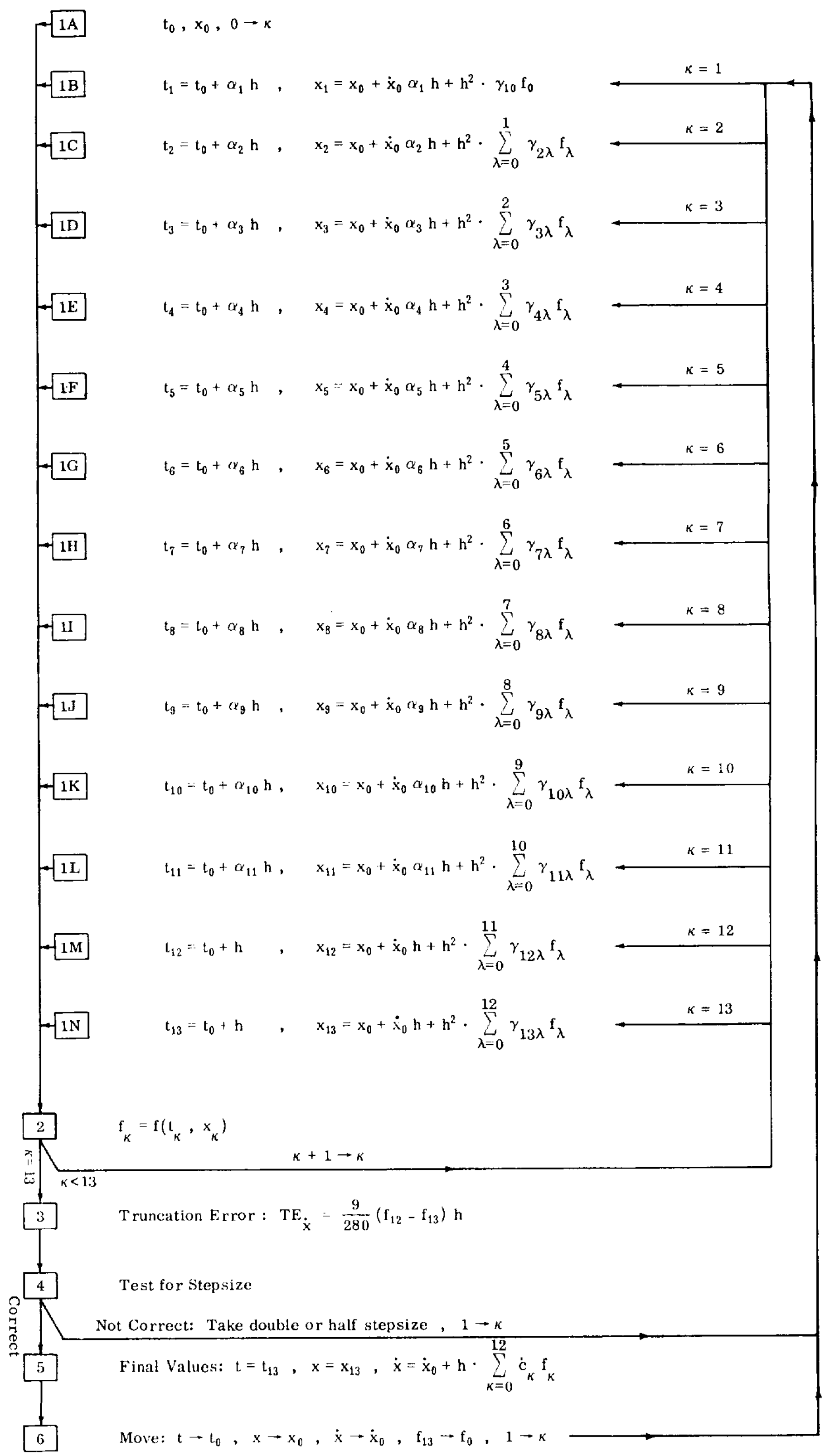

TABLE 5. PATTERN FOR RKN 7(8)-10-$\dot{x}$

	α_κ	$\gamma_{\kappa\lambda}$										c_κ	$\dot{c}_\kappa$
$\kappa \backslash \lambda$		0	1	2	3	4	5	6	7	8	9		
0	0	0										*	*
1	*	*										0	0
2	*	*	*									0	0
3	*	*	*	*								0	0
4	*	*	0	*	*							0	0
5	*	*	0	*	*	*						0	0
6	*	*	0	0	*	*	*					*	*
7	*	*	0	0	*	*	*	*				*	*
8	*	*	0	0	*	*	*	*	*			*	*
9	1	*	0	0	*	*	0	*	*	*		0	*
10	1	*	0	0	0	0	0	*	*	*	0		

TABLE 6. COEFFICIENTS (IN FRACTION FORM) FOR RKN 7(8)-10-$\dot{x}$

$$\alpha_1 = \frac{5}{21}\ ,\quad \alpha_2 = \frac{10}{21}\ ,\quad \alpha_3 = \frac{1}{7}\ ,\quad \alpha_4 = \frac{5}{7}\ ,\quad \alpha_5 = \frac{1}{14}\left(7+\sqrt{21}\right)\ ,\quad \alpha_6 = \frac{1}{2}\ ,\quad \alpha_7 = \frac{1}{14}\left(7-\sqrt{21}\right)\ ,$$

$$\alpha_8 = \frac{1}{14}\left(7+\sqrt{21}\right)\ ,\quad \alpha_9 = 1\ ,\quad \alpha_{10} = 1$$

$$\gamma_{10} = \frac{25}{882}$$

$$\gamma_{20} = \frac{50}{1323}\ ,\quad \gamma_{21} = \frac{100}{1323}$$

$$\gamma_{30} = \frac{73}{9800}\ ,\quad \gamma_{31} = \frac{17}{4900}\ ,\quad \gamma_{32} = -\frac{1}{1400}$$

$$\gamma_{40} = \frac{25}{1176}\ ,\quad \gamma_{41} = 0\ ,\quad \gamma_{42} = \frac{225}{2744}\ ,\quad \gamma_{43} = \frac{625}{4116}$$

$$\gamma_{50} = \frac{1}{42\,000}\left(661+73\sqrt{21}\right)\ ,\quad \gamma_{51} = 0\ ,\quad \gamma_{52} = \frac{9}{98\,000}\left(637+141\sqrt{21}\right)\ ,\quad \gamma_{53} = \frac{1}{11\,760}\left(1127+226\sqrt{21}\right)$$

$$\gamma_{54} = \frac{1}{42\,000}\left(357+76\sqrt{21}\right)$$

$$\gamma_{60} = \frac{1}{19\,200}\left(531-73\sqrt{21}\right)\ ,\quad \gamma_{61} = 0\ ,\quad \gamma_{62} = 0\ ,\quad \gamma_{63} = -\frac{7}{15\,360}\left(105-73\sqrt{21}\right)$$

$$\gamma_{64} = \frac{7}{230\,400}\left(399+73\sqrt{21}\right)\ ,\quad \gamma_{65} = \frac{73}{11\,520}\left(21-5\sqrt{21}\right)$$

$$\gamma_{70} = \frac{1}{29\,400}\left(789-89\sqrt{21}\right)\ ,\quad \gamma_{71} = 0\ ,\quad \gamma_{72} = 0\ ,\quad \gamma_{73} = -\frac{1}{8400}\left(105-17\sqrt{21}\right)$$

$$\gamma_{74} = -\frac{1}{25\,200}\left(77+103\sqrt{21}\right)\ ,\quad \gamma_{75} = \frac{13}{17\,640}\left(147-29\sqrt{21}\right)\ ,\quad \gamma_{76} = \frac{2}{11\,025}\left(325-51\sqrt{21}\right)$$

$$\gamma_{80} = \frac{1}{29\,400}\left(579+235\sqrt{21}\right)\ ,\quad \gamma_{81} = 0\ ,\quad \gamma_{82} = 0\ ,\quad \gamma_{83} = \frac{1}{4200}\left(315-191\sqrt{21}\right),\quad \gamma_{84} = -\frac{1}{12\,600}\left(161-15\sqrt{21}\right)$$

$$\gamma_{85} = -\frac{3}{490}\left(21-5\sqrt{21}\right)\ ,\quad \gamma_{86} = \frac{2}{11\,025}\left(395+51\sqrt{21}\right)\ ,\quad \gamma_{87} = \frac{1}{280}\left(43+9\sqrt{21}\right)$$

$$\gamma_{90} = \frac{1}{20} - \frac{4}{49}\gamma_{93}\ ,\quad \gamma_{91} = 0\ ,\quad \gamma_{92} = 0\ ,\quad \gamma_{93} = \gamma_{93}\ ,\quad \gamma_{94} = -\frac{1}{9}\gamma_{93}\ ,\quad \gamma_{95} = 0\ ,\quad \gamma_{96} = \frac{8}{45} + \frac{64}{441}\cdot\gamma_{93}$$

$$\gamma_{97} = \frac{7}{360}\left(7+\sqrt{21}\right) - \frac{10}{441}\left(21+5\sqrt{21}\right)\gamma_{93}\ ,\quad \gamma_{98} = \frac{7}{360}\left(7-\sqrt{21}\right) - \frac{10}{441}\left(21-5\sqrt{21}\right)\gamma_{93}$$

$$\gamma_{100} = c_0 = \frac{1}{20}\ ,\quad \gamma_{101} = c_1 = 0\ ,\quad \gamma_{102} = c_2 = 0\ ,\quad \gamma_{103} = c_3 = 0\ ,\quad \gamma_{104} = c_4 = 0\ ,\quad \gamma_{105} = c_5 = 0$$

$$\gamma_{106} = c_6 = \frac{8}{45}\ ,\quad \gamma_{107} = c_7 = \frac{7}{360}\left(7+\sqrt{21}\right)\ ,\quad \gamma_{108} = c_8 = \frac{7}{360}\left(7-\sqrt{21}\right),\quad \gamma_{109} = c_9 = 0$$

$$\dot{c}_0 = \frac{1}{20}\ ,\quad \dot{c}_1 = 0\ ,\quad \dot{c}_2 = 0\ ,\quad \dot{c}_3 = 0\ ,\quad \dot{c}_4 = 0\ ,\quad \dot{c}_5 = 0\ ,\quad \dot{c}_6 = \frac{16}{45}\ ,\quad \dot{c}_7 = \frac{49}{180}\ ,\quad \dot{c}_8 = \frac{49}{180}\ ,\quad \dot{c}_9 = \frac{1}{20}$$

Truncation Error in $\dot{x}$: $TE_{\dot{x}} = \frac{1}{20}\left(f_9 - f_{10}\right)h$

TABLE 7. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 7(8)-10-$\dot{x}$

$\alpha_1 =$	0.2380 9523 8095 2380 9523 8095 2380 9524	
$\alpha_2 =$	0.4761 9047 6190 4761 9047 6190 4761 9048	
$\alpha_3 =$	0.1428 5714 2857 1428 5714 2857 1428 5714	
$\alpha_4 =$	0.7142 8571 4285 7142 8571 4285 7142 8571	
$\alpha_5 =$	0.8273 2683 5353 9885 7189 9146 2281 2343	
$\alpha_6 =$	0.5	
$\alpha_7 =$	0.1726 7316 4646 0114 2810 0853 7718 7657	
$\alpha_8 =$	0.8273 2683 5353 9885 7189 9146 2281 2343	
$\alpha_9 =$	1	
$\alpha_{10} =$	1	
$\gamma_{10} =$	0.2834 4671 2018 1405 8956 9160 9977 3243	$\cdot\ 10^{-1}$
$\gamma_{20} =$	0.3779 2894 9357 5207 8609 2214 6636 4324	$\cdot\ 10^{-1}$
$\gamma_{21} =$	0.7558 5789 8715 0415 7218 4429 3272 8647	$\cdot\ 10^{-1}$
$\gamma_{30} =$	0.7448 9795 9183 6734 6938 7755 1020 4082	$\cdot\ 10^{-2}$
$\gamma_{31} =$	0.3469 3877 5510 2040 8163 2653 0612 2449	$\cdot\ 10^{-2}$
$\gamma_{32} =$	-0.7142 8571 4285 7142 8571 4285 7142 8571	$\cdot\ 10^{-3}$
$\gamma_{40} =$	0.2125 8503 4013 6054 4217 6870 7482 9932	$\cdot\ 10^{-1}$
$\gamma_{41} =$	0	
$\gamma_{42} =$	0.8199 7084 5481 0495 6268 2215 7434 4023	$\cdot\ 10^{-1}$
$\gamma_{43} =$	0.1518 4645 2866 8610 3012 6336 2487 8523	
$\gamma_{50} =$	0.2370 3048 2317 0896 0011 4506 5345 5765	$\cdot\ 10^{-1}$
$\gamma_{51} =$	0	
$\gamma_{52} =$	0.1178 3967 9152 0302 1396 2859 5090 6980	
$\gamma_{53} =$	0.1838 9983 9035 7159 7291 5722 6756 6178	
$\gamma_{54} =$	0.1679 2279 8289 6771 0488 1117 0444 5794	$\cdot\ 10^{-1}$
$\gamma_{60} =$	0.1023 2915 3264 6998 3308 2850 2889 8847	$\cdot\ 10^{-1}$
$\gamma_{61} =$	0	
$\gamma_{62} =$	0	
$\gamma_{63} =$	0.1046 0261 5893 3876 4605 2505 9971 3509	
$\gamma_{64} =$	0.2228 6007 7262 2584 3070 1670 6647 5673	$\cdot\ 10^{-1}$
$\gamma_{65} =$	-0.1212 1538 9460 8347 2430 9580 9250 9611	$\cdot\ 10^{-1}$
$\gamma_{70} =$	0.1296 4311 6717 3232 1068 4919 6597 8170	$\cdot\ 10^{-1}$
$\gamma_{71} =$	0	
$\gamma_{72} =$	0	
$\gamma_{73} =$	-0.3225 7396 6497 0323 7961 9085 6869 8362	$\cdot\ 10^{-2}$
$\gamma_{74} =$	-0.2178 5924 4674 7823 4947 5622 5638 7063	$\cdot\ 10^{-1}$
$\gamma_{75} =$	0.1039 5065 9297 9865 7455 5729 1428 3704	$\cdot\ 10^{-1}$
$\gamma_{76} =$	0.1656 0297 4253 5186 5698 6865 4750 4739	$\cdot\ 10^{-1}$
$\gamma_{80} =$	0.5632 3309 1263 4770 0732 9316 6974 5785	$\cdot\ 10^{-1}$
$\gamma_{81} =$	0	

TABLE 7. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 7(8)-10-$\dot{x}$
(Concluded)

γ_{82}	=	0	
γ_{83}	=	-0.1333 9808 5175 3727 2410 9123 0985 7192	
γ_{84}	=	-0.7322 3305 2187 7968 2461 2534 0642 3873	$\cdot 10^{-2}$
γ_{85}	=	0.1171 1500 8659 9510 2242 4912 4062 4327	$\cdot 10^{-1}$
γ_{86}	=	0.1140 5194 7472 6073 1797 4782 8402 5036	
γ_{87}	=	0.3008 6850 4480 7234 2878 3187 2312 2697	
γ_{90}	=	0.4183 6734 6938 7755 1020 4081 6326 5306	$\cdot 10^{-1}$
γ_{91}	=	0	
γ_{92}	=	0	
γ_{93}	=	0.1	
γ_{94}	=	-0.1111 1111 1111 1111 1111 1111 1111 1111	$\cdot 10^{-1}$
γ_{95}	=	0	
γ_{96}	=	0.1922 9024 9433 1065 7596 3718 8208 6168	
γ_{97}	=	0.1256 4106 1472 7145 7601 7125 3258 9020	
γ_{98}	=	0.5134 3065 5114 1240 8109 8588 0109 3928	$\cdot 10^{-1}$
$\gamma_{100}=c_0$	=	0.5	$\cdot 10^{-1}$
$\gamma_{101}=c_1$	=	0	
$\gamma_{102}=c_2$	=	0	
$\gamma_{103}=c_3$	=	0	
$\gamma_{104}=c_4$	=	0	
$\gamma_{105}=c_5$	=	0	
$\gamma_{106}=c_6$	=	0.1777 7777 7777 7777 7777 7777 7777 7778	
$\gamma_{107}=c_7$	=	0.2252 1674 9624 1413 3346 1434 2509 8916	
$\gamma_{108}=c_8$	=	0.4700 5472 5980 8088 8760 7879 7123 3067	$\cdot 10^{-1}$
$\gamma_{109}=c_9$	=	0	
$\dot{c}_0$	=	0.5	$\cdot 10^{-1}$
$\dot{c}_1$	=	0	
$\dot{c}_2$	=	0	
$\dot{c}_3$	=	0	
$\dot{c}_4$	=	0	
$\dot{c}_5$	=	0	
$\dot{c}_6$	=	0.3555 5555 5555 5555 5555 5555 5555 5556	
$\dot{c}_7$	=	0.2722 2222 2222 2222 2222 2222 2222 2222	
$\dot{c}_8$	=	0.2722 2222 2222 2222 2222 2222 2222 2222	
$\dot{c}_9$	=	0.5	$\cdot 10^{-1}$

TABLE 8. PATTERN FOR RKN 6(7)-8-$\dot{x}$

$\kappa \backslash \lambda$	α_κ	$\gamma_{\kappa\lambda}$ 0	1	2	3	4	5	6	7	c_κ	$\dot{c}_\kappa$
0	0	0								*	*
1	*	*								0	0
2	*	*	*							0	0
3	*	*	*	*						0	0
4	*	*	0	*	*					*	*
5	*	*	0	*	*	*				*	*
6	*	*	0	*	*	*	*			*	*
7	1	*	0	0	*	*	*	*		0	*
8	1	*	0	0	0	*	*	*	0		

TABLE 9. COEFFICIENTS (IN FRACTION FORM) FOR RKN 6(7)-8-$\dot{x}$

$\alpha_1 = \frac{1}{5}$, $\alpha_2 = \frac{2}{5}$, $\alpha_3 = \frac{1}{6}$, $\alpha_4 = \frac{1}{2}$, $\alpha_5 = \frac{1}{14}\left(7 - \sqrt{21}\right)$, $\alpha_6 = \frac{1}{14}\left(7 + \sqrt{21}\right)$,

$\alpha_7 = 1$, $\alpha_8 = 1$

$\gamma_{10} = \frac{1}{50}$

$\gamma_{20} = \frac{2}{75}$, $\gamma_{21} = \frac{4}{75}$

$\gamma_{30} = \frac{277}{31\,104}$, $\gamma_{31} = \frac{95}{15\,552}$, $\gamma_{32} = -\frac{35}{31\,104}$

$\gamma_{40} = \frac{5}{192}$, $\gamma_{41} = 0$, $\gamma_{42} = \frac{25}{1344}$, $\gamma_{43} = \frac{9}{112}$

$\gamma_{50} = \frac{1}{4116}\left(56 - 5\sqrt{21}\right)$, $\gamma_{51} = 0$, $\gamma_{52} = -\frac{25}{28\,812}\left(77 - 16\sqrt{21}\right)$, $\gamma_{53} = \frac{9}{9604}\left(133 - 27\sqrt{21}\right)$

$\gamma_{54} = \frac{1}{4116}\left(441 - 95\sqrt{21}\right)$

$\gamma_{60} = \frac{1}{11\,760}\left(781 + 103\sqrt{21}\right)$, $\gamma_{61} = 0$, $\gamma_{62} = -\frac{25}{279\,888}\left(1369 + 599\sqrt{21}\right)$

$\gamma_{63} = -\frac{9}{6860}\left(2389 + 513\sqrt{21}\right)$, $\gamma_{64} = \frac{1}{2940}\left(315 + 127\sqrt{21}\right)$, $\gamma_{65} = \frac{69}{4760}\left(225 + 49 \cdot \sqrt{21}\right)$

$\gamma_{70} = \frac{1}{20} - \frac{1}{54}\gamma_{73}$, $\gamma_{71} = 0$, $\gamma_{72} = 0$, $\gamma_{73} = \gamma_{73}$, $\gamma_{74} = \frac{8}{45} + \frac{1}{81}\gamma_{73}$

$\gamma_{75} = \frac{7}{360}\left(7 + \sqrt{21}\right) - \frac{7}{324}\left(23 + 5\sqrt{21}\right)\gamma_{73}$, $\gamma_{76} = \frac{7}{360}\left(7 - \sqrt{21}\right) - \frac{7}{324}\left(23 - 5\sqrt{21}\right)\gamma_{73}$

$\gamma_{80} = c_0 = \frac{1}{20}$, $\gamma_{81} = c_1 = 0$, $\gamma_{82} = c_2 = 0$, $\gamma_{83} = c_3 = 0$, $\gamma_{84} = c_4 = \frac{8}{45}$,

$\gamma_{85} = c_5 = \frac{7}{360}\left(7 + \sqrt{21}\right)$, $\gamma_{86} = c_6 = \frac{7}{360}\left(7 - \sqrt{21}\right)$, $\gamma_{87} = c_7 = 0$

$\dot{c}_0 = \frac{1}{20}$, $\dot{c}_1 = 0$, $\dot{c}_2 = 0$, $\dot{c}_3 = 0$, $\dot{c}_4 = \frac{16}{45}$, $\dot{c}_5 = \frac{49}{180}$, $\dot{c}_6 = \frac{49}{180}$, $\dot{c}_7 = \frac{1}{20}$

Truncation Error in $\dot{x}$: $TE_{\dot{x}} = \frac{1}{20}\left(f_7 - f_8\right) h$

TABLE 10. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 6(7)-8-$\dot{x}$

$\alpha_1 =$	0.2	
$\alpha_2 =$	0.4	
$\alpha_3 =$	0.1666 6666 6666 6666 6666 6666 6666 6667	
$\alpha_4 =$	0.5	
$\alpha_5 =$	0.1726 7316 4646 0114 2810 0853 7718 7657	
$\alpha_6 =$	0.8273 2683 5353 9885 7189 9146 2281 2343	
$\alpha_7 =$	1	
$\alpha_8 =$	1	
$\gamma_{10} =$	0.2	$\cdot 10^{-1}$
$\gamma_{20} =$	0.2666 6666 6666 6666 6666 6666 6666 6667	$\cdot 10^{-1}$
$\gamma_{21} =$	0.5333 3333 3333 3333 3333 3333 3333 3333	$\cdot 10^{-1}$
$\gamma_{30} =$	0.8905 6069 9588 4773 6625 5144 0329 2181	$\cdot 10^{-2}$
$\gamma_{31} =$	0.6108 5390 9465 0205 7613 1687 2427 9835	$\cdot 10^{-2}$
$\gamma_{32} =$	-0.1125 2572 0164 6090 5349 7942 3868 3128	$\cdot 10^{-2}$
$\gamma_{40} =$	0.2604 1666 6666 6666 6666 6666 6666 6667	$\cdot 10^{-1}$
$\gamma_{41} =$	0	
$\gamma_{42} =$	0.1860 1190 4761 9047 6190 4761 9047 6190	$\cdot 10^{-1}$
$\gamma_{43} =$	0.8035 7142 8571 4285 7142 8571 4285 7143	$\cdot 10^{-1}$
$\gamma_{50} =$	0.8038 6592 6268 7269 1853 8866 9589 7376	$\cdot 10^{-2}$
$\gamma_{51} =$	0	
$\gamma_{52} =$	-0.3192 0630 9932 1949 0963 7585 4592 1420	$\cdot 10^{-2}$
$\gamma_{53} =$	0.8687 4329 5769 7925 6975 3275 0096 2197	$\cdot 10^{-2}$
$\gamma_{54} =$	0.1373 9817 7337 1039 6924 5274 9415 8987	$\cdot 10^{-2}$
$\gamma_{60} =$	0.1065 4806 9437 1132 2454 7497 3521 2194	
$\gamma_{61} =$	0	
$\gamma_{62} =$	-0.3674 6509 6867 1886 7582 2671 9499 4454	
$\gamma_{63} =$	-0.6218 4769 6554 0978 6166 7886 4999 0441	$\cdot 10^{+1}$
$\gamma_{64} =$	0.3050 9765 7571 2216 6014 8531 2903 4131	
$\gamma_{65} =$	0.6516 5311 8164 8255 2651 8365 2849 1585	$\cdot 10^{+1}$
$\gamma_{70} =$	0.4981 4814 8148 1481 4814 8148 1481 4815	$\cdot 10^{-1}$
$\gamma_{71} =$	0	
$\gamma_{72} =$	0	
$\gamma_{73} =$	0.1	$\cdot 10^{-1}$
$\gamma_{74} =$	0.1779 0123 4567 9012 3456 7901 2345 6790	
$\gamma_{75} =$	0.2152 9730 0570 9482 9641 7280 4963 0457	
$\gamma_{76} =$	0.4698 6650 0463 3565 4200 0034 5431 2711	$\cdot 10^{-1}$

TABLE 10. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 6(7)-8-$\dot{x}$
(Concluded)

$\gamma_{80}=c_0=0.5$	$\cdot\ 10^{-1}$
$\gamma_{81}=c_1=0$	
$\gamma_{82}=c_2=0$	
$\gamma_{83}=c_3=0$	
$\gamma_{84}=c_4=0.1777\ 7777\ 7777\ 7777\ 7777\ 7777\ 7777\ 7778$	
$\gamma_{85}=c_5=0.2252\ 1674\ 9624\ 1413\ 3346\ 1434\ 2509\ 8916$	
$\gamma_{86}=c_6=0.4700\ 5472\ 5980\ 8088\ 8760\ 7879\ 7123\ 3067$	$\cdot\ 10^{-1}$
$\gamma_{87}=c_7=0$	
$\dot{c}_0=0.5$	$\cdot\ 10^{-1}$
$\dot{c}_1=0$	
$\dot{c}_2=0$	
$\dot{c}_3=0$	
$\dot{c}_4=0.3555\ 5555\ 5555\ 5555\ 5555\ 5555\ 5555\ 5556$	
$\dot{c}_5=0.2722\ 2222\ 2222\ 2222\ 2222\ 2222\ 2222\ 2222$	
$\dot{c}_6=0.2722\ 2222\ 2222\ 2222\ 2222\ 2222\ 2222\ 2222$	
$\dot{c}_7=0.5$	$\cdot\ 10^{-1}$

TABLE 11. PATTERN FOR RKN 5(6)-6-$\dot{x}$

κ	α_κ	$\gamma_{\kappa\lambda}$						c_κ	$\dot{c}_\kappa$
$\kappa \backslash \lambda$		0	1	2	3	4	5		
0	0							*	*
1	1	*						0	0
2	*	*	*					0	0
3	*	*	*	*				*	*
4	*	*	*	0	*			*	*
5	1	*	*	0	*	*		0	*
6	1	*	0	0	*	*	0		

TABLE 12. COEFFICIENTS (IN FRACTION FORM) FOR RKN 5(6)-6-$\dot{x}$

$$\alpha_1 = 1 \;, \quad \alpha_2 = \frac{1}{10}\left(5+\sqrt{5}\right) \;, \quad \alpha_3 = \frac{1}{10}\left(5-\sqrt{5}\right) \;,$$

$$\alpha_4 = \frac{1}{10}\left(5+\sqrt{5}\right) \;, \quad \alpha_5 = 1 \;, \quad \alpha_6 = 1$$

$$\gamma_{10} = \frac{1}{2}$$

$$\gamma_{20} = \frac{1}{300}\left(35 + 11 \cdot \sqrt{5}\right) \;, \quad \gamma_{21} = \frac{1}{150}\left(5 + 2 \cdot \sqrt{5}\right)$$

$$\gamma_{30} = \frac{1}{600}\left(25 - 3\sqrt{5}\right) \;, \quad \gamma_{31} = -\frac{1}{300}\sqrt{5} \;, \quad \gamma_{32} = \frac{1}{120}\left(13 - 5\sqrt{5}\right)$$

$$\gamma_{40} = \frac{1}{600}\left(25 + 3\sqrt{5}\right) \;, \quad \gamma_{41} = \frac{1}{300}\sqrt{5} \;, \quad \gamma_{42} = 0$$

$$\gamma_{43} = \frac{1}{120}\left(13 + 5\sqrt{5}\right)$$

$$\gamma_{50} = \frac{1}{12} - \gamma_{51} \;, \quad \gamma_{51} = \gamma_{51} \;, \quad \gamma_{52} = 0 \;,$$

$$\gamma_{53} = \frac{1}{24}\left(5+\sqrt{5}\right) + \sqrt{5} \cdot \gamma_{51} \;, \quad \gamma_{54} = \frac{1}{24}\left(5-\sqrt{5}\right) - \sqrt{5} \cdot \gamma_{51}$$

$$\gamma_{60} = c_0 = \frac{1}{12} \;, \quad \gamma_{61} = c_1 = 0 \;, \quad \gamma_{62} = c_2 = 0 \;,$$

$$\gamma_{63} = c_3 = \frac{1}{24}\left(5+\sqrt{5}\right) \;, \quad \gamma_{64} = c_4 = \frac{1}{24}\left(5-\sqrt{5}\right) \;, \quad \gamma_{65} = c_5 = 0$$

$$\dot{c}_0 = \frac{1}{12} \;, \quad \dot{c}_1 = 0 \;, \quad \dot{c}_2 = 0 \;, \quad \dot{c}_3 = \frac{5}{12} \;, \quad \dot{c}_4 = \frac{5}{12} \;, \quad \dot{c}_5 = \frac{1}{12}$$

Truncation Error in $\dot{x}$: $TE_{\dot{x}} = \frac{1}{12}\left(f_5 - f_6\right) h$

TABLE 13. COEFFICIENTS (IN DECIMAL FORM) FOR RKN 5(6)-6-$\dot{x}$

α_1	= 1	
α_2	= 0.7236 0679 7749 9789 6964 0917 3668 7313	
α_3	= 0.2763 9320 2250 0210 3035 9082 6331 2687	
α_4	= 0.7236 0679 7749 9789 6964 0917 3668 7313	
α_5	= 1	
α_6	= 1	
γ_{10}	= 0.5	
γ_{20}	= 0.1986 5582 5841 6589 5553 5003 0345 2015	
γ_{21}	= 0.6314 7573 0333 3052 9285 4556 4891 6417	$\cdot 10^{-1}$
γ_{30}	= 0.3048 6326 7791 6771 8184 6207 9832 3010	$\cdot 10^{-1}$
γ_{31}	=-0.7453 5599 2499 9298 9880 3057 8895 7709	$\cdot 10^{-2}$
γ_{32}	= 0.1516 3834 2708 4209 5982 9510 9713 6197	$\cdot 10^{-1}$
γ_{40}	= 0.5284 7006 5541 6561 5148 7125 3501 0323	$\cdot 10^{-1}$
γ_{41}	= 0.7453 5599 2499 9298 9880 3057 8895 7709	$\cdot 10^{-2}$
γ_{42}	= 0	
γ_{43}	= 0.2015 0283 2395 8245 7068 3715 5695 3047	
γ_{50}	= 0.8	$\cdot 10^{-1}$
γ_{51}	= 0.3333 3333 3333 3333 3333 3333 3333 3333	$\cdot 10^{-2}$
γ_{52}	= 0	
γ_{53}	= 0.3089 5639 2320 8238 6967 1746 1484 2624	
γ_{54}	= 0.1077 1027 4345 8427 9699 4920 5182 4043	
$\gamma_{60}=c_0=$	0.8333 3333 3333 3333 3333 3333 3333 3333	$\cdot 10^{-1}$
$\gamma_{61}=c_1=$	0	
$\gamma_{62}=c_2=$	0	
$\gamma_{63}=c_3=$	0.3015 0283 2395 8245 7068 3715 5695 3047	
$\gamma_{64}=c_4=$	0.1151 6383 4270 8420 9598 2951 0971 3620	
$\gamma_{65}=c_5=$	0	
$\dot{c}_0$	= 0.8333 3333 3333 3333 3333 3333 3333 3333	$\cdot 10^{-1}$
$\dot{c}_1$	= 0	
$\dot{c}_2$	= 0	
$\dot{c}_3$	= 0.4166 6666 6666 6666 6666 6666 6666 6667	
$\dot{c}_4$	= 0.4166 6666 6666 6666 6666 6666 6666 6667	
$\dot{c}_5$	= 0.8333 3333 3333 3333 3333 3333 3333 3333	$\cdot 10^{-1}$

TABLE 14. APPLICATION OF THE VARIOUS FORMULAS TO ORBIT NO. 1

Problem: $\begin{cases} \ddot{x} = -x/(\sqrt{x^2+y^2})^3 \\ \ddot{y} = -y/(\sqrt{x^2+y^2})^3 \end{cases}$ Initial Values: $t_0 = 0 \begin{cases} x_0 = \frac{49}{25}, & \dot{x}_0 = 0 \\ y_0 = 0, & \dot{y}_0 = \frac{1}{7} \end{cases}$

Results for $t = 2\pi$

Formula	Steps	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	7094 Time (min.)
RK 5(6)-8 [2]	4061	$0.940 \cdot 10^{-13}$	$0.269 \cdot 10^{-13}$	$0.824 \cdot 10^{-13}$	$-0.708 \cdot 10^{-14}$	0.57
RKN 5(6)-6-x [1]	5180	$-0.338 \cdot 10^{-12}$	$0.189 \cdot 10^{-12}$	$-0.180 \cdot 10^{-12}$	$0.248 \cdot 10^{-13}$	0.40
RKN 5(6)-6-$\dot{x}$	2564	$0.366 \cdot 10^{-14}$	$0.338 \cdot 10^{-13}$	$0.160 \cdot 10^{-13}$	$0.167 \cdot 10^{-15}$	0.20
RK 6(7)-10 [2]	2273	$-0.873 \cdot 10^{-13}$	$0.706 \cdot 10^{-13}$	$-0.421 \cdot 10^{-13}$	$0.655 \cdot 10^{-14}$	0.40
RKN 6(7)-7-x [1]	1231	$0.466 \cdot 10^{-14}$	$0.781 \cdot 10^{-13}$	$0.608 \cdot 10^{-14}$	$-0.486 \cdot 10^{-15}$	0.12
RKN 6(7)-8-$\dot{x}$	584	$0.403 \cdot 10^{-13}$	$-0.310 \cdot 10^{-14}$	$0.279 \cdot 10^{-13}$	$-0.448 \cdot 10^{-14}$	0.08
RK 7(8)-13 [2]	888	$0.980 \cdot 10^{-13}$	$-0.188 \cdot 10^{-13}$	$0.661 \cdot 10^{-13}$	$-0.697 \cdot 10^{-14}$	0.20
RKN 7(8)-9-x [1]	542	$-0.348 \cdot 10^{-12}$	$0.127 \cdot 10^{-12}$	$-0.213 \cdot 10^{-12}$	$0.170 \cdot 10^{-13}$	0.07
RKN 7(8)-10-$\dot{x}$	573	$0.344 \cdot 10^{-13}$	$-0.434 \cdot 10^{-15}$	$0.245 \cdot 10^{-13}$	$-0.230 \cdot 10^{-14}$	0.08
RK 8(9)-17 [2]	491	$0.704 \cdot 10^{-13}$	$-0.143 \cdot 10^{-13}$	$0.462 \cdot 10^{-13}$	$-0.518 \cdot 10^{-14}$	0.20
RKN 8(9)-11-x [1]	334	$-0.130 \cdot 10^{-13}$	$0.446 \cdot 10^{-13}$	$-0.100 \cdot 10^{-13}$	$0.736 \cdot 10^{-15}$	0.07
RKN 8(9)-13-$\dot{x}$	340	$0.592 \cdot 10^{-13}$	$-0.152 \cdot 10^{-13}$	$0.383 \cdot 10^{-13}$	$-0.429 \cdot 10^{-14}$	0.08

TABLE 15. APPLICATION OF THE VARIOUS FORMULAS TO ORBIT NO. 2

Problem:
$$\begin{cases} \ddot{x} = -x/(\sqrt{x^2+y^2})^3 \\ \ddot{y} = -y/(\sqrt{x^2+y^2})^3 \end{cases}$$

Initial Values: $t_0 = 0$
$$\begin{cases} x_0 = \frac{1849}{925}, & \dot{x}_0 = 0 \\ y_0 = 0, & \dot{y}_0 = \frac{43}{1849} \end{cases}$$

Results for $t = 2\pi$

Formula	Steps	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	7094 Time (min.)
RK 5(6)-8 [2]	6390	$0.555 \cdot 10^{-11}$	$-0.187 \cdot 10^{-12}$	$0.330 \cdot 10^{-11}$	$-0.645 \cdot 10^{-13}$	0.87
RKN 5(6)-6-x [1]	7363	$0.577 \cdot 10^{-11}$	$-0.172 \cdot 10^{-12}$	$0.343 \cdot 10^{-11}$	$-0.664 \cdot 10^{-13}$	0.57
RKN 5(6)-6-$\dot{x}$	4195	$-0.311 \cdot 10^{-11}$	$0.243 \cdot 10^{-12}$	$-0.182 \cdot 10^{-11}$	$0.364 \cdot 10^{-13}$	0.34
RK 6(7)-10 [2]	3630	$0.979 \cdot 10^{-11}$	$-0.460 \cdot 10^{-12}$	$0.579 \cdot 10^{-11}$	$-0.114 \cdot 10^{-12}$	0.62
RKN 6(7)-7-x [1]	1763	$0.189 \cdot 10^{-12}$	$0.257 \cdot 10^{-12}$	$0.118 \cdot 10^{-12}$	$-0.212 \cdot 10^{-14}$	0.17
RKN 6(7)-8-$\dot{x}$	956	$-0.589 \cdot 10^{-12}$	$0.486 \cdot 10^{-13}$	$-0.343 \cdot 10^{-12}$	$0.627 \cdot 10^{-14}$	0.11
RK 7(8)-13 [2]	1454	$0.159 \cdot 10^{-11}$	$-0.587 \cdot 10^{-13}$	$0.942 \cdot 10^{-12}$	$-0.184 \cdot 10^{-13}$	0.40
RKN 7(8)-9-x [1]	828	$0.243 \cdot 10^{-10}$	$-0.131 \cdot 10^{-11}$	$0.143 \cdot 10^{-10}$	$-0.287 \cdot 10^{-12}$	0.12
RKN 7(8)-10-$\dot{x}$	962	$-0.111 \cdot 10^{-11}$	$0.794 \cdot 10^{-13}$	$-0.652 \cdot 10^{-12}$	$0.129 \cdot 10^{-13}$	0.15
RK 8(9)-17 [2]	791	$-0.424 \cdot 10^{-12}$	$0.387 \cdot 10^{-13}$	$-0.247 \cdot 10^{-12}$	$0.488 \cdot 10^{-14}$	0.25
RKN 8(9)-11-x [1]	503	$-0.179 \cdot 10^{-11}$	$0.172 \cdot 10^{-12}$	$-0.105 \cdot 10^{-11}$	$0.208 \cdot 10^{-13}$	0.09
RKN 8(9)-13-$\dot{x}$	548	$0.352 \cdot 10^{-12}$	$-0.111 \cdot 10^{-13}$	$0.210 \cdot 10^{-12}$	$-0.406 \cdot 10^{-14}$	0.12

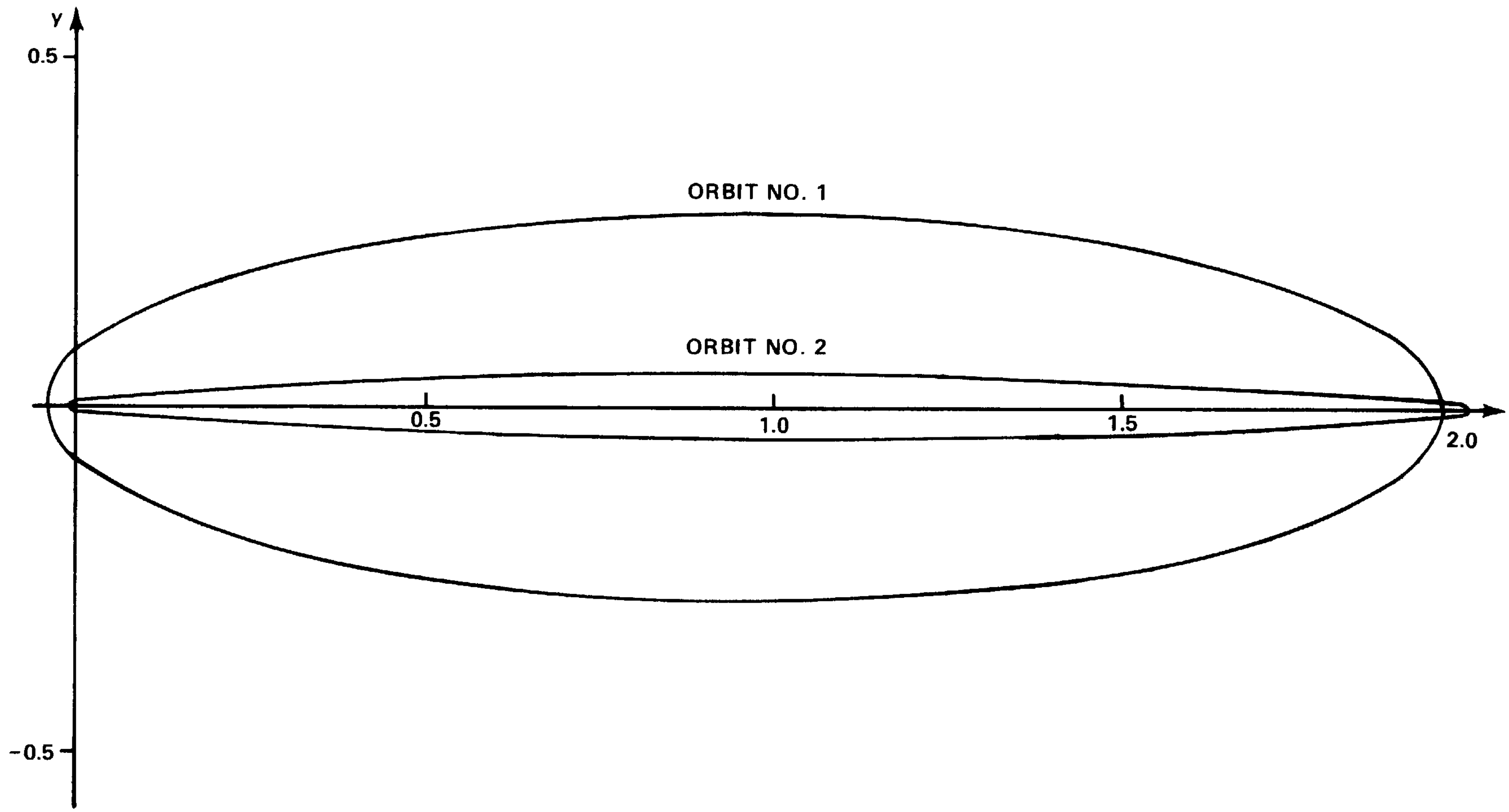

Figure 1. Central orbits of Section V.

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